Limited Memory, Time-varying Expectations and Asset Pricing

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Time-varying Expectation and Limited Memory

Motivation:

- Investors' expectations on *future* return in surveys tend to be *extrapolative*
- Novel fact: the mapping from past observations to expectation is *time-varying*
- But, full information rational expectations (FIRE) models with a unique forward-looking equilibrium preclude such dynamics

Question: *What if we perturb the full memory assumption?*

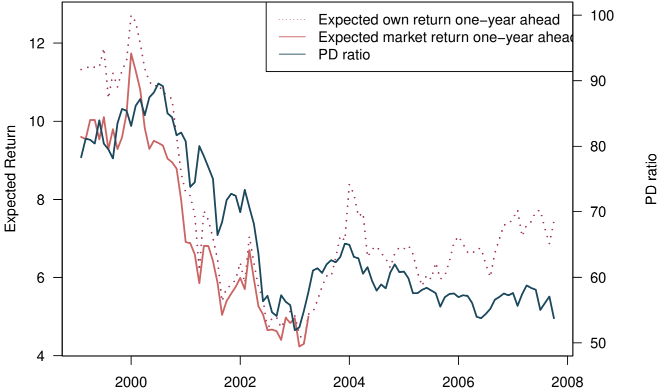
This paper:

A theory of asset pricing based on limited memory and time-varying expectations

$\Rightarrow$ time-varying equity premium and stochastic volatility arise endogenously

Past observations on Price *Matter* for Belief Formation

Figure 1: Price-Dividend (PD) ratio and expected return one year ahead



Data source: UBS/Gallup Survey. The figure contains the cross-sectional average of investors' one-year ahead expected return on the market portfolio and on respondents' own portfolio as well as the actual PD ratio.

Past observations on Price *Matter* for Belief Formation

- A regression of survey expected excess return (one-year ahead) on log PD ratio

$$\hat{\mathbb{E}}_t r_{s,t+1} - r_{f,t} = \beta_0 + \beta_t \log(PD_t) + \varepsilon_t$$

  - In constant parameter model ($\beta_t = \beta$), regress gives $\beta = 0.0269$

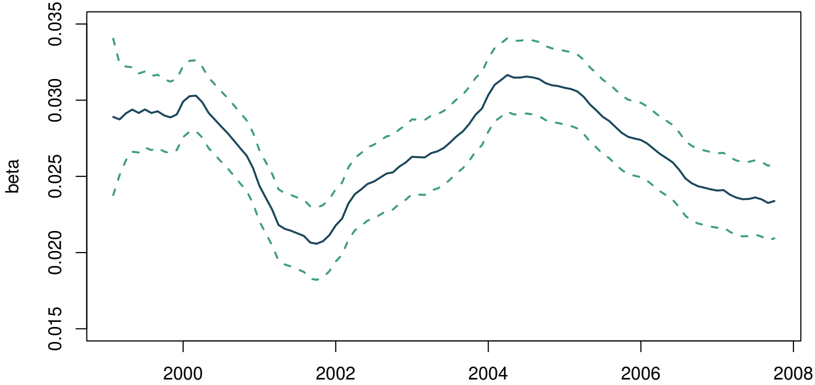
- To test the parameter stability, we augment the standard regression model with

$$\beta_t = \beta_{t-1} + \nu_t$$

where ε_t and $\nu_t \sim$ i.i.d $N(0, \tau^2 G)$ are uncorrelated

The Mapping from Past Observation to Expectations is *Time-varying*

Figure 2: Estimated coefficient β_t over sample period



The Figure plots the coefficient in a TVP regression of survey expected excess return on log PD ratio. Dashed lines show the 95% probability intervals standard error bands for the coefficient.

▸ Fitted Value

A Simple Example

- The forward-looking asset-pricing equation:

$$y_t = \theta \mathbb{E}_t y_{t+1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2)$$

  where $\theta \in (0, 1]$, with ε_t standing for the asset's dividend and y_t for its price

- Muth's (1961) RE: the forecast error $(\eta_t = y_t - \mathbb{E}_{t-1}\, y_t)$ cannot be predictable

$$\mathbb{E}_{t-1}(\eta_t) = 0$$

- The RE requirement is generally not enough to pin down a unique solution as any forecast error of the following form satisfies $\mathbb{E}_{t-1}(\eta_t) = 0$

$$\eta_t = b\varepsilon_t + \zeta_t$$

  - Free parameter b
  - Sunspot disturbance $\zeta_t \sim N(0, \sigma_\zeta^2)$, ζ_t, ε_t uncorrelated

Why Usual FIRE Model Fails?

- The forward-looking asset-pricing equation:

$$y_t = \theta \mathbb{E}_t y_{t+1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2)$$

where $\theta \in (0, 1]$, with ε_t standing for the asset's dividend and y_t for its price

- Under **Full information:** $h^t = \{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$, there exists **a continuum of equilibria satisfying RE** ▸ Derivation

$$y_t = (1 - b) y_t^B + b y_t^F$$

where $b \in \mathbb{R}$ is arbitrary scalar to serve **equilibrium selection**

$$\underbrace{y_t^B \equiv -\sum_{j=1}^{\infty} \frac{1}{\theta^j} \varepsilon_{t-j}}_{\text{backward-looking eq.}} \qquad \text{and} \qquad \underbrace{y_t^F \equiv \sum_{j=0}^{\infty} \theta^j \, \mathbb{E}_t \, \varepsilon_{t+j} = \varepsilon_t}_{\text{forward-looking eq.}}$$

Why Usual FIRE Model Fails?

Full Information: Stability condition $\Rightarrow b = 1 \Rightarrow$ a unique solution $y_t = y_t^F$.

- Histories do **not** matter for current price

$$y_t = y_t^F = \varepsilon_t$$

- Histories do **not** matter for price expectations (assume $\mathbb{E}_t\, \varepsilon_{t+j} = 0, \forall j$)

$$\mathbb{E}_t\, y_{t+1} = (b-1) \sum_{i=1}^{\infty} \left(\frac{1}{b\theta} \right)^i y_{t+1-i} \overset{\textbf{by } b=1}{=\!=\!=} 0$$

What's next: Small perturbations in memory assumption

Multiplicity under Limited Memory

Decay Memory: agents *naively* lose memory of past structural shocks at rate $\lambda < \theta$

- The period t information set of an agent

$$\mathbb{I}_t = \{\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \ldots\}$$

- With decay memory, stability condition cannot pin down a unique solution

Derivation

$$y_t = (1-b)\underbrace{\left(-\sum_{j=1}^{\infty}\left(\frac{\lambda}{\theta}\right)^j \varepsilon_{t-j}\right)}_{\text{bounded backward-looking eq.}} + b\sum_{j=0}^{\infty}\theta^j \mathbb{E}_t\varepsilon_{t+j}$$

  - FIRE corresponds to $\lambda = 1$
  - b is *not* constrained by stability condition, and thus can take any value
    - $\Rightarrow$ Multiplicity of bounded equilibrium

Time-varying Expectation Formation Process

- With decay memory, past observations matter for expectation formation

$$\mathbb{E}_t^* y_{t+1} = (b-1)\sum_{i=1}^{\infty}\left(\frac{1}{\theta b}\right)^i \lambda^{i-1} y_{t+1-i}$$

  - The closer the observation is, the greater its weight
- b defines how agents form their expectations. **How to choose b?**

- A **sunspot shock** to expectation parameter b_t

$$b_t = b_{t-1} + \xi_t$$

with $\xi_t \sim$ i.i.d $N(0, \sigma_b^2)$ being the sunspot shock
  - The sunspot shock captures the fact that how agents combine past data to form their expectations can change over time

▸ Updating of Expectations

Limited Memory and Time-varying Expectation Formation Process

The solution: is randomising among different admissible equilibria

$$y_t = (1 - b_t) \underbrace{\left(- \sum_{j=1}^{\infty} \left(\frac{\lambda}{\theta} \right)^j \varepsilon_{t-j} \right)}_{\text{bounded backward-looking eq.}} + b_t \underbrace{\sum_{j=0}^{\infty} \theta^j \mathbb{E}_t \varepsilon_{t+j}}_{\text{forward-looking eq.}}$$

with

$$b_t = b_{t-1} + \xi_t \qquad \xi_t \sim \text{i.i.d } N(0, \sigma_b^2)$$

Multiplicative sunspot $\Rightarrow$ time-varying parameter solution
$\Rightarrow$ endogenous stochastic volatility

An Economic Model for Asset Markets

Basic Setup: Incorporate limited memory and time-varying expectation in the Bansal and Yaron (2004) long-run risk model

- Representative agent with Epstein-Zin preferences

$$V_t = [(1-\beta)C_t^{1-1/\psi} + \beta(\mathbb{E}_t\, V_{t+1}^{1-\gamma})^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}}$$

γ: coefficient of risk aversion; ψ: intertemporal elasticity of substitution

- The asset pricing equation for any asset i

$$\mathbb{E}_t[\delta^\theta G_{c,t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1$$

where $\theta = \frac{1-\gamma}{1-1/\psi}$; $g_{c,t+1} = log(C_{t+1}/C_t)$; $R_{a,t+1}$ denotes the *unobservable* return on an asset that delivers aggregate consumption as its dividends each period

An Economic Model for Asset Markets

- Long-run risk in consumption/dividend process

$$x_{t+1} = \rho x_t + \varphi_e \sigma e_{t+1}$$
$$g_{c,t+1} = \mu + x_t + \sigma \eta_{t+1}$$
$$g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}$$

  where shocks $e_{t+1}, u_{t+1}, \eta_{t+1} \sim$ i.i.d $N(0,1)$ and independent to each other
- Do not impose stochastic volatility as a priori
- Expectation parameter $b_t = b_{t-1} + \xi_t$, where $\xi_t \sim$ i.i.d $N(0, \sigma_b^2)$ is uncorrelated with all other shocks

Solving the Model

Follows Bansal and Yaron (2004), the solution method involves two steps:

1. Solve the model using the approximation proposed by Campbell and Shiller

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + g_{d,t+1}$$

   - where $z_{m,t} \equiv log(P_t/D_t)$ is the log PD ratio and $\kappa_{1,m} = exp(\bar{z}_m)/(1 + exp(\bar{z}_m)) < 1$

2. Assume relevant state variable for deriving the solution for $z_{m,t}$ are the history of persistent component $\{x_t, \lambda x_{t-1}, \lambda^2 x_{t-2}, \ldots\}$, then apply the undetermined coefficient method ▸ Solve the Model

Price-dividend Ratio

The solution for log PD ratio takes form

$$z_{m,t} = b_t \underbrace{\left(A_{0,m} + \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho} x_t \right)}_{\text{fundamental eq., } z^{RE}_{m,t}} + (1 - b_t) \underbrace{\left(A_{0,m} - \sum_{j=1}^{\infty} \left(\frac{\lambda}{\kappa_{1,m}} \right)^j \left(\phi - \frac{1}{\psi} \right) x_{t-j} \right)}_{\text{bounded backward-looking eq.}}$$

Three key implications:

1. Log PD ratio can deviate from fundamental values (coincide with $b_t \neq 1$).
   - Usual RE solution coincides with $b_t = 1$
   - **Weak correlation** between log PD ratio and consumption growth

Price-dividend Ratio

2. **Persistent** under-and overvaluations of asset prices
   - Define $\hat{z}$ as the deviation from the usual RE solution, i.e.,

$$\hat{z}_{m,t} = z_{m,t} - z_{m,t}^{RE}$$

   - Then for $b_t \neq 1$:

$$\hat{z}_{m,t+1} = \frac{\lambda}{\kappa_{1,m}} \frac{b_{t+1}-1}{b_t - 1} \hat{z}_{m,t} + (b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right) \frac{1}{1-\kappa_{1,m}\rho} \varphi_e \sigma e_{t+1} + \underbrace{(1-\lambda)(b_{t+1}-1)\left(\phi - \frac{1}{\psi}\right) \frac{1}{1-\kappa_{1,m}\rho} \rho x_t}_{\text{Due to memory loss, approach to zero as } \lambda \to 1} .$$

   - $\hat{z}_{m,t+1}$ positively depends on the deviation in the last period when b_t and b_{t+1} on the same side relative to 1

Price-dividend Ratio

3. **Stochastic volatility** arises endogenously
   - The conditional variance of the log PD ratio (assume $\rho = 0$ for simplicity):

$$\mathbb{V}ar_t(z_{m,t+1}) = \left(\frac{z_{m,t} - z_{m,t}^{RE}}{b_t - 1}\right)^2 \sigma_b^2 + \left(\phi - \frac{1}{\psi}\right)^2 (b_t \varphi_e \sigma)^2, \qquad \text{for } b_t \neq 1$$

$$\mathbb{V}ar_t(z_{m,t+1}) = \left(\phi - \frac{1}{\psi}\right)^2 \varphi_e^2 \sigma^2, \qquad \text{for } b_t = 1$$

   - $z_{m,t} - z_{m,t}^{RE} \uparrow \;\Rightarrow \mathbb{V}ar_t(z_{m,t+1}) \uparrow$, price volatility increases in a bubbly market
   - b_t induces time-variation in how σ^2 feed into price-dividend volatility

Equity Premium

The equity premium is

$$\mathbb{E}_t(r_{m,t+1} - r_{f,t}) = \vartheta_{e,t}\sigma^2 + \vartheta_{\xi,t}\sigma_b^2 - 0.5\mathbb{V}ar_t(r_{m,t+1}),$$

with

$$\vartheta_{e,t} = (1-\theta)\left(1 - \frac{1}{\psi}\right)\frac{\kappa_1 b_t}{1-\kappa_1\rho}\left(\phi - \frac{1}{\psi}\right)\frac{\kappa_{1,m} b_t}{1-\kappa_{m,1}\rho}\varphi_e^2\sigma^2,$$

$$\vartheta_{\xi,t} = \frac{\vartheta_{e,t}}{b_t^2\varphi_e^2\sigma^2}\left[(1-\kappa_1\rho)\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_1})^j x_{t-j} + x_t\right]\left[(1-\kappa_{m,1})\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j x_{t-j} + x_t\right]$$

$$\mathbb{V}ar_t(r_{m,t+1}) = (\beta_{m,u} + \beta_{m,e,t+1})^2\sigma^2 + \beta_{m,\xi,t+1}^2\sigma_b^2$$

- One process for expectation formation affects both the **slope** and **intercept** of the equation for the equity premium

Quantitative Analysis

- Data
  - Sample period: 1929 to 2019
  - Real S&P 500 stock returns and dividends from Robert Shiller's website
  - Nominal return to one-month Treasury bills from CRSP
  - Consumption data is from Barro and Ursua (2012) and was extended to 2019 using data from BEA
  - All nominal terms deflated by CPI
  - Agents make decisions on a monthly basis. We compute moments at an annual frequency
- We estimate the parameters of our model using the Simulated Method of Moments SMMdetails

Parameter estimates

Table 1: Parameter Values

Parameter	Estimated value	Parameter	Estimated value
γ	3.9015	μ	0.0016
	(0.1551)		(0.0001)
δ	0.9961	ϕ	2.5344
	(0.0003)		(0.1851)
ψ	1.1148	φ_d	6.2188
	(0.0085)		(0.5413)
ρ	0.9915	σ_b	0.0245
	(0.0020)		(0.0051)
φ_e	0.0788	λ	0.9419
	(0.0121)		(0.0016)
σ	0.0040		
	(0.0002)		

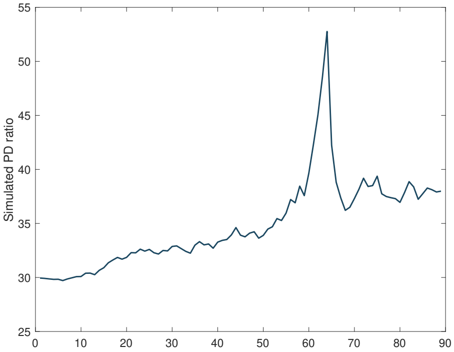
Quantitative Model Performance

	U.S. Data		Decay Memory		BY (with stochastic vol)	
	Data Moment	Std. Dev.	Model Moment	t-Stat.	Model Moment	t-Stat.
Mean stock return E_{rs}	7.79	1.83	6.27	0.83	5.31	1.43
Mean bond return E_{rb}	0.45	0.49	1.05	-1.20	-1.16	2.11
Mean PD ratio E_{PD}	31.42	1.43	34.80	-1.91	35.60	-0.71
Mean dividend growth $E_{\Delta D/D}$	1.74	1.12	2.56	-0.73	3.20	-1.82
Std. dev. stock return σ_{rs}	18.71	0.94	19.26	-0.56	14.64	2.23
Std. dev. PD ratio σ_{PD}	16.07	2.05	17.91	-0.69	3.24	3.42
Std. dev. Dividend Growth $\sigma_{\Delta D/D}$	10.67	1.60	11.08	-0.25	12.76	-0.79
Std. dev. bond return σ_{rb}	3.76	0.43	3.28	1.46	1.25	3.59
Autocorrel. PD ratio $\rho_{PD,-1}$	0.91	0.12	0.80	0.74	0.17	7.66
Mean consumption growth $E_{\Delta C/C}$	2.01	0.32	2.00	0.03	2.50	-1.44
Std. dev. consumption growth $\sigma_{\Delta C/C}$	2.93	0.32	2.93	0.88	2.70	0.42
Autocorrel. consumption growth $\rho_{\Delta C/C,-1}$	0.61	0.12	0.62	0.00	0.14	0.31
Autocorrel. dividend growth $\rho_{\Delta D/D,-1}$	0.24	0.37	0.79	-0.08	0.03	0.45
Corr. $corr_{\Delta C/C,\Delta D/D}$	0.47	0.13	0.50	-0.25	0.18	1.96
Predictability β_{PD}	-0.0110	0.0003	- 0.0090	-0.70	-0.0145	0.9524
Predictability R^2	0.1327	0.086	0.0756	0.66	0.0348	1.0622
Contemporaneous correlation between stock return and consumption growth	0.03	0.11	0.24	-1.88	0.14	-0.95
Correlation between stock return and one-period lag consumption growth	-0.13	0.27	0.10	-0.45	0.14	-0.95
Test statistic $\hat{W}_N$			7.7713		558	
p-value of $\hat{W}_N$			16.93%		0%	

Simulated Price-Dividend Ratio

The simulated time series can generate booms and busts as observed in the data

Figure 3: Simulated PD ratio using the estimated model



PD Negatively Predicts *Actual* Excess Returns

Regress real excess returns on equity over holding periods of one, three, and five years on the lagged price-dividend ratio, that is

$$r_{s,t+n} - r_{f,t} = c_n^1 + c_n^2 \log(PD_t) + u_{t,n}$$

- PD **negatively** predicts **actual** excess market returns

Table 2: Predictability of excess returns

	Slope coefficient			R^2	
	Data	Decay Memory		Data	Decay Memory
c_1^2	-0.0022	-0.0019	R_1^2	0.0391	0.0235
	(0.0010)	(-0.36)		(0.0375)	(0.42)
c_3^2	-0.0062	-0.0054	R_3^2	0.0890	0.0553
	(0.0025)	(-0.34)		(0.0872)	(0.52)
c_5^2	-0.0110	-0.0090	R_5^2	0.1327	0.0756
	(0.0034)	(-0.58)		(0.0872)	(0.66)

PD Positively Predicts *Expected* Excess Returns

How do survey return expectations relate to cycles in asset prices?

$$\hat{\mathbb{E}}_t r_{s,t+1} - r_{f,t} = \beta_0 + \beta_t \log(PD_t) + \varepsilon_t$$

- PD **positively** predicts **expected** excess market returns

Table 3: Survey Return Expectations and PD ratio

	Data Moment		Model		
	Estimate	(SE)	Mean	5%	95%
$\log(PD_t)$	0.0269	(0.009)	0.0240	0.0112	0.0368
R^2	0.08		0.22		

Data source: UBS/Gallup Survey.

Comparison of Alternative Models

- Comparison with the **long-run risk model** in Bansal and Yaron (2004):
  - BY: an *exogenous* AR(1) process for stochastic volatility:

$$\sigma_{t+1}^2 = \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}$$

  - This paper: *endogenous* stochastic volatility and time-varying equity premium
  - The quantitative performance of our model outperforms BY.
- Comparison with the **learning model** in Adam et al. (2016):
  - Learning model: Simple version of Lucas (1978) model, CRRA investors learn about price behaviour from past price observations
  - This paper: replicates a host of asset pricing moments without generating strong correlation between consumption and price

Conclusion

- We propose a novel mechanism for asset pricing models based on two features: (i) limited memory; (ii) time-varying expectations.
- Time-varying equity premium and stochastic volatility arise endogenously
- The model quantitatively replicates a host of asset-pricing features ...
  - Including equity premium, excessive volatility, persistence of price-dividend ratio, predictability of excess returns and the consumption correlation puzzle.
- ... as well as the positive correlation between PD ratio and return expectation

The Mapping from Past Observation to Expectations is *Time-varying*

Figure 4: Actual and fitted expected excess return

Derivation

Consider the following expectational difference equation

$$y_t = \theta \, \mathbb{E}_t \, y_{t+1} + \epsilon_t, \quad \epsilon_t \sim \text{i.i.d. } N(0, \sigma_\epsilon^2)$$

Muth (1961) original formulation states that the RE solution should be a function of all the past, present and expected future structural shocks

$$\sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_t \, \varepsilon_{t+j} =$$

$$\theta \, \mathbb{E}_t (\sum_{j=1}^{\infty} u_j \varepsilon_{t+1-j} + b\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_{t+1} \, \varepsilon_{t+1+j}) + \varepsilon_t$$

where u_j, c_j and b are coefficients to be determined.

Derivation

$$\sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_t \, \varepsilon_{t+j} =$$

$$\theta \, \mathbb{E}_t\Big(\sum_{j=1}^{\infty} u_j \varepsilon_{t+1-j} + b\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_{t+1} \, \varepsilon_{t+1+j}\Big) + \varepsilon_t$$

Equal coefficients of ε_{t-j} gives the expression for u's:

$$\begin{aligned}
\varepsilon_t : &\quad b = \theta u_1 + 1 \Rightarrow u_1 = \frac{1}{\theta}(b-1); \\
\varepsilon_{t-1} : &\quad u_1 = \theta u_2 \Rightarrow u_2 = \frac{1}{\theta} u_1; \\
&\vdots \\
\varepsilon_{t-T} : &\quad u_T = \theta u_{T+1} \Rightarrow u_{T+1} = \frac{1}{\theta} u_T; \\
&\vdots
\end{aligned}$$

Derivation

$$\sum_{j=1}^{\infty} u_j \varepsilon_{t-j} + b\varepsilon_t + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_t \, \varepsilon_{t+j} =$$

$$\theta \, \mathbb{E}_t \Big(\sum_{j=1}^{\infty} u_j \varepsilon_{t+1-j} + b\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_j \, \mathbb{E}_{t+1} \, \varepsilon_{t+1+j} \Big) + \varepsilon_t$$

and solve for c's:

$$\begin{aligned}
\varepsilon_{t+1} &: \quad c_1 = \theta b \\
\varepsilon_{t+2} &: \quad c_2 = \theta c_1 \\
&\vdots \\
\varepsilon_{t+T} &: \quad c_T = \theta c_{T-1} \\
&\vdots
\end{aligned}$$

Derivation under Decay Memory

Under decay memory, assume the solution has the following form

$$y_t = \sum_{j=1}^{\infty} u_{j,t} \lambda^j \varepsilon_{t-j} + b_t \varepsilon_t + \sum_{j=1}^{\infty} c_{j,t} \, \mathbb{E}_t \, \varepsilon_{t+j}$$

At time t, the information set of the agent is given by $\mathcal{I}_t = \{\varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \dots\}$. Based on this information set, she forms her expectations

$$\begin{aligned}
\mathbb{E}_t \, y_{t+1} &= \mathbb{E}(y_{t+1} | \mathcal{I}_t) = \mathbb{E}(y_{t+1} | \varepsilon_t, \lambda\varepsilon_{t-1}, \lambda^2\varepsilon_{t-2}, \dots) \\
&= \mathbb{E}_t \left(\sum_{j=1}^{\infty} u_{j,t+1} \lambda^{j-1} \varepsilon_{t+1-j} + b_{t+1} \varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1} \, \mathbb{E}_{t+1} \, \varepsilon_{t+1+j} \right)
\end{aligned}$$

Derivation under Decay Memory

Substitute for y_t and $\mathbb{E}_t\, y_{t+1}$ in the expectational difference equation,

$$\sum_{j=1}^{\infty} u_{j,t}\lambda^j \varepsilon_{t-j} + b_t\varepsilon_t + \sum_{j=1}^{\infty} c_{j,t}\, \mathbb{E}_t\, \varepsilon_{t+j} =$$

$$\theta\, \mathbb{E}_t(\sum_{j=1}^{\infty} u_{j,t+1}\lambda^{j-1}\varepsilon_{t+1-j} + b_{t+1}\varepsilon_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1}\, \mathbb{E}_{t+1}\, \varepsilon_{t+1+j}) + \varepsilon_t$$

Derivation under Decay Memory

Equal coefficients to find an expression for the u's:

$$\begin{aligned}
\varepsilon_t: &\quad b_t = \theta\,\mathbb{E}_t\, u_{1,t+1} + 1 \Rightarrow \mathbb{E}_t\, u_{1,t+1} = \frac{1}{\theta}(b_t - 1);\\
\varepsilon_{t-1}: &\quad \lambda u_{1,t} = \theta\lambda\,\mathbb{E}_t\, u_{2,t+1} \Rightarrow \mathbb{E}_t\, u_{2,t+1} = \frac{1}{\theta} u_{1,t};\\
\vdots &
\end{aligned}$$

and for the c's:

$$\begin{aligned}
\varepsilon_{t+1}: &\quad c_{1,t} = \theta\,\mathbb{E}_t\, b_{t+1}\\
\varepsilon_{t+2}: &\quad c_{2,t} = \theta\,\mathbb{E}_t\, c_{1,t+1}\\
\vdots &
\end{aligned}$$

Derivation under Decay Memory

For constant $b_t = b$, the coefficient for ε_{t-j}, $\forall j$ is $u_j = (b-1)\left(\frac{\lambda}{\theta}\right)^j$, and the coefficient for $\mathbb{E}_t\,\varepsilon_{t+j}$, $\forall j$ is $c_j = b\theta^j$.

$$y_t = (b-1)\sum_{j=1}^{\infty}(\frac{\lambda}{\theta})^j\varepsilon_{t-j} + b\varepsilon_t + b\sum_{j=1}^{\infty}\theta^j\mathbb{E}_t\varepsilon_{t+j}$$

For $b_t = b_{t-1} + \sigma_b\xi_t$ follows a random walk process, the solution is

$$y_t = (b_t-1)\sum_{j=1}^{\infty}(\frac{\lambda}{\theta})^j\varepsilon_{t-j} + b_t\varepsilon_t + b_t\sum_{j=1}^{\infty}\theta^j\mathbb{E}_t\varepsilon_{t+j}$$

Back

How Price Expectations are Updated?

The expectation in the decay memory case with time-varying b_t is

$$\bar{\mathbb{E}}_t y_{t+1} = (b_t - 1) \sum_{i=1}^{\infty} \left(\frac{\lambda^{i-1}}{\theta^i \prod_{j=0}^{i-1} b_{t-j}} \right) y_{t+1-i},$$

- It can be written recursively as

$$\mathbb{E}_t^* y_{t+1} = \frac{1}{\theta} \left[\frac{\nu_t}{\nu_{t-1}} \lambda \bar{\mathbb{E}}_{t-1} y_t + \nu_t \left(y_t - \lambda \bar{\mathbb{E}}_{t-1} y_t \right) \right]$$

  where $\nu_t = \frac{b_t - 1}{b_t}$ is the gain parameter.

- This expression reminds the updating implied by constant gain learning, employed by Adam et al. (2016) and Nagel and Xu (2021)

$$\bar{\mathbb{E}}_t y_{t+1} = \bar{\mathbb{E}}_{t-1} y_t + \nu \left(y_t - \bar{\mathbb{E}}_{t-1} y_t \right),$$

  where ν is the gain parameter.

Solving the Model

- Follows Campbell and Shiller (1988), the (approximated) log return on the wealth portfolio can by written as

$$r_{m,t+1} = \kappa_0 + \kappa_1 z_{m,t+1} - z_{m,t} + g_{d,t+1}$$

where $z_t \equiv log(P_t/D_t)$ is the log PD ratio and $\kappa_{1,m} = exp(\bar{z}_m)/(1 + exp(\bar{z}_m))$

- Assume relevant state variable for deriving the solution for $z_{m,t}$ are the history of persistent component $\{x_t, \lambda x_{t-1}, \lambda^2 x_{t-2}\}$, then

$$z_{m,t} = A_{0,m,t} + \left(\phi - \frac{1}{\psi}\right)\left(\sum_{j=1}^{\infty} u_{j,t}\lambda^j x_{t-j} + b_t x_t + \sum_{j=1}^{\infty} c_{j,t}\mathbb{E}_t x_{t+j}\right)$$

- Plug the approximation into the (log form) Euler equation

$$E_t\left[\exp\left(\theta\log(\delta) - \frac{\theta}{\psi}g_{c,t+1} + \theta r_{a,t+1}\right)\right] = 1$$

Solving the Model

- Guess and verify gives the equilibrium solution for $\log(P_t/D_t) \equiv z_{m,t}$:

$$z_{m,t} = A_{0,m,t} + (\phi - \frac{1}{\psi})\left[\sum_{j=1}^{\infty}(\frac{\lambda}{\kappa_{1,m}})^j(b_t - 1)x_{t-j} + b_t x_t + b_t \sum_{j=1}^{\infty}(\kappa_{1,m}\rho)^j x_t\right]$$

  - $\kappa_{1,m} < 1$, determines the strength of extrapolation. Model

Estimated parameters

- The parameter vector θ, includes the 11 parameters:
  - γ: coefficient of relative risk aversion;
  - ψ: elasticity of intertemporal substitution;
  - δ: rate of time preference;
  - μ: drift in the log consumption growth and log dividend growth;
  - ρ: persistence of expected growth rate process;
  - σ: volatility of innovation;
  - φ_e: captures the volatility of the persistent component;
  - ϕ: calibrate the correlation between consumption and dividend;
  - φ_d: captures the volatility of dividend;
  - σ_b: the volatility of innovation in expectation formation process;
  - λ: the decay rate of memory

Moments of interests

- Moments of interests:
  - Consumption growth: mean, standard deviations, and first-order autocorrelation.
  - Dividend growth: mean, standard deviations, and first-order autocorrelation.
  - Correlation between growth rate of dividends and growth rate of consumption.
  - Real stock returns: mean and standard deviations.
  - Price-dividend ratio: mean, standard deviations, and persistence.
  - Risk free rate: mean, standard deviations.
  - Excess return predictability: coefficient c^2 and R^2 in the regression

$$r_{s,t,t+n} - r_{f,t,t+n} = c_n^1 + c_n^2 \log(PD_t) + u_{t,n}$$

  - Correlation between stock returns and consumption growth.
  - Correlation between stock returns and one-period lagged consumption growth.

Which Moments to Match?

- Including all the moments listed above may violate the non-singularity of the covariance matrix and result in the estimation to vary greatly with small changes in the model or testing procedure
  - see Adda and Cooper (2003) and Davidson et al. (2004)
- We compute the variability of each statistic that cannot be explained by a linear combination of the remaining statistics, similarly to the R^2 coefficient of regression of each statistic on all the other statistics
- Test suggests to exclude *the coefficient of the excess return regression* and *the autocorrelation of consumption growth* ◂ SMM