Preference Conditions for Linear Demand Functions

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Characterizations in the literature

Let $\mathcal{N} := \{1, \ldots, n\}$ be the number of commodities. Take the nth commodity as the numeraire and $k \in \mathcal{N} \setminus \{n\}$. Let also $K := \{1, \ldots, k\}$, $M := \{k+1, \ldots, n\}$, and $M_0 := M \setminus \{n\}$.

Let B be a $k \times k$ matrix of constants and suppose that the (total) demand function $x : \mathbb{R}^n_{++} \supseteq Q \to X$ is such that

$$x_K(q) := \alpha(q_{M_0}, w) + Bq_K$$

with $\alpha : Q_M \to \mathbb{R}^k$ a continuous function.

La France (JET 1985): B symmetric and -ve semidefinite.
(i) $\alpha(q_{M_0}, w) = \alpha(q_{M_0}) \Rightarrow$ quasi-linear/quadratic indirect utility.
(ii) $\alpha(q_{M_0}, w) \Rightarrow$ Leontief indirect utility.

Amir et al. (JET 2017): B symmetric, -ve definite with non-zero diagonal entries
$\alpha(\cdot) = \alpha \Rightarrow x_K(\cdot)$ is generated by a quasi-linear/quadratic direct utility.

Linear demand functions more generally

Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on $X \subseteq \mathbb{R}^n_{++}$ (open and convex).

Let $\mathcal{N} := \{1, \ldots, n\}$ be the number of commodities. Take the nth commodity as the numeraire and $k \in \mathcal{N} \setminus \{n\}$. Let also $K := \{1, \ldots, k\}$, $M := \{k+1, \ldots, n\}$, and $M_0 := M \setminus \varnothing$.

Let B be a $k \times k$ matrix of constants and suppose that $\succsim$ generates the (total) demand function $x : \mathbb{R}^n_{++} \supseteq Q \to X$ where

$$x_K(q) := \alpha(q_M) + Bq_K$$

with $\alpha : Q_M \to \mathbb{R}^k$ a continuous function.

Main result

Theorem

The following are equivalent:

(i). $\succsim$ is differentiable.

(ii). Q is open in $\mathbb{R}^n_{++}$, B is non-singular, $M_0 = \varnothing$ while $\alpha(\cdot)$ is constant.

(iii). $M_0 = \varnothing$ and $\alpha(\cdot)$ is constant while $\succsim$ is represented by the utility function $u : X \to \mathbb{R}$ given by

$$u\left(x_K, x_n\right) := x_n - x_K \widetilde{B} \alpha - x_K \widetilde{B} x_K / 2$$

where $\alpha = \alpha(\cdot)$ and $\widetilde{B}$ is a non-singular $k \times k$ matrix of constants (specifically, $\widetilde{B} := B^{-1}$).

(iv). Q is open in $\mathbb{R}^n_{++}$, B is symmetric and negative definite while $M_0 = \varnothing$ and $\alpha(\cdot)$ is constant.

(v). Q is open in $\mathbb{R}^n_{++}$, $M_0 = \varnothing$, $\alpha(\cdot)$ is constant while $x_K(\cdot)$ satisfies the strict Law of Demand:

$$(q_K - \widetilde{q}_K)(x_K(q_K) - x_K(\widetilde{q}_K)) < 0 \qquad \forall q_K, \widetilde{q}_K \in Q_K : \; q_K \neq \widetilde{q}_K$$

Applications

Let $\mathcal{N} = \{1, 2, 3\}$ and take the 3rd commodity as the numeraire.

The demand function $x_1(q_1, q_2, w) = f(w) - 2q_1 + q_2$ *cannot* be rationalized with (weakly) smooth preferences.

The demand function $x_1(q_1, q_2, w) = 1 - 2q_1 + q_2$ *can* be rationalized with (weakly) smooth preferences *if* we also have $x_2(q_1, q_2, w) = \alpha_2 + q_1 + b_{22}q_2$ where

$$B := \begin{bmatrix} -2 & 1 \\ 1 & b_{22} \end{bmatrix}$$

is -ve definite.

Applications

Let $\mathcal{H} = \{1, \ldots, H\}$ be customers in a market, each with demand

$$x^h(q) = \alpha^h + B^h q, \quad h \in \mathcal{H}$$

These demands can be rationalized *only if* each B^h is symmetric and -ve definite. Hence, only if $B := \sum_{h \in \mathcal{H}} B^h$ is also symmetric and -ve definite.

The aggregate demand function

$$x(q) = \sum_{h \in \mathcal{H}} \left(\alpha^h + B^h q \right)$$

is rationalizable by a single hypothetical agent with preferences given by the Theorem.

Applications

Estimation of utilities that generate demand functions (Deaton EJ1978): The form $u(x) = (x - \alpha)A(x - \alpha)$ is consistent with linear demand *only if*
(i) the $(n-1)$th principal minor of A is symmetric and -ve definite, and
(ii).$A_{nn} = 0 = A_{nj+A_{jn}}$ for all $j = 1, \ldots, n-1$.

Additive preferences (Houthakker ECMA1960): An additive $u(\cdot)$ is consistent with linear demand *only if* it is of the form
$u(x) = x_n + \sum_{j=1}^{n-1}(\alpha_j x_j + b_j x_j^2)$ - i.e., B is diagonal.

Integrability of demand systems that are independent of income (Nocke and Schutz JET2017): $x(q)$ that satisfies the Law of Demand and there exists $v(\cdot)$ such that $\nabla_q v(q) = -x(q)$.

Differentiability and Linear Demand

Intuition   Income normalization

Key result

Taking $k \in \mathcal{N} \setminus \{n\}$, let $K := \{1, \ldots, k\}$, $M := \{k+1, \ldots, n\}$, $M_0 := M \setminus \{n\}$ and B be a $k \times k$ matrix of constants. Suppose also that the continuous, strictly convex, and strictly monotonic weak order $\succsim$ on X generates the (total) demand function $x : Q \to X$ where

$$x_K(q) := \alpha(q_M) + Bq_K$$

with $\alpha : Q_M \to \mathbb{R}^k$ a continuous function. The following are equivalent:
(i). $\succsim$ is differentiable.
(ii). Q is open in $\mathbb{R}^n_{++}$, B is non-singular, $M_0 = \varnothing$ while $\alpha(\cdot)$ is constant.

Differentiable preferences [Rubinstein (2006)]

$\succsim$ is *differentiable* at $x \in X$ if there exists $p_x \in \mathbb{R}^n \setminus \{0\}$ that makes the following sets

$$\{y \in \mathbb{R}^n : p_x y > 0\}$$

and

$$\{y \in \mathbb{R}^n : \exists \lambda_x^* > 0 \text{ such that } x + \lambda y \succ x \quad \forall \lambda \in (0, \lambda_x^*)\}$$

coincide.

If it exists, p_x will be referred to as a *preference gradient* at x.

Preference gradients graphically

Integrability

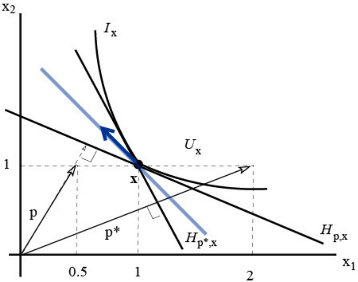

Figure: p, p^* support $\mathcal{U}_x$; neither is a preference gradient.

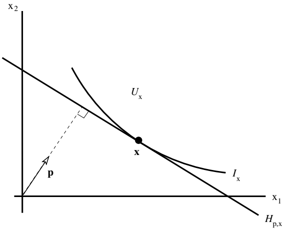

Figure: p is a preference gradient.

Key background result back

Diasakos and Gerasimou (AEJ: Micro 2022)

The following are equivalent for a continuous weak order $\succsim$ on X:
(i). $\succsim$ is strictly convex, strictly monotonic, and differentiable.
(ii). There is a unique, open set $Y \subseteq \mathbb{R}^n_{++}$ and a unique, homeomorphic demand function $x : Y \to X$ that is generated by $\succsim$.

Indifference-projection functions

Consider the *indifference-projection correspondence* $I_n(\cdot|x) : \mathcal{I}_x^{-n} \twoheadrightarrow \mathcal{I}_x^n$ for good n by requiring

$$y_n \in I_n(y_{-n}|x) \iff y \in \mathcal{I}_x,$$

The graph of this correspondence is the indifference set $\mathcal{I}_x$.

$I_i(\cdot|x)$ is actually a *function* that is locally convex and thus also continuous.

$I_i(\cdot|x)$ is differentiable at y_{-i} if and only if the local subdifferential $\partial I_i(y_{-i}|x)$ is a singleton, in which case the unique subgradient coincides with the gradient.

The indifference-projection gradient back

Let $q_{-n}(x)$ denote the -ve of the gradient (equivalently, the unique subgradient) of the indifference-projection function $I_n(\cdot|x)$ for good n at x.

The preference gradient p_x coincides with $p(x)$, the inverse demand at this bundle, and is determined by

$$\begin{aligned} q_{-n}(x) &= -\nabla I_n(x_{-n}|x), \\ q_n(x) &= \frac{1}{x_n + q_{-n}(x) \cdot x_{-n}}, \\ p(x) &= q_n(x)\big(1, q_{-n}(x)\big), \end{aligned}$$

where $q_{-n}(x) \gg 0$, $q_n(x) > 0$ and $p(x) \gg 0$.

$\succsim$ differentiable $\Rightarrow$ B non-singular back

If B is singular, there exist $\mathrm{v} \in \mathbb{R}^k \setminus \{0^k\}$: $B\mathrm{v} = 0^k$.
Let $\succsim$ be differentiable.
Taking $\mathrm{q} \in Q$ and $\varepsilon > 0$ sufficiently small, we have $(\mathrm{q}_K + \lambda\mathrm{v}, \mathrm{q}_M) \in Q$ for any $\lambda \in (-\varepsilon, \varepsilon)$. Define the function $q_K : (-\varepsilon, \varepsilon) \to Q_K$ by $q_K(\lambda) := \mathrm{q}_K + \lambda\mathrm{v}$. We have

$$x_K(\mathrm{q}_K + \lambda\mathrm{v}, \mathrm{q}_M) = \mathrm{a}(\mathrm{q}_M) + B\mathrm{q}_K + \lambda B\mathrm{v} = x_K(\mathrm{q})$$

and we can show that

$$\begin{aligned} x_n(q_K(\lambda), \mathrm{q}_M) + \mathrm{q}_{M_0} x_{M_0}(q_K(\lambda), \mathrm{q}_M) &= x_n(\mathrm{q}) + \mathrm{q}_{M_0} x_{M_0}(\mathrm{q}) \\ &- (q_K(\lambda) - \mathrm{q}_K) x_K(\mathrm{q}) \\ &= x_n(\mathrm{q}) + \mathrm{q}_{M_0} x_{M_0}(\mathrm{q}) \\ &- \lambda\mathrm{v} x_K(\mathrm{q}) \end{aligned}$$

$\succsim$ differentiable $\Rightarrow$ B non-singular back

Case I: There exists $x^0 \in X$: $vx_K^0 = 0$.
Then $q^0 \in Q$: $x^0 = x(q^0)$ wgives $q^0 x(q^0 + \lambda v) = (q^0 + \lambda v)x^0$. $\rightarrow | \leftarrow$

Case II: There exists $x^0 \in X$ $vx_K^0 = vx_K(q_K^0, q_{M_0}^0, w')$.
Letting $\lambda := -\left(w' - w^0\right)/vx^0$ we get that $q^0 x(q_K^0 + \lambda v, q_{M_0}^0, w^0) = w'$ while $(q_K^0 + \lambda v)x_K(q_K^0, q_{M_0}^0, w') + q_M^0 x_M x(q_K^0, q_{M_0}^0, w') = w^0$. $\rightarrow | \leftarrow$.

Case III: $vx_K(q_K, q_{M_0}, w) \neq vx_K(q_K, q_{M_0}, w')$ for all $(q_K, q_{M_0}, w), (q_K, q_{M_0}, w') \in Q$.
Consider the sets

$$\begin{aligned} Q^0 &:= \left\{q \in Q : \ x_M(q) = x_M^0\right\} \\ X^0 &:= \left\{x \in X : \ x_M = x_M^0\right\} \\ X_K^* &:= \left\{x_K \in X_K^0 : \ vx_K = vx_K^0\right\} \end{aligned}$$

Then income must remain constant along the hyperplane X_K^*. $\rightarrow | \leftarrow$.

$\succsim$ differentiable, B non-singular $\Rightarrow M_0 = \varnothing$ back

Let $\succsim$ be differentiable and $M_0 \neq \varnothing$. For $x \in X$ and $q_M \in Q_M$ consider the sets

$$X^{q_M} := \{\widetilde{x} \in X : \exists q_K \in Q_K,\ \widetilde{x} = x(q_K, q_M)\}$$
$$Q_M^x := \{q_M \in Q_M : \exists \widetilde{x} \in X^{q_M},\ \widetilde{x} \sim x\}$$

Take $x^0 \in X$ and notice that $\mathcal{I}_{x^0}$ can be partitioned as $\mathcal{I}_{x^0} = \cup_{q_M \in Q_M^{x^0}} \mathcal{I}_{x^0|q_M}$ where $\mathcal{I}_{x^0|q_M} := \mathcal{I}_{x^0} \cap X^{q_M}$. Restricting thus attention to the domain $X^{q_M^0}$, we trace $\mathcal{I}_{x^0|q_M^0}$ as long as we obey the following system of partial differential equations

$$\partial x_n / \partial x_j = -q_j^0, \qquad j \in M_0$$
$$\partial x_n / \partial x_j = \left(B^{-1}(\alpha\left(q_M^0\right) - x_K)\right)_j, \qquad j \in K$$

Integrating along $\mathcal{I}_{x^0|q_M^0}$ gives

$$x_n = x_K B^{-1} \alpha\left(q_M^0\right) - x_K B^{-1} x_K / 2 - q_{M_0}^0 x_{M_0} + c_0, \qquad x \in \mathcal{I}_{x^0|q_M^0}$$

for some integrating constant c_0.

$\succsim$ differentiable, B non-singular, $M_0 = \varnothing \Rightarrow \alpha(\cdot)$ is constant back

Let $\succsim$ be differentiable. As $M_0 = \varnothing$, $\alpha(\cdot)$ is a function only of income. Suppose that $\alpha(w) \neq \alpha(w_0)$ for all $w \in (w_0 - \lambda_0, w_0 + \lambda_0) \setminus \{w_0\}$ for sufficiently small $\lambda_0 \in \mathbb{R}_{++}$.

Define the $(-\lambda_0, \lambda_0) \to \mathbb{R}^K$ function $\epsilon(\lambda) := B^{-1}(\alpha(w_0 + \lambda) - \alpha(w_0))$.

For $\lambda \in (-\lambda_0, \lambda_0)$ consider the sets

$$\hat{Q}_K^{\lambda} := \left\{q_K \in Q_K : \ x_n(q_K - \epsilon(\lambda), w_0 + \lambda) = x_n\left(q_K^0 - \epsilon(\lambda), w_0 + \lambda\right)\right\}$$
$$\hat{X}^{\lambda} := \left\{x \in X : \ x_n = x_n\left(q_K^0 - \epsilon(\lambda), w_0 + \lambda\right)\right\}$$

We can show that there exist $\lambda_1 \in (-\lambda_0, \lambda_0)$ and $\rho_1 > 0$ such that

$$0 = \epsilon(\lambda_1)\left(x_K(q_K, w_0) - x_K^0\right) \qquad \forall q_K \in \mathcal{B}_{q_K^0}(\rho_1)$$

When prices are normalized w.r.t. income back

Proposition

Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X. Taking $k \in \mathcal{N} \setminus \{n\}$, let also $K := \{1, \ldots, k\}$, $M := \{k+1, \ldots, n\}$, and B be a $k \times k$ matrix of constants. Suppose finally that the generated demand function $x : Y \to X$ is such that

$$x_K(p) := \alpha(p_M) + Bp_K$$

with $\alpha : Y_M \to R^k$ a continuous function. The following are equivalent.
(i). $\succsim$ is differentiable.
(ii). Y is open in $\mathbb{R}^n_{++}$, B is non-singular and $M = \{n\}$.

When prices are normalized w.r.t. income back

Theorem

Let $x : Y \to X$ be such that

$$x_K(p) := \alpha + \gamma p_n + B p_{-n}, \qquad p \in Y$$

where $\alpha, \gamma \in \mathbb{R}^{n-1}$ are constants. The following are equivalent.
(i). $\succsim$ is differentiable.
(ii). Y is open in $\mathbb{R}^n_{++}$ and B is non-singular.
(iii). $\succsim$ is represented by the utility function $u : X \to \mathbb{R}$ given by

$$u(x) := \begin{cases} \left(x_n - x_{-n}\widetilde{B}\gamma\right) \exp\left(\int_{X_K^0} \frac{\widetilde{B}(x_{-n}-\alpha)}{1-x_{-n}\widetilde{B}(x_{-n}-\alpha)} \mathrm{d}x_{-n}\right) & x_{-n} \in X^0_{-n} \\ 0 & x_{-n} \in X_{-n} \setminus X^0_K \end{cases}$$

where $\widetilde{B}$ is a non-singular $(n-1) \times (n-1)$ matrix of constants (with $\widetilde{B}^{-1} = B$) while $X^0_{-n} =: \{x_{-n} \in X_{-n} : x_{-n}\widetilde{B}(x_{-n} - \alpha) \neq 1\}$.