Inefficiency of Random Serial Dictatorship under Incomplete Information

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Introduction

- Several allocation problems: impossible or impractical to use monetary transfers
- Allocating
  - students to public schools
  - course seats to students,
  - offices to faculty members
  - tasks to team members,

Introduction

- Allocating n indivisible goods to n agents in the absence of transfers.
- Each agent can get at most one object.
- **Incomplete information**: Each agent has private information regarding their preferences (cardinal values) over objects.
- Welfare comparison

Introduction

- One of the most popular methods: Random Serial Dictatorship (RSD) (Sometimes referred to as Random Priority)
- An order over agents is randomly determined.
- Following this order, each agent is assigned his favorite object among the available ones.

Introduction

- Incentives: RSD is strategy-proof.
- However, RSD may be inefficient:
- Bogomolnaia and Moulin (2001): Example in which another random allocation is unambiguously better than what RSD induces.
- Manea (2009): Such inefficiency is prevalent in large allocation problems.
- **Our main result:** Exhibit inefficiency of RSD by finding another method that *dominates* RSD under *incomplete information*.

Introduction

- Another method: Random Boston mechanism (RB) (with random tie breaking)—adapted from Boston mechanism known in school-choice literature.
- Each agent reports an ordinal ranking over the objects.
- Rank based: Allocate the object to the agent with the highest ranking for the object (randomly when necessary).

The Random Boston mechanism

- Each agent reports a ranking over objects and the following algorithm is performed:
- **Step 1:** Each object is allocated to an agent who ranks it as a *first* choice, randomly if necessary.
- **Step 2:** Each unassigned object is allocated to an agent who ranks it as a *second* choice, randomly if necessary.
- Stop when all objects are allocated.

Random Boston Mechanism

- Boston tries to give agents their first choice.
- What if an agent fails to get her first choice?
- Her later choices may already be assigned!
- Risk in ranking an object first if the chance of obtaining is low $\Longrightarrow$Open to strategic manipulation

RB vs RSD

- RSD has the advantage of strategy-proofness whereas Boston mechanism is manipulable (Abdulkadiroglu and Sonmez (2003)).
- But, how about welfare?

Symmetric Model

- *Incomplete* information regarding agents' preferences.
- Agents' preferences: Ex-ante uncorrelated.
- Random market

Main Result

n objects, n agents.

Theorem

When n is large enough, every agent, regardless of his preferences, has a strictly higher expected utility under the Random Boston mechanism than that under RSD (under some regularity conditions). This **strict** *dominance hold even in the limit as $n \to \infty$.*

Literature

- Relatively recent studies on welfare comparison of different assignment rules.

- School Choice: Deferred Acceptance (DA) vs Boston:
  - Miralles (2009), Abdulkadiroğlu, Che and Yasuda (2011), Troyan (2012) (perfectly correlated preferences)
  - Featherstone and Niederle (2016) (experimental, some theoretical results with ex-ante uncorrelated preferences), Akyol (2022) (3 school case, ex-ante uncorrelated preferences)

- **Random markets**: Pittel (1989), Knuth (1996), Roth and Rothblum(1999), Ehlers (2008), Ashlagi et al. (2017), Ashlagi and Nikzad (2020)

- Che and Tercieux (2018): Pareto efficient mechanisms are asymptotically payoff equivalent in large markets (applies to random markets considered here as well).

Model

- $n \geq 2$ agents, $\{i_1, ..., i_n\}$, $n \geq 2$ objects, $\{o_1, ..., o_n\}$
- Each agent i's valuation vector $\mathbf{v}^i = \left(v_j^i\right)_{j=1}^n$ is independently drawn from an *exchangeable* cumulative distribution function F over

$$V \subset \left\{\mathbf{v} = (v_j)_{j=1}^n \in [\underline{v}, \overline{v}]^n : v_j \neq v_k \text{ for any } j \neq k\right\}$$

- F is invariant under the permutations of its arguments so that $F(\mathbf{v}) = F(\mathbf{z})$ whenever $\mathbf{z}$ is a permutation of $\mathbf{v}$

$\Longrightarrow$ Each agent's ranking over objects is independently and uniformly drawn at random from the set of all possible orders over objects.

Induced Games

- Agents privately observe their types and submit a ranking over objects (may or may not be the true ranking).
- The corresponding mechanism is implemented.

Incentive Properties

- RSD is strategy-proof. (well-known in the literature.)
- In general, truthful reporting may not be an equilibrium under the Boston mechanism.

Symmetry: Truthtelling Equilibrium

Proposition: Truth-telling is a (Bayes-Nash) equilibrium under the Random Boston mechanism in our setting.

(Adapted from Featherstone and Niederle (2016))

Welfare Criteria

- Let $\left(P_k^n\right)^X$ is the interim probability that an agent receives their k^{th} choice under mechanism X.
- Any agent with type $\mathbf{v} = (v_j)_{j=1}^n$, (without loss say, $v_1 > v_2 > ... > v_n$) the *interim* expected payoff of this agent under mechanism $X \in \{\text{RSD, RB}\}$ is just

$$U^X(\mathbf{v}) = \sum_{k=1}^{n} \left(P_k^n\right)^X v_k,$$

- Mechanism X (strictly) *interim dominates* mechanism Y if the interim utility of any type of student is (strictly) higher under X than under Y.

Interim Probabilities

Lemma

For any $K \in \{1, 2, \dots\}$, *we have*

$$\lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{RB} > \lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{RSD} .$$

- As $n \to \infty$, $(P_k^n)^{RSD} \to \frac{1}{k(k+1)}$: $(P_1^n)^{RSD} \to \frac{1}{2}$, $(P_2^n)^{RSD} \to \frac{1}{6}$, $(P_3^n)^{RSD} \to \frac{1}{12}, \dots$ ▸ probRSD
- As $n \to \infty$, $(P_1^n)^{RB} \to 1 - \frac{1}{e} \approx 0.632\,12$,
  $(P_2^n)^{RB} \to \frac{1}{e}\left(1 - \frac{1}{e^{\frac{1}{e}}}\right) \approx 0.113\,23$,
  $(P_3^n)^{RB} \to \frac{1}{e}\frac{1}{e^{\frac{1}{e}}}\left(1 - \frac{1}{e^{\frac{1}{e}\frac{1}{e^{\frac{1}{e}}}}}\right) \approx 0.057247, \dots$
  (By using techniques from "occupancy problems") ▸ probRB

Main Result

Let V^n be the associated type space with market size n and consider a sequence of allocation problems with type spaces (V^n).

Assumption (A1). (Non-technical statement) There is *some* $k \geq 1$ such that the (expected) value difference between the k^{th} choice and the $(k+1)^{th}$ choice does *not* vanish even in the limit.

Example

Assume that for any n, V^n consists of all the permutations of $\left(1, \frac{1}{2n}, \frac{1}{3n}, \dots, \frac{1}{n^2}\right)$.

Example

Assume that for any n, V^n consists of all the permutations of $(1, 0, 0, \dots, 0)$

Main Result

Consider a sequence of allocation problems represented by (V^n, F^n), where each agent's valuation vector is independently drawn from an exchangeable cumulative distribution function F^n over V^n. Assume also that A1 holds.

Theorem

For sufficiently large n, the Random Boston mechanism **strictly** *interim dominates the Random Serial Dictatorship mechanism. Furthermore, this strict dominance holds even in the limit.*

Example

Example

Assume that for any n, V^n consists of all the permutations of $(1, 0, 0, ..., 0)$. For any $\mathbf{v} \in V^n$

$$U^{RSD}(\mathbf{v}) = \frac{n+1}{2n}$$

and

$$U^{RB}(\mathbf{v}) = 1 - \left(\frac{n-1}{n}\right)^n$$

$$1 - \left(\frac{n-1}{n}\right)^n > \frac{n+1}{2n} \text{ for any } n \geq 3$$

and as $n \to \infty$,

$$U^{RB}(\mathbf{v}) \to 1 - \frac{1}{e} \approx 0.632\,12, \; U^{RSD}(\mathbf{v}) \to \frac{1}{2}.$$

Conclusion

- In a symmetric setting with private information regarding preferences:
- Random Boston mechanism outperforms RSD in terms of welfare when preferences are ex-ante uncorrelated in a large market.

RSD Probabilities

Assume that there are n objects and n agents. For any $k \in \{1, ..., n\}$,

$$\left(P_k^n\right)^{RSD} = \left(\frac{n+1}{n}\right)\frac{1}{k\left(k+1\right)},$$

and hence for any $K \in \{1, 2, ...\}$,

$$\sum_{k=1}^{K}\left(P_k^n\right)^{RSD} = \left(\frac{n+1}{n}\right)\left(1-\frac{1}{K+1}\right),$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{K}\left(P_k^n\right)^{RSD} = 1-\frac{1}{K+1}.$$

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RSD Probabilities

- Random Serial Dictatorship (RSD)
- $$P_k^n = \frac{(n+1)}{k\,(k+1)\,n}$$
- Recursive formulation:

$$P_1^n = \underbrace{\frac{1}{n}}_{\substack{\text{chosen}\\ \text{as first}}} + \underbrace{\frac{n-1}{n}}_{\substack{\text{not chosen}\\ \text{as first}}} \left(\underbrace{\frac{n-1}{n}}_{\substack{\text{first picker's first}\\ \text{choice is different}}} P_1^{n-1} \right)$$

and for $k \geq 2$

$$P_k^n = \underbrace{\frac{n-1}{n}}_{\substack{\text{not chosen}\\ \text{as first}}} \left[\underbrace{\frac{k-1}{n}}_{\substack{\text{first picker's first}\\ \text{choice} \in \{1,..,(k-1)\}}} P_{k-1}^{n-1} + \underbrace{\frac{n-k}{n}}_{\substack{\text{first picker's first}\\ \text{choice } \in \{(k+1),...,n\}}} P_k^{n-1} \right]$$

RSD Probabilities (Continued)

- For $k = 1$, we claim

$$P_1^n = \frac{(n+1)}{k(k+1)n} = \frac{(n+1)}{2n}$$

- $$P_1^n = \frac{1}{n} + \frac{n-1}{n}\left(\frac{n-1}{n} P_1^{n-1}\right)$$

- Induction on n. Now, $P_1^1 = 1$. If true for $(n-1)$, true for n :

$$\begin{aligned} P_1^n &= \frac{1}{n} + \frac{n-1}{n}\left(\frac{n-1}{n} P_1^{n-1}\right) \\ &= \frac{1}{n} + \frac{n-1}{n}\left(\frac{n-1}{n} \frac{n}{2(n-1)}\right) \\ &= \frac{1}{n} + \frac{n-1}{2n} = \frac{n+1}{2n} \end{aligned}$$

RSD Probabilities (Continued)

- For $k \geq 2$, we claim
$$P_k^n = \frac{(n+1)}{k(k+1)n}$$
-
$$P_k^n = \frac{n-1}{n}\left[\frac{k-1}{n}P_{k-1}^{n-1} + \frac{n-k}{n}P_k^{n-1}\right]$$
- If true for $(n-1)$, true for n:

$$\begin{aligned} P_k^n &= \frac{n-1}{n}\left[\frac{k-1}{n}P_{k-1}^{n-1} + \frac{n-k}{n}P_k^{n-1}\right] \\ &= \frac{n-1}{n}\left[\frac{k-1}{n}\frac{n}{k(k-1)(n-1)} + \frac{n-k}{n}\frac{n}{k(k+1)(n-1)}\right] \\ &= \frac{n-1}{n}\left[\frac{1}{k(n-1)} + \frac{n-k}{k(k+1)(n-1)}\right] \\ &= \frac{(n+1)}{k(k+1)n} \end{aligned}$$

RSD Probabilities (Continued)

- For $k = 2$, we claim for $n \geq 2$

$$P_2^n = \frac{n+1}{6n}$$

- Note that $P_2^2 = 1 - P_1^2 = \frac{1}{4} \left(= \frac{2+1}{6*2}\right)$. Hence, by induction, we have the result.
- We next claim that for $n \geq 3$

$$P_3^n = \frac{n+1}{12n}$$

$P_3^3 = 1 - P_1^3 - P_2^3 = 1 - \frac{2}{3} - \frac{2}{9} = \frac{1}{9} \left(= \frac{3+1}{12*3}\right)$ and again by induction, we have the result.

RSD Probabilities (Continued)

- Continuing in this manner, for a general $k \geq 2$, we claim that for all $n \geq k$

$$P_k^n = \frac{n+1}{k\,(k+1)\,n}$$

Now,

$$\begin{aligned} P_k^k &= 1 - \sum_{j=1}^{k-1} P_j^k = 1 - \sum_{j=1}^{k-1} \frac{k+1}{j\,(j+1)\,k} \\ &= 1 - \frac{k+1}{k} \sum_{j=1}^{k-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \\ &= 1 - \frac{k+1}{k} \left(\frac{k-1}{k} \right) = \frac{1}{k^2} \left(= \frac{k+1}{(k+1) * k * k} \right) \end{aligned}$$

and hence by induction we have that $P_k^n = \frac{(n+1)}{k(k+1)n}$

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RB Probabilities

- Let $\alpha_0 = 0$, $\alpha_1 = 1$ and for any $k \in \{1, 2, \ldots\}$,
$$\alpha_{k+1} = \alpha_k e^{-\alpha_k},$$
where e is the base of the natural logarithm, and approximately equal to 2.71828.
Furthermore, for any $k \in \{0, 1, \ldots\}$, define
$$q_k = e^{-\alpha_k}.$$
- Assume that there are n objects and n agents. For any $K \in \{1, 2, \ldots\}$,
$$\lim_{n\to\infty} \left(P_K^n\right)^{RB} = \left(\prod_{k=0}^{K-1} q_k\right)(1 - q_K),$$
and
$$\lim_{n\to\infty} \sum_{k=1}^{K} \left(P_k^n\right)^{RB} = 1 - \left(\prod_{k=1}^{K} q_k\right),$$
for any $k \in \{0, 1, \ldots\}$.

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RB Probabilities (Continued)

- Consider step 1 of RB.
- For any object o_j, let $A^n(j)$ denote the event that *no agent* ranks o_j as a first choice. Define

$$I^n(j) = \begin{cases} 1 & \text{if } A^n(j) \text{ happens} \\ 0 & \text{otherwise} \end{cases}$$

- Let X^n denote the number of objects that no agent ranks as a first choice. Hence,

$$X^n = \sum_{j=1}^{n} I^n(j).$$

- Given the ex-ante symmetry of the agents, the probability that an agent is *not* assigned an object in step 1 is just $E\left(\frac{X^n}{n}\right)$ since there are n agents that are ex-ante symmetric, and X^n of them are unassigned. Thus, the probability that an agent is assigned an object at step 1 is just $1 - E\left(\frac{X^n}{n}\right)$.

RB Probabilities (Continued)

- The probability that an agent does *not* rank o_j as a first choice is $1 - \frac{1}{n}$. Therefore, we have

$$E\left[I^n(j)\right] = \Pr\left(A^n(j)\right) = \left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n.$$

Then, due to the linearity of expectation,

$$E\left(X^n\right) = n\left(\frac{n-1}{n}\right)^n$$

and hence

$$E\left(\frac{X_n}{n}\right) = \frac{1}{n}E\left(X^n\right) = \left(\frac{n-1}{n}\right)^n.$$

Thus, we have

$$\left(P_1^n\right)^{RB} = 1 - \left(\frac{n-1}{n}\right)^n,$$

and as $n \longrightarrow \infty$

$$\left(P_1^n\right)^{RB} \longrightarrow 1 - e^{-1}.$$

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