Optimal Monetary Policy in HANK *

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April 2022

Preliminary and incomplete

Abstract

In this project, I study optimal monetary policy in a New Keynesian model with heterogeneous households. I use the continuous-time formulation and the numerical techniques from Kaplan et al. (2018) and expand the dynamic programming approach to the optimal policy proposed by Dixit et al. (1994) and further developed by Nuño and Thomas (2016). I show that occasionally binding borrowing constraints significantly change the way Ramsey planner uses monetary policy relative to the economy without borrowing constraints. I find that both in response to contractionary TFP shock and shock that reduces the desired firms' markup the wage/dividends margin is no longer as important. Instead Ramsey planner finds it optimal to reduce the real interest rate in order to relax the borrowing constraint at least for some households. Relaxing the borrowing constraint by the means of monetary policy becomes the primary driver of the optimal response to a given negative shock to the economy regardless of the nature of the shock.

Keywords: heterogeneous agents, borrowing constraint, New Keynesian models.

JEL Codes: E12, E21, E24, E43, E52, E61, G51.

*I am grateful to my advisor Davide Debortoli for his invaluable support. I particularly thank Jordi Gali, Albert Marcet, Edouard Schaal, Donghai Zhang, as well as participants of CREI Macro Lunch. All errors are my own.

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Introduction

In this paper I study the optimal monetary policy in a New Keynesian model with heterogeneous households that insure against idiosyncratic risk by holding tradable nominal bonds. In such a setup in response to the unanticipated shocks, the Ramsey planner chooses a monetary policy that is substantially different from the representative household case. Moreover, I show that the presence of the occasionally binding borrowing limit substantially changes the optimal policy as well. Specifically, the deciding force that substantially affects the behavior of the Ramsey planner is the desire to relax the borrowing constraint for at least some of the households, following the contractionary shock.

Presence of the constrained households was shown to have a significant impact on the propagation of the shocks through the economy, for example in the papers by Algan and Ragot (2010) andDebortoli et al. (2017). I contribute to this literature by emphasizing that this channel also has a substantial effect on the optimal policy. In the model calibrated to the US economy this margin proves to be more important than all of the other channels. Specifically when looking at the optimal real interest rate in response to the negative TFP shock and negative shock to desired markup, the behavior is opposite to the those in the the representative agent (RA) case and substantially different with respect to the model with natural borrowing limit which is not binding in equilibrium, as assumed in the paper by Bhandari et al. (2021). This difference originates in the desire of the Ramsey planner to relax the borrowing constraint on some of the agents and is achieved by lowering real interest rate, thus relieving the pressure on borrowers and lowering the returns of lenders.

Heterogeneity in macro can be seen as the way to increase the relevance of the models, making them capture some of the important margins in actual economies, that representative agent models can't. Naturally, when facing the model with heterogeneity one can ask whether it significantly contributes to altering the behavior of the aggregates, and if so, through which channels. At the same time, heterogeneity allows studying the redistributive impact of the shocks in the economy. That is, between the two models win no heterogeneity and with it, both of which deliver identical aggregate dynamics, the latter yields the insights into identifying winners and losers in response to shocks, or from a change in the policy rule. Importantly when thinking about optimal policy in such models, both effects are crucially important. First, different propagation of the shocks due to heterogeneity can alter the way optimal policy is conducted, to achieve the desired goal. Second, redistributive motives can prove to be as much important as governing the aggregates or even more so, which is the case of the model I study in this paper, thus calling for the reassessment of the optimal policy altogether.

To tackle the optimal monetary policy problem I rely on the continuous-time formulation. Combining together the analytical part of the solution with the method of first differences allows me to substantially decrease the complexity of the problem and perform the numerical calculations much faster. The advantages of this method were proven to be substantial, as it allows to solve the model very fast without relying on local projections both in the steady state as well as computing the impulse responses to unexpected transitory shocks. To solve the optimal policy problem I use the optimal control in the dynamic programming setting approach and calculus of variations. The main idea of the method is to use the Lagrangian formulation of the problem in the infinite-horizon setting to find the optimal path for the control variables. I show that the set of equations that one gets in this setting, mirrors the structure of the model equations under a simple policy rule, which means that there is no substantial extra cost of solving for the optimal policy as long as the model can be solved for any given policy.

Literature review

The New Keynesian model has been analyzed in many papers and books that constitute the core of the results on the monetary policy in this setup, among them are Clarida et al. (1999), Woodford (2003), Galí (2015). The standard version is characterized by a representative agent, which allows solving for the optimal policy results in closed form and is exceptionally good for building intuition for the main forces acting in the monetary policy analysis.

The results though do rely on the assumption of a representative household, so everything that happens in the economy can be seen as influencing the welfare of this single individual. This unrealistic assumption was one of the motivations for the recent literature that asked the same questions but in the model with heterogeneous agents. The main complication that was restricting advancements in this field was the computational complexity of such models. In order not to solve the models that are numerically heavy, as well as to retain the relative simplicity of the results for analysis some authors have chosen to study the models with reduced heterogeneity.

Some of the examples include Debortoli et al. (2017), who study the New Keynesian model with two agents. One agent is representing the households on the borrowing limit and another agent represents households away from the borrowing constraint. This simplification allows for the calculation of the effects of the monetary policy on aggregates under limited heterogeneity. Another approach was used in the papers by Bilbiie and Ragot (2017) and Challe et al. (2017) where heterogeneity is restricted by limited history dependencies, which helps to simplify the numerical solution process.

One of the new powerful methods involves solving the approximation of the model around the equilibrium path as shown by Bhandari et al. (2021). The authors study the heterogeneous agent model with nominal rigidities (HANK) with both monetary and fiscal policy. They find that the Ramsey planner has a strong motive to increase the inflation and labor tax in the short run to redistribute labor income. Higher inflation leads to lower dividend payments and partially reduces the inequality caused by differential income from equity. Importantly though this holds under assumption of no binding borrowing constraint (BC), or the exogenous BC, which is the crucial restricting assumption, as I show in this paper.

A different approach is taken by McKay and Wolf (2022) where the idea is to capture the incentive of the monetary authority to change the monetary policy in the presence of the heterogeneous agents by finding the weight on the additional term in the quadratic welfare function that captures the change in consumption dispersion. The advantage of this method is in the relative computational simplicity of determining the change of the optimal policy in the presence of heterogeneous agents.

The method that I use in this work makes use of the continuous-time formulation of the New Keynesian model. This approach was developed in works by Kaplan and Violante (2014), Kaplan et al. (2018) in this recent work authors build the HANK model with two assets to study the effect of an exogenous monetary policy shock on the distribution. They find that in the setting with liquid and illiquid assets the effect of monetary policy shock on the aggregate outcomes that is explained by the indirect effects acting through the household distribution outweighs the direct effect that dominates in the representative household model.

In this work I build on the method for finding an optimal policy in continuous time models proposed by Dixit et al. (1994) and further developed by Nuño (2013) to calculate the optimal monetary policy in the heterogeneous agents continuous-time model. This method relies on deriving the Lagrangian associated with the optimization problem and calculating its partial derivatives using the calculus of variations. This method was used in Nuño and Thomas (2016) in the small open economy setting to derive the optimal policy. Though in the analyzed setting there was no effect of the changing wealth distribution on aggregate variables, the method can be used in a more complicated setting as I will show in this work.

In the recent working paper by González et al. (2021), authors use the method in the setting with the heterogeneous firms in the New Keynesian model and show how the optimal policy problem can be solved using Dynare. The similar approach is taken by Dávila and Schaab (2022) where authors analyze the optimal monetary policy in HANK setting, but the maximization of the Lagrangian by the Ramsey planner is performed already on the discretised model equations. In both papers the disadvantage of the optimization approach is in the inflexibility of the method. If one is to use any other method for solving the model, other than the finite difference, it will be impossible to apply the algorithm proposed in both papers, which is not a restriction when using my approach.

In this paper I contribute to both branches of the literature. Specifically, from the theoretical standpoint, I show that the presence of an occasionally binding borrowing constraint (with the resulting endogenous share of constrained households) significantly changes the optimal monetary policy. This naturally extends the work by Kaplan et al. (2018), and also highlights the differences with the setup with an exogenous fraction of constrained households, as in Bhandari et al. (2021). Moreover, I contribute to the growing literature on optimal policy in the continuous time approach, by extending the method to the models with non-trivial binding constraints.

The rest of the paper is organized as follows: In the second section, I describe the model setup, in the third section I show the Ramsey problem formulation and solution approach, in the fourth section I show the results of the calibrated model and finally the fifth section concludes.

Model

In this section I introduce the model used in the subsequent analysis. It is a standard New Keynesian model with monopolistic competition on intermediate goods market with sticky prices. The supply side of the economy, as well as the government block, are kept at a bare minimum to keep the model as tractable as possible. The household block is the one that has most of the complexity and one that crucially affects the way the optimal monetary policy is conducted. Households make a decision on their consumption, labor supply, and investment into nominal bonds, given the aggregate values in the economy as well as household-specific productivity $\varepsilon_{i,t}$ and bond holdings $\underline{b}_{i,t}$ of this household, that are constrained by the binding borrowing limit $\underline{b}$. The government corrects the inefficiency, caused by imperfect competition, by subsidizing wages while keeping balanced budget by collecting the lump-sum tax. Below I describe the model in more detail.

Supply side

The final good is produced by competitive firms that use the continuum of intermediate goods in the production process. Each intermediate good is supplied by a single producer. Intermediate producers have a unit mass and are indexed by $j \in [0,1]$. Consequently, the same indexing applies to intermediate goods as well. The production function has a CES structure with ϕ_t determining the price elasticity of demand for the intermediate goods $y_{j,t}$. The price elasticity is fixed in the steady state but can change unexpectedly, with subsequent convergence back to the initial level. This is a source of markup shock.

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\phi_t - 1}{\phi_t}} dj \right)^{\frac{\phi_t}{\phi_t - 1}} \tag{1}$$

Producers of the final good are competitive and take both the price P_t of the final good as well as prices of intermediate goods $p_{j,t}$ as given. Thus they maximize the following expression, subject to the production function above

$$\max_{\{y_{j,t}\}_{j \in [0,1]}} P_t Y_t - \int_0^1 p_{j,t} y_{j,t} dj \tag{2}$$

Notice that since the maximization problem is solved independently of past and future decisions of the firms, there is no difference between this specification in continuous time or in the discreet time formulation of the problem.

The outcome of the profit maximization, is a demand function

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t} \right)^{-\phi_t} Y_t \tag{3}$$

Zero profit condition arises from the assumption of CES demand for intermediate goods and perfect competition on the final good market. This condition together with the previous optimization result yields:

$$P_t = \left(\int_0^1 p_{j,t}^{1-\phi_t} dj \right)^{\frac{1}{1-\phi_t}} \tag{4}$$

Having derived the demand function facing the intermediate producers, we can now turn to their problem. These producers are monopolists with a linear production function that is the same for all $j \in [0,1]$. For simplicity, I assume that a given firm j in its' production process uses only labor $n_{j,t}$ as input and features linear production technology.

$$y_{j,t} = \theta_t n_{j,t} \tag{5}$$

The aggregate productivity level θ_t is common across all firms and is equal to 1 in the steady state, whereas when shock arrives in the economy, it temporarily deviates from the steady state level with gradual convergence back to the original level, more on this later.

Intermediate goods producers set prices optimally given the demand functions from the final good producers, thus competing monopolistically. Thus the j-th firm maximization problem is

$$\max_{\{\dot{p}_{j,t}\}_t} \mathbb{E}_t \int_0^\infty e^{-\int\limits_0^t r^b_\tau d\tau} \left[\left(\frac{p_{jt}}{P_t} - m_t \right) \left(\frac{p_{j,t}}{P_t} \right)^{-\phi_t} Y_t - \frac{\psi}{2} \left(\frac{\dot{p}_{j,t}}{p_{j,t}} \right)^2 Y_t \right] dt \tag{6}$$

Where $m_t = \dfrac{W_t}{\theta_t}$ is the marginal cost of production, with W_t denoting the real wage at time period t and $-\int\limits_0^t r^b_\tau d\tau$ is the discount factor[1].

The Hamilton-Jacobi-Bellman equation for the above problem has the following form

$$r^b_t \mathcal{J}_{j,t} = \max_{\dot{p}_{j,t}} \left(\frac{p_{j,t}}{P_t} - m_t \right) \left(\frac{p_{j,t}}{P_t} \right)^{-\phi_t} Y_t - \frac{\psi}{2} \left(\frac{\dot{p}_{j,t}}{p_{j,t}} \right)^2 Y_t + \dot{p}_{j,t} \frac{\partial \mathcal{J}_t}{\partial p} + \frac{\partial \mathcal{J}_t}{\partial t} \tag{7}$$

Taking derivative with respect to inflation $\pi_{j,t} = \frac{\dot{p}_{j,t}}{p_{j,t}}$ (maximization problem) and price $p_{j,t}$ (using the Envelope theorem) and using the symmetry of the solutions of all the firms in the equilibrium I get the PDE that describes the solution of the firms' problem

$$\pi_t = \frac{P_t}{\psi Y_t} \frac{\partial \mathcal{J}_t}{\partial p} \tag{8}$$

$$r^b_t \frac{\partial \mathcal{J}_t}{\partial p} = -\phi_t (1 - m_t) \frac{Y_t}{P_t} + \frac{Y_t}{P_t} + \pi_t \frac{\partial \mathcal{J}_t}{\partial p} + P_t \pi_t \frac{\partial^2 \mathcal{J}_t}{\partial p^2} + \frac{\partial^2 \mathcal{J}_t}{\partial p \partial t} \tag{9}$$

[1]Notice, that the firms' discounting factor is equal to the return on the bonds market. This is due to the assumption that the firms are owned by the mutual fund that all the households in the economy invest into. This means that if the distribution of households who have shares in the mutual fund is not the same as the distribution that participates in the bond market, the discount rate should be different. Despite of that I take this discount factor as a simplifying assumption even in the case when dividends are not distributed equally to all of the households.

This implies Phillips curve that links inflation with the markup gap in the economy

$$\left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1}m_t - 1\right) + \dot{\pi}_t \tag{10}$$

Moreover, I assume that intermediate firms are owned by a fund that distributes profits equally among all the households in the economy. This means that since both the intermediate firms as well as households have unit masses, the dividends that go to household are exactly equal to the firms' aggregate profits

$$d_t = \left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t - W_t N_t \tag{11}$$

Finally, total output is equal to a representative firm output in the symmetric equilibrium also because of the unit mass of the firms and because they are identical. This means as well, that the labor demand of each firm is equal to the aggregate labor demand $n_{j,t} = N_t$

$$Y_t = y_{j,t} = \theta_t n_{j,t} = \theta_t N_t \tag{12}$$

Household side

There is a continuum of infinitely-lived households indexed by $i \in [0, 1]$ that have a unit mass and differ in their bond holdings, labor productivity and equity shares. All households have the same discounting factor ρ. They maximize their expected utility over the infinite lifetime span subject to the budget constraint, choosing the consumption path, labor supply path, and bond investment path. Households have a borrowing constraint $b_{i,t} \geq \underline{b} < 0$

$$\mathcal{V}_{i,t} = \underset{\{c_{i,t}, l_{i,t}, \dot{b}_{i,t}\}_t}{max} \mathbb{E}_t \int_0^{+\infty} e^{-\rho t}\left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma}\right)dt \tag{13}$$

$$\text{s.t.} \quad c_{i,t} + \dot{b}_{i,t} = \lambda W_t l_{i,t}\varepsilon_{i,t} + d_t s_i + T_t + r_t^b b_{i,t} \tag{14}$$

$$b_{i,t} \geq \underline{b} \tag{15}$$

Where $b_{i,t}$ are the bond holdings by the household that are priced in nominal terms, thus their value depreciates at the inflation rate[2].

$$r_t^b = i_t - \pi_t \tag{16}$$

The significant assumption in my model is that the borrowing constraint is constant both across households and time. This simplification helps me with finding a solution, but clearly constraints the implications of the model. Importantly, if the borrowing constraint depends on the characteristics of the household, such as labor income or capital holdings, the results can change. On the other hand, as I argue later in the paper, the crucial factor is that there is a hard borrowing constraint that is binding for some households. Regardless of the amount of borrowing where this constraint starts to bind, the implication for the Ramsey planner is to relax this constraint as much as possible, to allow households take advantage of the financial markets to smooth the consumption path. This also implies that if there is a way for the monetary authority to effect directly the constraint, this mechanism is going to play a significant importance as well. In this paper the borrowing constraint is exogenous, but building the tractable model with endogenous borrowing constraint can shed more light on implications for optimal monetary policy in HANK.

The idiosyncratic productivity level $\varepsilon_{i,t}$ is the exponent of an Ornstein-Uhlenbeck process $e_{i,t}$ that has zero mean with mean reversion parameter ρ_e and Wiener process $dW_{e,i,t}$ multiplied by parameter for standard deviation of the shock σ_e

$$\varepsilon_{i,t} = exp\{e_{i,t}\};\ de_{i,t} = -\rho_e e_{i,t} dt + \sigma_e dW_{e,i,t} \tag{17}$$

Households receive labor subsidy λ that cancels the inefficiency associated with monopolistic competition and pay lump-sum transfer T_t to balance the government budget that finances the labor subsidy. Finally, households receive dividends from firms that are distributed equally among all households.

Share of equity holdings s_i that determine the dividend payments from firms are fixed and exogenous. Share of households have zero shares of equity, thus not receiving any dividend. Others have some positive shares that entitle them to more or less of the firm profit, depending

[2]In the rest of the paper I treat all the variables in real terms. This also means the effectively central bank is controlling the real rate directly, which lets me to omit the step with the Fischer equation for simplicity of exposition. This doesn't mean though that the nominal pricing in the economy is not important, as it remains the cornerstone of the model, causing indeterminacy and calling for the central bank policy as means of determining the equilibrium.

on the size of the share, but can't trade them.

Distribution of the households over the space of bond holdings and productivity levels is denoted by f_t. This distribution hence is a two-dimensional object that in a steady state is constant, due to the law of large numbers. Importantly, the dimension that is associated with the evolution of the idiosyncratic shock is evolving exogenously, according to the law of motion associated with the exponent of the Ornstein-Uhlenbeck process, whereas the evolution of the bond holdings follows the process determined by the optimizing behavior of the households.

The Hamilton-Jacobi-Bellman equation for the household problem takes the following form

$$\rho \mathcal{V}_{i,t} = \max_{c_{i,t}, l_{i,t}} \left\{ \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \left(\lambda W_t l_{i,t} \varepsilon_{i,t} + d_t s_i + T_t + r_t^b b_{i,t} - c_{i,t} \right) \frac{\partial \mathcal{V}}{\partial b} \right\} - \tag{18}$$

$$- \varepsilon_{i,t} \rho_e e_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_e^2}{2} \frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} \tag{19}$$

Where the first two terms represent instantaneous utility, third term governs the change of the value function because of the change of the bond holdings, next two terms correspond to the change associated with stochastic labor productivity process and finally last term is capturing the changes associated with the changes of the aggregate variables.

Solving this maximization problem for the optimal value of consumption and labor and combining it with the HJB equation I get the optimality conditions for the household problem in the form of the partial differential equation, which is the standard step in continuous-time literature[3].

$$c_{i,t} = \left(\frac{\partial \mathcal{V}_{i,t}}{\partial b} \right)^{-\frac{1}{\nu}} \tag{20}$$

$$l_{i,t} = \left(\frac{\lambda W_t \varepsilon_{i,t}}{\varphi} \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right)^{\frac{1}{\gamma}} \tag{21}$$

$$\rho \mathcal{V}_{i,t} = \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t} \mathcal{V}_{i,t} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} \tag{22}$$

Where $\mathcal{A}_{i,t}$ is the infinitesimal generator of process $\mathcal{V}_{i,t}$, that captures the changes in the value function due to the evolution of the household-specific state variables $\varepsilon_{i,t}$ and $b_{i,t}$

$$\mathcal{A}_{i,t} \mathcal{V}_{i,t} = \left(\lambda W_t l_{i,t} \varepsilon_{i,t} + d_t s_i + T_t + r_t^b b_{i,t} - c_{i,t} \right) \frac{\partial \mathcal{V}}{\partial b} - \varepsilon_{i,t} \rho_e e_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_e^2}{2} \frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} \tag{23}$$

[3]See the paper by Achdou et al. (2021) for the extensive explanation of the solution method of consumption-saving problem in continuous time.

Fokker–Planck / Kolmogorov forward equation

The law of motion of the joint distribution of bond holdings and labor productivities can be characterized by Fokker–Planck or Kolmogorov forward equation

$$\frac{\partial f_{i,t}}{\partial t} = -\frac{\partial}{\partial b}\left[\left(\lambda W_t l_{i,t}\varepsilon_{i,t} + d_t s_i + T_t + r_t^b b_{i,t} - c_{i,t}\right) f_{i,t}\right] + \tag{24}$$

$$+\frac{\partial}{\partial \varepsilon}\left[\varepsilon_{i,t}\rho_e e_{i,t} f_{i,t}\right] + \frac{\partial^2}{\partial \varepsilon^2}\left[\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2} f_{i,t}\right] = \mathcal{A}_{i,t}^* f_{i,t} \tag{25}$$

Notice that conceptually the evolution of the distribution should be tightly linked to the household problem. The saving decisions of the households change the distribution of bonds in the economy. This means that knowing the saving decision at any point of the state space it is possible to determine what is the corresponding stationary distribution of bonds.

Mathematically speaking, the law of motion of the household value function adjoins to the law of motion of distribution. See Appendix for the analytical derivation of this fact in the setting of my model. For the standard Aiyagari–Bewley–Huggett model see the paper by Achdou et al. (2021).

Markets clearing

In addition to the supply and household side there is also the government, that only provides the labor subsidy, as mentioned before. The subsidy λ to workers is such that it cancels out the inefficiency created by the monopolistic competition of the intermediate goods producers. Government runs the balanced budget and levies the lump-sum tax on the households to finance the labor subsidy

$$\lambda = \frac{\phi}{\phi - 1}, \quad T_t = \langle (1-\lambda) W_t l_{i,t}\varepsilon_{i,t}, f_t \rangle$$

Where $\langle \cdot, \cdot \rangle$ stands for summation over the distribution, see more detailed explanation in the Appendix. There are no additional redistributions done by the government, apart from the one described above.

In equilibrium bond market clears to zero aggregate bond supply[4].

$$\langle b_{i,t}, f_{i,t} \rangle = 0$$

The labor market clears with the wage W_t so that labor demand equals to effective labor supply

$$\frac{Y_t}{\theta_t} = N_t = \langle \varepsilon_{i,t} l_{i,t}, f_{i,t} \rangle$$

Dividends payed to the households are equal to the profits of the intermediate firms

$$d_t = Y_t - W_t \langle l_{i,t} \varepsilon_{i,t}, f_{i,t} \rangle - \frac{\psi}{2} \pi_t^2$$

Finally the last equation that determines the equilibrium is the goods market clearing condition

$$C_t = Y_t - \frac{\psi}{2} \pi_t^2$$

Notice that there is one equation that is still missing, which is the one that defines the monetary policy. Since the policy has to be optimal, it is determined by the optimality conditions of the Ramsey planner that are discussed in the next section.

Ramsey problem

The problem for the optimal monetary policy is solved by the standard Lagrangian method. The difficulty in this application is that all the objects with respect to which the derivatives have to be taken are actually functions, not variables. Partially this is due to the fact that in continuous time taking the derivative with respect to the control variable has to be done not period-by-period as in discreet time approach, but rather on all of the paths simultaneously. This is hardly a real complication because after using the method described below the analogy to the discreet time method is straightforward and conceptually doesn't differ at all. In Appendix, I include the derivation of the Ramsey problem for the standard RANK model in continuous time, without log-linearization both as a simpler example for the derivations, but also as a reference model for the comparison of the results. Moreover, some of the objects are functions even within one

[4] $\langle \cdot, \cdot \rangle$ stands for the inner product of two functions. In this case it's the aggregation of bond holdings for all the households in the distribution. See a more extensive explanation of this notation in the "Ramsey problem" section.

period. For example, the distribution is a two-dimensional object at every instant of time, which means, that again, a different approach to taking derivatives has to be pursued.

The seminal paper by Nuño and Thomas (2016) proposed the method for obtaining the Ramsey policy in continuous time with heterogeneous agents. Their method involves writing the social planner problem in form of Lagrangian and taking the Gateaux derivatives with respect to the state variables. In this paper, I slightly change the methodology by using Calculus of Variations instead of Gateaux differentials. On the one hand, the method is very similar in its essence, so the logic for the main results applies. On the other hand, using calculus of variations is suited much better for analysis of the behavior of the problem at the boundary constraints. Something that can be done with Gateaux differentials in the simplest cases, but as soon as the conditions on the constraint are more complex, which is the case for the binding borrowing constraint, it can't be used conveniently.

Calculus of variations is widely used in other fields of economics as well, e.g. wide range of continuous-time applications in finance can be found in books Malliavin and Thalmaier (2006), Clarke (2013). But to the best of my knowledge, this is the first work to show how it can be applied in macro, where it is particularly useful in applications with nontrivial binding constraints.

I apply the method to solve for the Ramsey problem in the general equilibrium setting of the HANK model described above and show how to perform the calculations to get the system of differential equations that can be solved numerically and that define the optimal policy in this model. This method proves to be very convenient, as the resulting system of differential equations for the costates is not just linear, which is something to be expected for any Lagrangian problem, but also has symmetry with respect to the original problem. See derivations and more detailed intuitions in the Appendix.

The Ramsey problem is a maximization of the utilitarian value function subject to all of the equations that define the competitive equilibrium

$$\mathcal{V}_{\mathcal{G}} = \max_{f,\mathcal{V},c,l,W,Y,\pi,r^b} \int_0^\infty e^{-\rho t} \left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma}, f_{i,t} \right\rangle dt \tag{26}$$

$$\text{s.t. } \{\text{competitive equilibrium equations}\} \tag{27}$$

Notation $\langle \cdot, \cdot \rangle$ represents the inner product of two functions, defined on the common domain D. In this case it is the space of productivity levels ε and bond holdings b. In case of the social planner instantaneous utility this means

$$\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma}, f_{i,t} \right\rangle = \int_{\underline{b}}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} \right) f_{i,t} \, d\varepsilon_{i,t} db_{i,t} ds_i \tag{28}$$

The full Ramsey problem has the following form

$$\mathcal{L}[f, \mathcal{V}, c, l, W, Y, \pi, r^b, T, d] = \tag{29}$$

$$= \int_{0}^{\infty} e^{-\rho t} \left[\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma}, f_{i,t} \right\rangle + \left\langle \zeta_{i,t}, \mathcal{A}_{i,t}^{*} f_{i,t} - \frac{\partial f_{i,t}}{\partial t} \right\rangle \right. \tag{30}$$

$$+ \left\langle \varrho_{i,t}, \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t} \mathcal{V}_{i,t} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} - \rho \mathcal{V}_{i,t} \right\rangle \tag{31}$$

$$+ \left\langle \mu_{i,t}, c_{i,t}^{-\nu} - \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \left\langle \kappa_{i,t}, l_{i,t}^{\gamma} c_{i,t}^{\nu} - \frac{\lambda W_t \varepsilon_{i,t}}{\varphi} \right\rangle \tag{32}$$

$$+ \eta_{b,t} \langle b_{i,t}, f_{i,t} \rangle + \eta_{Y,t} \left(Y_t - \theta_t \langle l_{i,t} \varepsilon_{i,t}, f_{i,t} \rangle \right) \tag{33}$$

$$+ \eta_{T,t} \left(T_t - (1-\lambda) \frac{W_t}{\theta_t} Y_t \right) + \eta_{d,t} \left(d_t - \left(1 - \frac{\psi}{2} \pi_t^2 - \frac{W_t}{\theta_t} \right) Y_t \right) \tag{34}$$

$$\left. + \eta_{\pi,t} \left(\frac{\phi - 1}{\psi} \left(\frac{\phi}{\phi - 1} \frac{W_t}{\theta_t} - 1 \right) + \dot{\pi}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t} \right) \pi_t \right) \right] dt \tag{35}$$

Where

$$\mathcal{A}_{i,t} \mathcal{V}_{i,t} = \dot{b}_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial b} - \rho_{\varepsilon} \varepsilon_{i,t} e_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_{\varepsilon}^2}{2} \frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} \tag{36}$$

$$\mathcal{A}_{i,t}^{*} f_{i,t} = -\frac{\partial}{\partial b} \dot{b}_{i,t} f_{i,t} + \frac{\partial}{\partial \varepsilon} \rho_{\varepsilon} \varepsilon_{i,t} e_{i,t} f_{i,t} + \frac{\partial^2}{\partial \varepsilon^2} \varepsilon_{i,t} \frac{\sigma_{\varepsilon}^2}{2} f_{i,t} \tag{37}$$

$$\dot{b}_{i,t} = \lambda W_t l_{i,t} \varepsilon_{i,t} + r_t^b b_{i,t} + T_t + d_t s_i - c_{i,t} \tag{38}$$

In Appendix, I show how to take the differentials with respect to control variables using calculus of variations and how to rearrange the equations to get the equations for updating the solution algorithm.

Solution algorithm

The solution algorithm aims to find the equilibrium response under the optimal monetary policy to an unexpected transitory shock (MIT). The shocks that I study are negative TFP and shock to demand price elasticity that lowers desired markup. After the arrival of the unexpected shock, agents form rational expectations about the changes of the aggregate variables from that moment onward. For households this is captured by the $\frac{\partial \mathcal{V}}{\partial t}$ which summarizes the changes of the households value due to the changes in the aggregate variables.

First, I conjecture some path for the aggregate variables that enter the household problem in response to the shock at hand. Specifically, I fix the path for the real interest rate r_t^b[5]. Given this path, I can solve the household problem. Starting at time T, very distant from the shock impact allows me to assume that the household value at this point coincides with the one in the steady state, so I can solve for the value of $\mathcal{V}_T$, given the state variables at that point. Then, using backward induction I solve for the path of $\mathcal{V}_t$ all the way back to the initial impact at $t = 0$. Having solved for the household choices for the whole path, I can now use forward induction to determine the corresponding path of the distribution. Specifically, starting at $t = 0$ and taking distribution f_0 to be the one from the steady state, I can use the Fokker–Planck / Kolmogorov forward equation to solve for the path of the distribution up to time T. Now, having determined the path of the distribution I can check if all the markets clear, and if they don't, update the corresponding prices and aggregate variables (notice that there is no update for the policy yet) and start the procedure again.

Second, I find the update for the monetary policy. For this update I have to solve the corresponding costate variables problem from the Lagrangian first-order conditions that have the form of linear differential equations, so technically is a bit easier than the first step. Specifically, I guess for the paths for the costate variables, in the same way, I have guessed for the path of state variables. Then I start at the time period T and under the assumption that by this time the steady state has been attained, I solve for the stationary value of the multipliers associated with the first-order condition on the distribution. Then, following the analogy with the value function, I use backward induction to solve backward for the path of this costate two-dimensional object until the impact at $t = 0$. Then I take the costate associated with the value function and take its values to be the same at the moment of impact as in the steady state. Notice, that this object is a two-dimensional distribution as well. Also, the assumption that the starting values are the values from the steady state has to do with the fact that I'm solving the problem under Woodford's timeless perspective. Specifically, if the Ramsey planner was able to ignore the previous commitments as soon as the shock hits, the value for this costate should have been taken to be zero. Now, following the analogy with the distribution, I solve forward for the values of this costate into the future until time T. Then, using the rest of the first-order conditions, I

[5] As mentioned above, the real interest rate has to be thought of as the monetary policy instrument. This allows me to omit the Fisher equation and think directly in terms of the real variables in the economy. At the same time this doesn't create indeterminacy, because the "missing" equation is still there and one can determine the nominal interest rate using Fisher equation, if needed.

update the guess for the costate variables. Finally, as in every Lagrangian problem, there is an extra equation that is associated with the policy function, that allows me to get an update on the policy itself. Now I need to repeat the process for costate variables until the convergence and use the final update on the policy function to change it and go to the first step again.

The solution is achieved when the residuals from the equilibrium and optimality conditions are smaller than the predefined convergence criterion. Notice that the first and second steps that respectively solve for the state and costate variables equilibria, do not have to be done separately. As a matter of fact, the updates can be done simultaneously, which can give a faster speed of convergence.

Calibration

To calibrate the model I use the parameters from the two papers that are closest to mine in the application. Specifically, for the supply side as well as a household preference I take the parameters from Kaplan et al. (2018) and for the labor income process and inequality I take the moments from Bhandari et al. (2021). Final good producers have a price elasticity parameter $\phi = 10$ which implies the markups for the intermediate firms equal to around 10% and the cost of price adjustment is $\psi = 100$. This means that the resulting slope of the Phillips curve is $\phi/\psi = 0.1$. Notice that in this setting the actual markup of the firms is not actually that important. Specifically, it governs primarily the amount of profits and thus the relative size of the dividends share in the incomes of the households. But since the dividends are distributed equally, the smaller size of profit share actually helps to reduce the bias of the results that come from this assumption.

Household utility function has the risk aversion $\nu = 1$ and the inverse Frisch elasticity of labor supply $\gamma = 1$ as well. The household discount factor is calibrated to match a yearly real interest rate of 3%. The borrowing limit is calibrated to deliver 30% of constrained households in steady state. The value of $\underline{b}$ for which this is achieved is approximately equal to three average yearly labor incomes. Finally, to calibrate the two parameters for the labor productivity process I match the variance of the yearly labor income change in one year (for the variance of the stochastic component) and the variance of the natural logarithm of the labor earnings in the cross-section (for the mean reversion parameter). All of the parameters and their values can be seen in the table below[6].

[6] I assume that the is no ZLB in this setup and solve the model under this assumption.

Fixed	Description	Value		
ν	Risk aversion	1		
$1/\gamma$	Frisch elasticity of labor supply	1		
ϕ	Price elasticity of demand	10	(slope of the Phillips Curve	
ψ	Price adjustment cost	100	$\phi/\psi = 0.1$)	
Fitted	**Description**	**Value**	**Moment**	**Value**
ρ	Discount rate	0.067	real return	3%
$\underline{b}$	Borrowing limit	-3.54	% constrained	30%
ρ_e	Mean reversion	0.1	var $log(LI)$	0.7
σ_e	Volatility	0.32	var $\Delta(LI)$	0.23

Table 1: Calibration

Results

Steady state

Solving the model first in the steady state and finding the optimal level of inflation for it reveals that the optimal inflation is actually not significantly different from zero. The intuition for this result is fairly simple. Since what happens in the steady state is perfectly anticipated by all of the agents, the only reason for the inflation to be non-zero is to change the balance between the way firms' sales are distributed between dividends and wages. But as this redistribution is anticipated by the households the main effect of this change will result in the more or less demand of bonds to compensate for the changes in labor income and dividend composition with changes in labor productivity. This means that by having non-zero inflation Ramsey planner has to sacrifice some efficiency for potential redistribution that is not taking place since households are able to effectively counteract it by changing the bond holdings in the equilibrium. In practice for the calibrated model optimal steady state level of inflation is less than 0.1%.

The steady state is characterized by the distribution of bonds that have a unit mass of 0.3 at $\underline{b}$ and the rest of the distribution as shown in the Figure 1. The plots display the average values for the given bond holdings, to reduce the dimensionality of plots. The 3D plots on the space of bond holdings and labor productivity can be found in the Appendix.

As households are able to optimally choose the consumption and labor supply, the following holds in the steady state. First, households with lower bond holdings are consuming less and supplying more labor in terms of hours worked. At the same time, because of the endogenous positive correlation of bond holdings and labor productivity, the effective labor supply measured in efficiency units is increasing in bond holdings, despite the fact that the labor supply decreases.

Composition of household income components in the steady state play the crucial role in

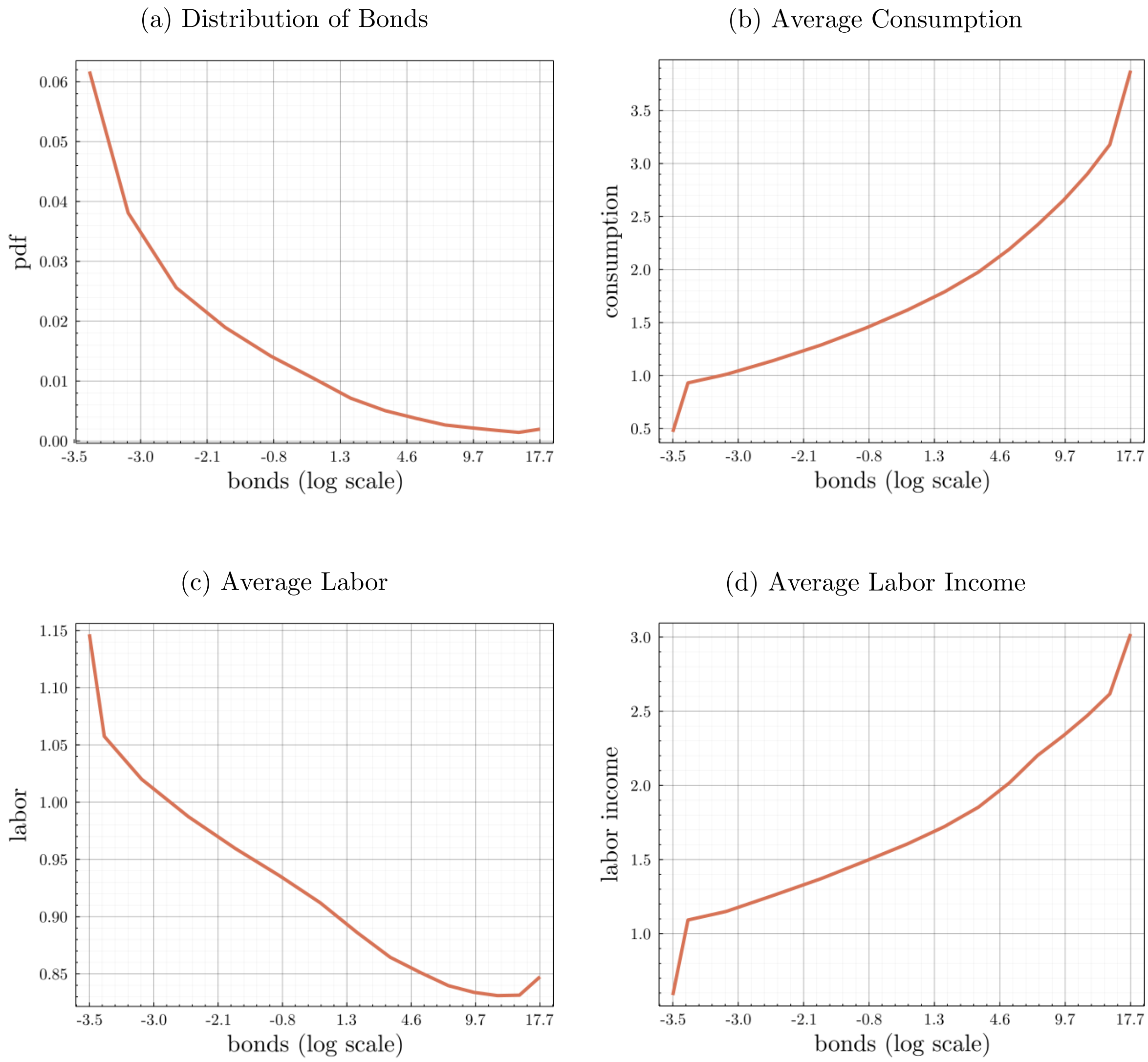
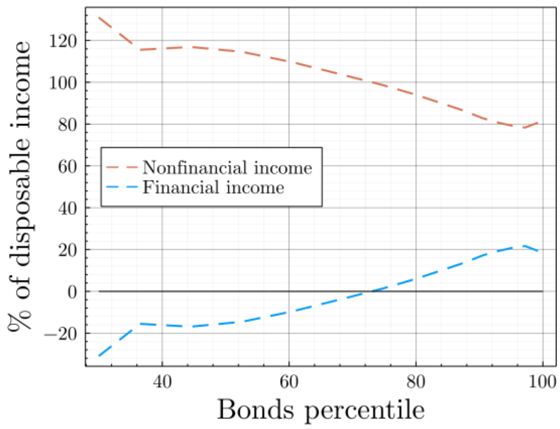


Figure 1: Steady state

determining propagation of the monetary policy. Recent empirical evidence by Holm et al. (2021), Andersen et al. (2020) and Flodén et al. (2021) shows that propagation of monetary policy shocks is largely driven by the direct effect of changing interest rates on income, and not by intertemporal shift of consumption. Explanation of this empirical fact relies on differential exposure of households to interest rates. Some of them are borrowers and receive the negative income shock when rates increase, some have positive net asset position and benefit from higher rates. This dimension of heterogeneity of exposure to the interest rates is at the core of the monetary policy transmission, as suggested by the empirical evidence. Since the key dimension of heterogeneity in my model is the bond position, the same mechanism will certainly play a significant role in my predictions as well. In order to have an understanding of similarities and differences between my model and empirical evidence, I provide a comparison of behavior of the model and empirical facts in the steady state and later in response to the monetary policy shock.

Starting with the steady state analysis, the most direct comparison to make is with Holm et al. (2021) who use data from Norway to capture differential effects of monetary policy conditional on liquidity holdings, which is exactly the dimension of heterogeneity in my model. In Figure 2 I plot the two separate sources of income for the households in the model, conditional on their bond holdings: financial income is the return on holding bonds and nonfinancial income is everything else. I find that the importance of financial income increases with higher liquidity and nonfinancial income share decreases, which is perfectly inline with the empirical evidence.

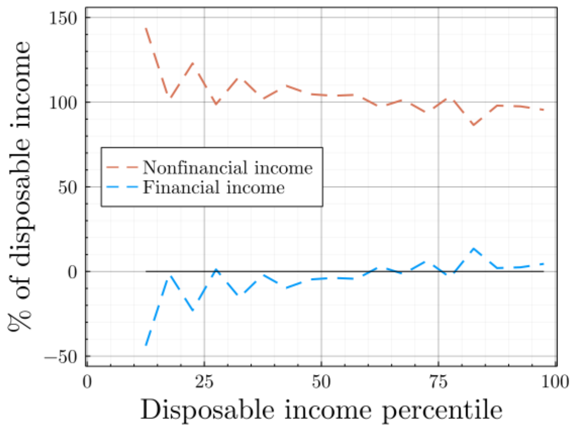
Figure 2: Cash flow shares



It is more difficult to directly compare my model with Andersen et al. (2020). The caveat is that in the paper the dimension of heterogeneity is the disposable income of the household. Since households with higher disposable income hold on average more debt, my model can't be directly compared, as I don't have wealthy hand-to-mouth agents, that are driving this empirical result. Moreover, since my model doesn't have redistributive taxation, I have to perform aggregation of income into financial and nonfinancial categories. Despite of these shortcomings, the values in the model have similar behavior to their counterparts in the data. Specifically when looking at Figure 3, I conclude that in the model, households with higher income have relatively lower importance of nonfinancial income, which is inline with the data.

Finally, comparing with the empirical evidence from Flodén et al. (2021), I conclude that in the same way, as it is documented in the paper based on the data from Sweden, the households with higher debt to income ratio have higher debt expenditures shares. Empirically, the cause of higher debt position of these households is the debt taken for the purchase of the house. Despite

Figure 3: Cash flow shares



of the fact that my model doesn't have housing, the primary implication of negative effect of increasing interest rates on indebted households is captured by my model and plays the crucial role in implications for the optimal monetary policy.

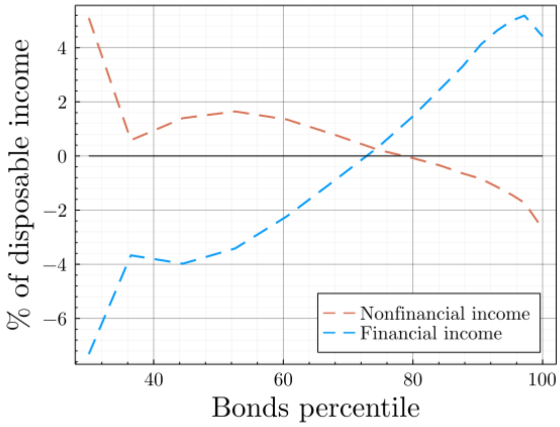
Dynamics

Before looking at the optimal monetary policy responses to TFP and markup shocks, I first provide an analysis of responses to the monetary policy shocks under the simple rule policy. As in the steady state analysis, I compare these responses to the evidence from the empirical data and show what are the effects of changes in the interest rates on the households in the model. Using the standard monetary policy shock is a natural way to see both the differential effect on the households income as well as their saving and consumption decisions. The insights that can be gathered from this exercise help in understanding economics behind the Ramsey policy later on.

I look at response of the model to 1% contractionary monetary policy shock. As optimal monetary policy can't exist under the monetary policy shocks, I use simple policy rule $i_t = r_t + 1.5\pi_t + \varepsilon_t$ to close the model, where ε_t is a persistent monetary policy shock. Under this simple rule, in Figure 4 I plot the change of the households income shares in response to the contractionary monetary policy shock.

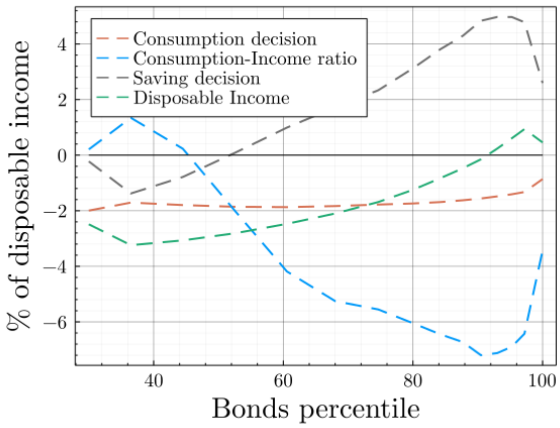
Contractionary monetary policy increases real interest rates. As a result of that financial income share decreases for borrowers and increases for lenders. At the same time, the change of the rest of the income components has the opposite effect. This implication of the model qualitatively is inline with the empirical evidence from Holm et al. (2021).

Figure 4: Cash flow shares change



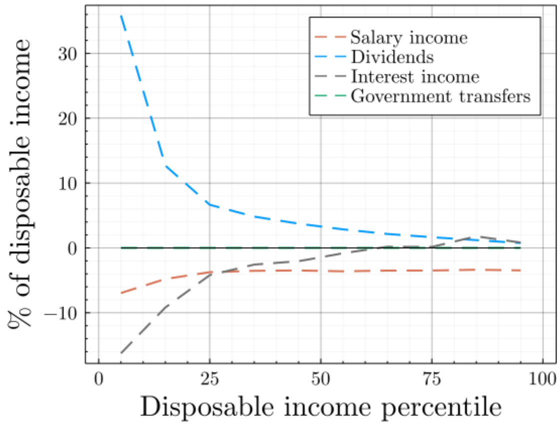
Moreover, when looking at the changes in disposable income, consumption and investment choices in Figure 5, the predictions of the model can be compared to the data as well. Disposable income decreases for all households, but more so for the poor. In the model reduction of consumption share is relatively the same, given different bond holdings. Disposable income change is transmitted to savings decision, and doesn't affect consumption as much as documented empirically. But crucially, poorer households suffer more from this shock, which is true both in the model and in the data.

Figure 5: Income and Consumption-Saving decision change



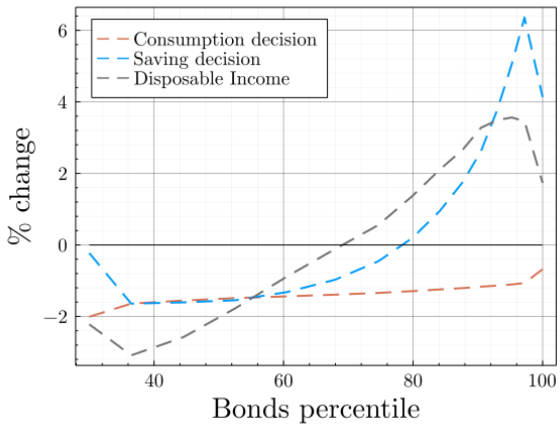
Comparison with empirical evidence from Andersen et al. (2020) can't deliver same results. The reason for this is that in the model there is no wealthy hand-to-mouth agents. Empirically there is a high proportion of households with high disposable income and high debt that have an increase of debt payments following the increase of real interest rates, which is documented empirically. Since in the model there is only one asset, wealthy individuals have an increase of income from increased returns on bonds, following the increase of the real rates. This dimension is at odds with the data, because of the simplifying assumption of having one asset in the economy. The labor income response, on the other hand, is captured quite precisely and matches the more significant reduction of labor income for lower income groups after contractionary monetary policy shock (see Figure 6).

Figure 6: Cash flow shares change



Finally, comparing model responses with empirical evidence from Flodén et al. (2021), reveals that households with higher exposure to interest rate shocks and higher debt have a more severe decline in consumption (see Figure7).

Figure 7: Consumption change



Effects of increase of the real rates due to the monetary policy intervention indicate that the model predicts a more detrimental effect on the borrowers than lenders. This means, that for the given shock that increases inequality the monetary authority should implement looser policy, compared to the SIT that is optimal in representative agent NK model. In what follows

I analyze the optimal Ramsey stabilization policy, that was calculated numerically using the procedure described in previous sections.

As discussed above, Ramsey policy is under the Woodford's timeless perspective. This means that the optimal policy that I get is an optimal stabilization policy. Target for such policy is to minimize deviations from the steady state, both in terms of aggregate variables as well as individual outcomes. This implies that for the shock that increases inequality, planner will aim to reduce inequality to initial levels, but the opposite also holds. This is one of the shortcomings of the method that comes from the fact that optimal monetary policy is time inconsistent in this setting due to inequality that is present in the steady state, and can be affected by the unexpected monetary policy intervention.

In this paper I take the approach of Woodford's timeless perspective, but the alternatively one can introduce the redistribution scheme that completely eliminates the time inconsistency and analyze the unconstrained Ramsey policy in response to shocks under such redistribution scheme. Arguably, the method that I'm using gives an advantage of studying implications for the model that has reasonable inequality in the steady state and allows the monetary policy to influence the inequality level. The disadvantage is that for the shocks that reduce inequality, optimal policy will aim for maintaining the initial level of inequality, hence taking action to increase it, after the impact of the shock. Taking this into account, for both TFP and markup shocks I choose the sign of the shock such that the overall effect increases inequality under the SIT policy and allows to illustrate the mechanisms at play for the reduction of inequality that the Ramsey planner chooses. At the same time, it is true that for the opposite shocks the optimal stabilization policy will imply the symmetrical and opposite response.

I first analyze the optimal responses to the TFP shock and perform the comparison with the RANK model. Specifically what I find is that in response to contractionary TFP shock the optimal RANK model policy, of course, results in zero inflation. Very much different from that, optimal policy in HANK model instructs to respond to the shock with positive inflation and then reduce inflation to negative values before returning to the steady state level. Notice, that despite the fact that inflation is negative for some period of time, the new nominal price level after the shock has passed is going to be different from the initial steady state level.

The reason behind this response can potentially have two explanations: either strict inflation targeting (SIT) policy in HANK fails to achieve the same behavior of the aggregates, and a different policy is needed to sustain the same optimal response, or the notion of optimal response is different. The different target for the optimal response can be explained by the new redistributive motive, that calls for a change in the SIT policy, that achieves zero output gap. When comparing responses of RANK model and HANK model with SIT to the same TFP shock in Figure 8, it is clear, that the effect of the same policy is almost identical in two models. Since the optimal policy in HANK is substantially different from SIT, and calls for non-zero response of inflation, it is clear, that having heterogeneous agents in the economy introduces a new source of inefficiency that has to be addressed by the optimal monetary policy as well.

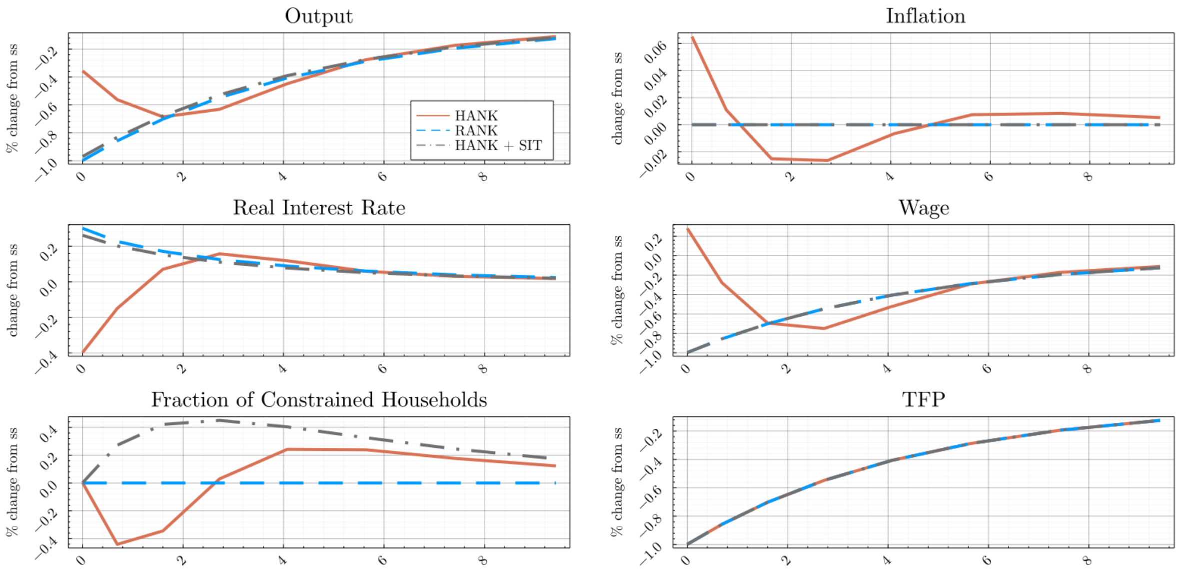


Figure 8: Response to the TFP shock. HANK vs RANK

Specifically when looking at the result of the policy, the crucial difference between RANK and HANK optimal responses is in the real interest rate responses. HANK model has the real interest rate go down, in contrast with the upward movement in RANK. The reason for this is to create redistribution towards the less wealthy individuals, and more specifically towards the constrained agents. The result of such redistribution can be seen in the fact that under the optimal policy in HANK the fraction of constrained agents is actually falling as opposed to implementing the strict inflation targeting in HANK.

To have a better understanding of the redistributional effect of the optimal monetary policy, I look at the heterogeneous response of household disposable income after the shock hits the economy. In Figure 9 I show the baseline redistribution in HANK model under SIT. Comparing disposable income components immediately after shock with the steady state values, predictably reveals that increase of the real interest rate creates redistribution from poor to rich and reduction of wages creates redistribution in the opposite direction. Importantly, redistribution created by real interest rate dominates and total redistributive effect favors the rich.

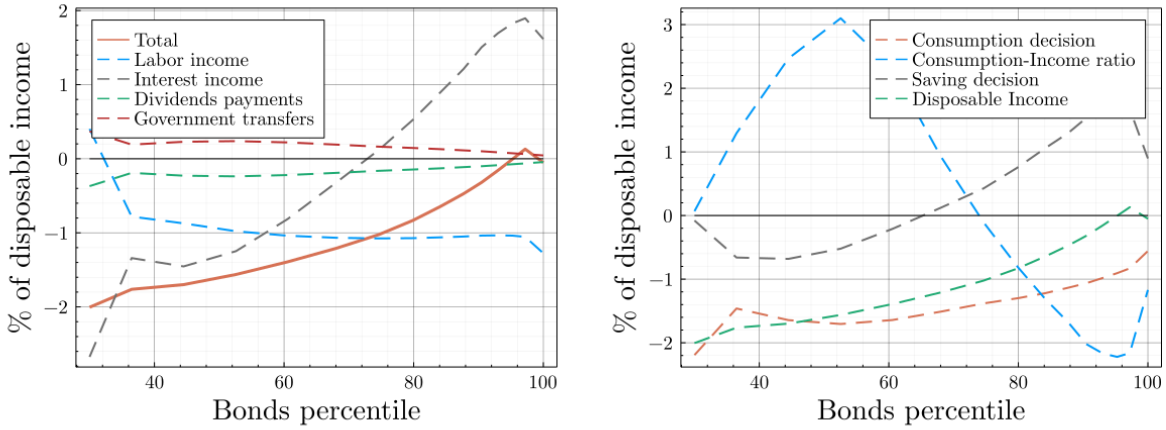


Figure 9: Disposable income and consumption-saving policy response under SIT

Now having the baseline in the form of SIT policy, I compare it to the redistribution achieved by optimal policy. Specifically, Figure 10 shows the redistribution under Ramsey policy and Figure 11 compares the disposable income of the households immediately after the TFP shock under the optimal policy with the disposable income under SIT.

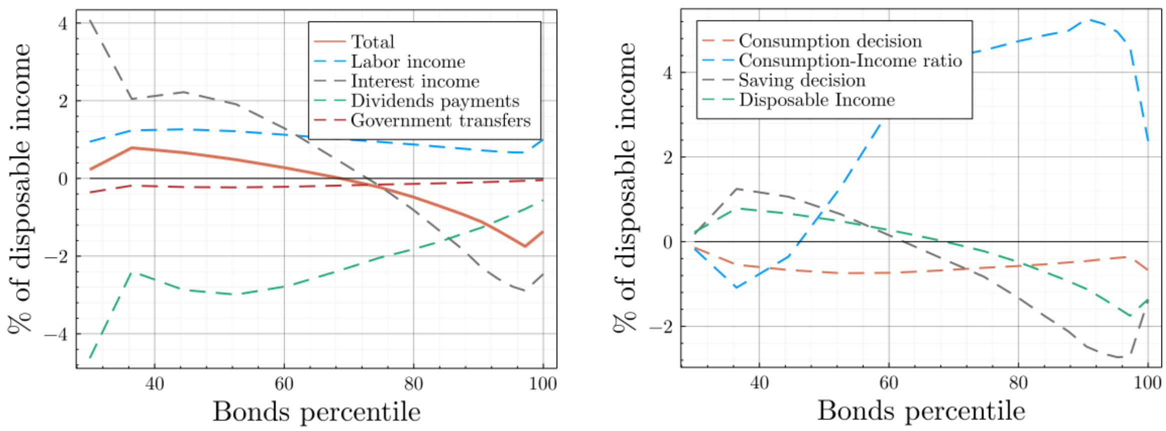


Figure 10: Disposable income and consumption-saving policy response under Ramsey

The graph on the left shows that the aggregate disposable income is redistributed more towards the households with higher debt which is achieved by implementing lower real interest rate. Notice that in the specification with the lump-sum distribution of dividends, the higher wages and lower dividends are working in favor of wealthy individuals, which is the opposite effect from what the Ramsey planner wants to achieve. This allows me to conclude that under any other distribution rule that implies a positive correlation between dividends and bond holdings, this

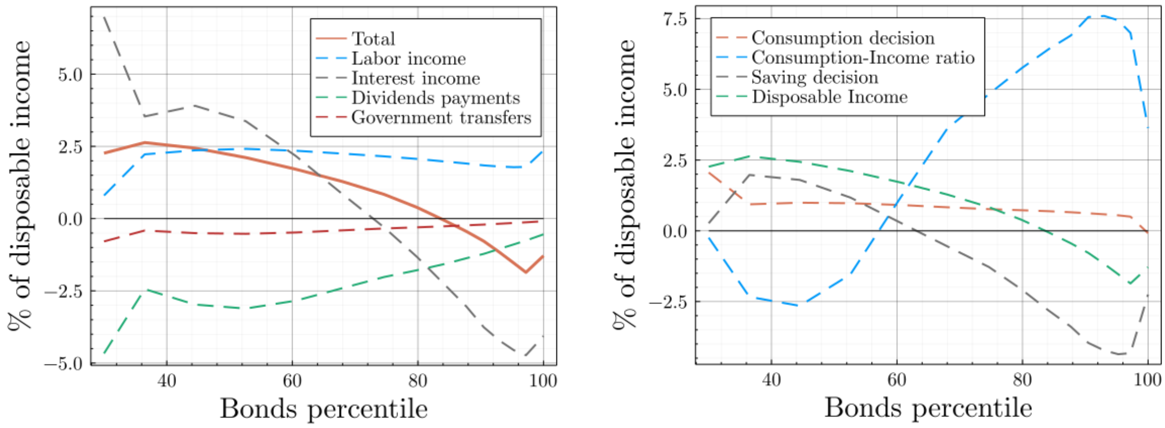


Figure 11: Disposable income and consumption-saving differential response (Ramsey - SIT)

negative effect is going to be smaller, or may even become positive, as in the case of Bhandari et al. (2021). On the right-hand side, I show how the change in disposable income translates to the changes in consumption-saving decision. Inline with the logic discussed above when looking at pure monetary policy shocks, the lower real interest rate under the Ramsey policy creates redistribution towards poorer individuals which translates into their higher consumption and higher investment.

At the same time one can try to understand how much of this effect is present in the models where the fraction of the constrained households is fixed, as in TANK models, or is zero, as in the HANK model with the natural borrowing constraint (NB HANK), as in Bhandari et al. (2021). Specifically, in the TANK model, since both agents have the same labor productivity but different bond holdings (positive for unconstrained and negative for constrained), the effect on income through opposite exposure to interest rates (Figure 12) can be achieved as well. Borrower pays interest on debt and lender receives the return. But since there is no incentive for Ramsey planner to affect bond holdings of the individuals, since they are fixed, the inflation does not differ much from the RANK optimal policy.

When comparing with the HANK model with the natural borrowing limit (NB_HANK), the effect is much more pronounced (Figure 13), but still lacks in magnitude, because despite of the fact that there is an effect on the households' bond holdings, there is no incentive as strong as relieving some agents from the borrowing constraint. This is why the real interest rate doesn't show as much of a decline, since all the changes of the disposable income that households experience, they can compensate by borrowing or saving.

Comparison of these three cases illustrates that having a fixed share of constrained agents or

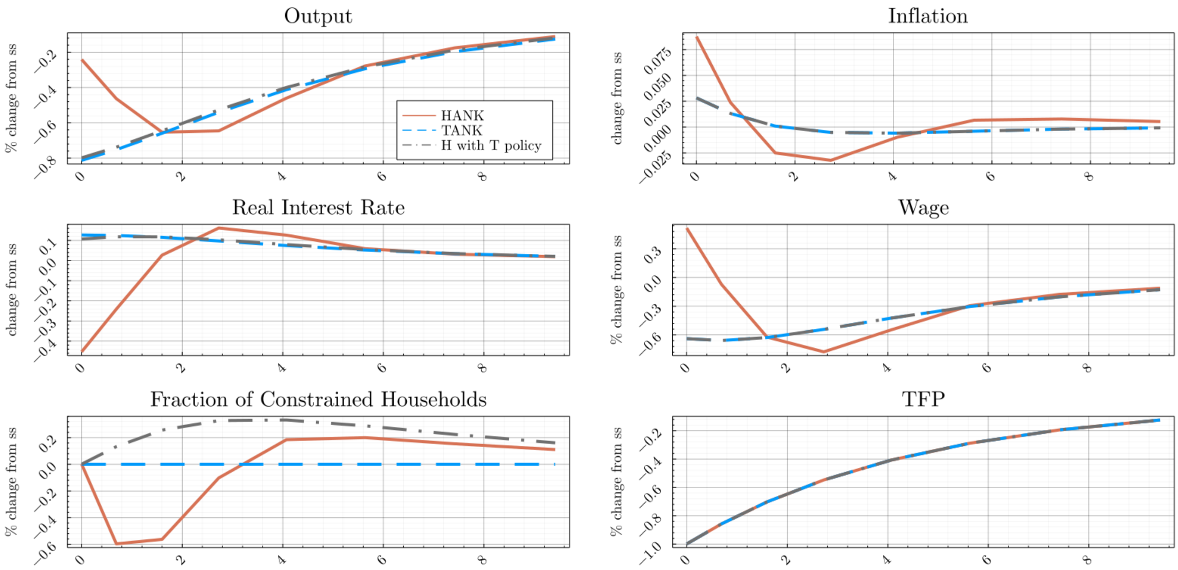


Figure 12: Response to the TFP shock. HANK vs TANK

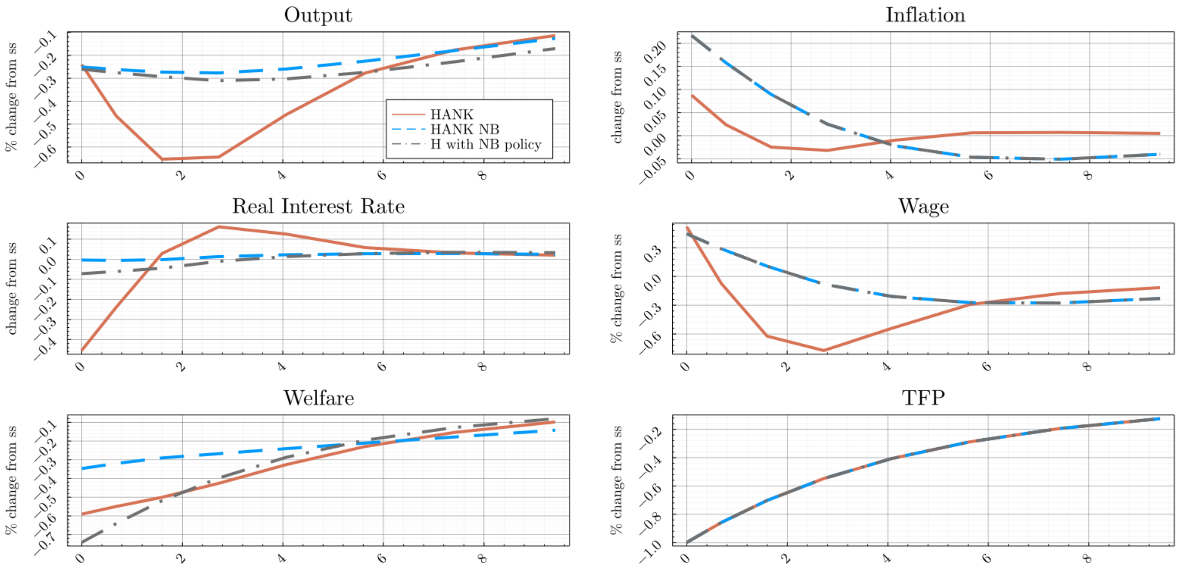


Figure 13: Response to the TFP shock. HANK vs NB_HANK

heterogeneous agents with natural borrowing constraint does not provide the same incentives for the Ramsey planner to reduce the real interest rate as HANK model with occasionally binding borrowing constraint. The reason for this is that in the case of TANK model reduction of real interest rates does reduce the pressure on the constrained share of households, but they can't react in the same way, by switching to unconstrained. As for the HANK model with natural borrowing limit, despite of the fact that effect of changing the real rates on financial income is strong, households are able to fully compensate for this effect by altering their savings decisions, so effectively they don't benefit from the loose monetary policy as much.

The same logic as I have just described for the case of the TFP shock also holds for the markup shock (Figure 14). Specifically, I study the response to a shock that temporarily increases the price elasticity of the demand for the intermediate goods by 1, thus reducing the desired markups by approximately 1%.

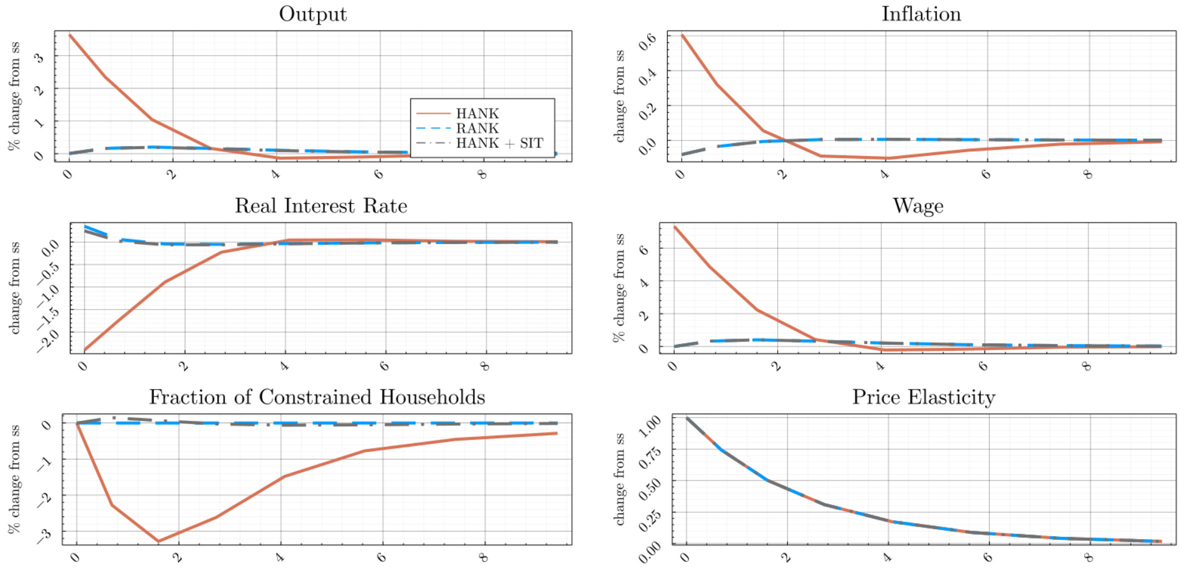


Figure 14: Response to the Markup shock. HANK vs RANK

In this case the redistributional motive has the same logic of relaxing the binding constraint for some of the households on it. The mechanism is similar to the response to the TFP shock. When comparing the difference in responses under optimal policy to SIT policy in Figure the mechanism of the optimal policy is again, to create relative redistribution of disposable income towards poorer households through raising real interest rates, which leads to higher consumption.

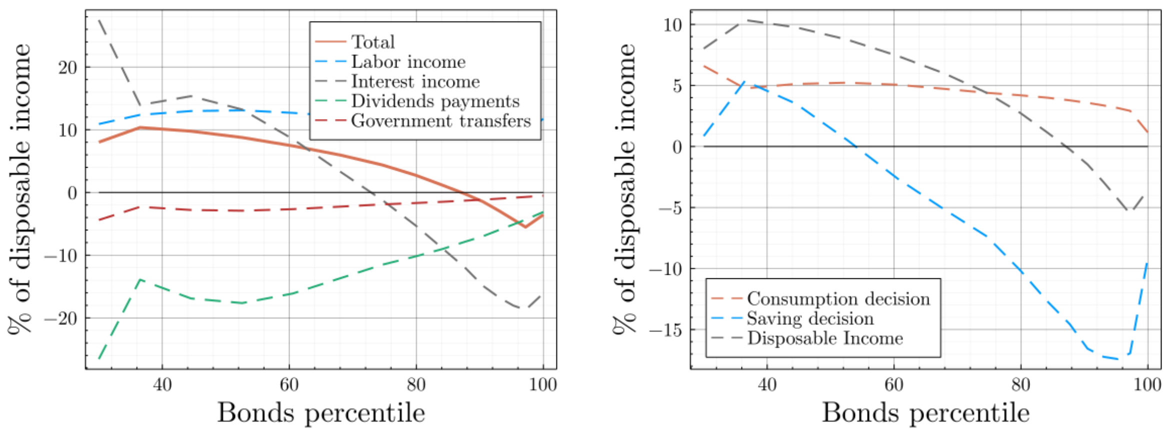


Figure 15: Disposable income and consumption-saving differential response (Ramsey - SIT)

Finally, comparing this result to the optimal policy in HANK with natural borrowing constraint and RANK in Figure 16, reveals that incentive for reducing the real interest rate is strongest in the case of occasionally binding borrowing constraint, for the same reason of reducing the fraction of constrained households.

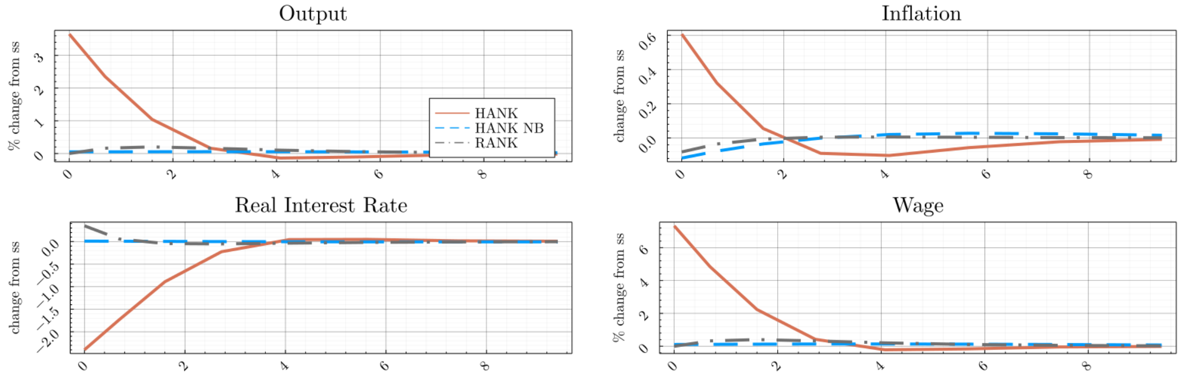


Figure 16: Response to the Markup shock. HANK vs HANK NB vs RANK

Conclusion

In this paper I study the optimal monetary policy in the HANK model with the occasionally binding borrowing constraints, when fraction of credit constrained households is endogenous. The solution to this problem makes a two-fold contribution to the literature. I extend the methods used in continuous-time optimal policy models to the application with the non-trivial binding constraint, which extends the set of models the method can be used in. Moreover, I use a different solution method that is different from the alternative approach that relies on Dynare, used in the working paper by González et al. (2021).

Using this extension I take the case from Debortoli et al. (2017), where authors argue that the presence of credit constrained agents was proved to be a crucial part in the understanding of the model, but do not have the optimal policy analysis. In this paper, I show that indeed, considering the HANK model with the endogenous fraction of constrained individuals significantly changes the optimal Ramsey policy with respect to the RANK model, which is something Bhandari et al. (2021) have already shown, but also with respect to their paper.

I conclude that the presence of an endogenous fraction of constrained households in the model introduces an important margin for the optimal monetary policy to consider. Specifically, the Ramsey planner optimally decides to relax the constraint for as many households as possible in response to the contractionary shocks to allow them to smooth their consumption. When looking at the model that is calibrated to US data, this channel proves to be most important channel, confirming the idea that the optimal policy should first target the biggest source of inefficiency in the model, which in this case is the borrowing constraint.

References

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2021, 04). Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach. *The Review of Economic Studies 89*(1), 45–86.

Algan, Y. and X. Ragot (2010). Monetary policy with heterogeneous agents and borrowing constraints. *Review of Economic Dynamics 13*(2), 295–316.

Andersen, A. L., N. Johannesen, M. Jørgensen, and J.-L. Peydró (2020). Monetary policy and inequality.

Bhandari, A., D. Evans, M. Golosov, and T. J. Sargent (2021). Inequality, business cycles, and monetary-fiscal policy. *Econometrica 89*(6), 2559–2599.

Bilbiie, F. O. and X. Ragot (2017). *Monetary Policy and Inequality when Aggregate Demand Depends on Liquidity.* Centre for Economic Policy Research.

Challe, E. et al. (2017). Uninsured unemployment risk and optimal monetary policy. *American Economic Journal: Macroeconomics*.

Clarida, R., J. Gali, and M. Gertler (1999). The science of monetary policy: a new keynesian perspective. *Journal of economic literature 37*(4), 1661–1707.

Clarke, F. (2013). *Functional analysis, calculus of variations and optimal control*, Volume 264. Springer.

Dávila, E. and A. Schaab (2022). Optimal monetary policy with heterogeneous agents: A timeless ramsey approach.

Debortoli, D., J. Galí, et al. (2017). Monetary policy with heterogeneous agents: Insights from tank models. *Manuscript, September*.

Dixit, A. K., R. K. Dixit, R. S. Pindyck, and R. Pindyck (1994). *Investment under uncertainty.* Princeton university press.

Flodén, M., M. Kilström, J. Sigurdsson, and R. Vestman (2021). Household debt and monetary policy: revealing the cash-flow channel. *The Economic Journal 131*(636), 1742–1771.

Galí, J. (2015). *Monetary policy, inflation, and the business cycle: an introduction to the new Keynesian framework and its applications.* Princeton University Press.

González, B., G. Nuño, D. Thaler, and S. Albrizio (2021). Firm heterogeneity, capital misallocation, and optimal monetary policy. *Working paper*.

Holm, M. B., P. Paul, and A. Tischbirek (2021). The transmission of monetary policy under the microscope. *Journal of Political Economy 129*(10), 2861–2904.

Kaplan, G., B. Moll, and G. L. Violante (2018). Monetary Policy According to HANK. *American Economic Review 108*(3), 697–743.

Kaplan, G. and G. L. Violante (2014). A model of the consumption response to fiscal stimulus payments. *Econometrica 82*(4), 1199–1239.

Malliavin, P. and A. Thalmaier (2006). *Stochastic calculus of variations in mathematical finance.* Springer.

McKay, A. and C. K. Wolf (2022). Optimal policy rules in hank.

Nuño, G. (2013). Optimal control with heterogeneous agents in continuous time.

Nuño, G. and C. Thomas (2016). Optimal Monetary Policy with Heterogeneous Agents. *Ssrn*.

Woodford, M. (2003). Interest and prices.

Appendix

Steady state plots

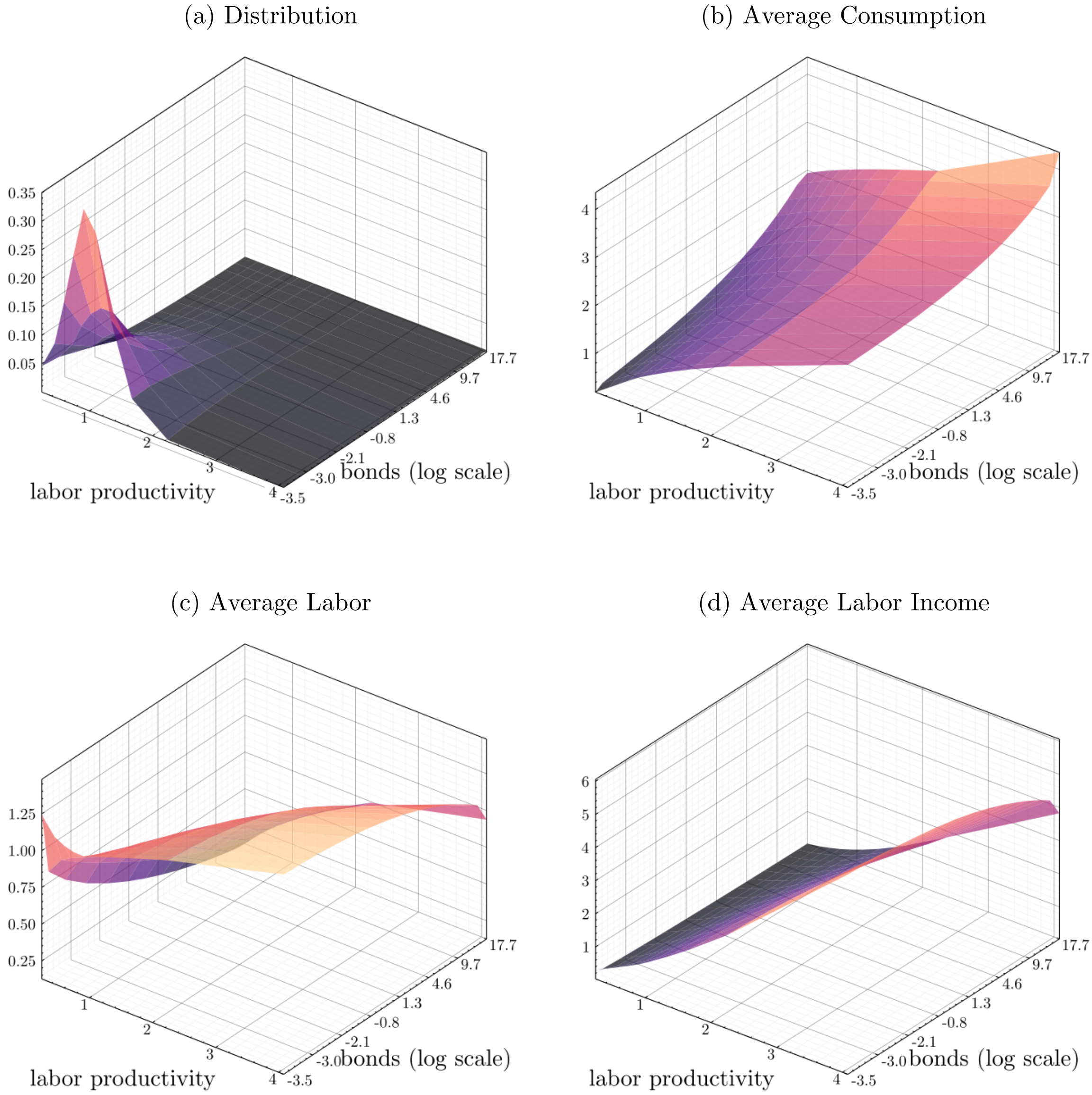


Figure 17: Steady state

Household law of motion adjoins to KF

In the appendix I will drop the time and household subscripts for the values. Though in some cases no subscripts means that the object is a function rather than a value of a function. The exact use will be clear each time and this simplification is done in order to shorten the expressions.

For the two given functions g, h the proof is the following

$$\langle g, \mathcal{A}^* h\rangle = \int\limits_s \int\limits_\varepsilon \int\limits_b g\mathcal{A}^* h db d\varepsilon ds = \tag{39}$$

$$= \int\limits_s \int\limits_\varepsilon \int\limits_b g\left(-\frac{\partial}{\partial b}\left\{\left(\lambda Wl\varepsilon + d + T + r^b b - c\right)h\right\} - \frac{\partial}{\partial \varepsilon}\rho_\varepsilon \varepsilon(\bar{e} - e)h + \frac{\partial^2}{\partial \varepsilon^2}\varepsilon\frac{\sigma_\varepsilon^2}{2}h\right) db d\varepsilon ds = \tag{40}$$

$$= -\int\limits_s \int\limits_\varepsilon \int\limits_b g\frac{\partial}{\partial b}\dot{b}h db d\varepsilon ds - \int\limits_s \int\limits_b \int\limits_\varepsilon g\frac{\partial}{\partial \varepsilon}\dot{\varepsilon}h d\varepsilon db ds + \int\limits_s \int\limits_b \int\limits_\varepsilon g\frac{\partial^2}{\partial \varepsilon^2}\frac{\sigma_\varepsilon^2}{2}h d\varepsilon db ds = \tag{41}$$

$$= \int\limits_s \int\limits_\varepsilon \int\limits_b \dot{b}h\frac{\partial}{\partial b}g\ db d\varepsilon ds - \int\limits_s \int\limits_\varepsilon \left[g\dot{b}f\Big|_{\underline{b}}^{\infty}\right] d\varepsilon ds + \int\limits_s \int\limits_b \int\limits_\varepsilon \dot{\varepsilon}h\frac{\partial}{\partial \varepsilon}g\ d\varepsilon db ds - \int\limits_s \int\limits_b \left[g\dot{\varepsilon}h|_0^\infty\right] db ds \tag{42}$$

$$+ \int\limits_s \int\limits_b \int\limits_\varepsilon \frac{\sigma_\varepsilon^2}{2}h\frac{\partial^2}{\partial \varepsilon^2}g\ d\varepsilon db ds + \int\limits_s \int\limits_b \left[g\frac{\partial}{\partial \varepsilon}\frac{\sigma_\varepsilon^2}{2}h\Big|_0^\infty\right] db ds - \int\limits_s \int\limits_b \left[\frac{\sigma_\varepsilon^2}{2}h\frac{\partial}{\partial \varepsilon}g\Big|_0^\infty\right] db ds = \tag{43}$$

$$= \langle \mathcal{A}g, h\rangle - \int\limits_s \int\limits_\varepsilon \left[g\dot{b}h\Big|_{\underline{b}}^{\infty}\right] d\varepsilon ds - \int\limits_s \int\limits_b \left[g\dot{\varepsilon}h|_0^\infty\right] db ds + \int\limits_s \int\limits_b \left[g\frac{\partial}{\partial \varepsilon}\frac{\sigma_\varepsilon^2}{2}h\Big|_0^\infty\right] db ds - \int\limits_s \int\limits_b \left[\frac{\sigma_\varepsilon^2}{2}h\frac{\partial}{\partial \varepsilon}g\Big|_0^\infty\right] db ds \tag{44}$$

Notice that when one of the functions is a distribution of the households, it must be that it is equal to zero on the limits of the domain. This implies that the only non-zero boundary integral is $\int \lim_\varepsilon g\dot{b}h\Big|_{b=\underline{b}} d\varepsilon$ This derivation together with the constraint on the density function on the limits of the domain is going to be crucially important for the derivation of the optimal policy.

Ramsey problem in RANK

This section shows how to solve the RANK optimal MP Ramsey problem using Calculus of Variations.

The Ramsey problem has the following form:

$$\mathcal{L}[c, l, W, Y, \pi, r^b] = \tag{45}$$

$$= \int_0^\infty e^{-\rho t}\left[Z_t\left(\frac{c_t^{1-\nu}}{1-\nu} - \varphi\frac{l_t^{1+\gamma}}{1+\gamma}\right) + \varrho_t\left(r_t^b - \rho + \frac{\dot{Z}_t}{Z_t} - \nu\frac{\dot{c}_t}{c_t}\right) + \mu_{i,t}\left(l_t^\gamma c_t^\nu - \frac{\lambda W_t}{\varphi}\right) + \right. \tag{46}$$

$$+ \eta_{Y,t}\left(Y_t - \theta_t l_t\right) + \eta_{T,t}\left(c_t - \left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t\right) + \tag{47}$$

$$\left. + \eta_{\pi,t}\left(\frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1}\frac{W_t}{\theta_t} - 1\right) + \dot{\pi}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t\right)\right] dt \tag{48}$$

Control over consumption

According to Calculus of Variations, I start with taking the first variation wrt the control function at hand. This implies taking the variation of the function and separately the variation of the variation of the derivative of this function. Notice that here I can take all the variations separately, as they do not interact between each other and the strong forms can be derived separately for each of the controls. This will not be the case in the HANK problem.

Weak form:

$$\frac{\delta\mathcal{L}}{\delta c} = \int_0^\infty e^{-\rho t}\left[Z_t c_t^{-\nu} v_{c,t} + \nu\frac{\varrho_t \dot{c}_t}{c_t^2}v_{c,t} - \nu\varrho_t\frac{\dot{v}_{c,t}}{c_t} + \nu\mu_t l_t^\gamma c_t^{\nu-1}v_{c,t} + \eta_{T,t}v_{c,t}\right] dt \tag{49}$$

To proceed further, I have to isolate the all the variations in their initial form without derivatives.

Rearranging the term with $\dot{v}_{c,t}$:

$$\int_0^\infty e^{-\rho t}\left[-\nu\varrho_t\frac{\dot{v}_{c,t}}{c_t}\right] dt = -e^{-\rho t}\nu\varrho_t\frac{v_{c,t}}{c_t}\bigg|_0^\infty + \int_0^\infty e^{-\rho t}\left[\nu\frac{\dot{\varrho}_t}{c_t} - \nu\varrho_t\frac{\dot{c}_t}{c_t^2} - \rho\nu\frac{\varrho_t}{c_t}\right] v_{c,t} dt \tag{50}$$

Finally weak form is:

$$\frac{\delta\mathcal{L}}{\delta c} = \int_0^\infty e^{-\rho t}\left[Z_t c_t^{-\nu} + \nu\frac{\dot{\varrho}_t}{c_t} - \rho\nu\frac{\varrho_t}{c_t} + \nu\mu_t l_t^\gamma c_t^{\nu-1} + \eta_{T,t}\right] v_{c,t} dt - e^{-\rho t}\nu\varrho_t\frac{v_{c,t}}{c_t}\bigg|_0^\infty = 0 \tag{51}$$

Now using the standard CV approach to allow for any admissible variation, we can conclude that at any point the term that multiplies the variation has to be zero. Thus the strong form:

$$\begin{cases} \rho\varrho_t = \dot{\varrho}_t + Z_t \dfrac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \dfrac{\eta_{T,t} c_t}{\nu} \\ \lim\limits_{t\to\infty} e^{-\rho t} \nu \dfrac{\varrho_t}{c_t} = 0 \\ \varrho_t \Big|_{t=0} = 0 \end{cases} \tag{52}$$

Control over labor

Weak form:

$$\frac{\delta \mathcal{L}}{\delta l} = \int_0^\infty e^{-\rho t} \left[-Z_t \varphi l_t^{\gamma} v_{l,t} + \gamma \mu_t l_t^{\gamma-1} c_t^{\nu} v_{l,t} - \eta_{Y,t} \theta_t v_{l,t} \right] dt = 0 \tag{53}$$

Strong form:

$$-Z_t \varphi l_t^{\gamma} + \gamma \mu_t l_t^{\gamma-1} c_t^{\nu} - \eta_{Y,t} \theta_t = 0 \tag{54}$$

Control over wage

Weak form:

$$\frac{\delta \mathcal{L}}{\delta W} = \int_0^\infty e^{-\rho t} \left[-\mu_t \frac{\lambda}{\varphi} v_{W,t} + \eta_{\pi,t} \frac{\phi_t}{\psi \theta_t} v_{W,t} \right] dt = 0 \tag{55}$$

Strong form:

$$-\mu_t \frac{\lambda}{\varphi} + \eta_{\pi,t} \frac{\phi_t}{\psi \theta_t} = 0 \tag{56}$$

Control over return

Weak form:

$$\frac{\delta \mathcal{L}}{\delta r^b} = \int_0^\infty e^{-\rho t} \left[\varrho_t v_{r^b,t} - \eta_\pi \pi_t v_{r^b,t} \right] dt = 0 \tag{57}$$

Strong form:

$$\varrho_t = \eta_\pi \pi_t \tag{58}$$

Control over output

Weak form:

$$\frac{\delta \mathcal{L}}{\delta Y} = \int_0^\infty e^{-\rho t} \left[-\eta_{T,t} \left(1 - \frac{\psi}{2} \pi_t^2 \right) v_{Y,t} + \eta_{Y,t} v_{Y,t} + \eta_{\pi,t} \pi_t \frac{\dot{v}_{Y,t}}{Y_t} - \eta_{\pi,t} \pi_t \frac{\dot{Y}_t}{Y_t^2} v_{Y,t} \right] dt \tag{59}$$

In the same way as in the control over the consumption, to proceed further, I have to isolate all the variations in their initial form without derivatives.

Rearranging the term with $\dot{v}_{Y,t}$:

$$\int_0^\infty e^{-\rho t} \left[\eta_{\pi,t} \pi_t \frac{\dot{v}_{Y,t}}{Y_t} \right] dt = e^{-\rho t} \eta_{\pi,t} \pi_t \frac{v_{Y,t}}{Y_t} \Big|_0^\infty - \int_0^\infty e^{-\rho t} \left[\frac{\frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t)}{Y_t} - \frac{\eta_{\pi,t}\pi_t \dot{Y}_t}{Y_t^2} - \rho \frac{\eta_{\pi,t}\pi_t}{Y_t} \right] v_{Y,t} dt \tag{60}$$

This implies the weak form to be:

$$\frac{\delta \mathcal{L}}{\delta Y} = \int_0^\infty e^{-\rho t} \left[-\eta_{T,t} \left(1 - \frac{\psi}{2} \pi_t^2 \right) + \eta_{Y,t} - \frac{\frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t)}{Y_t} + \rho \frac{\eta_{\pi,t}\pi_t}{Y_t} \right] v_{Y,t} dt + e^{-\rho t} \eta_{\pi,t} \pi_t \frac{v_{Y,t}}{Y_t} \Big|_0^\infty = 0 \tag{61}$$

Strong form:

$$\begin{cases} \rho \eta_{\pi,t} \pi_t = \frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t) + \eta_{T,t} \left(1 - \frac{\psi}{2} \pi_t^2 \right) Y_t - \eta_{Y,t} Y_t \\ \lim\limits_{t \to \infty} e^{-\rho t} \frac{\eta_{\pi,t}\pi_t}{Y_t} = 0 \\ \eta_{\pi,t} \pi_t \Big|_{t=0} = 0 \end{cases} \tag{62}$$

Control over inflation

Weak form:

$$\frac{\delta \mathcal{L}}{\delta \pi} = \int_0^\infty e^{-\rho t} \left[-\eta_{T,t} \psi \pi_t Y_t v_{\pi,t} - \eta_{\pi,t} \left(r_t^b - \frac{\dot{Y}_t}{Y_t} - \rho \right) - \dot{\eta}_{\pi,t} v_{\pi,t} \right] dt + e^{-\rho t} \eta_{\pi,t} v_{\pi,t} \Big|_0^\infty = 0 \tag{63}$$

Strong form:

$$\begin{cases} \dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{T,t}\psi\pi_t Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\ \eta_{\pi,t}\Big|_{t=0} = 0 \end{cases} \tag{64}$$

Optimal MP policy

Finally all conditions can be brought together in one system:

$$\begin{cases} \rho\varrho_t = \dot{\varrho}_t + Z_t\dfrac{c_t^{1-\nu}}{\nu} + \mu_t l_t^\gamma c_t^\nu + \dfrac{\eta_{T,t}c_t}{\nu} \\ \lim\limits_{t\to\infty} e^{-\rho t}\varrho_t = 0 \\ \varrho_t\Big|_{t=0} = 0 \\ -Z_t\varphi l_t^\gamma + \gamma\mu_t l_t^{\gamma-1}c_t^\nu - \eta_{Y,t}\theta_t = 0 \\ -\mu_t\frac{\lambda}{\varphi} + \eta_{\pi,t}\frac{\phi_t}{\psi\theta_t} = 0 \\ \varrho_t = \eta_\pi\pi_t \\ \rho\eta_{\pi,t}\pi_t = \frac{\partial}{\partial t}(\eta_{\pi,t}\pi_t) + \eta_{T,t}\left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t - \eta_{Y,t}Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\dfrac{\eta_{\pi,t}\pi_t}{Y_t} = 0 \\ \eta_{\pi,t}\pi_t\Big|_{t=0} = 0 \\ \dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{T,t}\psi\pi_t Y_t \\ \lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\ \eta_{\pi,t}\Big|_{t=0} = 0 \end{cases} \tag{65}$$

Combining differential equation for output control, inflation control and PC gives

$$\eta_{Y,t} = \eta_{T,t}\left(1 + \frac{\psi}{2}\pi_t^2\right) - \frac{\phi_t - 1}{\psi}\left(\frac{\phi_t}{\phi_t - 1}\frac{W_t}{\theta_t} - 1\right)\frac{1}{Y_t}\eta_{\pi,t} \tag{66}$$

And combining consumption control with output control and real return control

$$Z_t\frac{c_t^{1-\nu}}{\nu} + \mu_t l_t^\gamma c_t^\nu + \frac{\eta_{T,t}c_t}{\nu} = \eta_{T,t}\left(1 - \frac{\psi}{2}\pi_t^2\right)Y_t - \eta_{Y,t}Y_t \tag{67}$$

So the Optimal MP problem becomes:

$$
\begin{cases}
Z_t \dfrac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \dfrac{\eta_{T,t} c_t}{\nu} = \eta_{T,t}\left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t - \eta_{Y,t} Y_t \\
-Z_t \varphi l_t^{\gamma} + \gamma \mu_t l_t^{\gamma-1} c_t^{\nu} - \eta_{Y,t}\theta_t = 0 \\
-\mu_t \dfrac{\lambda}{\varphi} + \eta_{\pi,t} \dfrac{\phi_t}{\psi \theta_t} = 0 \\
\eta_{Y,t} = \eta_{T,t}\left(1 + \dfrac{\psi}{2}\pi_t^2\right) - \dfrac{\phi_t - 1}{\psi}\left(\dfrac{\phi_t}{\phi_t - 1}\dfrac{W_t}{\theta_t} - 1\right)\dfrac{1}{Y_t}\eta_{\pi,t} \\
\dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{T,t}\psi \pi_t Y_t \\
\lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\
\eta_{\pi,t}\Big|_{t=0} = 0
\end{cases}
\tag{68}
$$

Rearranging the first three equations:

$$
\begin{cases}
\mu_t = \dfrac{Z_t \varphi l_t}{\gamma c_t^{\nu}} + \eta_{Y,t}\dfrac{\theta_t}{\gamma l_t^{\gamma-1} c_t^{\nu}} \\
\eta_{\pi,t} = \mu_t \dfrac{\lambda \psi \theta_t}{\phi_t \varphi} \\
\eta_{T,t} = \dfrac{Z_t \frac{c_t^{1-\nu}}{\nu} + \mu_t l_t^{\gamma} c_t^{\nu} + \eta_{Y,t} Y_t}{\left(1 - \frac{1}{\nu}\right)\left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t} \\
\eta_{Y,t} = \eta_{T,t}\left(1 + \dfrac{\psi}{2}\pi_t^2\right) - \eta_{\pi,t}\dfrac{1}{Y_t}\dfrac{\phi_t - 1}{\psi}\left(\dfrac{\phi_t}{\phi_t - 1}\dfrac{W_t}{\theta_t} - 1\right) \\
\dot{\eta}_{\pi,t} = \left(\rho - r_t^b + \dfrac{\dot{Y}_t}{Y_t}\right)\eta_{\pi,t} + \eta_{T,t}\psi \pi_t Y_t \\
\lim\limits_{t\to\infty} e^{-\rho t}\eta_{\pi,t} = 0 \\
\eta_{\pi,t}\Big|_{t=0} = 0
\end{cases}
\tag{69}
$$

Ramsey problem in HANK

This section provides comprehensive scratch notes for the Ramsey problem solution using Calculus of Variations.

First, let's write the problem of the Ramsey planner:

$$\mathcal{L}[f,\mathcal{V},c,l,W,Y,\pi,r^b,T,d] = \tag{70}$$

$$= \int_0^\infty e^{-\rho t}\left[\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma}, f_{i,t}\right\rangle + \left\langle \zeta_{i,t}, \mathcal{A}^*_{i,t}f_{i,t} - \frac{\partial f_{i,t}}{\partial t}\right\rangle\right. \tag{71}$$

$$+ \left\langle \varrho_{i,t}, \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t}\mathcal{V}_{i,t} + \frac{\partial \mathcal{V}_{i,t}}{\partial t} - \rho\mathcal{V}_{i,t}\right\rangle \tag{72}$$

$$+ \left\langle \mu_{i,t}, c_{i,t}^{-\nu} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}\right\rangle + \left\langle \kappa_{i,t}, l_{i,t}^{\gamma}c_{i,t}^{\nu} - \frac{\lambda W_t\varepsilon_{i,t}}{\varphi}\right\rangle \tag{73}$$

$$+ \eta_{b,t}\left\langle b_{i,t}, f_{i,t}\right\rangle + \eta_{Y,t}\left(Y_t - \theta_t\left\langle l_{i,t}\varepsilon_{i,t}, f_{i,t}\right\rangle\right) \tag{74}$$

$$+ \eta_{T,t}\left(T_t - (1-\lambda)\frac{W_t}{\theta_t}Y_t\right) + \eta_{d,t}\left(d_t - \left(1 - \frac{\psi}{2}\pi_t^2 - \frac{W_t}{\theta_t}\right)Y_t\right) \tag{75}$$

$$\left. + \eta_{\pi,t}\left(\frac{\phi-1}{\psi}\left(\frac{\phi}{\phi-1}\frac{W_t}{\theta_t} - 1\right) + \dot{\pi}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t\right)\right] dt \tag{76}$$

Where

$$\mathcal{A}_{i,t}\mathcal{V}_{i,t} = \dot{b}_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial b} + \rho_\varepsilon\varepsilon_{i,t}(\bar{e} - e_{i,t})\frac{\partial \mathcal{V}_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial^2 \mathcal{V}_{i,t}}{\partial \varepsilon^2} \tag{77}$$

$$\mathcal{A}^*_{i,t}f_{i,t} = -\frac{\partial}{\partial b}\dot{b}_{i,t}f_{i,t} - \frac{\partial}{\partial \varepsilon}\rho_\varepsilon\varepsilon_{i,t}(\bar{e} - e_{i,t})f_{i,t} + \frac{\partial^2}{\partial \varepsilon^2}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}f_{i,t} \tag{78}$$

$$\dot{b}_{i,t} = \lambda W_t l_{i,t}\varepsilon_{i,t} + r_t^b b_{i,t} + T_t + d_t s_i - c_{i,t} \tag{79}$$

And for $\mathcal{E},\mathcal{B},\mathcal{S}$ defined as sets such that $\varepsilon \in \mathcal{E}, b \in \mathcal{B}, s \in \mathcal{S}$ and $\partial\mathcal{E},\partial\mathcal{B},\partial\mathcal{S}$ defined as boundaries of sets $\mathcal{E},\mathcal{B},\mathcal{S}$.

$$\langle\cdot,\cdot\rangle \equiv \int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}} \cdot\cdot\ db d\varepsilon ds \tag{80}$$

Using calculus of variations I will take the functional derivatives of $\mathcal{L}$ wrt all of the control functions. Notice that the key idea is to treat derivatives of the control functions in the Lagrangean as functions themselves.

One important aspect is that the variations should be taken all together, and not separately. This matters for the treatment of the boundary conditions. For the readability, I will write the variations one by one, but highlight the place in the derivation where I use the fact that variations are taken together to treat the boundary constraint on the bonds.

Control over distribution

Weak form:

$$\frac{\delta \mathcal{L}}{\delta f} = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi \frac{l_{i,t}^{1+\gamma}}{1+\gamma}, v_{i,t} \right\rangle + \left\langle \zeta_{i,t}, \mathcal{A}_{i,t}^{*} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right\rangle \right. \tag{81}$$

$$\left. + \eta_{b,t} \langle b_{i,t}, v_{i,t} \rangle + \eta_{Y,t} \left(-\theta_t \langle l_{i,t} \varepsilon_{i,t}, v_{i,t} \rangle \right) \right] dt = 0 \tag{82}$$

Now to move to the strong form some rearrangement should be done to the differential part of the weak form

$$\int_0^\infty e^{-\rho t} \left[\left\langle \zeta_{i,t}, \mathcal{A}_{i,t}^{*} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right\rangle \right] dt = \tag{83}$$

$$\int_0^\infty e^{-\rho t} \left[\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(\mathcal{A}_{i,t}^{*} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right) db d\varepsilon ds \right] dt = \tag{84}$$

$$\int_0^\infty e^{-\rho t} \left[\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(-\frac{\partial}{\partial b} \dot{b}_{i,t} v_{i,t} - \frac{\partial}{\partial \varepsilon} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} + \frac{\partial^2}{\partial \varepsilon^2} \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} v_{i,t} - \frac{\partial v_{i,t}}{\partial t} \right) db d\varepsilon ds \right] dt \tag{85}$$

Treating all four parts of the integral separately:

First:

$$- \int_0^\infty e^{-\rho t} \left[\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(\frac{\partial}{\partial b} \dot{b}_{i,t} v_{i,t} \right) db d\varepsilon ds \right] dt = \tag{86}$$

$$- \int_0^\infty e^{-\rho t} \int_{\mathcal{S}} \int_{\mathcal{E}} \left[\left(\zeta_{i,t} \dot{b}_{i,t} v_{i,t} \right) \Big|_{\partial \mathcal{B}} - \int_{\mathcal{B}} \dot{b}_{i,t} v_{i,t} \frac{\partial \zeta_{i,t}}{\partial b} db \right] d\varepsilon ds dt \tag{87}$$

Second:

$$- \int_0^\infty e^{-\rho t} \left[\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \zeta_{i,t} \left(\frac{\partial}{\partial \varepsilon} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \right) db d\varepsilon ds \right] dt = \tag{88}$$

$$- \int_0^\infty e^{-\rho t} \int_{\mathcal{S}} \int_{\mathcal{B}} \left[\left(\zeta_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \right) \Big|_{\partial \mathcal{E}} - \int_{\mathcal{E}} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \frac{\partial \zeta_{i,t}}{\partial \varepsilon} d\varepsilon \right] db ds dt \tag{89}$$

Third:

$$\int_0^\infty e^{-\rho t}\left[\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}}\zeta_{i,t}\left(\frac{\partial^2}{\partial\varepsilon^2}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)dbd\varepsilon ds\right]dt = \tag{90}$$

$$\int_0^\infty e^{-\rho t}\int_{\mathcal{S}}\int_{\mathcal{B}}\left[\zeta_{i,t}\left(\frac{\partial}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)\Bigg|_{\partial\mathcal{E}} - \int_{\mathcal{E}}\frac{\partial\zeta_{i,t}}{\partial\varepsilon}\left(\frac{\partial}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)d\varepsilon\right]dbdsdt = \tag{91}$$

$$\int_0^\infty e^{-\rho t}\int_{\mathcal{S}}\int_{\mathcal{B}}\left[\zeta_{i,t}\left(\frac{\partial}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)\Bigg|_{\partial\mathcal{E}} - \left(\frac{\partial\zeta_{i,t}}{\partial\varepsilon}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\right)\Bigg|_{\partial\mathcal{E}} + \int_{\mathcal{E}}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}v_{i,t}\frac{\partial^2\zeta_{i,t}}{\partial\varepsilon^2}d\varepsilon\right]dbdsdt \tag{92}$$

Fourth:

$$-\int_0^\infty e^{-\rho t}\left[\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}}\zeta_{i,t}\left(\frac{\partial v_{i,t}}{\partial t}\right)dbd\varepsilon ds\right]dt = -\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}}\left[\int_0^\infty e^{-\rho t}\zeta_{i,t}\frac{\partial v_{i,t}}{\partial t}dt\right]dbd\varepsilon ds = \tag{93}$$

$$-\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}}\left[e^{-\rho t}\zeta_{i,t}v_{i,t}\Big|_0^\infty - \int_0^\infty v_{i,t}\frac{\partial}{\partial t}e^{-\rho t}\zeta_{i,t}dt\right]dbd\varepsilon ds = \tag{94}$$

$$-\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}}\left[e^{-\rho t}\zeta_{i,t}v_{i,t}\Big|_0^\infty - \int_0^\infty v_{i,t}e^{-\rho t}\left(\frac{\partial\zeta_{i,t}}{\partial t} - \rho\zeta_{i,t}\right)dt\right]dbd\varepsilon ds \tag{95}$$

Finally, notice that the boundary constraint on $f_{i,t}$ implies that it has to be equal to zero on any boundary except $b = \underline{b}$ for values of $\dot{b} = 0$, as well as $\frac{\partial f_{i,t}}{\partial\varepsilon} = 0$ for $f \in \partial\mathcal{E}$. This implies that all the boundaries $\Big|_{\partial\mathcal{B}}$ and $\Big|_{\partial\mathcal{E}}$ actually cancel out. Moreover, since distribution is fixed at $t = 0$, the only boundary condition that is left is the one at $t \to \infty$. Then the term can be written as

$$\int_0^\infty e^{-\rho t}\left[\left\langle\zeta_{i,t}, \mathcal{A}_{i,t}^* v_{i,t} - \frac{\partial v_{i,t}}{\partial t}\right\rangle\right]dt = \tag{96}$$

$$\int_0^\infty e^{-\rho t}\left[\left\langle\mathcal{A}_{i,t}\zeta_{i,t} + \frac{\partial\zeta_{i,t}}{\partial t} - \rho\zeta_{i,t}, v_{i,t}\right\rangle\right]dt - \lim_{t\to\infty}\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}}e^{-\rho t}\zeta_{i,t}v_{i,t}dbd\varepsilon ds \tag{97}$$

Strong form:

$$\frac{c_{i,t}^{1-\nu}}{1-\nu} - \varphi\frac{l_{i,t}^{1+\gamma}}{1+\gamma} + \mathcal{A}_{i,t}\zeta_{i,t} + \frac{\partial\zeta_{i,t}}{\partial t} - \rho\zeta_{i,t} + \eta_{b,t}b_{i,t} - \eta_{Y,t}\theta_t l_{i,t}\varepsilon_{i,t} = 0 \tag{98}$$

$$\lim_{t\to\infty}e^{-\rho t}\zeta_{i,t} = 0 \tag{99}$$

Control over household value

Weak form:

$$\frac{\delta \mathcal{L}}{\delta \mathcal{V}} = \int_0^\infty e^{-\rho t} \left[\left\langle \varrho_{i,t}, \mathcal{A}_{i,t} v_{i,t} + \frac{\partial v_{i,t}}{\partial t} - \rho v_{i,t} \right\rangle + \left\langle \mu_{i,t}, -\frac{\partial v_{i,t}}{\partial b} \right\rangle \right] dt = 0 \tag{100}$$

Where

$$\mathcal{A}_{i,t} v_{i,t} = \dot{b}_{i,t} \frac{\partial v_{i,t}}{\partial b} + \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) \frac{\partial v_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial^2 v_{i,t}}{\partial \varepsilon^2} \tag{101}$$

$$\dot{b}_{i,t} = \lambda W_t l_{i,t} \varepsilon_{i,t} + r_t^b b_{i,t} + T_t + d_t s_i - c_{i,t} \tag{102}$$

For $\mathcal{A}_{i,t} v_{i,t}$ the following rearrangement can be done:

$$\langle \varrho_{i,t}, \mathcal{A}_{i,t} v_{i,t} \rangle = \int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \mathcal{A}_{i,t} v_{i,t} db d\varepsilon ds = \tag{103}$$

$$\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \left(\dot{b}_{i,t} \frac{\partial v_{i,t}}{\partial b} + \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) \frac{\partial v_{i,t}}{\partial \varepsilon} + \varepsilon_{i,t} \frac{\sigma_\varepsilon^2}{2} \frac{\partial^2 v_{i,t}}{\partial \varepsilon^2} \right) db d\varepsilon ds \tag{104}$$

As in the previous case, working the three terms separately.

First:

$$\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \dot{b}_{i,t} \frac{\partial v_{i,t}}{\partial b} db d\varepsilon ds = \int_{\mathcal{S}} \int_{\mathcal{E}} \varrho_{i,t} \dot{b}_{i,t} v_{i,t} \Big|_{\partial \mathcal{B}} d\varepsilon ds - \int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} v_{i,t} \frac{\partial}{\partial b} \varrho_{i,t} \dot{b}_{i,t} db d\varepsilon ds \tag{105}$$

Second:

$$\int_{\mathcal{S}} \int_{\mathcal{E}} \int_{\mathcal{B}} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) \frac{\partial v_{i,t}}{\partial \varepsilon} db d\varepsilon ds = \tag{106}$$

$$\int_{\mathcal{S}} \int_{\mathcal{B}} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) v_{i,t} \Big|_{\partial \mathcal{E}} db ds - \int_{\mathcal{S}} \int_{\mathcal{B}} \int_{\mathcal{E}} v_{i,t} \frac{\partial}{\partial \varepsilon} \varrho_{i,t} \rho_\varepsilon \varepsilon_{i,t} (\bar{e} - e_{i,t}) d\varepsilon db ds \tag{107}$$

Third:

$$\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}} \varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial^2 v_{i,t}}{\partial\varepsilon^2}dbd\varepsilon ds = \tag{108}$$

$$\int_{\mathcal{S}}\int_{\mathcal{B}} \varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial v_{i,t}}{\partial\varepsilon}\bigg|_{\partial\mathcal{E}} dbds - \int_{\mathcal{S}}\int_{\mathcal{B}}\int_{\mathcal{E}} \frac{\partial v_{i,t}}{\partial\varepsilon}\frac{\partial}{\partial\varepsilon}\varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}d\varepsilon dbds = \tag{109}$$

$$\int_{\mathcal{S}}\int_{\mathcal{B}} \varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial v_{i,t}}{\partial\varepsilon}\bigg|_{\partial\mathcal{E}} dbds - \int_{\mathcal{S}}\int_{\mathcal{B}} v_{i,t}\frac{\partial}{\partial\varepsilon}\varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\bigg|_{\partial\mathcal{E}} dbds + \int_{\mathcal{S}}\int_{\mathcal{B}}\int_{\mathcal{E}} v_{i,t}\frac{\partial^2}{\partial\varepsilon^2}\varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}d\varepsilon dbds \tag{110}$$

Notice, that because of Inada conditions, the values of $\frac{\partial\mathcal{V}}{\partial b}$ and $\frac{\partial\mathcal{V}}{\partial\varepsilon}$ are fixed to be zero for $b\to\infty$ and $\varepsilon\to\infty$. This implies that variation of partial derivatives has to be zero at these boundaries as well. This means that the expression becomes

$$\langle\varrho_{i,t},\mathcal{A}_{i,t}v_{i,t}\rangle = \left\langle\mathcal{A}^*_{i,t}\varrho_{i,t},v_{i,t}\right\rangle + \int_{\mathcal{S}}\int_{\mathcal{E}} \varrho_{i,t}\dot{b}_{i,t}v_{i,t}\bigg|_{b=\underline{b}} d\varepsilon ds + \int_{\mathcal{S}}\int_{\mathcal{B}} \varrho_{i,t}\rho_\varepsilon\varepsilon_{i,t}(\bar{e}-e_{i,t})v_{i,t}\bigg|_{\varepsilon=\underline{\varepsilon}} dbds+ \tag{111}$$

$$+\int_{\mathcal{S}}\int_{\mathcal{B}} \varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\frac{\partial v_{i,t}}{\partial\varepsilon}\bigg|_{\varepsilon=\underline{\varepsilon}} dbds - \int_{\mathcal{S}}\int_{\mathcal{B}} v_{i,t}\frac{\partial}{\partial\varepsilon}\varrho_{i,t}\varepsilon_{i,t}\frac{\sigma_\varepsilon^2}{2}\bigg|_{\varepsilon=\underline{\varepsilon}} dbds \tag{112}$$

Finally notice, that since the distribution of ε is a log-normal, then $\underline{\varepsilon}=0$, so that almost all the terms cancel out

$$\langle\varrho_{i,t},\mathcal{A}_{i,t}v_{i,t}\rangle = \left\langle\mathcal{A}^*_{i,t}\varrho_{i,t},v_{i,t}\right\rangle + \int_{\mathcal{S}}\int_{\mathcal{E}} \varrho_{i,t}\dot{b}_{i,t}v_{i,t}\bigg|_{b=\underline{b}} d\varepsilon ds - \int_{\mathcal{S}}\int_{\mathcal{B}} v_{i,t}\varrho_{i,t}\frac{\sigma_\varepsilon^2}{2}\bigg|_{\varepsilon=\underline{\varepsilon}} dbds \tag{113}$$

For the FOC constraints (utilizing Inada conditions on $b\to\infty$):

$$-\left\langle\mu_{i,t},\frac{\partial v_{i,t}}{\partial b}\right\rangle = -\int_{\mathcal{S}}\int_{\mathcal{E}} \mu_{i,t}v_{i,t}\bigg|_{b=\underline{b}} d\varepsilon ds + \int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}} v_{i,t}\frac{\partial\mu_{i,t}}{\partial b}dbd\varepsilon ds \tag{114}$$

For the time derivative:

$$\int_0^\infty e^{-\rho t}\left[\left\langle\varrho_{i,t},\frac{\partial v_{i,t}}{\partial t}-\rho v_{i,t}\right\rangle\right]dt = \left\langle\varrho_{i,t}e^{-\rho t}v_{i,t}\bigg|_0^\infty\right\rangle - \left\langle\int_0^\infty e^{-\rho t}v_{i,t}\frac{\partial\varrho_{i,t}}{\partial t}dt\right\rangle \tag{115}$$

Finally, this means that the strong form is:

$$\mathcal{A}^*_{i,t}\varrho_{i,t} - \frac{\partial \varrho_{i,t}}{\partial t} + \frac{\partial \mu_{i,t}}{\partial b} = 0 \tag{116}$$

$$\varrho_{i,0} = 0 \tag{117}$$

$$\varrho_{i,t}\Big|_{\varepsilon=\underline{\varepsilon}} = 0 \tag{118}$$

$$\varrho_{i,t}\dot{b}_{i,t} - \mu_{i,t}\Big|_{b=\underline{b}} = 0 \tag{119}$$

Control over household consumption

Weak form:

$$\frac{\delta \mathcal{L}}{\delta c} = \int_0^\infty e^{-\rho t}\Bigg[\Big\langle c_{i,t}^{-\nu}v_{i,t}, f_{i,t}\Big\rangle + \Big\langle \zeta_{i,t}, \frac{\partial}{\partial b}v_{i,t}f_{i,t}\Big\rangle + \Big\langle \varrho_{i,t}, c_{i,t}^{-\nu}v_{i,t} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}v_{i,t}\Big\rangle \tag{120}$$

$$+ \Big\langle \mu_{i,t}, -\nu c_{i,t}^{-\nu-1}v_{i,t}\Big\rangle + \Big\langle \kappa_{i,t}, \nu l_{i,t}^{\gamma}c_{i,t}^{\nu-1}v_{i,t}\Big\rangle\Bigg]dt = 0 \tag{121}$$

Rearranging the term with the derivative

$$\int_0^\infty e^{-\rho t}\Bigg[\Big\langle \zeta_{i,t}, \frac{\partial}{\partial b}v_{i,t}f_{i,t}\Big\rangle\Bigg]dt = \int_0^\infty e^{-\rho t}\Bigg[\int_{\mathcal{S}}\int_{\mathcal{E}}\int_{\mathcal{B}} \zeta_{i,t}\frac{\partial}{\partial b}v_{i,t}f_{i,t}\,db\,d\varepsilon\,ds\Bigg]dt = \tag{122}$$

$$\int_0^\infty e^{-\rho t}\int_{\mathcal{S}}\int_{\mathcal{E}}\Bigg[\zeta_{i,t}v_{i,t}f_{i,t}\Big|_{\partial\mathcal{B}} - \int_{\mathcal{B}} v_{i,t}f_{i,t}\frac{\partial \zeta_{i,t}}{\partial b}db\Bigg]d\varepsilon\,ds\,dt \tag{123}$$

The strong form is:

$$c_{i,t}^{-\nu}f_{i,t} - f_{i,t}\frac{\partial \zeta_{i,t}}{\partial b} + \varrho_{i,t}\left(c_{i,t}^{-\nu} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}\right) - \mu_{i,t}\nu c_{i,t}^{-\nu-1} + \kappa_{i,t}\nu l_{i,t}^{\gamma}c_{i,t}^{\nu-1} = 0 \tag{124}$$

$$\zeta_{i,t}f_{i,t}\Big|_{b=\underline{b}} = 0 \tag{125}$$

Since FOC wrt c holds, the condition becomes

$$\left(c_{i,t}^{-\nu} - \frac{\partial \zeta_{i,t}}{\partial b}\right)f_{i,t} - \mu_{i,t}\nu c_{i,t}^{-\nu-1} + \kappa_{i,t}\nu l_{i,t}^{\gamma}c_{i,t}^{\nu-1} = 0 \tag{126}$$

$$\zeta_{i,t}f_{i,t}\Big|_{b=\underline{b}} = 0 \tag{127}$$

Here I come back to the point that that the variations should be taken together. This means that the boundary conditions for $b = \underline{b}$ should be satisfied as a sum, not separately from each other. This means that the following should hold:

Combining the equation for μ with constraint from value control one can get the following conditions:

If $\dot{b}_{i,t} \neq 0$ at $b = \underline{b}$, then $f_{i,t} = 0$

$$\Rightarrow \kappa_{i,t} = 0 \Rightarrow \mu_{i,t} = 0 \Rightarrow \varrho_{i,t} = 0 \tag{128}$$

If $\dot{b}_{i,t} = 0$ at $b = \underline{b}$, then $f_{i,t} \neq 0 \Rightarrow \mu_{i,t} = 0$. Using FOC l from below:

$$\Rightarrow \left(\frac{c_{i,t}}{\nu} + \frac{l_{i,t}}{\gamma}\lambda W_t \varepsilon_{i,t}\right)\left(\frac{\partial \zeta_{i,t}}{\partial b} - \frac{\partial \mathcal{V}_{i,t}}{\partial b}\right) = \frac{\eta_{Y,t}\theta_t \varepsilon_{i,t} l_{i,t}}{\gamma} \tag{129}$$

Control over household labor

Weak form:

$$\frac{\delta \mathcal{L}}{\delta l} = \int_0^\infty e^{-\rho t}\left[\left\langle -\varphi l_{i,t}^{\gamma} v_{i,t}, f_{i,t}\right\rangle + \left\langle \zeta_{i,t}, -\frac{\partial}{\partial b}\lambda W_t v_{i,t}\varepsilon_{i,t} f_{i,t}\right\rangle\right. \tag{130}$$

$$+ \left\langle \varrho_{i,t}, -\varphi l_{i,t}^{\gamma} v_{i,t} + \lambda W_t v_{i,t}\varepsilon_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial b}\right\rangle \tag{131}$$

$$\left. + \left\langle \kappa_{i,t}, \gamma l_{i,t}^{\gamma-1} v_{i,t} c_{i,t}^{\nu}\right\rangle + \eta_{Y,t}\left(-\theta_t \left\langle v_{i,t}\varepsilon_{i,t}, f_{i,t}\right\rangle\right)\right] dt = 0 \tag{132}$$

After rearranging, I get the strong form:

$$\left(-\varphi l_{i,t}^{\gamma} + \lambda W_t \varepsilon_{i,t}\frac{\partial \zeta_{i,t}}{\partial b}\right) f_{i,t} + \kappa_{i,t}\gamma l_{i,t}^{\gamma-1} c_{i,t}^{\nu} - \eta_{Y,t}\theta_t \varepsilon_{i,t} f_{i,t} = 0 \tag{133}$$

Control over wage

Weak form:

$$\frac{\delta \mathcal{L}}{\delta W} = \int_0^\infty e^{-\rho t}\left[\left\langle \zeta_{i,t}, -\frac{\partial}{\partial b}\lambda v_t l_{i,t}\varepsilon_{i,t} f_{i,t}\right\rangle + \left\langle \varrho_{i,t}, \lambda v_t l_{i,t}\varepsilon_{i,t}\frac{\partial \mathcal{V}_{i,t}}{\partial b}\right\rangle\right. \tag{134}$$

$$\left. + \left\langle \kappa_{i,t}, -\frac{\lambda v_t \varepsilon_{i,t}}{\varphi}\right\rangle - \eta_{T,t}(1-\lambda)\frac{v_t}{\theta_t}Y_t + \eta_{d,t}\frac{v_t}{\theta_t}Y_t + \eta_{\pi,t}\left(\frac{\phi}{\psi}\frac{v_t}{\theta_t}\right)\right] dt = 0 \tag{135}$$

Strong form:

$$\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, \lambda l_{i,t} \varepsilon_{i,t} f_{i,t} \right\rangle + \left\langle \varrho_{i,t} \lambda l_{i,t} \varepsilon_{i,t}, \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle - \left\langle \kappa_{i,t}, \frac{\lambda \varepsilon_{i,t}}{\varphi} \right\rangle - \eta_{T,t} \left(1-\lambda\right) \frac{Y_t}{\theta_t} + \eta_{d,t} \frac{Y_t}{\theta_t} + \eta_{\pi,t} \frac{\phi}{\psi \theta_t} = 0 \tag{136}$$

Control over output

Weak form:

$$\frac{\delta \mathcal{L}}{\delta Y} = \int_0^\infty e^{-\rho t} \left[\eta_{Y,t} v_t - \eta_{T,t} \left(1-\lambda\right) \frac{W_t}{\theta_t} v_t - \eta_{d,t} \left(1 - \frac{\psi}{2} \pi_t^2 - \frac{W_t}{\theta_t}\right) v_t + \eta_{\pi,t} \frac{\dot{v}_t}{Y_t} \pi_t - \eta_{\pi,t} \frac{\dot{Y}_t}{Y_t} \frac{v_t}{Y_t} \pi_t \right] dt = 0 \tag{137}$$

Rearranging time variation:

$$\int_0^\infty e^{-\rho t} \eta_{\pi,t} \frac{\dot{v}_t}{Y_t} \pi_t dt = e^{-\rho t} \eta_{\pi,t} \frac{v_t}{Y_t} \pi_t \bigg|_0^\infty - \int_0^\infty v_t \frac{\partial}{\partial t} e^{-\rho t} \eta_{\pi,t} \frac{1}{Y_t} \pi_t dt \tag{138}$$

Strong form:

$$\eta_{Y,t} - \eta_{T,t} \left(1-\lambda\right) \frac{W_t}{\theta_t} - \eta_{d,t} \left(1 - \frac{\psi}{2} \pi_t^2 - \frac{W_t}{\theta_t}\right) - \eta_{\pi,t} \frac{\dot{Y}_t}{Y_t} \frac{1}{Y_t} \pi_t - \frac{\partial}{\partial t} \left(\eta_{\pi,t} \frac{1}{Y_t} \pi_t\right) + \rho \left(\eta_{\pi,t} \frac{1}{Y_t} \pi_t\right) = 0 \tag{139}$$

$$\eta_{\pi,0} \pi_0 = 0 \tag{140}$$

Which can be rearranged to

$$\eta_{Y,t} - \eta_{T,t} \left(1-\lambda\right) \frac{W_t}{\theta_t} - \eta_{d,t} \left(1 - \frac{\psi}{2} \pi_t^2 - \frac{W_t}{\theta_t}\right) - \left(\frac{\dot{\eta}_{\pi,t}}{\eta_{\pi,t}} + \frac{\dot{\pi}_t}{\pi_t} - \rho\right) \frac{\eta_{\pi,t} \pi_t}{Y_t} = 0 \tag{141}$$

$$\eta_{\pi,0} \pi_0 = 0 \tag{142}$$

Control over inflation

Weak form:

$$\frac{\delta \mathcal{L}}{\delta \pi} = \int_0^\infty e^{-\rho t} \left[\eta_{d,t} \psi \pi_t v_t Y_t + \eta_{\pi,t} \left(\dot{v}_t - \left(r_t^b - \frac{\dot{Y}_t}{Y_t} \right) v_t \right) \right] dt = 0 \tag{143}$$

$$\int_0^\infty e^{-\rho t} \left[\eta_{d,t} \psi \pi_t v_t Y_t - \eta_{\pi,t} \left(r_t^b - \frac{\dot{Y}_t}{Y_t} \right) v_t - \dot{\eta}_{\pi,t} v_t + \rho \eta_{\pi,t} v_t \right] dt + e^{-\rho t} \eta_{\pi,t} v_t \Big|_0^\infty = 0 \tag{144}$$

Strong form:

$$\eta_{d,t} \psi \pi_t Y_t - \eta_{\pi,t} \left(r_t^b - \frac{\dot{Y}_t}{Y_t} - \rho \right) - \dot{\eta}_{\pi,t} = 0 \tag{145}$$

$$\eta_{\pi,0} = 0 \tag{146}$$

Control over real return

Weak form:

$$\frac{\delta \mathcal{L}}{\delta r^b} = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, v_t b_{i,t} f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, v_t b_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle - \eta_{\pi,t} v_t \pi_t \right] dt = 0 \tag{147}$$

$$\zeta_{i,t} f_{i,t} b_{i,t} \Big|_{b=\underline{b}} = 0 \tag{148}$$

Strong form:

$$\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, b_{i,t} f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, b_{i,t} \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle - \eta_{\pi,t} \pi_t = 0 \tag{149}$$

Control over transfers

Weak form:

$$\frac{\delta \mathcal{L}}{\delta T} = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, v_t f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, v_t \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{T,t} v_t \right] dt = 0 \tag{150}$$

$$\zeta_{i,t} f_{i,t} \Big|_{b=\underline{b}} = 0 \tag{151}$$

Strong form:

$$\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{T,t} = 0 \tag{152}$$

Control over dividends

Weak form:

$$\frac{\delta \mathcal{L}}{\delta d} = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, v_t s_i f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, v_t s_i \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{d,t} v_t \right] dt = 0 \tag{153}$$

$$\zeta_{i,t} f_{i,t} s_i \Big|_{b=\underline{b}} = 0 \tag{154}$$

Strong form:

$$\left\langle \frac{\partial \zeta_{i,t}}{\partial b}, s_i f_{i,t} \right\rangle + \left\langle \varrho_{i,t}, s_i \frac{\partial \mathcal{V}_{i,t}}{\partial b} \right\rangle + \eta_{d,t} = 0 \tag{155}$$