Housing and Pecuniary Externalities*

Christopher Krause† Luca Pegorari‡

May 10, 2022

Abstract

We analyze the efficiency properties of a general equilibrium model that focuses on the interplay of housing and credit markets. In a setting with heterogeneous households and uninsurable idiosyncratic risk, we study the impact of the inherent illiquidity of housing and its collateralizable nature on optimal financial taxes. We characterize constrained efficiency by sufficient statistics and find that Pareto improvements can be achieved by taxing borrowings and savings nonuniformly, such that the capital stock and housing price are reduced. The illiquidity of houses limits how insurance can be implemented and determines that the socially efficient level of capital is below that of the laissez-faire outcome. Collateralizability, in turn, introduces a trade-off when alleviating the social costs of the two frictions at play: we find that the socially more important margin is to improve households' insurance, instead of enlarging their credit opportunities, achieved via less capital and a lower housing price.

JEL Codes: D62, E44, G18, G51.

Keywords: Housing, Collateral, Liquidity, Pecuniary Externalities, Incomplete Markets.

*We are grateful to Björn Brügemann and Johannes Brumm for extremely useful comments, and thank numerous participants at conferences and seminars for helpful discussions. Correspondence to L. Pegorari.

†Karlsruhe Institute of Technology (email: `christopher.krause@kit.edu`).

‡Karlsruhe Institute of Technology (email: `luca.pegorari@kit.edu`).

1 Introduction

The interrelation between housing and credit markets was at the core of the recent Great Recession. The collapse of the subprime mortgage market was identified as one of the crucial drivers of the financial crisis and the subsequent recession, triggering policy interventions in both the financial and housing sectors. Some of these measures, known as macroprudential policies, are aimed at decreasing the severity of future crises by affecting lending practices and household leverage. However, the evaluation of these ex ante interventions requires a deep understanding of whether and how intertwined housing and financial decisions lead to socially inefficient outcomes in general. This in turn would provide a rationale for the design and implementation of efficient policies that does not rely on the occurrence of adverse aggregate events alone.

In the spirit of Diamond (1967), we evaluate if and how a social planner could achieve a Pareto improvement upon the market allocation when facing the same constraints as the market participants. Our analysis revolves around the following policy questions: What implications does the presence of an asset such as housing have for the taxation of liquid financial assets? In particular, how do housing illiquidity and the occurrence of binding financial constraints affect taxation? Therefore, our study offers insight into the social desirability of households' illiquid and liquid assets decisions and, consequently, of the resulting wealth distribution.

In this paper, we analyze the normative properties of a two-period general equilibrium production economy populated by different household types that face both financial and housing decisions, and uninsurable idiosyncratic shocks to their labor earnings. Central to understanding the social efficiency of households' decisions, we focus on two fundamental attributes of houses. First, the illiquid nature of housing, modeled by households' inability to adjust housing positions unrestrictedly over time. Second, the collateral nature of houses, captured by households' ability to borrow up to a fraction of the value of acquired housing.

We use a perturbation approach to infer how deviations from laissez-faire financial decisions impact equilibrium prices and household types, in conjunction with the social planner problem, which allows us to identify efficiency conditions and derive optimal corrective taxation. Our investigation proceeds progressively, starting with the analysis of an environment with illiquid housing and a natural borrowing limit. Within this framework, we derive three results. First we identify sufficient statistics upon which constrained efficiency rests. These consist of households' net trading positions in the market for houses, the housing price sensitivity to changes in financial asset decisions, and the extent to which household types are affected by market incompleteness. Second, we find that a Pareto improvement can be unambiguously achieved by mandating lower borrowings and fewer savings, with the change in the latter being larger such that both aggregate capital and the housing price are lower than at the competitive equilibrium. Third, we provide formulas for financial taxes that implement the constrained efficient allocation.

The intuition behind this set of results is the following. The fundamental friction in this economy is a lack of assets, so the social planner's will to decrease borrowings and reduce savings to a larger extent reflects a desire to better insure households. Indeed, by lowering capital the planner can induce a risk rescaling of households' income components away from the stochastic labor earnings and toward the non-stochastic financial returns. While wealth-rich households are positively affected by these changes in financial positions, the wealth-poor are hurt. However, by letting the housing market work, the planner indirectly transfers part of the wealth-rich utility improvement from better insurance to the wealth-poor: Decreasing the housing price allows the planner to make all households better off. The planner can do so because, although reductions in borrowings and savings put opposite pressure on the housing price, the aggregate housing demand is more sensitive to shifts in debt.

To identify the impact of illiquidity, we modify the setting by introducing unrestrictedly adjustable houses. Within this framework, we cannot rule out that the competitive equilibrium might turn out to be constrained efficient; but these are non-generic cases. Under natural conditions, distortions from distributive externalities will by and large lead to non-zero corrective taxes. We identify sufficient statistics showing how the illiquid case is nested into this framework, retaining its unambiguous impact, and how unrestrictedly adjustable housing gives rise to additional externalities interplays that are ambiguously signed. Therefore, it is not clear that the planner would mandate changes in the competitive equilibrium allocation in the same way as for illiquid housing. Consequently, financial taxes implementing the constrained efficient allocation may flip signs. Yet we quantitatively show that even with unrestrictedly adjustable housing there is still over-borrowing and over-saving. The crucial difference now is that there is not necessarily either too much or too little capital in the competitive equilibrium. Thus, one of the main insights of our paper is that housing illiquidity plays a crucial role in defining the socially efficient level of aggregate capital.

Intuitively, the illiquidity of housing removes an effective indirect instrument that could otherwise be used to improve households' insurance. In particular, if houses were fully liquid, then Pareto improvements could be achieved by decreasing aggregate capital as before, but also by leaving it unchanged or even increasing it. This is because future housing returns, in contrast to labor earnings and financial returns, can induce a risk rescaling that ex ante impacts households in the same direction. Further, future housing returns are so significant for risk rescaling that it is possible to improve households' insurance even when enlarging the stochastic labor income component. The inability to unrestrictedly tap into these returns due to illiquidity hinders how the planner can improve households' insurance.

Finally, to assess how the collateral nature of housing affects efficiency, we revert to illiquid houses and introduce a collateral constraint. Whenever the latter is binding, we show that the sufficient statistics are now characterized by the interaction of collateral and distributive externalities and, more interestingly, how the additional collateral externalities are always antithetical to the housing market distributive externalities. Although it cannot be ruled out that they could offset each other, the competitive equilibrium is generally inefficient. Moreover, differently from the initial setting with a natural borrowing limit, we find that the laissez-faire equilibrium allocation might now be socially inefficient in either direction. In particular, the sign ambiguity of optimal financial taxes revolves around the importance of relaxing binding collateral constraints versus tilting the terms of trade in the housing market. Nevertheless, numerical experiments show that collateral externalities are generally dominated and interestingly so in the empirically relevant range for the collateral parameter, thus pointing to the prominent role of distributive externalities in setting optimal corrective taxes. In particular, even with binding borrowing constraints, for empirically plausible collateralizability the socially desirable distribution is one where there is less capital, with both lower borrowings and savings, and a lower housing price than at the competitive equilibrium.

Intuitively, the planner mandates changes to savings and borrowings gauging the interaction of the two fundamental frictions at play: Market incompleteness and binding debt constraints may introduce a trade-off between improving households' insurance and enlarging their credit opportunities. This trade-off is at play in the empirically relevant range for housing collateralizability, and the social planner chooses to reduce the risk that households face in their stochastic labor income over relaxing borrowing constraints. Therefore, another key insight of our paper is that the socially more important margin is to improve households' insurance, instead of enlarging their credit opportunities.

Methodologically we connect to the literature starting with Diamond (1967), notably extended by Hart (1975), Stiglitz (1982), Greenwald and Stiglitz (1986), Geanakoplos and Polemarchakis (1986), and Geanakoplos et al. (1990), where a competitive equilibrium is usually compared with allocations achieved by a social planner that can control only existing asset markets. That is, as the planner is not allowed to complete the markets, the relevant concept under analysis is that of second best, or whether a competitive equilibrium is constrained efficient. Our analysis follows this approach while, however, limiting the social planner's control over existing asset markets. In particular, by its multifaceted nature, housing is both a commodity and an asset, thus raising the question of whether to allow the planner control over the housing market. We choose to confine the planner's control to the financial asset market only, so that the housing market remains open for trading. We are not the first to study constrained inefficiency in partially controlled market economies. Citanna et al. (1998) have already drawn attention to the interesting question of whether the result of generic constrained suboptimality of competitive equilibria in incomplete market economies holds in intermediate cases where, for example, only a limited number of assets is regulated by the planner. However, differently from Citanna et al. (1998) we do not allow the social planner to distribute lump-sum transfers of the numeraire good among households.[1] It should be noted that this choice may result in constrained suboptimality of competitive equilibria only under an upper bound on the number of households, as in Geanakoplos and Polemarchakis (1986). Nevertheless, we view our approach as attractive since by restricting the planner's control and available instruments we are asking whether the minimal policy of only interfering in the financial asset market can outperform the competitive allocation without the use of tools for directly transferring goods in the first period, but rather letting the housing market price formation be the indirect instrument in the initial period.

Our work also connects to the more recent papers by Dávila et al. (2012) and Dávila and Korinek (2018). Dávila et al. (2012) contribute to the literature in both a qualitative and a quantitative dimension, delivering insights into the pecuniary externalities at work in a production general equilibrium economy with incomplete markets and uninsurable idiosyncratic risk. Our paper relates to their work in that it also considers a model that includes key elements of this workhorse macroeconomic framework, but it distinctly departs from their environment and analysis by introducing housing markets and debt restrictions. The latter, when binding, trigger a different form of pecuniary externalities that are not encompassed by these authors' setting, whereas the former induces additional distributive externalities. Moreover, while we align with the approach of Dávila et al. (2012) in that our social planner is not allowed to directly transfer goods between households in the initial period, we show that it is still possible to implement a Pareto improvement with initial wealth inequality—also in the absence of binding debt constraints, as in their framework. In particular, there is unanimity for the planner's desire to decrease the risk that households face in their stochastic income by reducing aggregate capital. This is because, while some households are negatively affected by the induced risk rescaling of their income components, the planner can more than compensate them by indirectly transferring part of the improved position of the better-off households via the workings of the housing market. Therefore, our results contribute to the literature by complementing the qualitative insights of Dávila et al. (2012).

The work of Dávila and Korinek (2018) provides a theoretical analysis, more in line with the typical work in the normative general equilibrium literature, focusing on macroprudential financial regulation. In particular, the authors contribute to the debate on possible inefficiencies related to fire sales and financial amplifications phenomena. Connected to our analysis, Dávila and Korinek (2018) show that the types of financial constraints that characterize both their and our framework induce two distinct kinds of pecuniary externalities: distributive externalities originating from incomplete markets and collateral externalities stemming from price-dependent borrowing constraints. The authors find that in general these externalities are ambiguously signed. Our paper departs from the work of Dávila and Korinek

[1] Citanna et al. (1998) consider both a case where asset markets are closed and a social planner can make lump-sum transfers of the numeraire good in period zero, as well as a case where all asset markets are open and a planner can impose lump-sum transfers among all households in period zero and only between two households in period one.

(2018) in several ways while complementing their results. First, specifically focusing on the interplay of housing and financial markets while accounting for key attributes of houses, we ask whether uninsurable idiosyncratic risk and borrowing constraints already justify policy intervention. Second, from a methodological standpoint we restrict the planner's control over asset markets and we do not allow the social planner to impose lump-sum transfers of the numeraire good among households in the initial period. Third, under some conditions we sign distributive and collateral externalities displaying the rationale behind optimal interventions. Although the housing market distributive externalities are always opposed by collateral externalities, driving ambiguity in optimal taxes, we can show analytically that the former type always dominates the latter in the interesting case of first-time buyers that can collateralize any fraction of the acquired housing, even fully. This is suggestive of the greater importance that distributive externalities may have in the setting of corrective financial taxes. By numerically investigating more general cases that simultaneously feature both types of externalities, we can confirm that for empirically relevant collateral parameters the planner's mandate to change financial positions is driven by distributive externalities.

The rest of this paper is organized as follows. Section 2 introduces the environment of the baseline model, where we consider the case of illiquid housing and natural borrowing limits. In Section 3 we analyze the efficiency properties of three different settings. In particular, Section 3.1 studies the efficiency properties of the baseline model and characterizes the associated set of optimal corrective taxes. In Section 3.2 we introduce unrestrictedly adjustable housing and investigate its impact, and in Section 3.3 we revert to the case of illiquid housing while introducing a collateral constraint. In Section 4 we provide the results of our numerical analysis, and Section 5 concludes.

2 Baseline Model

We consider a two-period production economy, in which time is given by $t = 1, 2$. The economy is populated by two types of households denoted by $i \in \{b, l\}$, which we label as borrowers and lenders. There is a unit measure of each household type, and they only differ in terms of their initial endowments.

Preferences. Preferences are defined over non-durable consumption c, the numeraire, and housing h, summarized by the following utility function,

$$U_i = \mathbb{E} \sum_{t=1}^{2} \beta^{t-1} \mathbf{u}(c_{ti}, h_{ti}), \tag{1}$$

where $\mathbf{u}(c, h) = u(c) + v(h)$, and both c and h are assumed to be normal goods. Moreover, we assume that the functions $u(\cdot)$ and $v(\cdot)$ satisfy the following conditions, (*i*) $u'(c) > 0,\ u''(c) < 0,$

(*ii*) $v'(h) > 0,\ v''(h) < 0,$

(*iii*) $\lim_{c\to 0} u'(c) = \infty,\ \lim_{h\to 0} v'(h) = \infty.$

Period 1. At the beginning of period 1, households are endowed with liquid wealth $\bar{\omega}_i$ (units of output) and housing $\bar{h}_i$, where we assume that $\bar{\omega}_l > \bar{\omega}_b$ and $\bar{h}_l > \bar{h}_b$. Aggregate housing $\bar{H}$ is exogenously given and constant over time.

In the first period, household i faces the following budget constraint,

$$c_{1i} + a_i + p_1 h_{1i} = \bar{\omega}_i + p_1 \bar{h}_i, \tag{2}$$

implying that he can use his initial resources to consume the non-durable good $c_{1i} \in \mathbb{R}_+$, purchase housing $h_{1i} \in \mathbb{R}_+$ at price p in units of c, or invest in a financial asset $a_i \in \mathbb{R}$.

Period 2. At the beginning of period 2, each household receives a type-independent idiosyncratic productivity shock, either e_1 with probability π or e_2 with probability $1-\pi$, with $0 < e_1 < e_2$. As shocks are independent across households, a law of large numbers holds so that probabilities reflect the shares of households in each group. There is no pure insurance instrument to reduce the idiosyncratic risk so that households feature a precautionary savings motive. Furthermore, households are endowed with one unit of time, which they inelastically supply on the labor market.

In the baseline model, we consider the case of fully illiquid houses: households are not able to adjust their housing position over time, and thus have to consume the same amount of housing in both periods. Therefore, financial income and labor income determine non-durable consumption in the second period. This is summarized by the following budget constraints,

$$c_{2si} = (1+r)a_i + we_s,\ \forall i \in \{b,l\}, s \in \{1,2\}, \tag{3}$$

where w is the wage rate per efficiency unit of labor and r is the interest rate on capital. It should be noted that housing does not feature in the above budget constraints since h_{2si} must be the same as h_{1i} for all s. To lighten notation, we will drop the time index on housing and the housing price in the following. Note that each household will see his financial income in period 2 as deterministic and equal to $(1+r)a$, whereas his labor income is random and equal to we_1 with probability π and we_2 with probability $1-\pi$.

Borrowings. In the baseline model we assume that the financial asset choice is bounded from below by the natural borrowing limit

$$\underline{a} := -\frac{we_1}{1+r}, \tag{4}$$

that is the amount of debt households can always repay with certainty. For borrowers to actually hold a negative financial asset position it would be enough that their initial endowments $\bar{\omega}_b$ and $\bar{h}_b$ are sufficiently small. Similarly, lenders must be characterized by sufficiently large initial endowments to hold a positive financial asset position. Furthermore, given our assumptions on the function u, it is ensured that the borrowing limit will never be binding and that $\underline{a} < a_b < 0 < a_l$ holds.[2]

Production. In the second period, production is carried out by perfectly competitive firms, which sell the output to households and rent capital and labor from them at rates r and w, respectively. There is a constant returns to scale technology $F(K, L)$ with the following properties,

(i) $F_K > 0,\ F_L > 0,$

(ii) $F_{KL} > 0,\ F_{KK} < 0,\ F_{LL} < 0.$

Factor prices are given by their marginal products,

$$w = F_L(K, L), \tag{5}$$

$$r = F_K(K, L) - \delta, \tag{6}$$

where δ is the depreciation rate of capital. Therefore, an aggregate investment of K units in the first period delivers $F(K, L) + (1 - \delta)K$ in the second period. Finally, aggregate labor is constant and given by

$$L = 2(\pi e_1 + (1 - \pi)e_2). \tag{7}$$

Definition 1.
A competitive equilibrium is a vector $(a_b, a_l, h_b, h_l, K, L, \bar{H}, r, w, p)$ such that:

1. *For $i \in \{b, l\}$, a_i and h_i solve household i's maximization problem*

$$\max_{\{a_i, h_i\}} u\big(\bar{\omega}_i + p(\bar{h}_i - h_i) - a_i\big) + (1+\beta)v(h_i) + \beta\mathbb{E}\Big[u((1+r)a_i + we)\Big]$$

$$\text{subject to} \quad a_i \geq -\frac{we_1}{1+r},$$

[2]Since each household type faces the same maximization problem, initially poor households will all make identical choices and, analogously, initially rich households will take identical decisions.

2. *the capital market clears according to* $a_b + a_l = K$*,*

3. *factor prices are given by* (5) *and* (6)*,*

4. *aggregate labor is given by* (7)*,*

5. *the period-1 housing market clears according to* $h_b + h_l = \bar{H}$*,*

6. *the period-2 housing market clears according to* $\sum_i \sum_s \pi_s h_i = \bar{H}$*.*

3 Analysis and Results

Within the framework studied in this paper the incompleteness of financial markets is due to the insufficient number of assets that can be traded. In such economies a competitive equilibrium is usually compared with allocations that can be achieved by a social planner that can control only existing asset markets. That is, as the planner is not allowed to complete the markets, the relevant concept under analysis is that of second best, or whether a competitive equilibrium is constrained efficient. Our analysis follows this approach while however limiting the social planner's control over existing asset markets. In particular, by its multifaceted nature, housing is both a commodity and an asset, thus raising the question of whether to allow the planner control over the housing market. We choose to confine the planner's control to the financial asset market only, so that the housing market remains open for trading. Moreover, we also restrict the planner's instruments by not allowing lump-sum transfers of the numeraire good among households in the initial period.

3.1 Illiquid Housing & Non-Collateralized Borrowings

Our analysis is centered around finding the impact of changes in the laissez-faire financial asset positions for borrowers and lenders.[3] That is, we ask whether modifications in savings and borrowings lead to a Pareto improvement when prices and allocations in the commodity, housing and factor markets move to maintain equilibrium. From the perspective of a social planner the question is whether commanding a different level of financial asset positions for borrowers and lenders, while being subject to the same constraints as the private market and letting competitive trade take place in the commodity, housing and factor markets, can make all households' types ex-ante better off.

To assess whether it is possible to improve on the market allocation, we start by differentiating the indirect utility of borrowers and lenders, thus obtaining for $i \in \{b, l\}$

$$dU_i = u'\left(\bar{\omega}_i + p(\bar{h}_i - h_i) - a_i\right) dc_{1i} + v'\left(h_i\right)(1+\beta) dh_i$$

[3] A change contemplates both separately modifying borrowings and savings or adjusting them jointly.

$$+\beta\Big[\pi u'\left((1+r)a_i+we_1\right)dc_{21i}+(1-\pi)\,u'\left((1+r)a_i+we_2\right)dc_{22i}\Big],$$

with $dc_{1i}=(\bar{h}_i-h_i)dp-pdh_i-da_i$ and $dc_{2si}=(1+r)da_i+a_idr+e_sdw$ for $s=1,2$. Using that the first-order conditions of household i are

$$\begin{aligned}&-u'\left(c_{1i}\right)p+v'\left(h_i\right)(1+\beta)=0,\\&-u'\left(c_{1i}\right)+\beta(1+r)\left[\pi u'\left(c_{21i}\right)+(1-\pi)\,u'\left(c_{22i}\right)\right]=0,\end{aligned}$$

we can rewrite the differentiated indirect utility of household i as

$$\begin{aligned}dU_i&=u'(\bar{\omega}_i+p(\bar{h}_i-h_i)-a_i)(\bar{h}_i-h_i)dp\\&+\beta\Big[\pi u'((1+r)a_i+we_1)(a_idr+e_1dw)\\&+(1-\pi)u'((1+r)a_i+we_2)(a_idr+e_2dw)\Big].\end{aligned}\tag{8}$$

Inspecting (8) we see that any effect on household i's utility of a marginal change in the market allocation works through prices – consequence of the envelope theorem. If markets are incomplete a social planner could improve on the laissez-faire allocation by affecting the relative prices at which households trade. When this is the case the economy features (relevant) distributive externalities, and a social planner can manipulate prices to benefit the under-insured households. The intuition, as already highlighted in Stiglitz (1982), is that when markets are incomplete prices not only have their conventional role in allocating resources, but they also perform a critical function in sharing and transferring risk.

Let us then turn to the effect of a change in the laissez-faire financial asset positions (a_b,a_l) on prices (p,r,w). Following the model's timeline, we will first investigate how perturbing the laissez-faire financial asset positions will impact the equilibrium housing price. To find dp in equation (8) we note that in equilibrium the housing market clearing condition must hold. That is, the equilibrium housing price solves

$$h_b\left(p,\mu_b\right)+h_l\left(p,\mu_l\right)=\bar{H},\tag{9}$$

where $h_i\left(p,\mu_i\right)$ is the implicit optimal housing demand function of agent i, and $\mu_i=\bar{\omega}_i+p\bar{h}_i-a_i$ are the initial resources held by household i for a given financial asset position. Clearly, at the laissez-faire financial asset positions the market clearing solution is the laissez-faire housing price. The first step to find dp entails asking how moving the financial asset positions affects the aggregate housing demand function $\sum_i h_i\left(p,\mu_i\right)$ on impact. Since an increase in the financial asset position a_i, i.e. a reduction in borrowings for $i=b$ or an increase in savings for $i=l$, decreases household i's initial resources and given that housing is assumed to be a normal good, we can conclude that the aggregate housing demand function will shift to the left.[4] More formally, for each $j \in \{b, l\}$ it holds that

$$\frac{\partial \sum_i h_i\left(p, \mu_i\right)}{\partial a_j} = -\frac{\partial h_j\left(p, \mu_j\right)}{\partial \mu_j} < 0, \tag{10}$$

where $\partial h_j(p, \mu_j)/\partial \mu_j$, i.e. the slope of agent j's housing Engel curve, is positive as the demand for housing is normal.[5] As we now know the impact response, we can infer the equilibrium price response dp from the properties of the aggregate housing demand function. It should be noted that, as highlighted in the literature on the law of demand,[6] it is in general unclear whether as the price of a good increases the aggregate quantity demanded of the good falls. Nevertheless, it is often possible to find conditions that are sufficient to guarantee that such relation is satisfied. In particular, the movement in aggregate housing demand following a housing price change can be decomposed as follows

$$\frac{\partial \sum_i h_i\left(p, \mu_i\right)}{\partial p} = \sum_i \mathcal{S}_i(p, \mu_i) - \sum_i \frac{\partial h_i\left(p, \mu_i\right)}{\partial \mu_i}(h_i\left(p, \mu_i\right) - \bar{h}_i), \tag{11}$$

where $\mathcal{S}_i(p, \mu_i)$ is the substitution effect for agent i, and $-\partial h_i\left(p, \mu_i\right)/\partial \mu_i \times \left(h_i\left(p, \mu_i\right) - \bar{h}_i\right)$ is the wealth effect for agent i, i.e. a combination of income and endowment effects. As both households are utility-maximizing, the substitution effect of each agent is negative. However, the sign of wealth effects depends on households' net trading positions. In particular, considering only the interesting case that borrowers are net buyers, i.e. $h_b - \bar{h}_b > 0$, entails that their wealth effect is negative, which implies that their optimal housing demand is monotonically decreasing in the price of houses. However, as lenders are net sellers, i.e. $h_l - \bar{h}_l < 0$, their wealth effect is positive, which implies that their optimal housing demand is not ensured to be either decreasing or monotone in the price of houses. Thus, in general, it is ambiguous how the aggregate housing demand adjusts following a housing price change. Nevertheless, to infer the latter, the properties of households' Engel curves would already suffice. This can be seen by rewriting (11) as

$$\frac{\partial \sum_i h_i\left(p, \mu_i\right)}{\partial p} = \sum_i \mathcal{S}_i(p, \mu_i) - \left(h_b\left(p, \mu_b\right) - \bar{h}_b\right)\left[\frac{\partial h_b\left(p, \mu_b\right)}{\partial \mu_b} - \frac{\partial h_l\left(p, \mu_l\right)}{\partial \mu_l}\right],$$

where we used that households' net trading positions are opposite. As it is clear from the above equation, as long as the slope of borrowers' housing Engel curve is equal or larger than the slope of lenders' housing Engel curve – at their respective levels of initial resources – then the aggregate housing demand is ensured to be monotonically decreasing in the price

[4] Putting the housing price on the y-axis and aggregate housing on the x-axis, otherwise it's a downward shift.

[5] Note that the positive income effect is $\partial h_j\left(p, \mu_j\right)/\partial \mu_j \times h_j\left(p, \mu_j\right)$.

[6] See for example, among others, Hildenbrand (1983, 1989) and Quah (1997, 2000).

of houses. It should be noted that as we assume identical preferences across households, the functional form of borrowers' optimal housing demand is the same as lenders' one, the function is evaluated at different levels of resources for the two types of households. Therefore, preferences that entail a linear or concave-in-resources housing Engel curve generate a market housing demand that is a monotonically decreasing function of its own price. Indeed, borrowers' and lenders' wealth effects annihilate each other under linear Engel curves,[7] leaving households' negative substitution effects to determine the aggregate housing demand adjustment. In the case of concave-in-resources housing Engel curves, borrowers' wealth effect will dominate lenders' wealth effect,[8] thus reinforcing the negative substitution effects.[9] Hence, under such conditions on preferences, the aggregate housing demand is monotonically decreasing in the price of houses. Since the housing market demand is decreasing in financial asset holdings, only a change in the housing price that is opposite in sign to the change in financial asset positions restores market clearing. Concluding, the negative relationship between the equilibrium housing price and financial asset holdings is asserted in Lemma 1 and illustrated in Figure 1.

Lemma 1.
Under preferences that entail linear or concave-in-resources housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dp = \Phi_{a,i}\, da_i < 0 \;\; \forall i,$$

with

$$\Phi_{a,i} := \frac{\dfrac{\partial h_i\left(p, \mu_i\right)}{\partial \mu_i}}{\sum_j \mathcal{S}_j(p, \mu_j) - \sum_j \dfrac{\partial h_j\left(p, \mu_j\right)}{\partial \mu_j}\left(h_j\left(p, \mu_j\right) - \bar{h}_j\right)}\,.$$

Proof. In the Appendix. □

To find the interest rate and wage rate responses, i.e. dr and dw in equation (8), induced by changes in the laissez-faire financial asset positions we note that the real interest rate and wage in equilibrium are given by $r = F_K(K, L) - \delta$ and $w = F_L(K, L)$, where K is determined by households' financial assets, i.e. $a_b + a_l = K$, and L by exogenous parameters. Thus,

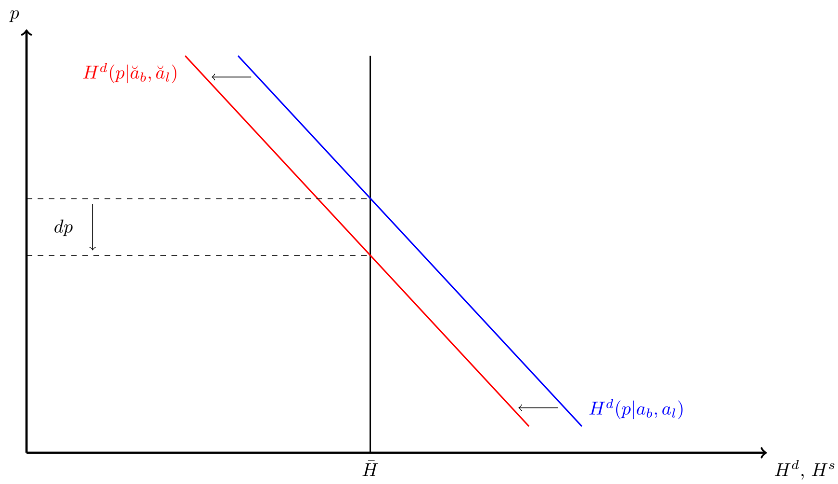
$$dr = F_{KK} dK = F_{KK}(da_b + da_l), \tag{12}$$

[7] Linear Engel curves could be attained by having, for example but not necessarily, homothetic preferences. Since households are characterized by additively separable preferences, imposing homotheticity would lead to the class of utility functions $u(x) = \sum_j \kappa_j x_j^{\rho}$ or $u(x) = \sum_j \kappa_j \ln(x_j)$.

[8] Note that $\mu_l > \mu_b$ holds.

[9] Note that preferences that entail a dominating lenders' wealth effect would still imply the law of demand for the housing market as long as households' combined substitution effects dominate. Thus, the conditions that we highlight are sufficient, but there are more classes of preferences that imply a decreasing market demand.

Figure 1: The equilibrium housing price response to a positive perturbation to either or both financial asset positions, from a_i to $\breve{a}_i$. Naturally, $H^d(\cdot)$ might not be linear.



$$dw = F_{LK}dK = F_{LK}(da_b + da_l). \tag{13}$$

Note that when $da_b > 0$ or $da_l > 0$ then $dr < 0$ and $dw > 0$, since $F_{KK} < 0$ and $F_{LK} > 0$ by the properties of the production function. Therefore, the interest rate drops when savings increase or borrowings decrease, where the wage rate increases for these financial asset positions changes.

With knowledge of prices' responses to changes in financial asset positions, we can now turn to the effects on households' utility of such changes. Starting with borrowers, substituting dp, dr and dw into (8), we can rewrite dU_b as

$$\begin{aligned}
&dU_b = \Psi^p_{a,b}da_b + \Psi^p_{a,l}da_l + \Psi^{r,w}_K(da_b + da_l),\\
&\text{with,}\\
&\Psi^p_{a,b} := u'(c_{1b})(\bar{h}_b - h_b)\Phi_{a,b},\\
&\Psi^p_{a,l} := u'(c_{1b})(\bar{h}_b - h_b)\Phi_{a,l},\\
&\Psi^{r,w}_K := \beta\Big[\pi u'\left(c_{21b}\right)\left(a_bF_{KK} + e_1F_{LK}\right) + (1-\pi)u'\left(c_{22b}\right)\left(a_bF_{KK} + e_2F_{LK}\right)\Big],
\end{aligned}$$

where the terms $\Psi^p_{a,b}$ and $\Psi^p_{a,l}$ capture how changes in the price of houses affect borrowers' utility by altering the terms of trade, i.e. the housing market distributive externalities, while the term $\Psi^{r,w}_K$ captures how borrowers' utility is affected by modifying the returns from investment and labor, i.e. the credit and labor market distributive externalities. The following lemma characterizes the change in borrowers' utility when the laissez-faire financial asset holdings are perturbed.

Lemma 2.
Under preferences that entail linear or concave-in-resources housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dU_b = \left(\Psi^p_{a,i} + \Psi^{r,w}_K\right) da_i > 0 \;\; \forall i.$$

Proof. From Lemma 1, the properties of the utility function, and borrowers' net housing position it trivially follows that $\Psi^p_{a,i} > 0$ for all i. Further, from the properties of the production function and borrowers' negative financial asset holdings it is also clear that $\Psi^{r,w}_K > 0$. □

A change in the laissez-faire equilibrium financial asset positions, so that the initially poor households borrow less and/or initially rich save more, implies an increase in borrowers' utility. This is due to the non-internalized changes in the equilibrium prices r, w and p which are induced by lower borrowings and higher savings. In particular, there are two distinct channels through which the initially poor are made better off. First, the term $\Psi^{r,w}_K$ is positive since lower borrowings or higher savings decrease the interest rate and increase the wage, which makes borrowers unambiguously better off by improving their credit and labor conditions. Second, the terms $\Psi^p_{a,b}$ and $\Psi^p_{a,l}$ are positive since a decrease in borrowings or increase in savings diminishes the housing price, which improves borrowers' terms of trade in the housing market.

Therefore, overall, lower borrowings and/or higher savings move equilibrium prices in a way that makes initially poor households better off. This result is only partly comparable to Dávila et al. (2012) insight that for those households with savings below aggregate capital a social planner would be able to increase their utility by choosing a higher aggregate capital. In particular, by incorporating housing we also show how increasing aggregate capital through only borrowings or savings affects the price of another asset, and in turn how this has a positive and distinct effect on those households' utility.

We now derive the implications of changes in the market equilibrium financial asset holdings for lenders' utility. Substituting dp, dr and dw into (8), we rewrite dU_l as

$$\begin{aligned}
&dU_l = \Theta^p_{a,b} da_b + \Theta^p_{a,l} da_l + \Theta^{r,w}_K (da_b + da_l),\\
&\text{with,}\\
&\Theta^p_{a,b} := u'(c_{1l})(\bar{h}_l - h_l)\Phi_{a,b},\\
&\Theta^p_{a,l} := u'(c_{1l})(\bar{h}_l - h_l)\Phi_{a,l},\\
&\Theta^{r,w}_K := \beta\Big[\pi u'\left(c_{21l}\right)\left(a_l F_{KK} + e_1 F_{LK}\right) + (1-\pi)u'\left(c_{22l}\right)\left(a_l F_{KK} + e_2 F_{LK}\right)\Big],
\end{aligned}$$

where the terms $\Theta^p_{a,b}$ and $\Theta^p_{a,l}$ capture how changes in the price of houses affect lenders' utility by altering the terms of trade, i.e. the housing market distributive externalities, while the term $\Theta^{r,w}_K$ captures how lenders' utility is affected by modifying the returns from investment and labor, i.e. the credit and labor market distributive externalities. The following lemma characterizes the change in lenders' utility when laissez-faire financial asset positions are perturbed.

Lemma 3.
Under preferences that entail linear or concave-in-resources housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dU_l = \left(\Theta^{p}_{a,i} + \Theta^{r,w}_{K}\right) da_i < 0 \;\; \forall i.$$

Proof. In the Appendix. □

Conversely to Lemma 2, a change in the laissez-faire financial holdings such that the initially poor households borrow less, the initially rich save less, or both implies a decrease in lenders' utility. As for borrowers, this is due to the non-internalized changes in the equilibrium prices. The two channels through which the initially rich are made worse off are the following. The term $\Theta^{r,w}_{K}$ is negative as the deterioration in the lenders' credit conditions dominates the improvement in their labor conditions induced by higher savings or fewer borrowings. The terms $\Theta^{p}_{a,b}$ and $\Theta^{p}_{a,l}$ are both negative as lenders' terms of trade in the housing market are weakened due to the decline in the housing price.

Therefore, overall, lower borrowings and/or higher savings move equilibrium prices in a way that makes initially rich households worse off. This result is only partly comparable to Dávila et al. (2012) insight that for those households with savings above aggregate capital a social planner would be able to increase their utility by choosing a lower aggregate capital. In particular, by incorporating housing we also show how increasing aggregate capital affects the price of another asset, and in turn how this has a negative and distinct effect on those households' utility.

3.1.1 Constrained Efficiency

Having determined how perturbations in borrowings and savings affect each household type's utility, we now study constrained efficiency through the lens of the social planner. In particular, we will show that if the system of first-order conditions, characterizing the planner's optimal solution, evaluated at the laissez-faire outcome does not have a non-trivial solution – in terms of welfare weights – then the competitive equilibrium is constrained inefficient.

Recollect that the social planner chooses the financial asset positions while being subject to the same constraints as the private market, letting competitive trade take place in the commodity, housing and factor markets – thus respecting that housing, capital and labor prices are market-determined –, and cannot implement a reallocation of initial wealth via date-zero revenue transfers. Formally, the constrained social planner problem is

$$\max \sum_{i\in\{b,l\}} \gamma_i \Big\{ \mathbf{u}(c_{1i}, h_i(p,\mu_i)) + \beta \Big[\pi\, \mathbf{u}(c_{21i}, h_i(p,\mu_i)) + (1-\pi)\, \mathbf{u}(c_{22i}, h_i(p,\mu_i)) \Big] \Big\}$$

subject to

$$c_{1i} + a_i + p h_i(p,\mu_i) = \bar{\omega}_i + p\bar{h}_i \ ,$$

$$a_i \geq -\frac{we_1}{1+r} \ ,$$

$$c_{2si} = (1+r)a_i + we_s \ , \text{ for } s = 1,2 \text{ and } prob(e = e_1) = \pi \ ,$$

$$r = F_K(K,L) - \delta \ , \ w = F_L(K,L) \ ,$$

$$K = a_b + a_l \ , \ L = 2(\pi e_1 + (1-\pi)e_2) \ ,$$

$$\bar{H} = h_b(p,\mu_b) + h_l(p,\mu_l) \ ,$$

where γ_b and γ_l are the welfare weights for borrowers and lenders respectively.

Evaluating the social planner first-order conditions at the laissez-faire equilibrium, and rewriting the system of equations in matrix form we obtain

$$\begin{bmatrix} \Psi^p_{a,b} + \Psi^{r,w}_K & \Theta^p_{a,b} + \Theta^{r,w}_K \\ \Psi^p_{a,l} + \Psi^{r,w}_K & \Theta^p_{a,l} + \Theta^{r,w}_K \end{bmatrix} \begin{bmatrix} \gamma_b \\ \gamma_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{14}$$

Let A be the 2×2 matrix on the left-hand side of (14), since the above system of equations is homogeneous, the only solution in terms of welfare weights is the trivial one – corresponding to zero welfare weights – if the system has a non-singular matrix A. If that is the case then the social planner would never optimally choose the market equilibrium for any non-zero welfare weights, which implies that the competitive allocation is constrained inefficient. In Proposition 1 we identify the three sufficient statics upon which constrained efficiency rests, and establish under which conditions the competitive equilibrium is inefficient.

Proposition 1.
Constrained efficiency of the competitive equilibrium can be characterized by the following three sufficient statistics:

$(A1)$ *households' net trading position in the housing market* $\left(\bar{h}_b - h_b\right)$, *or* $-\left(\bar{h}_l - h_l\right)$,

$(A2)$ *the difference between the housing price response from changing borrowings only and the housing price change when perturbing savings only* $[\Phi_{a,b} - \Phi_{a,l}]$,

$(A3)$ *the consumption-marginal-utility cross-weighted sum of credit and labor market distributive externalities* $\left\{u'(c_{1b})\Theta^{r,w}_K + u'(c_{1l})\Psi^{r,w}_K\right\}$.

In particular, and respectively connected to each of the above sufficient statistics, the competitive equilibrium is constrained inefficient as long as all the following three conditions hold:

$(C1)$ *there is trading in the market for houses,*

$(C2)$ *households' housing Engel curves exhibit curvature,*

$(C3)$ *market incompleteness is relevant, i.e.* $\pi \in]0, 1[$ *and* $e_2 > e_1$.

Proof. In the Appendix. □

3.1.2 Implementing a Pareto Improvement

Operating under the three natural conditions identified in Proposition 1, we now turn to assess which modifications to the laissez-faire financial asset positions lead to a Pareto improvement. In particular, to find out what signs da_b and da_l should have to achieve a Pareto improvement over the competitive equilibrium, we require that

$$dU_b = \Psi^p_{a,b} da_b + \Psi^p_{a,l} da_l + \Psi^{r,w}_K (da_b + da_l) > 0, \tag{15}$$

$$dU_l = \Theta^p_{a,b} da_b + \Theta^p_{a,l} da_l + \Theta^{r,w}_K (da_b + da_l) = 0, \tag{16}$$

that is we look for movements in borrowings and savings such that one group, in this case the lenders, is indifferent and the other group is better off, in this case the borrowers. By letting $da_l = da_b + \varepsilon$, with $\varepsilon \in \mathbb{R}$, and using (16) to find an expression for ε,

$$\varepsilon = -\frac{\Theta^p_{a,b} + \Theta^p_{a,l} + 2\Theta^{r,w}_K}{\Theta^p_{a,l} + \Theta^{r,w}_K} da_b, \tag{17}$$

we then investigate the sign of da_b that satisfies inequality (15). The following proposition characterizes how the social planner would perturb households' borrowings and savings in order to achieve a Pareto improvement.

Proposition 2.
Under concave housing Engel curves, a Pareto improvement can be achieved by mandating lower borrowings, i.e. $da_b > 0$, *and lower savings, i.e.* $da_l < 0$, *compared to the competitive equilibrium. In particular, the social planner would mandate a decrease in savings of larger magnitude than the reduction in borrowings, i.e.* $|da_l| > |da_b|$, *so to achieve:*

(i) *a decrease in the housing price,*

(ii) *an increase in the interest rate and a reduction in the wage.*

Proof. In the Appendix. □

In other words, households' borrowings and savings are socially inefficient, as wealth-poor over-borrow and wealth-rich over-save. Mechanically, the planner mandate to decrease savings to a larger extent than borrowings negatively impact wealth-poor households by increasing the interest rate and reducing the wage, and at the same time positively affect them by decreasing the housing price: borrowers are overall made better off as the positive housing market distributive externalities more than compensate the negative impact of the credit and labor market distributive externalities. In the case of lenders the positive impact due to the credit and labor market distributive externalities is just offset by the negative effect of housing market distributive externalities, making wealth-rich households indifferent.

Intuitively, the fundamental friction in this economy is that of a lack of assets, so that the social planner's will to decrease borrowings and reduce savings to a larger extent reflects a desire to better insure households. Indeed, in doing so the planner is able to reduce the risk that households face in their stochastic labor income, for which there is no direct market insurance, and at the same time increase financial returns, which is the deterministic component of households' income. As a corollary to Lemma 2 and 3, while this is good news for wealth-rich households, it also hurts the wealth-poor. However, by letting the housing market work, the planner is able to indirectly transfer income to the wealth-poor, in such a way that they are more than compensated. In a sense, the improvement to wealth-rich coming from the induced risk rescaling of their income components is transferred away to the wealth-poor via the housing market, so that the wealth-poor are made better off, while the wealth-rich are made indifferent.

Where in the above we have shown that it is possible to induce a Pareto improvement such that some households are indifferent and others are better-off, the planner can actually do better. That is, we can show that there is a region in the da_b-da_l plane where all households are strictly better-off. Clearly, all points in this region satisfy lower borrowings, i.e. $da_b > 0$, lower savings, i.e. $da_l < 0$, and a decrease in savings of larger magnitude than the reduction in borrowings, i.e. $|da_l| > |da_b|$. In our numerical section we illustrate the latter region, and show how the sign of the determinant of A directly implies how to implement a Pareto improvement, i.e. the relative magnitude and signs of da_b and da_l which improve on the competitive equilibrium.

As it is possible to implement a Pareto improvement via a direct mandate to each type of households on how to borrow and save, the same can be achieved through explicit tax incentives: a social planner would tax both borrowings and savings to different degrees. The following proposition characterizes how to implement constrained efficient allocations. Specifically, this will be achieved by a (tax) wedge on households' savings and borrowings decisions, which can be introduced as a proportional tax/subsidy on financial asset returns, accompanied by a (lump-sum) transfer so that the net transfer to each household is zero.

Proposition 3.
A social planner can implement any constrained efficient allocation by setting taxes on financial asset returns that satisfy

$$\tau_b^a = \frac{-1}{\beta r \gamma_b \sum_{s=1}^{2} \pi_s u'\left(c_{2sb}\right)} \left\{ \gamma_b \Psi_{a,b}^p + \gamma_l \Theta_{a,b}^p + \gamma_b \Psi_K^{r,w} + \gamma_l \Theta_K^{r,w} \right\},$$
$$\tau_l^a = \frac{-1}{\beta r \gamma_l \sum_{s=1}^{2} \pi_s u'\left(c_{2sl}\right)} \left\{ \gamma_b \Psi_{a,l}^p + \gamma_l \Theta_{a,l}^p + \gamma_b \Psi_K^{r,w} + \gamma_l \Theta_K^{r,w} \right\},$$

and conducting lump-sum transfers T_b^a and T_l^a, with $T_b^a = r\tau_b^a a_b$ and $T_l^a = r\tau_l^a a_l$, so that net transfers to households are zero and the government budget constraint is satisfied.

Proof. In the Appendix. □

Inspecting the tax formulas it is clear how they encompass both distributive externalities in the housing market and the credit and labor markets. Furthermore, uniform taxes across borrowers and lenders are in general not sufficient to attain constrained efficiency.

3.2 Liquid Housing & Non-Collateralized Borrowings

So far we analyzed the distributive externalities related to illiquid housing. To identify the impact of illiquidity we modify the model by introducing liquid houses. In particular, households are no longer constrained in their second-period housing decision, and can decide to continuously adjust their housing position contingent on the second-period idiosyncratic state. Thus, the optimization problem of household $i \in \{b, l\}$ is to choose the tuple $(c_{1i}, c_{21i}, c_{22i}, h_{1i}, h_{21i}, h_{22i}, a_i)$ to maximize the function

$$U_i = \mathbf{u}(c_{1i}, h_{1i}) + \beta \Big[\pi\, \mathbf{u}(c_{21i}, h_{21i}) + (1-\pi)\, \mathbf{u}(c_{22i}, h_{22i}) \Big]$$

subject to

$$c_{1i} + a_i + p_1 h_{1i} = \bar{\omega}_i + p_1 \bar{h}_i \ ,$$
$$a_i \geq -\frac{w e_1 + p_2 h_{1i}}{1+r} \ ,$$
$$c_{2si} + p_2 h_{2si} = (1+r) a_i + w e_s + p_2 h_{1i} \ ,$$

for $s \in \{1, 2\}$ with $e_s \in \{e_1, e_2\}$ and $prob(e = e_1) = \pi$. Therefore, the equilibrium definition modifies as follows.

Definition 2.
A competitive equilibrium is a vector $(a_b, a_l, h_{1b}, h_{1l}, h_{21b}, h_{22b}, h_{21l}, h_{22l}, K, L, \bar{H}, r, w, p_1, p_2)$ such that:

1. *For $i \in \{b, l\}$ and $s \in \{1, 2\}$, a_i, h_{1i} and h_{2si} solve household i's maximization problem*

$$\max_{\{a_i, h_{1i}, h_{2si}\}} \mathbf{u}\big(\bar{\omega}_i + p_1(\bar{h}_i - h_{1i}) - a_i, h_{1i}\big) + \beta \mathbb{E}\Big[\mathbf{u}\big((1+r)a_i + we + p_2(h_{1i} - h_{2i}), h_{2i}\big)\Big]$$

$$\textit{subject to} \quad a_i \geq -\frac{we_1 + p_2 h_{1i}}{1+r},$$

2. *the capital market clears according to $a_b + a_l = K$,*

3. *factor prices are given by* (5) *and* (6)*,*

4. *aggregate labor is given by* (7)*,*

5. *the period-1 housing market clears according to $h_{1b} + h_{1l} = \bar{H}$,*

6. *the period-2 housing market clears according to $\sum_i \sum_s \pi_s h_{2si} = \bar{H}$.*

The introduction of liquid houses is relevant for both borrowers and lenders. To assess whether it is possible to improve on the market allocation we differentiate their indirect utility to obtain

$$\begin{aligned} dU_i &= u'(c_{1i})(\bar{h}_i - h_{1i})dp_1 \\ &+ \beta \Big\{ \pi u'(c_{21i}) \Big[a_i dr + e_1 dw\Big] + (1-\pi) u'(c_{22i}) \Big[a_i dr + e_2 dw\Big] \Big\} \\ &+ \beta \Big\{ \pi u'(c_{21i})(h_{1i} - h_{21i}) + (1-\pi) u'(c_{22i})(h_{1i} - h_{22i}) \Big\} dp_2, \end{aligned} \tag{18}$$

where household i's first-order conditions were used to simplify the expression. As before, we see that any effect of a marginal change in the market allocation works through prices. Distributive externalities in the capital and labor markets are defined as under illiquid housing, as dr and dw are unchanged. However, although period-1 housing distributive externalities are identically characterized overall, we will show that their definition is partially altered by a different expression for dp_1. Furthermore, now that households can unrestrictedly adjust their housing position in the second period the additional dp_2 price differential arises.

As the effect of a change in the laissez-faire financial asset positions (a_b, a_l) on (r, w) are unchanged, we concentrate on the housing prices (p_1, p_2). To find dp_1 in equation (18) we can again use the period-1 housing market clearing condition. Since the housing decision is now meaningfully dynamic,[10] households' optimal initial housing choice will also be affected by how deviations in their financial asset positions directly impact on their future resources. Thus, the direct effect on the aggregate housing demand will now involve more than the slope of housing Engel curves. In particular, the relationship between the equilibrium period-1 housing price and financial asset holdings is asserted in Lemma 4.

[10] That is, the amount of initial housing carried into the future now matters since future housing does not need to coincide with the initial choice.

Lemma 4.
Under preferences that entail linear or concave-in-resources housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dp_1 = \widehat{\Phi}^1_{a,i}\, da_i < 0 \;\; \forall i,$$

with

$$\widehat{\Phi}^1_{a,i} := -\frac{\dfrac{\partial h_{1i}\left(p_1, \mu_i\right)}{\partial \mu_i}\dfrac{\partial \mu_i}{\partial a_i}}{\sum_j \mathcal{S}_{1j}(p_1, \mu_j) - \sum_j \dfrac{\partial h_{1j}\left(p_1, \mu_j\right)}{\partial \mu_j}(h_{1j}\left(p_1, \mu_j\right) - \bar{h}_j)}\,.$$

Proof. In the Appendix. □

A change in the laissez-faire equilibrium financial asset positions, so that the initially poor households borrow less and/or initially rich save more, induces a decrease in the equilibrium current housing price, as for illiquid houses. This is because although these changes directly increase households' future resources (as borrowers have less debt to pay off and lenders more savings to tap into) the initial negative impact on households' current resources is relatively more important, i.e. $\partial\mu_i/\partial a_i < 0$ still holds. Thus, as the period-1 aggregate housing demand function is negatively impacted by fewer borrowings and more savings, and since with linear or concave housing Engel curves the market housing demand is monotonically decreasing in its own price, only a reduction in p_1 restores equilibrium.

Turning to the future housing price response dp_2 in (18), in equilibrium p_2 must solve the market clearing condition

$$\sum_i \sum_s \pi_s h_{2si}(p_2, \mu_{2si}) = \bar{H}, \tag{19}$$

where $h_{2si}(p_2, \mu_{2si})$ is the implicit optimal future housing demand function of agent i in state s, and $\mu_{2si} := (1+r)a_i + we_s + p_2 h_{1i}$. The relationship between the equilibrium period-2 housing price and financial asset holdings is asserted in Lemma 5.

Lemma 5.
Under preferences that entail linear housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dp_2 = \widehat{\Phi}^2_{a,i}\, da_i > 0 \;\; \forall i,$$

where under preferences that entail concave-in-resources housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dp_2 = \widehat{\Phi}^2_{a,i}\, da_i \gtreqless 0 \;\; \forall i,$$

with

$$\widehat{\Phi}^2_{a,i} := -\frac{\sum_j \sum_s \pi_s \frac{\partial h_{2sj}(p_2,\mu_{2sj})}{\partial \mu_{2sj}} \frac{\partial \mu_{2sj}}{\partial a_i}}{\sum_j \sum_s \pi_s \mathcal{S}_{2sj}(p_2,\mu_{2sj}) - \sum_j \sum_s \pi_s \frac{\partial h_{2sj}(p_2,\mu_{2sj})}{\partial \mu_{2sj}} (h_{2sj} - h_{1j})}.$$

Proof. In the Appendix. □

The slope of period-2 housing Engel curves is the same at all incomes for all households under linearity. In this case, the numerator of $\widehat{\Phi}^2_{a,i}$ is positive since only the direct impact on household i's future resources of changing a_i does not wash out, and the denominator is negative as households' wealth effects cancel out and only substitution effects are left to determine the sign. In other words, a change in the laissez-faire equilibrium financial asset positions, so that the initially poor households borrow less and/or initially rich save more, induces an increase in the equilibrium future housing price under linear future housing Engel curves. This is because under linearity only the direct impact of future less debt and/or more savings affect the future aggregate housing demand. Thus, the period-2 aggregate housing demand function is positively impacted by fewer borrowings and more savings, and since with linear housing Engel curves the market housing demand is monotonically decreasing in its own price only an increase in p_2 restores equilibrium.

Instead, when households' period-2 housing Engel curves exhibit curvature the indirect effects on households' future resources and wealth effects do not wash out in the aggregate, so that both the numerator and denominator of $\widehat{\Phi}^2_{a,i}$ do not have a sign a priori. In particular, without additional assumptions, concavity of housing Engel curves is not enough to ensure that the indirect effects on households' future resources of less debt and/or more savings will not lead to an opposite shift in the future aggregate housing demand, and do not ensure that the market housing demand is monotonically decreasing in its own price. Conditions on the relative magnitude of households' future resources and net-housing positions across states would be sufficient to analytically sign dp_2. However, instead of imposing additional conditions, we will quantitatively explore in the numerical section the change in the equilibrium period-2 housing price.

With knowledge of prices' responses to changes in the market allocation, we can now turn to the effects on households' utility of such changes. For borrowers, substituting dp_1, dp_2, dr and dw into (18) we can rewrite dU_b as

$$dU_b = \widehat{\Psi}^{p_1}_{a,b} da_b + \widehat{\Psi}^{p_1}_{a,l} da_l + \widehat{\Psi}^{p_2}_{a,b} da_b + \widehat{\Psi}^{p_2}_{a,l} da_l + \Psi^{r,w}_K (da_b + da_l), \tag{20}$$

where for $i \in \{b, l\}$

$$\widehat{\Psi}^{p_1}_{a,i} := u'(c_{1b})(\bar{h}_b - h_{1b})\widehat{\Phi}^1_{a,i},$$

$$
\begin{aligned}
\widehat{\Psi}^{p_2}_{a,i} &:= \beta\left[\pi u'(c_{21b})(h_{1b}-h_{21b})+(1-\pi)u'(c_{22b})(h_{1b}-h_{22b})\right]\widehat{\Phi}^2_{a,i},\\
\Psi^{r,w}_K &:= \beta\left[\pi u'(c_{21b})(a_bF_{KK}+e_1F_{LK})+(1-\pi)u'(c_{22b})(a_bF_{KK}+e_2F_{LK})\right].
\end{aligned}
$$

The terms $\widehat{\Psi}^{p_1}_{a,i}$ capture how changes in the first-period price of housing affect borrowers' utility, i.e. the period-1 housing market distributive externalities, the terms $\widehat{\Psi}^{p_2}_{a,i}$ capture the period-2 housing market distributive externalities, while the term $\Psi^{r,w}_K$ captures how changes in the interest rate and wage affect borrowers' utility, i.e. the capital and labor market distributive externalities.

The following lemma characterizes the change in borrowers' utility when the planner perturbs the market allocation of financial asset positions.

Lemma 6.
Under preferences that entail linear or concave-in-resources housing Engel curves,

$$
da_i>0 \;\Rightarrow\; dU_b=\left(\widehat{\Psi}^{p_1}_{a,i}+\widehat{\Psi}^{p_2}_{a,i}+\Psi^{r,w}_K\right)da_i \gtreqless 0 \;\;\forall i.
$$

Proof. For either case highlighted in Lemma 5, it trivially follows from the second-period net housing positions in equation (20), which do not have a sign a priori. □

In words, under liquid housing, a change in the laissez-faire equilibrium financial asset positions, such that the initially poor households borrow less and/or the initially rich households save more, has an unclear effect on borrowers' utility. Comparing results in the above lemma with those under illiquid housing in Lemma 2, the considered changes in asset positions no longer have an unambiguously positive impact on borrowers' utility. This is because the degree of liquidity modifies distributive externalities in housing markets and might change the overall impact on households' utility. In particular, where period-1 housing market distributive externalities are congruently signed under liquid and illiquid houses – although differently defined in part –, full liquidity leads to an overall unclear effect as period-2 housing market distributive externalities do not have a sign a priori. In the numerical section we will quantitatively explore the competitive equilibrium, and show the overall impact of future housing market distributive externalities on borrowers' utility.

Now that we have derived the implications of changes in the market allocation for borrowers' utility, we are left with doing the same for lenders. Substituting dp_1, dp_2, dr and dw into (18) we can rewrite dU_l as

$$
dU_l=\widehat{\Theta}^{p_1}_{a,b}da_b+\widehat{\Theta}^{p_1}_{a,l}da_l+\widehat{\Theta}^{p_2}_{a,b}da_b+\widehat{\Theta}^{p_2}_{a,l}da_l+\Theta^{r,w}_K(da_b+da_l), \tag{21}
$$

where for $i\in\{b,l\}$

$$
\begin{aligned}
\widehat{\Theta}^{p_1}_{a,i} &:= u'(c_{1l})(\bar{h}_l-h_{1l})\widehat{\Phi}^1_{a,i},\\
\widehat{\Theta}^{p_2}_{a,i} &:= \beta\left[\pi u'(c_{21l})(h_{1l}-h_{21l})+(1-\pi)u'(c_{22l})(h_{1l}-h_{22l})\right]\widehat{\Phi}^2_{a,i},
\end{aligned}
$$

$$\Theta_K^{r,w} := \beta \left[\pi u'(c_{21l})(a_l F_{KK} + e_1 F_{LK}) + (1-\pi) u'(c_{22l})(a_l F_{KK} + e_2 F_{LK}) \right].$$

Similarly to the borrowers' case, the terms $\widehat{\Theta}_{a,i}^{p_1}$ capture the period-1 housing market distributive externalities, the terms $\widehat{\Theta}_{a,i}^{p_2}$ capture the period-2 housing market distributive externalities, while the term $\Theta_K^{r,w}$ captures the capital and labor market distributive externalities.

The following lemma characterizes the change in lenders' utility when the planner perturbs the market allocation of financial asset positions.

Lemma 7.
Under preferences that entail linear or concave-in-resources housing Engel curves,

$$da_i > 0 \;\Rightarrow\; dU_l = \left(\widehat{\Theta}_{a,i}^{p_1} + \widehat{\Theta}_{a,i}^{p_2} + \Theta_K^{r,w} \right) da_i \gtreqless 0 \;\; \forall i.$$

Proof. For either case highlighted in Lemma 5, it trivially follows from the second-period net housing positions in equation (21), which do not have a sign a priori. □

Under liquid housing, a change in the laissez-faire equilibrium financial asset positions, such that the initially poor households borrow less and/or the initially rich households save more, has an unclear effect on lenders' utility. If we compare results in the above lemma with those under illiquid housing in Lemma 3, we find that the considered changes in financial asset positions no longer have an unambiguously negative impact on lenders' utility: The degree of liquidity of houses affect distributive externalities in housing markets and might change the overall impact on households' utility. In the numerical section we will quantitatively show the overall impact of future housing market distributive externalities on lenders' utility.

3.2.1 Constrained Efficiency

Although we cannot analytically determine how perturbations to borrowings and savings affect each household type's utility under unrestrictedly adjustable housing, and thus gain indications on which modifications to the laissez-faire financial holdings might implement a Pareto improvement, we can characterize constrained efficiency following the methodology outlined in Section 3.1.1. Under liquid housing, the constrained social planner problem becomes

$$\max \sum_{i \in \{b,l\}} \gamma_i \Big\{ \mathbf{u}\left(c_{1i}, h_{1i}(p_1, \mu_i)\right) + \beta \Big[\pi\, \mathbf{u}\left(c_{21i}, h_{21i}(p_2, \mu_{21i})\right) + (1-\pi)\, \mathbf{u}\left(c_{22i}, h_{22i}(p_2, \mu_{22i})\right) \Big] \Big\}$$

subject to

$c_{1i} + a_i + p_1 h_{1i}(p_1, \mu_i) = \bar{\omega}_i + p_1 \bar{h}_i,$

$c_{2si} + p_2 h_{2si}(p_2, \mu_{2si}) = (1+r)a_i + p_2 h_{1i}(p_1, \mu_i) + w e_s, \quad$ for $s = 1,2$ and $\pi_1 = \pi,$

$a_i \geq -\dfrac{w e_1 + p_2 h_{1i}(p_1, \mu_i)}{1+r},$

$$
\begin{aligned}
&r = F_K(K, L) - \delta, \quad w = F_L(K, L), \\
&K = a_b + a_l, \quad L = 2(\pi e_1 + (1-\pi)e_2), \\
&\bar{H} = \sum_i h_{1i}(p_1, \mu_i) = \sum_i \sum_s \pi_s h_{2si}(p_2, \mu_{2si}).
\end{aligned}
$$

Evaluating the planner's system of first-order conditions at the laissez-faire allocation, and rewriting the system in matrix form we obtain

$$
\begin{bmatrix} \widehat{\Psi}^{p_1}_{a,b} + \widehat{\Psi}^{p_2}_{a,b} + \Psi^{r,w}_K & \widehat{\Theta}^{p_1}_{a,b} + \widehat{\Theta}^{p_2}_{a,b} + \Theta^{r,w}_K \\ \widehat{\Psi}^{p_1}_{a,l} + \widehat{\Psi}^{p_2}_{a,l} + \Psi^{r,w}_K & \widehat{\Theta}^{p_1}_{a,l} + \widehat{\Theta}^{p_2}_{a,l} + \Theta^{r,w}_K \end{bmatrix} \begin{bmatrix} \gamma_b \\ \gamma_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{22}
$$

Let $\widehat{A}$ be the 2×2 matrix on the left-hand side of (22), in Proposition 4 we identify three terms upon which constrained efficiency rests, and establish under which conditions the competitive equilibrium is inefficient by checking when the above system of equations has a non-singular matrix $\widehat{A}$.

Proposition 4.
Constrained efficiency of the competitive equilibrium can be characterized by the sum of the three following terms:

($\widehat{A}1$) *The cross-household interaction between period-1 housing and credit-labor market distributive externalities, composed of*

$$
(\bar{h}_b - h_{1b})(\widehat{\Phi}^1_{a,b} - \widehat{\Phi}^1_{a,l})\big(u'(c_{1b})\Theta^{r,w}_K + u'(c_{1l})\Psi^{r,w}_K\big),
$$

($\widehat{A}2$) *The cross-household interaction between period-2 housing and credit-labor market distributive externalities, composed of*

$$
\beta(\widehat{\Phi}^2_{a,b} - \widehat{\Phi}^2_{a,l})\Big[\Theta^{r,w}_K \mathbb{E}\Big\{u'(c_{2b})(h_{1b} - h_{2b})\Big\} - \Psi^{r,w}_K \mathbb{E}\Big\{u'(c_{2l})(h_{1l} - h_{2l})\Big\}\Big],
$$

($\widehat{A}3$) *The cross-household interaction between period-1 and period-2 housing markets distributive externalities, composed of*

$$
\beta(\bar{h}_b - h_{1b})\Big(\widehat{\Phi}^1_{a,b}\widehat{\Phi}^2_{a,l} - \widehat{\Phi}^1_{a,l}\widehat{\Phi}^2_{a,b}\Big)\Big[u'(c_{1b})\mathbb{E}\Big\{u'(c_{2l})(h_{1l} - h_{2l})\Big\} + u'(c_{1l})\mathbb{E}\Big\{u'(c_{2b})(h_{1b} - h_{2b})\Big\}\Big].
$$

In particular, and respectively connected to each of the above terms, the competitive equilibrium is generally constrained inefficient as long as at least one of the following three conditions hold:

($\widehat{C}1$) *there is trading in the period-1 market for houses, changes in borrowings and savings differently impact the period-1 aggregate housing demand, and market incompleteness is relevant,*

($\widehat{C}2$) *there is trading in the period-2 market for houses, changes in borrowings and savings differently impact the period-2 aggregate housing demand, and market incompleteness is relevant,*

($\widehat{C3}$) *there is trading in both period-1 and period-2 markets for houses, and changes in borrowings and savings differently impact at least one between the period-1 and period-2 aggregate housing demand.*

Proof. In the Appendix. □

Connecting back to Proposition 1, with illiquid housing and natural borrowing limits there are two externalities at play. The determinant, identical to ($\widehat{A1}$) in Proposition 4, consists of only one term which captures the cross-household interaction between the housing and capital-labor market distributive externalities. Under continuously adjustable houses and natural borrowing limits, there are three distributive externalities. The resulting determinant $|\widehat{A}|$ then consists of three terms, capturing the cross-household interaction between the first-period housing and the capital-labor market distributive externalities, the second-period housing and the capital-labor market distributive externalities, and the current and future housing markets distributive externalities.

Although we cannot rule out that the competitive equilibrium may turn out to be constrained efficient,[11] these are non-generic cases: As long as there is trading in at least one market for houses, borrowings and savings differently impact at least one between the current and future aggregate housing demand, and market incompleteness is relevant constrained inefficiency attains generally. The following proposition characterizes proportional taxes or subsidies on financial asset returns, accompanied by a lump-sum transfer, implementing the constrained-efficient allocation.

Proposition 5.
A social planner can implement any constrained-efficient allocation by setting taxes on financial asset returns that satisfy

$$\widehat{\tau}_b = \frac{-1}{\beta r \gamma_b \sum_{s=1}^{2} \pi_s u'(c_{2sb})} \left[\gamma_b \left(\widehat{\Psi}^{p_1}_{a,b} + \widehat{\Psi}^{p_2}_{a,b} + \Psi^{r,w}_{K} \right) + \gamma_l \left(\widehat{\Theta}^{p_1}_{a,b} + \widehat{\Theta}^{p_2}_{a,b} + \Theta^{r,w}_{K} \right) \right], \tag{23}$$

$$\widehat{\tau}_l = \frac{-1}{\beta r \gamma_l \sum_{s=1}^{2} \pi_s u'(c_{2sl})} \left[\gamma_b \left(\widehat{\Psi}^{p_1}_{a,l} + \widehat{\Psi}^{p_2}_{a,l} + \Psi^{r,w}_{K} \right) + \gamma_l \left(\widehat{\Theta}^{p_1}_{a,l} + \widehat{\Theta}^{p_2}_{a,l} + \Theta^{r,w}_{K} \right) \right], \tag{24}$$

in combination with lump-sum transfers $\widehat{T}_b = r\widehat{\tau}_b a_b$ and $\widehat{T}_l = r\widehat{\tau}_l a_l$, ensuring that net transfers to each agent are zero and the government budget constraint is satisfied.

Proof. In the Appendix. □

Until now, we have characterized efficiency without however identifying which modifications to the laissez-faire financial asset positions can implement a Pareto improvement. In

[11] Even if all three terms in Proposition 4 are non-zero, there is no guarantee that there aren't cases in which they offset each other.

particular, under liquid housing, is it still the case that wealth-poor over-borrow and wealth-rich over-save? Furthermore, if the latter continues to attain, would the social planner still mandate a reduction in savings of larger magnitude than the decrease in borrowings? As we do not impose additional conditions, we cannot provide analytical answers to these questions. Nevertheless, through our numerical analysis, we will illustrate the Pareto improving region in the da_b-da_l plane and show that only the first of these results carries through while gaining new findings. Intuitively, the way in which the planner mandates modifications to the laissez-faire financial asset positions rests upon whether and how the future housing price may be used as an additional indirect instrument to improve households' insurance ex-ante and/or as compensation for changes in the latter.

3.3 Illiquid Housing & Collateralized Borrowings

Until now we analyzed the distributive externalities related to housing and its degree of liquidity. To identify the impact of the collateral nature of housing we modify the model by reverting to illiquid houses while introducing a new constraint on borrowings. In particular, households no longer face the natural limit, but rather the constraint

$$a_i \geq -\xi p h_i, \quad \text{with } \xi \in [0,1], \text{ for } i \in \{b,l\}. \tag{25}$$

That is, housing provides collateral for borrowings, so that households cannot borrow if they do not acquire houses. This collateral constraint implies that the maximum debt a household can incur is a fraction of the owned housing value. Thus, the optimization problem of household i is to choose the tuple $(c_{1i}, c_{21i}, c_{22i}, h_i, a_i)$ to maximize the function

$$\begin{aligned}
&U_i = \mathbf{u}(c_{1i}, h_i) + \beta\Big[\pi\, \mathbf{u}(c_{21i}, h_i) + (1-\pi)\, \mathbf{u}(c_{22i}, h_i)\Big] \\
&\text{subject to} \\
&c_{1i} + a_i + p h_i = \bar{\omega}_i + p\bar{h}_i, \\
&a_i \geq -\xi p h_i, \quad \text{with } \xi \in [0,1], \\
&c_{2si} = (1+r)a_i + w e_s,
\end{aligned}$$

for $s = 1,2$ with $e_s \in \{e_1, e_2\}$ and $prob(e = e_1) = \pi$. Therefore, the equilibrium definition will adjust as follows.

Definition 3.
A competitive equilibrium is a vector $(a_b, a_l, h_b, h_l, K, L, \bar{H}, r, w, p)$ such that:

1. *For $i \in \{b,l\}$, a_i and h_i solve household i's maximization problem*

$$\max_{\{a_i,h_i\}} u\big(\bar{\omega}_i + p(\bar{h}_i - h_i) - a_i\big) + (1+\beta)v(h_i) + \beta\mathbb{E}\Big[u((1+r)a_i + we)\Big]$$

subject to $a_i \geq -\xi p h_i$,

2. *the capital market clears according to* $a_b + a_l = K$,
3. *factor prices are given by* (5) *and* (6),
4. *aggregate labor is given by* (7),
5. *the period-1 housing market clears according to* $h_b + h_l = \bar{H}$,
6. *the period-2 housing market clears according to* $\sum_i \sum_s \pi_s h_i = \bar{H}$.

The collateral constraint is irrelevant for lenders so that dU_l is defined as in (8). However, under a binding collateral constraint – the only interesting case – borrowers' differentiated indirect utility becomes

$$\begin{aligned} dU_b = & \left[u'(c_{1b})(\bar{h}_b - h_b) + \lambda\xi h_b\right] dp \\ & + \beta\left[\pi u'(c_{21b})(a_b dr + e_1 dw) + (1-\pi)u'(c_{22b})(a_b dr + e_2 dw)\right], \end{aligned} \tag{26}$$

where borrowers' first-order conditions were used to simplify the expression. Where dr and dw are unchanged, so that distributive externalities in the credit and labor markets are unaltered, dp is no longer defined as in Lemma 1, since condition (9) is modified by the binding collateral constraint. Thus, housing market distributive externalities are, in part, defined differently. Moreover, a binding collateral constraint introduces the new term $\lambda\xi h_b dp$, which captures that by internalizing changes in the price of houses a social planner takes into account how affecting the equilibrium allocation can relax or tighten the collateral constraint, and thus initially poor' s borrowings and, in turn, utility. In line with the literature, we label this new term collateral externalities.

Under a binding collateral constraint, to find the housing price response dp induced by changes in the laissez-faire financial asset holdings, we note that in equilibrium a modified version of the housing market clearing condition (9) must hold. In particular, the equilibrium housing price solves

$$\widetilde{h}_b\left(p, a_b\right) + h_l\left(p, \mu_l\right) = \bar{H}\ , \tag{27}$$

where $\widetilde{h}_b\left(p, a_b\right) = -a_b/(\xi p)$ is the housing demand function, for given borrowings, of initially poor households when the collateral constraint binds. The first step to find dp entails asking how moving the financial asset positions affects the aggregate housing demand function on impact. Since an increase in the financial asset position a_i, i.e. a reduction in borrowings or an increase in savings, directly translates into less housing for borrowers via the binding collateral constraint and a decrease in lenders' initial resources, we can again conclude that the aggregate housing demand function will shift to the left on impact. More formally, when perturbing borrowings the aggregate housing demand moves as follows on impact

$$\frac{\partial\left(\widetilde{h}_b\left(p,a_b\right)+h_l\left(p,\mu_l\right)\right)}{\partial a_b}=-\frac{1}{\xi p}<0, \tag{28}$$

and when perturbing savings

$$\frac{\partial\left(\widetilde{h}_b\left(p,a_b\right)+h_l\left(p,\mu_l\right)\right)}{\partial a_l}=-\frac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}<0, \tag{29}$$

where the slope of agent l's housing Engel curve is positive as the demand for housing is normal. Knowing the impact response, we can infer the equilibrium price response dp from the properties of the aggregate housing demand function. In particular, we identify sufficient conditions under which the law of demand in the housing market is satisfied. The movement in the aggregate housing demand following a housing price change can be decomposed as follows

$$\frac{\partial\left(\widetilde{h}_b\left(p,a_b\right)+h_l\left(p,\mu_l\right)\right)}{\partial p}=\frac{a_b}{\xi p^2}+\mathcal{S}_l(p,\mu_l)-\frac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}(h_l\left(p,\mu_l\right)-\bar{h}_l), \tag{30}$$

where the first term on the right hand side represents the mechanical change in borrowers' housing demand following a variation in p, which is clearly negative since $a_b<0$, and the other two terms are the substitution and wealth effects for lenders. As they are net sellers, their optimal housing demand is not ensured to be neither decreasing nor monotone in the price of houses. Thus, in general, it is ambiguous how the aggregate demand for houses adjusts following a housing price change. However, conversely from Section 3.1, only identifying key properties of households' housing Engel curves will not suffice in this case, as the change in borrowers' housing is mechanical in nature when the collateral constraint binds.[12] Nonetheless, by inspecting equation (30) it is clear that it would also be possible to attain the law of demand via a different set of preferences' properties: identifying conditions that ensure monotonicity of lenders' demand function. Specifically, preferences that generate a substitution effect that dominates the opposing wealth effect will lead to an aggregate housing demand that is monotonically decreasing in the price of houses. How does the latter compare to the condition on the properties of households' housing Engel curves? The following lemma elaborates on this point while stating the equilibrium housing price response to a change in the laissez-faire financial asset positions.

[12]It is thus unclear how it compares to lenders' wealth effect.

Lemma 8.

Under a binding collateral constraint, preferences that entail linear or concave-in-resources housing Engel curves imply that

$$da_i > 0 \;\Rightarrow\; dp = \widetilde{\Phi}_{a,i}\, da_i \gtreqless 0 \;\; \forall i.$$

However, under a binding collateral constraint, preferences entailing that the substitution effect dominates the wealth effect imply that

$$da_i > 0 \;\Rightarrow\; dp = \widetilde{\Phi}_{a,i}\, da_i < 0 \;\; \forall i,$$

with

$$\widetilde{\Phi}_{a,b} := \frac{\dfrac{1}{\xi p}}{\dfrac{a_b}{\xi p^2} + \mathcal{S}_l(p,\mu_l) - \dfrac{\partial h_l\,(p,\mu_l)}{\partial \mu_l}(h_l\,(p,\mu_l) - \bar{h}_l)},$$

$$\widetilde{\Phi}_{a,l} := \frac{\dfrac{\partial h_l\,(p,\mu_l)}{\partial \mu_l}}{\dfrac{a_b}{\xi p^2} + \mathcal{S}_l(p,\mu_l) - \dfrac{\partial h_l\,(p,\mu_l)}{\partial \mu_l}(h_l\,(p,\mu_l) - \bar{h}_l)}.$$

Proof. In the Appendix. □

Thus, with a binding collateral constraint housing distributive externalities will be defined distinctly via the altered $\widetilde{\Phi}_{a,i}$ coefficients. Interestingly, stricter conditions on preferences are required for the law of demand to attain.[13] Specifically, maintaining properties only on households' Engel curves allows, in principle, for a positive equilibrium price response following a decrease in borrowings or increase in savings.

Turning to the effects on borrowers' utility of such changes, by using the derived dp, dr and dw, we can rewrite dU_b as

$$dU_b = \left[\widetilde{\Psi}^p_{a,b} + \widetilde{\Psi}^p_{\lambda,b}\right] da_b + \left[\widetilde{\Psi}^p_{a,l} + \widetilde{\Psi}^p_{\lambda,l}\right] da_l + \Psi^{r,w}_K (da_b + da_l),$$

with,

$$\widetilde{\Psi}^p_{a,b} := u'(c_{1b})(\bar{h}_b - h_b)\widetilde{\Phi}_{a,b},$$

$$\widetilde{\Psi}^p_{\lambda,b} := \lambda \xi h_b \widetilde{\Phi}_{a,b},$$

$$\widetilde{\Psi}^p_{a,l} := u'(c_{1b})(\bar{h}_b - h_b)\widetilde{\Phi}_{a,l},$$

$$\widetilde{\Psi}^p_{\lambda,l} := \lambda \xi h_b \widetilde{\Phi}_{a,l},$$

[13] We interpret conditions on preferences that entail a dominating substitution effect as more stringent since they directly impose monotonicity at the household level.

$$\Psi_K^{r,w} := \beta\left[\pi u'\left(c_{21b}\right)\left(a_b F_{KK} + e_1 F_{LK}\right) + (1-\pi)u'\left(c_{22b}\right)\left(a_b F_{KK} + e_2 F_{LK}\right)\right],$$

where the terms $\widetilde{\Psi}^p_{a,b}$, $\widetilde{\Psi}^p_{a,l}$ and $\Psi_K^{r,w}$ capture as before the housing, and the credit and labor market distributive externalities respectively, while the new terms $\widetilde{\Psi}^p_{\lambda,b}$ and $\widetilde{\Psi}^p_{\lambda,l}$ capture how changes in the housing price affect borrowers' utility by relaxing or tightening the collateral constraint, i.e. the collateral externalities. Without stricter conditions on preferences, housing distributive externalities could now impact households inversely, depending on the equilibrium price response. This is already an implication induced by the presence of a binding collateral constraint. Furthermore, the additional pecuniary externalities triggered by the collateral constraint are always antithetical to housing market distributive externalities: a decrease (increase) in the housing price positively (negatively) affects borrowers' terms of trade while tightening (relaxing) the collateral constraint. The following lemma characterizes the change in borrowers' utility when the market-equilibrium financial asset holdings are changed.

Lemma 9.
With a binding collateral constraint, under preferences that entail linear or concave-in-resources housing Engel curves

$$da_i > 0 \;\Rightarrow\; dU_b = \left(\widetilde{\Psi}^p_{a,i} + \widetilde{\Psi}^p_{\lambda,i} + \Psi_K^{r,w}\right) da_i \gtreqless 0 \;\; \forall i.$$

Furthermore, with a binding collateral constraint, under preferences entailing that the substitution effect dominates the wealth effect

$$da_i > 0 \;\Rightarrow\; dU_b = \left(\widetilde{\Psi}^p_{a,i} + \widetilde{\Psi}^p_{\lambda,i} + \Psi_K^{r,w}\right) da_i \begin{cases} > 0 \;\; if \;\; \bar{h}_b \leq (1-\xi)h_b \\ \gtreqless 0 \;\; if \;\; \bar{h}_b > (1-\xi)h_b \end{cases} \forall i.$$

Proof. In the Appendix. □

The first part of the lemma states that, under the same conditions on housing Engel curves identified in Lemma 2, a change in the laissez-faire equilibrium financial asset positions has now an ambiguous effect on borrowers' utility. While the positive sign of $\Psi_K^{r,w}$ is unaffected, as the equilibrium housing price response does not have a sign a priori, $\widetilde{\Psi}^p_{a,i}$ and $\widetilde{\Psi}^p_{\lambda,i}$ could take either sign while always opposing each other. The second part of the lemma takes a step further by stating that even when more stringent conditions on households' preferences apply, so that the law of demand holds once again, there is another salient condition that must be met to analytically obtain the same conclusion as without a binding collateral constraint. In particular, changing the laissez-faire equilibrium financial asset positions, so that the initially poor households borrow less and/or initially rich save more, unambiguously implies an increase in borrowers' utility when households' down payment $p(1-\xi)h_b$

is at least as large as the value of the initially owned housing $p\bar{h}_b$. Alternatively, the condition can be interpreted as stating that borrowers' net amount of acquired housing $p(h_b - \bar{h}_b)$ is not more than financed by the collateralized debt extracted from the acquired housing $\xi p h_b$. In other words, reducing borrowings and/or increasing savings makes initially poor better-off whenever their housing down payment is sufficiently large, since under this scenario housing distributive externalities dominate collateral externalities and such changes in financial positions decrease the housing price and interest rate while increasing the wage. For example, in the case of financially constrained agents without any housing endowment all $\xi \in [0,1]$ satisfy the down payment requirement, and thus imply that reducing their borrowings and/or increasing lenders' savings would make first-time home buyers at the constraint better-off. Conversely, whenever a homeowner can collateralize a large enough fraction of the newly acquired housing or if the amount that a first-time home buyer can collateralize is strictly larger than the acquired housing value, i.e. $\xi > 1$, then collateral externalities *could* dominate housing distributive externalities. If so, a drop in the price of houses generates a negative impact on borrowers' utility as the tightening of the collateral constraint is relatively more important than the improved terms of trade in the market for houses. Nevertheless, even so the overall utility effect remains ambiguous since a reduction in borrowings and/or increase in savings continue to improve the terms of trade in the credit and labor markets. It is a quantitative question whether collateral externalities could be so large as to dominate all distributive externalities. In that case, increasing borrowings and/or reducing savings would increase borrowers' utility, as such movements in financial asset holdings would increase the housing price and thus relax the collateral constraint, the more important margin.

We now turn to the effects on lenders' utility of changes in the equilibrium financial asset holdings. By using the derived dp, dr and dw, we can rewrite dU_l as

$$
\begin{aligned}
&dU_l = \widetilde{\Theta}^p_{a,b} da_b + \widetilde{\Theta}^p_{a,l} da_l + \Theta^{r,w}_K (da_b + da_l), \\
&\text{with,} \\
&\widetilde{\Theta}^p_{a,b} := u'(c_{1l})(\bar{h}_l - h_l)\widetilde{\Phi}_{a,b}, \\
&\widetilde{\Theta}^p_{a,l} := u'(c_{1l})(\bar{h}_l - h_l)\widetilde{\Phi}_{a,l}, \\
&\Theta^{r,w}_K := \beta\Big[\pi u'\left(c_{21l}\right)\left(a_l F_{KK} + e_1 F_{LK}\right) + (1-\pi)u'\left(c_{22l}\right)\left(a_l F_{KK} + e_2 F_{LK}\right)\Big],
\end{aligned}
$$

where the terms $\widetilde{\Theta}^p_{a,b}$, $\widetilde{\Theta}^p_{a,l}$ and $\Theta^{r,w}_K$ capture the housing, and the credit and labor market distributive externalities respectively. As for borrowers, besides being differently defined, housing distributive externalities could now impact lenders inversely, depending on the equilibrium price response. The following lemma characterizes the change in lenders' utility when market-equilibrium financial asset holdings are perturbed.

Lemma 10.
With a binding collateral constraint, under preferences that entail linear or concave-in-resources housing Engel curves

$$da_i > 0 \;\Rightarrow\; dU_l = \left(\widetilde{\Theta}^{p}_{a,i} + \Theta^{r,w}_{K}\right) da_i \gtreqless 0 \;\; \forall i.$$

Furthermore, with a binding collateral constraint, under preferences entailing that the substitution effect dominates the wealth effect

$$da_i > 0 \;\Rightarrow\; dU_l = \left(\widetilde{\Theta}^{p}_{a,i} + \Theta^{r,w}_{K}\right) da_i < 0 \;\; \forall i.$$

Proof. Trivially follows from Lemma 8 and, as previously shown, $\Theta^{r,w}_{K} < 0$. □

The first part of the lemma states that, under the previously identified conditions on housing Engel curves, due to the ambiguous equilibrium housing price response a change in the laissez-faire equilibrium financial asset positions has now an unclear effect on lenders' utility too. In particular, $\widetilde{\Theta}^{p}_{a,i}$ could take either sign, where the negative sign of $\Theta^{r,w}_{K}$ is preserved. If, following a reduction in borrowings and/or increase in savings, the equilibrium price of houses increases then housing distributive externalities favor lenders' terms of trade. In that case, it remains a quantitative question whether the negative credit and labor market distributive externalities could be outweighed or still prevail – in the former (latter) case obtaining the opposite (same) conclusion in Lemma 3. When more stringent conditions on households' preferences apply, so that the law of demand holds, the same result obtained under natural borrowing limits attains: lenders' utility is negatively affected by a reduction in borrowings and/or increase in savings via the workings of both distributive externalities.

3.3.1 Constrained Efficiency

We characterize constrained efficiency as before through the lens of the social panner. Under illiquid houses and a collateral constraint, the constrained social planner problem becomes

$$\max \sum_{i\in\{b,l\}} \gamma_i \Big\{ \mathbf{u}(c_{1i}, h_i(p,\mu_i)) + \beta \Big[\pi\, \mathbf{u}(c_{21i}, h_i(p,\mu_i)) + (1-\pi)\, \mathbf{u}(c_{22i}, h_i(p,\mu_i)) \Big] \Big\}$$

subject to

$$c_{1i} + a_i + p h_i(p,\mu_i) = \bar{\omega}_i + p\bar{h}_i \;,$$
$$a_i \geq -\xi p h_i(p,\mu_i) \;,$$
$$c_{2si} = (1+r)a_i + w e_s \;, \text{ for } s = 1,2 \text{ and } prob(e = e_1) = \pi \;,$$
$$r = F_K(K,L) - \delta \;,\; w = F_L(K,L) \;,$$
$$K = a_b + a_l \;,\; L = 2(\pi e_1 + (1-\pi) e_2) \;,$$
$$\bar{H} = h_b(p,\mu_b) + h_l(p,\mu_l) \;.$$

Evaluating the planner's system of first-order conditions at the laissez-faire allocation, and rewriting the system in matrix form we obtain

$$\begin{bmatrix} \widetilde{\Psi}^{p}_{a,b} + \widetilde{\Psi}^{p}_{\lambda,b} + \Psi^{r,w}_{K} & \widetilde{\Theta}^{p}_{a,b} + \Theta^{r,w}_{K} \\ \widetilde{\Psi}^{p}_{a,l} + \widetilde{\Psi}^{p}_{\lambda,l} + \Psi^{r,w}_{K} & \widetilde{\Theta}^{p}_{a,l} + \Theta^{r,w}_{K} \end{bmatrix} \begin{bmatrix} \gamma_b \\ \gamma_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{31}$$

Let $\widetilde{A}$ be the 2×2 matrix on the left-hand side of (31), in Proposition 6 we identify sufficient statistics upon which constrained efficiency rests, and establish under which conditions the competitive equilibrium is inefficient by checking when the above system of equations has a non-singular matrix $\widetilde{A}$.

Proposition 6.
Constrained efficiency of the competitive equilibrium can be characterized by the following two sufficient statistics:

($\widetilde{A1}$) *the difference between the housing price response from changing borrowings only and the housing price change when perturbing savings only* $\left[\widetilde{\Phi}_{a,b} - \widetilde{\Phi}_{a,l}\right]$,

($\widetilde{A2}$) *and the sum of the following three terms:*

- ($\widetilde{A2.1}$) *the interaction of borrowers' housing linked externalities with lenders' credit and labor market externalities* $u'(c_{1b})\left(\bar{h}_b - h_b\right)\Theta^{r,w}_{K}$,
- ($\widetilde{A2.2}$) *the interaction of lenders' housing linked externalities with borrowers' credit and labor market externalities* $u'(c_{1l})\left(\bar{h}_b - h_b\right)\Psi^{r,w}_{K}$,
- ($\widetilde{A2.3}$) *and the interaction of borrowers' collateral linked externalities with lenders' credit and labor market externalities* $\lambda\xi h_b \Theta^{r,w}_{K}$.

In particular, and respectively connected to each of the above sufficient statistics, the competitive equilibrium is generally constrained inefficient as long as the following two conditions hold simultaneously:

($\widetilde{C1}$) *changes in borrowings and savings differently impact the aggregate housing demand,*

($\widetilde{C2}$) *at least one of the following two conditions hold:*

- ($\widetilde{C2.1}$) *market incompleteness is relevant and there is trading in the market for houses,*
- ($\widetilde{C2.2}$) *borrowers are debt-constrained.*

Proof. In the Appendix. □

It should be noted that when ($\widetilde{C1}$) and both ($\widetilde{C2.1}$) and ($\widetilde{C2.2}$) apply, there may be cases where the sum of the terms ($\widetilde{A2.1}$), ($\widetilde{A2.2}$) and ($\widetilde{A2.3}$) is zero. This would imply that, even with two frictions at work, the competitive allocation is socially efficient. However, these are non-generic cases.[14] Moreover, without additional assumptions we cannot sign $(\widetilde{A1})$ and $(\widetilde{A2})$. For $(\widetilde{A1})$, when the law of demand holds both $\widetilde{\Phi}_{a,b}$ and $\widetilde{\Phi}_{a,l}$ are negative. However, even if $(\widetilde{C1})$ applies it is unclear how the mechanical impact on the aggregate housing demand of a change in borrowings compares to the effect of a perturbation in savings, captured by the slope of lenders' housing Engel curves. Although it is clear that the impact induced by a perturbation in borrowings dominates that of a change in savings as ξ approaches zero, so that $(\widetilde{A1})$ becomes negative, it remains a quantitative question whether the latter is true more generally. We will explore this in the numerical section. For $(\widetilde{A2})$, we can sign each of its three terms. In particular, $(\widetilde{A2.1})$ is positive while both $(\widetilde{A2.2})$ and $(\widetilde{A2.3})$ are negative. In the numerical section we will show that the sign of $(\widetilde{A2})$ is largely driven by the interplay of borrowers' collateral and housing market distributive externalities. Analytically it is trivial that $(\widetilde{A2})$ is negative when collateral externalities dominate, since the absolute value of $(\widetilde{A2.3})$ is larger than the absolute value of $(\widetilde{A2.1})$ in that case. Furthermore, we can analytically show that the sign of $(\widetilde{A2})$ is positive when considering borrowers that are first-time homebuyers that can collateralize any fraction of the acquired housing, even entirely.[15] In all these cases the aforementioned condition on a large enough down payment is met, so housing distributive externalities dominate collateral externalities, and it turns out that $(\widetilde{A2.1})$ drives the sign of the second sufficient statistic. In Section 4 we study numerically the interplay of collateral and distributive externalities in $(\widetilde{A2})$ as a function of the collateral parameter. This, along with the investigation of $(\widetilde{A1})$, allows us to illustrate how the Pareto improving region in the da_b-da_l plane changes as we vary ξ, thus showing how different leverage degrees affect deviations from the competitive allocation.

Intuitively, the planner will mandate changes to savings and borrowings by accounting for the interaction of the two fundamental frictions at play. In particular, market incompleteness and binding debt constraints may introduce a trade-off between improving households' insurance and enlarging their credit opportunities: The social planner's desire to reduce the risk that households face in their stochastic labor income, for which there is no direct market insurance, may be in opposition to its desire of also relaxing borrowings constraints. In the numerical section we show when this trade-off is present and how it plays out.

Finally, the following proposition characterizes explicit tax incentives that implement constrained efficient allocations. Specifically, this will be achieved through a proportional tax or subsidy on financial asset returns, accompanied by a transfer.

[14]Under similar conditions, such scenarios never arise with a natural limit on borrowings since the comparable term $(A1) \times (A3)$ is always different from zero.

[15]See Lemma B1 in Appendix B.1 for the proof of this statement.

Proposition 7.
A social planner can implement any constrained efficient allocation by setting taxes on financial asset returns that satisfy

$$\widetilde{\tau}_b^a = \frac{-1}{\beta r \gamma_b \sum_{s=1}^{2} \pi_s u'\left(c_{2sb}\right)} \left\{ \gamma_b \widetilde{\Psi}_{a,b}^p + \gamma_b \widetilde{\Psi}_{\lambda,b}^p + \gamma_l \widetilde{\Theta}_{a,b}^p + \gamma_b \Psi_K^{r,w} + \gamma_l \Theta_K^{r,w} \right\},$$

$$\widetilde{\tau}_l^a = \frac{-1}{\beta r \gamma_l \sum_{s=1}^{2} \pi_s u'\left(c_{2sl}\right)} \left\{ \gamma_b \widetilde{\Psi}_{a,l}^p + \gamma_b \widetilde{\Psi}_{\lambda,l}^p + \gamma_l \widetilde{\Theta}_{a,l}^p + \gamma_b \Psi_K^{r,w} + \gamma_l \Theta_K^{r,w} \right\},$$

and conducting lump-sum transfers $\widetilde{T}_b^a$ and $\widetilde{T}_l^a$, with $\widetilde{T}_b^a = r\widetilde{\tau}_b^a a_b$ and $T_l^a = r\widetilde{\tau}_l^a a_l$, so that net transfers to households are zero and the government budget constraint is satisfied.

Proof. In the Appendix. □

4 Numerical Analysis

In this section, we parametrize the three environments and solve them numerically. This serves not only as an illustration of our theoretical results, but also leads to new insights on how Pareto improvements can be achieved for the cases of Sections 3.2 and 3.3.

Conceptually, we proceed as follows. First, we choose a subset of parameters that is exogenously determined and fixed across all model variants. Second, we determine the remaining parameters to match a set of long-run aggregate and distributional targets from U.S. data. The values of these parameters then vary from case to case.[16]

Exogenous Parameters. We use standard functional forms for the utility function and the production function. In particular, similarly to Díaz and Luengo-Prado (2010), we use a utility function that is separable in consumption and housing,

$$\mathbf{u}(c,h) = \theta \frac{c^{1-\sigma} - 1}{1-\sigma} + (1-\theta)\frac{h^{1-\gamma} - 1}{1-\gamma}, \tag{32}$$

where σ and γ are the inverse elasticities of intertemporal substitution with respect to non-durable consumption and housing consumption, respectively, and θ measures the relative taste for non-durable consumption. The production function is Cobb-Douglas and given by

$$F(K,L) = K^{\alpha} L^{1-\alpha}, \tag{33}$$

where α is the capital share of income.

[16] We provide a comparison of parameter values in Table B1. Note that we also choose parameters such that each model variant exhibits a given set of properties, which we explain below in detail.

Table 1: Exogenous Parameters

Description	Parameter	Value
Discount factor	β	0.93
Depreciation rate of capital	δ	0.26
IES – non-durable consumption	σ	3
IES – housing	γ	4
Relative share of non-durable consumption	θ	0.75
Capital share	α	0.33

We set $\sigma = 3$ and $\gamma = 4$, indicating a higher elasticity with respect to non-durable consumption. In line with values from the literature, such as Favilukis et al. (2017) and Díaz and Luengo-Prado (2010), θ is set to 0.75. One model period corresponds to two years. We set $\delta = 0.26$ to yield an annual depreciation rate of about 12 percent, and the discount factor $\beta = 0.93$ implying an annual real interest rate of about 5 percent.

Endogenous Parameters. We identify four targets from the data that our model should be able to match. The targets are the following: 1) the ratio of top half to bottom half income, 2) the housing-resources ratio of the bottom half income distribution, 3) the housing-resources ratio of the top half income distribution, and 4) the annualized ratio of aggregate capital to output. We define resources here as the sum of wealth and income. The parameter set that is chosen to match these targets is $\{e_1, e_2, \pi, \bar{\omega}_b, \bar{\omega}_l, \bar{h}_b, \bar{h}_l\}$, involving the parameters from the income process as well as parameters relating to initial endowments. Moreover, these parameters must imply that in equilibrium borrowers take a negative position in the financial asset and that they are net buyers of houses in the first period.

We use data from the Survey of Consumer Finance (SCF) to determine the three distributional targets. In particular, we construct an income variable consisting of labor and transfer income such that we capture only labor-related income and rule out cases of negative income caused by business failures. Across the SCF waves from 1992 to 2007, we find that the average ratio of top half income to bottom half income is about 4.2. For the housing-resources ratios, we find values of about 0.61 for the bottom half of the income distribution and 0.40 for the top half. Finally, we follow Díaz and Luengo-Prado (2010) who use the Fixed Asset Tables of the Bureau of Economic Analysis and determine the capital-output ratio to be 1.64.

4.1 Illiquid Housing & Non-Collateralized Borrowings

We first examine the case of illquid housing and non-collateralized borrowings, serving as a basis for the other cases. For the income process, we set $e_1 = 0.25$, $e_2 = 1.05$ and $\pi = 0.5$. The initial housing endowments $\bar{h}_b$ and $\bar{h}_l$ are 0.07 and 1.3, respectively, while the initial wealth endowments $\bar{\omega}_b$ and $\bar{\omega}_l$ are 0.075 and 2.75. These choices imply an income-ratio of 4.2, housing-resources ratios of 0.6 for the bottom half and 0.32 for the top half of the income distribution as well as a capital-output ratio of 1.79. Moreover, these parameter choices entail that in equilibrium wealth-poor households are net buyers of houses and have a negative position in the financial asset, while wealth-rich are net sellers and hold a positive amount of the financial asset.

To find out what signs and relative magnitude da_b and da_l should have to achieve a strict Pareto improvement over the competitive equilibrium, we require that

$$dU_b = \Psi^p_{a,b} da_b + \Psi^p_{a,l} da_l + \Psi^{r,w}_K (da_b + da_l) > 0, \tag{34}$$

$$dU_l = \Theta^p_{a,b} da_b + \Theta^p_{a,l} da_l + \Theta^{r,w}_K (da_b + da_l) > 0, \tag{35}$$

that is, we look for movements in borrowings and savings such that both groups are strictly better off. Rewriting the above, one arrives to

$$da_l > -\frac{\Psi^p_{a,b} + \Psi^{r,w}_K}{\Psi^p_{a,l} + \Psi^{r,w}_K} da_b, \tag{36}$$

$$da_l < -\frac{\Theta^p_{a,b} + \Theta^{r,w}_K}{\Theta^p_{a,l} + \Theta^{r,w}_K} da_b, \tag{37}$$

which form a system of linear inequalities whose borders are both downward sloping lines passing through the origin. Let Ψ be the coefficient in front of da_b in (36) and Θ the coefficient in (37). It can be shown that if and only if

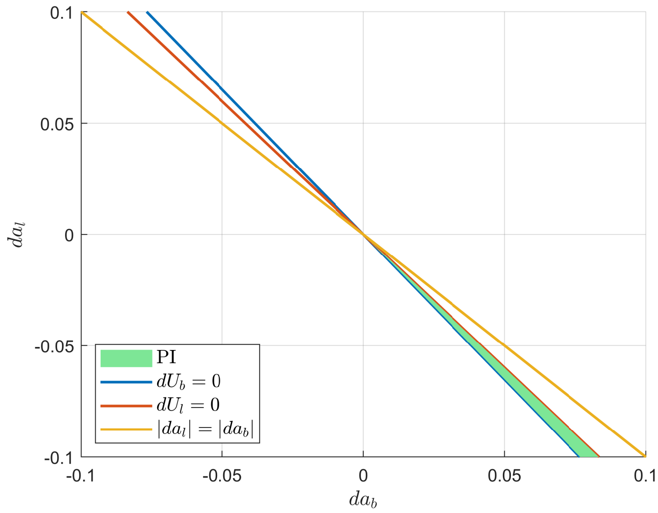
$$(\bar{h}_b - h_b)[\Phi_{a,b} - \Phi_{a,l}]\{u'(c_{1b})\Theta^{r,w}_K + u'(c_{1l})\Psi^{r,w}_K\} < 0, \tag{38}$$

i.e., the product of the three sufficient statistics characterizing constrained efficiency is negative, then $|\Psi|$ is larger than $|\Theta|$, implying that da_b has to be positive and da_l has to be negative. Otherwise, $|\Psi|$ is smaller than $|\Theta|$, which implies that da_b has to be negative and da_l has to be positive.

Figure 2 depicts, for given movements in borrowings and savings, the Pareto improving region. The yellow line implies changes in the lenders' assets that are of the same magnitude as changes in the borrowers' asset position but of opposite sign, such that there is no change in aggregate capital. Pareto improvements can be achieved below $dU_l = 0$ (red line) and above $dU_b = 0$ (blue line). We find Pareto improvements (green area) only in the fourth quadrant, where both borrowings and savings are reduced. Since the green area is below the yellow line, reductions in borrowings are always of smaller magnitude than those in savings, as we have shown in Section 3.1.

It should be noted that although all households are better off, the utility surplus does not need to be equally shared among households: The planner can influence how this surplus is distributed among them. Figure 2 shows that, moving from the blue to the red line, the

Figure 2: The Pareto Improving Region



borrowers' share of the overall utility surplus is highest when the reduction in aggregate capital is lowest. This is due to two channels. First, a lower reduction in capital generates a smaller loss in borrowers' utility via credit and labor market distributive externalities, as the decrease in the wage and the increase in the interest rate are smaller. Second, for a given reduction in savings, larger reductions in borrowings induce a larger drop in the housing price, so generating a bigger gain in borrowers' utility via the housing market distributive externalities.

4.2 Liquid Housing & Non-Collateralized Borrowings

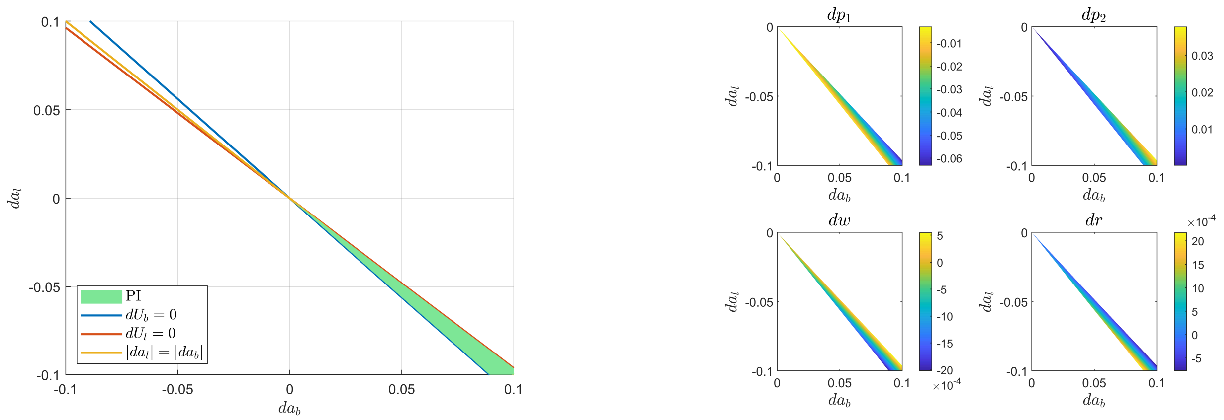
We now turn to the case of liquid housing and natural borrowing limits. The income process is parametrized as above, with $e_1 = 0.25$, $e_2 = 1.05$ and $\pi = 0.5$. For the initial housing endowments, we set $\bar{h}_b = 0.15$ and $\bar{h}_l = 1.3$, while the initial wealth endowments $\bar{\omega}_b$ and $\bar{\omega}_l$ are set to 0.15 and 2.9, respectively. This parametrization then leads to an income-ratio of 4.2, housing-resources ratios of 0.59 for the bottom half of the income distribution and 0.32 for the top half, and an implied capital-output ratio of 1.82. In the competitive equilibrium, borrowers have a negative asset position and are net buyers in the housing market in the first period, while lenders are net sellers in the first period and have a positive asset position. In the second period, all households that draw the low income shock endogenously become net sellers in the housing market, while households that draw the high income shock become net buyers of houses.

In the case of liquid housing and natural borrowing limits, a strict Pareto improvement over the competitive equilibrium is achieved if the following two inequalities are satisfied,

$$dU_b = \widehat{\Psi}^{p_1}_{a,b} da_b + \widehat{\Psi}^{p_1}_{a,l} da_l + \widehat{\Psi}^{p_2}_{a,b} da_b + \widehat{\Psi}^{p_2}_{a,l} da_l + \Psi^{r,w}_{K} (da_b + da_l) > 0, \tag{39}$$

$$dU_l = \widehat{\Theta}^{p_1}_{a,b} da_b + \widehat{\Theta}^{p_1}_{a,l} da_l + \widehat{\Theta}^{p_2}_{a,b} da_b + \widehat{\Theta}^{p_2}_{a,l} da_l + \Theta^{r,w}_{K} (da_b + da_l) > 0. \tag{40}$$

Figure 3: The Pareto Improving Region and Price Changes under Liquid Housing



(a) Pareto Improving Region

(b) Induced Change in Prices

Recall that we could already sign $\widehat{\Psi}^{p_1}_{a,i}$ and $\widehat{\Theta}^{p_1}_{a,i}$ in Section 3.2, while $\widehat{\Psi}^{p_2}_{a,i}$ and $\widehat{\Theta}^{p_2}_{a,i}$ do not have a sign a priori. At the competitive equilibrium, we find that $\widehat{\Psi}^{p_2}_{a,i} > 0$ and $\widehat{\Theta}^{p_2}_{a,i} > 0$, and moreover $\widehat{\Theta}^{p_1}_{a,i} + \widehat{\Theta}^{p_2}_{a,i} + \Theta^{r,w}_K < 0$. Thus, we can rewrite (39) and (40) in the following way,

$$da_l > -\frac{\widehat{\Psi}^{p_1}_{a,b} + \widehat{\Psi}^{p_2}_{a,b} + \Psi^{r,w}_K}{\widehat{\Psi}^{p_1}_{a,l} + \widehat{\Psi}^{p_2}_{a,l} + \Psi^{r,w}_K} da_b, \tag{41}$$

$$da_l < -\frac{\widehat{\Theta}^{p_1}_{a,b} + \widehat{\Theta}^{p_2}_{a,b} + \Theta^{r,w}_K}{\widehat{\Theta}^{p_1}_{a,l} + \widehat{\Theta}^{p_2}_{a,l} + \Theta^{r,w}_K} da_b, \tag{42}$$

where (41) and (42) form a system of linear inequalities whose borders are both downward sloping lines going through the origin. While the slope of the border of (41) is larger than one in absolute magnitude, the slope of the border of (42) is now smaller than one in absolute magnitude, differently from the illiquid housing case.

Figure 3a shows the Pareto improving region for given changes in the asset postions of borrowers and lenders. The first insight is that, even under continuously adjustable housing, Pareto improvements are located only in the fourth quadrant: The key result of over-borrowing and over-saving holds. As a consequence, a social planner would mandate less debt and lower savings. The second insight is that with fully liquid houses new implications for aggregate capital arise. While under illiquid housing capital has to decrease to achieve a Pareto improvement, the planner can now additionally implement a Pareto improvement by both increasing aggregate capital or by leaving it unchanged. These two new options correspond to all points on and above the yellow line, now within the green region. Put differently, the illiquid nature of houses only allows for one of the three options, in terms of affecting aggregate capital, that a planner would have with fully liquid assets.

To better understand these results, we first examine how equilibrium prices change within the Pareto improving region. Figure 3b shows that the induced housing price change in the first period is always negative, while the housing price change in the second period is always positive. Moreover, the absolute magnitude of both dp_1 and dp_2 is increasing in the aggregate capital stock, i.e., for a given reduction in savings, a larger reduction in borrowings results in a larger change of both housing prices. The changes in the wage and interest rate switch sign depending if capital rises or falls.

Knowing the equilibrium price dynamics, how does the planner achieve a Pareto improvement by leaving capital unchanged? In this case, only distributive externalities in the housing markets play a role since the wage and the interest rate do not change. Reducing income risk is therefore only possible via the second-period housing price. Given that low income households are net sellers in the second period, the corresponding price change ex-post benefits this group, while households that drew the high income state are worse off. Thus, from an ex-ante perspective, reductions in savings and borrowings of the same magnitude induce a Pareto improvement for two reasons. First, both lenders' and borrowers' utilities are positively impacted by the risk rescaling of their income via the second-period housing price, as $\widehat{\Psi}^{p_2}_{a,i}$ and $\widehat{\Theta}^{p_2}_{a,i}$ are positive, and second, the utility loss suffered by lenders via the first-period housing price drop is more than compensated. The former is because the utility impact in the adverse idiosyncratic event is relatively more important for all households. Note that a risk rescaling that affects the utilities of all households in the same direction cannot be implemented via the credit and labor market distributive externalities, as the wealth-rich and wealth-poor are antithetically affected through changes in wage and interest rate.

In the case of a change in aggregate capital, additional forces arise through credit and labor market distributive externalities. A rise in capital implies that the wage increases, the interest rate decreases, and the housing prices move as before but with higher absolute magnitude. Borrowers unambiguously benefit from these price changes, while the overall impact on lenders depends on the strength of the improved insurance implemented through the second-period housing market distributive externalities. Indeed, this is the only positive utility impact for the wealth-rich, whereas all other externalities adversely affect them. Thus, as long as they are sufficiently compensated via the second-period housing market, increasing capital constitutes a Pareto improvement over the competitive equilibrium. It is important to note that the part of future income that is directly related to the changes in wage and interest rate becomes riskier as the stochastic component grows. However, the planner can afford to increase aggregate capital as the future housing price can better insure households compared to changes in the wage and interest rate.

While contributing towards better households' insurance, lower aggregate capital benefits lenders and hurts borrowers via credit and labor market distributive externalities. On the other hand, with fully liquid houses, borrowers not only benefit through the first-period housing market distributive externalities, but also through the improved insurance via the future housing price change. As before, lenders are hurt due to the former but gain from the latter, such that two of the three channels work to their benefit.

To sum up, with unrestrictedly adjustable houses, it descends that capital is not necessarily too much or too little in the competitive equilibrium. The degree of housing illiquidity plays a crucial role in defining the socially efficient level of aggregate capital. Furthermore, as in the case of illiquid housing, the borrowers' fraction of the utility surplus is increasing in the aggregate capital stock: The planner can indirectly transfer the lenders' utility gain from improved insurance to borrowers by increasing capital.

4.3 Illiquid Housing & Collateralized Borrowings

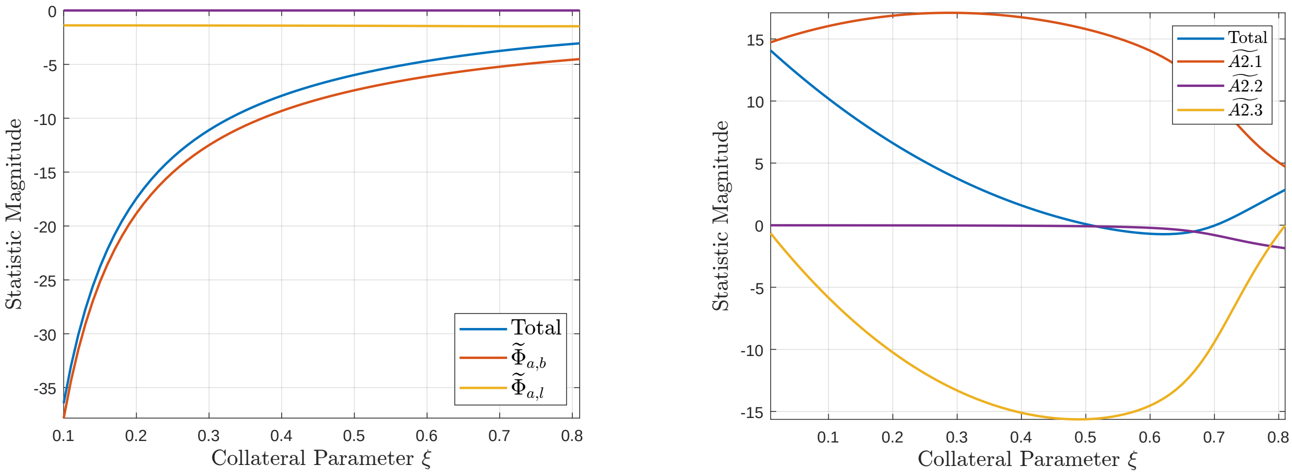
Finally, we evaluate the scenario encompassing both the illiquid and collateral nature of housing. To gain a comprehensive understanding of the interaction between collateral and distributive externalities, we solve the model for a wide range of values for the collateral parameter ξ. This allows us to determine whether different leverage degrees affect how the social planner would change the equilibrium allocation.

As the collateral parameter influences the equilibrium, we adjust our calibration strategy in the following way. We first determine an interval for ξ that contains the empirically relevant range of values. Then we choose parameters such that the collateral constraint is always binding and our calibration targets are met within this interval.

In line with values that are usually chosen in the housing literature, e.g. Favilukis et al. (2017) or Díaz and Luengo-Prado (2010), we set the range of relevant collateral parameter values to $\xi \in [0.75, 0.8]$. The income process is parametrized as in the other two cases with $e_1 = 0.25$, $e_2 = 1.05$ and $\pi = 0.5$. Moreover, initial housing endowments are $\bar{h}_b = 0.0425$ and $\bar{h}_l = 1.35$, while wealth endowments are set to $\bar{\omega}_b = 0.035$ and $\bar{\omega}_l = 2.9$. These parameters then imply an income-ratio of 4.2, housing-resources ratios of 0.6 for the bottom half and 0.31 for the top half of the income distribution, and the capital-output ratio is 1.85. Additionally, these parameter choices imply that borrowers are net buyers and are at the collateral constraint for $\xi \in]0, 0.81]$, while lenders are net sellers and hold a positive amount of the financial asset.

In Section 3.3.1 we characterize constrained efficiency of the competitive equilibrium by the sufficient statistics $(\widetilde{A1})$ and $(\widetilde{A2})$. We now explore numerically their sign, magnitude and composition as functions of ξ. The left panel of Figure 4 depicts the first sufficient statistic as well as its components. The blue line is the total $[\widetilde{\Phi}_{a,b} - \widetilde{\Phi}_{a,l}]$, where the red line and the yellow line are the changes in the housing price caused by only perturbing the financial asset positions of borrowers and lenders, respectively. Analytically, we show that it is not clear how the housing price moves with changes in the financial asset positions. In particular, it is unclear how the mechanical change in the borrowers' housing compares to the lenders' wealth effect. However, our numerical results reveal that both $\widetilde{\Phi}_{a,b}$ and $\widetilde{\Phi}_{a,l}$ are everywhere

Figure 4: Sufficient Statistics under Collateralized Borrowings as functions of ξ



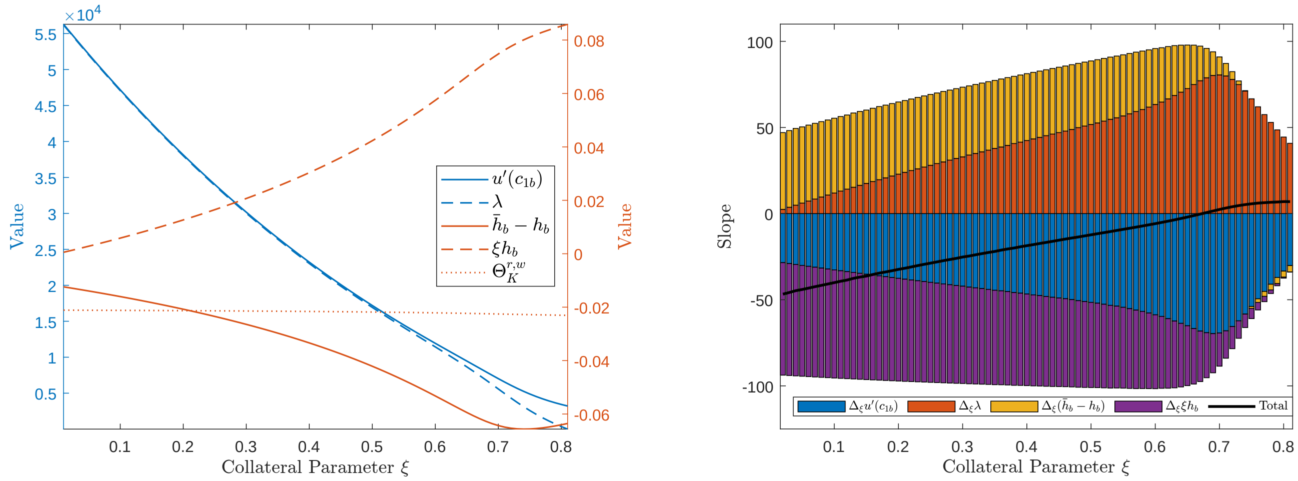
(a) Sufficient Statistic $(\widetilde{A1})$ (b) Sufficient Statistic $(\widetilde{A2})$

negative. Furthermore, the overall price impact, i.e. the first sufficient statistic, is increasing in ξ and always negative since changes in the borrowers' asset position dominate.

The right panel of Figure 4 shows the second sufficient statistic and its decomposition. In particular, the red line shows the interaction of borrowers' housing linked externalities with lenders' credit and labor market externalities, i.e. $(\widetilde{A2.1})$, the purple line is the interaction of lenders' housing linked externalities with borrowers' credit and labor market externalities, i.e. $(\widetilde{A2.2})$, and the yellow line is the interaction of borrowers' collateral linked externalities with lenders' credit and labor market externalities, i.e. $(\widetilde{A2.3})$. The blue line shows the total value of the sufficient statistic. Analytically, the sign of $(\widetilde{A2})$ and whether it changes in the collateral parameter space are unclear. As a first result, our numerical analysis shows that there are three relevant regions where the sign switches from positive to negative and then back to positive. Second, our decomposition demonstrates that the dynamics of this sufficient statistic are driven by $(\widetilde{A2.1})$ and $(\widetilde{A2.3})$, whereas $(\widetilde{A2.2})$ is negligible for a wide range of leverage degrees but the very top. In the region of lowest collateralizability, $\xi \in]0, 0.5]$, the total effect is always positive as the borrowers' housing market distributive externalities dominate the collateral externalities. The sign of the total effect then switches for smaller down-payments, $\xi \in]0.5, 0.7]$, as collateral externalities become relatively more important. However, for even higher leverage degrees, $\xi \in]0.7, 0.81]$, housing market distributive externalities prevail again. To sum up, we find that the competitive equilibrium is inefficient for almost all ξ, and importantly collateral externalities are dominated in the empirically relevant range for housing collateralizability. In the following, we explain these results and their implications in more detail.

Figure 5 provides a decomposition of $(\widetilde{A2.1})$ and $(\widetilde{A2.3})$. In Figure 5a, we show the value of each component for both terms: The product of the solid blue, solid red and dotted red line is $(\widetilde{A2.1})$, and the product of the dashed blue, dashed red and dotted red line is $(\widetilde{A2.3})$.

Figure 5: Decomposition of $(\widetilde{A2.1})$ and $(\widetilde{A2.3})$ as functions of ξ



(a) Components' magnitude

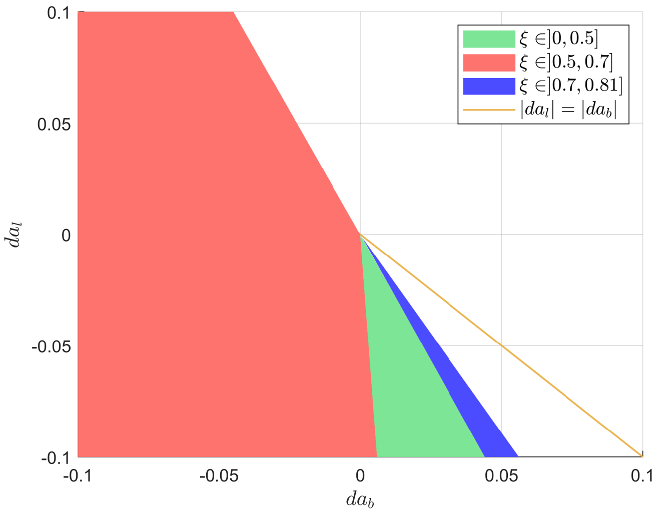
(b) Components' slope contribution

The contribution of each component to the slope of $(\widetilde{A2})$ is shown in Figure 5b.[17] Focusing on the shape of $(\widetilde{A2.1})$, Figure 5b shows that for $\xi < 0.3$ the yellow bars dominate the blue ones. This implies that the increase in housing from relaxing the collateral constraint dominates the increase in non-durable consumption in this region so that $(\widetilde{A2.1})$ is initially increasing. However, as the opportunity to lever rises further, the change in the marginal utility of consumption weighs more than the change in the net housing position so that $(\widetilde{A2.1})$ is decreasing for higher ξ. Turning to the shape of $(\widetilde{A2.3})$, an increase in leverage triggers two effects. On the one hand, there is the effect of slackening the collateral constraint, which allows to extract more debt from larger housing positions. On the other hand, a change in ξ also implies a change in the shadow value of the collateral constraint. Figure 5b shows that for $\xi < 0.5$ the purple bars dominate the orange ones. This entails that the mechanical change in ξ and the endogenous increase in housing dominate the decrease in λ in this region, thus explaining the decreasing part of $(\widetilde{A2.3})$. However, as ξ becomes larger, the drop in the shadow value outweighs the increase in ξh_b, so that $(\widetilde{A2.3})$ is increasing for higher leverage degrees.

Having understood the shape of $(\widetilde{A2.1})$ and $(\widetilde{A2.3})$ independently, we now turn to explain how the highlighted components drive the sign switches in $(\widetilde{A2})$. Figure 5b shows that extracting more debt from larger housing positions and changes in non-durable consumption jointly dominate for low and medium leverage positions, and are thus responsible for the first sign switch in $(\widetilde{A2})$. As we move to higher leverage degrees, larger debt and changes in housing become less and less important, so that the shape of $(\widetilde{A2})$ is determined by the opposing changes in the marginal utility of consumption and the shadow value of

[17] Total refers to the sum of the slopes of $(\widetilde{A2.1})$ and $(\widetilde{A2.3})$, which are the relevant terms to understand the dynamics of $(\widetilde{A2})$. Further, as Figure 5a shows $\partial\Theta_K^{r,w}/\partial\xi \approx 0$, we do not include its contribution in Figure 5b.

Figure 6: Pareto Improving Region



the collateral constraint. In particular, it is the rising importance of the latter that induces the second sign switch in $(\widetilde{A2})$.

Finally, we turn to how the planner affects the competitive equilibrium for different leverage degrees. Figure 6 depicts the Pareto improving regions in the (da_b, da_l) space for the three aforementioned intervals of the collateral parameter ξ. Connecting to Figure 4b, the values of ξ for which $(\widetilde{A2})$ is positive correspond to the green and the blue area in Figure 6. In these regions, where housing market distributive externalities dominate collateral externalities, a Pareto improvement can only be achieved by reducing both borrowings and savings in a way that decreases the equilibrium housing price. Furthermore, the reduction in savings needs to be larger than the reduction in borrowings so that aggregate capital is lower than at the competitive equilibrium. Therefore, the planner faces a trade-off in alleviating the consequences of the two frictions, choosing to prioritize the market incompleteness. Another key observation is that in the empirically relevant region for ξ the drop in capital needs to be smaller than for low levels of ξ. That is, for a given reduction in savings, borrowings need to be cut more in the blue area as compared to the green area. The intuition is as follows. For high levels of collateralizability, borrowers are more leveraged and are therefore hurt more by credit and labor market distributive externalities. Moreover, as the housing price is not as responsive in this region, it is more difficult for the planner to make them better off via the housing market. Thus, to induce a Pareto improvement, the planner cannot reduce aggregate capital as much as for low leverage degrees. In other words, the planner still prioritizes market incompleteness but accounts more for high-leverage binding borrowing constraints.

The values of ξ for which $(\widetilde{A2})$ is negative in Figure 4b, correspond to the red area in Figure 6. Note that different ξ values lead to different portions within the red area. In particular, there are some leverage degrees such that the planner can achieve a Pareto improvement by doing some or all of the following: increase both borrowings and savings, increase borrowings while reducing savings, still reduce both borrowings and savings, or leave savings unchanged (borrowings unchanged) and increase borrowings (decrease savings). The planner can mandate all these possible ways of deviating from the competitive equilibrium because they all induce an increase in the housing price. As collateral externalities dominate and lenders are net sellers of houses, all households are positively affected by this price change. Furthermore, Pareto improving deviations from the competitive equilibrium can now lead to changing aggregate capital in either direction or leaving it unaffected. Indeed, when capital is unchanged a Pareto improvement is achieved by relaxing credit constraints for borrowers while tilting the terms of trade in the housing market in favor of lenders. When aggregate capital is lower (higher), lenders benefit (lose) whereas borrowers lose (benefit) via the credit and labor market distributive externalities. Thus, for some levels of collateralizability achieving a Pareto improvement while enhancing insurance is possible as borrowers can be more than compensated by relaxing credit constraints. However, this is not the case for all $\xi \in]0.5, 0.7]$ as the planner may need to increase aggregate capital. This is because, for some leverage degrees, borrowers are only mildly affected by the housing price increase and thus cannot sustain losses via the credit and labor market distributive externalities. Lenders, on the other hand, are sufficiently compensated via the housing price to accept less insurance.

To sum up, when there are both market incompleteness and binding collateral constraints the planner may face a trade-off between improving insurance and enlarging credit opportunities. This trade-off is at play in the empirically relevant range for housing collateralizability. In particular, our analysis shows that the socially more important margin is to improve households' insurance, instead of relaxing borrowing constraints. In terms of distributing the utility surplus, the finding that the borrowers' share is the highest when capital is the least decreased or most increased holds throughout the collateral parameter space.

5 Conclusion

This paper analyzes the efficiency properties of a general equilibrium model that features housing markets and financial frictions, dissecting the impact of the inherent illiquidity of housing and its collateralizable nature. We characterize constrained efficiency by a set of sufficient statistics and find that competitive equilibria are in general inefficient due to distributive and collateral externalities: Pareto improvements over the competitive equilibrium can be achieved by taxing borrowings and savings to different degrees.

We disentangle the impact of illiquidity and of collateralizability by analyzing different environments. A high degree of illiquidity limits the way in which insurance can be implemented, and as a consequence, determines that the socially efficient level of aggregate capital is lower than that of the laissez-faire outcome. In the presence of a collateral constraint, a trade-off arises when alleviating the social costs of the two frictions at play. We find that the socially more important margin is to improve households' insurance, instead of enlarging their credit opportunities, and that this can be achieved through a lower housing price and less capital.

Connecting back to two relevant strands of the literature we touched upon in the introduction, we shed some light on the following questions. First, what do we learn about the social desirability of the wealth distribution? The macroeconomic literature emphasizes from a positive perspective the importance of liquid wealth heterogeneity, both as a key microeconomic feature and as a crucial determinant of aggregate consumption responses (Carroll et al., 2017; Kaplan and Violante, 2014; Kaplan et al., 2014). We find that the constrained efficient equilibrium is characterized by wealth-poor households holding more liquid wealth and wealth-rich households having more illiquid wealth. This suggests that the lower end of the distribution should hold less housing wealth and less debt, while the upper end should hold more housing wealth and less liquid assets. Second, would it be socially desirable to constrain lending and borrowing even in the absence of crises? We find this to be the case and our policy recommendation thus aligns with macroprudential regulations aiming to decrease the severity of future recessions (Bianchi and Mendoza, 2010; Jeanne and Korinek, 2019; Lorenzoni, 2008), although the rationale is different in our setup. An interesting takeaway that could be drawn from this is that implementing policies aimed at improving households' insurance against idiosyncratic risk could alleviate the need for or necessary extent of macroprudential policies.

Appendix

A Proofs

A.1 Illiquid Housing & Non-Collateralized Borrowings

Lemma 1

Proof. Totally differentiating, perturbing only a_b or a_l in the initial resources, the housing market clearing condition (9) gives

$$\frac{\partial \sum_i h_i\left(p, \mu_i\right)}{\partial a_j} da_j + \frac{\partial \sum_i h_i\left(p, \mu_i\right)}{\partial p} dp = 0 \,,$$

with $j \in \{b, l\}$. Rewriting,

$$-\frac{\partial h_j\left(p, \mu_j\right)}{\partial \mu_j} da_j + \left[\sum_i \mathcal{S}_i(p, \mu_i) - \sum_i \frac{\partial h_i\left(p, \mu_i\right)}{\partial \mu_i}\left(h_i\left(p, \mu_i\right) - \bar{h}_i\right)\right] dp = 0 \,,$$

and thus,

$$dp = \frac{\dfrac{\partial h_j\left(p, \mu_j\right)}{\partial \mu_j}}{\sum_i \mathcal{S}_i(p, \mu_i) - \sum_i \dfrac{\partial h_i\left(p, \mu_i\right)}{\partial \mu_i}\left(h_i\left(p, \mu_i\right) - \bar{h}_i\right)} da_j \,.$$

Let $\Phi_{a,j}$ define the coefficient in front of da_j, by the normality of housing the numerator of $\Phi_{a,j}$ is positive, and under linear or concave-in-resources housing Engel curves the denominator of $\Phi_{a,j}$ is negative. □

Lemma 3

Proof. From Lemma 1, the properties of the utility function, and lenders' net housing position it trivially follows that $\Theta^p_{a,i} < 0$ for all i. Conversely from borrowers, a few steps are needed to show the sign of $\Theta^{r,w}_K$. In particular, since $a_l > 0$, it is unclear what is the sign of the term

$$\pi u'\left(c_{21l}\right)\left(a_l F_{KK} + e_1 F_{LK}\right) + (1-\pi) u'\left(c_{22l}\right)\left(a_l F_{KK} + e_2 F_{LK}\right).$$

To determine its sign, we first note that as $F(K,L)$ is homogeneous of degree one, then $KF_{KK} + LF_{LK} = 0$. By using that $a_b + a_l = K$ and adding and subtracting LF_{LK} we can rewrite the above term as

$$
\begin{aligned}
&\pi u'(c_{21l})\left(KF_{KK} + LF_{LK} - a_b F_{KK} + (e_1 - L)F_{LK}\right) + \\
&(1-\pi)\, u'(c_{22l})\left(KF_{KK} + LF_{LK} - a_b F_{KK} + (e_2 - L)F_{LK}\right) \\
=&\pi u'(c_{21l})\left(-a_b F_{KK} + (e_1 - L)F_{LK}\right) + (1-\pi) u'(c_{22l})\left(-a_b F_{KK} + (e_2 - L)F_{LK}\right) \\
=&u'(c_{22l})\left(\pi \frac{u'(c_{21l})}{u'(c_{22l})}(-a_b F_{KK} + (e_1 - L)F_{LK}) + (1-\pi)(-a_b F_{KK} + (e_2 - L)F_{LK})\right) \\
=&u'(c_{22l})\left(\pi \frac{u'(c_{21l})}{u'(c_{l;2,2})}(e_1 - L)F_{LK} + (1-\pi)(e_2 - L)F_{LK} - \left(\pi \frac{u'(c_{21l})}{u'(c_{22l})} + (1-\pi)\right) a_b F_{KK}\right).
\end{aligned}
$$

By using that $L = 2(\pi e_1 + (1-\pi)e_2)$, then $(1-\pi)(e_2 - L) = -\frac{1}{2}L - \pi(e_1 - L)$ so that the above term becomes

$$
u'(c_{22l})\left(\pi\left[\frac{u'(c_{21l})}{u'(c_{22l})} - 1\right](e_1 - L)F_{LK} - \frac{1}{2}LF_{LK} - a_b F_{KK}\left(\pi\left[\frac{u'(c_{21l})}{u'(c_{22l})} - 1\right] + 1\right)\right).
$$

Let $\Xi \equiv \frac{u'(c_{21l})}{u'(c_{22l})} - 1$, then $\Xi > 0$ and we can rewrite the term as

$$
u'(c_{22l})\left(\pi \Xi F_{LK}(e_1 - L) - \frac{1}{2}LF_{LK} - (\pi\Xi + 1)\, a_b F_{KK}\right),
$$

which is clearly negative. □

Proposition 1

Proof. Substituting all the constraints in the planner's objective function, and since the natural borrowing limits never bind, the social planner problem can be rewritten as

$$
\begin{aligned}
\max_{\{a_b, a_l\}} \gamma_b \Bigg\{ & u\Big(\bar{\omega}_b + p\left(\bar{h}_b - h_b(p,\mu_b)\right) - a_b\Big) + v\big(h_b(p,\mu_b)\big)(1+\beta) \\
& + \beta\Big[\pi\, u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_b + F_L\big(K(a_b,a_l),L\big)e_1\Big) \\
& + (1-\pi)u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_b + F_L\big(K(a_b,a_l),L\big)e_2\Big)\Big]\Bigg\} + \\
\gamma_l \Bigg\{ & u\Big(\bar{\omega}_l + p\left(\bar{h}_l - h_l(p,\mu_l)\right) - a_l\Big) + v\big(h_l(p,\mu_l)\big)(1+\beta) \\
& + \beta\Big[\pi\, u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_l + F_L\big(K(a_b,a_l),L\big)e_1\Big) \\
& + (1-\pi)u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_l + F_L\big(K(a_b,a_l),L\big)e_2\Big)\Big]\Bigg\},
\end{aligned}
$$

where $\bar{H} = h_b(p, \mu_b) + h_l(p, \mu_l)$, $K(a_b, a_l) = a_b + a_l$ and $L = 2(\pi e_1 + (1-\pi)e_2)$.
Thus the social planner first-order conditions are

$$
\begin{aligned}
\frac{\partial SPF}{\partial a_b} &= \gamma_b \Bigg\{ u'(c_{1b}) \left[\frac{\partial p}{\partial a_b}(\bar{h}_b - h_b) - \frac{\partial h_b}{\partial a_b} p - 1 \right] + v'(h_b) \frac{\partial h_b}{\partial a_b}(1+\beta) \\
&\quad + \beta \Big[\pi u'(c_{21b}) \big((1 + F_K - \delta) + F_{KK} a_b + F_{LK} e_1 \big) \\
&\quad + (1-\pi) u'(c_{22b}) \big((1 + F_K - \delta) + F_{KK} a_b + F_{LK} e_2 \big) \Big] \Bigg\} + \\
&\quad \gamma_l \Bigg\{ u'(c_{1l}) \left[\frac{\partial p}{\partial a_b}(\bar{h}_l - h_l) \right] + \beta \Big[\pi u'(c_{21l}) \big(F_{KK} a_l + F_{LK} e_1 \big) + (1-\pi) u'(c_{22l}) \big(F_{KK} a_l + F_{LK} e_2 \big) \Big] \Bigg\} \\
&= \gamma_b \Bigg\{ \underbrace{-u'(c_{1b}) + \beta(1 + F_K - \delta) \sum_{s=1}^{2} \pi_s u'(c_{2sb})}_{\text{Agent-}b\text{ FOC}} + \frac{\partial h_b}{\partial a_b} \Big[\underbrace{-u'(c_{1b}) p + v'(h_b)(1+\beta)}_{\text{Agent-}b\text{ FOC}} \Big] \Bigg\} + \\
&\quad \underbrace{\sum_{i \in \{b,l\}} \gamma_i u'(c_{1i})(\bar{h}_i - h_i) \frac{\partial p}{\partial a_b} + \sum_{i \in \{b,l\}} \gamma_i \sum_{s=1}^{2} \beta \left[\pi_s u'(c_{2si}) \big(F_{KK} a_i + F_{LK} e_s \big) \right]}_{\gamma_b \Psi^p_{a,b} + \gamma_l \Theta^p_{a,b} + \gamma_b \Psi^{r,w}_K + \gamma_l \Theta^{r,w}_K} = 0 \, , \\
\frac{\partial SPF}{\partial a_l} &= \gamma_l \Bigg\{ \underbrace{-u'(c_{1l}) + \beta(1 + F_K - \delta) \sum_{s=1}^{2} \pi_s u'(c_{2sl})}_{\text{Agent-}l\text{ FOC}} + \frac{\partial h_l}{\partial a_l} \Big[\underbrace{-u'(c_{1l}) p + v'(h_l)(1+\beta)}_{\text{Agent-}l\text{ FOC}} \Big] \Bigg\} + \\
&\quad \underbrace{\sum_{i \in \{b,l\}} \gamma_i u'(c_{1i})(\bar{h}_i - h_i) \frac{\partial p}{\partial a_l} + \sum_{i \in \{b,l\}} \gamma_i \sum_{s=1}^{2} \beta \left[\pi_s u'(c_{2si}) \big(F_{KK} a_i + F_{LK} e_s \big) \right]}_{\gamma_b \Psi^p_{a,l} + \gamma_l \Theta^p_{a,l} + \gamma_b \Psi^{r,w}_K + \gamma_l \Theta^{r,w}_K} = 0 \, .
\end{aligned}
$$

Evaluating the above system of equations at the laissez-faire allocation, and rewriting the system in matrix form we obtain

$$
\begin{bmatrix} \Psi^p_{a,b} + \Psi^{r,w}_K & \Theta^p_{a,b} + \Theta^{r,w}_K \\ \Psi^p_{a,l} + \Psi^{r,w}_K & \Theta^p_{a,l} + \Theta^{r,w}_K \end{bmatrix} \begin{bmatrix} \gamma_b \\ \gamma_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .
$$

To verify whether the matrix A is non-singular we check its determinant:

$$
|A| = \left(\Psi^p_{a,b} + \Psi^{r,w}_K \right) \left(\Theta^p_{a,l} + \Theta^{r,w}_K \right) - \left(\Theta^p_{a,b} + \Theta^{r,w}_K \right) \left(\Psi^p_{a,l} + \Psi^{r,w}_K \right) .
$$

Manipulating the above, the determinant can be simplified to

$$
|A| = (\bar{h}_b - h_b) \left[\Phi_{a,b} - \Phi_{a,l} \right] \left\{ u'(c_{1b}) \Theta^{r,w}_K + u'(c_{1l}) \Psi^{r,w}_K \right\} .
$$

It is straightforward to see that as long as $(C1)$ holds then $(A1)$ is non-zero. For $(C2)$, recollect the definition for $\Phi_{a,i}$ in Lemma 1 to see that the numerators when moving borrowings and savings need to differ for $(A2)$ to be non-zero: since we assume identical preferences across types, the housing Engel curves' slope evaluated at different levels of resources needs to vary. To realize the third condition we refer the reader to the next section on implementing a Pareto improvement, proof of Proposition 2, where we show the sign of the third sufficient statistics and that $(C3)$ is needed for $(A3)$ to be non-zero.

□

Proposition 2

Proof. To show that $da_b > 0$ and $da_l < 0$ implement a Pareto improvement, first note that in (17) each Θ term is negative. Since $|\Theta^p_{a,l}| = |u'(c_{1l})(\bar{h}_l - h_l)\Phi_{a,l}| < |u'(c_{1l})(\bar{h}_l - h_l)\Phi_{a,b}| = |\Theta^p_{a,b}|$ under concave-in-resources housing Engel curves (see Lemma 1), it follows that

$$\frac{\Theta^p_{a,b} + \Theta^p_{a,l} + 2\Theta^{r,w}_K}{\Theta^p_{a,l} + \Theta^{r,w}_K} > 2$$

so that we can infer that ε is more than twice as large as da_b in absolute value. Moreover, since both the numerator as well as the denominator are negative, it follows that ε and da_b have opposite signs. These two results indicate that not only da_l and da_b have to have opposite signs to generate a Pareto improvement, but also that da_l has to be larger than da_b in absolute value.

Inserting (17) into (15), we get

$$\left[\frac{\left(\Psi^p_{a,b} + \Psi^p_{a,l} + 2\Psi^{r,w}_K\right)\left(\Theta^p_{a,l} + \Theta^{r,w}_K\right) - \left(\Psi^p_{a,l} + \Psi^{r,w}_K\right)\left(\Theta^p_{a,b} + \Theta^p_{a,l} + 2\Theta^{r,w}_K\right)}{\Theta^p_{a,l} + \Theta^{r,w}_K}\right] da_b > 0, \tag{43}$$

where it's clear that the denominator of the term in front of da_b is negative. Let us analyze the numerator. Expanding and using that $\Psi^p_{a,b}\Theta^p_{a,l} = \Theta^p_{a,b}\Psi^p_{a,l}$, yields

$$\Psi^{r,w}_K\Theta^p_{a,l} + \Psi^p_{a,b}\Theta^{r,w}_K - \Psi^p_{a,l}\Theta^{r,w}_K - \Theta^p_{a,b}\Psi^{r,w}_K.$$

Regrouping these terms yields

$$\Theta^{r,w}_K\left(\Psi^p_{a,b} - \Psi^p_{a,l}\right) + \Psi^{r,w}_K\left(\Theta^p_{a,l} - \Theta^p_{a,b}\right).$$

Note that $\Theta_K^{r,w} < 0$, $\Psi_K^{r,w} > 0$, and that both terms in parentheses are positive, i.e.

$$\Psi^p_{a,b} - \Psi^p_{a,l} = u'(c_{1b})\left(\bar{h}_b - h_b\right)\left(\Phi_{a,b} - \Phi_{a,l}\right) > 0,$$
$$\Theta^p_{a,l} - \Theta^p_{a,b} = u'(c_{1l})\left(\bar{h}_l - h_l\right)\left(\Phi_{a,l} - \Phi_{a,b}\right) > 0.$$

Using the two expressions together with $\bar{h}_b - h_b = -\left(\bar{h}_l - h_l\right)$ we can rewrite the numerator as

$$\left(\bar{h}_l - h_l\right)\left(\Phi_{a,l} - \Phi_{a,b}\right)\left[u'(c_{1b})\Theta_K^{r,w} + u'(c_{1l})\Psi_K^{r,w}\right].$$

Note that $\left(\bar{h}_l - h_l\right) > 0$ and $\left(\Phi_{a,l} - \Phi_{a,b}\right) > 0$ under concave-in-resources housing Engel curves (see Lemma 1), so that we can focus on the sign of the last term to figure out the sign of the numerator.

To determine the sign of $\left[u'(c_{1l})\Psi_K^{r,w} + u'(c_{1b})\Theta_K^{r,w}\right]$, let us first rewrite more explicitly the consumption-marginal-utility cross-weighted sum of credit and labor markets distributive externalities by using the definitions of $\Psi_K^{r,w}$ and $\Theta_K^{r,w}$,

$$\begin{aligned} u'(c_{1l})\beta\Big[&\pi u'(c_{21b})\big(F_{KK}a_b + F_{LK}e_1\big) + (1-\pi)u'(c_{22b})\big(F_{KK}a_b + F_{LK}e_2\big)\Big] \\ + u'(c_{1b})\beta\Big[&\pi u'(c_{21l})\big(F_{KK}a_l + F_{LK}e_1\big) + (1-\pi)u'(c_{22l})\big(F_{KK}a_l + F_{LK}e_2\big)\Big]. \end{aligned}$$

By using that $K = a_b + a_l$ and $KF_{KK} + LF_{LK} = 0$, we can rewrite the above as

$$\begin{aligned} &\beta F_{KK}a_b\Big[u'(c_{1l})\Big(\pi u'(c_{21b}) + (1-\pi)u'(c_{22b})\Big) - u'(c_{1b})\Big(\pi u'(c_{21l}) + (1-\pi)u'(c_{22l})\Big)\Big] \\ &+\beta F_{LK}\Big[u'(c_{1l})\Big(\pi u'(c_{21b})e_1 + (1-\pi)u'(c_{22b})e_2\Big) \\ &\qquad + u'(c_{1b})\Big(\pi u'(c_{21l})(e_1 - L) + (1-\pi)u'(c_{22l})(e_2 - L)\Big)\Big]. \end{aligned}$$

Since at the competitive equilibrium $u'(c_{1i}) = \beta(1+r)\sum_s \pi_s u'(c_{2si})$ for all i, then it holds $u'(c_{1l})\sum_s \pi_s u'(c_{2sb}) - u'(c_{1b})\sum_s \pi_s u'(c_{2sl}) = 0$ and we can simplify $u'(c_{1l})\Psi_K^{r,w} + u'(c_{1b})\Theta_K^{r,w}$ to

$$\begin{aligned} \beta F_{LK}\Big[&u'(c_{1l})\Big(\pi u'(c_{21b})e_1 + (1-\pi)u'(c_{22b})e_2\Big) \\ &+ u'(c_{1b})\Big(\pi u'(c_{21l})(e_1 - L) + (1-\pi)u'(c_{22l})(e_2 - L)\Big)\Big]. \end{aligned}$$

Now, by letting $e_2 = e_1 + h$ with $h > 0$, and using $L = 2(\pi e_1 + (1-\pi)e_2) = 2(e_1 + (1-\pi)h)$, we arrive to $(e_1 - L) = -(e_1 + 2(1-\pi)h)$ and $(e_2 - L) = -(e_1 + 2(1-\pi)h - h)$. Then we can rewrite the above expression as

$$\beta F_{LK}\Big[e_1\Big\{u'(c_{1l})\Big(\pi u'(c_{21b}) + (1-\pi)u'(c_{22b})\Big) - u'(c_{1b})\Big(\pi u'(c_{21l}) + (1-\pi)u'(c_{22l})\Big)\Big\} +$$

$$u'(c_{1l})(1-\pi)u'(c_{22b})h - u'(c_{1b})\Big(\pi u'(c_{21l})2(1-\pi)h + (1-\pi)u'(c_{22l})(1-2\pi)h\Big)\Big].$$

Using again that $u'(c_{1l})\sum_s \pi_s u'(c_{2sb}) - u'(c_{1b})\sum_s \pi_s u'(c_{2sl}) = 0$, and collecting h we obtain

$$\beta F_{LK} h\Big[u'(c_{1l})(1-\pi)u'(c_{22b}) - u'(c_{1b})\Big(\pi u'(c_{21l})2(1-\pi) + (1-\pi)u'(c_{22l})(1-2\pi)\Big)\Big].$$

Then, using again that at the competitive equilibrium $u'(c_{1i}) = \beta(1+r)\sum_s \pi_s u'(c_{2si})$, we arrive to

$$\beta F_{LK} h\beta(1+r)\Big[-\pi u'(c_{21b})\Big(\pi u'(c_{21l})2(1-\pi) + (1-\pi)u'(c_{22l})(1-2\pi)\Big) \\ -(1-\pi)u'(c_{22b})\Big(\pi u'(c_{21l})(1-2\pi) + (1-\pi)u'(c_{22l})(-2\pi)\Big)\Big].$$

Continuing, after repeating the above expression, we expand the terms within squared brackets and then collect π and $(1-\pi)$,

$$\underbrace{\beta^2 F_{LK}(1+r)h\pi(1-\pi)}_{>0\,,\ \forall\pi\in]0,1[}\Big[\underbrace{2\pi\Big(\underbrace{u'(c_{21l}) - u'(c_{22l})}_{>0}\Big)\Big(\underbrace{u'(c_{22b}) - u'(c_{21b})}_{<0}\Big)}_{\leq 0\,,\ \forall\pi\in[0,1]} \\ \underbrace{-u'(c_{22b})\Big(\underbrace{u'(c_{21l}) - u'(c_{22l})}_{>0}\Big)}_{<0}\underbrace{-u'(c_{22l})\Big(\underbrace{u'(c_{21b}) - u'(c_{22b})}_{>0}\Big)}_{<0}\Big] < 0\ \forall\,\pi\in]0,1[.$$

Note that distributive externalities in the credit and labor markets do not matter, i.e. are null, whenever $\pi = 0$, or $\pi = 1$ or $h = 0$. That is, when market incompleteness is not relevant. Therefore, as we can show that $\left[u'(c_{1b})\Theta_K^{r,w} + u'(c_{1l})\Psi_K^{r,w}\right] < 0$, we can conclude that the numerator in (43) is negative. Since the denominator is also negative, only a $da_b > 0$ will satisfy inequality (43). Note that then it must be that $da_l < 0$.
Turning to the induced movements in the equilibrium prices, it follows from (12) and (13) that $dr > 0$ and $dw < 0$. To determine the sign of dp, note that the change in the housing price is given by

$$dp = \Phi_{a,b}da_b + \Phi_{a,l}da_l = \frac{\dfrac{\partial h_b\,(p,\mu_b)}{\partial\mu_b}da_b + \dfrac{\partial h_l\,(p,\mu_l)}{\partial\mu_l}da_l}{\sum_j \mathcal{S}_j(p,\mu_j) - \sum_j \dfrac{\partial h_j\,(p,\mu_j)}{\partial\mu_j}(h_j\,(p,\mu_j) - \bar{h}_j)},$$

where we used the definition of $\Phi_{a,i}$ in Lemma 1, and $\dfrac{\partial h_i\left(p,\mu_i\right)}{\partial \mu_i}$ is the slope of agent i's housing Engel curve. Using that $da_l = da_b + \varepsilon$, with ε defined as in (17), we can rewrite the change in the housing price as

$$dp = \frac{\dfrac{\partial h_b\left(p,\mu_b\right)}{\partial \mu_b}(\Theta^p_{a,l} + \Theta^{r,w}_K) - \dfrac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}(\Theta^p_{a,b} + \Theta^{r,w}_K)}{\left(\Theta^p_{a,l} + \Theta^{r,w}_K\right)\left(\sum_j \mathcal{S}_j(p,\mu_j) - \sum_j \dfrac{\partial h_j\left(p,\mu_j\right)}{\partial \mu_j}(h_j\left(p,\mu_j\right) - \bar{h}_j)\right)} da_b.$$

Using that $\dfrac{\partial h_b\left(p,\mu_b\right)}{\partial \mu_b}\Theta^p_{a,l} = \dfrac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}\Theta^p_{a,b}$ we can rewrite dp as

$$dp = \underbrace{\frac{\left(\dfrac{\partial h_b\left(p,\mu_b\right)}{\partial \mu_b} - \dfrac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}\right)\Theta^{r,w}_K}{\left(\Theta^p_{a,l} + \Theta^{r,w}_K\right)\left(\sum\limits_j \mathcal{S}_j(p,\mu_j) - \sum\limits_j \dfrac{\partial h_j\left(p,\mu_j\right)}{\partial \mu_j}(h_j\left(p,\mu_j\right) - \bar{h}_j)\right)}}_{>0} da_b\ ,$$

where we have signed the coefficient in front of da_b under preferences that entail concave-in-resources housing Engel curves.[18] As we know that $da_b > 0$ to achieve a Pareto improvement then $dp < 0$, i.e. the housing price will be lower than at the laissez-faire equilibrium. □

Proposition 3

Proof. Introducing taxation, to generate the needed incentives for agents to satisfy the social planner first-order conditions, modifies households' optimization problem as follows. Household $i = b$ chooses $(c_{1b}, c_{21b}, c_{22b}, h_b, a_b)$ to maximize

$$U_b(c_{1b}, c_{21b}, c_{22b}, h_b) = \mathbf{u}(c_{1b}, h_b) + \beta\Big[\pi\,\mathbf{u}(c_{21b}, h_b) + (1-\pi)\,\mathbf{u}(c_{22b}, h_b)\Big]$$

subject to

$$c_{1b} + a_b + p h_b = \bar{\omega}_b + p\bar{h}_b\ ,$$

$$a_b \geq -\frac{we_1 + T^a_b}{1 + r(1-\tau^a_b)}\ ,$$

$$c_{2sb} = (1 + r(1-\tau^a_b))a_b + we_s + T^a_b\ ,$$

[18] Note that if preferences would entail convex-in-resources housing Engel curves, while still satisfying the law of demand in the housing market, then the coefficient would be positive instead of negative.

for $s = 1,2$ with $e_s \in \{e_1, e_2\}$ and $prob(e = e_1) = \pi$, and where T^a_b is a lump-sum transfer (taken as given by households) that must equal $T^a_b = r\tau^a_b a_b$. Household $i = l$ chooses $(c_{1l}, c_{21l}, c_{22l}, h_l, a_l)$ to maximize

$$\begin{aligned}
&U_l(c_{1l}, c_{21l}, c_{22l}, h_l) = \mathbf{u}(c_{1l}, h_l) + \beta\Big[\pi\,\mathbf{u}(c_{21l}, h_l) + (1-\pi)\,\mathbf{u}(c_{22l}, h_l)\Big] \\
&\text{subject to} \\
&c_{1l} + a_l + ph_l = \bar{\omega}_l + p\bar{h}_l\ , \\
&a_l \geq -\frac{we_1 + T^a_l}{1 + r(1-\tau^a_l)}\ , \\
&c_{2sl} = (1 + r(1-\tau^a_l))a_l + we_s + T^a_l\ ,
\end{aligned}$$

for $s = \{1,2\}$, $prob(e = e_1) = \pi$, and where T^a_l is a lump-sum transfers (taken as given by households) that must equal $T^a_l = r\tau^a_l a_l$. Then, the first-order conditions of households $i = b, l$ are

$$\begin{aligned}
&-u'(c_{1b})\,p + v'(h_b)(1+\beta) = 0, \\
&-u'(c_{1b}) + \beta(1 + r(1-\tau^a_b))\left[\pi u'(c_{21b}) + (1-\pi)\,u'(c_{22b})\right] = 0, \\
&-u'(c_{1l})\,p + v'(h_l)(1+\beta) = 0, \\
&-u'(c_{1l}) + \beta(1 + r(1-\tau^a_l))\left[\pi u'(c_{21l}) + (1-\pi)\,u'(c_{22l})\right] = 0.
\end{aligned}$$

Then, comparing the above equations with the planner first-order conditions, to ensure that agents' incentives are aligned with the planner's will it must hold that

$$-\beta r\tau^a_b \sum_{s=1}^{2} \pi_s u'(c_{2sb}) = \frac{\partial p}{\partial a_b}\sum_{i\in\{b,l\}}\frac{\gamma_i}{\gamma_b}u'(c_{1i})(\bar{h}_i - h_i) + \beta\sum_{i\in\{b,l\}}\sum_{s=1}^{2}\frac{\gamma_i}{\gamma_b}\pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big)\ ,$$
$$-\beta r\tau^a_l \sum_{s=1}^{2} \pi_s u'(c_{2sl}) = \frac{\partial p}{\partial a_l}\sum_{i\in\{b,l\}}\frac{\gamma_i}{\gamma_l}u'(c_{1i})(\bar{h}_i - h_i) + \beta\sum_{i\in\{b,l\}}\sum_{s=1}^{2}\frac{\gamma_i}{\gamma_l}\pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big)\ ,$$

thus we arrive to the following set of taxes

$$\tau^a_b = \frac{-1}{\beta r\sum_{s=1}^{2}\pi_s u'(c_{2sb})}\left\{\frac{\partial p}{\partial a_b}\sum_{i\in\{b,l\}}\frac{\gamma_i}{\gamma_b}u'(c_{1i})(\bar{h}_i - h_i) + \beta\sum_{i\in\{b,l\}}\sum_{s=1}^{2}\frac{\gamma_i}{\gamma_b}\pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big)\right\},$$
$$\tau^a_l = \frac{-1}{\beta r\sum_{s=1}^{2}\pi_s u'(c_{2sl})}\left\{\frac{\partial p}{\partial a_l}\sum_{i\in\{b,l\}}\frac{\gamma_i}{\gamma_l}u'(c_{1i})(\bar{h}_i - h_i) + \beta\sum_{i\in\{b,l\}}\sum_{s=1}^{2}\frac{\gamma_i}{\gamma_l}\pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big)\right\}.$$

□

A.2 Liquid Housing & Non-Collateralized Borrowings

Lemma 4

Proof. Totally differentiating the period-1 housing market clearing condition

$$h_{1b}\left(p_1, \mu_b\right) + h_{1l}\left(p_1, \mu_l\right) = \bar{H},$$

while perturbing only a_b or a_l in the initial resources, gives

$$\sum_j \frac{\partial h_{1j}\left(p_1, \mu_j\right)}{\partial \mu_j} \frac{\partial \mu_j}{\partial a_i} da_i + \sum_j \frac{\partial h_{1j}\left(p_1, \mu_j\right)}{\partial p_1} dp_1 = 0,$$

with $j \in \{b, l\}$. Since μ_j only moves when changing a_j, we arrive to

$$dp_1 = -\frac{\dfrac{\partial h_{1i}\left(p_1, \mu_i\right)}{\partial \mu_i} \dfrac{\partial \mu_i}{\partial a_i}}{\sum_j \mathcal{S}_{1j}(p_1, \mu_j) - \sum_j \dfrac{\partial h_{1j}\left(p_1, \mu_j\right)}{\partial \mu_j}\left(h_{1j}\left(p_1, \mu_j\right) - \bar{h}_j\right)} da_i.$$

Let $\widehat{\Phi}^1_{a,i}$ denote the coefficient in front of da_i. As we assume linear or concave-in-housing Engel curves, the denominator is negative. To determine the sign of $\widehat{\Phi}^1_{a,i}$, we have to derive an expression for its numerator.

We start with the households' first order conditions given by

$$f_1 := -p_1 u'\left(\bar{\omega} + p_1 \bar{h} - a - p_1 h_1\right) + v'(h_1) + p_2 \beta \sum_s \pi_s u'\left((1+r)a + we_s + p_2 h_1 - p_2 h_{2s}\right) = 0,$$

$$f_2 := v'(h_{21}) - u'((1+r)a + we_1 + p_2 h_1 - p_2 h_{21}) p_2 = 0,$$

$$f_3 := v'(h_{22}) - u'((1+r)a + we_2 + p_2 h_1 - p_2 h_{22}) p_2 = 0.$$

Totally differentiating the above conditions and rewriting in matrix form leads to:

$$\mathcal{M} \begin{bmatrix} dh_1 \\ dh_{21} \\ dh_{22} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial f_1}{\partial r} dr - \dfrac{\partial f_1}{\partial a} da - \dfrac{\partial f_1}{\partial w} dw - \dfrac{\partial f_1}{\partial p_1} dp_1 - \dfrac{\partial f_1}{\partial p_2} dp_2 \\ -\dfrac{\partial f_2}{\partial r} dr - \dfrac{\partial f_2}{\partial a} da - \dfrac{\partial f_2}{\partial w} dw - \dfrac{\partial f_2}{\partial p_2} dp_2 \\ -\dfrac{\partial f_3}{\partial r} dr - \dfrac{\partial f_3}{\partial a} da - \dfrac{\partial f_3}{\partial w} dw - \dfrac{\partial f_3}{\partial p_2} dp_2 \end{bmatrix},$$

with

$$\mathcal{M} = \begin{bmatrix} \partial f_1 / \partial h_1 & \partial f_1 / \partial h_{21} & \partial f_1 / \partial h_{22} \\ \partial f_2 / \partial h_1 & \partial f_2 / \partial h_{21} & 0 \\ \partial f_3 / \partial h_1 & 0 & \partial f_3 / \partial h_{22} \end{bmatrix}.$$

The inverse of $\mathcal{M}$ is then given by

$$\mathcal{M}^{-1} = \frac{1}{|\mathcal{M}|}\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix},$$

with

$$\begin{aligned}
m_{11} &= \Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big) > 0, \\
m_{21} &= u''(c_{21})p_2^2\Big(u''(c_{22}p_2^2 + v''(h_{22})\Big), \\
m_{31} &= -\Big(u''(c_{21})p_2^2 + v''(h_{22})\Big)\Big(- u''(c_{22})p_2^2\Big), \\
m_{12} &= -\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big)\Big(- \beta\pi u''(c_{21})p_2^2\Big) > 0, \\
m_{22} &= \frac{\partial f_1}{\partial h_1}\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big) - \Big(- \beta(1-\pi)u''(c_{22})p_2^2\Big(- u''(c_{22})p_2^2\Big)\Big), \\
m_{32} &= -u''(c_{22})p_2^2\Big(- \beta\pi u''(c_{21})p_2^2\Big), \\
m_{13} &= -\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)\Big(- \beta(1-\pi)u''(c_{22})p_2^2\Big) > 0, \\
m_{23} &= -u''(c_{21})p_2^2\Big(- \beta(1-\pi)u''(c_{22})p_2^2\Big), \\
m_{33} &= \frac{\partial f_1}{\partial h_1}\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big) - \Big(- \beta\pi u''(c_{21})p_2^2\Big(- u''(c_{21})p_2^2\Big)\Big), \\
\frac{\partial f_1}{\partial h_1} &= u''(c_1)p_1^2 + v''(h_1) + \beta\Big[\pi u''(c_{21})p_2^2 + (1-\pi)u''(c_{22})p_2^2\Big] < 0.
\end{aligned}$$

The determinant of $\mathcal{M}$ is given by

$$\begin{aligned}
|\mathcal{M}| = u''(c_{22})p_2^2\Big[&u''(c_{21})p_2^2\Big(u''(c_1)p_1^2 + v''(h_1)\Big) \\
&+ v''(h_{21})\Big(u''(c_1)p_1^2 + v''(h_1) + \beta\pi u''(c_{21})p_2^2\Big)\Big] \\
&+ v''(h_{22})\Big[u''(c_{21})p_2^2\Big(u''(c_1)p_1^2 + v''(h_1) + \beta(1-\pi)u''(c_{22})p_2^2\Big) + \frac{\partial f_1}{\partial h_1}v''(h_{21})\Big].
\end{aligned}$$

By the properties of $u(\cdot)$ and $v(\cdot)$, it follows that the terms in square brackets are unambiguously positive and thus $|\mathcal{M}| < 0$.
Now that we have $|\mathcal{M}|$, we can compute the change in the first period housing demand induced by a change in the asset position as

$$\begin{aligned}
\frac{dh_1}{da} &= \frac{1}{|\mathcal{M}|}\Big[- m_{11}\frac{\partial f_1}{\partial a} - m_{12}\frac{\partial f_2}{\partial a} - m_{13}\frac{\partial f_3}{\partial a}\Big] \\
&= \frac{1}{|\mathcal{M}|}\Big[- m_{11}\Big(u''(c_1)p_1 + \beta\pi u''(c_{21})(1+r)p_2 + \beta(1-\pi)u''(c_{22})(1+r)p_2\Big)
\end{aligned}$$

$$+ m_{12}u''(c_{21})(1+r)p_2 + m_{13}u''(c_{22})(1+r)p_2\Big]$$

$$= \frac{1}{|\mathcal{M}|}\Big[u''(c_{21})(1+r)p_2\Big(m_{12} - \beta\pi m_{11}\Big) + u''(c_{22})(1+r)p_2\Big(m_{13} - \beta(1-\pi)m_{11}\Big)$$

$$- u''(c_1)p_1 m_{11}\Big]$$

Using the fact that

$$\begin{aligned} m_{12} - \beta\pi m_{11} &= -\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big)\Big(-\beta\pi u''(c_{21})p_2^2\Big) \\ &\quad - \beta\pi\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big) \\ &= \Big(u''(c_{22})p_2^2 + v''(h_{22})\Big)\Big[\beta\pi u''(c_{21})p_2^2 - \beta\pi u''(c_{21})p_2^2 - \beta\pi v''(h_{21})\Big] \\ &= -\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big)\beta\pi v''(h_{21}) < 0, \end{aligned}$$

and

$$\begin{aligned} m_{13} - \beta(1-\pi)m_{11} &= -\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)\Big(-\beta(1-\pi)u''(c_{22})p_2^2\Big) \\ &\quad - \beta(1-\pi)\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)\Big(u''(c_{22})p_2^2 + v''(h_{22})\Big) \\ &= \beta(1-\pi)\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)\Big[u''(c_{22})p_2^2 - u''(c_{22})p_2^2 - v''(h_{22})\Big] \\ &= -\beta(1-\pi)\Big(u''(c_{21})p_2^2 + v''(h_{21})\Big)v''(h_{22}) < 0, \end{aligned}$$

we can conclude that $dh_1/da < 0$, as the three terms in the squared brackets are all positive and the determinant $|\mathcal{M}|$ is negative. This leads us to the following expression for $\widehat{\Phi}^1_{a,i}$,

$$\widehat{\Phi}^1_{a,i} := -\frac{\frac{1}{|\mathcal{M}(i)|}\Big[-m_{11}(i)\frac{\partial f_1}{\partial a}(i) - m_{12}(i)\frac{\partial f_2}{\partial a}(i) - m_{13}(i)\frac{\partial f_3}{\partial a}(i)\Big]}{\sum_j \mathcal{S}_{1j}(p_1,\mu_j) - \sum_j \frac{\partial h_{1j}(p_1,\mu_j)}{\partial \mu_j}(h_{1j}(p_1,\mu_j) - \bar{h}_j)} < 0.$$

$\square$

Lemma 5

Proof. We start from second-period housing market clearing condition,

$$\sum_j \sum_s \pi_s h_{2sj}(p_2, \mu_{2sj}) = \bar{H}, \tag{44}$$

where $\mu_{2sj} := (1+r)a_j + we_s + p_2 h_{1j}$. Totally differentiating, perturbing only a_b or a_l in the initial resources, the above condition gives

$$\left(\sum_j \sum_s \pi_s \frac{\partial h_{2sj}(p_2, \mu_{2sj})}{\partial \mu_{2sj}} \frac{\partial \mu_{2sj}}{\partial a_i}\right) da_i + \left(\sum_j \sum_s \pi_s \frac{\partial h_{2sj}(p_2, \mu_{2sj})}{\partial p_2}\right) dp_2 = 0, \tag{45}$$

so that

$$dp_2 = -\frac{\sum_j \sum_s \pi_s \frac{\partial h_{2sj}(p_2, \mu_{2sj})}{\partial \mu_{2sj}} \frac{\partial \mu_{2sj}}{\partial a_i}}{\sum_j \sum_s \pi_s \frac{\partial h_{2sj}(p_2, \mu_{2sj})}{\partial p_2}} da_i, \tag{46}$$

where $i \in \{b, l\}$. Let $\widehat{\Phi}^2_{a,i}$ denote the coefficient in front of da_i.

Case 1: Linear Engel curves. We start with the numerator in (46). Assuming linear Engel curves implies

$$\frac{\partial h_2}{\partial \mu_2} = \frac{\partial h_{2si}}{\partial \mu_{2si}}, \quad \forall i \in \{b, l\} \text{ and } s \in \{1, 2\}, \tag{47}$$

so that, after inserting the respective $\partial \mu_{2si}/\partial a_i$ and $\partial \mu_{2sj}/\partial a_i$, we end up with the following expression,

$$\begin{aligned}
&\frac{\partial h_2}{\partial \mu_2}\left[\sum_s \pi_s \left(1 + r + \frac{\partial r}{\partial a_i} a_i + \frac{\partial w}{\partial a_i} e_s + \frac{\partial h_{1i}}{\partial a_i} p_2\right) + \sum_s \pi_s \left(\frac{\partial r}{\partial a_i} a_j + \frac{\partial w}{\partial a_i} e_s + \frac{\partial h_{1j}}{\partial a_i} p_2\right)\right] \\
&= \frac{\partial h_2}{\partial \mu_2}\left[1 + r + \frac{\partial r}{\partial a_i}(a_i + a_j) + \frac{\partial w}{\partial a_i} \sum_s 2\pi_s e_s + \left(\frac{\partial h_{1i}}{\partial a_i} + \frac{\partial h_{1j}}{\partial a_i}\right) p_2\right] \\
&= \frac{\partial h_2}{\partial \mu_2}\left[1 + r + \frac{\partial r}{\partial a_i} K + \frac{\partial w}{\partial a_i} L + \left(\frac{\partial h_{1i}}{\partial a_i} + \frac{\partial h_{1j}}{\partial a_i}\right) p_2\right] = \frac{\partial h_2}{\partial \mu_2}(1 + r) > 0.
\end{aligned}$$

The denominator can be decomposed as follows,

$$\sum_i \sum_s \pi_s \frac{\partial h_{2si}}{\partial p_2} = \sum_i \sum_s \pi_s \mathcal{S}_{2si}(p_2, \mu_{2si}) - \sum_i \sum_s \pi_s \frac{\partial h_{2si}}{\partial \mu_{2si}} (h_{2si} - h_{1i}), \tag{48}$$

where the first term on the right hand side is the sum of substitution effects and the second term is the sum of all wealth effects for all types and states, triggered by a change in p_2. Note that, if Engel curves are linear, (47) holds, and thus the sum of wealth effects are zero,

$$\sum_i \sum_s \pi_s \frac{\partial h_{2si}}{\partial \mu_{2si}} (h_{2si} - h_{1i}) = \frac{\partial h_2}{\partial \mu_2} \sum_i \sum_s \pi_s (h_{2si} - h_{1i}) = \frac{\partial h_2}{\partial \mu_2} (\bar{H} - \bar{H}) = 0. \tag{49}$$

Since substitution effects are negative, we conclude that the denominator is negative and thus, under the assumption of linear Engel curves, $\widehat{\Phi}^2_{a,i}$ is positive.

Case 2: Concave Engel curves. For concave Engel curves, condition (47) does not hold, and we would have to compute the partial derivatives in (46) for all types and states. However, without further assumptions, we neither can unambiguously sign the numerator nor the denominator. In particular, when decomposing the denominator as in (48), the sum of wealth effects can take both signs as the endogenous second-period net housing positions are scaled by the annexed Engel curve slope. Therefore, the denominator in (46) does not have a sign a priori and $\widehat{\Phi}^2_{a,i} \lesseqgtr 0$.

$\square$

Proposition 4

Proof. Substituting all the constraints in the objective function, and since the natural borrowing limits never bind, the social planner problem can be written as

$$\begin{aligned}
\max_{\{a_b,a_l\}} \gamma_b \Bigg\{ & u\Big(\bar{\omega}_b + p_1\left(\bar{h}_b - h_{1b}(p_1,\mu_{1b})\right) - a_b\Big) + v\big(h_{1b}(p_1,\mu_{1b})\big) \\
& + \beta\Big[\pi u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_b + F_L\big(K(a_b,a_l),L\big)e_1 \\
& - p_2\Big(h_{21b}(p_2,\mu_{21b}) - h_{1b}(p_1,\mu_{1b})\Big)\Big) + \pi v\Big(h_{21b}(p_2,\mu_{21b})\Big) \\
& + (1-\pi)u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_b + F_L\big(K(a_b,a_l),L\big)e_2 \\
& - p_2\big(h_{22b}(p_2,\mu_{22b}) - h_{1b}(p_1,\mu_{1b})\big)\Big) + (1-\pi)v\Big(h_{22b}(p_2,\mu_{22b})\Big)\Big]\Bigg\} \\
& + \gamma_l\Bigg\{ u\Big(\bar{\omega}_l + p_1\left(\bar{h}_l - h_{1l}(p_1,\mu_{1l})\right) - a_l\Big) + v\big(h_{1l}(p_1,\mu_{1l})\big) \\
& + \beta\Big[\pi u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_l + F_L\big(K(a_b,a_l),L\big)e_1 \\
& - p_2\Big(h_{21l}(p_2,\mu_{21l}) - h_{1l}(p_1,\mu_{1l})\Big)\Big) + \pi v\Big(h_{21l}(p_2,\mu_{21l})\Big) \\
& + (1-\pi)u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_l + F_L\big(K(a_b,a_l),L\big)e_2 \\
& - p_2\big(h_{22l}(p_2,\mu_{22l}) - h_{1l}(p_1,\mu_{1l})\big)\Big) + (1-\pi)v\Big(h_{22l}(p_2,\mu_{22l})\Big)\Big]\Bigg\}.
\end{aligned}$$

The social planner first-order conditions are

$$\frac{\partial SPF}{\partial a_b} = \gamma_b \Bigg\{ \underbrace{-u'(c_{1b}) + \beta(1+F_K-\delta)\sum_{s=1}^{2}\pi_s u'(c_{2sb})}_{\text{Agent-}b\text{ FOC}}$$

$$+ \frac{\partial h_b}{\partial a_b} \underbrace{\left[-u'(c_{1b})p_1 + v'(h_{1b}) + \beta p_2\left(\pi u'(c_{21b}) + (1-\pi)u'(c_{22b})\right)\right]}_{\text{Agent-}b\text{ FOC}} \Bigg\}$$

$$+ \gamma_b\beta\sum_{s=1}^{2}\pi_s \underbrace{\left(v'(h_{2sb}) - u'(c_{2sb})p_2\right)}_{\text{Agent-}b\text{ FOC}} \frac{\partial h_{2sb}}{\partial a_b} + \underbrace{\sum_{i\in\{b,l\}}\gamma_i\sum_{s=1}^{2}\beta\pi_s u'(c_{2si})(h_{1i}-h_{2si})\frac{\partial p_2}{\partial a_b}}_{\gamma_b\widehat{\Psi}^{p_2}_{a,b}+\gamma_l\widehat{\Theta}^{p_2}_{a,b}}$$

$$+ \underbrace{\sum_{i\in\{b,l\}}\gamma_i u'(c_{1i})(\bar{h}_i - h_{1i})\frac{\partial p_1}{\partial a_b}}_{\gamma_b\widehat{\Psi}^{p_1}_{a,b}+\gamma_l\widehat{\Theta}^{p_1}_{a,b}} + \underbrace{\sum_{i\in\{b,l\}}\gamma_i\sum_{s=1}^{2}\beta\left[\pi_s u'(c_{2si})\left(F_{KK}a_i + F_{LK}e_s\right)\right]}_{\gamma_b\Psi^{r,w}_K+\gamma_l\Theta^{r,w}_K} = 0,$$

$$\frac{\partial SPF}{\partial a_l} = \gamma_l \Bigg\{ \underbrace{-u'(c_{1l}) + \beta(1+F_K-\delta)\sum_{s=1}^{2}\pi_s u'(c_{2sl})}_{\text{Agent-}l\text{ FOC}}$$

$$+ \frac{\partial h_l}{\partial a_l} \underbrace{\left[-u'(c_{1l})p_1 + v'(h_{1l}) + \beta p_2\left(\pi u'(c_{21l}) + (1-\pi)u'(c_{22l})\right)\right]}_{\text{Agent-}l\text{ FOC}} \Bigg\}$$

$$+ \gamma_l\beta\sum_{s=1}^{2}\pi_s \underbrace{\left(v'(h_{2sl}) - u'(c_{2sl})p_2\right)}_{\text{Agent-}l\text{ FOC}} \frac{\partial h_{2sl}}{\partial a_l} + \underbrace{\sum_{i\in\{b,l\}}\gamma_i\sum_{s=1}^{2}\beta\pi_s u'(c_{2si})(h_{1i}-h_{2si})\frac{\partial p_2}{\partial a_l}}_{\gamma_b\widehat{\Psi}^{p_2}_{a,l}+\gamma_l\widehat{\Theta}^{p_2}_{a,l}}$$

$$+ \underbrace{\sum_{i\in\{b,l\}}\gamma_i u'(c_{1i})(\bar{h}_i - h_{1i})\frac{\partial p_1}{\partial a_l}}_{\gamma_b\widehat{\Psi}^{p_1}_{a,l}+\gamma_l\widehat{\Theta}^{p_1}_{a,l}} + \underbrace{\sum_{i\in\{b,l\}}\gamma_i\sum_{s=1}^{2}\beta\left[\pi_s u'(c_{2si})\left(F_{KK}a_i + F_{LK}e_s\right)\right]}_{\gamma_b\Psi^{r,w}_K+\gamma_l\Theta^{r,w}_K} = 0.$$

Evaluating the above system of equations at the laissez-faire allocation, and rewriting the system in matrix form we obtain

$$\underbrace{\begin{bmatrix} \widehat{\Psi}^{p_1}_{a,b} + \widehat{\Psi}^{p_2}_{a,b} + \Psi^{r,w}_K & \widehat{\Theta}^{p_1}_{a,b} + \widehat{\Theta}^{p_2}_{a,b} + \Theta^{r,w}_K \\ \widehat{\Psi}^{p_1}_{a,l} + \widehat{\Psi}^{p_2}_{a,l} + \Psi^{r,w}_K & \widehat{\Theta}^{p_1}_{a,l} + \widehat{\Theta}^{p_2}_{a,l} + \Theta^{r,w}_K \end{bmatrix}}_{\widehat{A}} \begin{bmatrix} \gamma_b \\ \gamma_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To verify whether the matrix $\widehat{A}$ is non-singular we check its determinant:

$$|\widehat{A}| = \widehat{\Psi}^{p_1}_{a,b}\widehat{\Theta}^{p_1}_{a,l} + \widehat{\Psi}^{p_1}_{a,b}\Theta^{r,w}_K + \Psi^{r,w}_K\widehat{\Theta}^{p_1}_{a,l} - \widehat{\Psi}^{p_1}_{a,l}\widehat{\Theta}^{p_1}_{a,b} - \Psi^{r,w}_k\widehat{\Theta}^{p_1}_{a,b} - \widehat{\Psi}^{p_1}_{a,l}\Theta^{r,w}_K$$

$$
\begin{aligned}
&+\widehat{\Psi}^{p_2}_{a,b}\Theta^{r,w}_K+\Psi^{r,w}_K\widehat{\Theta}^{p_1}_{a,l}-\widehat{\Psi}^{p_2}_{a,l}\Theta^{r,w}_K-\Psi^{r,w}_K\widehat{\Theta}^{p_2}_{a,b}\\
&+\widehat{\Psi}^{p_1}_{a,b}\widehat{\Theta}^{p_2}_{a,l}+\widehat{\Psi}^{p_2}_{a,b}\widehat{\Theta}^{p_1}_{a,l}-\widehat{\Psi}^{p_1}_{a,l}\widehat{\Theta}^{p_2}_{a,b}-\widehat{\Psi}^{p_2}_{a,l}\widehat{\Theta}^{p_1}_{a,b}\\
&=(\bar{h}_b-h_{1b})(\widehat{\Phi}^1_{a,b}-\widehat{\Phi}^1_{a,l})(u'(c_{1b})\Theta^{r,w}_K+u'(c_{1l})\Psi^{r,w}_K)\\
&+\beta(\widehat{\Phi}^2_{a,b}-\widehat{\Phi}^2_{a,l})\Big[\Theta^{r,w}_K(\pi u'(c_{21b})(h_{1b}-h_{21b})+(1-\pi)u'(c_{22b})(h_{1b}-h_{22b}))\\
&-\Psi^{r,w}_K(\pi u'(c_{21l})(h_{1l}-h_{21l})+(1-\pi)u'(c_{22l})(h_{1l}-h_{22l})\Big]\\
&+\beta(\bar{h}_b-h_{1b})\Big[u'(c_{1b})\Big(\pi u'(c_{21l})(h_{1l}-h_{21l})+(1-\pi)u'(c_{22l})(h_{1l}-h_{22l})\Big)\\
&+u'(c_{1l})\Big(\pi u'(c_{21b})(h_{1b}-h_{21b})+(1-\pi)u'(c_{22b})(h_{1b}-h_{22b})\Big)\Big]\Big(\widehat{\Phi}^1_{a,b}\widehat{\Phi}^2_{a,l}-\widehat{\Phi}^1_{a,l}\widehat{\Phi}^2_{a,b}\Big)\\
&=|A|+\beta(\widehat{\Phi}^2_{a,b}-\widehat{\Phi}^2_{a,l})\Big[\Theta^{r,w}_K\mathbb{E}\Big\{u'(c_{2b})(h_{1b}-h_{2b})\Big\}-\Psi^{r,w}_K\mathbb{E}\Big\{u'(c_{2l})(h_{1l}-h_{2l})\Big\}\Big]\\
&+\beta(\bar{h}_b-h_{1b})\Big[u'(c_{1b})\mathbb{E}\Big\{u'(c_{2l})(h_{1l}-h_{2l})\Big\}+u'(c_{1l})\mathbb{E}\Big\{u'(c_{2b})(h_{1b}-h_{2b})\Big\}\Big]\\
&\times\Big(\widehat{\Phi}^1_{a,b}\widehat{\Phi}^2_{a,l}-\widehat{\Phi}^1_{a,l}\widehat{\Phi}^2_{a,b}\Big)
\end{aligned}
$$

It is straightforward to see that each term $(\widehat{A1})$-$(\widehat{A3})$ is non-zero if the respective condition $(\widehat{C1})$-$(\widehat{C3})$ holds.

□

Proposition 5

Proof. Introducing taxation modifies the households' optimization problem as follows. Household b chooses $(c_{1b}, c_{21b}, c_{22b}, h_{1b}, h_{21b}, h_{22b}, a_b)$ to maximize

$$
\begin{aligned}
&U_b(c_{1b}, c_{21b}, c_{22b}, h_{1b}, h_{21b}, h_{22b}) = \mathbf{u}(c_{1b}, h_{1b}) + \beta\Big[\pi\,\mathbf{u}(c_{21b}, h_{21b}) + (1-\pi)\mathbf{u}(c_{22b}, h_{22b})\Big]\\
&\text{subject to}\\
&c_{1b} + a_b + p_1 h_{1b} = \bar{\omega}_b + p_1\bar{h}_b,\\
&a_b \geq -\frac{we_1 + p_2 h_{1b} + \widehat{T}^a_b}{1 + r(1-\widehat{\tau}^a_b)},\\
&c_{2sb} + p_2 h_{2sb} = (1 + r(1-\widehat{\tau}^a_b))a_b + we_s + p_2 h_{1b} + \widehat{T}^a_b,
\end{aligned}
$$

for $s = \{1,2\}$, $prob(e = e_1) = \pi$, and where $\widehat{T}^a_b$ is a lump-sum transfer that must equal $\widehat{T}^a_b = r\widehat{\tau}^a_b a_b$. Accordingly, household l chooses $(c_{1l}, c_{21l}, c_{22l}, h_{1l}, h_{21l}, h_{22l}, a_l)$ to maximize

$$
\begin{aligned}
&U_l(c_{1l}, c_{21l}, c_{22l}, h_{1l}, h_{21l}, h_{22l}) = \mathbf{u}(c_{1l}, h_{1l}) + \beta\Big[\pi\mathbf{u}(c_{21l}, h_{21l}) + (1-\pi)\mathbf{u}(c_{22l}, h_{22l})\Big]\\
&\text{subject to}\\
&c_{1l} + a_l + p_1 h_{1l} = \bar{\omega}_l + p_1\bar{h}_l,\\
&a_l \geq -\frac{we_1 + p_2 h_{1l} + \widehat{T}^a_l}{1 + r(1-\widehat{\tau}^a_l)},
\end{aligned}
$$

$$c_{2sl} + p_2 h_{2sl} = (1 + r(1 - \widehat{\tau}_l^a))a_l + we_s + p_2 h_{1l} + \widehat{T}_l^a,$$

where $\widehat{T}_l^a = r\widehat{\tau}_l^a a_l$. Then, given that the natural borrowing limits are never binding, the first-order conditions are

$$\begin{aligned}
&- u'(c_{1b})\, p_1 + v'(h_{1b}) + p_2\beta\left[\pi u'(c_{21b}) + (1-\pi)u'(c_{22b})\right] = 0,\\
&- u'(c_{1b}) + \beta(1 + r(1 - \widehat{\tau}_b^a))\left[\pi u'(c_{21b}) + (1-\pi)\, u'(c_{22b})\right] = 0,\\
&- u'(c_{2sb})p_2 + v'(h_{2sb}) = 0, \quad \text{for } s = 1, 2,\\
&- u'(c_{1l})\, p + v'(h_{1l}) + p_2\beta\left[\pi u'(c_{21l}) + (1-\pi)u'(c_{22l})\right] = 0,\\
&- u'(c_{1l}) + \beta(1 + r(1 - \widehat{\tau}_l^a))\left[\pi u'(c_{21l}) + (1-\pi)\, u'(c_{22l})\right] = 0,\\
&- u'(c_{2sl})p_2 + v'(h_{2sl}) = 0, \quad \text{for } s = 1, 2.
\end{aligned}$$

To ensure that agents' incentives are aligned with the planner's will, it must hold that

$$\begin{aligned}
-\beta r\widehat{\tau}_b^a \sum_{s=1}^{2} \pi_s u'(c_{2sb}) &= \frac{\partial p_1}{\partial a_b} \sum_{i\in\{b,l\}} \frac{\gamma_i}{\gamma_b} u'(c_{1i})(\bar{h}_i - h_{1i}) + \beta\frac{\partial p_2}{\partial a_b} \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \pi_s \frac{\gamma_i}{\gamma_b} u'(c_{2si})(h_{1i} - h_{2si})\\
&\quad + \beta \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_b} \pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big),\\
-\beta r\widehat{\tau}_l^a \sum_{s=1}^{2} \pi_s u'(c_{2sl}) &= \frac{\partial p_1}{\partial a_l} \sum_{i\in\{b,l\}} \frac{\gamma_i}{\gamma_l} u'(c_{1i})(\bar{h}_i - h_{1i}) + \beta\frac{\partial p_2}{\partial a_l} \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \pi_s \frac{\gamma_i}{\gamma_l} u'(c_{2si})(h_{1i} - h_{2si})\\
&\quad + \beta \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_l} \pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big).
\end{aligned}$$

Thus, we arrive to the following set of taxes

$$\begin{aligned}
\widehat{\tau}_b^a &= \frac{-1}{\beta r \sum_{s=1}^{2} \pi_s u'(c_{2sb})} \Bigg\{ \frac{\partial p_1}{\partial a_b} \sum_{i\in\{b,l\}} \frac{\gamma_i}{\gamma_b} u'(c_{1i})(\bar{h}_i - h_{1i}) + \beta\frac{\partial p_2}{\partial a_b} \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \pi_s \frac{\gamma_i}{\gamma_b} u'(c_{2si})(h_{1i} - h_{2si})\\
&\quad + \beta \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_b} \pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big) \Bigg\},\\
\widehat{\tau}_l^a &= \frac{-1}{\beta r \sum_{s=1}^{2} \pi_s u'(c_{2sl})} \Bigg\{ \frac{\partial p}{\partial a_l} \sum_{i\in\{b,l\}} \frac{\gamma_i}{\gamma_l} u'(c_{1i})(\bar{h}_i - h_i) + +\beta\frac{\partial p_2}{\partial a_l} \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \pi_s \frac{\gamma_i}{\gamma_l} u'(c_{2si})(h_{1i} - h_{2si})\\
&\quad + \beta \sum_{i\in\{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_l} \pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big) \Bigg\}.
\end{aligned}$$

□

A.3 Illiquid Housing & Collateralized Borrowings

Lemma 8

Proof. Totally differentiating the housing market clearing condition (27) gives

$$\frac{\partial \tilde{h}_b(p,a_b)}{\partial p}dp + \frac{\partial \tilde{h}_b(p,a_b)}{\partial a_b}da_b + \frac{\partial h_l(p,\mu_l)}{\partial p}dp - \frac{\partial h_l(p,\mu_l)}{\partial \mu_l}da_l = 0,$$

where we used (29) for the last term. Inserting (28) and (30), and rearranging terms yields

$$-\frac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}da_l - \frac{1}{\xi p}da_b + \left[\frac{a_b}{\xi p^2} + \mathcal{S}_l(p,\mu_l) - \frac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}\left(h_l\left(p,\mu_l\right) - \bar{h}_l\right)\right]dp = 0.$$

Then, depending on whether borrowings or savings are perturbed, the respective coefficient is given by

$$\widetilde{\Phi}_{a,b} = \frac{\dfrac{1}{\xi p}}{\dfrac{a_b}{\xi p^2} + \mathcal{S}_l(p,\mu_l) - \dfrac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}\left(h_l\left(p,\mu_l\right) - \bar{h}_l\right)},$$

$$\widetilde{\Phi}_{a,l} = \frac{\dfrac{\partial h_l(p,\mu_l)}{\partial \mu_l}}{\dfrac{a_b}{\xi p^2} + \mathcal{S}_l(p,\mu_l) - \dfrac{\partial h_l\left(p,\mu_l\right)}{\partial \mu_l}\left(h_l\left(p,\mu_l\right) - \bar{h}_l\right)}.$$

□

Lemma 9

Proof. To prove the second part of the lemma we first use the definitions of all externalities:

$$\begin{aligned} dU_b &= \left(\widetilde{\Psi}^p_{a,i} + \widetilde{\Psi}^p_{\lambda,i} + \Psi^{r,w}_K\right)da_i \\ &= \Big[\left(u'(c_{1b})(\bar{h}_b - h_b) + \lambda\xi h_b\right)\widetilde{\Phi}_{a,i} + \Psi^{r,w}_K\Big]da_i. \end{aligned}$$

To see that $dU_b > 0$ when $\bar{h}_b \leq (1-\xi)h_b$, it remains to be shown that the term in front of $\tilde{\Phi}_{a,i}$ is less than or equal to zero, since we already know that $\Psi^{r,w}_K > 0$ and $\widetilde{\Phi}_{a,i} < 0$ – as we are considering preferences that entail a dominating substitution effect. It follows from borrowers' first-order conditions that

$$u'(c_{1b})(\bar{h}_b - h_b) + \lambda\xi h_b < \lambda(\bar{h}_b - h_b) + \lambda\xi h_b,$$

and rewriting,

$$u'(c_{1b})(\bar{h}_b - h_b) + \lambda\xi h_b < \lambda(\bar{h}_b - (1-\xi)h_b) \leq 0,$$

when $\bar{h}_b \leq (1-\xi)h_b$. If the latter applies then $dU_b > 0$. Instead, if $\bar{h}_b > (1-\xi)h_b$ then $u'(c_{1b})(\bar{h}_b - h_b) + \lambda\xi h_b$ cannot be signed without additional conditions, and thus $dU_b \gtreqless 0$. □

Proposition 6

Proof. Substituting all the constraints in the objective function, and since the collateral constraint only matters for borrowers, the social planner problem can be rewritten as

$$\begin{aligned}
\max_{\{a_b,a_l\}} \gamma_b \Bigg\{ & u\Big(\bar{\omega}_b + p\left(\bar{h}_b - h_b(p,\mu_b)\right) - a_b\Big) + v\big(h_b(p,\mu_b)\big)(1+\beta) \\
& + \beta\Big[\pi\, u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_b + F_L\big(K(a_b,a_l),L\big)e_1\Big) \\
& + (1-\pi)u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_b + F_L\big(K(a_b,a_l),L\big)e_2\Big)\Big] \\
& + \lambda\Big(a_b + \xi p h_b(p,\mu_b)\Big)\Bigg\} + \\
\gamma_l \Bigg\{ & u\Big(\bar{\omega}_l + p\left(\bar{h}_l - h_l(p,\mu_l)\right) - a_l\Big) + v\big(h_l(p,\mu_l)\big)(1+\beta) \\
& + \beta\Big[\pi\, u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_l + F_L\big(K(a_b,a_l),L\big)e_1\Big) \\
& + (1-\pi)u\Big(\left(1 + F_K\big(K(a_b,a_l),L\big) - \delta\right) a_l + F_L\big(K(a_b,a_l),L\big)e_2\Big)\Big]\Bigg\},
\end{aligned}$$

where $\bar{H} = h_b(p,\mu_b) + h_l(p,\mu_l)$, $K(a_b,a_l) = a_b + a_l$ and $L = 2(\pi e_1 + (1-\pi)e_2)$. Thus the social planner first-order conditions are

$$\begin{aligned}
\frac{\partial SPF}{\partial a_b} = {} & \gamma_b \Bigg\{ \underbrace{\lambda - u'(c_{1b}) + \beta(1 + F_K - \delta)\sum_{s=1}^{2}\pi_s u'(c_{2sb})}_{\text{Agent-}b\text{ FOC}} + \frac{\partial h_b}{\partial a_b}\Big[\underbrace{\lambda\xi p - u'(c_{1b})p + v'(h_b)(1+\beta)}_{\text{Agent-}b\text{ FOC}}\Big]\Bigg\} \\
& + \underbrace{\sum_{i\in\{b,l\}} \gamma_i u'(c_{1i})(\bar{h}_i - h_i)\frac{\partial p}{\partial a_b} + \gamma_b \lambda\xi h_b \frac{\partial p}{\partial a_b} + \sum_{i\in\{b,l\}} \gamma_i \sum_{s=1}^{2}\beta\left[\pi_s u'(c_{2si})\big(F_{KK}a_i + F_{LK}e_s\big)\right]}_{\gamma_b\tilde{\Psi}^p_{a,b} + \gamma_b\tilde{\Psi}^p_{\lambda,b} + \gamma_l\tilde{\Theta}^p_{a,b} + \gamma_b\Psi^{r,w}_K + \gamma_l\Theta^{r,w}_K} = 0\,,
\end{aligned}$$

$$\frac{\partial SPF}{\partial a_l} = \gamma_l \Bigg\{ \underbrace{-u'(c_{1l}) + \beta(1 + F_K - \delta)\sum_{s=1}^{2}\pi_s u'(c_{2sl})}_{\text{Agent-}l\text{ FOC}} + \frac{\partial h_l}{\partial a_l}\Big[\underbrace{-u'(c_{1l})p + v'(h_l)(1+\beta)}_{\text{Agent-}l\text{ FOC}}\Big]\Bigg\}$$

$$+\underbrace{\sum_{i\in\{b,l\}}\gamma_i u'(c_{1i})(\bar{h}_i-h_i)\frac{\partial p}{\partial a_l}+\gamma_b\lambda\xi h_b\frac{\partial p}{\partial a_l}+\sum_{i\in\{b,l\}}\gamma_i\sum_{s=1}^{2}\beta\left[\pi_s u'(c_{2si})\big(F_{KK}a_i+F_{LK}e_s\big)\right]}_{\gamma_b\widetilde{\Psi}^p_{a,l}+\gamma_b\widetilde{\Psi}^p_{\lambda,l}+\gamma_l\widetilde{\Theta}^p_{a,l}+\gamma_b\Psi^{r,w}_K+\gamma_l\Theta^{r,w}_K}=0\,,$$

Evaluating the above system of equations at the laissez-faire allocation, and rewriting the system in matrix form we obtain

$$\underbrace{\begin{bmatrix}\widetilde{\Psi}^p_{a,b}+\widetilde{\Psi}^p_{\lambda,b}+\Psi^{r,w}_K & \widetilde{\Theta}^p_{a,b}+\Theta^{r,w}_K\\ \widetilde{\Psi}^p_{a,l}+\widetilde{\Psi}^p_{\lambda,l}+\Psi^{r,w}_K & \widetilde{\Theta}^p_{a,l}+\Theta^{r,w}_K\end{bmatrix}}_{\widetilde{A}}\begin{bmatrix}\gamma_b\\ \gamma_l\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Once again, we verify whether the matrix $\widetilde{A}$ is non-singular checking its determinant:

$$|\widetilde{A}|=\left(\widetilde{\Psi}^p_{a,b}+\widetilde{\Psi}^p_{\lambda,b}+\Psi^{r,w}_K\right)\left(\widetilde{\Theta}^p_{a,l}+\Theta^{r,w}_K\right)-\left(\widetilde{\Theta}^p_{a,b}+\Theta^{r,w}_K\right)\left(\widetilde{\Psi}^p_{a,l}+\widetilde{\Psi}^p_{\lambda,l}+\Psi^{r,w}_K\right).$$

Manipulating the above, the determinant can be simplified to

$$|\widetilde{A}|=\left[\widetilde{\Phi}_{a,b}-\widetilde{\Phi}_{a,l}\right]\left\{u'(c_{1b})\left(\bar{h}_b-h_b\right)\Theta^{r,w}_K+\lambda\xi h_b\Theta^{r,w}_K+u'(c_{1l})\left(\bar{h}_b-h_b\right)\Psi^{r,w}_K\right\}.$$

It is straightforward to see that as long as $(\widetilde{C1})$ holds then $(\widetilde{A1})$ is non-zero. For $(\widetilde{A2})$ to be non-zero, either $(\widetilde{C2.1})$ has to hold so that $(\widetilde{A2.1})$ and $(\widetilde{A2.2})$ are non-zero, or $(\widetilde{C2.2})$ holds to guarantee that $(\widetilde{A2.3})$ is non-zero.

□

Proposition 7

Proof. Introducing taxation modifies the households' optimization problem as follows. Household b chooses $(c_{1b}, c_{21b}, c_{22b}, h_b, a_b)$ to maximize

$$U_b(c_{1b}, c_{21b}, c_{22b}, h_b)=\mathbf{u}(c_{1b},h_b)+\beta\Big[\pi\mathbf{u}(c_{21b},h_b)+(1-\pi)\mathbf{u}(c_{22b},h_b)\Big]$$

subject to

$$c_{1b}+a_b+ph_b=\bar{\omega}_b+p\bar{h}_b,$$

$$a_b\geq-\xi ph_b,$$

$$c_{2sb}=(1+r(1-\widetilde{\tau}^a_b))a_b+we_s+\widetilde{T}^a_b,$$

for $s=1,2$ with $e_s\in\{e_1,e_2\}$ and $prob(e=e_1)=\pi$, and where $\widetilde{T}^a_b$ is a lump-sum transfer that must equal $\widetilde{T}^a_b=r\widetilde{\tau}^a_b a_b$. Household $i=l$ chooses $(c_{1l}, c_{21l}, c_{22l}, h_l, a_l)$ to maximize

$$U_l(c_{1l}, c_{21l}, c_{22l}, h_l)=\mathbf{u}(c_{1l},h_l)+\beta\Big[\pi\mathbf{u}(c_{21l},h_l)+(1-\pi)\mathbf{u}(c_{22l},h_l)\Big]$$

subject to

$$
\begin{aligned}
& c_{1l} + a_l + ph_l = \bar{\omega}_l + p\bar{h}_l, \\
& a_l \geq -\xi p h_l, \\
& c_{2sl} = (1 + r(1 - \widetilde{\tau}_l^a))a_l + we_s + \widetilde{T}_l^a,
\end{aligned}
$$

for $s = 1, 2$ and $prob(e = e_1) = \pi$, and where $\widetilde{T}_l^a = r\widetilde{\tau}_l^a a_l$. Then, the first-order conditions are

$$
\begin{aligned}
& - u'(c_{1b})\, p + v'(h_b)(1+\beta) = 0, \\
& - u'(c_{1b}) + \beta(1 + r(1 - \widetilde{\tau}_b^a)) \left[\pi u'(c_{21b}) + (1-\pi) u'(c_{22b})\right] + \lambda = 0, \\
& - u'(c_{1l})\, p + v'(h_l)(1+\beta) = 0, \\
& - u'(c_{1l}) + \beta(1 + r(1 - \widetilde{\tau}_l^a)) \left[\pi u'(c_{21l}) + (1-\pi) u'(c_{22l})\right] = 0,
\end{aligned}
$$

where λ is the multiplier on the borrowers' binding borrowing constraint. To ensure that agents' incentives are aligned with the planner's will it must hold that

$$
\begin{aligned}
-\beta r \widetilde{\tau}_b^a \sum_{s=1}^{2} \pi_s u'(c_{2sb}) = & \frac{\partial p}{\partial a_b} \left[\sum_{i \in \{b,l\}} \frac{\gamma_i}{\gamma_b} u'(c_{1i})(\bar{h}_i - h_i) + \lambda \xi h_b \right] \\
& + \beta \sum_{i \in \{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_b} \pi_s u'(c_{2si}) \big(F_{KK} a_i + F_{LK} e_s\big), \\
-\beta r \widetilde{\tau}_l^a \sum_{s=1}^{2} \pi_s u'(c_{2sl}) = & \frac{\partial p}{\partial a_l} \left[\sum_{i \in \{b,l\}} \frac{\gamma_i}{\gamma_l} u'(c_{1i})(\bar{h}_i - h_i) + \frac{\gamma_b}{\gamma_l} \lambda \xi h_b \right] \\
& + \beta \sum_{i \in \{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_l} \pi_s u'(c_{2si}) \big(F_{KK} a_i + F_{LK} e_s\big),
\end{aligned}
$$

thus we arrive to the following set of taxes

$$
\begin{aligned}
\widetilde{\tau}_b^a = & \frac{-1}{\beta r \sum_{s=1}^{2} \pi_s u'(c_{2sb})} \left\{ \frac{\partial p}{\partial a_b} \left[\sum_{i \in \{b,l\}} \frac{\gamma_i}{\gamma_b} u'(c_{1i})(\bar{h}_i - h_i) + \lambda \xi h_b \right] \right. \\
& \left. + \beta \sum_{i \in \{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_b} \pi_s u'(c_{2si}) \big(F_{KK} a_i + F_{LK} e_s\big) \right\}, \\
\widetilde{\tau}_l^a = & \frac{-1}{\beta r \sum_{s=1}^{2} \pi_s u'(c_{2sl})} \left\{ \frac{\partial p}{\partial a_l} \left[\sum_{i \in \{b,l\}} \frac{\gamma_i}{\gamma_l} u'(c_{1i})(\bar{h}_i - h_i) + \frac{\gamma_b}{\gamma_l} \lambda \xi h_b \right] \right. \\
& \left. + \beta \sum_{i \in \{b,l\}} \sum_{s=1}^{2} \frac{\gamma_i}{\gamma_l} \pi_s u'(c_{2si}) \big(F_{KK} a_i + F_{LK} e_s\big) \right\}.
\end{aligned}
$$

□

B Miscellaneous

B.1 Illiquid Housing & Collateralized Borrowings

Let us consider the case of borrowers that are first-time home buyers who can collateralize any fraction of the acquired housing, i.e. $\bar{h}_b = 0$ and $\xi \in [0,1]$. Under $\widetilde{C1}$ and $\widetilde{C2}$,

$$|\widetilde{A}| = \big[\widetilde{\Phi}_{a,b} - \widetilde{\Phi}_{a,l}\big]\big(-h_b\big)\Big\{\big(u'(c_{1b}) - \lambda\xi\big)\Theta_K^{r,w} + u'(c_{1l})\Psi_K^{r,w}\Big\} \neq 0,$$

as we can show that $\left(u'(c_{1b}) - \lambda\xi\right)\Theta_K^{r,w} + u'(c_{1l})\Psi_K^{r,w} < 0$ for all $\xi \in [0,1]$ (see the below proof of Lemma B1). That is, for first-time home buyers who can collateralize any fraction of the acquired housing, distributive externalities in the market for houses directly dominate collateral externalities for any considered value of the collateral parameter, so that the negative term $u'(c_{1b})\Theta_K^{r,w}$ drives the sign of the statistic – note that both $-\lambda\xi\Theta_K^{r,w}$ and $u'(c_{1l})\Psi_K^{r,w}$ are positive. Thus, the planner mandates changes to financial holdings according to the impact of distributive externalities alone. Note that the way it will do so is pinned down by the sign of $|\widetilde{A}|$, which rests upon the difference between the numerator of $\widetilde{\Phi}_{a,b}$ and $\widetilde{\Phi}_{a,l}$.

Lemma B1.
In the case of borrowers that are first-time home buyers who can collateralize any amount of the acquired housing, i.e. $\bar{h}_b = 0$ and $\xi \in [0,1]$, the following condition holds:

$$\left(u'(c_{1b}) - \lambda\xi\right)\Theta_K^{r,w} + u'(c_{1l})\Psi_K^{r,w} < 0.$$

Proof. Let us rewrite more explicitly the weighted sum by using the definitions of $\Psi_K^{r,w}$ and $\Theta_K^{r,w}$, adding and subtracting λ in the first parentheses, and separating the sum into three terms:

$$\begin{aligned}
&\left(u'(c_{1b}) - \lambda\right)\beta\Big[\pi u'(c_{21l})\big(F_{KK}a_l + F_{LK}e_1\big) + (1-\pi)u'(c_{22l})\big(F_{KK}a_l + F_{LK}e_2\big)\Big] \\
&+ (1-\xi)\,\lambda\beta\Big[\pi u'(c_{21l})\big(F_{KK}a_l + F_{LK}e_1\big) + (1-\pi)u'(c_{22l})\big(F_{KK}a_l + F_{LK}e_2\big)\Big] \\
&+ u'(c_{1l})\beta\Big[\pi u'(c_{21b})\big(F_{KK}a_b + F_{LK}e_1\big) + (1-\pi)u'(c_{22b})\big(F_{KK}a_b + F_{LK}e_2\big)\Big].
\end{aligned}$$

We first show that the sum of the first and the third term is negative. By using that $K = a_b + a_l$ and $KF_{KK} + LF_{LK} = 0$, we can rewrite the sum of the first and the third term as

$$\begin{aligned}
&\beta F_{KK}a_b\Big[u'(c_{1l})\Big(\pi u'(c_{21b}) + (1-\pi)u'(c_{22b})\Big) - \left(u'(c_{1b}) - \lambda\right)\Big(\pi u'(c_{21l}) + (1-\pi)u'(c_{22l})\Big)\Big] \\
&+\beta F_{LK}\Big[u'(c_{1l})\Big(\pi u'(c_{21b})e_1 + (1-\pi)u'(c_{22b})e_2\Big) + \left(u'(c_{1b}) - \lambda\right) \\
&\qquad\times\Big(\pi u'(c_{21l})(e_1 - L) + (1-\pi)u'(c_{22l})(e_2 - L)\Big)\Big].
\end{aligned}$$

Since $u'(c_{1b}) - \lambda = \beta(1+r)\sum_s \pi_s u'(c_{2sb})$ and $u'(c_{1l}) = \beta(1+r)\sum_s \pi_s u'(c_{2sl})$ at the competitive equilibrium, then it holds $u'(c_{1l})\sum_s \pi_s u'(c_{2sb}) - (u'(c_{1b}) - \lambda)\sum_s \pi_s u'(c_{2sl}) = 0$ and we can simplify $u'(c_{1l})\Psi_K^{r,w} + (u'(c_{1b}) - \lambda)\,\Theta_K^{r,w}$ to

$$\beta F_{LK}\Big[u'(c_{1l})\Big(\pi u'(c_{21b})e_1 + (1-\pi)u'(c_{22b})e_2\Big) \\ + \big(u'(c_{1b}) - \lambda\big)\,\Big(\pi u'(c_{21l})(e_1 - L) + (1-\pi)u'(c_{22l})(e_2 - L)\Big)\Big].$$

Now, by letting $e_2 = e_1 + h$ with $h > 0$, and using $L = 2(\pi e_1 + (1-\pi)e_2) = 2(e_1 + (1-\pi)h)$, we arrive to $(e_1 - L) = -(e_1 + 2(1-\pi)h)$ and $(e_2 - L) = -(e_1 + 2(1-\pi)h - h)$. Then we can rewrite the above expression as

$$\beta F_{LK}\Big[u'(c_{1l})\Big(\pi u'(c_{21b})e_1 + (1-\pi)u'(c_{22b})(e_1 + h)\Big) \\ - \big(u'(c_{1b}) - \lambda\big)\,\Big(\pi u'(c_{21l})(e_1 + 2(1-\pi)h) + (1-\pi)u'(c_{22l})(e_1 + (1-2\pi)h)\Big)\Big].$$

Manipulating further by collecting the e_1 terms, we can rewrite the above as

$$\beta F_{LK}\Big[e_1\Big\{u'(c_{1l})\Big(\pi u'(c_{21b}) + (1-\pi)u'(c_{22b})\Big) - \big(u'(c_{1b}) - \lambda\big)\,\Big(\pi u'(c_{21l}) + (1-\pi)u'(c_{22l})\Big)\Big\} \\ + u'(c_{1l})(1-\pi)u'(c_{22b})h - \big(u'(c_{1b}) - \lambda\big)\,\Big(\pi u'(c_{21l})2(1-\pi)h + (1-\pi)u'(c_{22l})(1-2\pi)h\Big)\Big].$$

Using again that $u'(c_{1l})\sum_s \pi_s u'(c_{2sb}) - (u'(c_{1b}) - \lambda)\sum_s \pi_s u'(c_{2sl}) = 0$, and collecting h we obtain

$$\beta F_{LK}h\Big[u'(c_{1l})(1-\pi)u'(c_{22b}) - \big(u'(c_{1b}) - \lambda\big)\,\Big(\pi u'(c_{21l})2(1-\pi) + (1-\pi)u'(c_{22l})(1-2\pi)\Big)\Big].$$

Then, using again that $u'(c_{1b}) - \lambda = \beta(1+r)\sum_s \pi_s u'(c_{2sb})$ and $u'(c_{1l}) = \beta(1+r)\sum_s \pi_s u'(c_{2sl})$ at the competitive equilibrium, we arrive to

$$\beta F_{LK}h\beta(1+r)\Big[-\pi u'(c_{21b})\Big(\pi u'(c_{21l})2(1-\pi) + (1-\pi)u'(c_{22l})(1-2\pi)\Big) \\ -(1-\pi)u'(c_{22b})\Big(\pi u'(c_{21l})(1-2\pi) + (1-\pi)u'(c_{22l})(-2\pi)\Big)\Big].$$

Continuing, after repeating the above expression, we expand the terms within squared brackets and then collect π and $(1-\pi)$,

$$\underbrace{\beta^2 F_{LK}(1+r)h\pi(1-\pi)}_{>0\,,\ \forall\pi\in]0,1[}\Big[\underbrace{2\pi\Big(\underbrace{u'(c_{21l}) - u'(c_{22l})}_{>0}\Big)\Big(\underbrace{u'(c_{22b}) - u'(c_{21b})}_{<0}\Big)}_{\leq 0\,,\ \forall\pi\in[0,1]} \\ \underbrace{-u'(c_{22b})\Big(\underbrace{u'(c_{21l}) - u'(c_{22l})}_{>0}\Big)}_{<0}\underbrace{-u'(c_{22l})\Big(\underbrace{u'(c_{21b}) - u'(c_{22b})}_{>0}\Big)}_{<0}\Big] < 0\ \forall\,\pi\in]0,1[.$$

Now it only remains to show that

$$(1-\xi)\,\lambda\beta\Big[\pi u'(c_{21l})\big(F_{KK}a_l + F_{LK}e_1\big) + (1-\pi)u'(c_{22l})\big(F_{KK}a_l + F_{LK}e_2\big)\Big] \leq 0.$$

This is straightforward as $\xi \in [0,1]$, $\lambda > 0$, and $\Theta_K^{r,w} < 0$ as previously shown.

$\square$

B.2 Numerical Analysis

Table B1: Parameters and Targets Comparison Across Cases

Parameter	Illiquid Housing Natural BL	Liquid Housing Natural BL	Illiquid Housing Collateral	
e_1	0.25	0.25	0.25	
e_2	1.05	1.05	1.05	
π	0.50	0.50	0.50	
$\bar{\omega}_b$	0.075	0.15	0.035	
$\bar{\omega}_l$	2.75	2.90	2.90	
$\bar{h}_b$	0.07	0.15	0.0425	
$\bar{h}_l$	1.30	1.30	1.35	
Target				Data
Income ratio Top/Bottom	4.20	4.20	4.20	4.20
Housing-resources ratio Bottom	0.60	0.59	0.60	0.61
Housing-resources ratio Top	0.32	0.32	0.31	0.40
Capital-output ratio	1.79	1.82	1.85	1.64

Notes: The four targets in more detailed form are 1) the ratio of top half to bottom half income, 2) the housing-resources ratio of the bottom half income distribution, 3) the housing-resources ratio of the top half income distribution, and 4) the annualized ratio of aggregate capital to output. Resources are defined as the sum of wealth and income.

References

Bianchi, Javier and Enrique G. Mendoza (2010). “Overborrowing, Financial Crises and ’Macro-Prudential’ Taxes”. NBER Working Paper 16091.

Carroll, Christopher, Jiri Slacalek, Kiichi Tokuoka, and Matthew N. White (2017). “The Distribution of Wealth and the Marginal Propensity to Consume”. *Quantitative Economics 8*, 977–1020.

Citanna, Alessandro, Atsushi Kajii, and Antonio Villanacci (1998). “Constrained suboptimality in incomplete markets: a general approach and two applications”. *Economic Theory 11*, 495–521.

Dávila, Eduardo and Anton Korinek (2018). “Pecuniary Externalities in Economies with Financial Frictions”. *Review of Economic Studies 85*, 352–395.

Dávila, Julio, Jay H. Hong, Per Krusell, and José-Víctor Ríos-Rull (2012). “Constrained Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks”. *Econometrica 80* (6), 2431–2467.

Diamond, Peter A. (1967). “The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty”. *American Economic Review 57* (4), 759–776.

Díaz, Antonia and María José Luengo-Prado (2010). “The Wealth Distribution with Durable Goods”. *International Economic Review 51* (1), 143–170.

Favilukis, Jack, Sydney C. Ludvigson, and Stijn Van Nieuwerburgh (2017). “The Macroeconomic Effects of Housing Wealth, Housing Finance, and Limited Risk Sharing in General Equilibrium”. *Journal of Political Economy 125* (1), 140–223.

Geanakoplos, John D., Michael Magill, Martine Quinzii, and Jacques H. Drèze (1990). “Generic Inefficiency of Stock Market Equilibrium When Markets Are Incomplete”. *Journal of Mathematical Economics 19*, 113–151.

Geanakoplos, John D. and Heraklis M. Polemarchakis (1986). “Existence, Regularity, and Constrained Suboptimality of Competitive Allocations When the Asset Market Is Incomplete”. In: *Essays in Honor of Kenneth J. Arrow*. Ed. by Walter P. Heller, Ross M. Starr, and David A. Starrett. Vol. 3. Cambridge University Press, 65–95.

Greenwald, Bruce C. and Joseph E. Stiglitz (1986). “Externalities in Economies with Imperfect Information and Incomplete Markets”. *Quarterly Journal of Economics 101* (2), 229–264.

Hart, Oliver D. (1975). “On the Optimality of Equilibrium when the Market Structure is Incomplete”. *Journal of Economic Theory 11*, 418–443.

Hildenbrand, Werner (1983). “One the “Law of Demand””. *Econometrica 51* (4), 997–1019.

——— (1989). “Facts and Ideas in Micoreconomic Theory”. *European Economic Review 33*, 251–276.

Jeanne, Olivier and Anton Korinek (2019). “Managing credit booms and busts: A Pigouvian taxation approach”. *Journal of Monetary Economics 107*, 2–17.

Kaplan, Greg and Giovanni L. Violante (2014). “A Model of the Consumption Response to Fiscal Stimulus Payments”. *Econometrica 82* (4), 1199–1239.

Kaplan, Greg, Giovanni L. Violante, and Justin Weidner (2014). “The Wealthy Hand-to-Mouth”. *Brookings Papers on Economic Activity 48* (1), 77–153.

Lorenzoni, Guido (2008). “Inefficient Credit Booms”. *Review of Economic Studies 75*, 809–833.

Quah, John K.-H. (1997). “The Law of Demand When Income is Price Dependent”. *Econometrica 65* (6), 1421–1442.

——— (2000). “The Monotonicity of Individual and Market Demand”. *Econometrica 68* (4), 911–930.

Stiglitz, Joseph E. (1982). “The Inefficiency of the Stock Market Equilibrium”. *Review of Economic Studies 49* (2), 241–261.