Constrained Conditional Moment Restriction Models*

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Abstract

Shape restrictions have played a central role in economics as both testable implications of theory and sufficient conditions for obtaining informative counterfactual predictions. In this paper we provide a general procedure for inference under shape restrictions in identified and partially identified models defined by conditional moment restrictions. Our test statistics and proposed inference methods are based on the minimum of the generalized method of moments (GMM) objective function with and without shape restrictions. Uniformly valid critical values are obtained through a bootstrap procedure that approximates a subset of the true local parameter space. In an empirical analysis of the effect of childbearing on female labor supply, we show that employing shape restrictions in linear instrumental variables (IV) models can lead to shorter confidence regions for both local and average treatment effects. Other applications we discuss include inference for the variability of quantile IV treatment effects and for bounds on average equivalent variation in a demand model with general heterogeneity. We find in Monte Carlo examples that the critical values are conservatively accurate and that tests about objects of interest have good power relative to unrestricted GMM.

Keywords: Shape restrictions, inference on functionals, conditional moment (in)equality restrictions, instrumental variables, nonparametric and semiparametric models, Banach space, Banach lattice, Koltchinskii coupling.

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1 Introduction

Shape restrictions have played a central role in economics as both testable implications of classical theory and sufficient conditions for obtaining informative counterfactual predictions (Topkis, 1998). A long tradition in applied and theoretical econometrics has as a result studied shape restrictions, their ability to aid in identification, estimation, and inference, and the possibility of testing for their validity (Matzkin, 1994; Chetverikov et al., 2018). A canonical example of this interplay between theory and practice is consumer demand analysis, where theoretical predictions such as Slutsky conditions have been extensively tested for and employed in estimation (Hausman and Newey, 1995, 2016; Blundell et al., 2012; Dette et al., 2016). The empirical analysis of shape restrictions, however, goes well beyond this important application with recent examples including studies into the monotonicity of the state price density (Jackwerth, 2000; Art-Sahalia and Duarte, 2003), the presence of ramp-up and start-up costs (Wolak, 2007; Reguant, 2014), and the existence of complementarities in demand (Gentzkow, 2007) and organizational design (Athey and Stern, 1998; Kretschmer et al., 2012).

Shape restrictions are often equivalent to inequality restrictions on parameters of interest and on certain unknown functions. For example, Slutsky negative semi-definiteness and monotonicity require that certain functions satisfy inequality restrictions. Inference with inequality restrictions is difficult. Such restrictions lead to discontinuities in (pointwise) limiting distributions where the inequality restrictions are "close" to binding, which makes inference challenging due to non-pivotal and potentially unreliable pointwise asymptotic approximations (Andrews, 2000, 2001). Limit discontinuities further make it difficult to construct confidence intervals with uniform coverage.

We address these challenges by obtaining critical values through a bootstrap procedure that uniformly approximates a subset of the local parameter space. The proposed critical values simultaneously deliver uniformly valid inference and pointwise limiting rejection probabilities that equal the nominal level of the test in many applications. Our results apply to a class of conditional moment restriction models (Ai and Chen, 2007, 2012) that encompasses parametric (Hansen, 1982), semiparametric (Ai and Chen, 2003), and nonparametric (Newey and Powell, 2003) instrumental variable (IV) models, as well as panel data applications (Chamberlain, 1992), and the study of plug-in functionals. For parametric IV our results deliver novel uniformly valid tests of inequality and equality restrictions as well as confidence intervals for parameters of interest in the presence of inequality restrictions in both identified and partially identified models.

Our test statistics and proposed inference methods are based on the difference of the minimum of a generalized method of moments (GMM) objective function with and without inequality restrictions. The value of the test statistic increases when more binding constraints are imposed. To ensure uniform validity, critical values are obtained through a bootstrap procedure that acknowledges that some inequalities that do not bind in the sample could have bound under a different draw of the sample. Intuitively, in the bootstrap, we impose the inequalities that are within a region of the boundary that shrinks slightly slower than the convergence rate of the shape restricted estimator. The bootstrap procedure can further be set to ignore inequalities that are outside this shrinking region, leading to pointwise rejection probabilities that equal the nominal level in many applications. As always, uniformity is essential for confidence intervals to be asymptotically valid over a set of unknown parameter values. The resulting inference is powerful in exploiting the large amount of information that inequality restrictions can provide in many cases relevant for applications.

Our tests and confidence intervals remain valid under partial identification. In this setting, the tests and confidence intervals give an accurate and computationally feasible method of doing inference for a subvector of parameters under partial identification. Indeed, these methods have already been used by Torgovitsky (2019) to construct informative confidence intervals for various partially identified state dependence parameters in the presence of unobserved heterogeneity. Also, Kline and Walters (2021) used these methods to test shape constraints implied by a model of callback probabilities for employment applications. By incorporating nuisance parameters into the definition of the parameter space, our results can further be applied to partially identified semi(non)-parametric models defined by conditional moment inequalities.

We demonstrate the usefulness of this approach in an empirical application. Specifically, we conduct inference on the causal effect of childbearing on female labor force participation by relying on the instrumental variables approach of Angrist and Evans (1998). We find that monotonicity of the local average treatment effect (LATE) in education is not rejected by the data and neither is monotonicity and negativity – these restrictions were discussed, but not formally tested, by Angrist and Evans (1998). We further find that imposing these shape restrictions yields narrower confidence intervals for the LATE at different schooling levels. Finally, we obtain similar results for the partially identified average treatment effect (ATE), though the data is less informative about the ATE because of the low proportion of compliers.

The inequalities associated with nonparametric shape restrictions necessitate consideration of parameter spaces that are sufficiently general yet endowed with enough structure to ensure a fruitful asymptotic analysis. An important theoretical insight of this paper is that this simultaneous flexibility and structure is possessed by sets defined by inequality restrictions on Abstract M (AM) spaces; i.e. Banach lattices whose norm obeys a condition discussed in Section 3. We also introduce potentially regularized approximations to the local parameter spaces in order to account for the curvature present in nonlinear constraints. While aspects of our analysis are specific to models defined by conditional moment restrictions, the role of the local parameter space is solely dictated

by the shape restrictions. As such, we expect the insights of the set up here to be applicable to the study of shape restrictions in alternative models as well. The critical values are shown to be uniformly asymptotically valid by developing strong approximations to both the test and bootstrap statistics. Sufficient conditions are provided by adapting the coupling of Koltchinskii (1994). Our coupling arguments and the use of AM spaces are key features of the theory that enable us to show that inference is uniformly valid and that partial identification is permitted.

We illustrate the general applicability of our analysis by obtaining novel uniformly valid inference results in a variety of problems. Specifically, we: (i) Conduct inference about partially identified sets of average equivalent variation and other objects of interest in demand estimation with general heterogeneity and smooth demand functions; (ii) Test and impose shape restrictions on structural functions identified through quantile conditional moment restrictions; and (iii) Impose the Slutsky restrictions to conduct inference in a linear conditional moment restriction model. Additionally, while we do not pursue further examples in detail for conciseness, we note our results may be applied to conduct tests of homogeneity, supermodularity, and economies of scale or scope.

In a small Monte Carlo study, we examine instrumental variables estimation of a nonlinear structural function and consider the power of imposing monotonicity and/or convexity on the structural function. We find rejection frequencies for our test that are conservatively accurate when testing a point null hypothesis about the value or derivative of the structural function. In addition, we find that imposing shape restrictions leads to large increases in power relative to employing an unrestricted estimator, in moderately large samples. Our Monte Carlo analysis further examines the performance of our test in a partially identified parametric IV model with discrete data. In that context, we find that shape restrictions have substantial identifying power and that our test provides valid inference on the value of a function at a point. A similar partially identified IV setting was previously studied by Freyberger and Horowitz (2015), who also provide an inference procedure. However, their procedure is based on limiting distributions that are discontinuous in true parameters leading to nonuniform inference.

Our paper contributes to an extensive literature studying semiparametric and non-parametric models under partial identification (Manski, 2003; Molinari, 2020). When specialized to finite dimensional models, our results enable us to conduct inference on functionals of the identified set in models defined by moment (in)equalities (Canay and Shaikh, 2017; Ho and Rosen, 2017). In that context, our results are complementary to those of Bugni et al. (2017) and Kaido et al. (2019), who provide uniformly valid procedures for subvector inference. Their analysis is focused on convex models and can thus be invalid or conservative when conducting inference on nonlinear functionals or imposing non-convex restrictions – we emphasize, however, that their analysis is also motivated by a different set of models than the ones we consider. Our analysis is further related

to Hong (2017), Santos (2012), Tao (2014), and Chen et al. (2011) who study inference on functionals of potentially partially identified structural functions, but do not allow for shape constraints as we do.

Following the original version of this paper, Zhu (2019) and Fang and Seo (2019) have proposed inference methods for convex restrictions which, while applicable to an important class of problems, rule out inference on nonlinear functionals or tests of certain shape restrictions. Also related is Freyberger and Reeves (2018) who have more recently developed uniform inference for functionals under shape restrictions while imposing point identification. Our paper is of course related to a large literature on shape restrictions; see Samworth and Sen (2018) and Chetverikov et al. (2018) for recent reviews. We highlight here an important literature on linear Gaussian models focused on adaptivity (which we do not establish), but not applicable to many of the models that motivate us (Dumbgen and Spokoiny, 2001; Cai et al., 2013; Armstrong, 2015).

The results here are also highly complementary to Chetverikov and Wilhelm (2017) in providing inference for nonparametric IV under shape restrictions while they showed that imposing monotonicity can greatly improve the convergence rate of the estimator – an observation that additionally motivates our use of test statistics based on shape constrained (instead of unconstrained) estimators. Finally, we note that our results do not lend themselves computationally for the construction of uniform confidence bands for shape restricted functions – a problem that has been addressed in different contexts by Chernozhukov et al. (2009) and Horowitz and Lee (2017).

The remainder of the paper is organized as follows. In Section 2 we show how to implement our tests in a linear instrumental variables model with inequality restrictions under both point and partial identification. Section 2 further illustrates our results by revisiting the analysis of Angrist and Evans (1998). Section 3 contains our main theoretical results, while Section 4 applies them to conduct inference in the heterogenous demand model of Hausman and Newey (2016). Finally, Section 5 contains a brief simulation study. All mathematical derivations are included in a series of appendices; see in particular Appendix A.2 for applications of our general results and Appendix S.6 for coupling results based on Koltchinskii (1994).

2 Application for Linear Instrumental Variables

To fix ideas, we first describe our test in a linear instrumental variables model and illustrate its implementation by revisiting the analysis of Angrist and Evans (1998). We reserve until later the full mathematical framework and focus here on implementation.

2.1 Linear Instrumental Variables

As perhaps the simplest possible example, we first consider a linear instrumental variable model in which $\theta_0 \in \Theta \subseteq \mathbf{R}^{d_\theta}$ is identified through the moment conditions

$$E_P[(Y - W'\theta_0)Z] = 0,$$

where Y is a scalar, W and Z are vectors, and P denotes the distribution of $V \equiv (Y, W, Z)$. We are interested in testing whether θ_0 belongs to a set R characterized by

$$R = \{ \theta \in \mathbf{R}^{d_{\theta}} : F\theta = f, \ G\theta \le g \}, \tag{1}$$

for known matrices F and G and known vectors f and g.

We consider tests based on minimizing the norm of the weighted sample moments as in Sargan (1958) and Hansen (1982). To this end, we define the criterion

$$Q_n(\theta) \equiv \|\hat{\Sigma}_n \{ \frac{1}{n} \sum_{i=1}^n (Y_i - W_i'\theta) Z_i \} \|_2,$$
 (2)

where $\|\cdot\|_2$ is the standard Euclidean norm and $\hat{\Sigma}_n$ is consistent for $(E[ZZ'U^2])^{-1/2}$ for $U \equiv Y - W'\theta_0$. Our analysis then enables us to employ tests based on the statistics

$$I_n(R) \equiv \min_{\theta \in \Theta \cap R} \sqrt{n} Q_n(\theta)$$
 $I_n(\Theta) \equiv \min_{\theta \in \Theta} \sqrt{n} Q_n(\theta);$ (3)

e.g., we may consider a test that rejects for large values of $I_n(R) - I_n(\Theta)$. In what follows it will also be helpful to let $\hat{\theta}_n$ and $\hat{\theta}_n^{\rm u}$ denote the minimizers of Q_n over $\Theta \cap R$ and Θ respectively – i.e. $\hat{\theta}_n$ and $\hat{\theta}_n^{\rm u}$ are the constrained and unconstrained estimators.

We construct critical values by relying on the multiplier bootstrap (Ledoux and Talagrand, 1988). Specifically, let $b \in \{1, ..., B\}$ index a bootstrap draw, $\{\omega_i^b\}_{i=1}^n$ be i.i.d. independent of the data with $\omega_i^b \sim N(0, 1)$, and for any $\theta \in \mathbf{R}^{d_\theta}$ define

$$\hat{\mathbb{W}}_{n}^{b}(\theta) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_{i}^{b} \{ (Y_{i} - W_{i}'\theta) Z_{i} - \frac{1}{n} \sum_{i=1}^{n} (Y_{j} - W_{j}'\theta) Z_{j} \},$$

which is a simulated draw of the true (centered) moment functions.¹ We also require an estimator of the derivative of the moment conditions, and to this end we set

$$\hat{\mathbb{D}}_n[h] \equiv -\frac{1}{n} \sum_{i=1}^n Z_i W_i' h.$$

¹We follow previous work (Lewbel, 1995; Hansen, 1996) in considering Gaussian weights $\{\omega_i\}_{i=1}^n$ because it simplifies the proofs of our main results in Section 3. We expect our analysis extends to alternative specifications for the distribution of $\{\omega_i\}_{i=1}^n$ – e.g., for ω_i following an exponential distribution, which results in a version of the Bayesian bootstrap advocated by Chamberlain and Imbens (2003).

Here, we can think of h as a local parameter, representing the possible values that the random variable $\sqrt{n}\{\hat{\theta}_n - \theta_0\}$ may take (recall $\hat{\theta}_n$ is the minimizer of Q_n over $\Theta \cap R$).

Finally, we need to enforce the inequality constraints in the bootstrap in a way that delivers a uniformly valid critical value. To this end, we account for the variation in $G_j\hat{\theta}_n - g_j$ for each j, where G_j is the j^{th} row of G and g_j the j^{th} coordinate of g. That is, we account for the likelihood that a constraint will bind at the restricted estimator $\hat{\theta}_n$ when computing $I_n(R) = \sqrt{n}Q_n(\hat{\theta}_n)$. For this purpose we introduce the set

$$\hat{V}_n(\hat{\theta}_n, R) \equiv \{ h \in \mathbf{R}^{d_{\theta}} : Fh = 0, \ G_j h \le \sqrt{n} \max\{0, -(r_n + G_j \hat{\theta}_n - g_j) \} \text{ for all } j \},$$
 (4)

where $r_n > 0$ is a slackness parameter whose choice we discuss shortly. The set $\hat{V}_n(\hat{\theta}_n, R)$ can be thought of as a local version of R, approximating the set of values h that could equal $\sqrt{n}\{\hat{\theta}_n - \theta_0\}$. Our bootstrap approximations to $I_n(R)$ and $I_n(\Theta)$ are then

$$\hat{U}_{n}^{b}(R) \equiv \min_{h \in \hat{V}_{n}(\hat{\theta}_{n}, R)} \|\hat{\Sigma}_{n}\{\hat{\mathbb{W}}_{n}^{b}(\hat{\theta}_{n}) + \hat{\mathbb{D}}_{n}[h]\}\|_{2}$$
 (5)

$$\hat{U}_n^b(\Theta) \equiv \min_{h \in \mathbf{R}^{d_\theta}} \|\hat{\Sigma}_n \{\hat{\mathbb{W}}_n^b(\hat{\theta}_n^{\mathbf{u}}) + \hat{\mathbb{D}}_n[h]\}\|_2.$$
 (6)

Thus, we may obtain a level α test by rejecting whenever the test statistic $I_n(R) - I_n(\Theta)$ exceeds the $1 - \alpha$ quantile of $\hat{U}_n^b(R) - \hat{U}_n^b(\Theta)$ across the B bootstrap draws. The main assumption required for the test to be asymptotically valid is that θ_0 be strongly identified – i.e. θ_0 can be consistently estimated uniformly in P.

The critical value depends on the choice of r_n . When applied to linear instrumental variables, our asymptotic theory requires that r_n tend to zero slower than the convergence rate of the restricted estimator, which is $1/\sqrt{n}$. Heuristically, when r_n tends to zero any constraint that is not binding at θ_0 will also not be binding in the bootstrap with probability approaching one (under pointwise in P asymptotics). Consequently inference is not asymptotically conservative for a fixed data generating process. Setting $r_n \to 0$ while satisfying $r_n \sqrt{n} \to \infty$ leads to uniformly valid inference with constraints only being conservatively enforced when they are within order $1/\sqrt{n}$ of binding at θ_0 . Setting $r_n = +\infty$ is always theoretically valid, but it may be conservative and result in a loss of power. Other, smaller choices of r_n can lead to smaller, valid critical values and so may result in more powerful tests and tighter confidence intervals than $r_n = +\infty$.

Intuitively, r_n is meant to quantify the sampling uncertainty in $G\{\hat{\theta}_n - \theta_0\}$. Since the distribution of $\hat{\theta}_n$ cannot be uniformly consistently estimated, we suggest linking r_n to the degree of sampling uncertainty in $G\{\hat{\theta}_n^{\rm u} - \theta_0\}$ instead. Specifically, for $\hat{\theta}_n^{\rm u*}$ a "bootstrap" analogue of $\hat{\theta}_n^{\rm u}$ and some $\gamma_n \to 0$, we recommend setting r_n to satisfy

$$P(\max_{i} G_{j}\{\hat{\theta}_{n}^{\mathbf{u}} - \hat{\theta}_{n}^{\mathbf{u}\star}\} \le r_{n}|\mathrm{Data}) = 1 - \gamma_{n}.$$
 (7)

This approach changes the problem of selecting r_n into the problem of selecting γ_n . However, γ_n is more interpretable: If we employed $\hat{V}_n(\hat{\theta}_n^u, R)$ in place of $\hat{V}_n(\hat{\theta}_n, R)$ in (5), then a Bonferroni bound implies that the test that rejects whenever $I_n(R) - I_n(\Theta)$ exceeds the $1-\alpha$ quantile of $\hat{U}_n^b(R) - \hat{U}_n^b(\Theta)$ has asymptotic size at most $\alpha + \gamma_n$ even if γ_n is fixed with n.² In particular, if we employed the $1-\alpha+\gamma_n$ quantile of $\hat{U}_n^b(R) - \hat{U}_n^b(\Theta)$ as a critical value instead, then the resulting test would have asymptotic size at most α (even if γ_n is fixed). In simulations, however, we find the described bound to be pessimistic in that, when setting r_n according to (7), our test has a rejection probability under the null hypothesis of at most α for a wide range of choices of γ_n .

Remark 2.1. Our results may be employed to obtain confidence regions for a coordinate of θ_0 while imposing restrictions of the form $G\theta_0 \leq g$ on θ_0 (e.g., sign or monotonicity restrictions on $w \mapsto w'\theta_0$). For example, for θ_k the k^{th} coordinate of $\theta \in \mathbf{R}^{d_{\theta}}$ let

$$R_{\lambda} = \{ \theta \in \mathbf{R}^{d_{\theta}} : \theta_k = \lambda, \ G\theta \le g \},$$

which is a special case of (1). We may then obtain a confidence region for the k^{th} coordinate of θ_0 by conducting test inversion in λ employing the test based on $I_n(R_{\lambda}) - I_n(\Theta)$; see also Remark 3.1 for alternative constructions based on our analysis.

Remark 2.2. In certain applications it may be desirable to studentize the constraints in our bootstrap approximation – i.e. replace G_j and g_j by $G_j/\hat{\sigma}_j$ and $g_j/\hat{\sigma}_j$ everywhere in (4) (and in (7) if employed). In the empirical analysis below we proceed in this manner by setting $\hat{\sigma}_j^2$ to be an estimate of the asymptotic variance of $\sqrt{n}G_j\{\hat{\theta}_n^{\rm u}-\theta_0\}$.

2.1.1 Fertility and Labor Supply: LATE

We illustrate the preceding discussion by revisiting the study by Angrist and Evans (1998) on the causal effect of childbearing on female labor force participation. Like Angrist and Evans (1998), we employ the 1980 Census Public Use Micro Sample restricted to mothers aged 21-35 with at least two children, and set: (i) $D \in \{0,1\}$ to indicate whether a mother has more than two children (the treatment); (ii) $Y \in \{0,1\}$ to indicate whether a mother is employed (the outcome of interest); and (iii) $Z \in \{0,1\}$ to indicate whether the first two children are of the same sex (the instrument). We further adopt the heterogeneous treatment effects model of Imbens and Angrist (1994) and let Y_d denote the potential outcome under treatment status $d \in \{0,1\}$ and employ "C," "NT," and "AT" to denote compliers, never takers, and always takers.

Angrist and Evans (1998) document that the impact of childbearing on labor force participation depends on observable characteristics. In particular, their two stage least

²While we may replace $\hat{V}_n(\hat{\theta}_n, R)$ with $\hat{V}_n(\hat{\theta}_n^{\mathrm{u}}, R)$ in identified models, in partially identified models we employ $\hat{V}_n(\hat{\theta}_n, R)$ due to the identified set potentially not being a subset of R under the null hypothesis.

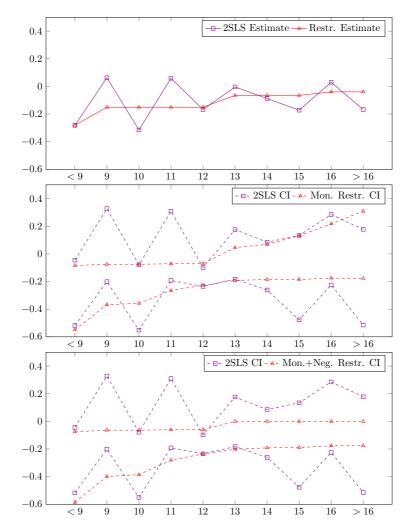


Figure 1: First Panel: Unconstrained and shape restricted LATE estimates (imposing monotonicity or monotonicity and negativity yield the same estimates). Second and Third Panels: 95% Confidence intervals for LATE at different education levels.

squares (2SLS) estimates suggest a negative impact of child bearing on labor force participation across different levels of schooling, but that the magnitude of the impact decreases with schooling – a phenomenon that may reflect that more educated mothers have a stronger attachment to the labor force. To formally examine this claim, we introduce dummy variables S for each year of schooling between 9 and 16 and for the categories "less than 9" and "more than 16." Defining the local average treatment effects

$$LATE(S) \equiv E[Y_1 - Y_0|S, C]$$

we then test whether: (i) LATE(·) is increasing in schooling, and (ii) LATE(·) is increasing in schooling and nonpositive. Both hypotheses fall within the framework of the preceding section because LATE(·) is identified through linear moment restrictions and the hypothesized restrictions are linear in LATE(·). Employing five thousand bootstrap

replications and setting $r_n = +\infty$ or r_n as suggested in (7) with $\gamma_n = 0.05$ yields in this case equal *p*-values that fail to reject either null hypothesis. The *p*-values for LATE(·) being nondecreasing is 0.21 and for it being nondecreasing and nonpositive is 0.394.

In Figure 1 we study the values of LATE(S) at different schooling levels S. The first panel displays the unconstrained 2SLS estimates and their monotonicity restricted counterparts – the latter are negative and hence additionally demanding nonpositivity does not change the estimates. Unfortunately, two sided confidence regions based on the (pointwise in P) asymptotic distribution of the shape-restricted 2SLS estimator can asymptotically undercover the true parameter. In the second panel of Figure 1 we instead proceed as in Remark 2.1 to obtain 95% confidence intervals while imposing monotonicity and again selecting r_n by setting $\gamma_n = 0.05$ in (7). Employing the monotonicity restriction in this manner yields confidence intervals that are sometimes substantially shorter than their 2SLS counterparts. Notably, we observe lower upper ends for the restricted confidence intervals at the lower education levels and higher lower ends at higher education levels. As shown in the third panel of Figure 1, additionally imposing that LATE(·) be nonpositive mostly reduces the upper bound of our confidence intervals at higher education levels.

2.2 Partial Identification

We next illustrate the implementation of our results in a partially identified setting. With an eye towards extending the preceding empirical analysis to study average treatment effects (ATEs), we maintain that the parameter of interest $\theta_0 \in \mathbf{R}^{d_\theta}$ satisfies

$$E_P[(Y - W'\theta_0)Z] = 0, (8)$$

but no longer assume θ_0 is identified by (8). Instead, we define the identified set

$$\Theta_0 \equiv \{ \theta \in \Theta : E_P[(Y - W'\theta)Z] = 0 \}$$
(9)

and consider the problem of testing whether the intersection of Θ_0 and R is nonempty (i.e. $\Theta_0 \cap R \neq \emptyset$). Such hypotheses can be employed, for instance, to build confidence regions for functionals of the identified set; see Remark 2.3 below. We also now set

$$R = \{ \theta \in \mathbf{R}^{d_{\theta}} : \Upsilon_F(\theta) = 0, \ G\theta \le g \}, \tag{10}$$

for Υ_F a known possibly nonlinear function – e.g., $\Upsilon_F(\theta) = F\theta - f$ recovers (1).

We continue to rely on the statistics $I_n(R)$ and $I_n(\Theta)$ (as in (3)) for inference. However, since in many settings in which θ_0 fails to be identified by (8) we will have that the dimension of Z is smaller than that of W, in what follows we assume for ease of exposition that $I_n(\Theta) = 0$ (almost surely); see Section 3.2.2 for a general discussion. Another distinction relative to Section 2.1 is that the choice of $\hat{\Sigma}_n$ (as in (2)) may need to be modified in settings in which $U \equiv Y - W'\theta_0$ cannot be consistently estimated due to θ_0 being partially identified. In such instances we may, for example, set

$$\hat{\Sigma}_n \equiv (\frac{1}{n} \sum_{i=1}^n Z_i Z_i' (Y_i - W_i' \hat{\theta}_n^{\mathrm{u}})^2)^{-1/2},$$

where we now interpret $\hat{\theta}_n^{\mathbf{u}}$ as the minimum norm minimizer of Q_n over Θ . While the choice of $\hat{\Sigma}_n$ has an impact on how local power is directed, we note that the test has correct asymptotic size provided $\hat{\Sigma}_n$ converges in probability to a non-stochastic limit.

Our bootstrap procedure requires two modifications relative to our preceding discussion. First, because in (10) we consider nonlinear equality constraints, we now set

$$\hat{V}_n(\theta, R) \equiv \{ h \in \mathbf{R}^{d_{\theta}} : \Upsilon_F(\theta + \frac{h}{\sqrt{n}}) = 0, \ G_j h \le \sqrt{n} \max\{0, -(r_n + G_j \theta - g_j)\} \text{ for all } j \}$$

(notice that if $\Upsilon_F(\theta) = F\theta - f$, then we recover (4)). A distinction with Section 2.1 is that if one aims to employ (7) to select r_n , then an alternative to an unrestricted estimator $\hat{\theta}_n^{\text{u}}$ may be necessary; see Section 2.2.1 for an example. Second, our bootstrap approximation employs an estimator $\hat{\Theta}_n^{\text{r}}$ for $\Theta_0 \cap R$. To this end, we set

$$\hat{\Theta}_{n}^{\mathrm{r}} \equiv \{ \theta \in \Theta \cap R : Q_{n}(\theta) \leq \inf_{\theta \in \Theta \cap R} Q_{n}(\theta) + \tau_{n} \}$$

where $\tau_n \geq 0$ is a bandwidth whose choice we discuss shortly – i.e. $\hat{\Theta}_n^{\rm r}$ is the set of "near" minimizers of Q_n over $\Theta \cap R$. Our bootstrap approximation to $I_n(R)$ then equals

$$\hat{U}_n^b(R) \equiv \min_{\theta \in \hat{\Theta}_n^r} \min_{h \in \hat{V}_n(\theta, R)} \|\hat{\Sigma}_n \{\hat{\mathbb{W}}_n^b(\theta) + \hat{\mathbb{D}}_n[h]\}\|_2.$$

Thus, to obtain a level α test we reject the null hypothesis whenever $I_n(R)$ exceeds the $1-\alpha$ quantile of $\hat{U}_n^b(R)$ across bootstrap draws. Paralleling Section 2.1, a principal assumption for the test to be asymptotically valid is that Θ_0 be strongly identified.

When specialized to the current setting, our asymptotic theory requires that τ_n tend to zero. It is theoretically valid to set $\tau_n = 0$, which simplifies the computation of our bootstrap statistic – e.g., let $\hat{\Theta}_n^{\rm r} = \{\hat{\theta}_n\}$ for any $\hat{\theta}_n$ minimizing Q_n over $\Theta \cap R$ to recover (5). However, setting $\tau_n = 0$ can result in lower power in applications for which the corresponding $\hat{\Theta}_n^{\rm r}$ is not consistent for $\Theta_0 \cap R$ (in the Hausdorff metric) – to ensure consistency, τ_n must in addition satisfy $\tau_n \sqrt{n} \to \infty$. For applications in which it is desirable to set $\tau_n > 0$, we propose a procedure inspired by Romano and Shaikh (2010).

Specifically, for any set $K \subseteq \Theta \cap R$ we define the quantile $\hat{q}_n(K)$ according to

$$P(\sup_{\theta \in K} \|\hat{\Sigma}_n \hat{\mathbb{W}}_n(\theta)\|_2 \le \hat{q}_n(K)|\text{Data}) = 1 - \gamma_n$$

where $\gamma_n \in (0,1)$. Letting $S_1 \equiv \Theta \cap R$, we then inductively define $S_{j+1} \equiv \{\theta \in \Theta \cap R : \sqrt{n}Q_n(\theta) \leq \hat{q}_n(S_j)\}$ noting that by construction $S_{j+1} \subseteq S_j$. To select τ_n , we proceed inductively until we find $S_j = \emptyset$, in which case we set $\tau_n = 0$, or $S_{j+1} = S_j \neq \emptyset$, in which case we set $\tau_n = \hat{q}_n(S_j)$. Heuristically, under such a choice of τ_n , the set $\hat{\Theta}_n^r$ may be interpreted as a $1 - \gamma_n$ confidence region for $\Theta_0 \cap R$. While power considerations suggest setting γ_n to tend to zero, for practical considerations we suggest simply setting $1 - \gamma_n$ to be a high quantile fixed with n (e.g., $1 - \gamma_n = 0.8$).

Remark 2.3. The introduced test can be employed to obtain confidence regions for functionals of the identified set satisfying the coverage requirement advocated by Imbens and Manski (2004). Specifically, given a functional $\Upsilon_F: \Theta \to \mathbf{R}$ we may set

$$R_{\lambda} = \{ \theta \in \mathbf{R}^{d_{\theta}} : \Upsilon_F(\theta) = \lambda, G\theta \le g \}$$

and obtain the desired confidence region by conducting test inversion in λ of the null hypothesis that the set $\Theta_0 \cap R_{\lambda}$ is not empty.

2.2.1 Fertility and Labor Supply: ATE

Returning to our analysis of the causal impact of fertility on female labor force participation, we next turn to estimating the average treatment effect at different education levels S (denoted ATE(S)). Following the literature, we decompose ATE(S) into

$$LATE(S)P(C|S) + E[Y_1 - Y_0|S, AT]P(AT|S) + E[Y_1 - Y_0|NT, S]P(NT|S),$$
(11)

where recall C, AT, and NT denote "compliers," "always takers," and "never takers." With the exception of $E[Y_0|AT, S]$ and $E[Y_1|NT, S]$, all terms in (11) can be identified through linear moment restrictions.³ Because S has ten support points, we obtain sixty moments and eighty parameters so that $I_n(\Theta) = 0$ almost surely.

Following our analysis of LATE(S) we conduct inference on ATE(S) under three increasingly stringent set of (linear) restrictions: (i) The logical bounds implied by $Y_d \in \{0,1\}$; (ii) Adding to (i) that the average treatment effect be increasing in schooling among all types (i.e. C, NT, and AT); (iii) Adding to (ii) that average treatment effects be nonpositive for all levels of education and types. Figure 2 reports the resulting

³Technically, the moment equations have the structure $E_P[(Y_j - W'_j\theta_0)Z_j] = 0$ with the instruments Z_j not being common across all $1 \leq j \leq \mathcal{J}$ equations. The bootstrap implementation in this case, formally studied in Section 3, is identical with only $\hat{\mathbb{W}}_n$ and $\hat{\mathbb{D}}_n$ being modified in the natural way.

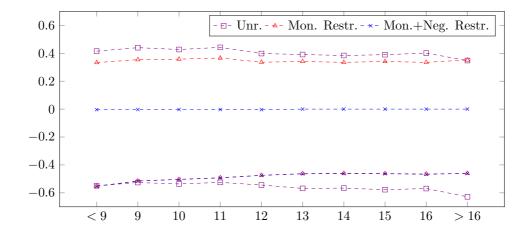


Figure 2: 95% Confidence intervals for ATE at different education levels. "Unr." uses bounds implied by $Y_d \in \{0,1\}$; "Mon. Restr." adds that average treatment effects be increasing in education for all types; "Mon.+Neg. Restr." also requires they be negative.

95% confidence regions obtained through the approach described in Remark 2.3 – here, the restriction $G\theta \leq g$ imposes the described shape constraints while the nonlinear restriction $\Upsilon_F(\theta) = 0$ corresponds to imposing a hypothesized value for ATE(S) through (11). In our bootstrap approximation, we set $\tau_n = 0$ and selected r_n according to (7) with $\gamma_n = 0.05$ and where, when necessary, we used the distribution of estimators of identified parameters for their partially identified counterparts.⁴ We do not report estimates of the identified sets for ATE(S) as they are very close to the obtained confidence intervals: On average the bounds of the confidence intervals exceed the bounds of estimates of the identified set by 0.011. Nonetheless, the unrestricted confidence intervals are large as the estimates for the identified set are themselves large – a result driven by the low proportion of compliers (5% on average across schooling levels). Imposing monotonicity across types carries identifying information on the upper end of the identified set at low levels of education and on the lower end of the identified set at high levels of education. Additionally imposing nonpositivity sharpens the upper bound of the identified set at all schooling levels. The resulting confidence regions sign ATE(S) at all education levels (weakly) smaller than 12 as strictly negative, though very close to zero.

Finally, as a preview of our general analysis in Section 3, in Table 1 we employ the same shape restrictions to report estimates and 95% confidence intervals for the identified sets of the average treatment effects for: High School Dropouts (edu \in [9,12)), College Dropouts (edu \in [13,15)), College Graduates (edu \geq 16) and the overall average treatment effect. These confidence regions are obtained through test inversion after noting that a hypothesized value for the average treatment effect of a subgroup can be written as a nonlinear moment restriction in θ_0 through (11) – nonlinear moment restrictions fall within our general framework but outside the scope of Section 2.2. Overall

⁴E.g., for the constraint $E[Y_1|NT, S] \leq 1$ we substituted the corresponding $G_j\{\hat{\theta}_n^{\mathbf{u}} - \hat{\theta}_n^{\mathbf{u}^*}\}$ term in (7) with a mean zero normal distribution with the variance of the estimator for $E[Y_0|NT, S]$.

	Unrestricted		Mon.	Restr.	Mon.+Neg Restr.		
Subgroup	Estimate	95% CI	Estimate	95% CI	Estimate	95% CI	
HS Drop	[-0.520, 0.426]	[-0.526, 0.432]	[-0.489,0.346]	[-0.500, 0.356]	[-0.489,-0.008]	[-0.501,-0.003]	
Coll. Drop	[-0.561, 0.380]	[-0.566, 0.385]	[-0.447, 0.325]	[-0.460, 0.337]	[-0.447, -0.004]	[-0.462, 0.000]	
Coll. Grad	[-0.579, 0.375]	[-0.586, 0.382]	[-0.446, 0.328]	[-0.462, 0.339]	[-0.446,-0.002]	[-0.464, 0.000]	
All	[-0.545, 0.395]	[-0.547,0.398]	[-0.467, 0.328]	[-0.477, 0.338]	[-0.467,-0.008]	[-0.478,-0.003]	

Table 1: Point Estimates and 95% confidence intervals for the average treatment effect at different groups defined by schooling levels under different shape restrictions.

the impact of imposing shape restrictions parallels the results in Figure 2.

3 General Analysis

We next develop a general inferential framework that encompasses the tests discussed in Section 2. The class of models we consider are those in which the parameter of interest $\theta_0 \in \Theta$ satisfies a finite number \mathcal{J} of conditional moment restrictions

$$E_P[\rho_{\jmath}(X,\theta_0)|Z_{\jmath}] = 0 \text{ for } 1 \leq \jmath \leq \mathcal{J}$$

with $\rho_{\jmath}: \mathbf{X} \times \Theta \to \mathbf{R}$, $X \in \mathbf{X}$, and $Z_{\jmath} \in \mathbf{Z}_{\jmath}$. For notational simplicity, we also let $Z \equiv (Z_1, \dots, Z_{\mathcal{J}})$ and $V \equiv (X, Z)$ with $V \sim P \in \mathbf{P}$. In some of the applications that motivate us, the parameter θ_0 is not identified. We therefore define the identified set

$$\Theta_0 \equiv \{\theta \in \Theta : E_P[\rho_j(X,\theta)|Z_j] = 0 \text{ for } 1 \le j \le \mathcal{J}\}$$

and employ it as the basis of our statistical analysis – we emphasize that Θ_0 depends on P, but leave such dependence implicit to simplify notation. For a set R of parameters satisfying a conjectured restriction, we develop a test for the hypothesis

$$H_0: \Theta_0 \cap R \neq \emptyset$$
 $H_1: \Theta_0 \cap R = \emptyset;$ (12)

i.e. we devise a test of whether at least one element of the identified set satisfies the posited constraint. In what follows, we denote the set of distributions $P \in \mathbf{P}$ satisfying the null hypothesis in (12) by \mathbf{P}_0 . We also note that in an identified model, a test of (12) is equivalent to a test of whether θ_0 itself satisfies the hypothesized constraint.

The defining elements determining the type of applications encompassed by (12) are the choices of Θ and R. In imposing restrictions on Θ and R we therefore aim to allow for a general framework while simultaneously ensuring enough structure for a fruitful asymptotic analysis. To this end, we require Θ to be a subset of a complete vector space

B with norm $\|\cdot\|_{\mathbf{B}}$ (i.e. $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a Banach space) and consider sets R satisfying

$$R = \{ \theta \in \mathbf{B} : \Upsilon_F(\theta) = 0 \text{ and } \Upsilon_G(\theta) \le 0 \}, \tag{13}$$

where $\Upsilon_F : \mathbf{B} \to \mathbf{F}$ and $\Upsilon_G : \mathbf{B} \to \mathbf{G}$ are known maps. Our first assumption formalizes the basic structure of the hypothesis testing problem we study.

Assumption 3.1. (i) $\{V_i\}_{i=1}^n$ is i.i.d. with $V \sim P \in \mathbf{P}$; (ii) $\Theta \subseteq \mathbf{B}$, where $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a Banach space; (iii) $\Upsilon_F : \mathbf{B} \to \mathbf{F}$ and $\Upsilon_G : \mathbf{B} \to \mathbf{G}$, where $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ is a Banach space and $(\mathbf{G}, \|\cdot\|_{\mathbf{G}})$ is an AM space with order unit $\mathbf{1}_{\mathbf{G}}$.

Through Assumption 3.1(i) we focus on the i.i.d. setting, though extensions to other sampling frameworks are feasible. Assumption 3.1(ii) allows us to address parametric, semiparametric, and nonparametric models, while Assumption 3.1(iii) allows Υ_F to impose both finite dimensional or infinite dimensional equality restrictions. Assumption 3.1(iii) further requires that Υ_G take values in an AM space \mathbf{G} – we provide an overview of AM spaces in the supplemental appendix. Heuristically, the key properties of \mathbf{G} are: (i) \mathbf{G} is a vector space equipped with a partial order " \leq "; (ii) The partial order and the vector space operations interact in the same manner they do on \mathbf{R} (e.g. if $\theta_1 \leq \theta_2$, then $\theta_1 + \theta_3 \leq \theta_2 + \theta_3$); and (iii) The order unit $\mathbf{1}_{\mathbf{G}} \in \mathbf{G}$ is an element such that for any $\theta \in \mathbf{G}$ there exists a scalar $\lambda > 0$ satisfying $|\theta| \leq \lambda \mathbf{1}_{\mathbf{G}}$ (e.g. when $\mathbf{G} = \mathbf{R}^d$ we may set $\mathbf{1}_{\mathbf{G}} \equiv (1,\ldots,1)' \in \mathbf{R}^d$). These properties of an AM space will prove instrumental in our analysis. In particular, the order unit $\mathbf{1}_{\mathbf{G}}$ will provide a crucial link between the partial order " \leq ", the norm $\|\cdot\|_{\mathbf{G}}$, and (through smoothness of Υ_G) allow us to leverage a rate of convergence in \mathbf{B} to build a suitable sample analogue to the local parameter space.

3.1 Main Results

Our analysis centers around a statistic $I_n(R)$ that constitutes a "building block" for different tests of (12) – e.g., it may be employed to implement a generalization of the Jtest of Sargan (1958) and Hansen (1982) or the incremental J-test of Eichenbaum et al. (1988). In this section we first introduce $I_n(R)$, obtain an approximation to its finite sample distribution, and devise a bootstrap procedure for estimating its quantiles. Together, these results allow us to establish the asymptotic validity of different tests.

3.1.1 The Building Block

We first introduce the statistic $I_n(R)$ that we employ to build different tests. To this end, for each instrument Z_j we consider transformations $\{q_{k,j}\}_{k=1}^{k_{n,j}}$ and let $q_j^{k_{n,j}}(z_j) \equiv (q_{1,j}(z_j), \ldots, q_{k_{n,j},j}(z_j))'$. Recalling that $Z \equiv (Z_1, \ldots, Z_{\mathcal{J}})$, we further set $k_n \equiv \sum_{j=1}^{\mathcal{J}} k_{n,j}$,

$$q^{k_n}(z) \equiv (q_1^{k_{n,1}}(z_1)', \dots, q_{\mathcal{J}}^{k_{n,\mathcal{J}}}(z_{\mathcal{J}})')', \ \rho(x,\theta) \equiv (\rho_1(x,\theta), \dots, \rho_{\mathcal{J}}(x,\theta))',$$
 and let

$$\rho(X_i, \theta) * q^{k_n}(Z_i) \equiv \begin{pmatrix} \rho_1(X_i, \theta) q_1^{k_{n,1}}(Z_{i,1}) \\ \vdots \\ \rho_{\mathcal{J}}(X_i, \theta) q_{\mathcal{J}}^{k_{n,\mathcal{J}}}(Z_{i,\mathcal{J}}) \end{pmatrix};$$

i.e. for each θ we take the product of each "residual" $\rho_{j}(X,\theta)$ with the transformations of its respective instrument Z_{j} . For a $k_{n} \times k_{n}$ matrix $\hat{\Sigma}_{n}$, we then define

$$Q_n(\theta) \equiv \left\| \frac{1}{n} \sum_{i=1}^n \rho(X_i, \theta) * q^{k_n}(Z_i) \right\|_{\hat{\Sigma}_n, p},$$

where $||a||_{\hat{\Sigma}_n,p} \equiv ||\hat{\Sigma}_n a||_p$ and $||\cdot||_p$ is the *p*-norm on \mathbf{R}^{k_n} for any $p \geq 2$ – i.e. $||a||_p \equiv (\sum_{i=1}^d |a^{(i)}|^p)^{1/p}$ for any $a \equiv (a^{(1)}, \dots, a^{(d)})' \in \mathbf{R}^d$. Letting $\Theta_n \cap R$ be a finite dimensional subset of $\Theta \cap R$ that grows dense in $\Theta \cap R$ (Chen, 2007), we then define $I_n(R)$ to equal

$$I_n(R) \equiv \inf_{\theta \in \Theta_n \cap R} \sqrt{n} Q_n(\theta).$$

We note that setting p = 2 is often computationally attractive. However, we allow for p > 2 because higher values of p enable us to establish distributional approximations under weaker conditions on the number of unconditional moments k_n .

Heuristically, $\sqrt{n}Q_n$ should diverge to infinity when evaluated at any $\theta \notin \Theta_0$ and remain "stable" when evaluated at a $\theta \in \Theta_0$. Thus, examining the minimum of $\sqrt{n}Q_n$ over R should reveal whether there is a θ that simultaneously makes $\sqrt{n}Q_n(\theta)$ "stable" $(\theta \in \Theta_0)$ and satisfies the conjectured restriction $(\theta \in R)$. This intuition suggests $I_n(R)$ may be employed as a test statistic that is similar in spirit to the J-statistic of Hansen (1982). Alternatively, we may build a test by considering the recentered test statistic

$$I_n(R) - I_n(\Theta),$$

which aims power in a different direction than $I_n(R)$ (Chen and Santos, 2018). Conceptually, it is important to note that $I_n(\Theta)$ is a special case of $I_n(R)$ (i.e. set $R = \Theta$). We refer to $I_n(R)$ as a "building block" in the sense that, together with closely related variants like $I_n(\Theta)$, it may be employed to obtain a variety of different tests.

3.1.2 Strong Approximation

We first obtain a strong approximation to statistics of the form $I_n(R)$. Before proceeding, we introduce some additional notation. First, we define the class

$$\mathcal{F}_n \equiv \{ \rho_{\jmath}(\cdot, \theta) : \theta \in \Theta_n \cap R \text{ and } 1 < \jmath < \mathcal{J} \}. \tag{14}$$

The "size" of \mathcal{F}_n plays a crucial role, and we control it through the bracketing integral

$$J_{[]}(\delta, \mathcal{F}_n, \|\cdot\|_{P,2}) \equiv \int_0^\delta \sqrt{1 + \log N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2})} d\epsilon,$$

where $||f||_{P,2}^2 \equiv E_P[f^2(V)]$ and $N_{[]}(\epsilon, \mathcal{F}_n, ||\cdot||_{P,2})$ is the smallest number of ϵ -brackets (under $||\cdot||_{P,2}$) required to cover \mathcal{F}_n . Finally, we denote the empirical process by

$$\mathbb{G}_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \{ \rho(X_i, \theta) * q^{k_n}(Z_i) - E_P[\rho(X, \theta) * q^{k_n}(Z)] \}.$$

Our next assumptions imposes requirements on $\Theta_n \cap R$ and the transformation $q^{k_n}(Z)$.

Assumption 3.2. (i) $\max_{1 \leq j \leq \mathcal{J}} \max_{1 \leq k \leq k_{n,j}} \|q_{k,j}\|_{\infty} \leq B_n$ with $B_n \geq 1$; (ii) The eigenvalues of $E_P[q_j^{k_{n,j}}(Z_j)q_j^{k_{n,j}}(Z_j)']$ are bounded uniformly in $k_{n,j}$ and $P \in \mathbf{P}$; (iii) \mathcal{F}_n has envelope F_n , $\sup_{P \in \mathbf{P}} \|F_n\|_{P,2} < \infty$, and $\sup_{P \in \mathbf{P}} J_{[]}(\|F_n\|_{P,2}, \mathcal{F}_n, \|\cdot\|_{P,2}) \leq J_n$ with $J_n < \infty$.

Assumption 3.3. (i) $\sup_{\theta \in \Theta_n \cap R} \|\mathbb{G}_n(\theta) - \mathbb{W}_P(\theta)\|_p = o_P(a_n)$ uniformly in $P \in \mathbf{P}$ for some $a_n = o(1)$ and Gaussian \mathbb{W}_P satisfying $E[\mathbb{W}_P(\theta)] = 0$ and $\operatorname{Cov}\{\mathbb{W}_P(\theta), \mathbb{W}_P(\theta')\} = \operatorname{Cov}_P\{\mathbb{G}_n(\theta), \mathbb{G}_n(\theta')\}$; (ii) There is a norm $\|\cdot\|_{\mathbf{E}}$, $\kappa_\rho > 0$, and $K_\rho < \infty$ such that $E_P[\|\rho(X, \theta_1) - \rho(X, \theta_2)\|_2^2] \le K_\rho^2 \|\theta_1 - \theta_2\|_{\mathbf{E}}^{2\kappa_\rho}$ for all $\theta_1, \theta_2 \in \Theta_n \cap R$ and $P \in \mathbf{P}$.

Assumptions 3.2(i)(ii) impose standard requirements on the transformations q^{k_n} – e.g., Assumption 3.2(i) holds with $B_n=1$ for trigonometric series and $B_n \approx \sqrt{k_n}$ for normalized B-splines. Assumption 3.2(iii) controls the "size" of \mathcal{F}_n . We allow J_n to depend on n to accommodate non-compact parameter spaces (Chen and Pouzo, 2012, 2015). Assumption 3.3(i) requires that the empirical process be approximately Gaussian. The sequence $\{a_n\}_{n=1}^{\infty}$ denotes a bound on the rate of coupling, which in turn characterizes the rate of convergence of our strong approximation. In the appendix, we verify Assumption 3.3(i) by relying on existing results (Yurinskii, 1977; Zhai, 2018) or a novel extension of Koltchinskii (1994). Assumption 3.3(ii) imposes a mild restriction on the moment functions that ensures \mathbb{W}_P is equicontinuous with respect to $\|\cdot\|_{\mathbf{E}}$.

In establishing our strong approximation to $I_n(R)$, it is helpful to derive the rate of convergence of the minimizer of Q_n over $\Theta_n \cap R$. To this end, we follow the literature on set estimation (Chernozhukov et al., 2007; Beresteanu and Molinari, 2008; Santos, 2011; Kaido and Santos, 2014) and for any sets A and B we define

$$\overrightarrow{d}_{H}(A, B, \|\cdot\|_{\mathbf{E}}) \equiv \sup_{a \in A} \inf_{b \in B} \|a - b\|_{\mathbf{E}},$$

which is known as the directed Hausdorff distance. For each $\theta \in \Theta \cap R$, we further let

 $\Pi_n \theta$ denote its approximation on $\Theta_n \cap R$ and denote the approximation to $\Theta_0 \cap R$ by

$$\Theta_{0n}^{\mathbf{r}} \equiv \{ \Pi_n \theta : \theta \in \Theta_0 \cap R \}. \tag{15}$$

Our next assumption enables us to obtain a rate of convergence (under $\|\cdot\|_{\mathbf{E}}$) to $\Theta_{0n}^{\mathbf{r}}$.

Assumption 3.4. There are $V_n(P) \subseteq \Theta_n \cap R$ and a sequence constants $\{v_n\}$ with $0 < v_n^{-1} = O(1)$ such that (i) For any $\theta \in V_n(P)$ it holds that

$$\nu_n^{-1} \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \le \sup_{\tilde{\theta} \in \Theta_{0n}^r} \|E_P[(\rho(X, \theta) - \rho(X, \tilde{\theta})) * q^{k_n}(Z)]\|_{\Sigma_P, p};$$

(ii) There is a $\hat{\theta}_n \in \mathcal{V}_n(P)$ satisfying $Q_n(\hat{\theta}_n) \leq \inf_{\theta \in \Theta_n \cap R} Q_n(\theta) + o(a_n/\sqrt{n})$ with probability tending to one uniformly in $P \in \mathbf{P}_0$.

Assumption 3.4(ii) requires that an approximate minimum of Q_n over $\Theta_n \cap R$ be attained at a point $\hat{\theta}_n$ in a set $\mathcal{V}_n(P)$ with high probability. Typically, $\mathcal{V}_n(P)$ may be taken to equal the entire sieve in convex models, or it may be taken to equal a local neighborhood of Θ_{0n}^r after establishing the consistency of $\hat{\theta}_n$ through standard arguments; see, e.g., Lemma S.1.1 in the appendix. Assumption 3.4(i) introduces a local identification condition on $\mathcal{V}_n(P)$ by requiring that the moments "change" at a rate ν_n^{-1} as θ moves away from Θ_{0n}^r . The parameter ν_n^{-1} , which implicitly depends on k_n and the choice of sieve $\Theta_n \cap R$, is conceptually related to sieve measure of ill-posedness (Blundell et al., 2007).

By employing Assumption 3.4, we are able to show that with arbitrarily high probability, $\hat{\theta}_n$ is contained in a $\|\cdot\|_{\mathbf{E}}$ -neighborhood of Θ_{0n}^r that shrinks at a rate

$$\mathcal{R}_n \equiv \nu_n \{ \frac{k_n^{1/p} \sqrt{\log(1 + k_n)} J_n B_n}{\sqrt{n}} \}, \tag{16}$$

where recall B_n and J_n where introduced in Assumption 3.2. Under assumptions on the (Hausdorff) distance between $\Theta_{0n}^{\rm r}$ and $\Theta_0 \cap R$, the triangle inequality can yield a rate of convergence of $\hat{\theta}_n$ to $\Theta_0 \cap R$. Heuristically, we focus on convergence to $\Theta_{0n}^{\rm r}$ (instead of $\Theta_0 \cap R$) because our strong approximation will rely on undersmoothing.

In our final assumptions, we follow the literature and accommodate non-differentiable moment functions by requiring that their conditional expectations be differentiable (Chen and Pouzo, 2009, 2012). Specifically, for each $1 \le j \le \mathcal{J}$ and $\theta \in \Theta$ we set

$$m_{P,j}(\theta)(Z_j) \equiv E_P[\rho_j(X,\theta)|Z_j];$$

i.e. $m_{P,j}$ maps each $\theta \in \Theta$ to a square integrable function of Z_j . Letting \mathbf{B}_n denote the vector subspace generated by $\Theta_n \cap R$, we then impose the following:

Assumption 3.5. There is a norm $\|\cdot\|_{\mathbf{L}}$ on \mathbf{B}_n , linear maps $\nabla m_{P,j}(\theta): \mathbf{B} \to L_P^2$, and constants $\epsilon > 0$ and $K_m, M < \infty$ such that for all $P \in \mathbf{P}$, $h \in \mathbf{B}_n$, and elements $\theta_1, \theta_2 \in \{\theta \in \Theta_n \cap R: \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \leq \epsilon\}$ we have: (i) $\|m_{P,j}(\theta_1) - m_{P,j}(\theta_2) - \nabla m_{P,j}(\theta_2)[\theta_1 - \theta_2]\|_{P,2} \leq K_m \|\theta_1 - \theta_2\|_{\mathbf{L}} \|\theta_1 - \theta_2\|_{\mathbf{E}}$; (ii) $\|\nabla m_{P,j}(\theta_1)[h] - \nabla m_{P,j}(\theta_2)[h]\|_{P,2} \leq K_m \|\theta_1 - \theta_2\|_{\mathbf{L}} \|h\|_{\mathbf{E}}$; (iii) $\|\nabla m_{P,j}(\theta_2)[h]\|_{P,2} \leq M \|h\|_{\mathbf{E}}$.

Assumption 3.6. (i) $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}(\mathcal{R}_n^{\kappa_{\rho}}, \mathcal{F}_n, \| \cdot \|_{P,2}) = o(a_n);$ (ii) $\sup_{P \in \mathbf{P}_0} \sup_{\theta \in \Theta_{0n}^r} \sqrt{n} \|E_P[\rho(X, \theta) * q^{k_n}(Z)]\|_{\Sigma_{P,p}} = o(a_n).$

Assumption 3.7. (i) For each $P \in \mathbf{P}$ there is a $k_n \times k_n$ matrix $\Sigma_P > 0$ such that $\|\hat{\Sigma}_n - \Sigma_P\|_{o,p} = o_P(1 \wedge a_n \{k_n^{1/p} \sqrt{\log(1 + k_n)} B_n J_n\}^{-1})$ uniformly in $P \in \mathbf{P}$; (ii) $\|\Sigma_P\|_{o,p}$ and $\|\Sigma_P^{-1}\|_{o,p}$ are uniformly bounded in k_n and $P \in \mathbf{P}$.

Assumption 3.5(i) ensures $m_{P,j}$ is approximated by linear maps $\nabla m_{P,j}$ with an approximation error that is controlled by $\|\cdot\|_{\mathbf{E}}$ and a potentially stronger norm $\|\cdot\|_{\mathbf{L}}$. In turn, Assumptions 3.5(ii)(iii) impose continuity conditions on $\nabla m_{P,j}$ – these assumptions are not used in this section, but will be needed for our bootstrap results. Assumption 3.6 contains our key rate restrictions. Assumption 3.6(i) ensures the rate of convergence \mathcal{R}_n (as in (16)) is sufficiently fast to overcome an asymptotic loss of equicontinuity – a requirement that can hold even when \mathcal{R}_n is slower than the traditional $o(n^{-1/4})$ rate employed to linearize nonlinear models. Assumption 3.6(ii) is an undersmoothing assumption, which ensures that $I_n(R)$ is properly centered under the null hypothesis. Finally, Assumption 3.7 requires $\hat{\Sigma}_n$ to converge to an invertible matrix Σ_P at a suitable rate – here, $\|\cdot\|_{o,p}$ denotes the operator norm when \mathbf{R}^{k_n} is endowed with $\|\cdot\|_p$.

The introduced assumptions suffice for obtaining a strong approximation through a local reparametrization. Formally, we denote the local deviations from $\theta \in \Theta_n \cap R$ by

$$V_n(\theta, R|\ell) \equiv \{h \in \mathbf{B}_n : \theta + \frac{h}{\sqrt{n}} \in \Theta_n \cap R \text{ and } \|\frac{h}{\sqrt{n}}\|_{\mathbf{E}} \le \ell\}.$$

Recall \mathbf{B}_n denotes the vector subspace generated by $\Theta_n \cap R$ and for any $h \in \mathbf{B}_n$ set

$$\mathbb{D}_{P}(\theta)[h] \equiv E_{P}[\nabla m_{P}(\theta)[h](Z) * q^{k_{n}}(Z)],$$

where $\nabla m_P(\theta)[h](Z) \equiv (\nabla m_{P,1}(\theta)[h](Z_1), \dots, \nabla m_{P,\mathcal{J}}(\theta)[h](Z_{\mathcal{J}}))'$. For any given sequence ℓ_n , we then define a sequence of random variables $U_P(R|\ell_n)$ to be given by

$$U_P(R|\ell_n) \equiv \inf_{\theta \in \Theta_{0n}^r} \inf_{h \in V_n(\theta, R|\ell_n)} \| \mathbb{W}_P(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p}.$$
 (17)

As a final piece of notation, for any two norms $\|\cdot\|_{\mathbf{A}_1}$ and $\|\cdot\|_{\mathbf{A}_2}$ defined on \mathbf{B}_n , we set

$$S_n(\mathbf{A}_1, \mathbf{A}_2) \equiv \sup_{b \in \mathbf{B}_n} \frac{\|b\|_{\mathbf{A}_1}}{\|b\|_{\mathbf{A}_2}},$$

which we note depends on the sample size n only through the choice of sieve $\Theta_n \cap R$.

The next result establishes the relation between $U_P(R|\ell_n)$ and $I_n(R)$. It is helpful to recall here that the norm $\|\cdot\|_{\mathbf{L}}$ and constants K_m , introduced in Assumption 3.5, control the linearization of the moments and that $K_m = 0$ for linear models.

Theorem 3.1. Let Assumptions 3.1(i), 3.2, 3.3, 3.4, 3.5(i), 3.6, and 3.7 hold. Then: (i) For any $\ell_n \downarrow 0$ satisfying $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ and $K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ it follows uniformly in $P \in \mathbf{P}_0$ that:

$$I_n(R) \le U_P(R|\ell_n) + o_P(a_n).$$

(ii) If in addition $K_m \mathcal{R}_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$, then ℓ_n may be additionally chosen to satisfy $\mathcal{R}_n = o(\ell_n)$, in which case it follows uniformly in $P \in \mathbf{P}_0$ that:

$$I_n(R) = U_P(R|\ell_n) + o_P(a_n).$$

Theorem 3.1 is perhaps best understood as establishing the validity of a family (indexed by $\{\ell_n\}$) of strong approximations that differ on the size of the local neighborhoods of Θ_{0n}^{r} that they employ. Its proof crucially relies on the linearization

$$\mathbb{D}_{P}(\theta)[h] \approx \sqrt{n} \{ E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] - E_{P}[\rho(X, \theta) * q^{k_n}(Z)] \}, \tag{18}$$

which holds for nonlinear moments $(K_m \neq 0)$ when h/\sqrt{n} is sufficiently small. In particular, if the infimum defining $I_n(R)$ is attained at a point $\hat{\theta}_n$ that converges to $\Theta_{0n}^{\rm r}$ sufficiently fast, then we may apply (18) to establish Theorem 3.1(ii). Regrettably, in certain models the rate of convergence of $\hat{\theta}_n$ may be too slow to apply the approximation in (18) to $\hat{\theta}_n$. In such instances, we may instead rely on the inequality

$$I_n(R) = \inf_{\theta \in \Theta_n \cap R} \sqrt{n} Q_n(\theta) \le \inf_{(\theta, h) \in (\Theta_{0n}^r, V_n(\theta, R|\ell_n))} \sqrt{n} Q_n(\theta + \frac{h}{\sqrt{n}})$$
(19)

and successfully couple the right hand side of (19) by restricting attention to sequences ℓ_n for which (18) is accurate. Thus, by regularizing the local parameter space through a norm bound, we obtain in Theorem 3.1(i) a distributional approximation that, while potentially conservative, holds under weaker requirements on the rate of convergence.

3.1.3 Bootstrap Approximation

Theorem 3.1 shows that the distribution of $U_P(R|\ell_n)$ is a suitable approximation for the distribution of $I_n(R)$. We next develop a bootstrap procedure for estimating the distribution of $U_P(R|\ell_n)$ with the goal of obtaining valid critical values. We estimate the distribution of $U_P(R|\ell_n)$ by replacing population parameters with suitable sample analogues. The key ingredients are: (i) A random variable $\hat{\mathbb{W}}_n$ whose distribution conditional on the data is consistent for the distribution of \mathbb{W}_P ; (ii) An estimator $\hat{\mathbb{D}}_n(\theta)$ for $\mathbb{D}_P(\theta)$; (iii) An estimator $\hat{\Theta}_n^r$ for Θ_{0n}^r (as in (15)); and (iv) A sample analogue $\hat{V}_n(\theta, R|\ell_n)$ for the local parameter space $V_n(\theta, R|\ell_n)$. We then approximate the distribution of $U_P(R|\ell_n)$ by the distribution (conditional on the data) of

$$\hat{U}_n(R|\ell_n) \equiv \inf_{\theta \in \hat{\Theta}_n^{\mathrm{r}}} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \|\hat{\mathbb{W}}_n(\theta) + \hat{\mathbb{D}}_n(\theta)[h]\|_{\hat{\Sigma}_n, p}.$$

For concreteness, we employ the following sample analogues in our construction.

Gaussian Distribution: We estimate the distribution of W_P with the multiplier bootstrap. Specifically, for i.i.d. $\{\omega_i\}_{i=1}^n$ with $\omega_i \sim N(0,1)$ independent of $\{V_i\}_{i=1}^n$ we let

$$\hat{\mathbb{W}}_n(\theta) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ \rho(X_i, \theta) * q^{k_n}(Z_i) - \frac{1}{n} \sum_{j=1}^n \rho(X_j, \theta) * q^{k_n}(Z_j) \}.$$

We focus on the multiplier bootstrap due to its theoretical tractability, though we note that alternative bootstrap approaches can also be valid. ■

The Derivative: We estimate $\mathbb{D}_P(\theta)$ by employing a construction that is applicable to non-differentiable moments. Specifically, for any $\theta \in \Theta_n \cap R$ and $h \in \mathbf{B}_n$ we set

$$\widehat{\mathbb{D}}_n(\theta)[h] \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\rho(X_i, \theta + \frac{h}{\sqrt{n}}) - \rho(X_i, \theta)) * q^{k_n}(Z_i).$$

We employ $\hat{\mathbb{D}}_n(\theta)$ due to its general applicability, though alternative approaches may be preferable in some applications. In particular, if moments are differentiabile, then using

$$\frac{1}{n}\sum_{i=1}^{n} \nabla_{\theta} \rho(X_i, \theta)[h] * q^{k_n}(Z_i)$$

as an estimator for $\mathbb{D}_P(\theta)[h]$ leads to a computationally simpler bootstrap statistic.

The Identified Set: We estimate the identified set by employing the set of (approximate) minimizers of Q_n on $\Theta_n \cap R$. Formally, for a sequence $\tau_n \downarrow 0$, we let

$$\hat{\Theta}_n^{\mathrm{r}} \equiv \{ \theta \in \Theta_n \cap R : Q_n(\theta) \le \inf_{\theta \in \Theta_n \cap R} Q_n(\theta) + \tau_n \}.$$
 (20)

We may set $\tau_n = 0$ in identified models, in which case $\hat{\Theta}_n^r$ reduces to the minimizer of Q_n . In partially identified models, $\hat{\Theta}_n^r$ can be shown to asymptotically lie in a shrinking neighborhood of Θ_{0n}^r provided $\tau_n \to 0$. In order for $\hat{\Theta}_n^r$ to additionally be Hausdorff consistent for Θ_{0n}^r , however, τ_n must not tend to zero too fast; see Lemma S.1.1.

Local Parameter Space: We account for the role inequality constraints play in determining the local parameter space by estimating "binding" sets in analogy to approaches pursued in the moment inequalities literature (Chernozhukov et al., 2007; Andrews and Soares, 2010). Specifically, for a sequence r_n and any $\theta \in \Theta_n \cap R$ we define

$$G_n(\theta) \equiv \{ h \in \mathbf{B}_n : \Upsilon_G(\theta + \frac{h}{\sqrt{n}}) \le (\Upsilon_G(\theta) - K_g r_n \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \mathbf{1}_{\mathbf{G}}) \lor (-r_n \mathbf{1}_{\mathbf{G}}) \},$$

where recall $\mathbf{1}_{\mathbf{G}}$ is the order unit in \mathbf{G} and $g_1 \vee g_2$ represents the supremum of any $g_1, g_2 \in \mathbf{G}$. The constant K_g , formally introduced in Assumption 3.8 below, is related to the curvature of Υ_G and equals zero for linear Υ_G . For any ℓ_n we then define

$$\hat{V}_n(\theta, R|\ell_n) \equiv \{ h \in \mathbf{B}_n : h \in G_n(\theta), \ \Upsilon_F(\theta + \frac{h}{\sqrt{n}}) = 0 \text{ and } \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} \le \ell_n \},$$
 (21)

i.e. in comparison to $V_n(\theta, R|\ell_n)$ we: (i) Replace $\Upsilon_G(\theta + h/\sqrt{n}) \leq 0$ by $h \in G_n(\theta)$; (ii) Retain $\Upsilon_F(\theta + h/\sqrt{n}) = 0$; and (iii) Substitute $\|h/\sqrt{n}\|_{\mathbf{E}} \leq \ell_n$ with $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$.

Before establishing the asymptotic validity of the proposed bootstrap procedure, we require some additional notation. For any set $A \subseteq \mathbf{B}_n$, we denote its ϵ -neighborhood by

$$(A)^{\epsilon} \equiv \{\theta \in \mathbf{B}_n : \inf_{a \in A} \|a - \theta\|_{\mathbf{B}} \le \epsilon\}.$$

We further denote the closure of the linear span of $\Upsilon_F(\mathbf{B}_n)$ by \mathbf{F}_n , and for any linear map Γ on \mathbf{B} we let $\mathcal{N}(\Gamma) \equiv \{h \in \mathbf{B} : \Gamma(h) = 0\}$ denote its null space. In the assumptions that follow, it is helpful to recall that $\Theta_{0n}^{\mathbf{r}}$ is implicitly a function of P.

Assumption 3.8. For some $K_g, M < \infty$, $\epsilon > 0$ and all $n, P \in \mathbf{P}_0$, $\theta_1, \theta_2 \in (\Theta_{0n}^{\mathbf{r}})^{\epsilon}$ (i) Υ_G is Fréchet differentiable with $\|\Upsilon_G(\theta_1) - \Upsilon_G(\theta_2) - \nabla \Upsilon_G(\theta_1)[\theta_1 - \theta_2]\|_{\mathbf{G}} \leq K_g \|\theta_1 - \theta_2\|_{\mathbf{B}}^2$; (ii) $\|\nabla \Upsilon_G(\theta_1) - \nabla \Upsilon_G(\theta_2)\|_o \leq K_g \|\theta_1 - \theta_2\|_{\mathbf{B}}^2$; (iii) $\|\nabla \Upsilon_G(\theta_1)\|_o \leq M$.

Assumption 3.9. For some $K_f, M < \infty$, $\epsilon > 0$ and all $n, P \in \mathbf{P}_0$, $\theta_1, \theta_2 \in (\Theta_{0n}^{\mathbf{r}})^{\epsilon}$ (i) Υ_F is Fréchet differentiable with $\|\Upsilon_F(\theta_1) - \Upsilon_F(\theta_2) - \nabla \Upsilon_F(\theta_1)[\theta_1 - \theta_2]\|_{\mathbf{F}} \le K_f \|\theta_1 - \theta_2\|_{\mathbf{B}}^2$; (ii) $\|\nabla \Upsilon_F(\theta_1) - \nabla \Upsilon_F(\theta_2)\|_o \le K_f \|\theta_1 - \theta_2\|_{\mathbf{B}}^2$; (iii) $\|\nabla \Upsilon_F(\theta_1)\|_o \le M$; (iv) $\nabla \Upsilon_F(\theta_1) : \mathbf{B}_n \to \mathbf{F}_n$ admits a right inverse $\nabla \Upsilon_F(\theta_1)^-$ with $K_f \|\nabla \Upsilon_F(\theta_1)^-\|_o \le M$.

Assumption 3.10. Either (i) $\Upsilon_F : \mathbf{B} \to \mathbf{F}$ is affine, or (ii) There are constants $\epsilon > 0$, $M < \infty$ such that for every $P \in \mathbf{P}_0$, n, and $\theta \in \Theta_{0n}^r$ there exists a $h \in \mathbf{B}_n \cap \mathcal{N}(\nabla \Upsilon_F(\theta))$ satisfying $\Upsilon_G(\theta) + \nabla \Upsilon_G(\theta)[h] \leq -\epsilon \mathbf{1}_{\mathbf{G}}$ and $\|h\|_{\mathbf{B}} \leq M$.

Assumption 3.8 imposes that Υ_G be Fréchet differentiable. The constant K_g , employed in the construction of $\hat{V}_n(\theta, R|\ell_n)$, may be interpreted as a bound on the second derivative of Υ_G and equals zero when Υ_G is linear. Assumptions 3.9 and 3.10 mark an important difference between hypotheses in which Υ_F is linear and those in which Υ_F is nonlinear – note linear Υ_F automatically satisfy Assumptions 3.9 and 3.10. This

distinction reflects that when Υ_F is linear its impact on the local parameter space is known and need not be estimated.⁵ Thus, while Assumptions 3.9(i)-(iii) impose conditions analogous to those required of Υ_G , Assumption 3.9(iv) additionally demands that $\nabla \Upsilon_F(\theta)$ posses a norm bounded right inverse on $(\Theta_{0n}^r)^{\epsilon}$ – the existence of a right inverse is equivalent to a classical rank condition.⁶ Finally, for nonlinear Υ_F , Assumption 3.10(ii) requires the existence of a local perturbation to any $\theta \in \Theta_{0n}^r$ that relaxes "active" inequality constraints without a first order effect on the equality restrictions.

We impose a final set of assumptions in order to couple our bootstrap statistic.

Assumption 3.11. $\sup_{\theta \in \Theta_n \cap R} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^{\star}(\theta)\|_p = o_P(a_n)$ uniformly in $\Phi \times P$ with $P \in \mathbf{P}$ for Φ the standard normal distribution, $a_n = o(1)$, and \mathbb{W}_P^{\star} independent of $\{V_i\}_{i=1}^n$ and having the same distribution as \mathbb{W}_P .

Assumption 3.12. (i) For some $M < \infty$, $||h||_{\mathbf{E}} \le M||h||_{\mathbf{B}}$ for all $h \in \mathbf{B}_n$; (ii) There is an $\epsilon > 0$ such that $P((\hat{\Theta}_n^{\mathbf{r}})^{\epsilon} \subseteq \Theta_n)$ tends to one uniformly in $P \in \mathbf{P}_0$; (iii) For $\mathcal{V}_n(P)$ as in Assumption 3.4, $P(\hat{\Theta}_n^{\mathbf{r}} \subseteq \mathcal{V}_n(P))$ tends to one uniformly in $P \in \mathbf{P}_0$.

Assumption 3.13. (i) Either Υ_F and Υ_G are affine or $(\mathcal{R}_n + \nu_n \tau_n) \times \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(1)$; (ii) The sequences ℓ_n , τ_n satisfy $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_p} \vee (\nu_n \tau_n)^{\kappa_p}, \mathcal{F}_n, \| \cdot \|_{P,2}) = o(a_n)$, $K_m \ell_n (\ell_n + \mathcal{R}_n + \nu_n \tau_n) \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$, and $\ell_n (\ell_n + \{\mathcal{R}_n + \nu_n \tau_n\} \times \mathcal{S}_n(\mathbf{B}, \mathbf{E})) 1\{K_f > 0\} = o(a_n n^{-1/2})$; (iii) The sequence r_n satisfies $\limsup_{n \to \infty} 1\{K_g > 0\} \ell_n / r_n < 1/2$ and $(\mathcal{R}_n + \nu_n \tau_n) \times \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(r_n)$.

Assumption 3.11 demands that $\hat{\mathbb{W}}_n$ be coupled with a Gaussian \mathbb{W}_P^* independent of $\{V_i\}_{i=1}^n$. This condition implies the multiplier bootstrap is valid in our potentially non-Donsker setting; see Appendix S.7 for sufficient conditions. More generally, we note that our analysis remains valid if the multiplier bootstrap is replaced with any other resampling scheme (e.g., nonparametric bootstrap) satisfying a coupling requirement like Assumption 3.11. Assumption 3.12(i) ensures that $\|\cdot\|_{\mathbf{B}}$ is (weakly) stronger than $\|\cdot\|_{\mathbf{E}}$. Assumption 3.12(ii) demands that $\hat{\Theta}_n^{\mathbf{r}}$ be asymptotically contained in the interior of Θ_n . This requirement does not rule out that parameter space restrictions be binding at $\Theta_{0n}^{\mathbf{r}}$ — instead, Assumption 3.12(ii) requires that all such restrictions be stated through R. Together with Assumption 3.4(i), Assumption 3.12(iii) enables us to obtain a rate of convergence for $\hat{\Theta}_n^{\mathbf{r}}$ and may be verified in the same manner as Assumption 3.4(ii).

Assumption 3.13 contains our main rate requirements. In particular, Assumption 3.13(i) ensures the one sided Hausdorff convergence of $\hat{\Theta}_n^{\rm r}$ to $\Theta_{0n}^{\rm r}$ under $\|\cdot\|_{\bf B}$ whenever Υ_F or Υ_G are nonlinear. The main conditions on ℓ_n , employed in constructing

⁵For linear Υ_F , the requirement $\Upsilon_F(\theta + h/\sqrt{n}) = 0$ is equivalent to $\Upsilon_F(h) = 0$ for any $\theta \in R$.

⁶Recall for a linear map $\Gamma: \mathbf{B}_n \to \mathbf{F}_n$, its right inverse is a map $\Gamma^-: \mathbf{F}_n \to \mathbf{B}_n$ such that $\Gamma\Gamma^-(h) = h$ for any $h \in \mathbf{B}_n$. The right inverse Γ^- need not be unique if Γ is not bijective, in which case Assumption 3.9(iv) is satisfied as long as it holds for some right inverse of $\nabla \Upsilon_F(\theta): \mathbf{B}_n \to \mathbf{F}_n$.

 $\hat{V}_n(\theta, R|\ell_n)$, are contained in Assumption 3.13(ii). These conditions ensure the consistency of $\hat{\mathbb{D}}_n(\theta)[h]$, the applicability of Theorem 3.1, and that $\hat{V}_n(\theta, R|\ell_n)$ be well approximated by the true local parameter space. Heuristically, whenever the rate of convergence \mathcal{R}_n is too slow, regularizing the local parameter space by selecting a small ℓ_n can ensure the asymptotic validity of the test. As in Section 2, however, we note that whenever the rate of convergence \mathcal{R}_n is sufficiently fast such regularization is unnecessary and it is possible to set $\ell_n = +\infty$ – in such applications, setting ℓ_n to be too small can lead to a loss of power. In turn, Assumption 3.13(iii) requires that r_n not decrease to zero faster than the $\|\cdot\|_{\mathbf{B}}$ -rate of convergence of $\hat{\Theta}_n^r$. Assumption 3.13(iii) is always satisfied if $r_n = +\infty$, though setting $r_n \to 0$ can improve power against certain alternatives. Similarly, we note that the requirements on τ_n imposed by Assumption 3.13 can always be satisfied by setting $\tau_n = 0$, but such a choice can lead to a loss of power in certain partially identified models (recall the discussion in Section 2.2).

Our next result provides a coupling result for our bootstrap statistic. In its statement, $U_P^{\star}(R|\ell_n)$ is defined identically to $U_P(R|\ell_n)$ but with \mathbb{W}_P^{\star} in place of \mathbb{W}_P .

Theorem 3.2. If Assumptions 3.1, 3.2, 3.3, 3.4(i), 3.5, 3.6(ii), 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13 hold, then there is $\tilde{\ell}_n \approx \ell_n$ so that uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n).$$

Theorem 3.2 shows that with unconditional probability tending to one uniformly on $P \in \mathbf{P}_0$ our bootstrap statistic is bounded from below by a random variable that is independent of the data. The significance of this result lies in that the lower bound is equal in distribution to the coupling to $I_n(R)$ obtained in Theorem 3.1. Thus, Theorems 3.1 and 3.2 provide the basis for constructing tests that employ increasing functions of $I_n(R)$ as a test statistic and the analogous bootstrap quantiles of $\hat{U}_n(R|\ell_n)$ as critical values. The resulting tests may be conservative, however, whenever the inequalities in Theorems 3.1 and 3.2 are not "sharp." In particular, in order for the pointwise (in P) rejection probability to equal the nominal level of the test under the null hypothesis we require: (i) The rate of convergence \mathcal{R}_n must be sufficiently fast for Theorem 3.1(ii) to apply (in which case setting $\ell_n = +\infty$ is often valid); (ii) We should select r_n to tend to zero with the sample size; and (iii) In partially identified settings, τ_n must tend to zero sufficiently slowly so that $\hat{\Theta}_n^{\rm r}$ is Hausdorff consistent for $\Theta_{0n}^{\rm r}$.

3.2 The Tests

We next employ the theoretical results of Section 3.1 to establish the asymptotic validity of different tests of the null hypothesis defined in (12). In what follows, for any statistic \hat{T}_n that is a function of $\{V_i\}_{i=1}^n$ and the bootstrap weights $\{\omega_i\}_{i=1}^n$, we let $\hat{q}_{\tau}(\hat{T}_n)$ denote

its conditional τ quantile given $\{V_i\}_{i=1}^n$. For example, we have that

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|\ell_n)) = \inf\{u : P(\hat{U}_n(R|\ell_n) \le u | \{V_i\}_{i=1}^n) \ge 1 - \alpha\}.$$

3.2.1 Tests Based on $I_n(R)$

We first examine a test that employs $I_n(R)$ as a test statistic and a bootstrap quantile of $\hat{U}_n(R|\ell_n)$ as a critical value. As has been shown in the literature, uniform consistent estimation of approximating distributions is not sufficient for characterizing the asymptotic size of a test (Romano and Shaikh, 2012). Heuristically, to establish the asymptotic validity of a test the approximating distributions must additionally be suitably uniformly continuous. Our next assumption suffices for verifying this final requirement.

Assumption 3.14. There is $\eta \geq 0$ and $\varrho_n = o(a_n^{-1})$ such that for $\hat{c}_n = \hat{q}_{1-\alpha}(\hat{U}_n(R|\ell_n))$ and any $\tilde{\ell}_n \approx \ell_n$: (i) $P(I_n(R) > \hat{c}_n) = P(I_n(R) > \hat{c}_n \vee \eta) + o(1)$ uniformly in $P \in \mathbf{P}_0$, and (ii) $\sup_{P \in \mathbf{P}_0} \sup_{t \in (\eta - a_n, +\infty)} P(|U_P(R|\tilde{\ell}_n) - t| \leq \epsilon) \leq \varrho_n(\epsilon \wedge 1) + o(1)$.

Assumption 3.14(i) trivially holds with $\eta=0$ since both $I_n(R)$ and $\hat{U}_n(R|\ell_n)$ are (weakly) positive almost surely. However, in some applications it is possible to verify Assumption 3.14(i) in fact holds with $\eta>0$ by arguing that the bootstrap quantiles of $\hat{U}_n(R|\ell_n)$ are suitably bounded away from zero when $I_n(R)$ is strictly positive. Establishing Assumption 3.14(i) holds with $\eta>0$ eases the verification of Assumption 3.14(ii), which intuitively requires that $U_P(R|\tilde{\ell}_n)$ be continuously distributed on $(\eta-a_n,+\infty)$ with a density bounded by a, possibly diverging, ϱ_n . Because $U_P(R|\tilde{\ell}_n)$ is a functional of the Gaussian measure \mathbb{W}_P , Assumption 3.14(ii) can in some applications be verified using available results in the literature (Davydov et al., 1998). For instance, when $U_P(R|\tilde{\ell}_n)$ is a convex function of \mathbb{W}_P , as in the application of Section 2.1.1, the distribution of $U_P(R|\tilde{\ell}_n)$ can readily be shown to be continuous in $(0,+\infty)$. We refer the reader to Chernozhukov et al. (2014) for further discussion and motivation of conditions such as Assumption 3.14(ii), called anti-concentration conditions.

The next result establishes the asymptotic validity of a test based on $I_n(R)$.

Corollary 3.1. Let Assumption 3.14 hold and the conditions of Theorem 3.1(i) and Theorem 3.2 be satisfied. If $\hat{c}_n = \hat{q}_{1-\alpha}(\hat{U}_n(R|\ell_n))$, then it follows that:

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(I_n(R) > \hat{c}_n) \le \alpha.$$

In Algorithm 1 below we describe how to obtain p-values for the test described in Corollary 3.1 when the moments are differentiable. We note that if there are no inequality constraints, then it is possible to show that the test in Corollary 3.1 is similar and its asymptotic size equals the nominal level α whenever the conditions of Theorem

3.1(ii) are satisfied. The consistency of the test against any $P \in \mathbf{P} \setminus \mathbf{P}_0$ for which $\max_{j} ||E_P[\rho_j(X,\theta)|Z_j]||_{P,2}$ is bounded away from zero (in $\theta \in \Theta \cap R$) is also straightforward to establish under suitable conditions. Finally, we also note that if we instead employ the critical value $\hat{c}_n = \hat{q}_{1-\alpha+\delta}(\hat{U}_n(R|\ell_n)) + \delta$ for any $\delta > 0$, then the conclusion of Corollary 3.1 holds without needing to impose Assumption 3.14; see Corollary S.3.1. This modification to the critical value was originally proposed in a different context by Andrews and Shi (2013), who suggest setting $\delta = 10^{-6}$.

Algorithm 1 Computing p-values for test based on $I_n(R)$

```
Require: \Theta_n, \Upsilon_F, \Upsilon_G, \{\rho(X_i,\theta) * q^{k_n}(Z_i)\}_{i=1}^n, \hat{\Sigma}_n, r_n, \tau_n, \ell_n
          ▷ Compute the Test Statistic
  1: Q_n(\theta) \leftarrow \|\hat{\Sigma}_n\{\frac{1}{n}\sum_{i=1}^n \rho(X_i, \theta) * q^{k_n}(Z_i)\}\|_p
2: R \leftarrow \{\theta : \Upsilon_F(\theta) = 0, \Upsilon_G(\theta) \leq 0\}
                                                                                                                                                                                        ▷ Criterion function
                                                                                                                                                                                                  ▷ Constraint Set
   3: I_n(R) \leftarrow \min_{\theta \in \Theta_n} \sqrt{n} Q_n(\theta) s.t. \theta \in R

→ Test Statistic

          ▶ Prepare variables for bootstrap problem
  4: \hat{\mathbb{D}}_{n}(\theta)[h] \leftarrow \frac{1}{n} \sum_{i=1}^{n} \nabla_{\theta} \rho(X_{i}, \theta)[h] * q^{k_{n}}(Z_{i}) \Rightarrow Moments Derivative

5: \hat{\Theta}_{n}^{r} \leftarrow \{\theta \in \Theta_{n} \cap R : Q_{n}(\theta) \leq I_{n}(R)/\sqrt{n} + \tau_{n}\} \Rightarrow Boot Constraint \theta

6: G_{n}(\theta) \leftarrow \{h : \Upsilon_{G}(\theta + h/\sqrt{n}) \leq (\Upsilon_{G}(\theta) - K_{g}r_{n}\|h/\sqrt{n}\|\mathbf{B}\mathbf{1}_{G}) \vee (-r_{n}\mathbf{1}_{G})\}

7: \hat{V}_{n}(\theta, R|\ell_{n}) \leftarrow \{h \in G_{n}(\theta) : \Upsilon_{F}(\theta + h/\sqrt{n}) = 0, \|h\|\mathbf{B} \leq \ell_{n}\sqrt{n}\} \Rightarrow Boot Constraint h
          \triangleright Compute B bootstrap statistics and obtain p-value
   8: for b = 1 to B do
                    \{\omega_i^b\}_{i=1}^n \leftarrow \text{Generate i.i.d. sample of } N(0,1) \text{ variables} 
 \hat{\mathbb{W}}_n^b(\theta) \leftarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i^b \{ \rho(X_i, \theta) * q^{k_n}(Z_i) - \frac{1}{n} \sum_{j=1}^n \rho(X_j, \theta) * q^{k_n}(Z_j) \} 
                   F_n^b(\theta, h) \leftarrow \|\hat{\Sigma}_n\{\hat{\mathbb{W}}_n^b(\theta) + \hat{\mathbb{D}}_n(\theta)[h]\}\|_p
                                                                                                                                                                                                  ▶ Boot Criterion
11:
                   Boot[b] \leftarrow \min_{\theta,h} F^b(\theta,h) \text{ s.t. } \theta \in \hat{\Theta}_n^r, h \in \hat{V}_n(\theta,R|\ell_n)
                                                                                                                                                                                                    ▶ Boot Statistic
13: end for
14: pval \leftarrow \frac{1}{B} \sum_{b=1}^{B} 1\{I_n(R) \leq \text{Boot}[b]\}
                                                                                                                                                                                           ▶ Compute p-value
```

Remark 3.1. Suppose θ_0 is identified, we aim to test whether $\Upsilon_F(\theta_0) = 0$, and we are confident θ_0 satisfies $\Upsilon_G(\theta_0) \leq 0$. We could then set R to equal R_1 or R_2 , where

$$R_1 = \{ \theta \in \mathbf{B} : \Upsilon_G(\theta) \le 0 \text{ and } \Upsilon_F(\theta) = 0 \}$$

$$R_2 = \{ \theta \in \mathbf{B} : \Upsilon_F(\theta) = 0 \}.$$

The power functions of the corresponding tests are not necessarily ranked. As a result, it can be desirable to combine both tests by, for instance, using the test statistic $T_n \equiv \max\{F_1(I_n(R_1)), F_2(I_n(R_2))\}$ for F_1, F_2 increasing functions, and the quantiles of $\max\{F_1(\hat{U}_n(R_1|\ell_n)), F_2(\hat{U}_n(R_2|\ell_n))\}$ as critical values – e.g., F_j may be c.d.f. of $\hat{U}_n(R_j|\ell_n)$ conditional on the data. The asymptotic validity of such a test follows from Theorems 3.1 and 3.2 under a suitable modification of Assumption 3.14.

3.2.2 Tests Based on $I_n(R) - I_n(\Theta)$

We next establish the asymptotic validity of a test based on $I_n(R) - I_n(\Theta)$ by also relying on Theorems 3.1 and 3.2. In what follows, we signify parameters associated with setting $R = \Theta$ by a "u" superscript – e.g. $\mathcal{F}_n^{\mathrm{u}}$ is understood to be as in (14) but with $R = \Theta$.

In order to obtain a distributional approximation to the recentered statistic, we may simply apply Theorem 3.1(i) to $I_n(R)$ and Theorem 3.1(ii) to $I_n(\Theta)$ to conclude that

$$I_n(R) - I_n(\Theta) \le U_P(R|\ell_n) - U_P(\Theta|\ell_n^{\mathbf{u}}) + o_P(a_n). \tag{22}$$

Moreover, by Theorem 3.2 we may approximate the distribution of $U_P(R|\ell_n)$ by using $\hat{U}_n(R|\ell_n)$. Similarly, to obtain a bootstrap approximation to $U_P(\Theta|+\infty)$, we define

$$\hat{\Theta}_n^{\mathrm{u}} \equiv \{ \theta \in \Theta_n : Q_n(\theta) \le \inf_{\theta \in \Theta_n} Q_n(\theta) + \tau_n^{\mathrm{u}} \};$$

i.e. $\hat{\Theta}_n^{\mathrm{u}}$ is simply the set estimator in (20) applied with $\Theta = R$. For $\mathbf{B}_n^{\mathrm{u}}$ the closed linear span of Θ_n , we then approximate the law of $U_P(\Theta|\ell_n^{\mathrm{u}})$ by employing

$$\hat{U}_n(\Theta|+\infty) \equiv \inf_{\theta \in \hat{\Theta}_n^{\mathrm{u}}} \inf_{h \in \mathbf{B}_n^{\mathrm{u}}} \|\hat{\mathbb{W}}_n(\theta) + \hat{\mathbb{D}}_n(\theta)[h]\|_{\hat{\Sigma}_n, p};$$

i.e. the bootstrap approximation equals that of Theorem 3.2, with the local parameter space being unconstrained due to the absence of equality or inequality restrictions.

The preceding discussion suggests that the quantiles of $\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty)$ conditional on the data provide valid critical values for the recentered statistic. Our next result formally establishes that the resulting test is indeed asymptotically valid.

Corollary 3.2. Let the conditions of Theorems 3.1(i) and 3.2 hold with R as in (13), the conditions of Theorems 3.1(ii) and 3.2 hold with $R = \Theta$, and Assumption 3.14 hold with $I_n(R) - I_n(\Theta)$, $\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty)$, and $U_P(R|\tilde{\ell}_n) - U_P(\Theta|\tilde{\ell}_n^{\mathrm{u}})$ in place of $I_n(R)$, $\hat{U}_n(R|\ell_n)$, and $U_P(R|\tilde{\ell}_n)$ with $\tilde{\ell}_n^{\mathrm{u}}$ satisfying $\mathcal{R}_n^{\mathrm{u}} = o(\tilde{\ell}_n^{\mathrm{u}})$ and Assumption 3.13(ii) with $R = \Theta$. If $\tau_n^{\mathrm{u}} \downarrow 0$ satisfies $J_n^{\mathrm{u}} B_n k_n^{1/p} \sqrt{\log(1+k_n)/n} = o(\tau_n^{\mathrm{u}})$ and $\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}} \times \mathcal{S}_n^{\mathrm{u}}(\mathbf{B}, \mathbf{E}) = o(1)$, then for $\hat{c}_n \equiv \hat{q}_{1-\alpha}(\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty))$ it follows that

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}_0} P(I_n(R) - I_n(\Theta) > \hat{c}_n) \le \alpha.$$

It is worth emphasizing that in coupling $I_n(\Theta)$ we must rely on Theorem 3.1(ii) instead of Theorem 3.1(i) in order to ensure that (22) holds. As a result, whenever moments are nonlinear, Corollary 3.2 requires the rate of convergence of the unconstrained estimator to be sufficiently fast for Theorem 3.1(ii) to apply. Similarly, in coupling $\hat{U}_n(\Theta|+\infty)$ it is important that $\hat{\Theta}_n^{\rm u}$ be consistent in the Hausdorff metric. Thus, while we may set $\tau_n^{\rm u}=0$ in identified models, in partially identified models we require that $\tau_n^{\rm u}$

not tend to zero too fast; see Theorem S.1.1. Finally, we note that in identified models, it is possible to employ either $\hat{\mathbb{W}}_n(\hat{\theta}_n)$ or $\hat{\mathbb{W}}_n(\hat{\theta}_n^{\mathrm{u}})$ in constructing both $\hat{U}_n(R|\ell_n)$ and $\hat{U}_n(\Theta|+\infty)$ – a change that results in an asymptotically equivalent coupling but ensures that the bootstrap statistic $\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty)$ is (weakly) positive.

4 Heterogeneity and Demand Analysis

For our final example, we illustrate how to conduct inference in the heterogeneous demand model of Hausman and Newey (2016) – alternative models of demand under conditional moment restrictions include the analysis in Hausman and Newey (1995), Blundell et al. (2012), and Chen and Christensen (2018). Specifically, for $Y \in [0, 1]$ equal to the expenditure share on a commodity, $W \in \mathbf{W}$ a vector of prices, income, and covariates, and η representing unobserved individual heterogeneity we suppose

$$Y = g(W, \eta) \tag{23}$$

where g is a known function of (W, η) . The unobserved heterogeneity η can potentially be infinite dimensional. For instance, Hausman and Newey (2016) set $\eta = \{\beta_j\}_{j=1}^{\infty}$ to be a random variable in the sequence space $\ell^2 \equiv \{\{a_j\}_{j=1}^{\infty} : \sum_j a_j^2 < \infty\}$, and let

$$g(W,\eta) = \sum_{j=1}^{\infty} \psi_j(W)\beta_j,$$
(24)

where $\{\psi_j\}_{j=1}^{\infty}$ is a known basis satisfying $\sum_{j=1}^{\infty} \psi_j^2(W) < \infty$ almost surely (in W).

If the covariates W are independent of η , then for any $c \in \mathbf{R}$ it follows that

$$P(Y \le c|W) = P(g(W, \eta) \le c|W) = \int 1\{g(W, \eta) \le c\} \mu_0(d\eta)$$
 (25)

where μ_0 denotes the unknown distribution of η . Result (25) restricts the possible values of μ_0 and hence the identified set for functionals of μ_0 , such as average exact consumer surplus or average share. Specifically, for $\Psi(g,\eta)$ an object of interest for preferences denoted by η , such as equivalent variation, Hausman and Newey (2016) study functionals

$$\int \Psi(g,\eta)\mu_0(d\eta),\tag{26}$$

which is the average across individuals. By evaluating the set of values of (26) which can be generated by a distribution μ_0 satisfying (25) at a grid $\{c_j\}_{j=1}^{\mathcal{J}}$, Hausman and Newey (2016) provide estimates of the identified set for the functional of interest. We further note bounds on the distribution of $\Psi(g,\eta)$ under μ_0 can be obtained by replacing $\Psi(g,\eta)$ in (26) with an indicator that $\Psi(g,\eta)$ be less than or equal to some number.

In what follows, we apply our results to conduct inference on functionals as in (26). To this end, we let $F_P(c_j|W) \equiv P(Y \leq c_j|W)$ for a given grid $\{c_j\}_{j=1}^{\mathcal{J}}$. To define \mathbf{B} , we suppose $\eta \in \Omega$ for some known Hausdorff space Ω , set \mathcal{B} to be the Borel σ -algebra on Ω , let \mathcal{M} be the space of regular signed Borel measures on Ω , and let $\|\cdot\|_{TV}$ denote the total variation norm. Assuming $F_P(c_j|\cdot) \in C_B(\mathbf{W})$ for $C_B(\mathbf{W})$ the space of continuous and bounded functions on \mathbf{W} , we set $\mathbf{B} = (\bigotimes_{j=1}^{\mathcal{J}} C_B(\mathbf{W})) \times \mathcal{M}$, for any $(\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \mathbf{B}$ let $\|\theta\|_{\mathbf{B}} = \sum_{j=1}^{\mathcal{J}} \|F(c_j|\cdot)\|_{\infty} + \|\mu\|_{TV}$, and set

$$\Theta = \{ (\{F(c_{j}|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \mathbf{B} : \max_{1 \le j \le \mathcal{J}} \|F(c_{j}|\cdot)\|_{\infty} \le 2 \},$$
 (27)

where the "2" norm bound is simply selected to ensure Θ_0 is in the interior of Θ .

Letting X = (Y, W) and setting $Z_{j} = W$ for every $1 \leq j \leq \mathcal{J}$ we then define

$$\rho_{1}(X,\theta) = 1\{Y \le c_{1}\} - F(c_{1}|W), \tag{28}$$

which yields conditional moment restrictions that identify $F_P(c_j|W)$ – note, however, that μ_0 is potentially partially identified. For a grid $\{w_l\}_{l=1}^{\mathcal{L}} \subseteq \mathbf{W}$ we test whether a hypothesized value λ belongs to the identified set for the functional in (26) by setting

$$R = \left\{ (\{F(c_{\jmath}|\cdot)\}_{\jmath=1}^{\mathcal{J}}, \mu) : \mu(\Omega) = 1, \ \mu(B) \ge 0 \text{ for all } B \in \mathcal{B}, \ \int \Psi(g, \eta) \mu(d\eta) = \lambda, \right.$$

$$\text{and } F(c_{\jmath}|w_{l}) = \int 1\{g(w_{l}, \eta) \le c_{\jmath}\} \mu(d\eta) \text{ for all } 1 \le \jmath \le \mathcal{J}, 1 \le l \le \mathcal{L} \right\}. \tag{29}$$

Thus, the null hypothesis that $\Theta_0 \cap R$ be nonempty corresponds to requiring that there exist a distribution μ for η satisfying the restrictions in (25) at the points (c_j, w_l) and yielding a value for the functional in (26) of λ . By conducting test inversion in λ we can obtain a confidence region for the desired functional. To map R into the framework of Section 3, we set $\mathbf{G} = \ell^{\infty}(\mathcal{B})$ for $\ell^{\infty}(\mathcal{B})$ the set of bounded functions on \mathcal{B} and for any $(\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \mathbf{B}$ let $\Upsilon_G : \mathbf{B} \to \ell^{\infty}(\mathcal{B})$ be given by

$$\Upsilon_G(\theta)(B) = -\mu(B). \tag{30}$$

Finally, we set $\Upsilon_F : \mathbf{B} \to \mathbf{R}^{\mathcal{JL}+2}$ to equal $\Upsilon_F(\theta) = (\Upsilon_F^{(e)}(\theta), \Upsilon_F^{(\mu)}(\theta), \Upsilon_F^{(s)}(\theta))$, where

$$\Upsilon_F^{(e)}(\theta) = \{ F(c_j|w_l) - \int 1\{ g(w_l, \eta) \le c_j \} \mu(d\eta) \}_{1 \le j \le \mathcal{J}, 1 \le l \le \mathcal{L}}
\Upsilon_F^{(\mu)}(\theta) = \mu(\Omega) - 1
\Upsilon_F^{(s)}(\theta) = \int \Psi(g, \eta) \mu(d\eta) - \lambda.$$
(31)

Given these definitions, we may then map R (as introduced in (29)) into the framework of Section 3 by noting that $R = \{\theta \in \mathbf{B} : \Upsilon_F(\theta) = 0 \text{ and } \Upsilon_G(\theta) \leq 0\}.$

As in Hausman and Newey (2016), we can impose utility maximization by requiring that the support Ω consist only of η such that $g(\cdot, \eta)$ satisfies the Slutsky conditions. One may sample from Ω by drawing randomly from sets of η that satisfy Slutsky symmetry and only keeping those where the compensated price effects matrix is negative semidefinite on a grid. This is the procedure followed in Hausman and Newey (2016) for two goods. Importantly, we emphasize that because the utility maximization restrictions are imposed through Ω , they do not affect the basic structure of Υ_F and Υ_G – i.e., Υ_F and Υ_G remain linear maps satisfying Assumptions 3.8-3.10. In this sense, as long as they are imposed through the support Ω of η , our procedure allows us to accommodate a wide array of shape restrictions on individual demand $g(\cdot, \eta)$.

Given a collection of orthogonal probability measures $\{\delta_s\}_{s=1}^{s_n} \subseteq \mathcal{M}$ we employ

$$\mathcal{M}_n = \{ \mu \in \mathcal{M} : \mu = \sum_{s=1}^{s_n} \alpha_s \delta_s \text{ for some } \{\alpha_s\}_{s=1}^{s_n} \in \mathbf{R}^{s_n} \}$$

as a sieve for \mathcal{M} . Employing orthogonal measures, such as distinct Dirac measures, is computationally attractive as it simplifies imposing the nonnegativity constraint on any $\mu \in \mathcal{M}_n$. As a sieve for $\{F_P(c_j|\cdot)\}_{j=1}^{\mathcal{J}}$, we employ approximating functions $\{p_j\}_{j=1}^{j_n}$. In particular, setting $p^{j_n}(w) = (p_1(w), \dots, p_{j_n}(w))'$, we set as our sieve

$$\Theta_n = \{(\{p^{j_n\prime}\beta_j\}_{j=1}^{\mathcal{J}}, \mu) : \mu \in \mathcal{M}_n \text{ and } \max_{1 \le j \le \mathcal{J}} \|p^{j_n\prime}\beta_j\|_{\infty} \le 2\}.$$

Similarly, for a sequence $\{q_k\}_{k=1}^{k_n}$ and $k_n \times k_n$ positive definite matrices $\{\hat{\Sigma}_{j,n}\}_{j=1}^{\mathcal{J}}$, we set $q^{k_n}(w) = (q_1(w), \dots, q_{k_n}(w))'$ and for any $(\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta$ define

$$Q_n(\theta) = \{ \sum_{j=1}^{\mathcal{J}} \| \frac{1}{n} \sum_{i=1}^{n} (1\{Y_i \le c_j\} - F(c_j|W_i)) q^{k_n}(W_i) \|_{\hat{\Sigma}_{j,n},2}^2 \}^{1/2}.$$
 (32)

The statistics $I_n(R)$ and $I_n(\Theta)$ then equal the minimums of $\sqrt{n}Q_n$ over $\Theta_n \cap R$ and Θ_n .

Our next set of assumptions enable us to couple $I_n(R)$ and $I_n(R) - I_n(\Theta)$.

Assumption 4.1. (i) $\{Y_i, W_i\}_{i=1}^n$ is i.i.d. with $(Y, W) \sim P \in \mathbf{P}$; (ii) $\sup_w \|p^{j_n}(w)\|_2 \lesssim \sqrt{j_n}$; (iii) $E_P[p^{j_n}(W)p^{j_n}(W)']$ has eigenvalues bounded away from zero and infinity uniformly in $P \in \mathbf{P}$ and j_n ; (iv) For each $P \in \mathbf{P}_0$ and $\theta \in \Theta_0 \cap R$, there exists a $\Pi_n \theta = (\{F_n(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_n) \in \Theta_n \cap R$ such that $\sum_{j=1}^{\mathcal{J}} \|E_P[(F_n(c_j|W) - F_P(c_j|W))q^{k_n}(W)]\|_2 = O((n\log(n))^{-1/2})$ uniformly in $P \in \mathbf{P}_0$ and $\theta \in \Theta_0 \cap R$.

Assumption 4.2. (i) $\max_{1 \le k \le k_n} \|q_k\|_{\infty} \lesssim \sqrt{k_n}$; (ii) $E_P[q^{k_n}(W)q^{k_n}(W)']$ has eigenvalues bounded uniformly in $P \in \mathbf{P}$ and k_n ; (iii) $E_P[q^{k_n}(W)p^{j_n}(W)']$ has singular values bounded away from zero uniformly in $P \in \mathbf{P}$ and (k_n, j_n) ; (iv) $k_n^2 j_n \log^3(n) = o(n^{1/2})$.

Assumption 4.3. For all
$$1 \leq j \leq \mathcal{J}$$
: (i) $\|\hat{\Sigma}_{j,n} - \Sigma_{j,P}\|_{o,2} = o_P(1/k_n\sqrt{j_n}\log^2(n))$

uniformly in $P \in \mathbf{P}$; (ii) The $k_n \times k_n$ matrices $\Sigma_{j,P}$ are invertible and $\|\Sigma_{j,P}\|_{o,2}$ and $\|\Sigma_{j,P}^{-1}\|_{o,2}$ are bounded uniformly in $P \in \mathbf{P}$ and k_n .

Assumptions 4.1(ii)-(iv) state the conditions on Θ_n , with Assumptions 4.1(ii)(iii) being satisfied by standard choices such as B-Splines or wavelets. Assumption 4.1(iv) is an asymptotic unbiasedness requirement – a condition that is eased by noting no requirements are imposed on the approximating space for μ_0 . The requirements on $\{q_k\}_{k=1}^{k_n}$ are imposed in Assumption 4.2(i)(iii) and are again satisfied by standard choices. Assumption 4.2(iv) states a rate condition that suffices for verifying the coupling requirements of Theorem 3.1. Assumption 4.3 imposes the requirements on the weighting matrices.

Our next result employs Theorem 3.1(ii) to obtain strong approximations.

Theorem 4.1. Let Assumptions 4.1, 4.2, 4.3 hold, $a_n = (\log(n))^{-1/2}$, and for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) \in \mathbf{B}$ let $\|\theta\|_{\mathbf{E}} = \sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}$. If $\ell_n, \ell_n^{\mathrm{u}} \downarrow 0$ satisfy $k_n \sqrt{j_n} \log^2(n) (\ell_n \vee \ell_n^{\mathrm{u}}) = o(1)$, $k_n \sqrt{j_n} \log(n) / \sqrt{n} = o(\ell_n \wedge \ell_n^{\mathrm{u}})$, then uniformly in $P \in \mathbf{P}_0$:

$$I_n(R) = U_P(R|\ell_n) + o_P(a_n)$$

$$I_n(R) - I_n(\Theta) = U_P(R|\ell_n) - U_P(\Theta|\ell_n^{\mathbf{u}}) + o_P(a_n).$$

In order to conduct inference, we next aim to estimate the distributions of $U_P(R|\ell_n)$ and $U_P(\Theta|\ell_n^u)$. To this end, we note that Θ_{0n}^r (as in (15)) is potentially non-singleton and we therefore employ a set estimator $\hat{\Theta}_n^r$ (as in (20)) to estimate the distribution of $U_P(R|\ell_n)$. In contrast, since $U_P(\Theta|\ell_n^u)$ only depends on the identified component $\{F_P(c_j|\cdot)\}_{j=1}^{\mathcal{J}}$, for the unconstrained problem we employ any minimizer $\hat{\theta}_n^u$ of Q_n over Θ_n . With regards to the local parameter space, we note that in this application

$$G_n(\theta) = \{ (\{p^{jn'}\beta_{j,h}\}_{j=1}^{\mathcal{J}}, \mu_h) : \mu_h(B) \ge \sqrt{n} \min\{r_n - \mu(B), 0\} \text{ for all } B \in \mathcal{B} \}$$
 (33)

for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu)$. Computationally, since any $\mu, \mu_h \in \mathcal{M}_n$ has the structure $\mu = \sum_{s=1}^{s_n} \alpha_s \delta_s$ and $\mu_h = \sum_{s=1}^{s_n} \alpha_{sh} \delta_s$ it follows that the constraints in (33) reduce to $\alpha_{sh} \geq \min\{r_n - \alpha_s, 0\}$ for all $1 \leq s \leq s_n$ whenever $\{\delta_s\}_{s=1}^{s_n}$ are orthogonal. Furthermore, since moments and restrictions are linear, we may let $\ell_n = +\infty$ and set

$$\hat{V}_n(\theta, R| + \infty) = \{ (\{p^{j_n} \beta_{j,h}\}_{j=1}^{\mathcal{J}}, \mu_h) : h \in G_n(\theta), \ \Upsilon_F(h) = 0 \}.$$
 (34)

For each $\theta \in \Theta_n$, we denote the bootstrap process for the j^{th} conditional moment by

$$\hat{\mathbb{W}}_{j,n}(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_i \{ \rho_j(X_i, \theta) q^{k_n}(W_i) - \frac{1}{n} \sum_{i=1}^{n} \rho_j(X_j, \theta) q^{k_n}(W_j) \}.$$

Similarly, we set $\hat{\mathbb{D}}_{j,n}[h] = -\sum_{i=1}^n q^{k_n}(W_i)p^{j_n}(W_i)'\beta_{j,h}/n$ for any $h = (\{p^{j_n'}\beta_{j,h}\}_{j=1}^{\mathcal{J}}, \mu_h)$.

Thus, the estimators of the strong approximations obtained in Theorem 4.1 equal

$$\hat{U}_{n}(R|+\infty) = \inf_{\theta \in \hat{\Theta}_{n}^{r}} \inf_{h \in \hat{V}_{n}(\theta, R|+\infty)} \{ \sum_{j=1}^{\mathcal{J}} \|\hat{\mathbb{W}}_{j,n}(\theta) + \hat{\mathbb{D}}_{j,n}[h]\|_{\hat{\Sigma}_{j,n},2} \}^{1/2}$$

$$\hat{U}_{n}(\Theta|+\infty) = \inf_{h} \{ \sum_{j=1}^{\mathcal{J}} \|\hat{\mathbb{W}}_{j,n}(\hat{\theta}_{n}^{u}) + \hat{\mathbb{D}}_{j,n}[h]\|_{\hat{\Sigma}_{j,n},2} \}^{1/2}.$$

Before stating our final assumption, we need an auxiliary result. To this end, define

$$\Gamma_n(\theta) \equiv \{ \tilde{\mu} \in \mathcal{M}_n : \tilde{\theta} = (\{ F(c_j | \cdot) \}_{j=1}^{\mathcal{J}}, \tilde{\mu}) \text{ satisfies } \Upsilon_F(\tilde{\theta}) = 0, \ \Upsilon_G(\tilde{\theta}) \le 0 \}$$
 (35)

for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu)$ – i.e. $\Gamma_n(\theta)$ is the set of distributions of η that agree with the restrictions implied by $\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}$. Our next result bounds the $\|\cdot\|_{TV}$ -Hausdorff distance between $\Gamma_n(\theta_1)$ and $\Gamma_n(\theta_2)$, which we denote by $d_H(\Gamma_n(\theta_1), \Gamma_n(\theta_2), \|\cdot\|_{TV})$.

Lemma 4.1. If the probability measures $\{\delta_s\}_{s=1}^{s_n}$ are orthogonal, then for every n there exists a constant $\zeta_n < \infty$ independent of \mathbf{P} such that

$$d_H(\Gamma_n(\theta_1), \Gamma_n(\theta_2), \|\cdot\|_{TV}) \le \zeta_n \sum_{j=1}^{\mathcal{J}} \|F_1(c_j|\cdot) - F_2(c_j|\cdot)\|_{\infty}$$

for any
$$(\{F_1(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_1) = \theta_1 \in \Theta_n \cap R \text{ and } (\{F_2(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_2) = \theta_2 \in \Theta_n \cap R.$$

We introduce our final assumption to show the validity of our bootstrap procedure.

Assumption 4.4. (i) $\Psi(g,\cdot)$ is bounded on Ω ; (ii) The probability measures $\{\delta_s\}_{s=1}^{s_n}$ are orthogonal; (iii) $k_n^4 j_n^5 \log^5(n)/n = o(1)$; (iv) $\Pi_n \theta = (\{F_n(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_n)$ satisfies $\|F_n(c_j|\cdot) - F_P(c_j|\cdot)\|_{\infty} = o(1)$ uniformly in $\theta \in \Theta_0 \cap R$ and $P \in \mathbf{P}_0$; (v) $k_n \sqrt{j_n} \log^2(n) \tau_n = o(1)$, and $\zeta_n(k_n j_n \log(n)/\sqrt{n} + \sqrt{j_n} \tau_n) = o(r_n)$.

The boundedness of $\Psi(g,\cdot)$ on Ω ensures $\Upsilon_F^{(s)}$ (as in (31)) is continuous, while Assumption 4.4(ii) allows us to apply Lemma 4.1. Assumption 4.4(iii) is a low level sufficient condition for verifying the bootstrap coupling requirement of Assumption 3.11. These rate requirements could be improved under smoothness conditions on $F_P(c_j|\cdot)$. Finally, Assumption 4.4(iv) imposes a mild requirement on the sieve, while Assumption 4.4(v) states conditions on τ_n and r_n – note $\tau_n = 0$ and $r_n = +\infty$ are always valid, though such choices can lead to lower local power against certain alternatives.

Our final result obtains a coupling for our bootstrap approximations.

Theorem 4.2. Let the conditions of Theorem 4.1 hold and Assumption 4.4 be satisfied. Then: there are sequences $\ell_n, \ell_n^{\mathrm{u}} \downarrow 0$ satisfying $k_n \sqrt{j_n} \log(n) / \sqrt{n} = o(\ell_n \wedge \ell_n^{\mathrm{u}})$ and

 $k_n\sqrt{j_n}\log^2(n)(\ell_n\vee\ell_n^{\mathrm{u}})=o(1)$ such that uniformly in $P\in\mathbf{P}_0$

$$\hat{U}_n(R|+\infty) \ge U_P^{\star}(R|\ell_n) + o_P(a_n)$$
$$\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty) \ge U_P^{\star}(R|\ell_n) - U_P^{\star}(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n).$$

In particular, since the conditions on ℓ_n and $\ell_n^{\rm u}$ imposed in Theorems 4.1 and 4.2 are the same, it follows that we may employ the quantiles of $\hat{U}_n(R|+\infty)$ and $\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty)$ conditional on the data as critical values for $I_n(R)$ and $I_n(R) - I_n(\Theta)$.

5 Simulation Evidence

To conclude, we study the finite sample performance of our inference procedure by revisiting the simulation design in Chetverikov and Wilhelm (2017).

5.1 Identified Model

We first consider a nonparametric instrumental variable model in which, for some unknown function θ_0 , the distribution of $(Y, W, Z) \in \mathbf{R}^3$ satisfies the restriction

$$Y = \theta_0(W) + \varepsilon$$
 $E[\varepsilon|Z] = 0;$ (36)

see Appendix A.2 for a formal study of this model. Following Chetverikov and Wilhelm (2017), we set $\theta_0(w) \equiv 0.2w + w^2$ and for (ϵ, ζ, ν) independent standard normal random variables we let $Z = \Phi(\zeta)$, $W = \Phi(0.3\zeta + \sqrt{1 - (0.3)^2}\epsilon)$, and $\varepsilon = (0.3\epsilon + \sqrt{1 - (0.3)^2}\nu)/2$ for Φ the cumulative distribution function of a standard normal. All reported results are based on five thousand replications employing five hundred bootstrap draws each.

In what follows, we utilize the restriction $\Upsilon_F(\theta_0) = 0$ to impose a hypothesized value on the the level or the derivative of θ_0 at the point $w_0 = 0.5$ and use $\Upsilon_G(\theta_0) \leq 0$ to impose that θ_0 be either monotonically increasing or monotonically increasing and convex. We employ the test statistic $I_n(R) - I_n(\Theta)$ with p = 2 and $\hat{\Sigma}_n$ an estimate of the optimal weighting matrix based on a first stage unconstrained estimator. The implementation of the test is similar to that of the linear model of Section 2.1, with the difference that we must select the sieve $\Theta_n = \{p^{j_n} \beta : \beta \in \mathbf{R}^{j_n}\}$ and q^{k_n} . We follow Chetverikov and Wilhelm (2017) in employing continuously differentiable piecewise quadratic splines with equally spaced knots for both p^{j_n} and q^{k_n} .

In computing critical values we set $\ell_n = +\infty$ since the model is linear and $\tau_n = 0$ since the model is identified. We select r_n by proceeding as in Section 2.1. Specifically, the choice of sieve implies that, for any $\theta = p^{j_n \prime} \beta$, the restriction $\Upsilon_G(\theta) \leq 0$ is equivalent

		Imposed: Mon.			I	Imposed: Mon.+ Conv.				
		Level		Derivative			Level		Derivative	
	$r_n/(j_n,k_n)$	(4,4)	(4,6)	(4,4)	(4,6)	(4)	,4)	(4,6)	(4,4)	(4,6)
n = 500	∞	1.90	1.72	1.88	2.02	1.	44	1.52	2.74	2.84
	95%	1.74	1.68	1.90	2.08	1.	46	1.54	2.68	2.84
	50%	1.74	1.70	1.90	2.10	1.	46	1.54	2.68	2.84
	5%	2.18	2.90	2.20	2.96	1.	52	1.82	2.74	2.98
	0	5.30	5.10	4.62	4.48	5.4	42	5.36	5.08	4.84
n = 1000	∞	1.56	1.82	1.68	1.94	1.	40	1.54	2.26	2.32
	95%	1.52	1.84	1.64	1.86	1.	36	1.44	2.04	2.26
	50%	1.52	1.86	1.64	1.86	1.	36	1.44	2.04	2.26
	5%	2.02	2.84	2.06	3.06	1.	44	1.86	2.14	2.38
	0	4.54	4.56	4.58	4.68	4.	62	4.78	4.38	4.20
n = 5000	∞	1.34	1.58	1.26	1.52	1.	04	1.36	1.36	1.58
	95%	1.40	1.50	1.32	1.62	1.	06	1.42	1.36	1.62
	50%	1.42	1.52	1.32	1.62	1.	06	1.42	1.36	1.62
	5%	2.20	3.62	2.36	3.36	1.	42	2.38	1.46	1.86
	0	3.98	4.56	4.68	4.50	4.	10	4.74	3.98	4.06

Table 2: Empirical rejection probabilities for 5%-level tests based on $I_n(R) - I_n(\Theta)$. Value of r_n set to a percentile corresponds to choice of $1 - \gamma_n$ in (37).

to $G\beta \leq 0$ for a known matrix G. For $p^{j_n'}\hat{\beta}_n^{\mathrm{u}}$ the minimizer of $I_n(\Theta)$ and $p^{j_n'}\hat{\beta}_n^{\mathrm{u}*}$ its score bootstrap analogue (Kline and Santos, 2012), we therefore set r_n to satisfy

$$P(\max_{j} G_{j} \{ \hat{\beta}_{n}^{u \star \prime} - \hat{\beta}_{n} \} \le r_{n} | \{V_{i}\}_{i=1}^{n}) = 1 - \gamma_{n}$$
(37)

where $\gamma_n \in (0,1)$ and the vectors $G_j \in \mathbf{R}^{j_n}$ depend on the shape restriction being imposed. We emphasize that the sequence γ_n must tend to zero in order for r_n to satisfy our assumptions. Finally, we employ the minimizer of $I_n(R)$ in obtaining bootstrap draws for both $\hat{U}_n(R|+\infty)$ and $\hat{U}_n(\Theta|+\infty)$; see discussion following Corollary 3.2.

Table 2 reports empirical rejection probabilities under the null hypothesis for 5%-level tests on the derivative and level of θ_0 at $w_0 = 0.5$ under different shape restrictions. With regards to r_n , we examine the extreme possible values (0 and ∞) and choices corresponding to (37) for different γ_n . In accord to theory, which requires $\gamma_n \downarrow 0$, we find that the rejection probability is no larger than the nominal level except for very small values of $1 - \gamma_n$. Overall, we find the general lack of sensitivity to different choices of bandwidths to be reassuring for empirical practice.

In Figure 3 we report power curves for different 5%-level tests concerning the value of θ_0 and its derivative at $w_0 = 0.5$. For conciseness, we focus on the sample sizes $n \in \{1000, 5000\}$ and r_n chosen as in (37) with $1 - \gamma_n = 0.95$. The curves labeled "Mon" and "Mon+Conv" correspond to tests based on $I_n(R) - I_n(\Theta)$ with R imposing monotonicity and monotonicity and convexity while changing the conjectured value of

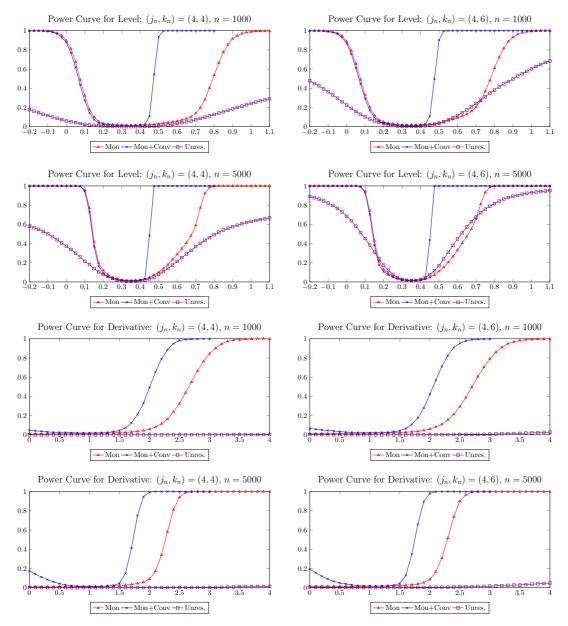


Figure 3: Rejection probabilities for 5%-level tests on conjectured value of $\theta_0(0.5)$ (true value 0.35) and $\theta'_0(0.5)$ (true value 1.2). Tests implemented with $1 - \gamma_n = 0.05$ in (37).

 θ_0 and its derivative at $w_0 = 0.5$. The curve labeled "Unres." corresponds to a Wald test based on the unrestricted estimator. For all designs we find that imposing shape restrictions can improve power. The effect of imposing shape restrictions, however, depend on both the sampling uncertainty and how "close" the shape restrictions are to binding (Chetverikov et al., 2018). Since our design is fixed with n and θ_0 is strictly increasing and convex, in our simulations we see the advantages of imposing shape restrictions decrease with n as sample uncertainty decreases. Similarly, since estimating the derivative is a harder than estimating the level, we observe larger power gains when imposing shape restrictions in the former problem.

5.2 Partially Identified Model

We next examine the performance of our test in a partially identified setting by discretizing the simulation design in Chetverikov and Wilhelm (2017). Concretely, we generate $(W, Z, \epsilon) \in [0, 1]^2 \times \mathbf{R}$ as in Section 5.1, divide [0, 1] into S_w and S_z equally spaced segments, and generate dummy variables D_w and D_z for the segment to which W and Z belong – e.g. if $(S_w, S_z) = (3, 2)$, then $D_w(W) \equiv (1\{W \in [0, 1/3]\}, 1\{W \in (1/3, 2/3]\}, 1\{W \in (2/3, 1]\})'$ and $D_z(Z) \equiv (1\{Z \in [0, 1/2]\}, 1\{Z \in (1/2, 1]\})'$. The outcome Y is generated according to (36) but employing D_w in place of W.

The discretized design is characterized by S_z linear unconditional moment restrictions in S_w unknowns. For conciseness, we focus on imposing that θ_0 be monotonically increasing and convex while conducting inference on the value of θ_0 at the point $d_0 \equiv D_w(0.5)$ – e.g, if $S_w = 3$, then $d_0 = (0,1,0)'$. The parameter $\theta_0(d_0)$ is generically not identified whenever $S_w > S_z$ but, as we report in Table 3, imposing a shape restriction on θ_0 partially identifies $\theta_0(d_0)$. A similar setting was previously studied by Freyberger and Horowitz (2015) who develop confidence regions for parameters such as $\theta_0(d_0)$. Their leading procedure is computationally simpler than ours, but can suffer from size distortions, for example, when the identified set for $\theta_0(d_0)$ is "small."

	(S_w, S_z)					
Restriction on θ_0	(3, 2)	(4, 2)	(3, 2)			
Mon.+Convex	[0.059, 0.252]	[0.100, 0.412]	[0.310, 0.388]			
No Restriction	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$			

Table 3: Identified sets for $\theta_0(d_0)$ with and without shape restrictions.

We test whether a value λ belongs to the identified set for $\theta_0(d_0)$ by setting $\Upsilon_F(\theta) = \theta(d_0) - \lambda$ and employ the constraint $\Upsilon_G(\theta) \leq 0$ to impose that θ be monotonically increasing and convex. We base inference on $I_n(R)$ with p=2, $\hat{\Sigma}_n$ the sample analogue to $E[D_zD_z']$, all moment restrictions $(k_n=S_z)$, and a saturated model for θ_0 $(j_n=S_w)$. To compute critical values we set $\ell_n=+\infty$ and $\tau_n=0$ —though note $\hat{\Theta}_n^r$ need not be a singleton when $\tau_n=0$ because $j_n>k_n$. We select r_n by modifying the approach employed in Section 5.1. Specifically, we note that the constraint $\Upsilon_G(\theta)\leq 0$ may be written as $G\theta\leq 0$ for some matrix G, and for $\hat{\theta}_n^L$ and $\hat{\theta}_n^U$ the minimizer and maximizers of $\theta(d_0)$ over the set of θ that are monotonically increasing, convex, and minimize $\|\sum_{i=1}^n (Y_i-\theta(D_{w,i}))D_{z,i}/n\|_{\infty}$, we set r_n according to

$$P(\max_{j} \max\{G_{j}(\hat{\theta}_{n}^{L\star} - \hat{\theta}_{n}^{L}), G_{j}(\hat{\theta}_{n}^{U\star} - \hat{\theta}_{n}^{U})\} \le r_{n}|\{V_{i}\}_{i=1}^{n}) = 1 - \gamma_{n}, \tag{38}$$

where $\hat{\theta}_n^{L\star}$ and $\hat{\theta}_n^{U\star}$ are again computed employing the score bootstrap. As in our previous analysis, γ_n must tend to zero with n in order for r_n to satisfy our assumptions.

		Lower Endpoint				Midpoint			Upper Endpoint		
		(S_w, S_z)				(S_w, S_z)			(S_w, S_z)		
	r_n	(3,2)	(4,2)	(4,3)	(3,2)	(4,2)	(4,3)	(3,2)	(4,2)	(4,3)	
n = 500	∞	1.96	3.34	1.48	0.10	0.02	1.48	1.88	3.10	2.00	
	95%	3.64	4.70	1.46	0.10	0.02	1.46	2.26	3.12	1.98	
	50%	5.34	5.24	1.46	0.50	0.06	1.50	5.22	5.02	2.04	
	5%	5.36	5.24	3.56	0.50	0.06	3.44	5.24	5.02	3.54	
	0	5.34	5.26	4.64	0.50	0.06	4.48	5.24	5.16	4.60	
n = 1000	∞	1.84	3.06	1.12	0.00	0.00	1.10	1.96	2.90	1.34	
	95%	4.98	4.84	1.12	0.02	0.00	1.08	2.98	2.90	1.34	
	50%	5.10	4.88	1.20	0.12	0.00	1.14	5.00	4.86	1.44	
	5%	5.10	4.88	3.48	0.12	0.00	3.12	5.00	4.86	2.78	
	0	5.28	4.88	4.42	0.08	0.00	4.14	5.10	4.86	3.82	
n = 5000	∞	1.98	4.40	1.34	0.00	0.00	1.22	1.98	2.80	1.36	
	95%	5.08	6.76	1.34	0.00	0.00	1.26	4.56	4.86	1.34	
	50%	5.08	8.30	1.48	0.00	0.00	1.44	4.58	4.84	1.52	
	5%	5.08	9.00	4.28	0.00	0.00	4.14	4.58	4.84	3.58	
	0	4.96	8.84	4.70	0.00	0.00	4.38	4.64	5.02	4.46	

Table 4: Empirical rejection probabilities for 5%-level tests based on $I_n(R)$ for different points in the null hypothesis. Lower and upper endpoints correspond to Table 3.

Table 4 reports empirical rejection rates for testing whether λ belongs to the identified set, with the lower and upper endpoint columns corresponding to setting λ to equal the lower and upper endpoints in Table 3. All tests are conducted at a 5% nominal level. Across designs, we find that setting $r_n = +\infty$ always delivers tests with rejection probabilities below their nominal level. Setting r_n according to (38) with $1 - \gamma_n = 0.95$ also delivers adequate size control, with the exception of n = 5000 and $(S_w, S_z) = (4, 2)$ where we see a modest over-rejection at the lower endpoint of the identified set. Overall, the degree of sensitivity to the choice of r_n is similar to that found in Section 5.1.

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Supplemental Appendix I

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This Supplemental Appendix to "Constrained Conditional Moment Restriction Models" is organized as follows. Sections A.1 provides a review of AM spaces. Section A.2 specializes the general results of Section 3 to three additional examples: (i) GMM, (ii) Quantile Treatment Effects, and (iii) The Slutsky restriction in a partially linear model. The proofs for all results are included in Supplmental Appendix II.

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A.1 AM Spaces

We provide a brief introduction to AM spaces and refer the reader to Chapters 8 and 9 of Aliprantis and Border (2006) for a more detailed exposition. Before proceeding, we first recall the definitions of a partially ordered set and a lattice.

Definition A.1.1. A partially ordered set (\mathbf{G}, \geq) is a set \mathbf{G} with a partial order relationship \geq defined on it – i.e. \geq is a transitive $(x \geq y \text{ and } y \geq z \text{ implies } x \geq z)$, reflexive $(x \geq x)$, and antisymmetric $(x \geq y \text{ implies the negation of } y \geq x)$ relation.

Definition A.1.2. A *lattice* is a partially ordered set (\mathbf{G}, \geq) such that any pair $x, y \in \mathbf{G}$ has a least upper bound (denoted $x \vee y$) and a greatest lower bound (denoted $x \wedge y$).

Whenever **G** is both a vector space and a lattice, it is possible to define objects that depend on both the vector space and lattice operations. In particular, for $x \in \mathbf{G}$ we define the positive part $x^+ \equiv x \vee 0$, the negative part $x^- \equiv (-x) \vee 0$, and the absolute value $|x| \equiv x \vee (-x)$. It is also natural to demand that the order relation \geq interact with the algebraic operations in a manner analogous to that of \mathbf{R} – i.e. to have

$$x \ge y \text{ implies } x + z \ge y + z \text{ for each } z \in \mathbf{G}$$
 (A.1)

$$x \ge y$$
 implies $\alpha x \ge \alpha y$ for each $0 \le \alpha \in \mathbf{R}$. (A.2)

A complete normed vector space that shares these familiar properties of \mathbf{R} under a given order relation \geq is referred to as a *Banach lattice*. Formally, we define:

Definition A.1.3. A Banach space **G** with norm $\|\cdot\|_{\mathbf{G}}$ is a *Banach lattice* if (i) **G** is a lattice under \geq , (ii) $\|x\|_{\mathbf{G}} \leq \|y\|_{\mathbf{G}}$ when $|x| \leq |y|$, (iii) (A.1) and (A.2) hold.

An AM space is a Banach lattice in which the maximum of the norms of any two positive elements is equal to the norm of the maximums of the two elements.

Definition A.1.4. A Banach lattice **G** is called an AM space if for any elements $0 \le x, y \in \mathbf{G}$ it follows that $||x \vee y||_{\mathbf{G}} = \max\{||x||_{\mathbf{G}}, ||y||_{\mathbf{G}}\}$.

In certain Banach lattices there may exist an element $\mathbf{1}_{\mathbf{G}} > 0$ called an *order unit* such that for any $x \in \mathbf{G}$ there exists a $0 < \lambda \in \mathbf{R}$ for which $|x| \le \lambda \mathbf{1}_{\mathbf{G}}$ – for example, in \mathbf{R}^d the vector $(1, \ldots, 1)'$ is an order unit. The order unit $\mathbf{1}_{\mathbf{G}}$ can be used to define

$$||x||_{\infty} \equiv \inf\{\lambda > 0 : |x| \le \lambda \mathbf{1}_{\mathbf{G}}\},\tag{A.3}$$

which is a norm on \mathbf{G} . In principle, $\|\cdot\|_{\infty}$ need not be related to the original norm $\|\cdot\|_{\mathbf{G}}$. However, if \mathbf{G} is an AM space, then $\|\cdot\|_{\mathbf{G}}$ and $\|\cdot\|_{\infty}$ are equivalent in that they generate the same topology. Hence, we refer to \mathbf{G} as an AM space with unit $\mathbf{1}_{\mathbf{G}}$ if: (i) \mathbf{G} is an AM space, (ii) $\mathbf{1}_{\mathbf{G}}$ is an order unit in \mathbf{G} , and (iii) The norm of \mathbf{G} equals $\|\cdot\|_{\infty}$.

A.2 Illustrative Examples

In this Section, we examine special cases of our general analysis and illustrate both how to implement our procedure and verify the assumptions in the main text.

A.2.1 Generalized Method of Moments

Our first example concerns the generalized method of moments (GMM) model of Hansen (1982). We assume the parameter of interest θ_0 is identified as the unique solution to

$$E_P[\rho(X,\theta_0)] = 0, (A.4)$$

where $X \in \mathbf{X}$ is distributed according to $P \in \mathbf{P}$ and $\rho : \mathbf{X} \times \Theta \to \mathbf{R}^{\mathcal{J}}$. This model maps into our general framework by letting $Z_{\jmath} = 1$ for all $1 \leq \jmath \leq \mathcal{J}$. Moreover, since we have assumed θ_0 is identified, the hypothesis testing problem simplifies to

$$H_0: \theta_0 \in R$$
 $H_1: \theta_0 \notin R$.

The set R is, as in the main text, defined by equality and inequality restrictions. In particular, for known functions $\Upsilon_F: \mathbf{R}^{d_\theta} \to \mathbf{R}^{d_F}$ and $\Upsilon_G: \mathbf{R}^{d_\theta} \to \mathbf{R}^{d_G}$ we set

$$R \equiv \{ \theta \in \mathbf{R}^{d_{\theta}} : \Upsilon_F(\theta) = 0 \text{ and } \Upsilon_G(\theta) \le 0 \}.$$
(A.5)

To verify Assumptions 3.1(ii)(iii), note \mathbf{R}^d is a Banach space under any norm $\|\cdot\|_p$ with $1 \leq p \leq \infty$, so for concreteness we set $\mathbf{B} = \mathbf{R}^{d_\theta}$, $\mathbf{F} = \mathbf{R}^{d_F}$, and $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{F}} = \|\cdot\|_2$. The space \mathbf{R}^d is in addition a lattice under the standard pointwise partial order

$$a \le b$$
 if and only if $a_i \le b_i$ for all $1 \le i \le d$ (A.6)

for any $(a_1, \ldots, a_d)' = a$ and $(b_1, \ldots, b_d)' = b$ in \mathbf{R}^d , while the least upper bound equals

$$a \vee b = (\max\{a_1, b_1\}, \dots, \max\{a_d, b_d\})'.$$

The vector (1, ..., 1)' is an order unit in \mathbf{R}^d under the partial order in (A.6). As discussed in Section A.1 of this Supplemental Appendix, the order unit induces the norm

$$\{\inf \lambda > 0 : |a| \le \lambda(1, \dots, 1)'\} = \max_{1 \le i \le d} |a_i|,$$

which corresponds to the usual $\|\cdot\|_{\infty}$ norm on \mathbf{R}^d . Hence, by setting $\mathbf{G} = \mathbf{R}^{d_G}$, $\|\cdot\|_{\mathbf{G}} = \|\cdot\|_{\infty}$, and $\mathbf{1}_{\mathbf{G}} = (1, \dots, 1)'$ we verify the requirements of Assumption 3.1(ii)(iii).

Since the parameter space Θ is finite dimensional and all moment restrictions are

unconditional, we may set $\Theta_n = \Theta$ and $k_n = \mathcal{J}$ for all n. We base our test statistic on quadratic forms in the moments (p = 2), which implies $Q_n(\theta)$ is given by

$$Q_n(\theta) \equiv \|\hat{\Sigma}_n\{\frac{1}{n}\sum_{i=1}^n \rho(X_i, \theta)\}\|_2.$$

In what follows, we consider tests based on both the un-centered statistic $I_n(R)$ and the re-centered statistic $I_n(R) - I_n(\Theta)$. To this end, we impose the following:

Assumption A.2.1. (i) $\{X_i\}_{i=1}^n$ is i.i.d. with $X_i \sim P \in \mathbf{P}$; (ii) For each $P \in \mathbf{P}_0$ there exists a unique $\theta_0 \in \Theta$ solving (A.4); (iii) Θ is convex and compact.

Assumption A.2.2. (i) The function $\rho(x,\cdot): \Theta \to \mathbf{R}^{\mathcal{J}}$ is twice differentiable for all x; (ii) $E_P[\sup_{\theta \in \Theta} \|\rho(X,\theta)\|_2^3]$, $E_P[\sup_{\theta \in \Theta} \|\nabla_{\theta}\rho(X,\theta)\|_{o,2}^2]$, $E_P[\sup_{\theta \in \Theta} \|\nabla_{\theta}^2\rho_{\jmath}(X,\theta)\|_{o,2}^{1+\delta}]$ are finite and bounded uniformly in $P \in \mathbf{P}$ for some $\delta > 0$.

Assumption A.2.3. (i) $\inf_{P \in \mathbf{P}_0} \inf_{\theta \in \Theta: \|\theta - \theta_0\|_2 \ge \epsilon} \|E_P[\rho(X, \theta)]\|_2 > 0$ for all $\epsilon > 0$; (ii) The singular values of $E_P[\nabla_{\theta} \rho(X, \theta_0)]$ are bounded away from zero in $P \in \mathbf{P}_0$.

Assumption A.2.4. (i) $\|\hat{\Sigma}_n - \Sigma_P\|_{o,2} = O_P(n^{-1/2})$ uniformly in $P \in \mathbf{P}$; (ii) Σ_P is invertible and $\|\Sigma_P\|_{o,2}$ and $\|\Sigma_P^{-1}\|_{o,2}$ are bounded uniformly in $P \in \mathbf{P}$.

In Assumption A.2.2 we focus on differentiable moments for simplicity. Assumption A.2.3 essentially imposes strong identification of θ_0 and hence guarantees that θ_0 can be consistently estimated uniformly in $P \in \mathbf{P}_0$ – recall that θ_0 depends on P through (A.4), though the dependence is left implicit in the notation. Finally, Assumption A.2.4 states the requirements on the $\mathcal{J} \times \mathcal{J}$ weighting matrix $\hat{\Sigma}_n$.

In what follows, we set the local parameter spaces $V_n(\theta, R|\ell)$ and $V_n(\theta, \Theta|\ell)$ to equal

$$V_n(\theta, R|\ell) = \{ h \in \mathbf{R}^{d_{\theta}} : \theta + h/\sqrt{n} \in \Theta \cap R \text{ and } \|h/\sqrt{n}\|_2 \le \ell \}$$
$$V_n(\theta, \Theta|\ell) = \{ h \in \mathbf{R}^{d_{\theta}} : \theta + h/\sqrt{n} \in \Theta \text{ and } \|h/\sqrt{n}\|_2 \le \ell \}.$$

Setting $\mathbb{D}_P(\theta_0)[h] \equiv E_P[\nabla_{\theta}\rho(X,\theta_0)]h$ and letting $\mathbb{W}_P(\theta_0) \sim N(0, \operatorname{Var}_P\{\rho(X,\theta_0)\})$ we then denote the variables to which $I_n(R)$ and $I_n(\Theta)$ will be coupled to by

$$U_P(R|\ell_n) \equiv \inf_{h \in V_n(\theta_0, R|\ell_n)} \| \mathbb{W}_P(\theta_0) + \mathbb{D}_P(\theta_0)[h] \|_{\Sigma_P, 2}$$
$$U_P(\Theta|\ell_n) \equiv \inf_{h \in V_n(\theta_0, \Theta|\ell_n)} \| \mathbb{W}_P(\theta_0) + \mathbb{D}_P(\theta_0)[h] \|_{\Sigma_P, 2}.$$

Our distributional approximations follow immediately from Theorem 3.1(ii).

Theorem A.2.1. Let Assumptions A.2.1, A.2.2, A.2.3, and A.2.4 hold, Υ_F and Υ_G be continuous, and set $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d_\theta}}$. Then: For any $\ell_n, \ell_n^u \downarrow 0$ satisfying

 $(\ell_n \vee \ell_n^{\mathrm{u}})\sqrt{\log(1/\ell_n \vee \ell_n^{\mathrm{u}})} = o(a_n)$ and $n^{-1/2} = o(\ell_n \vee \ell_n^{\mathrm{u}})$ we have uniformly in $P \in \mathbf{P}_0$

$$I_n(R) = U_P(R|\ell_n) + o_P(a_n)$$

$$I_n(R) - I_n(\Theta) = U_P(R|\ell_n) - U_P(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n).$$

The rate of coupling $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d_{\theta}}}$ obtained in Theorem A.2.1 suffices for both the empirical process and bootstrap coupling; see Lemmas S.4.12 and S.4.13 in Supplemental Appendix II. While the rate is adequate for our purposes, it can be improved under additional moment restrictions. Here, we rely in Yurinskii (1977) both to illustrate the diversity of coupling arguments that can be employed to verify Assumption 3.3(i) and to impose only the weak third moment restriction of Assumption A.2.2(ii).

Our next goal is to obtain bootstrap approximations to the distributions of $U_P(R|\ell_n)$ and $U_P(\Theta|\ell_n^u)$. To this end, we write $\Upsilon_F(\theta) = (\Upsilon_{F,1}(\theta), \dots, \Upsilon_{F,d_F}(\theta))'$ and $\Upsilon_G(\theta) = (\Upsilon_{G,1}(\theta), \dots, \Upsilon_{G,d_G}(\theta))'$, for any $\epsilon > 0$ we define $B^{\epsilon} \equiv \bigcup_{P \in \mathbf{P}_0} \{\theta : \|\theta - \theta_0\|_2 \le \epsilon\}$ (where recall θ_0 implicitly depends on P through (A.4)), and impose:

Assumption A.2.5. For some $\epsilon > 0$: (i) $B^{\epsilon} \subseteq \Theta$; (ii) Υ_F and Υ_G are twice differentiable on B^{ϵ} ; (iii) $\|\nabla \Upsilon_F(\theta)\|_{o,2}$ and $\|\nabla \Upsilon_G(\theta)\|_{o,2}$ are bounded on B^{ϵ} ; (iv) $\|\nabla^2 \Upsilon_{F,j}(\theta)\|_{o,2}$ is bounded on B^{ϵ} for $1 \leq j \leq d_F$; (v) $\|\nabla^2 \Upsilon_{G,j}(\theta)\|_{o,2}$ is bounded on B^{ϵ} for $1 \leq j \leq d_G$; (vi) $\nabla \Upsilon_F(\theta)$ has full row-rank on B^{ϵ} .

Assumption A.2.6. Either (i) $\Upsilon_F : \mathbf{R}^{d_{\theta}} \to \mathbf{R}^{d_F}$ is affine, or (ii) There is an $\epsilon > 0$ and $M < \infty$ such that the singular values of $\nabla \Upsilon_F(\theta)'$ are bounded away from zero uniformly in $\theta \in B^{\epsilon}$, and for every $P \in \mathbf{P}_0$ there is an $h \in \mathcal{N}(\nabla \Upsilon_F(\theta_0))$ with $||h||_2 \leq M$ satisfying $\Upsilon_{G,j}(\theta_0) + \nabla \Upsilon_{G,j}(\theta_0)[h] \leq -\epsilon$ for all $1 \leq j \leq d_G$.

In order to describe our bootstrap procedure in this application, we let $\hat{\theta}_n$ and $\hat{\theta}_n^{\mathrm{u}}$ denote the minimizers of Q_n over $\Theta \cap R$ and Θ respectively. Employing $\hat{\theta}_n$ and $\hat{\theta}_n^{\mathrm{u}}$ we obtain estimators for the distribution of $\mathbb{W}_P(\theta_0)$ and for $\mathbb{D}_P(\theta_0)$ by evaluating

$$\hat{\mathbb{W}}_n(\theta) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ \rho(X_i, \theta) - \frac{1}{n} \sum_{j=1}^n \rho(X_j, \theta) \}$$
(A.7)

$$\hat{\mathbb{D}}_n(\theta) \equiv \frac{1}{n} \sum_{i=1}^n \nabla_{\theta} \rho(X_i, \theta), \tag{A.8}$$

at $\theta = \hat{\theta}_n$ and $\theta = \hat{\theta}_n^{\text{u}}$, where recall $\{\omega_i\}_{i=1}^n$ is an i.i.d. sample independent of $\{X_i\}_{i=1}^n$ with $\omega_i \sim N(0,1)$. We note that because moments are differentiable, we employ an analytical derivative in (A.8) instead of the numerical derivative studied in Section 3.

With regards to the local parameter space, we note that the construction of $\hat{V}_n(\theta, R|\ell)$ requires the bound K_g on the second derivative of Υ_G (as specified in Assumption 3.8).

In particular, Assumption A.2.5(v) implies Assumption 3.8 is satisfied with

$$K_g \equiv \max_{1 \le j \le d_G} \sup_{\theta \in R^{\epsilon}} \|\nabla_{\theta}^2 \Upsilon_{G,j}(\theta)\|_{o,2}$$

(see Lemma S.4.14). If an a-priory bound on the second derivative is not available, then it is also possible to simply substitute K_g with the data driven choice

$$\hat{K}_g \equiv \max_{1 \le j \le d_G} \sup_{\theta \in \Theta: \|\theta - \hat{\theta}_n\|_2 < r_n} \|\nabla_{\theta}^2 \Upsilon_{G,j}(\theta)\|_{o,2},$$

where we discuss the choice of r_n below. Given K_g (or \hat{K}_g), we set $G_n(\theta)$ to equal

$$G_n(\theta) = \{ h \in \mathbf{R}^{d_{\theta}} : \Upsilon_{j,G}(\theta + \frac{h}{\sqrt{n}}) \le \max\{\Upsilon_{j,G}(\theta) - K_g r_n \| \frac{h}{\sqrt{n}} \|_2, -r_n \} \text{ for all } j \}$$

In this application we may additionally specify ℓ_n to be infinite, and hence we set

$$\hat{V}_n(\theta, R| + \infty) = \{ h \in \mathbf{R}^{d_\theta} : h \in G_n(\theta) \text{ and } \Upsilon_F(\theta + \frac{h}{\sqrt{n}}) = 0 \}.$$

The approximations to the distributions of $I_n(R)$ and $I_n(\Theta)$ are then given by the laws of $\hat{U}_n(R|+\infty)$ and $\hat{U}_n(\Theta|+\infty)$ conditional on the data, where

$$\hat{U}_n(R|+\infty) \equiv \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|+\infty)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n(\hat{\theta}_n)[h]\|_{\hat{\Sigma}_n, 2}$$
$$\hat{U}_n(\Theta|+\infty) \equiv \inf_{h \in \mathbf{R}^{d_\theta}} \|\hat{\mathbb{W}}_n(\hat{\theta}_n^{\mathbf{u}}) + \hat{\mathbb{D}}_n(\hat{\theta}_n^{\mathbf{u}})[h]\|_{\hat{\Sigma}_n, 2}.$$

The validity of these distributional approximations follows from Theorem 3.2.

Theorem A.2.2. Let Assumptions A.2.1, A.2.2, A.2.3, A.2.4, A.2.5, and A.2.6 hold, set $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d_{\theta}}}$, and let $n^{-1/2} = o(r_n)$. Then: there are sequences $\ell_n, \ell_n^{\mathrm{u}} \downarrow 0$ satisfying $(\ell_n \vee \ell_n^{\mathrm{u}})^2 \sqrt{\log(1/(\ell_n \vee \ell_n^{\mathrm{u}}))} = o(a_n n^{-\frac{1}{2}})$, $\ell_n = o(r_n)$, and $n^{-\frac{1}{2}} = o(\ell_n \wedge \ell_n^{\mathrm{u}})$ for which it follows uniformly in $P \in \mathbf{P}_0$ that

$$\hat{U}_n(R|+\infty) \ge U_P^{\star}(R|\ell_n) + o_P(a_n)$$

$$\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty) \ge U_P^{\star}(R|\ell_n) - U_P^{\star}(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n).$$

Crucially, note that any sequences ℓ_n and $\ell_n^{\rm u}$ satisfying the conditions of Theorem A.2.2 also satisfy the conditions of Theorem A.2.1. Therefore, Theorems A.2.2 and A.2.1 together establish the validity of employing the laws of $\hat{U}_n(R|+\infty)$ and $\hat{U}_n(\Theta|+\infty)$ conditional on the data to approximate the laws of $I_n(R)$ and $I_n(\Theta)$. In particular, for a level α test we may compare the test statistic $I_n(R)$ to the critical value

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|+\infty)) \equiv \inf\{c : P(\hat{U}_n(R|+\infty) \le c|\{X_i\}_{i=1}^n) \ge 1-\alpha\}.$$

Similarly, for the re-centered statistic $I_n(R) - I_n(\Theta)$, valid critical values are given by:

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty))$$

$$\equiv \inf\{c : P(\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty) \le c | \{X_i\}_{i=1}^n) \ge 1 - \alpha\}.$$

These approximations are valid under the requirement that r_n satisfy $r_n\sqrt{n} \to \infty$. Intuitively, the bandwidth r_n is meant to reflect a bound on the distance between $\hat{\theta}_n$ and θ_0 . For a data driven choice of r_n we may therefore employ a bootstrap estimate of an upper quantile of the distribution of the *unconstrained* estimator. Specifically, for $\hat{\theta}_n^{\text{ux}}$ the bootstrapped version of $\hat{\theta}_n^{\text{u}}$, we may set \hat{r}_n to be given by

$$\hat{r}_n \equiv \inf\{c : P(\|\hat{\theta}_n^{u\star} - \hat{\theta}_n^u\|_2 \le c | \{X_i\}_{i=1}^n)\} \ge 1 - \gamma_n$$

for $\gamma_n \to 0$ as the sample size n tends to infinity, and employ \hat{r}_n in place of r_n .

A.2.2 Consumer Demand

We base our next example on a long-standing literature aiming to replace parametric assumptions with shape restrictions implied by economic theory (Matzkin, 1994). Specifically, suppose that quantity demanded by individual i, denoted Q_i , satisfies

$$Q_i = g_0(S_i, Y_i) + W_i' \gamma_0 + U_i,$$

where $S_i \in \mathbf{R}_+$ denotes price, $Y_i \in \mathbf{R}_+$ denotes income, and $W_i \in \mathbf{R}^{d_w}$ is a set of covariates. In addition, we assume there is an instrument Z_i yielding the restriction

$$E_P[Q - g_0(S, Y) - W'\gamma_0|Z] = 0. (A.9)$$

For instance, under exogeneity of prices we may let Z = (S, Y, W')' as in Blundell et al. (2012). Alternatively, if there is a concern that prices are endogenous, then we may set Z = (I, Y, W')' for I an instrument for S, as in Blundell et al. (2017).

Our goal is to conduct inference on the level of demand at particular price income pair (s_0, y_0) while imposing that the function g_0 satisfies the Slutsky restriction

$$\frac{\partial}{\partial s}g_0(s,y) + g_0(s,y)\frac{\partial}{\partial y}g_0(s,y) \le 0. \tag{A.10}$$

To map this problem into our framework, we assume that for some set Ω , $(S, Y) \in \Omega \subseteq \mathbb{R}^2_+$ with probability one for all $P \in \mathbb{P}$ and impose that $g_0 \in C_B^1(\Omega)$, where

$$C_B^m(\Omega) \equiv \{g : \Omega \to \mathbf{R} \text{ s.t. } \|g\|_{m,\infty} < \infty\} \qquad \|g\|_{m,\infty} \equiv \sup_{0 \le \alpha \le m} \sup_{(s,y) \in \Omega} |\nabla^{\alpha} g(s,y)|.$$

Since $\theta_0 \equiv (g_0, \gamma_0)$ with $\gamma_0 \in \mathbf{R}^{d_w}$, we set $\mathbf{B} = C_B^1(\Omega) \times \mathbf{R}^{d_w}$ and for any $(g, \gamma) = \theta \in \mathbf{B}$ let $\|\theta\|_{\mathbf{B}} = \max\{\|g\|_{1,\infty}, \|\gamma\|_2\}$. We also note that X = (Q, S, Y, W) and

$$\rho(X,\theta) = Q - g(S,Y) - W'\gamma. \tag{A.11}$$

We will assume $\theta_0 \equiv (g_0, \gamma_0)$ is identified by (A.9). Hence, we can think of θ_0 as a function of P through (A.9), though we leave such dependence implicit in the notation.

In order to impose the Slutsky restriction in (A.10) we let $\mathbf{G} = C_B^0(\Omega)$ and $\|\cdot\|_{\mathbf{G}} = \|\cdot\|_{\infty}$, where with some abuse of notation we write $\|\cdot\|_{\infty}$ in place of $\|\cdot\|_{0,\infty}$. The space $C_B^0(\Omega)$ is a Banach lattice under the standard pointwise ordering given by

$$a \le b$$
 if and only if $a(s, y) \le b(s, y)$ for all $(s, y) \in \Omega$ (A.12)

for any $a, b \in C_B^0(\Omega)$. The constant function $\mathbf{c} \in C_B^0(\Omega)$ satisfying $\mathbf{c}(s, y) = 1$ for all $(s, y) \in \Omega$ is an order unit under the partial ordering in (A.12). Its induced norm is

$$\{\inf \lambda > 0 : |a| \le \lambda \mathbf{c}\} = \sup_{(s,y) \in \Omega} |a(s,y)|,$$

which coincides with the norm $\|\cdot\|_{\infty}$ on $C_B^0(\Omega)$, and we therefore set $\mathbf{1}_{\mathbf{G}} = \mathbf{c}$. To encode the Slutsky restriction in (A.10) we then let the map $\Upsilon_G : \mathbf{B} \to \mathbf{G}$ equal

$$\Upsilon_G(\theta)(s,y) = \frac{\partial}{\partial s}g(s,y) + g(s,y)\frac{\partial}{\partial y}g(s,y)$$
 (A.13)

for any $\theta = (g, \gamma) \in \mathbf{B}$. Finally, to test whether the level of demand at a prescribed price s_0 and income y_0 equals a hypothesized value c_0 , we set $\mathbf{F} = \mathbf{R}$, $\|\cdot\|_{\mathbf{F}} = |\cdot|$, and

$$\Upsilon_F(\theta) = g(s_0, y_0) - c_0 \tag{A.14}$$

for any $\theta = (g, \gamma) \in \mathbf{B}$. By setting $R = \{\theta \in \mathbf{B} : \Upsilon_G(\theta) \leq 0 \text{ and } \Upsilon_F(\theta) = 0\}$ and conducting test inversion (over different values of c_0) of the null hypothesis

$$H_0: \theta_0 \in R$$
 $H_1: \theta_0 \notin R$

we may obtain a confidence region for the level of demand at price s_0 and income y_0 .

We set the parameter space to be a ball in **B** under $\|\cdot\|_{\mathbf{B}}$ by letting Θ be equal to

$$\Theta = \{ (g, \gamma) \in C_R^1(\Omega) \times \mathbf{R}^{d_w} : ||g||_{1,\infty} \le C_0 \text{ and } ||\gamma||_2 \le C_0 \}$$
 (A.15)

for some $C_0 < \infty$. Given a sequence of approximating functions $\{p_j\}_{j=1}^{j_n}$, we then let $p^{j_n}(s,y) \equiv (p_1(s,y),\ldots,p_{j_n}(s,y))'$ and set the sieve Θ_n to equal

$$\Theta_n \equiv \{(p^{j_n}{}'\beta, \gamma) : \|p^{j_n}{}'\beta\|_{1,\infty} \le C_0 \text{ and } \|\gamma\|_2 \le C_0\}.$$

Similarly, for a sequence $\{q_k\}_{k=1}^{k_n}$ of transformations of the conditioning variable Z, we let $q^{k_n}(z) \equiv (q_1(z), \dots, q_{k_n}(z))'$. We base our test statistic on the quadratic forms

$$Q_n(\theta) \equiv \|\hat{\Sigma}_n\{\frac{1}{n}\sum_{i=1}^n (Q_i - g(S_i, Y_i) - W_i'\gamma)q^{k_n}(Z_i)\}\|_2$$

for some $k_n \times k_n$ weighting matrix $\hat{\Sigma}_n$ and every $(g, \gamma) = \theta \in \Theta$. The statistics $I_n(R)$ and $I_n(\Theta)$ simply equal the minimums of $\sqrt{n}Q_n(\theta)$ over $\Theta_n \cap R$ and Θ_n respectively.

The next assumptions suffice for obtaining a strong approximation. In their statement, the notation $sing\{A\}$ denotes the smallest singular value of a matrix A.

Assumption A.2.7. (i) $\{X_i, Z_i\}_{i=1}^n$ is i.i.d. with (X, Z) distributed according to $P \in \mathbf{P}$; (ii) For Θ as in (A.15) and each $P \in \mathbf{P}_0$ there exists a unique $\theta_0 \in \Theta$ satisfying $E_P[\rho(X, \theta_0)|Z] = 0$; (iii) The support of (Q, W) is bounded uniformly in $P \in \mathbf{P}$.

Assumption A.2.8. (i) $\sup_{(s,y)} \|p^{j_n}(s,y)\|_2 \lesssim \sqrt{j_n}$; (ii) $\sup_{(s,y)} \|\partial_a p^{j_n}(s,y)\|_2 \lesssim j_n^{3/2}$ for $a \in \{s,y\}$; (iii) The eigenvalues of $E_P[p^{j_n}(S,Y)p^{j_n}(S,Y)']$ are bounded away from zero and infinity uniformly in $P \in \mathbf{P}$ and j_n ; (iv) For each $P \in \mathbf{P}_0$ there is a $\prod_n \theta_0 = (g_n, \gamma_0) \in \Theta_n \cap R$ with $\sup_{P \in \mathbf{P}_0} \|E_P[(g_0(S,Y) - g_n(S,Y))q^{k_n}(Z)\|_2 = o((n\log(n))^{-1/2})$.

Assumption A.2.9. (i) $\max_{1 \le k \le k_n} \|q_k\|_{\infty} \lesssim \sqrt{k_n}$; (ii) $E_P[q^{k_n}(Z)q^{k_n}(Z)']$ has eigenvalues bounded uniformly in $P \in \mathbf{P}$, k_n ; (iii) $\mathbf{s}_n \equiv \inf_{P \in \mathbf{P}} \underline{\sup} \{ E_P[q^{k_n}(Z)(p^{j_n}(S,Y)' \ W')] \}$ satisfies $0 < \mathbf{s}_n = O(1)$; (iv) $j_n^2 k_n^3 \log^3(n) = o(n)$ and $k_n^2 j_n \log^{3/2}(1 + k_n)/(\mathbf{s}_n \sqrt{n})(1 \vee \sqrt{\log(\mathbf{s}_n \sqrt{n}/k_n)}) = o((\log(n))^{-1/2})$.

Assumption A.2.10. (i) $\|\hat{\Sigma}_n - \Sigma_P\|_{o,2} = o_P((k_n\sqrt{j_n}\log^{3/2}(n))^{-1})$ uniformly in $P \in \mathbf{P}$; (ii) Σ_P is invertible and $\|\Sigma_P\|_{o,2}$ and $\|\Sigma_P^{-1}\|_{o,2}$ are bounded in $P \in \mathbf{P}$ and k_n .

Assumption A.2.7(iii) requires (Q, W) to be bounded, which enables us to apply the recent coupling results by Zhai (2018). Alternatively, Assumption A.2.7(iii) can be relaxed under appropriate tail conditions. Assumptions A.2.8(i)-(iii) are standard requirements on Θ_n that can be satisfied by, e.g., tensor product wavelets or B-splines (Newey, 1997; Chen, 2007; Belloni et al., 2015; Chen and Christensen, 2018). Assumption A.2.8(iv) pertains the approximating requirements on the sieve; see Remarks A.2.1 and A.2.2 below. In turn, Assumption A.2.9(i)(ii) imposes standard requirements on $\{q_k\}_{k=1}^{k_n}$. Assumption A.2.9(iii)(iv) contains the required rate conditions, which are governed by \mathbf{s}_n – a parameter that is proportional to ν_n^{-1} (as in Assumption 3.4) and is closely linked the degree of ill-posedness; see Remark A.2.2 below. Finally, Assumption A.2.10 states the conditions on the weighting matrix $\hat{\Sigma}_n$.

In this application, we may set $\|\theta\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|g\|_{P,2} + \|\gamma\|_2$ for any $(g, \gamma) \in \Theta$.

Since in addition any $\theta = (g, \gamma) \in \Theta_n \cap R$ has the structure $g = p^{j_n \prime} \beta$, we have

$$V_n(\theta, R|\ell) = \left\{ (p^{j_n \prime} \beta_h, \gamma_h) : \|g + \frac{p^{j_n \prime} \beta_h}{\sqrt{n}} \|_{1,\infty} \le C_0 \text{ and } \|\gamma + \frac{\gamma_h}{\sqrt{n}} \|_2 \le C_0 \right.$$
 (A.16)

$$p^{j_n}(s_0, y_0)'\beta_h = 0 (A.17)$$

$$\frac{\partial}{\partial s} \left(g + \frac{p^{j_n \prime} \beta_h}{\sqrt{n}}\right) + \left(g + \frac{p^{j_n \prime} \beta_h}{\sqrt{n}}\right) \frac{\partial}{\partial y} \left(g + \frac{p^{j_n \prime} \beta_h}{\sqrt{n}}\right) \le 0 \quad (A.18)$$

$$\sup_{P \in \mathbf{P}} \|p^{j_{n'}} \beta_h\|_{P,2} + \|\gamma_h\|_2 \le \ell \sqrt{n} \Big\}, \tag{A.19}$$

where constraint (A.16) corresponds to $(\theta + h/\sqrt{n}) \in \Theta_n$, constraints (A.17) and (A.18) impose $\theta + h/\sqrt{n} \in R$, and constraint (A.19) imposes $||h/\sqrt{n}||_{\mathbf{E}} \leq \ell$. Similarly,

$$V_n(\theta, \Theta | \ell) = \left\{ (p^{j_n \prime} \beta_h, \gamma_h) : \|g + \frac{p^{j_n \prime} \beta_h}{\sqrt{n}} \|_{1,\infty} \le C_0 \text{ and } \|\gamma + \frac{\gamma_h}{\sqrt{n}} \|_2 \le C_0 \right.$$
 (A.20)

$$\sup_{P \in \mathbf{P}} \|p^{j_n \prime} \beta_h\|_{P,2} + \|\gamma_h\|_2 \le \ell \sqrt{n} \right\}. \tag{A.21}$$

Finally, recall that $\mathbb{W}_P(\theta) \sim N(0, \operatorname{Var}_P\{\rho(X, \theta)q^{k_n}(Z)\})$ and define \mathbb{D}_P to be given by

$$\mathbb{D}_P[h] \equiv -E_P[q^{k_n}(Z)(p^{j_n}(S,Y)'\beta_h + W'\gamma_h)]$$

for any $h = (p^{j_n} \beta_h, \gamma_h)$. Given these definitions, note that for any ℓ_n we have that

$$U_P(R|\ell_n) \equiv \inf_{h \in V_n(\Pi_n \theta_0, R|\ell_n)} \| \mathbb{W}_P(\Pi_n \theta_0) + \mathbb{D}_P[h] \|_{\Sigma_P, 2}$$

$$U_P(\Theta|\ell_n) \equiv \inf_{h \in V_n(\Pi_n \theta_0, \Theta|\ell_n)} \| \mathbb{W}_P(\Pi_n \theta_0) + \mathbb{D}_P[h] \|_{\Sigma_P, 2}.$$

Theorem 3.1(ii) immediately yields the following distributional approximations.

Theorem A.2.3. Let Assumptions A.2.7-A.2.10 hold, and $a_n = (\log(n))^{-1/2}$. Then: for any $\ell_n, \ell_n^{\mathrm{u}} \downarrow 0$ satisfying $k_n \sqrt{j_n \log(1+k_n)} (\ell_n \vee \ell_n^{\mathrm{u}}) \sqrt{\log(\sqrt{j_n}/(\ell_n \vee \ell_n^{\mathrm{u}}))} = o(a_n)$ and $k_n \sqrt{j_n} \log(1+k_n)/s_n \sqrt{n} = o(\ell_n \wedge \ell_n^{\mathrm{u}})$ it follows uniformly in $P \in \mathbf{P}_0$ that

$$I_n(R) = U_P(R|\ell_n) + o_P(a_n)$$

$$I_n(R) - I_n(\Theta) = U_P(R|\ell_n) - U_P(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n).$$

To obtain bootstrap estimates of the distributional approximations in Theorem A.2.3 we let $\hat{\theta}_n$ and $\hat{\theta}_n^{\rm u}$ denote the minimizers of Q_n over $\Theta_n \cap R$ and Θ_n respectively. For $\rho(\cdot, \theta)$ as in (A.11), we approximate the law of $\mathbb{W}_P(\Pi_n \theta_0)$ by evaluating

$$\hat{\mathbb{W}}_n(\theta) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ q^{k_n}(Z_i) \rho(X_i, \theta) - \frac{1}{n} \sum_{j=1}^n q^{k_n}(Z_j) \rho(X_j, \theta) \},$$

at $\theta = \hat{\theta}_n$ and $\theta = \hat{\theta}_n^{\mathrm{u}}$, where $\{\omega_i\}_{i=1}^n$ is an i.i.d. sample independent of the data satisfying $\omega_i \sim N(0,1)$. As our estimator for $\mathbb{D}_P[h]$, for any $h = (p^{j_n \prime} \beta_h, \gamma_h)$, we let

$$\hat{\mathbb{D}}_n[h] = -\frac{1}{n} \sum_{i=1}^n q^{k_n}(Z_i) (W_i' \gamma_h + p^{j_n}(S_i, Y_i)' \beta_h).$$

With regards to the local parameter space, we note that in this application Assumptions 3.8(i)(ii) are satisfied with $K_g = 2$ (see Lemma S.4.20). Therefore, we have

$$G_{n}(\hat{\theta}_{n}) = \left\{ h : \frac{\partial}{\partial s} p^{j_{n}}(s, y)'(\hat{\beta}_{n} + \frac{\beta_{h}}{\sqrt{n}}) + p^{j_{n}}(s, y)'(\hat{\beta}_{n} + \frac{\beta_{h}}{\sqrt{n}}) \frac{\partial}{\partial y} p^{j_{n}}(s, y)'(\hat{\beta}_{n} + \frac{\beta_{h}}{\sqrt{n}}) \right\}$$

$$\leq \max \left\{ \frac{\partial}{\partial s} p^{j_{n}}(s, y)'\hat{\beta}_{n} + p^{j_{n}}(s, y)'\hat{\beta}_{n} \frac{\partial}{\partial y} p^{j_{n}}(s, y)'\hat{\beta}_{n} - 2r_{n} \| \frac{p^{j_{n}'}\beta_{h}}{\sqrt{n}} \|_{1, \infty}, -r_{n} \right\}. \quad (A.22)$$

Moreover, because $\rho(X,\cdot)$ and Υ_F are linear, we may set $\ell_n=+\infty$ and obtain that

$$\hat{V}_n(\hat{\theta}_n, R| + \infty) = \{ h = (p^{j_n} \beta_h, \gamma_h) : h \in G_n(\hat{\theta}_n) \text{ and } p^{j_n}(s_0, y_0)' \beta_h = 0 \}.$$

Given the introduced notation, we define the statistics $\hat{U}_n(R|+\infty)$ and $\hat{U}_n(\Theta|+\infty)$ by

$$\hat{U}_n(R|+\infty) \equiv \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|+\infty)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n[h]\|_{\hat{\Sigma}_{n,2}}$$
$$\hat{U}_n(\Theta|+\infty) \equiv \inf_{h = (p^{j_n}'\beta_h, \gamma_h)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n^{\mathrm{u}}) + \hat{\mathbb{D}}_n[h]\|_{\hat{\Sigma}_{n,2}}.$$

We impose one final assumption to establish the validity of the bootstrap. In the requirements below, it is helpful to recall θ_0 is implicitly a function of P through (A.9).

Assumption A.2.11. (i) There is an $\epsilon > 0$ such that $||g_0||_{1,\infty} \vee ||\gamma_0||_2 \leq C_0 - \epsilon$ for all $P \in \mathbf{P}_0$; (ii) $\Pi_n \theta_0 = (g_n, \gamma_0) \in \Theta_n \cap R$ satisfies $||g_n - g_0||_{1,\infty} = o(1)$ uniformly in $P \in \mathbf{P}_0$; (iii) The sequence $r_n \downarrow 0$ satisfies $k_n j_n^2 \sqrt{\log(1 + k_n)} / s_n \sqrt{n} = o(r_n/\sqrt{\log(n)})$; (iv) $k_n j_n^{3/4} (\mathcal{E}_n \vee \sqrt{\log(k_n)}) \log^{1/4} (1 + k_n) = o(n^{1/4}/\sqrt{\log(n)})$, where $\mathcal{E}_n \equiv \int_0^\infty \sqrt{\log(\epsilon, C_n, ||\cdot||_2)} d\epsilon$ and $C_n \equiv \{\beta : ||p^{j_n'}\beta||_{1,\infty} \leq C_0\}$.

Assumptions A.2.11(i)(ii) suffice for verifying Assumption 3.12(ii). These requirements may be dropped at the expense of modifying $\hat{V}_n(\hat{\theta}_n, R|+\infty)$ to reflect the possible impact of $\Pi_n\theta_0$ being "near" the boundary of Θ_n . Assumption A.2.11(iii) imposes the rate conditions on r_n . Finally, Assumption A.2.11(iv) controls the "size" of the set of coefficients β corresponding to elements $p^{j_{n'}}\beta \in \Theta_n$ and suffices for verifying the bootstrap coupling requirement of Assumption 3.11. For instance, $\mathcal{E}_n \approx j_n^{1/4}$ for tensor product B-splines (see Lemma S.4.23), which implies a sufficient condition for Assumption A.2.11(iv) is that $k_n^4 j_n^4 \log^4(k_n) = o(n/\log^2(n))$. The rate requirements for a bootstrap coupling can be weakened if the test statistic is based on the $\|\cdot\|_{\infty}$ -norm (see Lemma S.4.19) or under additional smoothness assumptions (see Theorem S.7.1(ii)).

Our next result characterizes the properties of the proposed bootstrap statistics.

Theorem A.2.4. Let Assumptions A.2.7, A.2.8, A.2.9, A.2.10, A.2.11 hold, and $a_n = (\log(n))^{-1/2}$. Then: there are sequences $\ell_n, \ell_n^{\mathrm{u}} \downarrow 0$ satisfying $k_n j_n^2 \log(1 + k_n) / s_n \sqrt{n} = o(\ell_n \wedge \ell_n^{\mathrm{u}})$, $\ell_n = o(r_n)$, and $k_n \sqrt{j_n \log(1 + k_n)} (\ell_n \vee \ell_n^{\mathrm{u}}) \sqrt{\log(\sqrt{j_n}/(\ell_n \vee \ell_n^{\mathrm{u}}))} = o(a_n)$ for which it follows that uniformly in $P \in \mathbf{P}_0$ we have

$$\hat{U}_n(R|+\infty) \ge U_P^{\star}(R|\ell_n) + o_P(a_n)$$
$$\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty) \ge U_P^{\star}(R|\ell_n) - U_P^{\star}(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n).$$

Importantly, any sequences ℓ_n and $\ell_n^{\rm u}$ satisfying the requirements of Theorem A.2.4 also satisfy the requirements of Theorem A.2.3. Hence, we may employ

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|+\infty)) \equiv \inf\{c : P(\hat{U}_n(R|+\infty) \le c | \{V_i\}_{i=1}^n) \ge 1-\alpha\}$$

as a critical value for $I_n(R)$. Similarly, for the statistic $I_n(R) - I_n(\Theta)$ we may employ

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty))$$

$$\equiv \inf\{c : P(\hat{U}_n(R|+\infty) - \hat{U}_n(\Theta|+\infty) \le c|\{V_i\}_{i=1}^n) \ge 1 - \alpha\}.$$

Remark A.2.1. Suppose for notational simplicity that there are no covariates W and let the marginal distribution of (S,Y,Z) be constant in $P \in \mathbf{P}$. If Z = (S,Y) (i.e. (S,Y) is exogenous), we may set $q^{k_n}(Z) = p^{k_n}(S,Y)'$ for some $k_n \geq j_n$. The singular value s_n can then be assumed to be bounded away from zero, and a sufficient condition for Assumption A.2.9(iv) is that $k_n^4 j_n^2 \log^5(n) = o(n)$. In order to appreciate the content of Assumption A.2.8(iv), suppose $\{p_j\}_{j=1}^{\infty}$ is an orthonormal basis such that

$$g_0 = \sum_{j=1}^{\infty} \beta_j p_j$$
 with $|\beta_j| = O(j^{-\gamma_\beta})$.

Setting $\Pi_n^{\mathbf{u}} g_0 = \sum_{j=1}^{j_n} p_j \beta_j$, we obtain from a standard integral bound for a sum that

$$||E_P[(g_0(S,Y) - \Pi_n^{\mathrm{u}}g_0(S,Y))q^{k_n}(Z)]||_2^2 \lesssim \sum_{j=j_n+1}^{k_n} \frac{1}{j^{2\gamma_\beta}} \lesssim \frac{1}{j_n^{2\gamma_\beta - 1}} - \frac{1}{k_n^{2\gamma_\beta - 1}}.$$
 (A.23)

For instance, if $k_n - j_n = O(1)$, then the bound in (A.23) is of order $1/j_n^{2\gamma_\beta}$. Hence, provided the approximation error by $\Pi_n^{\rm u}g_0$ and g_n (as in Assumption A.2.8(iv)) are of the same order when $g_0 \in R$, we obtain that Assumption A.2.8(iv) is equivalent to $\sqrt{n\log(n)}/j_n^{\gamma_\beta} = o(1)$ when $k_n - j_n = O(1)$. This approximation requirement is compatible with the condition $k_n^4 j_n^2 \log^5(n) = o(n)$ provided $\gamma_\beta > 3$.

Remark A.2.2. Building on Remark A.2.1, suppose again there are no covariates W

and the marginal distribution of (S, Y, Z) is constant in $P \in \mathbf{P}$, but now let (S, Y) be endogenous. A standard benchmark for nonparametric models with endogeneity is to assume the operator $g \mapsto E_P[g(S, Y)|Z]$ is compact, in which case there are orthonormal sequences of functions $\{\phi_j\}_{j=1}^{\infty}$ of (S, Y) and $\{\psi_j\}_{j=1}^{\infty}$ of Z satisfying

$$E_P[\phi_i(S,Y)|Z] = \lambda_i \psi_i(Z)$$
 $E_P[\psi_i(Z)|S,Y] = \lambda_i \phi_i(S,Y)$

where $\lambda_j > 0$ tends to zero. In addition suppose g_0 admits for an expansion satisfying

$$g_0 = \sum_{j=1}^{\infty} \beta_j \phi_j \text{ with } |\beta_j| = O(j^{-\gamma_\beta}),$$

and let $p^{j_n} = (\phi_1, \dots, \phi_{j_n})'$, $q^{k_n} = (\psi_1, \dots, \psi_{k_n})'$ with $k_n \geq j_n$ and $k_n - j_n = O(1)$, and set $\Pi_n^u g_0 = \sum_{j=1}^{j_n} \phi_j \beta_j$. Provided the approximation error of $\Pi_n^u g_0$ and g_n (as in Assumption A.2.8(iv)) are of the same order when $g_0 \in R$, we then obtain

$$||E_P[(g_0(S,Y) - g_n(S,Y))q^{k_n}(Z)]||_2 \lesssim \frac{\lambda_{j_n}}{i_n^{\gamma_{\beta}}}.$$

Moreover, direct calculation shows s_n , which is proportional to ν_n^{-1} as in Assumption 3.4, satisfies $s_n = \lambda_{j_n}$ and hence equals the reciprocal of the sieve measure of ill-posedness (Blundell et al., 2007). It follows that if $\lambda_j \simeq j^{-\gamma_\lambda}$, and $\gamma_\beta > 3$, then Assumptions A.2.8(iv) and A.2.9(iv) can be satisfied by setting $j_n \simeq n^{\kappa}$ with $(\gamma_\lambda + \gamma_\beta)^{-1} < 2\kappa < (3 + \gamma_\lambda)^{-1}$ and $k_n - j_n = O(1)$. Alternatively, if $\lambda_j = \exp\{-\gamma_\lambda j\}$, then Assumption A.2.8(iv) and A.2.9(iv) can be satisfied when $\gamma_\beta > 4$ by setting, for example, $j_n = (\log(n) - \kappa \log(\log(n)))/2\gamma_\lambda$ with $7 < \kappa < 2\gamma_\beta - 1$ and $k_n - j_n = O(1)$.

A.2.3 Quantile Treatment Effects

For our next example, we study a nonparametric quantile treatment effect (QTE) model. Specifically, for an outcome $Y \in \mathbf{R}$, treatment $D \in [0,1]$, instrument $Z \in \mathbf{R}$, and quantile $\tau \in (0,1)$, we assume the parameter of interest θ_0 satisfies

$$P(Y \le \theta_0(D)|Z) = \tau. \tag{A.24}$$

If D is randomly assigned, then we may set D = Z and interpret $\nabla \theta_0$ as the τ^{th} quantile treatment effect (QTE). Alternatively, if $D \neq Z$, then we obtain the QTE model of Chernozhukov and Hansen (2005). To map (A.24) into our framework, we set

$$\rho(X,\theta) = 1\{Y \le \theta(D)\} - \tau, \tag{A.25}$$

where $X = (Y, D) \in \mathbf{X} \equiv \mathbf{R} \times [0, 1]$. In order to illustrate our conditions in a number of different settings, we focus on conducting inference on a nonlinear function of θ_0 .

Specifically, we conduct inference on the variance of the quantile treatment effects:

$$\int_{0}^{1} (\nabla \theta_{0}(u))^{2} du - (\int_{0}^{1} \nabla \theta_{0}(u) du)^{2}$$

while imposing that the QTE be increasing in treatment intensity (i.e. $d \mapsto \nabla \theta_0(d)$ is increasing). To map this problem into our framework we define

$$C_B^m([0,1]) \equiv \{\theta: [0,1] \to \mathbf{R} \text{ s.t. } \|\theta\|_{m,\infty} < \infty\} \qquad \|\theta\|_{m,\infty} \equiv \sup_{0 \le \alpha \le m} \sup_{d \in [0,1]} |\nabla^\alpha \theta(d)|,$$

and set $\mathbf{B} = C_B^2([0,1])$ and $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{2,\infty}$. We impose the restriction that the quantile treatment effect be increasing in the intensity of treatment by letting $\mathbf{G} = C_B^0([0,1])$, $\|\cdot\|_{\mathbf{G}} = \|\cdot\|_{\infty}$ (where we write $\|\cdot\|_{\infty}$ in place of $\|\cdot\|_{0,\infty}$), and defining

$$\Upsilon_G(\theta) \equiv -\nabla^2 \theta. \tag{A.26}$$

As shown in Section A.2.2, **G** is a lattice with order unit $\mathbf{1}_{\mathbf{G}} = \mathbf{c}$ for **c** the constant function $\mathbf{c}(d) = 1$ for all $d \in [0, 1]$. Setting $\mathbf{F} = \mathbf{R}$ with $\|\cdot\|_{\mathbf{F}} = |\cdot|$, we test whether the variance of the quantile treatment effects equals a hypothesized value $\lambda \neq 0$ by setting

$$\Upsilon_F(\theta) = \int_0^1 (\nabla \theta(u))^2 du - (\int_0^1 \nabla \theta(u) du)^2 - \lambda. \tag{A.27}$$

For the parameter space for θ_0 we employ a ball in **B** and we thus set Θ to equal

$$\Theta = \{ \theta \in C_B^2([0,1]) \text{ s.t. } \|\theta\|_{2,\infty} \le C_0 \}$$
(A.28)

for some $C_0 < \infty$. For a sequence of approximating functions $\{p_j\}_{j=1}^{j_n}$ defined on [0,1] we then let $p^{j_n}(d) \equiv (p_1(d), \dots, p_{j_n}(d))'$ and define Θ_n to equal

$$\Theta_n \equiv \{ p^{j_n \prime} \beta \in C_B^2([0, 1]) : \| p^{j_n \prime} \beta \|_{2, \infty} \le C_0 \}.$$
(A.29)

Similarly for a sequence $\{q_k\}_{k=1}^{k_n}$, we set $q^{k_n}(z) \equiv (q_1(z), \dots, q_{k_n}(z))'$ and define

$$Q_n(\theta) \equiv \|\hat{\Sigma}_n\{\frac{1}{n}\sum_{i=1}^n (1\{Y_i \le \theta(D_i)\} - \tau)q^{k_n}(Z_i)\}\|_p$$

for some $2 \leq p \leq \infty$ and weighting matrix $\hat{\Sigma}_n$. The statistics $I_n(R)$ and $I_n(\Theta)$ then equal the minimums of $\sqrt{n}Q_n$ over $\Theta_n \cap R$ and Θ_n respectively.

In what follows, we will assume for simplicity that θ_0 is identified. As a result, we may think of θ_0 as a function of P through (A.24), though we leave such dependence implicit in the notation. We next impose the following assumptions:

Assumption A.2.12. (i) $\{Y_i, D_i, Z_i\}_{i=1}^n$ is i.i.d. with $(Y, D, Z) \in \mathbf{R} \times [0, 1] \times \mathbf{R}$ dis-

tributed according to $P \in \mathbf{P}$; (ii) For Θ as in (A.28) and each $P \in \mathbf{P}_0$ there exists a unique $\theta_0 \in \Theta$ satisfying (A.24); (iii) The distribution of Y conditional on (D, Z) is absolutely continuous with density $f_{Y|DZ,P}(\cdot|D,Z)$ that is bounded and Lipschitz uniformly in (D,Z) and $P \in \mathbf{P}$; (iv) Assumptions S.6.1 and S.6.2 hold.

Assumption A.2.13. (i) $\sup_d \|p^{j_n}(d)\|_2 \lesssim \sqrt{j_n}$; (ii) $E_P[p^{j_n}(D)p^{j_n}(D)']$ has eigenvalues bounded away from zero and infinity uniformly in $P \in \mathbf{P}$ and j_n ; (iii) For each $P \in \mathbf{P}_0$ there is a $\Pi_n \theta_0 \in \Theta_n \cap R$ satisfying $\sup_{P \in \mathbf{P}_0} \|E_P[(1\{Y \leq \Pi_n \theta_0(D)\} - 1\{Y \leq \theta_0(D)\})q^{k_n}(Z)]\|_p = O((n\log(n))^{-1/2})$ and $\sup_{P \in \mathbf{P}_0} \|\theta_0 - \Pi_n \theta_0\|_{1,\infty} = o(1)$.

Assumption A.2.14. (i) $\inf_{P \in \mathbf{P}_0} \inf_{\theta \in \Theta: \|\theta - \theta_0\|_{1,\infty} \ge \epsilon} E_P[(P(Y \le \theta(D)|Z) - \tau)^2] > 0$ for every $\epsilon > 0$; (ii) There are ϵ and $\mathbf{s}_n > 0$ satisfying for all $P \in \mathbf{P}_0$ and $\|\theta - \Pi_n \theta_0\|_{1,\infty} \le \epsilon$, $\mathbf{s}_n \le \inf\{E_P[f_{Y|D,Z}(\theta(D)|D,Z)q^{k_n}(Z)p^{j_n}(D)']\}$ and $\mathbf{s}_n = O(1)$.

Assumption A.2.15. (i) $\max_{1 \le k \le k_n} \|q_k\|_{\infty} = O(1)$; (ii) $\max_{1 \le k \le k_n} \|q_k\|_{1,\infty} = O(k_n)$; (iii) $E_P[q^{k_n}(Z)q^{k_n}(Z)']$ has eigenvalues bounded away from zero and infinity uniformly in $P \in \mathbf{P}$ and k_n ; (iv) For each $\theta \in \Theta$ there is a $\pi_n(\theta) \in \mathbf{R}^{k_n}$ with $E_P[(E_P[\rho(X,\theta)|Z] - q^{k_n}(Z)'\pi_n(\theta))^2] = o(1)$ uniformly in $P \in \mathbf{P}$ and $\theta \in \Theta$; (v) $k_n^{1/p} \sqrt{j_n} \log^{3/2}(n) (n^{1/6} \vee k_n)/n^{1/3} = o(1)$ and $j_n \log^{3/2}(1 + k_n)k_n^{2/p+1/2}/s_n\sqrt{n} = o((\log(n))^{-2})$.

Assumption A.2.16. (i) $\|\hat{\Sigma}_n - \Sigma_P\|_{o,p} = o_P((k_n^{1/p}\log(n))^{-1})$ uniformly in $P \in \mathbf{P}$; (ii) Σ_P is invertible and $\|\Sigma_P\|_{o,p}$ and $\|\Sigma_P^{-1}\|_{o,p}$ are bounded in $P \in \mathbf{P}$ and k_n .

Assumption A.2.12 imposes regularity conditions on the distribution P that enable us to apply the empirical process coupling results of Appendix S.6. Assumption A.2.13 states the requirements on Θ_n , including demanding an asymptotically negligible bias in Assumption A.2.13(iii). Assumption A.2.14(i) holds pointwise in $P \in \mathbf{P}_0$ due to Θ being compact under $\|\cdot\|_{1,\infty}$, and hence the uniformity in $P \in \mathbf{P}_0$ demanded by Assumption A.2.14(i) corresponds to imposing strong identification. Assumption A.2.14(ii) enables us to obtain a uniform rate of convergence under $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$. As in Section A.2.2, \mathbf{s}_n can be shown to be related to the degree of ill-posedness. Assumptions A.2.15(i)-(iv) impose conditions on $\{q_k\}_{k=1}^{k_n}$ including that they be bounded – this requirement can be relaxed at the cost of more stringent rate restrictions to ensure a coupling of the empirical process (see Lemma S.4.28). Finally, Assumption A.2.15(v) states our rate restrictions, which we note are easier to satisfy for higher values of p.

For any $\theta = p^{j_n \prime} \beta \in \Theta_n \cap R$, in this application the local parameter space equals

$$V_{n}(\theta, R|\ell) = \left\{ h = p^{j_{n'}} \beta_{h} : \|\theta + \frac{h}{\sqrt{n}}\|_{2,\infty} \le C_{0}, \sup_{P \in \mathbf{P}} \|h\|_{P,2} \le \ell \sqrt{n}, \right.$$

$$\int_{0}^{1} (\nabla \theta(u) + \frac{\nabla h(u)}{\sqrt{n}})^{2} du - \left(\int_{0}^{1} \{\nabla \theta(u) + \frac{\nabla h(u)}{\sqrt{n}}\} du\right)^{2} = \lambda,$$

$$- \nabla^{2} \theta(d) - \frac{\nabla^{2} h(d)}{\sqrt{n}} \le 0 \text{ for all } d \in [0, 1] \right\}, \tag{A.30}$$

where the first two constraints impose that $\theta + h/\sqrt{n} \in \Theta_n$ and $||h/\sqrt{n}||_{\mathbf{E}} \leq \ell$, while the final two constraints require that $\theta + h/\sqrt{n} \in R$. Similarly, here

$$V_n(\theta, \Theta | \ell) = \left\{ h = p^{j_n \prime} \beta_h : \|\theta + \frac{h}{\sqrt{n}}\|_{2,\infty} \le C_0 \text{ and } \sup_{P \in \mathbf{P}} \|h\|_{P,2} \le \ell \sqrt{n} \right\}.$$

Also recall that $\mathbb{W}_P(\theta) \sim N(0, \operatorname{Var}_P\{\rho(X, \theta)q^{k_n}(Z)\})$ and for any $h = p^{j_n'}\beta_h$ define

$$\mathbb{D}_{P}(\theta)[h] \equiv E_{P}[q^{k_n}(Z)f_{Y|DZ,P}(\theta(D)|D,Z)p^{j_n}(D)'\beta_h]. \tag{A.31}$$

The random variables to which $I_n(R)$ and $I_n(\Theta)$ will be coupled are then given by

$$U_P(R|\ell_n) \equiv \inf_{h \in V_n(\Pi_n\theta_0, R|\ell_n)} \| \mathbb{W}_P(\Pi_n\theta_0) + \mathbb{D}_P(\Pi_n\theta_0)[h] \|_{\Sigma_P, 2}$$

$$U_P(\Theta|\ell_n) \equiv \inf_{h \in V_n(\Pi_n \theta_0, \Theta|\ell_n)} \| \mathbb{W}_P(\Pi_n \theta_0) + \mathbb{D}_P(\Pi_n \theta_0)[h] \|_{\Sigma_P, 2}.$$

Our next result obtains distributional approximations by applying Theorem 3.1.

Theorem A.2.5. Let Assumptions A.2.12, A.2.13, A.2.14, A.2.15, and A.2.16 hold, $a_n = (\log(n))^{-1/2}$, and $\ell_n \downarrow 0$ satisfy $k_n^{1/p} \sqrt{j_n \ell_n \log(1 + k_n) \log(1/\ell_n)} = o((\log(n))^{-1/2})$ and $\ell_n^2 \sqrt{n j_n \log(n)} = o(1)$. Then: (i) Uniformly in $P \in \mathbf{P}_0$ it follows that

$$I_n(R) \le U_P(R|\ell_n) + o_P(a_n).$$

(ii) If in addition $k_n \log(1+k_n)\sqrt{j_n \log(n)}/s_n^2\sqrt{n} = o(1)$, then for any $\ell_n^{\rm u} \downarrow 0$ satisfying $k_n^{1/p}\sqrt{j_n\ell_n^{\rm u}\log(1+k_n)\log(1/\ell_n^{\rm u})} = o((\log(n))^{-1/2})$, $(\ell_n^{\rm u})^2\sqrt{nj_n\log(n)} = o(1)$, and $\sqrt{k_n\log(1+k_n)}/s_n\sqrt{n} = o(\ell_n^{\rm u})$, it follows uniformly in $P \in \mathbf{P}_0$ that

$$I_n(R) - I_n(\Theta) \le U_P(R|\ell_n) - U_P(\Theta|\ell_n^{\mathbf{u}}) + o_P(a_n).$$

Theorem A.2.5(i) obtains an upper bound for $I_n(R)$ by relying on Theorem 3.1(i). In order to approximate the re-centered statistic $I_n(R) - I_n(\Theta)$, we cannot rely on an upper bound for $I_n(\Theta)$ as the resulting approximation could fail to control size. Therefore, Theorem A.2.5(ii) instead relies on Theorem 3.1(ii). Applying Theorem 3.1(ii), however, requires an additional rate condition in order to establish the linearization of the moment conditions is asymptotically valid. We also note that the conclusion of Theorem A.2.5(ii) in fact holds with equality if ℓ_n satisfies the same rate restrictions as $\ell_n^{\rm u}$.

In order to provide bootstrap estimates for these distributional approximations, we let $\hat{\theta}_n$ and $\hat{\theta}_n^{\mathrm{u}}$ denote minimizers of Q_n over $\Theta_n \cap R$ and Θ_n respectively. Our bootstrap approximation estimates the law of $\mathbb{W}_P(\theta_0)$ and the derivative $\mathbb{D}_P(\theta_0)$ by evaluating

$$\hat{\mathbb{W}}_n(\theta) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ q^{k_n}(Z_i) (1\{Y_i \le \theta(D_i)\} - \tau) - \frac{1}{n} \sum_{j=1}^n q^{k_n}(Z_j) (1\{Y_j \le \theta(D_j)\} - \tau) \}$$

$$\hat{\mathbb{D}}_n(\theta)[h] \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n q^{k_n}(Z_i) (1\{Y_i \le \theta(D_i) + \frac{h(D_i)}{\sqrt{n}}\} - 1\{Y_i \le \theta(D_i)\})$$

at $\hat{\theta}_n$ and $\hat{\theta}_n^{\text{u}}$. An unappealing feature of $\hat{\mathbb{D}}_n(\theta)$ is that it is not linear in h, which complicates computation. Alternatively, a plug-in estimator based on (A.31) could be used, though at the expense of having to estimate the density $f_{Y|DZ,P}$.

With regards to the local parameter space, we note that in this application

$$G_n(\hat{\theta}_n) \equiv \{h = p^{j_n \prime} \beta_h : -\nabla^2 \hat{\theta}_n(d) - \frac{\nabla^2 h(d)}{\sqrt{n}} \le \max\{-\nabla^2 \hat{\theta}_n(d) \lor -r_n\} \text{ for all } d \in [0, 1]\}.$$

Employing that $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{2,\infty}$ and the expression for Υ_F in (A.27), we obtain that

$$\hat{V}_n(\hat{\theta}_n, R|\ell_n) = \Big\{ h = p^{j_n \prime} \beta_h : h \in G_n(\hat{\theta}_n), \ \|\frac{h}{\sqrt{n}}\|_{2,\infty} \le \ell_n$$

$$\int_0^1 (\nabla \hat{\theta}_n(u) + \frac{\nabla h(u)}{\sqrt{n}})^2 du - (\int_0^1 (\nabla \hat{\theta}_n(u) + \frac{\nabla h(u)}{\sqrt{n}}) du)^2 = \lambda \Big\},$$

where ℓ_n is chosen to satisfy conditions stated below. The bootstrap statistics $\hat{U}_n(R|\ell_n)$ and $\hat{U}_n(\Theta|+\infty)$ for approximating the distributions in Theorem A.2.5 are then

$$\hat{U}_n(R|\ell_n) \equiv \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n(\hat{\theta}_n)[h]\|_{\hat{\Sigma}_n, p}$$
$$\hat{U}_n(\Theta|+\infty) \equiv \inf_{h=p^{j_n}/\beta_h} \|\hat{\mathbb{W}}_n(\hat{\theta}_n^{\mathrm{u}}) + \hat{\mathbb{D}}_n(\hat{\theta}_n^{\mathrm{u}})[h]\|_{\hat{\Sigma}_n, p}.$$

The following final assumption will enable us to establish bootstrap validity. In the requirements below, it is helpful to recall θ_0 is implicitly a function of P through (A.24).

Assumption A.2.17. (i) The functions $\theta(d) = 1$, $\theta(d) = d^2$ are in \mathbf{B}_n ; (ii) $\|\theta_0 - \Pi_n \theta_0\|_{2,\infty} = o(1)$ uniformly in $P \in \mathbf{P}_0$ and $\sup_{P \in \mathbf{P}_0} \|\theta_0\|_{2,\infty} < C_0$; (iii) k_n satisfies $k_n^{1/p+12/26} = o(n^{1/26}/\log(n))$; (iv) $\sup_d \|\nabla^2 p^{j_n}(d)\|_2 \vee \|\nabla p^{j_n}(d)\|_2 \lesssim j_n^{5/2}$; (v) r_n, ℓ_n satisfy $k_n^{1/p} \sqrt{j_n \ell_n \log(1 + k_n) \log(1/\ell_n)} = o((\log(n))^{-1/2})$, $j_n^{5/2} \sqrt{k_n \log(1 + k_n)}/s_n \sqrt{n} = o(1 \wedge r_n)$, and $\ell_n(\sqrt{j_n n}\ell_n + j_n^{5/2} \sqrt{k_n \log(1 + k_n)}/s_n) = o((\log(n))^{-1/2})$.

Assumption A.2.17(i) requires that the quadratic functions belong to \mathbf{B}_n – a condition that holds if quadratic functions belong to the span of $\{p_j\}_{j=1}^{j_n}$. Assumption A.2.17(ii) implies that θ_0 and its approximation $\Pi_n\theta_0$ belong to the interior of Θ . Assumption A.2.17(iii) enables us to verify the bootstrap coupling requirement of Assumption 3.11 by applying the results in Appendix S.7 to a Haar basis expansion. While condition A.2.17(iii) suffices for verifying Assumption 3.11 in both the endogenous $(Z \neq D)$ and exogenous (Z = D) settings, we note that in both cases better rate conditions can be obtained. Finally, Assumption A.2.17(iv) ensures $\mathcal{S}_n(\mathbf{B}, \mathbf{E}) \approx j_n^{5/2}$, while Assumption

¹For instance under endogeneity, a better rate could be obtained by conducting a basis expansion using the tensor product of a Haar Basis for (Y, D) and the functions $\{q_k\}_{k=1}^{k_n}$.

A.2.17(v) imposes the requirements on ℓ_n and r_n .

The next theorem establishes the validity of the bootstrap procedure.

Theorem A.2.6. Let Assumptions A.2.12, A.2.13, A.2.14, A.2.15, A.2.16, and A.2.17 hold and $a_n = (\log(n))^{-1/2}$. Then, there is a sequence $\tilde{\ell}_n \approx \ell_n$ satisfying

$$\hat{U}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n)$$

uniformly in $P \in \mathbf{P}_0$. (ii) If in addition $k_n \log(1 + k_n) \sqrt{j_n \log(n)} / s_n^2 \sqrt{n} = o(1)$, then for any $\tilde{\ell}_n^{\mathrm{u}}$ satisfying the conditions of Theorem A.2.5(ii) we have uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty) \ge U_P^{\star}(R|\tilde{\ell}_n) - U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n).$$

Theorems A.2.5(i) and A.2.6(i) imply that as critical value for $I_n(R)$ we may employ

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|\ell_n)) \equiv \inf\{c : P(\hat{U}_n(R|\ell_n) \le c | \{V_i\}_{i=1}^n) \ge 1 - \alpha\}.$$

If in addition $k_n \log(1 + k_n) \sqrt{j_n \log(n)} / s_n^2 \sqrt{n} = o(1)$, then Theorems A.2.5(ii) and A.2.6(ii) imply a valid test can be obtained by rejecting whenever $I_n(R) - I_n(\Theta)$ exceeds

$$\hat{q}_{1-\alpha}(\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty)) \equiv \inf\{c : P(\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty) \le c|\{V_i\}_{i=1}^n) \ge 1 - \alpha\}.$$

Our critical values depend on the choices of r_n and ℓ_n . The slackness parameter r_n again measures sampling uncertainty in whether constraints "bind." Following the discussion in Section 2.1, for $\hat{\theta}_n^{\mathrm{u}\star}$ a "bootstrap" analogue to $\hat{\theta}_n^{\mathrm{u}}$, we may thus set

$$P(\max_{d \in [0,1]} \nabla^2 \hat{\theta}_n^{u}(d) - \nabla^2 \hat{\theta}_n^{u*}(d) \le r_n |\{V_i\}_{i=1}^n) = 1 - \gamma_n$$
(A.32)

with $\gamma_n \to 0$. With regards to ℓ_n , we note that its main role in this application is to ensure that $\hat{V}_n(\hat{\theta}_n, R|\ell_n)$ is well approximated by the true local parameter space despite the nonlinearity of Υ_F . To this end, the requirements on ℓ_n imposed in Assumption A.2.6 ensure $\sqrt{n}\ell_n\|\hat{\theta}_n - \Pi_n\theta_0\|_{\mathbf{B}} = o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$. Since $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{2,\infty}$ in this application, we may select ℓ_n in a data driven way by setting it to satisfy

$$P(\max_{d \in [0,1]} |\nabla^2 \hat{\theta}_n^{\mathbf{u}}(d) - \nabla^2 \hat{\theta}_n^{\mathbf{u}^*}(d)| \le \frac{1}{\sqrt{n\ell_n}} |\{V_i\}_{i=1}^n) = 1 - \gamma_n$$
 (A.33)

for some $\gamma_n \to 0$. While we set γ_n in (A.32) and (A.33) to be the same, it is worth noting they could be different. In fact, r_n and ℓ_n do not "interact" in the requirements of Assumption A.2.17(v) and, in this sense, can be set independently. We also note that in settings in which the rate of convergence is sufficiently fast, (A.33) should deliver a "large" ℓ_n in the sense that $\hat{U}_n(R|\ell_n)$ and $\hat{U}_n(R|+\infty)$ are asymptotically equiva-

lent. Moreover, in applications in which we expect the rate of convergence of $\hat{\theta}_n$ to be sufficiently fast, we may directly set $\ell_n = +\infty$; see Lemma S.3.1.

Remark A.2.3. To illustrate the role of ℓ_n , it is helpful to conduct a pointwise (in P) analysis, set p=2, and connect our assumptions to the literature on estimation of conditional moment restriction models (Chen and Pouzo, 2012). We follow the literature in imposing a local curvature assumption, which in our application, corresponds to

$$||E_P[(P(Y \le h(D)|Z) - \tau)q^{k_n}(Z)]||_2$$

$$\approx ||E_P[f_{Y|DZ,P}(\bar{\theta}(D)|D,Z)(\theta_0(D) - h(D))q^{k_n}(Z)]||_2 \quad (A.34)$$

for all $h \in \Theta_n$ and $\bar{\theta} \in \Theta$ that are in a neighborhood of θ_0 . We further suppose the linear operator $h \mapsto E_P[f_{Y|DZ,P}(\theta_0(D)|D,Z)h(D)|Z]$ is compact, in which case there exist orthonormal bases $\{\psi_j\}$ and $\{\phi_k\}$ and a sequence $\lambda_j \downarrow 0$ satisfying

$$E_P[f_{Y|DZ,P}(\theta_0(D)|D,Z)\phi_j(D)|Z] = \lambda_j \psi_j(Z). \tag{A.35}$$

Setting $k_n \geq j_n$ with $k_n - j_n = O(1)$, $p^{j_n} = (\phi_1, \dots, \phi_{j_n})'$, $q^{k_n} = (\psi_1, \dots, \psi_{k_n})'$, and $\prod_n^u \theta_0 = \sum_{j=1}^{j_n} \phi_j \beta_j$, we also suppose θ_0 admits an expansion

$$\theta_0 = \sum_{j=1}^{\infty} \beta_j \phi_j \text{ with } |\beta_j| = O(j^{-\gamma_\beta}).$$
 (A.36)

Provided that the approximation error of $\Pi_n\theta_0$ (as in Assumption A.2.13(iii)) and $\Pi_n^{\rm u}\theta_0$ are of the same order, it then follows from (A.34) and (A.35) that

$$||E_P[(1\{Y \le \Pi_n \theta_0(D)\} - 1\{Y \le \theta_0(D)\})q^{k_n}(Z)]||_2 \lesssim \frac{\lambda_{j_n}}{j_n^{\gamma_{\beta}}}$$
(A.37)

and $s_n \simeq \lambda_{j_n}$ – i.e. s_n is proportional to the reciprocal of the sieve measure of ill-posedness (Chen and Pouzo, 2012). As a result, if $\lambda_j \simeq j^{-\gamma_\lambda}$ and $\gamma_\beta > \max\{5/2, 3 - \gamma_\lambda\}$, then Theorem A.2.5 may be applied to couple $I_n(R)$ by setting $j_n \simeq n^{\kappa}$ with $(2(\gamma_\lambda + \gamma_\beta))^{-1} < \kappa < \min\{(5 + 2\gamma_\lambda)^{-1}, 1/6\}$, while coupling $I_n(R) - I_n(\Theta)$ additionally requires $\gamma_\beta > 3/2 + \gamma_\lambda$ and $\kappa < (3 + 4\gamma_\lambda)^{-1}$. In contrast, in the severely ill-posed case in which $\lambda_j \simeq \exp\{-\gamma_\lambda j\}$, the conditions of Theorem A.2.5 for coupling $I_n(R) - I_n(\Theta)$ are not satisfied. However, the conditions for coupling $I_n(R)$ can still be met provided $\gamma_\beta > 4$ by setting $j_n = (\log(n) - \kappa(\log(\log(n))))/2\gamma_\lambda$ with $7 < \kappa < 2\gamma_\beta - 1$. Thus, while in the ill-posed case the rate of convergence is too slow to apply Theorem A.2.5(ii), Theorem A.2.5(i) is still able to deliver a coupling upper bound for suitable ℓ_n .

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Supplemental Appendix II Not Intended For Publication

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This Supplemental Appendix to "Constrained Conditional Moment Restriction Models" contains the proofs for all results. Section S.1 derives rate of convergence results that are employed in our strong and bootstrap approximations. In Section S.2 we establish Theorem 3.1, while the proofs for all remaining results concerning our bootstrap approximation and test are contained in Section S.3. Section S.4 includes the proofs of the results stated in Section 4 and the examples discussed in Supplemental Appendix I. Finally, Sections S.5, S.6, and S.7 develop results that may be of independent interest, and include the analysis of the local parameter space, empirical process coupling results based on Koltchinskii (1994), and bootstrap coupling results.

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S.1 Rate of Convergence

This section contains consistency and rate of convergence results for $\hat{\Theta}_n^r$. The assumptions in the main text, which are designed to deliver a strong approximation, are stronger than needed for deriving the results in this section. We therefore next introduce a weaker set of assumptions that suffice for obtaining a rate of convergence. To this end, we set

$$Q_P(\theta) \equiv ||E_P[\rho(X,\theta) * q^{k_n}(Z)]||_{\Sigma_P,p}; \tag{S.1}$$

i.e. Q_P is the population analogue to the criterion function Q_n . In addition, we define

$$\overrightarrow{d}_{H}(A, B, \|\cdot\|_{\mathbf{E}}) \equiv \sup_{a \in A} \inf_{b \in B} \|a - b\|_{\mathbf{E}}$$

$$d_{H}(A, B, \|\cdot\|_{\mathbf{E}}) \equiv \max\{\overrightarrow{d}_{H}(A, B, \|\cdot\|_{\mathbf{E}}), \overrightarrow{d}_{H}(B, A, \|\cdot\|_{\mathbf{E}})\},$$

which constitute the directed Hausdorff and the Hausdorff distance (under $\|\cdot\|_{\mathbf{E}}$) between two sets A and B. Given these definition, we introduce the following requirements:

Assumption S.1.1. (i) There are $k_n \times k_n$ matrices $\Sigma_P > 0$ with $\|\hat{\Sigma}_n - \Sigma_P\|_{o,p} = o_P(1)$ uniformly in $P \in \mathbf{P}$; (ii) $\|\Sigma_P\|_{o,p} \vee \|\Sigma_P^{-1}\|_{o,p}$ is uniformly bounded in k_n and $P \in \mathbf{P}$.

Assumption S.1.2. Define the sequence $\eta_n \equiv J_n B_n k_n^{1/p} \sqrt{\log(1+k_n)/n}$. Then: (i) $\sup_{\theta \in \Theta_{0n}^r} Q_P(\theta) \times \|\hat{\Sigma}_n - \Sigma_P\|_{o,p} = O_P(\eta_n)$ uniformly in $P \in \mathbf{P}_0$; (ii) $\sup_{\theta \in \Theta_{0n}^r} Q_P(\theta) = \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) + O(\eta_n)$ uniformly in $P \in \mathbf{P}_0$.

Assumption S.1.3. There are sets $V_n(P) \subseteq \Theta_n \cap R$ and a sequence $\{\nu_n\}_{n=1}^{\infty}$ with $\nu_n^{-1} = O(1)$, such that $\hat{\Theta}_n^{\mathrm{r}} \subseteq V_n(P)$ with probability tending to one uniformly in $P \in \mathbf{P}_0$ and for any $\theta \in V_n(P)$ and $\eta_n \equiv J_n B_n k_n^{1/p} \sqrt{\log(1+k_n)/n}$ it follows that

$$\nu_n^{-1} \overrightarrow{d}_H(\{\theta\}, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \le \{Q_P(\theta) - \inf_{\tilde{\theta} \in \Theta_n \cap R} Q_P(\tilde{\theta})\} + O(\eta_n).$$

In particular, note Assumption S.1.1 is implied by Assumption 3.7. Similarly, Assumption S.1.2 follows from Assumptions 3.7(i) and 3.6(ii), while Assumption S.1.3 will be verified by relying on Assumptions 3.4(i), 3.4(ii) or 3.12(iii) (depending on the choice of τ_n), and 3.6(ii). Given these assumptions, we next establish a consistency (Lemma S.1.1) and rate of convergence results (Theorem S.1.1) for $\hat{\Theta}_n^{\rm r}$.

Lemma S.1.1. Let Assumptions 3.1(i), 3.2(i)(iii), S.1.1, S.1.2(i), $\|\cdot\|_{\mathbf{A}}$ be a norm on \mathbf{B}_n and for $\epsilon > 0$ let $\mathcal{V}_n(P) \equiv \{\theta \in \Theta_n \cap R : \overrightarrow{d}_H(\{\theta\}, \Theta_{0n}^r, \|\cdot\|_{\mathbf{A}}) \leq \epsilon\}$, and define

$$S_n(\epsilon) \equiv \inf_{P \in \mathbf{P}_0} \{ \inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} Q_P(\theta) - \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) \}.$$

(i) If
$$\eta_n \vee \tau_n = o(S_n(\epsilon))$$
 for $\eta_n \equiv k_n^{1/p} \sqrt{\log(1+k_n)} J_n B_n / \sqrt{n}$, then $\hat{\Theta}_n^r \subseteq \mathcal{V}_n(P)$ with

probability tending to one uniformly in $P \in \mathbf{P}_0$. (ii) If Assumption S.1.2(ii) holds and $\eta_n = o(\tau_n)$, then $\Theta_{0n}^{\mathbf{r}} \subseteq \hat{\Theta}_n^{\mathbf{r}}$ with probability tending to one uniformly in $P \in \mathbf{P}_0$.

PROOF: For a given $\epsilon > 0$ first notice that by definition of $\hat{\Theta}_n^{\rm r}$ and $\mathcal{V}_n(P)$ we have

$$P(\overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \|\cdot\|_{\mathbf{A}}) > \epsilon) \leq P(\inf_{\theta \in (\Theta_{n} \cap R) \setminus \mathcal{V}_{n}(P)} Q_{n}(\theta) \leq \inf_{\theta \in \Theta_{n} \cap R} Q_{n}(\theta) + \tau_{n}) \quad (S.2)$$

for all $P \in \mathbf{P}_0$. Setting $\hat{Q}_P(\theta) \equiv \|E_P[\rho(X,\theta) * q^{k_n}(Z)]\|_{\hat{\Sigma}_n,p}$ then note that Lemma S.1.2 and $\|\hat{\Sigma}_n\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}_0$ by Lemma S.1.4 allow us to conclude

$$\inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} \hat{Q}_P(\theta) \le \inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} Q_n(\theta) + O_P(\eta_n)$$
 (S.3)

uniformly in $P \in \mathbf{P}_0$. In addition, by similar arguments we obtain uniformly in $P \in \mathbf{P}_0$

$$\inf_{\theta \in \Theta_n \cap R} Q_n(\theta) \le \inf_{\theta \in \Theta_n \cap R} \hat{Q}_P(\theta) + O_P(\eta_n). \tag{S.4}$$

Next note that for any $a \in \mathbf{R}^{k_n}$ we have $\|\Sigma_P a\|_p \leq \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p} \|\hat{\Sigma}_n a\|_p$, and therefore

$$\inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} \hat{Q}_P(\theta) \ge \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p}^{-1} \inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} Q_P(\theta)$$

$$\ge \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p}^{-1} \{S_n(\epsilon) + \inf_{\theta \in \Theta_n \cap R} Q_P(\theta)\}$$
(S.5)

by definition of $S_n(\epsilon)$. Similarly, employing that $\|\hat{\Sigma}_n a\|_p \leq \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \|\Sigma_P a\|_p$ yields

$$\inf_{\theta \in \Theta_{n} \cap R} \hat{Q}_{P}(\theta) - \|\Sigma_{P} \hat{\Sigma}_{n}^{-1}\|_{o,p}^{-1} \inf_{\theta \in \Theta_{n} \cap R} Q_{P}(\theta)
\leq \{ \|\hat{\Sigma}_{n} \Sigma_{P}^{-1}\|_{o,p} - \|\Sigma_{P} \hat{\Sigma}_{n}^{-1}\|_{o,p}^{-1} \} \inf_{\theta \in \Theta_{n} \cap R} Q_{P}(\theta). \quad (S.6)$$

For I_{k_n} the $k_n \times k_n$ identity matrix, then note that $||I_{k_n}||_{o,p} = 1$ implies the bound

$$|\|\Sigma_{P}\hat{\Sigma}_{n}^{-1}\|_{o,p} - 1| = |\|\Sigma_{P}\hat{\Sigma}_{n}^{-1}\|_{o,p} - \|I_{k_{n}}\|_{o,p}| \leq \|(\Sigma_{P} - \hat{\Sigma}_{n})\hat{\Sigma}_{n}^{-1}\|_{o,p}$$

$$\leq \|\hat{\Sigma}_{n}^{-1}\|_{o,p}\|\Sigma_{P} - \hat{\Sigma}_{n}\|_{o,p} = O_{P}(\|\Sigma_{P} - \hat{\Sigma}_{n}\|_{o,p}), \quad (S.7)$$

where the final equality holds uniformly in $P \in \mathbf{P}_0$ by Lemma S.1.4. By identical arguments it follows that $|\|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} - 1| = O_P(\|\hat{\Sigma}_n - \Sigma_P\|_{o,p})$ uniformly in $P \in \mathbf{P}_0$, and therefore (S.6), $\Theta_{0n}^r \subseteq \Theta_n \cap R$, and Assumption S.1.2(i) imply that

$$\inf_{\theta \in \Theta_n \cap R} \hat{Q}_P(\theta) - \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p}^{-1} \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) \le O_P(\eta_n)$$
 (S.8)

uniformly in $P \in \mathbf{P}_0$. Therefore, (S.2), (S.3), (S.4), (S.5), and (S.8) yield that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(\overrightarrow{d}_H(\hat{\Theta}_n^{\mathrm{r}}, \Theta_{0n}^{\mathrm{r}}, \| \cdot \|_{\mathbf{A}}) > \epsilon)$$

$$\leq \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(S_n(\epsilon) \leq \|\Sigma_P\|_{o,p} \|\hat{\Sigma}_n^{-1}\|_{o,p} M(\eta_n + \tau_n)) = 0,$$

where the equality follows from Lemma S.1.4, Assumption S.1.1(ii), and $\eta_n \vee \tau_n = o(S_n(\epsilon))$ by hypothesis. Part (i) of the lemma then follows by definition of $\mathcal{V}_n(P)$.

In order to establish part (ii) of the lemma, note that the definition of $\hat{\Theta}_n^{\mathrm{r}}$ implies

$$P(\Theta_{0n}^{\mathrm{r}} \subseteq \hat{\Theta}_{n}^{\mathrm{r}}) \ge P(\sup_{\theta \in \Theta_{0n}^{\mathrm{r}}} Q_{n}(\theta) \le \inf_{\theta \in \Theta_{n} \cap R} Q_{n}(\theta) + \tau_{n})$$
(S.9)

for all $P \in \mathbf{P}_0$. Moreover, applying Lemmas S.1.2 and S.1.4 together with $\|\hat{\Sigma}_n a\|_p \le \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \|\Sigma_P a\|_p$ for any $a \in \mathbf{R}^{k_n}$ implies that uniformly in $P \in \mathbf{P}_0$

$$\sup_{\theta \in \Theta_{0n}^{\mathbf{r}}} Q_n(\theta) \leq \sup_{\theta \in \Theta_{0n}^{\mathbf{r}}} \hat{Q}_P(\theta) + O_P(\eta_n)
\leq \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \sup_{\theta \in \Theta_{0n}^{\mathbf{r}}} Q_P(\theta) + O_P(\eta_n) = \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) + O_P(\eta_n), \quad (S.10)$$

where the final equality follows from Assumption S.1.2(ii), identical arguments to those in (S.7) implying $|||\hat{\Sigma}_n\Sigma_P^{-1}||_{o,p} - 1| = O_P(||\hat{\Sigma}_n - \Sigma_P||_{o,p})$ uniformly in $P \in \mathbf{P}$, and Assumption S.1.2(i). Similarly, Lemmas S.1.2 and S.1.4, $||\Sigma_P a||_p \leq ||\Sigma_P \hat{\Sigma}_n^{-1}||_{o,p} ||\hat{\Sigma}_n a||_p$ for any $a \in \mathbf{R}^{k_n}$, Assumption S.1.2(i), and result (S.7) imply that uniformly in $P \in \mathbf{P}_0$

$$\inf_{\theta \in \Theta_n \cap R} Q_n(\theta) \ge \inf_{\theta \in \Theta_n \cap R} \hat{Q}_P(\theta) - O_P(\eta_n)$$

$$\ge \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p}^{-1} \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) - O_P(\eta_n) = \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) - O_P(\eta_n). \quad (S.11)$$

Part (ii) of the lemma thus follows from (S.9), (S.10), (S.11), and $\eta_n = o(\tau_n)$.

Theorem S.1.1. Let Assumptions 3.1(i), 3.2(i)(iii), S.1.1, S.1.2, S.1.3 hold, and

$$\mathcal{R}_n \equiv \nu_n \left\{ \frac{k_n^{1/p} \sqrt{\log(1+k_n)} J_n B_n}{\sqrt{n}} \right\}. \tag{S.12}$$

Then uniformly in $P \in \mathbf{P}_0$: (i) $\overrightarrow{d}_H(\hat{\Theta}_n^r, \Theta_{0n}^r, \| \cdot \|_{\mathbf{E}}) = O_P(\mathcal{R}_n + \nu_n \tau_n)$; and (ii) $d_H(\hat{\Theta}_n^r, \Theta_{0n}^r, \| \cdot \|_{\mathbf{E}}) = O_P(\nu_n \tau_n)$ provided $J_n B_n k_n^{1/p} \sqrt{\log(1 + k_n)/n} = o(\tau_n)$.

PROOF: Let $\eta_n \equiv k_n^{1/p} \sqrt{\log(1+k_n)} J_n B_n / \sqrt{n}$, $\delta_n^{-1} \equiv \nu_n (\eta_n + \tau_n)$, and $Q_P(\theta) \equiv$

 $||E_P[\rho(X,\theta)*q^{k_n}(Z)]||_{\Sigma_P,p}$. In addition, we define $A_n\equiv A_{n1}\cap A_{n2}\cap A_{n3}$ where

$$A_{n1} \equiv \{\hat{\Theta}_{n}^{r} \subseteq \mathcal{V}_{n}(P)\}$$

$$A_{n2} \equiv \{\hat{\Sigma}_{n}^{-1} \text{ exists and } \|\hat{\Sigma}_{n}^{-1}\|_{o,p} \vee \|\hat{\Sigma}_{n}\|_{o,p} \vee \|\Sigma_{P}^{-1}\|_{o,p} \vee \|\Sigma_{P}\|_{o,p} < B\}$$

$$A_{n3} \equiv \{\sup_{\theta \in \Theta_{0n}^{r}} Q_{P}(\theta) \times \|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p} \leq B\eta_{n} \text{ and } \|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p} \leq \frac{1}{2B}\}.$$
 (S.13)

Moreover, note that for any $\epsilon > 0$ and B sufficiently large we can conclude that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(A_n^c) < \epsilon \tag{S.14}$$

due to Lemma S.1.4 and Assumptions S.1.1(i), S.1.2(i), and S.1.3. Therefore, we obtain

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\delta_{n} \overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \| \cdot \|_{\mathbf{E}}) > 2^{M})$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\delta_{n} \overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \| \cdot \|_{\mathbf{E}}) > 2^{M}; \ A_{n}) + \epsilon \quad (S.15)$$

for any M. For each $P \in \mathbf{P}_0$, next partition $\mathcal{V}_n(P)$ into subsets $S_{n,j}(P)$ defined by

$$S_{n,j}(P) \equiv \{\theta \in \mathcal{V}_n(P) : 2^{j-1} < \delta_n \overrightarrow{d}_H(\{\theta\}, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \le 2^j\}.$$

Since $\hat{\Theta}_n^{\mathrm{r}} \subseteq \mathcal{V}_n(P)$ under A_n , it follows from the definition of $\hat{\Theta}_n^{\mathrm{r}}$, and (S.15) that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\delta_{n} \overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \| \cdot \|_{\mathbf{E}}) > 2^{M})$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} \sum_{i > M}^{\infty} P(\inf_{\theta \in S_{n,j}(P)} Q_{n}(\theta) \leq \inf_{\theta \in \Theta_{n} \cap R} Q_{n}(\theta) + \tau_{n}; A_{n}) + \epsilon. \quad (S.16)$$

Letting $\hat{Q}_P(\theta) \equiv ||E_P[\rho(X,\theta) * q^{k_n}(Z)]||_{\hat{\Sigma}_n,p}$, we then obtain from Lemma S.1.2 that

$$\inf_{\theta \in \Theta_n \cap R} Q_n(\theta) \le \inf_{\theta \in \Theta_n \cap R} \hat{Q}_P(\theta) + \|\hat{\Sigma}_n\|_{o,p} \mathcal{Z}_{n,P} \le \inf_{\theta \in \Theta_n \cap R} \hat{Q}_P(\theta) + B\mathcal{Z}_{n,P}$$
 (S.17)

where the final inequality holds under the event A_n by (S.13). Moreover, since for any $a \in \mathbf{R}^{k_n}$ we have $\|\hat{\Sigma}_n a\|_p \leq \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \|\Sigma_P a\|_p$, we obtain from $\Theta_{0n}^{\mathbf{r}} \subseteq \Theta_n \cap R$ and the inequality $\|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \leq \|\{\hat{\Sigma}_n - \Sigma_P\}\Sigma_P^{-1}\|_{o,p} + 1$ that under the event A_n we have

$$\inf_{\theta \in \Theta_n \cap R} \hat{Q}_P(\theta) \le \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \inf_{\theta \in \Theta_n \cap R} Q_P(\theta)$$

$$\le \{1 + \|\Sigma_P^{-1}\|_{o,p} \|\hat{\Sigma}_n - \Sigma_P\|_{o,p} \} \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) \le \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) + B^2 \eta_n. \quad (S.18)$$

In addition, note that by similar arguments we also obtain from Lemma S.1.2 and

 $\|\Sigma_P a\|_p \leq \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p} \|\hat{\Sigma}_n a\|_p$ that under the event A_n we must have

$$\inf_{\theta \in S_{n,j}(P)} Q_n(\theta) \ge \inf_{\theta \in S_{n,j}(P)} \hat{Q}_P(\theta) - \|\hat{\Sigma}_n\|_{o,p} \mathcal{Z}_{n,P}$$

$$\ge \|\Sigma_P \hat{\Sigma}_n^{-1}\|_{o,p}^{-1} \inf_{\theta \in S_{n,j}(P)} Q_P(\theta) - B\mathcal{Z}_{n,P}. \quad (S.19)$$

Next, we note the triangle inequality, $\|(\Sigma_P - \hat{\Sigma}_n)\hat{\Sigma}_n^{-1}\|_{o,p} \leq \|\hat{\Sigma}_n^{-1}\|_{o,p}\|\hat{\Sigma}_n - \Sigma_P\|_{o,p}$, and $\|\hat{\Sigma}_n^{-1}\|_{o,p} \leq B$ under the event A_n by (S.13) yield the inequality

$$\|\Sigma_{P}\hat{\Sigma}_{n}^{-1}\|_{o,p}^{-1} - 1 \ge (\|(\Sigma_{P} - \hat{\Sigma}_{n})\hat{\Sigma}_{n}^{-1}\|_{o,p} + 1)^{-1} - 1$$

$$\ge -\|(\Sigma_{P} - \hat{\Sigma}_{n})\hat{\Sigma}_{n}^{-1}\|_{o,p} \ge -B\|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p}. \quad (S.20)$$

Therefore, combining results (S.19) and (S.20), together with Assumption S.1.3 and the definition of $S_{n,j}(P)$ we obtain for B sufficiently large that under the event A_n we have

$$\inf_{\theta \in S_{n,j}(P)} Q_n(\theta) \ge (1 - B \|\hat{\Sigma}_n - \Sigma_P\|_{o,p}) \times \inf_{\theta \in S_{n,j}(P)} Q_P(\theta) - B \mathcal{Z}_{n,P}$$

$$\ge (1 - B \|\hat{\Sigma}_n - \Sigma_P\|_{o,p}) \left(\inf_{\theta \in \Theta_n \cap R} Q_P(\theta) + \frac{2^{j-1}}{\nu_n \delta_n} - B\eta_n\right) - B \mathcal{Z}_{n,P}$$

$$\ge \inf_{\theta \in \Theta_n \cap R} Q_P(\theta) + \frac{2^{j-2}}{\nu_n \delta_n} - B(\mathcal{Z}_{n,P} + 2B\eta_n), \tag{S.21}$$

where the final inequality follows from $\Theta_{0n}^{r} \subseteq \Theta_{n} \cap R$ and the definition of the evnet A_{n} in (S.13). Hence, results (S.16), (S.17), (S.18), and (S.21) yield

$$\limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\delta_{n} \overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \| \cdot \|_{\mathbf{E}}) > 2^{M})$$

$$\leq \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} \sum_{j \geq M}^{\infty} P(\frac{2^{j-2}}{\nu_{n} \delta_{n}} \leq 3B(B\eta_{n} + \mathcal{Z}_{n,P}) + \tau_{n}; A_{n}) + \epsilon$$

$$\leq \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} \sum_{j \geq M}^{\infty} P(2^{(j-3)}(\eta_{n} + \tau_{n}) \leq 3B\mathcal{Z}_{n,P}) + \epsilon, \qquad (S.22)$$

where in the final inequality we employed that we had defined $\delta_n^{-1} \equiv \nu_n(\eta_n + \tau_n)$. Therefore, $\mathcal{Z}_{n,P} \in \mathbf{R}_+$, Lemma S.1.2, $\tau_n \geq 0$, and Markov's inequality yield

$$\limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} \sum_{j \ge M}^{\infty} P(2^{(j-3)}(\eta_n + \tau_n) \le 3B \mathcal{Z}_{n,P})$$

$$\lesssim \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sum_{j \ge M} 2^{-j} \times \frac{1}{\eta_n} \frac{k_n^{1/p} \sqrt{\log(1 + k_n)} J_n B_n}{\sqrt{n}} = 0, \quad (S.23)$$

where in the final result we employed that $\eta_n \equiv k_n^{1/p} \sqrt{\log(1+k_n)} J_n B_n / \sqrt{n}$. The first claim of the theorem therefore follows from (S.22), (S.23), and ϵ being arbitrary.

To establish the second claim of the theorem, next define the event $A_{n4} \equiv \{\Theta_{0n}^{\rm r} \subseteq \hat{\Theta}_n^{\rm r}\}$. Since $\overrightarrow{d}_H(\Theta_{0n}^{\rm r}, \hat{\Theta}_n^{\rm r}, \|\cdot\|_{\mathbf{E}}) = 0$ whenever the event A_{n4} occurs, we can conclude from Lemma S.1.1(ii) and part (i) of this theorem that

$$\limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\delta_{n} d_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \| \cdot \|_{\mathbf{E}}) > 2^{M})$$

$$= \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\delta_{n} \overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \| \cdot \|_{\mathbf{E}}) > 2^{M}) = 0, \quad (S.24)$$

and thus the theorem follows from $\delta_n^{-1} = \nu_n(\eta_n + \tau_n)$ and $\eta_n = o(\tau_n)$.

Corollary S.1.1. If Assumptions 3.1(i), 3.2(i)(iii), 3.3(i), 3.4, 3.6(ii), and 3.7 hold, then $\overrightarrow{d}_H(\hat{\theta}_n, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) = O_P(\mathcal{R}_n)$ uniformly in $P \in \mathbf{P}_0$.

PROOF: Follows from Theorem S.1.1(i) applied with $\tau_n \equiv a_n/\sqrt{n}$ after noting that $a_n = o(1)$ (by Assumption 3.3(i)) implies $\nu_n a_n/\sqrt{n} = o(\mathcal{R}_n)$ and: (i) Assumption S.1.1 holds by Assumption 3.7; (ii) Assumption S.1.2(i) holds by Assumptions 3.6(ii) and 3.7(i); (iii) Assumption S.1.2(ii) holds by $Q_P(\theta) \geq 0$ and Assumption 3.6(ii); and (iv) Assumption S.1.3 holds with $\tau_n \equiv a_n/\sqrt{n}$ by Assumptions 3.4 and 3.6(ii), the triangle inequality, and $\inf_{\theta \in \Theta_n \cap R} Q_P(\theta) \leq \sup_{\theta \in \Theta_{0n}^r} Q_P(\theta)$ due to $\Theta_{0n}^r \subseteq \Theta_n \cap R$.

Corollary S.1.2. Let Assumptions 3.1(i), 3.2(i)(iii), 3.3(i), 3.4(i), 3.6(ii), 3.7, and 3.12(iii) hold. Then uniformly in $P \in \mathbf{P}_0$: (i) $\overrightarrow{d}_H(\hat{\Theta}_n^r, \Theta_{0n}^r, \| \cdot \|_{\mathbf{E}}) = O_P(\mathcal{R}_n + \nu_n \tau_n)$; and (ii) $d_H(\hat{\Theta}_n^r, \Theta_{0n}^r, \| \cdot \|_{\mathbf{E}}) = O_P(\nu_n \tau_n)$ provided $J_n B_n k_n^{1/p} \sqrt{\log(1 + k_n)/n} = o(\tau_n)$.

PROOF: Follows from Theorem S.1.1 after noting that $a_n = o(1)$ (by Assumption 3.3(i)) implies: (i) Assumption S.1.1 holds by Assumption 3.7; (ii) Assumption S.1.2(i) holds by Assumptions 3.6(ii) and 3.7(i); (iii) Assumption S.1.2(ii) holds by $Q_P(\theta) \geq 0$ and Assumption 3.6(ii); and (iv) Assumption S.1.3 holds by Assumptions 3.4(i), 3.6(ii), 3.12(iii), the triangle inequality, and $\Theta_{0n}^r \subseteq \Theta_n \cap R$.

Lemma S.1.2. Let $\hat{Q}_P(\theta) \equiv \|E_P[\rho(X,\theta)*q^{k_n}(Z)]\|_{\hat{\Sigma}_n,p}$, and Assumptions 3.1(i), 3.2(i), and 3.2(iii) hold. Then, for each $P \in \mathbf{P}$ there are random $\mathcal{Z}_{n,P} \in \mathbf{R}_+$ with

$$|Q_n(\theta) - \hat{Q}_P(\theta)| \le ||\hat{\Sigma}_n||_{o,p} \times \mathcal{Z}_{n,P},$$

for all $\theta \in \Theta_n \cap R$ and in addition $\sup_{P \in \mathbf{P}} E_P[\mathcal{Z}_{n,P}] = O(k_n^{1/p} \sqrt{\log(1+k_n)} J_n B_n / \sqrt{n})$.

PROOF: Let $\mathcal{G}_n \equiv \{fq_{k,j} : f \in \mathcal{F}_n, 1 \leq j \leq \mathcal{J} \text{ and } 1 \leq k \leq k_{n,j}\}$. Note that by Assumption 3.2(i), $\|q_{k,j}\|_{\infty} \leq B_n$ for all $1 \leq j \leq \mathcal{J}$ and $1 \leq k \leq k_{n,j}$. Hence, letting F_n be the envelope for \mathcal{F}_n , as in Assumption 3.2(iii), it follows that $G_n \equiv B_n F_n$ is an envelope for \mathcal{G}_n satisfying $\sup_{P \in \mathbf{P}} E_P[G_n^2(V)] < \infty$. Thus, we obtain

$$\sup_{P \in \mathbf{P}} E_P[\sup_{g \in \mathcal{G}_n} | \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(V_i) - E_P[g(V)]) |] \lesssim \sup_{P \in \mathbf{P}} J_{[]}(\|G_n\|_{P,2}, \mathcal{G}_n, \| \cdot \|_{P,2})$$
 (S.25)

by Theorem 2.14.2 in van der Vaart and Wellner (1996). Moreover, also notice that Lemma S.1.3, the change of variables $u = \epsilon/B_n$, and $B_n \ge 1$ imply

$$\sup_{P \in \mathbf{P}} J_{[]}(\|G_n\|_{P,2}, \mathcal{G}_n, \|\cdot\|_{P,2}) \leq \sup_{P \in \mathbf{P}} \int_0^{\|G_n\|_{P,2}} \sqrt{1 + \log(k_n N_{[]}(\epsilon/B_n, \mathcal{F}_n, \|\cdot\|_{P,2}))} d\epsilon$$

$$\leq (1 + \sqrt{\log(k_n)}) B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\|F_n\|_{P,2}, \mathcal{F}_n, \|\cdot\|_{P,2}) = O(\sqrt{\log(1 + k_n)} B_n J_n), \quad (S.26)$$

where the final equality follows from Assumption 3.2(iii). Next define $\mathcal{Z}_{n,P} \in \mathbf{R}_+$ by

$$\mathcal{Z}_{n,P} \equiv \frac{k_n^{1/p}}{\sqrt{n}} \times \sup_{g \in \mathcal{G}_n} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(V_i) - E_P[g(V)]) \right|$$

and note (S.25) and (S.26) imply $\sup_{P\in\mathbf{P}} E_P[\mathcal{Z}_{n,P}] = O(k_n^{1/p} \sqrt{\log(1+k_n)} B_n J_n/\sqrt{n})$ as desired. Moreover, for any $\theta \in \Theta_n \cap R$, the definitions of $\mathbb{G}_n(\theta)$, \mathcal{G}_n , and $\mathcal{Z}_{n,P}$ yield

$$|Q_{n}(\theta) - \hat{Q}_{P}(\theta)| \leq \frac{\|\hat{\Sigma}_{n}\|_{o,p}}{\sqrt{n}} \times \|\mathbb{G}_{n}(\theta)\|_{p}$$

$$\leq \|\hat{\Sigma}_{n}\|_{o,p} \times \frac{k_{n}^{1/p}}{\sqrt{n}} \times \sup_{g \in \mathcal{G}_{n}} |\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g(V_{i}) - E_{P}[g(V)])| \equiv \|\hat{\Sigma}_{n}\|_{o,p} \times \mathcal{Z}_{n,P},$$

which establishes the claim of the lemma.

Lemma S.1.3. Let $\{g_j\}_{j=1}^J$ be functions satisfying $\max_{1 \leq j \leq J} \|g_j\|_{\infty} \leq C < \infty$ and define $\mathcal{G}_n \equiv \{fg_j : f \in \mathcal{F}_n, \ 1 \leq j \leq J\}$. Then for any P it follows that

$$N_{[]}(\epsilon, \mathcal{G}_n, \|\cdot\|_{P,2}) \leq J \times N_{[]}(\epsilon/C, \mathcal{F}_n, \|\cdot\|_{P,2}).$$

PROOF: First define $g_j^+ \equiv g_j \vee 0$ and $g_j^- \equiv g_j \wedge 0$, where \vee and \wedge denote the pointwise maximums and minimums. If $\{[f_{i,l}, f_{i,u}]\}_i$ is a collection of brackets for \mathcal{F}_n satisfying

$$\int (f_{i,u} - f_{i,l})^2 dP \le \epsilon^2 \tag{S.27}$$

for all i, then it follows that the following collection of brackets covers the class \mathcal{G}_n :

$$\{[g_j^+ f_{i,l} + g_j^- f_{i,u}, g_j^- f_{i,l} + g_j^+ f_{i,u}]\}_{i,j}.$$
 (S.28)

Moreover, since $|g_j| = g_j^+ - g_j^-$ by construction, we also obtain from result (S.27) that

$$\int (g_j^+ f_{i,u} + g_j^- f_{i,l} - g_j^+ f_{i,l} - g_j^- f_{i,u})^2 dP = \int (f_{i,u} - f_{i,l})^2 |g_j|^2 dP \le \epsilon^2 C^2.$$
 (S.29)

Since there are $J \times N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2})$ brackets in (S.28), we conclude from (S.29) that $N_{[]}(\epsilon, \mathcal{G}_n, \|\cdot\|_{P,2}) \leq J \times N_{[]}(\epsilon/C, \mathcal{F}_n, \|\cdot\|_{P,2})$, which establishes the lemma.

Lemma S.1.4. If Assumption S.1.1 holds, then there is a constant $B < \infty$ such that

$$\liminf_{n \to \infty} \inf_{P \in \mathbf{P}} P(\hat{\Sigma}_n^{-1} \text{ exists and } \max\{\|\hat{\Sigma}_n\|_{o,p}, \|\hat{\Sigma}_n^{-1}\|_{o,p}\} < B) = 1.$$

PROOF: Recall that $\hat{\Sigma}_n$ and Σ_P are $k_n \times k_n$ matrices, though the dependence on k_n was suppressed from the notation. Then note that by Assumption S.1.1(ii) there exists a constant $B < \infty$ such that for all $P \in \mathbf{P}$ and k_n we have that

$$\max\{\|\Sigma_P\|_{o,p}, \|\Sigma_P^{-1}\|_{o,p}\} < \frac{B}{2}.$$
 (S.30)

Next, let I_{k_n} denote the $k_n \times k_n$ identity matrix and for each $P \in \mathbf{P}$ rewrite $\hat{\Sigma}_n$ as

$$\hat{\Sigma}_n = \Sigma_P \{ I_{k_n} - \Sigma_P^{-1} (\Sigma_P - \hat{\Sigma}_n) \}. \tag{S.31}$$

By Theorem 2.9 in Kress (1999), the matrix $\{I_{k_n} - \Sigma_P^{-1}(\Sigma_P - \hat{\Sigma}_n)\}$ is invertible and the operator norm of its inverse is bounded by two when $\|\Sigma_P^{-1}(\Sigma_P - \hat{\Sigma}_n)\|_{o,p} < 1/2$. Since Σ_P is invertible by Assumption S.1.1(i), result (S.31) implies that $\hat{\Sigma}_n$ is invertible if and only if $\{I_{k_n} - \Sigma_P^{-1}(\Sigma_P - \hat{\Sigma}_n)\}$ is invertible, which yields that

$$P(\hat{\Sigma}_{n}^{-1} \text{ exists and } \|\{I_{k_{n}} - \Sigma_{P}^{-1}(\Sigma_{P} - \hat{\Sigma}_{n})\}^{-1}\|_{o,p} < 2)$$

$$\geq P(\|\Sigma_{P}^{-1}(\hat{\Sigma}_{n} - \Sigma_{P})\|_{o,p} < \frac{1}{2}) \geq P(\|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p} < \frac{1}{B}), \quad (S.32)$$

where we employed $\|\Sigma_P^{-1}(\hat{\Sigma}_n - \Sigma_P)\|_{o,p} \leq \|\Sigma_P^{-1}\|_{o,p} \|\hat{\Sigma}_n - \Sigma_P\|_{o,p}$ and (S.30). Hence, since (S.31) implies $\hat{\Sigma}_n^{-1} = \{I_{k_n} - \Sigma_P^{-1}(\Sigma_P - \hat{\Sigma}_n)\}^{-1}\Sigma_P^{-1}$ whenever $\{I_{k_n} - \Sigma_P^{-1}(\Sigma_P - \hat{\Sigma}_n)\}^{-1}$ exists, the bound $\|\Sigma_P^{-1}\|_{o,p} < B/2$ and result (S.32) allow us to conclude

$$P(\hat{\Sigma}_n^{-1} \text{ exists and } \|\hat{\Sigma}_n^{-1}\|_{o,p} < B) \ge P(\|\hat{\Sigma}_n - \Sigma_P\|_{o,p} < \frac{1}{B}).$$
 (S.33)

Finally, since $\|\hat{\Sigma}_n\|_{o,p} \leq B/2 + \|\hat{\Sigma}_n - \Sigma_P\|_{o,p}$ by (S.30), result (S.33) implies that

$$\liminf_{n\to\infty}\inf_{P\in\mathbf{P}}P(\hat{\Sigma}_n^{-1}\text{ exists and }\max\{\|\hat{\Sigma}_n\|_{o,p},\|\hat{\Sigma}_n^{-1}\|_{o,p}\}< B)$$

$$\geq \liminf_{n \to \infty} \inf_{P \in \mathbf{P}} P(\|\hat{\Sigma}_n - \Sigma_P\|_{o,p} < \min\{\frac{B}{2}, \frac{1}{B}\}) = 1,$$

where the equality, and hence the lemma, follows from Assumption S.1.1(i). ■

Corollary S.1.3. If Assumption 3.7 holds, then for some $B < \infty$ it follows that:

$$\liminf_{n \to \infty} \inf_{P \in \mathbf{P}} P(\hat{\Sigma}_n^{-1} \text{ exists and } \max\{\|\hat{\Sigma}_n\|_{o,p}, \|\hat{\Sigma}_n^{-1}\|_{o,p}\} < B) = 1.$$

PROOF: Follows from Lemma S.1.4 and Assumption 3.7 together with $a_n = o(1)$, which

is imposed by Assumption 3.3(i) (or 3.11), implying Assumption S.1.1 holds. ■

Lemma S.1.5. If
$$a \in \mathbf{R}^d$$
, then $||a||_{\tilde{p}} \leq d^{(\frac{1}{\tilde{p}} - \frac{1}{p})_+} ||a||_p$ for any $\tilde{p}, p \in [2, \infty]$.

PROOF: The case $p \leq \tilde{p}$ trivially follows from $||a||_{\tilde{p}} \leq ||a||_{p}$ for all $a \in \mathbf{R}^{d}$. For the case $p > \tilde{p}$, let $a = (a_{1}, \ldots, a_{d})'$ and note that by Hölder's inequality we obtain

$$||a||_{\tilde{p}}^{\tilde{p}} = \sum_{i=1}^{d} \{|a_{i}|^{\tilde{p}} \times 1\} \leq \{\sum_{i=1}^{d} (|a_{i}|^{\tilde{p}})^{\frac{p}{\tilde{p}}}\}^{\frac{\tilde{p}}{\tilde{p}}} \{\sum_{i=1}^{d} 1^{\frac{p}{p-\tilde{p}}}\}^{1-\frac{\tilde{p}}{\tilde{p}}} = \{\sum_{i=1}^{d} |a_{i}|^{p}\}^{\frac{\tilde{p}}{\tilde{p}}} d^{1-\frac{\tilde{p}}{\tilde{p}}}. \quad (S.34)$$

Thus, the claim of the lemma for $p > \tilde{p}$ follows from taking the $1/\tilde{p}$ power in (S.34).

S.2 Strong Approximation

This Section contains the proof of Theorem 3.1 and supporting results.

PROOF OF THEOREM 3.1: First note that by Assumption 3.7(ii) there is a constant $C_0 < \infty$ such that $\|\Sigma_P\|_{o,p} \le C_0$ for all $P \in \mathbf{P}_0$. Hence, Assumption 3.6(ii) and Lemma S.1.5 imply that for all $P \in \mathbf{P}_0$, $\theta \in \Theta_{0n}^r$, and $h \in V_n(\theta, R|\ell_n)$ we have

$$\|\sqrt{n}E_{P}[\rho(X,\theta+\frac{h}{\sqrt{n}})*q^{k_{n}}(Z)] - \mathbb{D}_{P}(\theta)[h]\|_{\Sigma_{P},p}$$

$$\leq C_{0}\|\sqrt{n}E_{P}[(\rho(X,\theta+\frac{h}{\sqrt{n}})-\rho(X,\theta))*q^{k_{n}}(Z)] - \mathbb{D}_{P}(\theta)[h]\|_{2} + o(a_{n}). \quad (S.35)$$

Moreover, Lemma S.2.5 and the maps $m_{P,j}$ satisfying Assumption 3.5(i) imply that

$$\sum_{j=1}^{\mathcal{J}} \sum_{k=1}^{k_{n,j}} \langle \sqrt{n} \{ m_{P,j}(\theta + \frac{h}{\sqrt{n}}) - m_{P,j}(\theta) \} - \nabla m_{P,j}(\theta) [h], q_{k,j} \rangle_{L_{P}^{2}}^{2}$$

$$\leq \sum_{j=1}^{\mathcal{J}} C_{1} \| \sqrt{n} \{ m_{P,j}(\theta + \frac{h}{\sqrt{n}}) - m_{P,j}(\theta) - \nabla m_{P,j}(\theta) [\frac{h}{\sqrt{n}}] \} \|_{P,2}^{2}$$

$$\leq \sum_{j=1}^{\mathcal{J}} C_{1} K_{m}^{2} \times n \times \| \frac{h}{\sqrt{n}} \|_{\mathbf{L}}^{2} \times \| \frac{h}{\sqrt{n}} \|_{\mathbf{E}}^{2}$$
(S.36)

for some constant $C_1 < \infty$ and all $P \in \mathbf{P}_0$, $\theta \in \Theta_{0n}^{\mathrm{r}}$, and $h \in V_n(\theta, R|\ell_n)$. Therefore, by results (S.35) and (S.36), the law of iterated expectations, the definition of $\mathcal{S}_n(\mathbf{L}, \mathbf{E})$, and $K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ by hypothesis, we obtain that

$$\sup_{P \in \mathbf{P}_0} \sup_{\theta \in \Theta_{0n}^{\mathrm{r}}} \sup_{h \in V_n(\theta, R|\ell_n)} \|\sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] - \mathbb{D}_P(\theta)[h]\|_{\Sigma_P, p}$$

$$\lesssim K_m \times \sqrt{n} \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) + o(a_n) = o(a_n). \quad (S.37)$$

Next, note that since $k_n^{1/p}\sqrt{\log(1+k_n)}B_n \times \sup_{P\in\mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho},\mathcal{F}_n,\|\cdot\|_{P,2}) = o(a_n)$, Assumption 3.6(i) implies there is a sequence $\tilde{\ell}_n$ satisfying the conditions of Lemma S.2.1 and $\ell_n = o(\tilde{\ell}_n)$. Therefore, applying Lemma S.2.1 we obtain uniformly in $P \in \mathbf{P}_0$

$$I_n(R) = \inf_{\theta \in \Theta_{0n}^r} \inf_{h \in V_n(\theta, R|\tilde{\ell}_n)} \| \mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] \|_{\Sigma_P, p} + o_P(a_n).$$
 (S.38)

Moreover, since $\ell_n = o(\tilde{\ell}_n)$ implies that $V_n(\theta, R|\tilde{\ell}_n) \subseteq V_n(\theta, R|\ell_n)$ for all $\theta \in \Theta_n \cap R$ for n sufficiently large, we obtain uniformly in $P \in \mathbf{P}_0$ that

$$\inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R|\tilde{\ell}_{n})} \|\mathbb{W}_{P}(\theta) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)]\|_{\Sigma_{P}, p}$$

$$\leq \inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R|\ell_{n})} \|\mathbb{W}_{P}(\theta) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)]\|_{\Sigma_{P}, p}$$

$$= \inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R|\ell_{n})} \|\mathbb{W}_{P}(\theta) + \mathbb{D}_{P}(\theta)[h]\|_{\Sigma_{P}, p} + o(a_{n}), \tag{S.39}$$

where the final equality following from (S.37), Assumption 3.7(ii) and Lemma S.2.6. Thus, the first claim of the Theorem follows from (S.38) and (S.39), while the second follows by noting that if $K_m \mathcal{R}_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$, then we may set ℓ_n to simultaneously satisfy the conditions of Lemma S.2.1 and $K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$, which obviates the need to introduce $\tilde{\ell}_n$ in (S.38) and (S.39).

Lemma S.2.1. Let Assumptions 3.1(i), 3.2(i), 3.2(iii), 3.3, 3.4, 3.6, and 3.7 hold. Then, for any sequence $\{\ell_n\}$ satisfying $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_p}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ and $\mathcal{R}_n = o(\ell_n)$, we have uniformly in $P \in \mathbf{P}_0$ that:

$$I_n(R) = \inf_{\theta \in \Theta_{0n}^r} \inf_{h \in V_n(\theta, R | \ell_n)} \| \mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] \|_{\Sigma_P, p} + o_P(a_n).$$

PROOF: First note that the required sequence $\{\ell_n\}$ exists by Assumption 3.6(i). Next, note that by Assumption 3.4(ii) and Corollary S.1.1 there is a $\hat{\theta}_n \in \Theta_n \cap R$ satisfying

$$Q_n(\hat{\theta}_n) \le \inf_{\theta \in \Theta_n \cap R} Q_n(\theta) + o(a_n/\sqrt{n}) \tag{S.40}$$

and $\overrightarrow{d}_H(\hat{\theta}_n, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) = O_P(\mathcal{R}_n)$ uniformly in $P \in \mathbf{P}_0$. Hence, defining $(\Theta_{0n}^r)^{\ell_n} \equiv \{\theta \in \Theta_n \cap R : \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \leq \ell_n\}$, which implicitly depends on $P \in \mathbf{P}_0$, we obtain

$$I_n(R) = \inf_{\theta \in (\Theta_{0n}^r)^{\ell_n}} \sqrt{n} Q_n(\theta) + o_P(a_n)$$
 (S.41)

uniformly in $P \in \mathbf{P}_0$ due to $\mathcal{R}_n = o(\ell_n)$, $\overrightarrow{d}_H(\hat{\theta}_n, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) = O_P(\mathcal{R}_n)$, $(\Theta_{0n}^r)^{\ell_n} \subseteq \Theta_n \cap R$ by construction, result (S.40), and the definition of $I_n(R)$. Next, note that by

Assumption 3.3(i), Corollary S.1.3, and Lemma S.2.6 it follows that

$$\begin{aligned}
&|\inf_{\theta \in (\Theta_{0n}^{\mathrm{r}})^{\ell_n}} \sqrt{n} Q_n(\theta) - \inf_{\theta \in (\Theta_{0n}^{\mathrm{r}})^{\ell_n}} \|\mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta) * q^{k_n}(Z)]\|_{\hat{\Sigma}_n, p}| \\
&\leq \|\hat{\Sigma}_n\|_{o, p} \times \sup_{\theta \in \Theta_n \cap R} \|\mathbb{G}_n(\theta) - \mathbb{W}_P(\theta)\|_p = o_P(a_n) \quad (S.42)
\end{aligned}$$

uniformly in $P \in \mathbf{P}_0$. Similarly, employing Corollary S.1.3, Lemmas S.2.2, S.2.6, and ℓ_n satisfying $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_p}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ yields

$$\inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R|\ell_{n})} \| \mathbb{W}_{P}(\theta + \frac{h}{\sqrt{n}}) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)] \|_{\hat{\Sigma}_{n}, p}$$

$$= \inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R|\ell_{n})} \| \mathbb{W}_{P}(\theta) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)] \|_{\hat{\Sigma}_{n}, p} + o_{P}(a_{n})$$

uniformly in $P \in \mathbf{P}_0$, which together with results (S.41) and (S.42), and Lemma S.2.3 establish the claim of the lemma.

Lemma S.2.2. Let Assumptions 3.2(i) and 3.3(ii) hold. If $\{\delta_n\}$ is a sequence satisfying $k_n^{1/p}\sqrt{\log(1+k_n)}B_n\times\sup_{P\in\mathbf{P}}J_{[]}(\delta_n^{\kappa_\rho},\mathcal{F}_n,\|\cdot\|_{P,2})=o(a_n)$, then uniformly in $P\in\mathbf{P}$:

$$\sup_{\theta \in \Theta_{0n}^r} \sup_{h \in V_n(\theta, R | \delta_n)} \| \mathbb{W}_P(\theta + \frac{h}{\sqrt{n}}) - \mathbb{W}_P(\theta) \|_p = o_P(a_n).$$

PROOF: Since $||q_{k,j}||_{\infty} \leq B_n$ for all $1 \leq j \leq \mathcal{J}$ and $1 \leq k \leq k_{n,j}$ by Assumption 3.2(i), Assumption 3.3(ii) yields for any $P \in \mathbf{P}$, $\theta \in \Theta_n \cap R$, and $h \in V_n(\theta, R | \delta_n)$ that

$$E_{P}[\|\rho(X,\theta+\frac{h}{\sqrt{n}})-\rho(X,\theta)\|_{2}^{2}q_{k,j}^{2}(Z)] \leq K_{\rho}^{2}B_{n}^{2}\|\frac{h}{\sqrt{n}}\|_{\mathbf{E}}^{2\kappa_{\rho}} \leq K_{\rho}^{2}B_{n}^{2}\delta_{n}^{2\kappa_{\rho}}.$$
 (S.43)

Set $\mathcal{G}_n \equiv \{fq_{k,j} \text{ for some } f \in \mathcal{F}_n, \ 1 \leq j \leq \mathcal{J}, \ 1 \leq k \leq k_{n,j}\}$ and let \mathbb{G}_P be a Gaussian process on \mathcal{G}_n satisfying $E[\mathbb{G}_P(g_1)] = 0$ and $E[\mathbb{G}_P(g_1)\mathbb{G}_P(g_2)] = \operatorname{Cov}_P\{g_1(V), g_2(V)\}$ for any $g_1, g_2 \in \mathcal{G}_n$. Since $\|a\|_p \leq k_n^{1/p} \|a\|_{\infty}$ for any $a \in \mathbb{R}^{k_n}$, result (S.43) yields

$$E_{P}\left[\sup_{\theta\in\Theta_{0n}^{r}}\sup_{h\in V_{n}(\theta,R|\delta_{n})}\|\mathbb{W}_{P}(\theta+\frac{h}{\sqrt{n}})-\mathbb{W}_{P}(\theta)\|_{p}\right]$$

$$\leq k_{n}^{1/p}\times E\left[\sup_{g_{1},g_{2}\in\mathcal{G}_{n}:\|g_{1}-g_{2}\|_{P,2}\leq K_{\rho}B_{n}\delta_{n}^{\kappa_{\rho}}}|\mathbb{G}_{P}(g_{1})-\mathbb{G}_{P}(g_{2})|\right]. \quad (S.44)$$

Moreover, Corollary 2.2.8 in van der Vaart and Wellner (1996) additionally implies that

$$\sup_{P \in \mathbf{P}} E_{P}\left[\sup_{g_{1},g_{2} \in \mathcal{G}_{n}: \|g_{1} - g_{2}\|_{P,2} \leq K_{\rho} B_{n} \delta_{n}^{\kappa_{\rho}}} |\mathbb{G}_{P}(g_{1}) - \mathbb{G}_{P}(g_{2})|\right]$$

$$\lesssim \sup_{P \in \mathbf{P}} \int_{0}^{K_{\rho} B_{n} \delta_{n}^{\kappa_{\rho}}} \sqrt{\log N_{[]}(\epsilon, \mathcal{G}_{n}, \|\cdot\|_{P,2})} d\epsilon$$

$$\lesssim \sup_{P \in \mathbf{P}} \sqrt{\log(1 + k_{n})} B_{n} \int_{0}^{K_{\rho} \delta_{n}^{\kappa_{\rho}}} \sqrt{1 + \log N_{[]}(u, \mathcal{F}_{n}, \|\cdot\|_{P,2})} du, \quad (S.45)$$

where the second inequality follows from Lemma S.1.3 and the change of variables $u = \epsilon/B_n$. However, note that since $N_{[]}(u, \mathcal{F}_n, \|\cdot\|_{P,2})$ is decreasing in u, it follows that $J_{[]}(K_\rho \delta_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) \leq K_\rho J_{[]}(\delta_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2})$. Therefore, the lemma follows from results (S.44) and (S.45), the definition of $J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2})$, and $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\delta_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ by hypothesis. \blacksquare

Lemma S.2.3. Let Assumptions 3.2(i), 3.2(iii), 3.6(ii), and 3.7 hold with $a_n = o(1)$. For any positive sequence δ_n it then follows that uniformly in $P \in \mathbf{P}_0$ we have

$$\inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R | \delta_{n})} \| \mathbb{W}_{P}(\theta) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)] \|_{\Sigma_{P}, p}
= \inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R | \delta_{n})} \| \mathbb{W}_{P}(\theta) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)] \|_{\hat{\Sigma}_{n}, p} + o_{P}(a_{n}).$$

PROOF: First note that by Assumption 3.7(ii) there is a $C_0 < \infty$ such that $\|\Sigma_P\|_{o,p} \vee \|\Sigma_P^{-1}\|_{o,p} \le C_0$ for all $P \in \mathbf{P}$. Since $\|\hat{\Sigma}_n a\|_p \le \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \|\Sigma_P a\|_p$ for any $a \in \mathbf{R}^{k_n}$, and $\|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \le \|\Sigma_P^{-1}\|_{o,p} \|\hat{\Sigma}_n - \Sigma_P\|_{o,p} + 1$ by the triangle inequality, we obtain

$$\begin{aligned} \{C_0 \| \hat{\Sigma}_n - \Sigma_P \|_{o,p} + 1 \} \| \mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] \|_{\Sigma_P, p} \\ &\geq \| \mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] \|_{\hat{\Sigma}_n, p} \end{aligned}$$
 (S.46)

for any $\theta \in \Theta_{0n}^{r}$ and $h \in V_n(\theta, R|\delta_n)$. Moreover, $\|\Sigma_P\|_{o,p} \leq C_0$, $0 \in V_n(\theta, R|\delta_n)$ for any $\theta \in \Theta_n \cap R$, and Assumption 3.6(ii) imply uniformly in $P \in \mathbf{P}$ that

$$\inf_{\theta \in \Theta_{0n}^{r}} \inf_{h \in V_{n}(\theta, R \mid \delta_{n})} \| \mathbb{W}_{P}(\theta) + \sqrt{n} E_{P}[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_{n}}(Z)] \|_{\Sigma_{P}, p}
\lesssim \sup_{\theta \in \Theta_{n} \cap R} \| \mathbb{W}_{P}(\theta) \|_{p} + o(a_{n}) = O_{P}(k_{n}^{1/p} \sqrt{\log(1 + k_{n})} B_{n} J_{n}) + o(a_{n}) \quad (S.47)$$

where the final equality holds uniformly in $P \in \mathbf{P}_0$ by Lemma S.2.4 and Markov's

inequality. Therefore, results (S.46), (S.47), and Assumption 3.7(i) imply

$$\inf_{\theta \in \Theta_{0n}^{\mathbf{r}}} \inf_{h \in V_n(\theta, R \mid \delta_n)} \| \mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] \|_{\Sigma_P, p} + o_P(a_n)
\geq \inf_{\theta \in \Theta_{0n}^{\mathbf{r}}} \inf_{h \in V_n(\theta, R \mid \delta_n)} \| \mathbb{W}_P(\theta) + \sqrt{n} E_P[\rho(X, \theta + \frac{h}{\sqrt{n}}) * q^{k_n}(Z)] \|_{\hat{\Sigma}_n, p} \quad (S.48)$$

uniformly in $P \in \mathbf{P}_0$. Next, note that Assumption 3.7 implies Assumption S.1.1 and therefore Lemma S.1.4 yields that $\|\hat{\Sigma}_n\|_{o,p} \vee \|\hat{\Sigma}_n^{-1}\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}$. The lemma then follows from (S.48) and noting that the reverse inequality also holds by identical arguments but relying on $\|\hat{\Sigma}_n\|_{o,p} \vee \|\hat{\Sigma}_n^{-1}\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}$ rather than on $\|\Sigma_P\|_{o,p} \vee \|\Sigma_P^{-1}\|_{o,p} \leq C_0$.

Lemma S.2.4. If Assumptions 3.2(i) and 3.2(iii) hold, then for some $C < \infty$ we have:

$$\sup_{P \in \mathbf{P}} E_P[\sup_{\theta \in \Theta_n \cap R} \| \mathbb{W}_P(\theta) \|_p] \le C k_n^{1/p} \sqrt{\log(1 + k_n)} B_n J_n.$$

PROOF: Let $\mathcal{G}_n \equiv \{fq_{k,j} : f \in \mathcal{F}_n, \ 1 \leq j \leq \mathcal{J}, \text{ and } 1 \leq k \leq k_{n,j}\}$ and \mathbb{G}_P be a Gaussian process on \mathcal{G}_n satisfying $E[\mathbb{G}_P(g_1)] = 0$ and $E[\mathbb{G}_P(g_1)\mathbb{G}_P(g_2)] = \operatorname{Cov}_P\{g_1(V), g_2(V)\}$ for any $g_1, g_2 \in \mathcal{G}_n$. Then note $||a||_p \leq d^{1/p}||a||_{\infty}$ for any $a \in \mathbb{R}^d$ implies that

$$E_{P}[\sup_{\theta \in \Theta_{n} \cap R} \| \mathbb{W}_{P}(\theta) \|_{p}] \leq k_{n}^{1/p} E_{P}[\sup_{g \in \mathcal{G}_{n}} |\mathbb{G}_{P}(g)|]$$

$$\leq k_{n}^{1/p} \{ E_{P}[|\mathbb{G}_{P}(g_{0})|] + C_{1} \int_{0}^{\infty} \sqrt{\log N_{[]}(\epsilon, \mathcal{G}_{n}, \| \cdot \|_{P,2})} d\epsilon \}, \quad (S.49)$$

where the final inequality holds for any $g_0 \in \mathcal{G}_n$ and some $C_1 < \infty$ by Corollary 2.2.8 in van der Vaart and Wellner (1996). Next, let $G_n \equiv B_n F_n$ for F_n as in Assumption 3.2(iii) and note Assumption 3.2(i) implies G_n is an envelope for \mathcal{G}_n . Thus, $[-G_n, G_n]$ is a bracket of size $2\|G_n\|_{P,2}$ covering \mathcal{G}_n , and as a result we obtain

$$\int_{0}^{\infty} \sqrt{\log N_{[]}(\epsilon, \mathcal{G}_{n}, \|\cdot\|_{P,2})} d\epsilon
\leq \int_{0}^{2\|G_{n}\|_{P,2}} \sqrt{1 + \log N_{[]}(\epsilon, \mathcal{G}_{n}, \|\cdot\|_{P,2})} d\epsilon \leq C_{2} \sqrt{\log(1 + k_{n})} B_{n} J_{n}, \quad (S.50)$$

where the final inequality holds for some $C_2 < \infty$ by result (S.26) and $N_{[]}(u, \mathcal{G}_n, \|\cdot\|_{P,2})$ being decreasing in u. Furthermore, since $E_P[|\mathbb{G}_P(g_0)|] \le \|g_0\|_{P,2} \le \|G_n\|_{P,2}$ we have

$$E_P[|\mathbb{G}_P(g_0)|] \le ||G_n||_{P,2} \le \int_0^{||G_n||_{P,2}} \sqrt{1 + \log N_{[]}(u, \mathcal{G}_n, ||\cdot||_{P,2})} du. \tag{S.51}$$

Thus, the claim of the lemma follows from (S.49), (S.50), and (S.51).

Lemma S.2.5. Let Assumption 3.2(ii) hold. It then follows that there exists a constant

 $C<\infty$ such that for all $P\in \mathbf{P}$, $n\geq 1$, $1\leq j\leq \mathcal{J}$, and functions $f\in L_P^2$ we have

$$\sum_{k=1}^{k_{n,j}} \langle f, q_{k,j} \rangle_{L_P^2}^2 \le C E_P[(E_P[f(V)|Z_j])^2]. \tag{S.52}$$

PROOF: Let $L_P^2(Z_j)$ denote the subspace of L_P^2 consisting of functions depending on Z_j only, and set $\ell^2(\mathbb{N}) \equiv \{\{c_k\}_{k=1}^{\infty} : c_k \in \mathbf{R} \text{ and } \|\{c_k\}\|_{\ell^2(\mathbb{N})} < \infty\}$, where $\|\{c_k\}\|_{\ell^2(\mathbb{N})}^2 \equiv \sum_k c_k^2$. For any sequence $\{c_k\} \in \ell^2(\mathbb{N})$, then define the map $J_{j,n} : \ell^2(\mathbb{N}) \to L_P^2(Z_j)$ by

$$J_{j,n}(\{c_k\}) = \sum_{k=1}^{k_{n,j}} c_k q_{k,j}.$$

Clearly, the maps $J_{j,n}: \ell^2(\mathbb{N}) \to L_P^2(Z_j)$ are linear and, moreover, by Assumption 3.2(ii) there is a $C < \infty$ such that the largest eigenvalue of $E_P[q_j^{k_{n,j}}(Z_j)q_j^{k_{n,j}}(Z_j)']$ is bounded by C for all $n \geq 1$ and $P \in \mathbf{P}$. Therefore, we can conclude that

$$\sup_{P \in \mathbf{P}} \sup_{n \ge 1} \|J_{j,n}\|_{o}^{2} = \sup_{P \in \mathbf{P}} \sup_{n \ge 1} \sup_{\{c_{k}\}: \sum_{k} c_{k}^{2} = 1} \|J_{j,n}(\{c_{k}\})\|_{P,2}^{2}$$

$$= \sup_{P \in \mathbf{P}} \sup_{n \ge 1} \sup_{\{c_{k}\}: \sum_{k} c_{k}^{2} = 1} E_{P}[(\sum_{k=1}^{k_{n,j}} c_{k} q_{k,j}(Z_{j}))^{2}] \le \sup_{\{c_{k}\}: \sum_{k} c_{k}^{2} = 1} C \sum_{k=1}^{\infty} c_{k}^{2} = C \quad (S.53)$$

which implies $J_{j,n}$ is continuous. Next, define $J_{j,n}^*:L_P^2(Z_j)\to\ell^2(\mathbb{N})$ to be given by

$$J_{j,n}^*(g) = \{a_k(g)\}_{k=1}^{\infty} \qquad a_k(g) \equiv \begin{cases} \langle g, q_{k,j} \rangle_{L_P^2} & \text{if } k \le k_{n,j} \\ 0 & \text{if } k > k_{n,j} \end{cases},$$

and note $J_{j,n}^*$ is the adjoint of $J_{j,n}$. Therefore, since $||J_{j,n}||_o = ||J_{j,n}^*||_o$ by Theorem 6.5.1 in Luenberger (1969), we obtain for any $P \in \mathbf{P}$, $n \ge 1$, and $g \in L_p^2(Z_j)$ that

$$\sum_{k=1}^{k_{n,j}} \langle g, q_{k,j} \rangle_{L_P^2}^2 = \|J_{j,n}^*(g)\|_{\ell^2(\mathbb{N})}^2 \le \|J_{j,n}^*\|_o^2 \|g\|_{P,2}^2 = \|J_{j,n}\|_o^2 \|g\|_{P,2}^2. \tag{S.54}$$

Therefore, since $E_P[f(V)q_{k,j}(Z_j)] = E_P[E_P[f(V)|Z_j]q_{k,j}(Z_j)]$ for any $f \in L_P^2$, setting $g(Z_j) = E_P[f(V)|Z_j]$ in (S.54) and employing (S.53) yields the lemma.

Lemma S.2.6. If Λ is a set, $A: \Lambda \to \mathbf{R}^k$, $B: \Lambda \to \mathbf{R}^k$, and W is a $k \times k$ matrix, then

$$|\inf_{\lambda \in \Lambda} \|WA(\lambda)\|_p - \inf_{\lambda \in \Lambda} \|WB(\lambda)\|_p| \le \|W\|_{o,p} \times \sup_{\lambda \in \Lambda} \|A(\lambda) - B(\lambda)\|_p.$$

PROOF: Fix $\eta > 0$, and let $\lambda_a \in \Lambda$ satisfy $\|WA(\lambda_a)\|_p \leq \inf_{\lambda \in \Lambda} \|WA(\lambda)\|_p + \eta$. Then,

$$\inf_{\lambda \in \Lambda} \|WB(\lambda)\|_{p} - \inf_{\lambda \in \Lambda} \|WA(\lambda)\|_{p} \le \|WB(\lambda_{a})\|_{p} - \|WA(\lambda_{a})\|_{p} + \eta$$

$$\le \|W(B(\lambda_{a}) - A(\lambda_{a}))\|_{p} + \eta \le \|W\|_{o,p} \times \sup_{\lambda \in \Lambda} \|A(\lambda) - B(\lambda)\|_{p} + \eta, \quad (S.55)$$

where the second result follows from the triangle inequality, and the final result from $||Wv||_p \le ||W||_{o,p} ||v||_p$ for any $v \in \mathbf{R}^k$. By identical manipulations we also have

$$\inf_{\lambda \in \Lambda} \|WA(\lambda)\|_p - \inf_{\lambda \in \Lambda} \|WB(\lambda)\|_p \le \|W\|_{o,p} \times \sup_{\lambda \in \Lambda} \|A(\lambda) - B(\lambda)\|_p + \eta. \tag{S.56}$$

Thus, since η was arbitrary, the lemma follows from results (S.55) and (S.56).

S.3 Bootstrap Approximation

This appendix contains the proof of all results concerning the bootstrap approximation. We first introduce two assumptions that generalize Assumption 3.13 (at the cost of introducing additional notation) and deliver a stronger version of Theorem 3.2.

Assumption S.3.1. There is an $\epsilon > 0$ and scalars $\mathcal{D}_n(\mathbf{L}, \mathbf{E})$ and $\mathcal{D}_n(\mathbf{B}, \mathbf{E})$ such that for any $P \in \mathbf{P}$, $\theta \in \Theta_{0n}^r$, and $\theta_1 \in \Theta_n \cap R$ satisfying $\|\theta_1 - \theta\|_{\mathbf{E}} \leq \epsilon$, there exists $\tilde{\theta} \in \Theta_{0n}^r$ such that $\|\theta - \tilde{\theta}\|_{\mathbf{E}} = 0$, $\|\tilde{\theta} - \theta_1\|_{\mathbf{L}} \leq \mathcal{D}_n(\mathbf{L}, \mathbf{E})\|\tilde{\theta} - \theta_1\|_{\mathbf{E}}$, and $\|\tilde{\theta} - \theta_1\|_{\mathbf{B}} \leq \mathcal{D}_n(\mathbf{B}, \mathbf{E})\|\tilde{\theta} - \theta_1\|_{\mathbf{E}}$.

Assumption S.3.2. (i) Either Υ_F and Υ_G are affine or $(\mathcal{R}_n + \nu_n \tau_n) \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(1)$; (ii) $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_{\rho}} \vee (\nu_n \tau_n)^{\kappa_{\rho}}, \mathcal{F}_n, ||\cdot||_{P,2}) = o(a_n), K_m \ell_n^2 \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-\frac{1}{2}}), K_m \ell_n (\mathcal{R}_n + \nu_n \tau_n) \mathcal{D}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-\frac{1}{2}}), \ell_n (\ell_n + \{\mathcal{R}_n + \nu_n \tau_n\} \mathcal{D}_n(\mathbf{B}, \mathbf{E})) 1\{K_f > 0\} = o(a_n n^{-\frac{1}{2}})$; (iii) $\limsup 1\{K_g > 0\}\ell_n/r_n < 1/2$ and $(\mathcal{R}_n + \nu_n \tau_n) \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(r_n)$.

In particular, note Assumption S.3.1 holds with $\mathcal{D}_n(\mathbf{L}, \mathbf{E}) = \mathcal{S}_n(\mathbf{L}, \mathbf{E})$, $\mathcal{D}_n(\mathbf{B}, \mathbf{E}) = \mathcal{S}_n(\mathbf{B}, \mathbf{E})$, and $\tilde{\theta} = \theta$. Hence, Assumption 3.13 implies Assumptions S.3.1 and S.3.2. In general, however, $\mathcal{D}_n(\mathbf{L}, \mathbf{E})$ and $\mathcal{D}_n(\mathbf{B}, \mathbf{E})$ can be smaller than $\mathcal{S}_n(\mathbf{L}, \mathbf{E})$ and $\mathcal{S}_n(\mathbf{B}, \mathbf{E})$ while the introduction of a $\tilde{\theta} \neq \theta$ eases requirements in partially identified models.

Our next theorem consists of two parts. The first part, which replaces Assumption 3.13 with S.3.1 and S.3.2, can by the preceding discussion be seen as a generalization of Theorem 3.2. The second part shows that, under additional restrictions, it is possible to replace the norm $\|\cdot\|_{\mathbf{B}}$ in the definition of $\hat{V}_n(\theta, R|\ell)$ (as in (21)) with the norm $\|\cdot\|_{\mathbf{E}}$ – an observation that is sometimes helpful in easing rate restrictions.

Theorem S.3.1. Let Assumptions 3.1, 3.2, 3.3, 3.4(i), 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12(i)(iii), S.3.1, and S.3.2 hold. Then, the following statements hold:

(i) If Assumption 3.12(ii) holds, then there is a $\tilde{\ell}_n \simeq \ell_n$ such that uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n).$$

(ii) In addition, suppose for some $\epsilon > 0$ and $\|\cdot\|_{\mathbf{I}}$ satisfying $\|h\|_{\mathbf{E}} \leq \|h\|_{\mathbf{I}}$ for all $h \in \mathbf{B}_n$, we have that for all $P \in \mathbf{P}_0$, $\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \|\cdot\|_{\mathbf{I}}) \leq \epsilon\} \subseteq \Theta_n$ and $P(\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\theta, \hat{\Theta}_n^r, \|\cdot\|_{\mathbf{I}}) \leq \epsilon\} \subseteq \Theta_n)$ tends to one uniformly in $P \in \mathbf{P}_0$. If Υ_F and Υ_G are affine, then part (i) holds with $\hat{U}_n(R|\ell_n)$ as in (17) but with

$$\hat{V}_n(\theta, R|\ell) \equiv \{ h \in \mathbf{B}_n : h \in G_n(\theta), \ \Upsilon_F(\theta + \frac{h}{\sqrt{n}}) = 0, \ and \ \|\frac{h}{\sqrt{n}}\|_{\mathbf{I}} \le \ell \}. \quad (S.57)$$

PROOF: First note Assumptions 3.6(i) and S.3.2(ii) imply $\mathcal{R}_n \vee \nu_n \tau_n = o(1)$. Hence, by Assumption S.3.2(ii) we may apply Lemma S.3.2 to obtain uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) = \inf_{\theta \in \hat{\Theta}_n^r} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p} + o_P(a_n). \tag{S.58}$$

Thus, we may select $\hat{\theta}_n \in \hat{\Theta}_n^r$ and $\hat{h}_n \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)$ so that uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) = \|\mathbb{W}_P^*(\hat{\theta}_n) + \mathbb{D}_P(\hat{\theta}_n)[\hat{h}_n]\|_{\Sigma_{P,P}} + o_P(a_n). \tag{S.59}$$

Next note that by Assumptions 3.6(i), S.3.1, and S.3.2 there is a δ_n so that $\delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(r_n)$, $\delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(1)$ if either Υ_F or Υ_G are not affine, $\mathcal{R}_n + \nu_n \tau_n = o(\delta_n)$, and

$$\ell_n \delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}) 1\{K_f > 0\} = o(a_n n^{-\frac{1}{2}})$$
(S.60)

$$K_m \delta_n \ell_n \mathcal{D}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-\frac{1}{2}}) \tag{S.61}$$

$$k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\delta_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n).$$
 (S.62)

Next, notice that Corollary S.1.2(i) implies that there exists a $\theta_{0n} \in \Theta_{0n}^{r}$ such that

$$\|\hat{\theta}_n - \theta_{0n}\|_{\mathbf{E}} = o_P(\delta_n) \tag{S.63}$$

uniformly in $P \in \mathbf{P}_0$ due to $(\mathcal{R}_n + \nu_n \tau_n) = o(\delta_n)$. Furthermore, by Assumption S.3.1 we can assume without loss of generality that θ_{0n} in addition satisfies

$$\|\hat{\theta}_n - \theta_{0n}\|_{\mathbf{L}} = o_P(\mathcal{D}_n(\mathbf{L}, \mathbf{E})\delta_n) \qquad \|\hat{\theta}_n - \theta_{0n}\|_{\mathbf{B}} = o_P(\mathcal{D}_n(\mathbf{B}, \mathbf{E})\delta_n)$$
 (S.64)

uniformly in $P \in \mathbf{P}_0$. In addition note that since $||q_{k,j}||_{\infty} \leq B_n$ for all $1 \leq j \leq \mathcal{J}$ and $1 \leq k \leq k_{n,j}$ by Assumption 3.2(i), we obtain from Assumption 3.3(ii) together with result (S.63) that with probability tending to one uniformly in $P \in \mathbf{P}_0$ we have

$$E_{P}[\|\rho(X,\hat{\theta}_{n}) - \rho(X,\theta_{0n})\|_{2}^{2}q_{k,j}^{2}(Z_{j})] \le B_{n}^{2}K_{\rho}^{2}\delta_{n}^{2\kappa_{\rho}}.$$
(S.65)

Set $\mathcal{G}_n \equiv \{fq_{k,j} : f \in \mathcal{F}_n, 1 \leq j \leq \mathcal{J}, 1 \leq k \leq k_{n,j}\}$ and let \mathbb{G}_P be a Gaussian process on \mathcal{G}_n satisfying $E[\mathbb{G}_P(g_1)\mathbb{G}_P(g_2)] = \operatorname{Cov}_P\{g_1(V), g_2(V)\}$ and $E[\mathbb{G}_P(g_1)] = 0$ for any $g_1, g_2 \in \mathcal{G}_n$. Since (S.65) holds with probability tending to one uniformly in $P \in \mathbf{P}_0$,

Assumption 3.7(ii), result (S.45), and δ_n satisfying (S.62) imply for any $\epsilon > 0$ that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\| \mathbb{W}_{P}^{\star}(\hat{\theta}_{n}) - \mathbb{W}_{P}^{\star}(\theta_{0n}) \|_{\Sigma_{P}, p} > a_{n} \epsilon)$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} \frac{1}{a_{n} \epsilon} k_{n}^{1/p} E_{P}[\sup_{g_{1}, g_{2} \in \mathcal{G}_{n}: \|g_{1} - g_{2}\|_{P, 2} \leq B_{n} K_{\rho} \delta_{n}^{\kappa_{\rho}}} |\mathbb{G}_{P}(g_{1}) - \mathbb{G}_{P}(g_{2})|] = 0. \quad (S.66)$$

Similarly, result (S.63) implies $\overrightarrow{d}_H(\hat{\theta}_n, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \leq \epsilon$ with probability tending to one uniformly in $P \in \mathbf{P}_0$ for any $\epsilon > 0$. Hence, Lemma S.3.4 yields uniformly in $P \in \mathbf{P}_0$

$$\|\mathbb{D}_{P}(\theta_{0n})[\hat{h}_{n}] - \mathbb{D}_{P}(\hat{\theta}_{n})[\hat{h}_{n}]\|_{\Sigma_{P},p} \lesssim \|\Sigma_{P}\|_{o,p} \times K_{m}\|\hat{\theta}_{n} - \theta_{0n}\|_{\mathbf{L}}\|\hat{h}_{n}\|_{\mathbf{E}} + o_{P}(a_{n})$$
$$\lesssim \|\Sigma_{P}\|_{o,p} \times K_{m}\mathcal{D}_{n}(\mathbf{L}, \mathbf{E})\delta_{n}\ell_{n}\sqrt{n} + o_{P}(a_{n}) = o_{P}(a_{n}), \quad (S.67)$$

where the second inequality follows from $\|\hat{h}_n/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ due to $\hat{h}_n/\sqrt{n} \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)$, Assumption 3.12(i), and (S.64). In turn, the final result in (S.67) follows from (S.61) and Assumption 3.7(ii). Next, note the conditions of Theorem S.5.1(i) hold because: Either Υ_F and Υ_G are affine (implying $K_f = K_g = 0$) or $\delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(1)$, and $\delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(r_n)$ and $\limsup \ell_n/r_n 1\{K_g > 0\} < 1/2$ by Assumption S.3.2(iii) imply

$$r_n \ge 2(\ell_n + \delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E})) 1\{K_q > 0\}$$

for n sufficiently large. Hence, Theorem S.5.1(i), Assumption 3.12(ii), and $||h||_{\mathbf{E}} \lesssim ||h||_{\mathbf{B}}$ for all $h \in \mathbf{B}_n$ by Assumption 3.12(i), imply that there is a constant $M < \infty$ for which with probability tending to one uniformly in $P \in \mathbf{P}_0$ we have that

$$\inf_{h \in V_n(\theta_{0n}, R|M\ell_n)} \|\frac{\hat{h}_n}{\sqrt{n}} - \frac{h}{\sqrt{n}}\|_{\mathbf{B}} \le M\ell_n(\ell_n + \delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}))1\{K_f > 0\}.$$

It follows from Assumption S.3.2(ii) and (S.60) that there is a $h_{0n} \in V_n(\theta_{0n}, R|M\ell_n)$ such that $||h_{0n} - \hat{h}_n||_{\mathbf{B}} = o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$, and hence Assumption 3.7(ii), Lemma S.3.4, and $||h||_{\mathbf{E}} \lesssim ||h||_{\mathbf{B}}$ by Assumption 3.12(i) yield

$$\|\mathbb{D}_{P}(\theta_{0n})[\hat{h}_{n}] - \mathbb{D}_{P}(\theta_{0n})[h_{0n}]\|_{\Sigma_{P,p}} \lesssim \|\Sigma_{P}\|_{o,p} \times \|\hat{h} - h_{0n}\|_{\mathbf{E}} = o_{P}(a_{n})$$
 (S.68)

uniformly in $P \in \mathbf{P}_0$. Therefore, combining results (S.59), (S.66), (S.67), and (S.68) together with $\theta_{0n} \in \Theta_{0n}^{r}$ and $h_{0n} \in V_n(\theta_{0n}, R|M\ell_n)$ yields

$$\hat{U}_n(R|\ell_n) = \|\mathbb{W}_P^{\star}(\theta_{0n}) + \mathbb{D}_P(\theta_{0n})[h_{0n}]\|_{\Sigma_P,p} + o_P(a_n)
\geq \inf_{\theta \in \Theta_{n,n}^{\star}} \inf_{h \in V_n(\theta,R|M\ell_n)} \|\mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h]\|_{\Sigma_P,p} + o_P(a_n)$$

uniformly in $P \in \mathbf{P}_0$. The first part of theorem then follows by setting $\tilde{\ell}_n = M\ell_n$.

In order to establish the second part of the theorem, note that the only assumptions

that potentially require the norm $\|\cdot\|_{\mathbf{B}}$ to be stronger than $\|\cdot\|_{\mathbf{I}}$ are Assumptions 3.8, 3.9, 3.10 (pertaining to the differentiability of Υ_F and Υ_G) and Assumption 3.12(ii) (since a stronger norm $\|\cdot\|_{\mathbf{B}}$ makes $(\hat{\Theta}_n^{\mathbf{r}})^{\epsilon}$ smaller). We therefore establish part (ii) of the theorem by repeating the arguments employed in showing part (i) while carefully re-examining the role played by the norm $\|\cdot\|_{\mathbf{B}}$. To this end, note that since

$$\liminf_{n \to \infty} \inf_{P \in \mathbf{P}_0} P(\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\theta, \hat{\Theta}_n^r, \| \cdot \|_{\mathbf{I}}) \le \epsilon\} \subseteq \Theta_n) = 1, \tag{S.69}$$

we may apply Lemma S.3.2 with $\|\cdot\|_{\mathbf{B}}$ set to equal $\|\cdot\|_{\mathbf{I}}$ to still obtain that

$$\hat{U}_n(R|\ell_n) = \inf_{\theta \in \hat{\Theta}_n^r} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p} + o_P(a_n). \tag{S.70}$$

Letting $\hat{\theta}_n$ and \hat{h}_n be defined as in (S.59) (but with $\hat{V}_n(\theta, R|\ell)$ as defined in (S.57)), then observe that since results (S.66) and (S.67) do not rely on Assumptions 3.8, 3.9, 3.10 or 3.12(ii), we can conclude from result (S.70) that uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) = \|\mathbb{W}_P^*(\theta_{0n}) + \mathbb{D}_P(\theta_{0n})[\hat{h}_n]\|_{\Sigma_{P,p}} + o_P(a_n)$$
(S.71)

for some $\theta_{0n} \in \Theta_{0n}^{r}$. Next, note $\delta_n \mathcal{D}_n(\mathbf{B}, \mathbf{E}) = o(r_n)$ and $K_f = K_g = 0$ due to Υ_F and Υ_G being affine, together with Theorem S.5.1(ii) imply that

$$\hat{V}_n(\hat{\theta}_n, R|\ell_n) \equiv \{h \in \mathbf{B}_n : h \in G_n(\hat{\theta}_n), \ \Upsilon_F(\hat{\theta}_n + \frac{h}{\sqrt{n}}) = 0, \ \|\frac{h}{\sqrt{n}}\|_{\mathbf{I}} \le \ell_n\}
\subseteq \{h \in \mathbf{B}_n : \Upsilon_G(\theta_{0n} + \frac{h}{\sqrt{n}}) \le 0, \ \Upsilon_F(\theta_{0n} + \frac{h}{\sqrt{n}}) = 0, \ \|\frac{h}{\sqrt{n}}\|_{\mathbf{I}} \le \ell_n\}
\subseteq V_n(\theta_{0n}, R|\ell_n),$$

with probability tending to one uniformly in $P \in \mathbf{P}_0$, and where the final inequality follows from $\ell_n \downarrow 0$, $\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \| \cdot \|_{\mathbf{I}}) \leq \epsilon\} \subseteq \Theta_n$ and $\| \cdot \|_{\mathbf{E}} \leq \| \cdot \|_{\mathbf{I}}$ by hypothesis. Therefore, we can conclude that $\hat{h}_n \in V_n(\theta_{0n}, R|\ell_n)$ with probability tending to one uniformly in $P \in \mathbf{P}_0$, which by (S.71) yields

$$\hat{U}_n(R|\ell_n) \ge \inf_{\theta \in \Theta_{l_n}^{\star}} \inf_{h \in V_n(\theta, R|\ell_n)} \|\mathbb{W}_P^{\star}(\theta_{0n}) + \mathbb{D}_P(\theta_{0n})[h]\|_{\Sigma_P, p} + o_P(a_n),$$

and hence establishes the second claim of the theorem.

PROOF OF THEOREM 3.2: Follows from immediately from Theorem S.3.1(i) and Assumption 3.13 implying Assumptions S.3.1 and S.3.2 are satisfied by setting $\mathcal{D}_n(\mathbf{B}, \mathbf{E}) = \mathcal{S}_n(\mathbf{B}, \mathbf{E}), \, \mathcal{D}_n(\mathbf{L}, \mathbf{E}) = \mathcal{S}_n(\mathbf{L}, \mathbf{E})$ and $\theta = \tilde{\theta}$.

Proof of Corollary 3.1: We establish the corollary by appealing to Lemmas S.3.5

and S.3.6. To this end, we first note Theorem 3.2 allows us to conclude that

$$\hat{U}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n) \tag{S.72}$$

uniformly in $P \in \mathbf{P}_0$ with $\ell_n \simeq \tilde{\ell}_n$, while Assumption 3.13(ii) implies $K_m \tilde{\ell}_n^2 \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-\frac{1}{2}})$ and $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\tilde{\ell}_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$, and hence

$$I_n(R) \le U_P(R|\tilde{\ell}_n) + o_P(a_n) \tag{S.73}$$

uniformly in $P \in \mathbf{P}_0$ by Theorem 3.1(i). Moreover, applying Lemma S.3.5 with $B_n = \hat{U}_n(R|\ell_n)$, $D_n = \{V_i\}_{i=1}^n$, and $C_{P,n}^{\star} = U_P^{\star}(R|\tilde{\ell}_n)$ yields, for some $\delta_n = o(1)$, that

$$\liminf_{n \to \infty} \inf_{P \in \mathbf{P}_0} P(\hat{c}_n + \frac{a_n}{2} > q_{1-\alpha-\delta_n,P}(U_P^{\star}(R|\tilde{\ell}_n))) = 1, \tag{S.74}$$

where $q_{\tau,P}(U_P^{\star}(R|\tilde{\ell}_n))$ denotes the τ quantile of $U_P^{\star}(R|\tilde{\ell}_n)$. Since $U_P^{\star}(R|\tilde{\ell}_n) \stackrel{d}{=} U_P(R|\tilde{\ell}_n)$, results (S.73), (S.74), and Assumption 3.14 verify the conditions of Lemma S.3.6 (applied with $T_n = I_n(R)$ and $C_{P,n} = U_P(R|\tilde{\ell}_n)$) and therefore the corollary follows.

PROOF OF COROLLARY 3.2: In what follows, we use a "u" superscript for parameters associated with setting $R = \Theta - \text{e.g.}$, \mathbf{B}_n^{u} denotes the vector subspace generated by Θ_n . First note Theorem 3.1(i) (for R as in (13)) and Theorem 3.1(ii) (for $R = \Theta$) imply that for any $\ell_n, \ell_n^{\text{u}} \downarrow 0$ satisfying $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \{\sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho}, \mathcal{F}_n, \| \cdot \|_{P,2}) + \sup_{P \in \mathbf{P}} J_{[]}((\ell_n^{\text{u}})^{\kappa_\rho}, \mathcal{F}_n^{\text{u}}, \| \cdot \|_{P,2})\} = o(a_n), K_m(\ell_n^{\text{u}})^2 \times \mathcal{S}_n^{\text{u}}(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2}), K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ and $\mathcal{R}_n^{\text{u}} = o(\ell_n^{\text{u}})$ it follows uniformly in $P \in \mathbf{P}_0$ that

$$I_n(R) - I_n(\Theta) \le U_P(R|\ell_n) - U_P(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n). \tag{S.75}$$

Next note that we may apply Theorem 3.2 to obtain that uniformly in $P \in \mathbf{P}_0$ we have

$$\hat{U}_n(R|\ell_n) \ge U_P^*(R|\tilde{\ell}_n) + o_P(a_n) \tag{S.76}$$

with $\tilde{\ell}_n \simeq \ell_n$. Similarly, also note that Lemma S.3.7 implies uniformly in $P \in \mathbf{P}_0$ that

$$\hat{U}_n(\Theta|+\infty) \le U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n), \tag{S.77}$$

for $\tilde{\ell}_n^{\mathrm{u}} \downarrow 0$ satisfying Assumption 3.13(ii) and $\mathcal{R}_n^{\mathrm{u}} = o(\tilde{\ell}_n^{\mathrm{u}})$. In particular, it follows from results (S.76) and (S.77) that uniformly in $P \in \mathbf{P}_0$ we have

$$\hat{U}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty) \ge U_P^{\star}(R|\tilde{\ell}_n) - U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n)$$
(S.78)

for sequences $\tilde{\ell}_n, \tilde{\ell}_n^{\rm u} \downarrow 0$ satisfying the rate requirements needed for (S.75) to hold (i.e. with $\ell_n, \ell_n^{\rm u}$ replaced by $\tilde{\ell}_n, \tilde{\ell}_n^{\rm u}$). The corollary then follows by the same arguments as in Corollary 3.1 but employing (S.75) and (S.78) in place of (S.72) and (S.73).

Lemma S.3.1. Suppose there is a $\mathcal{A}_n(P) \subseteq \Theta_n \cap R$ such that $||h||_{\mathbf{E}} \leq \nu_n ||\mathbb{D}_P(\theta)[h]||_p$ for all $\theta \in \mathcal{A}_n(P)$ and $h \in \sqrt{n} \{\mathbf{B}_n \cap R - \theta\}$. If the estimator $\hat{\mathbb{D}}_n(\theta)$ satisfies

$$\sup_{\theta \in \mathcal{A}_n(P)} \sup_{h \in \sqrt{n} \{\mathbf{B}_n \cap R - \theta\} : \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} \ge \ell_n} \frac{\|\hat{\mathbb{D}}_n(\theta)[h] - \mathbb{D}_P(\theta)[h]\|_p}{\|h\|_{\mathbf{E}}} = o_P(\nu_n^{-1})$$
(S.79)

and $\hat{\Theta}_n^{\rm r} \subseteq \mathcal{A}_n(P)$ with probability tending to one uniformly in $P \in \mathbf{P}_0$, Assumptions 3.2(i)(iii), 3.7, 3.11 hold, and $\mathcal{S}_n(\mathbf{B}, \mathbf{E})\mathcal{R}_n = o(\ell_n)$, then uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(R|\ell_n) = \inf_{\theta \in \hat{\Theta}_n^r} \inf_{h \in \hat{V}_n(\theta, R| + \infty)} \|\hat{\mathbb{W}}_n(\theta) + \hat{\mathbb{D}}_n(\theta)[h]\|_{\hat{\Sigma}_n, p} + o_P(a_n).$$
 (S.80)

PROOF: In the following arguments, we note that the only requirement on $\hat{\mathbb{D}}_n(\theta)$ is that it satisfy condition (S.79). As a result, the lemma applies to estimators $\hat{\mathbb{D}}_n(\theta)$ besides the numerical derivative examined in the main text.

In order to establish the result, we first let $\hat{\theta}_n \in \hat{\Theta}_n^r$ and $\hat{h}_n \in \hat{V}_n(\hat{\theta}_n, R| + \infty)$ satisfy

$$\inf_{\theta \in \hat{\Theta}_n^{\mathrm{r}}} \inf_{h \in \hat{V}_n(\theta, R| + \infty)} \|\hat{\mathbb{W}}_n(\theta) + \hat{\mathbb{D}}_n(\theta)[h]\|_{\hat{\Sigma}_n, p} = \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n(\hat{\theta}_n)[\hat{h}_n]\|_{\hat{\Sigma}_n, p} + o(a_n).$$

Then note that in order to establish the claim of the lemma it suffices to show that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(\|\frac{\hat{h}_n}{\sqrt{n}}\|_{\mathbf{B}} \ge \ell_n) = 0.$$
 (S.81)

To this end, note $0 \in \hat{V}_n(\theta, R|+\infty)$ for all $\theta \in \Theta_n \cap R$, the triangle inequality, $\|\hat{\Sigma}_n\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}$ by Corollary S.1.3, and Assumption 3.11 yield

$$\|\hat{\mathbb{D}}_{n}(\hat{\theta}_{n})[\hat{h}_{n}]\|_{\hat{\Sigma}_{n},p} \leq \|\hat{\mathbb{W}}_{n}(\hat{\theta}_{n}) + \hat{\mathbb{D}}_{n}(\hat{\theta}_{n})[\hat{h}_{n}]\|_{\hat{\Sigma}_{n},p} + \|\hat{\mathbb{W}}_{n}(\hat{\theta}_{n})\|_{\hat{\Sigma}_{n},p}$$

$$\leq 2\|\hat{\Sigma}_{n}\|_{o,p}\|\mathbb{W}_{P}^{\star}(\hat{\theta}_{n})\|_{p} + o_{P}(a_{n})$$
(S.82)

uniformly in $P \in \mathbf{P}$. Hence, since $\hat{\theta}_n \in \hat{\Theta}_n^{\mathrm{r}} \subseteq \Theta_n \cap R$ almost surely, we obtain from result (S.82), $\|\hat{\Sigma}_n\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}$, and Lemma S.2.4 that

$$\|\hat{\mathbb{D}}_{n}(\hat{\theta}_{n})[\hat{h}_{n}]\|_{\hat{\Sigma}_{n},p} \leq 2\|\hat{\Sigma}_{n}\|_{o,p} \sup_{\theta \in \Theta_{n} \cap R} \|\mathbb{W}_{P}^{\star}(\theta)\|_{p} + o_{P}(a_{n}) = O_{P}(k_{n}^{1/p}\sqrt{\log(1+k_{n})}B_{n}J_{n})$$
(S.83)

uniformly in $P \in \mathbf{P}$. Since $\hat{h}_n \in \hat{V}_n(\hat{\theta}_n, R| + \infty)$ implies $\hat{h}_n \in \sqrt{n}\{\mathbf{B}_n \cap R - \hat{\theta}_n\}$ and $\hat{\theta}_n \in \hat{\Theta}_n^{\mathrm{r}} \subseteq \mathcal{A}_n(P)$ with probability tending to one uniformly in $P \in \mathbf{P}_0$, we obtain from the first hypothesis of the lemma that $\|\hat{h}_n\|_{\mathbf{E}} \leq \nu_n \|\mathbb{D}_P(\hat{\theta}_n)[\hat{h}_n]\|_p$ with probability

tending to one uniformly in $P \in \mathbf{P}_0$. Therefore, it follows that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\ell_{n} \leq \| \frac{\hat{h}_{n}}{\sqrt{n}} \|_{\mathbf{B}})$$

$$= \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\ell_{n} \leq \| \frac{\hat{h}_{n}}{\sqrt{n}} \|_{\mathbf{B}} \text{ and } \| \hat{h}_{n} \|_{\mathbf{E}} \leq \nu_{n} \| \mathbb{D}_{P}(\hat{\theta}_{n}) [\hat{h}_{n}] \|_{p})$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(\ell_{n} \leq \| \frac{\hat{h}_{n}}{\sqrt{n}} \|_{\mathbf{B}} \text{ and } \| \hat{h}_{n} \|_{\mathbf{E}} \leq 2\nu_{n} \| \hat{\mathbb{D}}_{n}(\hat{\theta}_{n}) [\hat{h}_{n}] \|_{p}), \qquad (S.84)$$

where the inequality follows from condition (S.79). Hence, results (S.83) and (S.84), the definitions of $S_n(\mathbf{B}, \mathbf{E})$ and \mathcal{R}_n , and $S_n(\mathbf{B}, \mathbf{E})\mathcal{R}_n = o(\ell_n)$ by hypothesis yield

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(\ell_n \le \|\frac{\hat{h}_n}{\sqrt{n}}\|_{\mathbf{B}})$$

$$\le \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(\ell_n \le 2\frac{\nu_n}{\sqrt{n}} \mathcal{S}_n(\mathbf{B}, \mathbf{E}) \|\hat{\mathbb{D}}_n(\hat{\theta}_n)[\hat{h}_n]\|_p) = 0, \quad (S.85)$$

which establishes (S.81) and hence the claim of the lemma.

Lemma S.3.2. Let Assumptions 3.1(i), 3.2, 3.3, 3.4(i), 3.5(i), 3.6(ii), 3.7, 3.11, 3.12 hold and $\mathcal{R}_n \vee \nu_n \tau_n = o(1)$. If $\ell_n \downarrow 0$ satisfies $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_p}, \mathcal{F}_n, \| \cdot \|_{P,2}) = o(a_n)$ and $K_m \ell_n^2 \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-\frac{1}{2}})$, then uniformly in $P \in \mathbf{P}_0$ we have

$$\hat{U}_n(R|\ell_n) = \inf_{\theta \in \hat{\Theta}_n^r} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p} + o_P(a_n).$$

PROOF: First note that Corollary S.1.2(i) implies $\overrightarrow{d}_H(\hat{\Theta}_n^{\mathrm{r}}, \Theta_{0n}^{\mathrm{r}}, \|\cdot\|_{\mathbf{E}}) = O_P(\mathcal{R}_n + \nu_n \tau_n)$ uniformly in $P \in \mathbf{P}_0$. Hence, since $\mathcal{R}_n \vee \nu_n \tau_n = o(1)$, for any $\epsilon > 0$ it follows that

$$\liminf_{n \to \infty} \inf_{P \in \mathbf{P}_0} P(\hat{\Theta}_n^{\mathbf{r}} \subseteq \{ \theta \in \Theta_n \cap R : \overrightarrow{d}_H(\theta, \Theta_{0n}^{\mathbf{r}}, \| \cdot \|_{\mathbf{E}}) \le \epsilon \}) = 1.$$
 (S.86)

Furthermore, for any $\theta \in \hat{\Theta}_n^r$ and $h \in \hat{V}_n(\theta, R|\ell_n)$ note that $\Upsilon_G(\theta + h/\sqrt{n}) \leq 0$ and $\Upsilon_F(\theta + h/\sqrt{n}) = 0$ by definition of $\hat{V}_n(\theta, R|\ell_n)$. Thus, $\theta + h/\sqrt{n} \in R$ for any $\theta \in \hat{\Theta}_n^r$ and $h \in \hat{V}_n(\theta, R|\ell_n)$, and hence Assumption 3.12(ii) allows us to conclude

$$\lim_{n \to \infty} \inf_{P \in \mathbf{P}_0} P(\theta + \frac{h}{\sqrt{n}} \in \Theta_n \cap R \text{ for all } \theta \in \hat{\Theta}_n^{\mathrm{r}} \text{ and } h \in \hat{V}_n(\theta, R|\ell_n))$$

$$= \lim_{n \to \infty} \inf_{P \in \mathbf{P}_0} P(\theta + \frac{h}{\sqrt{n}} \in \Theta_n \text{ for all } \theta \in \hat{\Theta}_n^{\mathrm{r}} \text{ and } h \in \hat{V}_n(\theta, R|\ell_n)) = 1 \quad (S.87)$$

due to $||h/\sqrt{n}||_{\mathbf{B}} \leq \ell_n \downarrow 0$ for any $h \in \hat{V}_n(\theta, R|\ell_n)$. In particular, note that result (S.87) and Assumption 3.12(i) imply that for some $M < \infty$ we have $\hat{V}_n(\theta, R|\ell_n) \subseteq V_n(\theta, R|\ell_n/M)$ for all $\theta \in \hat{\Theta}_n^{\mathbf{r}}$ with probability tending to one uniformly in $P \in \mathbf{P}_0$.

Thus, (S.86) and Lemma S.3.3 allow us to conclude that uniformly in $P \in \mathbf{P}_0$ we have

$$\sup_{\theta \in \hat{\Theta}_n^r} \sup_{h \in \hat{V}_n(\theta, R|\ell_n)} \|\hat{\mathbb{D}}_n(\theta)[h] - \mathbb{D}_P(\theta)[h]\|_p = o_P(a_n). \tag{S.88}$$

Moreover, since $\hat{\Theta}_n^{\rm r} \subseteq \Theta_n \cap R$ almost surely, we also have from Assumption 3.11 that

$$\sup_{\theta \in \hat{\Theta}_{p}^{\star}} \|\hat{\mathbb{W}}_{n}(\theta) - \mathbb{W}_{P}^{\star}(\theta)\|_{p} = o_{P}(a_{n})$$
(S.89)

uniformly in $P \in \mathbf{P}$. Therefore, since $\|\hat{\Sigma}_n\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}$ by Corollary S.1.3, we obtain from results (S.88) and (S.89) and Lemma S.2.6 that

$$\hat{U}_n(R|\ell_n) = \inf_{\theta \in \hat{\Theta}_n^r} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\hat{\Sigma}_n, p} + o_P(a_n)$$
 (S.90)

uniformly in $P \in \mathbf{P}_0$. Next, note that by Assumption 3.7(ii) there exists a constant $C_0 < \infty$ such that $\|\Sigma_P^{-1}\|_{o,p} \le C_0$ for all $P \in \mathbf{P}$. Thus, using that $\|\hat{\Sigma}_n a\|_p \le \|\hat{\Sigma}_n \Sigma_P^{-1}\|_{o,p} \|\Sigma_P a\|_p$ for any $a \in \mathbf{R}^{k_n}$ and the triangle inequality we obtain

$$\|\mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h]\|_{\hat{\Sigma}_{n},p} \le \{C_{0}\|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p} + 1\}\|\mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h]\|_{\Sigma_{P},p}$$
 (S.91)

for any $\theta \in \Theta_n \cap R$, $h \in \mathbf{B}_n$, and $P \in \mathbf{P}$. In particular, since $0 \in \hat{V}_n(\theta, R | \ell_n)$ for any $\theta \in \Theta_n \cap R$, Assumption 3.7, Markov's inequality, and Lemma S.2.4 yield

$$\|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p} \times \inf_{\theta \in \hat{\Theta}_{n}^{r}} \inf_{h \in \hat{V}_{n}(\theta, R|\ell_{n})} \|\mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h]\|_{\Sigma_{P},p}$$

$$\leq \|\hat{\Sigma}_{n} - \Sigma_{P}\|_{o,p} \times \sup_{\theta \in \Theta_{n} \cap R} \|\mathbb{W}_{P}^{\star}(\theta)\|_{\Sigma_{P},p} = o_{P}(a_{n}) \quad (S.92)$$

uniformly in $P \in \mathbf{P}$. It then follows from (S.91) and (S.92) that uniformly in $P \in \mathbf{P}$

$$\inf_{\theta \in \hat{\Theta}_{n}^{r}} \inf_{h \in \hat{V}_{n}(\theta, R|\ell_{n})} \| \mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h] \|_{\hat{\Sigma}_{n}, p} \\
\leq \inf_{\theta \in \hat{\Theta}_{n}^{r}} \inf_{h \in \hat{V}_{n}(\theta, R|\ell_{n})} \| \mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h] \|_{\Sigma_{P}, p} + o_{P}(a_{n}). \quad (S.93)$$

The reverse inequality to (S.93) can be obtained by identical arguments but employing $\max\{\|\hat{\Sigma}_n\|_{o,p}, \|\hat{\Sigma}_n^{-1}\|_{o,p}\} = O_P(1)$ uniformly in $P \in \mathbf{P}$ by Corollary S.1.3 instead of $\|\Sigma_P\|_{o,p} \vee \|\Sigma_P^{-1}\|_{o,p}$ being bouded uniformly in $P \in \mathbf{P}$. The claim of the Lemma then follows from (S.90) and (S.93) (and its reverse inequality).

Lemma S.3.3. Let Assumptions 3.2(i)(ii), 3.3, and 3.5(i) hold, and define the sets

$$V_n(\theta, R|\ell_n) \equiv \{h \in \mathbf{B}_n : \theta + \frac{h}{\sqrt{n}} \in \Theta_n \cap R \text{ and } \|\frac{h}{\sqrt{n}}\|_{\mathbf{E}} \le \ell_n\}.$$
 (S.94)

If $\ell_n \downarrow 0$ satisfies $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ and $K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-\frac{1}{2}})$, then there is an $\epsilon > 0$ such that uniformly in $P \in \mathbf{P}_0$:

$$\sup_{\theta \in \Theta_n \cap R: \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \le \epsilon} \sup_{h \in V_n(\theta, R|\ell_n)} \|\hat{\mathbb{D}}_n(\theta)[h] - \mathbb{D}_P(\theta)[h]\|_p = o_P(a_n).$$
 (S.95)

PROOF: By definition of $V_n(\theta, R|\ell_n)$, we have $\theta + h/\sqrt{n} \in \Theta_n \cap R$ for any $\theta \in \Theta_n \cap R$, $h \in V_n(\theta, R|\ell_n)$. Therefore, since $\|h/\sqrt{n}\|_{\mathbf{E}} \leq \ell_n$ for all $h \in V_n(\theta, R|\ell_n)$ we obtain

$$\sup_{\theta \in \Theta_{n} \cap R} \sup_{h \in V_{n}(\theta, R \mid \ell_{n})} \|\hat{\mathbb{D}}_{n}(\theta)[h] - \sqrt{n} E_{P}[(\rho(X, \theta + \frac{h}{\sqrt{n}}) - \rho(X, \theta)) * q^{k_{n}}(Z)]\|_{p}$$

$$\leq \sup_{\theta_{1}, \theta_{2} \in \Theta_{n} \cap R: \|\theta_{1} - \theta_{2}\|_{\mathbf{E}} \leq \ell_{n}} \|\mathbb{G}_{n}(\theta_{1}) - \mathbb{G}_{n}(\theta_{2})\|_{p}$$

$$\leq \sup_{\theta_{1}, \theta_{2} \in \Theta_{n} \cap R: \|\theta_{1} - \theta_{2}\|_{\mathbf{E}} \leq \ell_{n}} \|\mathbb{W}_{P}(\theta_{1}) - \mathbb{W}_{P}(\theta_{2})\|_{p} + o_{P}(a_{n}) \quad (S.96)$$

uniformly in $P \in \mathbf{P}$ by Assumption 3.3(i). Next note Assumptions 3.2(i) and 3.3(ii) imply that for any $1 \leq j \leq \mathcal{J}$ and $1 \leq k \leq k_{n,j}$ we must have

$$\sup_{P \in \mathbf{P}} \sup_{\theta_1, \theta_2 \in \Theta_n \cap R: \|\theta_1 - \theta_2\|_{\mathbf{E}} \le \ell_n} E_P[\|\rho(X, \theta_1) - \rho(X, \theta_2)\|_2^2 q_{k, j}^2(Z_j)] \le B_n^2 K_\rho^2 \ell_n^{2\kappa_\rho}.$$
 (S.97)

Define $\mathcal{G}_n \equiv \{fq_{k,j} : f \in \mathcal{F}_n, 1 \leq j \leq \mathcal{J} \text{ and } 1 \leq k \leq k_{n,j}\}$ and let \mathbb{G}_P be a Gaussian process on \mathcal{G}_n satisfying $E[\mathbb{G}_P(g_1)\mathbb{G}_P(g_2)] = \operatorname{Cov}_P\{g_1(V), g_2(V)\}$ and $E[\mathbb{G}_P(g_1)] = 0$ for any $g_1, g_2 \in \mathcal{G}_n$. By result (S.97) and $\|a\|_p \leq k_n^{1/p} \|a\|_{\infty}$ for any $a \in \mathbf{R}^{k_n}$ we obtain

$$E\left[\sup_{\theta_{1},\theta_{2}\in\Theta_{n}\cap R:\|\theta_{1}-\theta_{2}\|_{\mathbf{E}}\leq\ell_{n}}\|\mathbb{W}_{P}(\theta_{1})-\mathbb{W}_{P}(\theta_{2})\|_{p}\right]$$

$$\leq k_{n}^{1/p}\times E\left[\sup_{g_{1},g_{2}\in\mathcal{G}_{n}:\|g_{1}-g_{2}\|_{P,2}\leq B_{n}K_{\rho}\ell_{n}^{\kappa_{\rho}}}|\mathbb{G}_{P}(g_{1})-\mathbb{G}_{P}(g_{2})|\right]. \quad (S.98)$$

Therefore, the calculations in (S.45), Markov's inequality, and $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ by hypothesis, yield that

$$\sup_{\theta \in \Theta_n \cap R} \sup_{h \in V_n(\theta, R | \ell_n)} \|\hat{\mathbb{D}}_n(\theta)[h] - \sqrt{n} E_P[(\rho(X, \theta + \frac{h}{\sqrt{n}}) - \rho(X, \theta)) * q^{k_n}(Z)]\|_p = o_P(a_n)$$
(S.99)

uniformly in $P \in \mathbf{P}$. Next, let $\epsilon > 0$ be sufficiently small for Assumption 3.5(i) to hold and define the neighborhood $\mathcal{N}_n \equiv \{\theta \in \Theta_n \cap R : \overrightarrow{d}_H(\{\theta\}, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \leq \epsilon\}$. We can

then conclude from Lemmas S.1.5 and S.2.5, and Assumption 3.5(i) that

$$\sup_{\theta \in \mathcal{N}_n} \sup_{h \in V_n(\theta, R|\ell_n)} \|\sqrt{n} E_P[(\rho(X, \theta + \frac{h}{\sqrt{n}}) - \rho(X, \theta)) * q^{k_n}(Z)] - \mathbb{D}_P(\theta)[h]\|_p$$

$$\lesssim \sup_{\theta \in \mathcal{N}_n} \sup_{h \in V_n(\theta, R|\ell_n)} \{K_m \times \sqrt{n} \|\frac{h}{\sqrt{n}} \|_{\mathbf{E}} \|\frac{h}{\sqrt{n}} \|_{\mathbf{L}}\} = o(a_n), \quad (S.100)$$

where the final equality follows from $K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ by hypothesis. Hence, the Lemma follows from results (S.99) and (S.100).

Lemma S.3.4. Let Assumptions 3.2(ii) and 3.5(ii)(iii) hold. Then there are constants $\epsilon > 0$ and $C < \infty$ such that for all $n, P \in \mathbf{P}, \theta_0 \in \Theta_{0n}^r, \theta_1 \in \Theta_n \cap R$ satisfying $\overrightarrow{d}_H(\theta_1, \Theta_{0n}^r, \|\cdot\|_{\mathbf{E}}) \leq \epsilon$, and $h_0, h_1 \in \mathbf{B}_n$ it follows that

$$\|\mathbb{D}_{P}(\theta_{0})[h_{0}] - \mathbb{D}_{P}(\theta_{1})[h_{1}]\|_{p} \le C\{\|h_{0} - h_{1}\|_{\mathbf{E}} + K_{m}\|\theta_{0} - \theta_{1}\|_{\mathbf{L}}\|h_{1}\|_{\mathbf{E}}\}.$$

PROOF: We first fix $\epsilon > 0$ such that Assumptions 3.5(ii)(iii) are satisfied. Then note that by Lemmas S.1.5 and S.2.5 it follows that there is a constant $C_0 < \infty$ with

$$\|\mathbb{D}_{P}(\theta_{0})[h_{0}] - \mathbb{D}_{P}(\theta_{1})[h_{1}]\|_{p} \leq \{\sum_{j=1}^{\mathcal{J}} C_{0} \|\nabla m_{P,j}(\theta_{0})[h_{0}] - \nabla m_{P,j}(\theta_{1})[h_{1}]\|_{P,2}^{2}\}^{1/2}.$$

Moreover, since $(h_0 - h_1) \in \mathbf{B}_n$, we can also conclude from Assumptions 3.5(ii)(iii) that

$$\begin{split} \|\nabla m_{P,j}(\theta_0)[h_0] - \nabla m_{P,j}(\theta_1)[h_1]\|_{P,2} \\ &\leq \|\nabla m_{P,j}(\theta_0)[h_0 - h_1]\|_{P,2} + \|\nabla m_{P,j}(\theta_0)[h_1] - \nabla m_{P,j}(\theta_1)[h_1]\|_{P,2} \\ &\leq M\|h_0 - h_1\|_{\mathbf{E}} + K_m\|\theta_1 - \theta_0\|_{\mathbf{L}}\|h_1\|_{\mathbf{E}} \end{split}$$

for some $M < \infty$, and therefore the claim of the lemma follows.

Lemma S.3.5. Let B_n and D_n be observable random variables, $C_{P,n}^{\star}$ be a potentially unobservable random variable depending on $P \in \mathbf{P}$, and for any $\alpha \in (0,1)$ define

$$\hat{q}_{\alpha} \equiv \inf\{u : P(B_n \le u | D_n) \ge \alpha\} \qquad q_{\alpha,P} \equiv \inf\{u : P(C_{P,n}^* \le u) \ge \alpha\}.$$

If $B_n \geq C_{P,n}^{\star} + o_P(a_n)$ (with $a_n > 0$) uniformly in $P \in \mathbf{P}$ and $C_{P,n}^{\star}$ is independent of D_n , then there exists a $\delta_n \downarrow 0$ such that $\liminf_{n \to \infty} \inf_{P \in \mathbf{P}} P(\hat{q}_{\alpha} + a_n \geq q_{\alpha - \delta_n, P}) = 1$.

Proof: In the statement of the lemma, \mathbf{P} and a_n represent a generic set of distributions and positive sequence – i.e. they need not be the same as in the main text. To establish the result, note Markov's inequality and the law of iterated expectations yield

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} P(P(C_{P,n}^{\star} > B_n + a_n | D_n) > \epsilon) \le \limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \frac{1}{\epsilon} P(C_{P,n}^{\star} > B_n + a_n) = 0,$$

where the final equality follows from $B_n \geq C_{P,n}^{\star} + o_P(a_n)$ uniformly in $P \in \mathbf{P}$ by hypothesis. Thus, we conclude there exists some sequence $\delta_n \downarrow 0$ such that the event

$$\Omega_n(P) \equiv \{D_n | P(C_{P,n}^* > B_n + a_n | D_n) \le \delta_n \}$$

satisfies $P(\Omega_n(P)^c) = o(1)$ uniformly in $P \in \mathbf{P}$. Hence, for any $t \in \mathbf{R}$ we obtain that

$$P(B_n \le t | D_n) 1\{D_n \in \Omega_n(P)\} \le P(B_n \le t \text{ and } C_{P,n}^* \le B_n + a_n | D_n) + \delta_n$$

$$\le P(C_{P,n}^* \le t + a_n) + \delta_n, \tag{S.101}$$

where in the final inequality we employed that $C_{P,n}^{\star}$ is independent of D_n . Therefore, setting $t = \hat{q}_{\alpha}$ in (S.101) implies that, under $\Omega_n(P)$, we have $\hat{q}_{\alpha} + a_n \geq q_{\alpha - \delta_n, P}$. Since $\sup_{P \in \mathbf{P}} P(\Omega_n(P)^c) = o(1)$, the claim of the lemma follows.

Lemma S.3.6. Let $T_n \leq C_{P,n} + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$ with $0 < a_n = o(1)$, define $q_{\alpha,P} \equiv \inf\{u : P(C_{P,n} \leq u) \geq \alpha\}$, and suppose that, for some $\delta_n \downarrow 0$, $\hat{c}_n + a_n/2 \geq q_{1-\alpha-\delta_n,P}$ with probability tending to one uniformly in $P \in \mathbf{P}_0$. If for some $\eta_n \geq 0$

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(T_n > \hat{c}_n) = \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(T_n > \hat{c}_n \vee \eta_n)$$
 (S.102)

and for some sequence ϱ_n satisfying $\varrho_n a_n = o(1)$ we have $\sup_{P \in \mathbf{P}_0} P(|C_{P,n} - t| \le \epsilon) \le \varrho_n(\epsilon \wedge 1) + o(1)$ for all $t \in (\eta_n - a_n, +\infty)$, then it follows that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(T_n > \hat{c}_n) \le \alpha.$$

PROOF: First note that by condition (S.102), $T_n \leq C_{P,n} + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$ and the maintained hypothesis on \hat{c}_n we can conclude that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(T_n > \hat{c}_n) = \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(T_n > \hat{c}_n \vee \eta_n)$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(C_{P,n} + \frac{a_n}{2} > (q_{1-\alpha-\delta_n,P} - \frac{a_n}{2}) \vee \eta_n)$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(C_{P,n} + a_n > q_{1-\alpha-\delta_n,P} \vee \eta_n). \quad (S.103)$$

Next observe that by direct calculation we also have the following inequalities

$$P(C_{P,n} + a_n > q_{1-\alpha-\delta_n,P} \vee \eta_n) - P(C_{P,n} > q_{1-\alpha-\delta_n,P})$$

$$\leq \begin{cases} 0 & \text{if } \eta_n - a_n \geq q_{1-\alpha-\delta_n,P} \\ P(|C_{P,n} - q_{1-\alpha-\delta_n,P}| \leq a_n) & \text{if } \eta_n - a_n < q_{1-\alpha-\delta_n,P} \end{cases} . (S.104)$$

Therefore, combining results (S.103) and (S.104) together with $\sup_{P\in\mathbf{P}_0} P(|C_{P,n}-t| \leq$

 $\epsilon \leq \varrho_n(\epsilon \wedge 1) + o(1)$ for all $t \in (\eta_n - a_n, +\infty)$ implies that

 $\limsup_{n\to\infty} \sup_{P\in\mathbf{P}_0} P(T_n > \hat{c}_n)$ $\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(C_{P,n} > q_{1-\alpha-\delta_n,P}) + \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} \sup_{t > \eta_n - a_n} P(|C_{P,n} - t| \leq a_n)$ $< \alpha + \delta_n + \rho_n(a_n \wedge 1).$

The claim of the lemma therefore follows from $\delta_n = o(1)$ and $\varrho_n a_n = o(1)$.

Lemma S.3.7. Let the conditions of Theorems 3.1(ii) and 3.2 hold with $R = \Theta$ and suppose that ℓ_n^{u} satisfies $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}((\ell_n^{\mathrm{u}})^{\kappa_\rho} \vee (\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}})^{\kappa_\rho}, \mathcal{F}_n^{\mathrm{u}}, \|\cdot\|_{P,2}) =$ $o(a_n)$, $K_m \ell_n^{\mathrm{u}}(\ell_n^{\mathrm{u}} + \mathcal{R}_n^{\mathrm{u}} + \nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}}) \times \mathcal{S}_n^{\mathrm{u}}(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$, and $\mathcal{R}_n^{\mathrm{u}} = o(\ell_n^{\mathrm{u}})$. (i) If $\tau_n^{\mathrm{u}} \downarrow 0$ satisfies $J_n^{\mathrm{u}} B_n k_n^{1/p} \sqrt{\log(1 + k_n)/n} = o(\tau_n^{\mathrm{u}})$ and $\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}} \times \mathcal{S}_n^{\mathrm{u}}(\mathbf{B}, \mathbf{E}) = o(1)$, then

$$\hat{U}_n(\Theta|+\infty) \le U_P^{\star}(\Theta|\ell_n^{\mathrm{u}}) + o_P(a_n)$$

uniformly in $P \in \mathbf{P}_0$. (ii) If $\mathcal{S}_n^{\mathrm{u}}(\mathbf{B}, \mathbf{E}) \times \mathcal{R}_n^{\mathrm{u}} = o(1)$ and Θ_{0n}^{u} is a singleton for all $P \in \mathbf{P}_0$ and n sufficiently large, then part (i) of the lemma continues to hold if $\tau_n^{\rm u} = 0$.

PROOF: First note that since we required $J_n^{\rm u}B_nk_n^{1/p}\sqrt{\log(1+k_n)/n}=o(\tau_n^{\rm u})$ and we assumed all other conditions of Corollary S.1.2(ii) are satisfied when $\Theta = R$, it follows

$$d_H(\hat{\Theta}_n^{\mathbf{u}}, \Theta_{0n}^{\mathbf{u}}, \| \cdot \|_{\mathbf{E}}) = O_P(\nu_n^{\mathbf{u}} \tau_n^{\mathbf{u}})$$
(S.106)

(S.105)

uniformly in $P \in \mathbf{P}_0$. Therefore, Lemma S.3.3 yields, uniformly in $P \in \mathbf{P}_0$, that

$$\sup_{\theta \in \hat{\Theta}_n^{\mathrm{u}}} \sup_{h \in V_n(\theta, \Theta | \ell_n^{\mathrm{u}})} \|\hat{\mathbb{D}}_n(\theta)[h] - \mathbb{D}_P(\theta)[h]\|_p = o_P(a_n).$$
 (S.107)

We further note that since $\hat{\Theta}_n^{\mathrm{u}} \subseteq \Theta_n$, Assumption 3.11 holding with $R = \Theta$ implies

$$\sup_{\theta \in \hat{\Theta}_n^{\mathrm{u}}} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^{\star}(\theta)\|_p = o_P(a_n)$$
 (S.108)

uniformly in $P \in \mathbf{P}_0$. Hence, by results (S.107) and (S.108), $\|\hat{\Sigma}_n\|_{o,p} = O_P(1)$ uniformly in $P \in \mathbf{P}_0$ by Corollary S.1.3, and $V_n(\theta, \Theta | \ell_n^{\mathrm{u}}) \subseteq \hat{V}_n(\theta, \Theta | + \infty)$ imply that

$$\hat{U}_{n}(\Theta|+\infty) \leq \inf_{\theta \in \hat{\Theta}_{n}^{u}} \inf_{h \in V_{n}(\theta,\Theta|\ell_{n}^{u})} \|\mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h]\|_{\hat{\Sigma}_{n},p} + o_{P}(a_{n})$$

$$= \inf_{\theta \in \hat{\Theta}_{n}^{u}} \inf_{h \in V_{n}(\theta,\Theta|\ell_{n}^{u})} \|\mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h]\|_{\Sigma_{P},p} + o_{P}(a_{n}) \tag{S.109}$$

uniformly in $P \in \mathbf{P}_0$, and where the equality can be established by employing identical arguments to those used in Lemma S.3.2 (see, in particular, (S.91)-(S.93)). Also note that, by hypothesis, there is an $\eta_n \downarrow 0$ satisfying $\nu_n^{\rm u} \tau_n^{\rm u} \times \mathcal{S}_n^{\rm u}(\mathbf{B}, \mathbf{E}) = o(\eta_n)$ and define

$$\mathcal{E}_n(\theta) \equiv V_n(\theta, \Theta | \ell_n^{\mathrm{u}}) \cap \{ h \in \mathbf{B}_n^{\mathrm{u}} : \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \le \eta_n \}.$$

Next, select $\theta_{0n} \in \Theta_{0n}^{\mathrm{u}}$ and $h_{0n} \in \mathcal{E}_n(\theta_{0n})$ so that the following equality is satisfied

$$\inf_{\theta \in \Theta_{0n}^{\mathbf{u}}} \inf_{h \in \mathcal{E}_n(\theta)} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p} = \| \mathbb{W}_P^{\star}(\theta_{0n}) + \mathbb{D}_P(\theta_{0n})[h_{0n}] \|_{\Sigma_P, p} + o(a_n).$$
 (S.110)

Assumption 3.13 holding with $R = \Theta$ implies $K_m \ell_n^{\mathrm{u}}(\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}}) \mathcal{S}_n^{\mathrm{u}}(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ and $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}((\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}})^{\kappa_\rho}, \mathcal{F}_n^{\mathrm{u}}, \|\cdot\|_{P,2}) = o(a_n)$. Hence, there is δ_n with

$$K_m \delta_n \ell_n^{\mathrm{u}} \mathcal{S}_n^{\mathrm{u}}(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$$
(S.111)

$$k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\delta_n^{\kappa_\rho}, \mathcal{F}_n^{\mathbf{u}}, \|\cdot\|_{P,2}) = o(a_n),$$
 (S.112)

and $\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}} = o(\delta_n)$. Moreover, note result (S.106) implies there is a $\hat{\theta}_{0n}$ in $\hat{\Theta}_n^{\mathrm{u}}$ such that

$$\|\theta_{0n} - \hat{\theta}_{0n}\|_{\mathbf{E}} = O_P(\nu_n^{\mathbf{u}} \tau_n^{\mathbf{u}})$$

uniformly in $P \in \mathbf{P}_0$. Thus, $\nu_n^{\mathbf{u}} \tau_n^{\mathbf{u}} = o(\delta_n)$ and $\hat{\theta}_{0n} \in \hat{\Theta}_n^{\mathbf{u}} \subseteq \Theta_n$ implies that $\sqrt{n}(\hat{\theta}_{0n} - \theta_{0n}) \in V_n(\theta_{0n}, \Theta|\delta_n)$ with probability tending to one uniformly in $P \in \mathbf{P}_0$. Hence, applying Lemma S.2.2 with $\Theta_{0n}^{\mathbf{u}}$ and $V_n(\theta, \Theta|\delta_n)$ in place of $\Theta_{0n}^{\mathbf{r}}$ and $V_n(\theta, R|\delta_n)$, yields

$$\|\mathbb{W}_{P}^{\star}(\hat{\theta}_{0n}) - \mathbb{W}_{P}^{\star}(\theta_{0n})\|_{p} = o_{P}(a_{n})$$
(S.113)

uniformly in $P \in \mathbf{P}_0$. Furthermore, Lemma S.3.4, $h_{0n} \in \mathcal{E}_n(\theta_{0n})$ and result (S.111) imply that with probability tending to one uniformly in $P \in \mathbf{P}_0$ we must have

$$\|\mathbb{D}_{P}(\hat{\theta}_{0n})[h_{0n}] - \mathbb{D}_{P}(\theta_{0n})[h_{0n}]\|_{p} \le K_{m}\mathcal{S}_{n}^{\mathsf{u}}(\mathbf{L}, \mathbf{E})\delta_{n}\ell_{n}^{\mathsf{u}}\sqrt{n} = o(a_{n}).$$
 (S.114)

Therefore, Assumption 3.7(ii), $\hat{\theta}_{0n} \in \hat{\Theta}_n^{\mathrm{u}}$, $h_{0n} \in \mathcal{E}_n(\theta_{0n})$, $\mathcal{E}_n(\theta_{0n}) \subseteq V_n(\hat{\theta}_{0n}, \Theta | \ell_n^{\mathrm{u}})$ by Assumption 3.12(ii), and results (S.110), (S.113), and (S.114) yield that

$$\inf_{\theta \in \Theta_{0n}^{\mathbf{u}}} \inf_{h \in \mathcal{E}_{n}(\theta)} \| \mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h] \|_{\Sigma_{P}, p}$$

$$\geq \inf_{\theta \in \hat{\Theta}_{n}^{\mathbf{u}}} \inf_{h \in V_{n}(\theta, \Theta | \ell_{n}^{\mathbf{u}})} \| \mathbb{W}_{P}^{\star}(\theta) + \mathbb{D}_{P}(\theta)[h] \|_{\Sigma_{P}, p} + o_{P}(a_{n}) \quad (S.115)$$

uniformly in $P \in \mathbf{P}_0$. To conclude, note that Assumption 3.4 holding with $R = \Theta$, Corollary S.1.1, and $\mathcal{R}_n^{\mathrm{u}} \times \mathcal{S}_n^{\mathrm{u}}(\mathbf{B}, \mathbf{E}) = o(\eta_n)$ due to $\mathcal{R}_n^{\mathrm{u}} = o(\tau_n^{\mathrm{u}} \nu_n^{\mathrm{u}})$ and $\nu_n^{\mathrm{u}} \tau_n^{\mathrm{u}} \times \mathcal{S}_n^{\mathrm{u}}(\mathbf{B}, \mathbf{E}) =$ $o(\eta_n)$ allow us to conclude that uniformly in $P \in \mathbf{P}_0$ we have

$$I_{n}(\Theta) = \inf_{\theta \in \Theta_{0n}^{u}} \inf_{h \in \mathcal{E}_{n}(\theta)} \sqrt{n} Q_{n}(\theta + \frac{h}{\sqrt{n}}) + o_{P}(a_{n})$$

$$= \inf_{\theta \in \Theta_{0n}^{u}} \inf_{h \in \mathcal{E}_{n}(\theta)} \| \mathbb{W}_{P}(\theta) + \mathbb{D}_{P}(\theta)[h] \|_{\Sigma_{P}, p} + o_{P}(a_{n}), \tag{S.116}$$

where the second equality follows by identical arguments to those employed in Theorem 3.1(ii). Combining result (S.116) with Theorem 3.1(ii) and employing the fact that \mathbb{W}_P^* and \mathbb{W}_P share the same distribution we thus obtain, uniformly in $P \in \mathbf{P}_0$, that

$$\inf_{\theta \in \Theta_{0n}^{\mathbf{u}}} \inf_{h \in V_n(\theta, \Theta | \ell_n^{\mathbf{u}})} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p}$$

$$= \inf_{\theta \in \Theta_{0n}^{\mathbf{u}}} \inf_{h \in \mathcal{E}_n(\theta)} \| \mathbb{W}_P^{\star}(\theta) + \mathbb{D}_P(\theta)[h] \|_{\Sigma_P, p} + o_P(a_n). \quad (S.117)$$

The claim of part (i) of the lemma therefore follows (S.109), (S.115), and (S.117). To establish part (ii) note that if $\Theta_{0n}^{\mathbf{u}}$ is a singleton, then $\overrightarrow{d}_{H}(\hat{\Theta}_{n}^{\mathbf{u}}, \Theta_{0n}^{\mathbf{u}}, \| \cdot \|_{\mathbf{E}}) = d_{H}(\hat{\Theta}_{n}^{\mathbf{u}}, \Theta_{0n}^{\mathbf{u}}, \| \cdot \|_{\mathbf{E}})$ and therefore Corollary S.1.2(i) implies $d_{H}(\hat{\Theta}_{n}^{\mathbf{u}}, \Theta_{0n}^{\mathbf{u}}, \| \cdot \|_{\mathbf{E}}) = O_{P}(\mathcal{R}_{n}^{\mathbf{u}})$ uniformly in $P \in \mathbf{P}_{0}$. Part (ii) of the lemma can then be established by replacing $\nu_{n}^{\mathbf{u}} \tau_{n}^{\mathbf{u}}$ with $\mathcal{R}_{n}^{\mathbf{u}}$ in the arguments employed in establishing part (i).

Corollary S.3.1. Suppose that $I_n(R) \leq U_P(R|\tilde{\ell}_n) + o_P(a_n)$ and $\hat{U}_n(R|\ell_n) \geq U_P^*(R|\tilde{\ell}_n) + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$ with $0 < a_n = o(1)$, $U_P(R|\tilde{\ell}_n) \stackrel{d}{=} U_P^*(R|\tilde{\ell}_n)$, and $U_P^*(R|\tilde{\ell}_n)$ independent of $\{V_i\}_{i=1}^n$. Then for any constant $\eta \in (0, \alpha)$ it follows that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} P(I_n(R) > \hat{q}_{1-\alpha+\eta}(\hat{U}_n(R|\ell_n)) + \eta) \le \alpha.$$

PROOF: Since $I_n(R) \leq U_P(R|\tilde{\ell}_n) + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$ by hypothesis, we obtain

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(I_{n}(R) > \hat{q}_{1-\alpha+\eta}(\hat{U}_{n}(R|\ell_{n})) + \eta)$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(U_{P}(R|\tilde{\ell}_{n}) + a_{n} > \hat{q}_{1-\alpha+\eta}(\hat{U}_{n}(R|\ell_{n})) + \eta)$$

$$\leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_{0}} P(U_{P}(R|\tilde{\ell}_{n}) > q_{1-\alpha+\eta-\delta_{n},P}(U_{P}^{\star}(R|\tilde{\ell}_{n})) + \eta - 2a_{n})$$

$$\leq \alpha, \tag{S.118}$$

where the second inequality holds for $q_{1-\alpha+\eta-\delta_n,P}(U_P^{\star}(R|\tilde{\ell}_n))$ the $1-\alpha+\eta-\delta_n$ quantile of $U_P^{\star}(R|\tilde{\ell}_n)$ and some $\delta_n=o(1)$ by Lemma S.3.5 applied with $B_n=\hat{U}_n(R|\ell_n),\ C_{P,n}^{\star}=U_P^{\star}(R|\tilde{\ell}_n)$, and $D_n=\{V_i\}_{i=1}^n$. In turn, the final inequality in (S.118) follows from $\eta>0$, $a_n=o(1),\ \delta_n=o(1),\ and\ U_P(R|\tilde{\ell}_n)\stackrel{d}{=}U_P^{\star}(R|\tilde{\ell}_n)$.

S.4 Illustrative Examples

In this Section, we include the proofs for all the examples discussed in the main text and Supplemental Appendix I – i.e., the results stated in Section 4 of the main text and in Section A.2 of Supplemental Appendix I.

S.4.1 Proofs for Section 4

PROOF OF THEOREM 4.1: We establish the claim of the theorem by verifying the conditions of Theorem 3.1(ii) for both R as in (29) (to couple $I_n(R)$) and $R = \Theta$ (to couple $I_n(\Theta)$). To this end, note that Assumption 3.1(i) is imposed in Assumption 4.1(i), Assumption 3.2(i) holds with $B_n \simeq \sqrt{k_n}$ by Assumption 4.2(i), Assumption 3.2(ii) is directly imposed in Assumption 4.2(ii), and Assumption 3.2(iii) is satisfied with $J_n \simeq \sqrt{j_n \log(1+j_n)}$ by Lemma S.4.2 and $||f||_{\infty} \leq 3$ for any $f \in \mathcal{F}_n$. The coupling requirement of Assumption 3.3(i) is satisfied for $R = \Theta$, and hence also for R as in (29), with $a_n = (\log(n))^{-1/2}$ by Lemma S.4.4 and Assumption 4.2(iv). Moreover, Assumptions 3.3(ii), 3.4, and 3.5 also hold by Lemmas S.4.1 and S.4.3. To verify Assumption 3.6, we first note that Assumption 3.6(ii) is implied by Assumptions 4.1(iv) and 4.3(ii). Furthermore, as argued, $B_n \simeq \sqrt{k_n}$, $J_n \simeq \sqrt{j_n \log(1+j_n)}$, and $\nu_n \approx 1$ by Lemma S.4.1, which yields that $\mathcal{R}_n \lesssim k_n \sqrt{j_n} \log(1+k_n)/\sqrt{n}$ since $k_n \geq j_n$ by Assumption 4.2(iii). Thus, $\kappa_{\rho}=1$ by Lemma S.4.3 and Lemma S.4.2 imply that Assumption 3.6(i) holds by Assumption 4.2(iv). By similar arguments, it also follows that Assumption 3.7 is implied by Assumption 4.3, and that the requirements $k_n^{1/p}\sqrt{\log(1+k_n)}B_n\sup_{P\in\mathbf{P}}J_{[]}(\ell_n^{\kappa_\rho},\mathcal{F}_n,\|\cdot\|_{P,2})=o(a_n)\text{ and }\mathcal{R}_n=o(\ell_n)\text{ are implied by }$ $k_n\sqrt{j_n}\log^2(n)\ell_n=o(1)$ and $k_n\sqrt{j_n}\log(n)/\sqrt{n}=o(\ell_n)$. Since $K_m=0$ in this application, it follows all the conditions of Theorem 3.1(ii) hold for both $R = \Theta$ and R as in (29), and hence the theorem follows.

PROOF OF LEMMA 4.1: The result essentially follows from Theorem 1 in Walkup and Wets (1969). To map our problem into their setting, note that since $\{\delta_s\}_{s=1}^{s_n}$ are orthogonal, every $\mu \in \mathcal{M}_n$ can be identified with a unique $(\alpha_1, \ldots, \alpha_{s_n}) \equiv \alpha \in \mathbf{R}^{s_n}$ through the relation $\mu = \sum_{s=1}^{s_n} \alpha_s \delta_s - \text{e.g.}$, by $\alpha_s = \mu(S_s)$ for S_s the support of δ_s . With some abuse of notation, for the remaining of the proof we therefore employ α and μ interchangeably. Further note that, for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu)$, the restrictions $\Upsilon_G(\theta) \leq 0$, $\Upsilon_F^{(\mu)}(\theta) = 0$, and $\Upsilon_F^{(s)}(\theta) = 0$ depend only on μ and define a closed convex polyhedron on \mathbf{R}^{s_n} , which we denote by K_n . Next, define the map $\Lambda_n : \mathbf{R}^{s_n} \to \mathbf{R}^{\mathcal{JL}}$ to be given by

$$\Lambda_n(\alpha) = \{ \sum_{s=1}^{s_n} \alpha_s (\int 1\{g(w_l, \eta) \le c_j\} \delta_s(d\eta)) \}_{1 \le j \le \mathcal{J}, 1 \le l \le \mathcal{L}}$$
 (S.119)

and note that for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \Theta_n \cap R$, it follows by (35) that

$$\Gamma_n(\theta) = K_n \cap \Lambda_n^{-1}(\{F(c_j|w_l)\}_{1 \le j \le \mathcal{J}, 1 \le l \le \mathcal{L}}). \tag{S.120}$$

Let d_n denote the dimension of the null space of Λ_n , and note that if $d_n = s_n$, then $\Gamma_n(\theta_1) = \Gamma_n(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta_n \cap R$ by result (S.120), and hence the conclusion of the lemma is immediate. On the other hand, if $1 \leq d_n \leq s_n - 1$, then Theorem 1 in Walkup and Wets (1969) implies there is a C_n such that for any $\theta_1, \theta_2 \in \Theta_n \cap R$ we have

$$d_{H}(\Gamma_{n}(\theta_{1}), \Gamma_{n}(\theta_{2}), \|\cdot\|_{2}) \leq C_{n} \{ \sum_{j=1}^{\mathcal{J}} \sum_{l=1}^{\mathcal{L}} (F_{1}(c_{j}|w_{l}) - F_{2}(c_{j}|w_{l}))^{2} \}^{1/2}$$

$$\lesssim C_{n} \sum_{j=1}^{\mathcal{J}} \|F_{1}(c_{j}|\cdot) - F_{2}(c_{j}|\cdot)\|_{\infty},$$
(S.121)

and where the norm $\|\cdot\|_2$ on $\Gamma_n(\theta)$ is understood as the usual Euclidean norm on the corresponding $\alpha \in \mathbf{R}^{s_n}$. Similarly, we note that if $d_n = 0$, then Λ_n is invertible and (S.121) holds with $C_n = \|\Lambda_n^{-1}\|_o$. Also note that for any $\mu = \sum_{s=1}^{s_n} \alpha_s \delta_s$ and $\tilde{\mu} = \sum_{s=1}^{s_n} \tilde{\alpha}_s \delta_s$ we have $\|\mu - \tilde{\mu}\|_{TV} = \|\alpha - \tilde{\alpha}\|_1$ due to the measures $\{\delta_s\}_{s=1}^{s_n}$ being orthogonal. Hence, since $\|a\|_1 \leq \sqrt{s_n} \|a\|_2$ for any $a \in \mathbf{R}^{s_n}$, result (S.121) yields

$$d_H(\Gamma_n(\theta_1), \Gamma_n(\theta_2), \|\cdot\|_{TV}) \lesssim \sqrt{s_n} C_n \sum_{j=1}^{\mathcal{J}} \|F_1(c_j|\cdot) - F_2(c_j|\cdot)\|_{\infty},$$

which establishes the claim of the lemma by setting $\zeta_n \asymp C_n \sqrt{s_n}$.

PROOF OF THEOREM 4.2: Let $\hat{V}_n(\theta, R|\ell) \equiv \hat{V}_n(\theta, R|+\infty) \cap \{h \in \mathbf{B}_n : \|h/\sqrt{n}\|_{\mathbf{E}} \leq \ell\}$, recall $\|\theta\|_{\mathbf{E}} = \sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}$ for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu)$, define

$$\hat{E}_n(R|\ell_n) = \inf_{\theta \in \hat{\Theta}_n^r} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \{ \sum_{j=1}^{\mathcal{J}} \|\hat{\mathbb{W}}_{j,n}(\theta) + \hat{\mathbb{D}}_{j,n}[h]\|_{\hat{\Sigma}_{j,n},2} \}^{1/2},$$

and note that for any ℓ_n satisfying the conditions of the theorem, Assumption 4.4(iii) and Lemma S.4.9 imply $\hat{U}_n(R|+\infty) = \hat{E}_n(R|\ell_n) + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$. Hence, to establish the theorem it suffices to show there are $\ell_n \simeq \tilde{\ell}_n$ and $\ell_n^{\mathrm{u}} \simeq \tilde{\ell}_n^{\mathrm{u}}$ such that

$$\hat{E}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n)$$

$$\hat{E}_n(R|\ell_n) - \hat{U}_n(\Theta|+\infty) \ge U_P^{\star}(R|\tilde{\ell}_n) - U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n)$$
(S.122)

uniformly in $P \in \mathbf{P}_0$. To this end, we rely on Theorem S.3.1(ii) (for $\hat{E}_n(R|\ell_n)$) and Lemma S.3.7. Also note that in the proof of Theorem 4.1 we showed Assumptions 4.1, 4.2, and 4.3 imply Assumptions 3.1-3.7 hold with $B_n \asymp \sqrt{k_n}$, $J_n \asymp \sqrt{j_n \log(1+j_n)}$, $\nu_n \asymp 1$, $\mathcal{R}_n \asymp k_n \sqrt{j_n \log(1+k_n) \log(1+j_n)/n}$, $a_n = (\log(n))^{-1/2}$, $\kappa_\rho = 1$, $\|\theta\|_{\mathbf{L}} = 1$

 $\|\theta\|_{\mathbf{E}} = \sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}$, and $\|\theta\|_{\mathbf{B}} = \sum_{j=1}^{\mathcal{J}} \|F(c_j|\cdot)\|_{\infty} + \|\mu\|_{TV}$ for $R = \Theta$ and R as in (29).

In order to apply Theorem S.3.1(ii), we set $\|\theta\|_{\mathbf{I}} = \max_{1 \leq j \leq \mathcal{J}} \mathcal{J} \|F(c_j|\cdot)\|_{\infty}$ for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) \in \mathbf{B}_n$ and note Assumption 4.4(i) and Lemma S.4.5 verify Assumptions 3.8, 3.9, and 3.10 are satisfied with $K_g = 0$ and $K_f = 0$. Also note Assumption 4.4(iii) and Lemma S.4.8 verify Assumption 3.11 and Assumptions 3.12(i)(iii) are immediate given the definitions of $\|\cdot\|_{\mathbf{E}}$ and $\|\cdot\|_{\mathbf{B}}$ and $\mathcal{V}_n(P) = \Theta_n \cap R$ by Lemma S.4.1. Also note $\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \|\cdot\|_{\mathbf{I}}) \leq 1/2\} \subseteq \Theta_n$ for n sufficiently large by Assumption 4.4(iv) and the definitions of Θ_n and $\|\cdot\|_{\mathbf{I}}$. Moreover, Assumptions 4.1(ii)(iii) imply

$$\sup_{h \in \mathbf{B}_n} \frac{\|h\|_{\mathbf{I}}}{\|h\|_{\mathbf{E}}} = \sup_{h \in \mathbf{B}_n} \frac{\max_{1 \le j \le \mathcal{J}} \mathcal{J} \|F(c_j|\cdot)\|_{\infty}}{\sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}} \lesssim \sqrt{j_n}, \tag{S.123}$$

and hence Corollary S.1.2(i), $\nu_n \approx 1$, and $\mathcal{R}_n \approx k_n \sqrt{j_n \log(1+k_n) \log(1+j_n)/n}$ yield

$$\overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \|\cdot\|_{\mathbf{I}}) \lesssim \sqrt{j_{n}} \overrightarrow{d}_{H}(\hat{\Theta}_{n}^{r}, \Theta_{0n}^{r}, \|\cdot\|_{\mathbf{E}}) = O_{P}(\frac{k_{n} j_{n} \log(n)}{\sqrt{n}} + \sqrt{j_{n}} \tau_{n})$$

uniformly in $P \in \mathbf{P}_0$. In particular, Assumptions 4.4(iii)(v) imply $\overrightarrow{d}_H(\hat{\Theta}_n^r, \Theta_{0n}^r, \| \cdot \|_{\mathbf{I}}) = o_P(1)$ uniformly in $P \in \mathbf{P}_0$, and therefore since, as argued, we have $\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\theta, \Theta_{0n}^r, \| \cdot \|_{\mathbf{I}}) \le 1/2\} \subseteq \Theta_n$ for n sufficiently large, we obtain

$$\liminf_{n \to \infty} \inf_{P \in \mathbf{P}} P(\{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\{\theta\}, \hat{\Theta}_n^{\mathrm{r}}, \| \cdot \|_{\mathbf{I}}) \le 1/4\} \subseteq \Theta_n) = 1.$$
 (S.124)

Next, observe Lemma 4.1, Assumption 4.1(ii) and the definitions of $\|\cdot\|_{\mathbf{E}}$, $\|\cdot\|_{\mathbf{L}}$, and $\|\cdot\|_{\mathbf{B}}$ imply Assumption S.3.1 holds with $\mathcal{D}_n(\mathbf{B},\mathbf{E}) \simeq \zeta_n \sqrt{j_n}$ and $\mathcal{D}_n(\mathbf{L},\mathbf{E}) = 1$. Since $K_m = K_g = K_f = 0$ and Υ_F and Υ_G are affine, the only requirements imposed by Assumption S.3.2 are that $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho} \vee (\nu_n \tau_n)^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ and $(\mathcal{R}_n + \nu_n \tau_n) \mathcal{D}_n(\mathbf{B},\mathbf{E}) = o(r_n)$, which are implied by Assumption 4.4(v), Lemma S.4.2, and $k_n \sqrt{j_n} \log^2(n) \ell_n = o(1)$ by hypothesis. Hence, all the conditions of Theorem S.3.1(ii) hold, which implies there is a $\tilde{\ell}_n \simeq \ell_n$ such that uniformly in $P \in \mathbf{P}_0$

$$\hat{E}_n(R|\ell_n) \ge U_P^*(R|\tilde{\ell}_n) + o_P(a_n). \tag{S.125}$$

Finally, to apply Lemma S.3.7 to $\hat{U}_n(\Theta|+\infty)$, note that we can set the norm $\|\cdot\|_{\mathbf{B}}$ to equal $\|\theta\|_{\mathbf{B}} = \max_{1 \leq j \leq \mathcal{J}} \|F(c_j|\cdot)\|_{\infty}$ and interpret Υ_G and Υ_F as satisfying $\Upsilon_G(\theta) = \Upsilon_F(\theta) = 0$ for all $\theta \in \mathbf{B}$ (since $R = \Theta$). Hence, Assumptions 3.8, 3.9, and 3.10, 3.12(i) are immediate, while Assumption 3.11 is satisfied by Assumption 4.4(iii) and Lemma S.4.8. Further note since Θ_0 is an equivalence class under $\|\cdot\|_{\mathbf{E}}$ and $\|\cdot\|_{\mathbf{B}}$, when studying the unconstrained statistic we can treat the model as identified. As a result, we may set $\tau_n^{\mathrm{u}} = 0$ and Assumption 3.12(ii) holds by the

same arguments employed in (S.124), while Assumption 3.12(iii) is immediate since $\mathcal{V}_n(P) = \Theta_n \cap R$. In order to apply Lemma S.3.7(ii), it therefore only remains to verify that $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\mathrm{u}}, \mathcal{F}_n^{\mathrm{u}}, \| \cdot \|_{P,2}) = o(a_n), \, \mathcal{R}_n^{\mathrm{u}} = o(\ell_n^{\mathrm{u}}), \, \text{and} \, \mathcal{S}_n^{\mathrm{u}}(\mathbf{B}, \mathbf{E}) \times \mathcal{R}_n^{\mathrm{u}} = o(1), \, \text{which are implied by } k_n \sqrt{j_n} \log^2(n) \ell_n^{\mathrm{u}} = o(1), \, k_n \sqrt{j_n} \log(n) / \sqrt{n} = o(\ell_n^{\mathrm{u}}), \, \text{and Assumption 4.4(iii) respectively. Thus, (S.125) and Lemma S.3.7(ii) verify (S.122) with <math>\tilde{\ell}_n^{\mathrm{u}} = \ell_n^{\mathrm{u}}$ and $\tilde{\ell}_n \times \ell_n$, which in turn establishes the theorem.

Lemma S.4.1. If Assumptions 4.1(iii), 4.2(iii), and 4.3(ii) hold, then Assumption 3.4 holds with $R = \Theta$ and R as in (29), $\mathcal{V}_n(P) = \Theta_n \cap R$, $\|\theta\|_{\mathbf{E}} = \sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}$ for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) \in \mathbf{B}$, and $\nu_n^{-1} \approx 1$.

PROOF: First note that since we are setting $\mathcal{V}_n(P) = \Theta_n \cap R$, Assumption 3.4(ii) is immediate. To verify Assumption 3.4(i), let $\|\theta\|_{\mathbf{E}} = \sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}$ for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) \in \mathbf{B}$. Then note that any $(\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \Theta_n$ must be such that $F(c_j|\cdot) = p^{jn'}\beta_{j,\theta}$ for some $\beta_{j,\theta} \in \mathbf{R}^{jn}$ and, similarly, $\Pi_n\theta_0 = (\{F_n(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_n)$ must satisfy $F_n(c_j|\cdot) = p^{jn'}\beta_{j,n}$. The Cauchy Schwarz inequality, and Assumptions 4.1(iii) and 4.2(iii) then yield that uniformly in $P \in \mathbf{P}_0$ we must have

$$\|\theta - \Pi_n \theta_0\|_{\mathbf{E}} \lesssim \sum_{j=1}^{\mathcal{J}} \|\beta_{j,\theta} - \beta_{j,n}\|_2 \lesssim \sum_{j=1}^{\mathcal{J}} \|E_P[q^{k_n}(W)p^{j_n}(W)'(\beta_{j,\theta} - \beta_{j,n})]\|_2$$
$$\lesssim \{\sum_{j=1}^{\mathcal{J}} \|E_P[(F(c_j|W) - F_n(c_j|W))q^{k_n}(W)]\|_{\Sigma_{j,P},2}^2\}^{1/2}, \quad (S.126)$$

where the final inequality holds due to $\|\Sigma_{j,P}^{-1}\|_{o,2}$ being uniformly bounded by Assumption 4.3(ii) and $\sum_{j=1}^{\mathcal{J}} |a^{(j)}| \leq \sqrt{\mathcal{J}} \|a\|_2$ for any $(a^{(1)}, \ldots, a^{(\mathcal{J})}) = a \in \mathbf{R}^{\mathcal{J}}$. Result (S.126) and the definition of $\rho_j(X, \theta)$ in (28) verify Assumption 3.4(i) holds with $\nu_n^{-1} \approx 1$.

Lemma S.4.2. Define the class $\mathcal{F}_n \equiv \{f : f(v) = (1\{y \leq c_j\} - p^{j_n}(w)'\beta) \text{ for some } 1 \leq j \leq J \text{ and } \|p^{j_n'}\beta\|_{\infty} \leq 2\}$ and suppose that Assumptions 4.1(ii)(iii) hold. Then, it follows that $\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim (1 \vee (\sqrt{j_n}K/\epsilon)^{j_n}) \text{ for some } K < \infty, \text{ and in addition } \sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim \epsilon \sqrt{j_n} (1 + \sqrt{\log(1 \vee (\sqrt{j_n}/\epsilon))}).$

PROOF: First note that for any $p^{j_n\prime}\beta_1$ and $p^{j_n\prime}\beta_2$, the Cauchy-Schwarz inequality yields

$$|p^{j_n}(w)'\beta_1 - p^{j_n}(w)'\beta_2| \le \sup_{w} ||p^{j_n}(w)||_2 ||\beta_1 - \beta_2||_2 \lesssim \sqrt{j_n} ||\beta_1 - \beta_2||_2,$$

where in the final inequality we employed Assumption 4.1(ii). Hence, Theorem 2.7.11 in van der Vaart and Wellner (1996), $\|\beta\|_2 \approx \sup_{P \in \mathbf{P}} \|p^{j_n'}\beta\|_{P,2}$ by Assumption 4.1(iii), and $\sup_{P \in \mathbf{P}} \|p^{j_n'}\beta\|_{P,2} \leq \|p^{j_n'}\beta\|_{\infty} \leq 2$ for any $p^{j_n'}\beta \in \Theta_n$ imply

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim 1 \vee (\frac{K\sqrt{j_n}}{\epsilon})^{j_n}, \tag{S.127}$$

for some $K < \infty$, which establishes the first claim of the lemma. For the second claim of the lemma, we employ (S.127) and the change of variables $v = u/\epsilon$ to obtain

$$\sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim \epsilon + \int_0^{\epsilon} (\log(1 \vee (\frac{K\sqrt{j_n}}{u})^{j_n}))^{1/2} du$$

$$= \epsilon (1 + \sqrt{j_n} \int_0^1 (\log(1 \vee (\frac{K\sqrt{j_n}}{v\epsilon})))^{1/2} dv) \lesssim \sqrt{j_n} \epsilon (1 + \sqrt{\log(1 \vee (\sqrt{j_n}/\epsilon))}),$$

where the final inequality follows from $(1 \vee ab) \leq (1 \vee a)(1 \vee b)$ for any $a, b \in \mathbf{R}_+$.

Lemma S.4.3. Let $\rho_{\jmath}: \mathbf{R} \times \mathbf{W} \times \Theta$ be as defined in (28). It then follows Assumptions 3.3(ii) and 3.5 hold with $\kappa_{\rho} = 1$, $K_{\rho} = 1$, $K_{m} = 0$, $M < \infty$, $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\mathbf{E}}$, and $\|\theta\|_{\mathbf{E}} = \sum_{\jmath=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_{\jmath}|\cdot)\|_{P,2}$ for any $\theta = (\{F(c_{\jmath}|\cdot)\}_{\jmath=1}^{\mathcal{J}}, \mu) \in \mathbf{B}$.

PROOF: First note that for any $(\{F_1(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_1) = \theta_1 \in \mathbf{B}$ and $(\{F_2(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_2) = \theta_2 \in \mathbf{B}$, we obtain from (28) and the definition of $\|\cdot\|_{\mathbf{E}}$ that for all $P \in \mathbf{P}$

$$E_P[\|\rho(X,\theta_1) - \rho(X,\theta_2)\|_2^2] = \sum_{j=1}^{\mathcal{J}} E_P[(F_1(c_j|W) - F_2(c_j|W))^2] \le \|\theta_1 - \theta_2\|_{\mathbf{E}}^2,$$

which verifies Assumption 3.3(ii) holds with $\kappa_{\rho} = 1$ and $K_{\rho} = 1$. Next, for any $P \in \mathbf{P}$ define $\nabla m_{P,j}(\theta)[h] = -F_h(c_j|W)$ for all $\theta \in \mathbf{B}$ and $(\{F_h(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu_h) = h \in \mathbf{B}$. Since $m_{P,j}(\theta) = P(Y \le c_j|W) - F(c_j|W)$ for any $\theta = (\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) \in \mathbf{B}$, direct calculation verifies Assumption 3.5 holds with $K_m = 0$, M = 1, and $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\mathbf{E}}$.

Lemma S.4.4. If $k_n^3 j_n^2 \log^2(n) = o(n)$, Assumptions 4.1(i)-(iii) and 4.2(i) hold, then Assumption 3.3(i) holds with $R = \Theta$ for any a_n with $k_n^3 j_n^2 \log^2(n)/n = o(a_n^2)$.

PROOF: We establish the result by applying Lemma S.4.6. To this end, we let $\tilde{j}_n = \mathcal{J} + j_n$ set $\{r_j\}_{j=1}^{\tilde{j}_n} = \{1\{y \leq c_j\}\}_{j=1}^{\mathcal{J}} \cup \{p_j\}_{j=1}^{j_n}$ and let $r^{\tilde{j}_n}(x) \equiv (r_1(x), \dots, r_{\tilde{j}_n}(x))'$. Next note that any $f \in \mathcal{F}_n$ may be written as $r^{\tilde{j}_n'}\beta$ for some $\beta \in \mathbf{R}^{\tilde{j}_n}$. Moreover, since $\sup_{P \in \mathbf{P}} \max_{1 \leq j \leq \mathcal{J}} \|F(c_j|\cdot)\|_{P,2} \leq \max_{1 \leq j \leq \mathcal{J}} \|F(c_j|\cdot)\|_{\infty} \leq 2$ for any $(\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \Theta_n$, Assumption 4.1(iii) implies that there exists a $C_0 < \infty$ (independent of \tilde{j}_n) such that $\|\beta\|_2 \leq C_0$ whenever $r^{\tilde{j}_n'}\beta \in \mathcal{F}_n$. Hence, by Assumptions 4.1(ii) and 4.2(i), we may apply Lemma S.4.6 with $b_{1n} \asymp \sqrt{j_n}$, $b_{2n} \asymp k_n$, and $C_n = O(1)$, from which the claim of the present lemma immediately follows.

Lemma S.4.5. Let $\mathbf{B} = (\bigotimes_{j=1}^{\mathcal{J}} C_B(\mathbf{W})) \times \mathcal{M}$ and Θ , Υ_G , and Υ_F be as defined in (27), (30), and (31). If $\Psi(g,\cdot)$ is bounded on Ω , then Assumptions 3.8, 3.9, and 3.10 are satisfied with $K_g = 0$, $\nabla \Upsilon_G(\theta)[h] = \Upsilon_G(h)$, $K_f = 0$, and $\nabla \Upsilon_F(\theta)[h]$ equal to

$$\nabla \Upsilon_F(\theta)[h] = (\Upsilon_F^{(e)}(h), \Upsilon_F^{(\mu)}(h) + 1, \Upsilon_F^{(s)}(h) + \lambda). \tag{S.128}$$

PROOF: For any measure $\mu \in \mathcal{M}$ let $\mu = \mu^+ - \mu^-$ denote its Jordan decomposition, $|\mu| = \mu^+ + \mu^-$, and recall the total variation of μ equals $\|\mu\|_{TV} = |\mu|(\Omega)$. Since $\Upsilon_G : \mathbf{B} \to \ell^\infty(\mathcal{B})$ is linear, in order to verify Assumption 3.8 we need only show that Υ_G is continuous. To this end, recall that for any $(\{F(c_j|\cdot)\}_{j=1}^{\mathcal{J}}, \mu) = \theta \in \mathbf{B}$ we had defined $\|\theta\|_{\mathbf{B}} = \sum_{j=1}^{\mathcal{J}} \|F(c_j|\cdot)\|_{\infty} + \|\mu\|_{TV}$. Hence, employing the definition of Υ_G we obtain

$$\|\Upsilon_G\|_o = \sup_{\|\theta\|_{\mathbf{B}} = 1} \|\Upsilon_G(\theta)\|_{\infty} = \sup_{\mu: \|\mu\|_{TV} = 1} \sup_{B \in \mathcal{B}} |\mu(B)| \le \sup_{\mu: \|\mu\|_{TV} = 1} |\mu|(\Omega) = 1,$$

which, by linearity of Υ_G , implies Assumption 3.8 holds with $\nabla \Upsilon_G = \Upsilon_G$ and $K_g = 0$. By similar arguments, note that $\Upsilon_F^{(e)} : \mathbf{B} \to \mathbf{R}^{\mathcal{JL}}$, as defined in (31), is linear and

$$\|\Upsilon_{F}^{(e)}\|_{o}^{2} = \sup_{\|\theta\|_{\mathbf{B}}=1} \sum_{j=1}^{\mathcal{J}} \sum_{l=1}^{\mathcal{L}} (F(c_{j}|w_{l}) - \int 1\{g(w_{l}, \eta) \leq c_{j}\}\mu(d\eta))^{2}$$

$$\leq \sum_{j=1}^{\mathcal{J}} \sum_{l=1}^{\mathcal{L}} \{2 \sup_{\|F(c_{j}|\cdot)\|_{\infty}=1} (F(c_{j}|w_{l}))^{2} + 2 \sup_{\|\mu\|_{TV}=1} (|\mu|(\Omega))^{2}\} = 4\mathcal{J}\mathcal{L}. \quad (S.129)$$

Moreover, note that for any bounded $f: \Omega \to \mathbf{R}$ and $\mu_1, \mu_2 \in \mathcal{M}$ it follows that

$$\int_{\Omega} f(\eta)(\mu_1(d\eta) - \mu_2(d\eta)) \le ||f||_{\infty} |\mu_1 - \mu_2|(\Omega) = ||f||_{\infty} ||\mu_1 - \mu_2||_{TV},$$

which implies $\Upsilon_F^{(\mu)}$ and $\Upsilon_F^{(s)}$ are Fréchet differentiable with $\nabla \Upsilon_F^{(\mu)} = \Upsilon_F^{(\mu)} + 1$, $\nabla \Upsilon_F^{(s)} = \Upsilon_F^{(s)} + \lambda$, $\|\nabla \Upsilon_F^{(\mu)}\|_o \leq 1$, and $\|\nabla \Upsilon_F^{(s)}\|_o \leq \|\Psi(g,\cdot)\|_{\infty}$. By (S.129) we may therefore conclude Assumptions 3.9(i)(ii)(iii) are satisfied with $\nabla \Upsilon_F$ as in (S.128) and $K_f = 0$. Furthermore, note that (provided $\Theta_n \cap R \neq \emptyset$) there is a $\theta^* \in \mathbf{B}_n$ such that $\Upsilon_F(\theta^*) = 0$, which together with (S.128) implies the range of $\nabla \Upsilon_F$ equals \mathbf{F}_n and hence Assumption 3.9(iv) holds. Finally, we note Assumption 3.10 is immediate due to Υ_F being affine.

Lemma S.4.6. Let $\{r_j\}_{j=1}^{j_n}$ be functions of X, $r^{j_n}(x) = (r_1(x), \ldots, r_{j_n}(x))'$, define the class $\mathcal{G}_n = \{r^{j_n'}\beta \text{ for some } \beta \text{ with } \|\beta\|_2 \leq C_n\}$, and suppose $b_{1n} \equiv \sup_x \|r^{j_n}(x)\|_2$ and $b_{2n} \equiv \sup_z \|q^{k_n}(z)\|_2$ are finite. If $\{X_i, Z_i\}_{i=1}^n$ is i.i.d. with $(X, Z) \sim P \in \mathbf{P}$, then there is an isonormal Gaussian process \mathbb{G}_P such that uniformly in $P \in \mathbf{P}$

$$\sup_{g \in \mathcal{G}_n} \| \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) q^{k_n}(Z_i) - E_P[g(X) q^{k_n}(Z)]) - \mathbb{G}_P(g q^{k_n}) \|_2$$

$$= O_P(\frac{C_n \sqrt{k_n j_n} b_{1n} b_{2n} \log(n)}{\sqrt{n}}). \quad (S.130)$$

PROOF: For notational simplicity, we first define a $k_n \times j_n$ matrix $\mathbb{E}_n^{(1)}$ to be given by

$$\mathbb{E}_n^{(1)} \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \{ q^{k_n}(Z_i) r^{j_n}(X_i)' - E_P[q^{k_n}(Z) r^{j_n}(X)'] \}.$$

For any matrix A let $\text{vec}\{A\}$ denote a column vector consisting of the unique elements of A and set $\mathbb{E}_n \equiv \text{vec}\{\mathbb{E}_n^{(1)}\}$, noting that \mathbb{E}_n has dimension (at most) $j_n k_n$. Our first step is to couple \mathbb{E}_n to a normal vector \mathbb{N}_P . To this end, we note that

$$\sup_{z,x} \|\operatorname{vec}\{q^{k_n}(z)r^{j_n}(x)' - E_P[q^{k_n}(Z)r^{j_n}(X)']\}\|_2^2$$

$$\leq \sup_{z,x} 4\operatorname{trace}\{q^{k_n}(z)r^{j_n}(x)'r^{j_n}(x)q^{k_n}(z)'\} \leq 4b_{1n}^2b_{2n}^2$$

by definition of b_{1n} and b_{2n} . Since the dimension of \mathbb{E}_n is at most $j_n k_n$, Theorem 1.1 in Zhai (2018) and Markov's inequality imply, provided the underlying probability space is suitably rich, that there is a Gaussian vector \mathbb{N}_P such that

$$\|\mathbb{E}_n - \mathbb{N}_P\|_2 = O_P(\frac{\sqrt{k_n j_n} b_{1n} b_{2n} \log(n)}{\sqrt{n}})$$
 (S.131)

uniformly in $P \in \mathbf{P}$. Next observe that for any $g \in \mathcal{G}_n$ there exists a $\beta \in \mathbf{R}^{j_n}$ such that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g(X_i)q^{k_n}(Z_i) - E_P[g(X)q^{k_n}(Z)]) = \mathbb{E}_n^{(1)}\beta.$$

Hence, letting $\mathbb{N}_P^{(1)}$ denote the $k_n \times j_n$ matrix built from the corresponding entries of the normal vector \mathbb{N}_P , we define the Gaussian process \mathbb{G}_P by setting

$$\mathbb{G}_P(gq^{k_n}) = \mathbb{N}_P^{(1)}\beta$$

for any $r^{j_n'}\beta = g \in \mathcal{G}_n$. Therefore, since $\|\beta\|_2 \leq C_n$ by definition of \mathcal{G}_n , and the operator norm is bounded by the Frobenius norm, we obtain from result (S.131) that

$$\sup_{g \in \mathcal{G}_n} \| \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) q^{k_n}(Z_i) - E_P[g(X) q^{k_n}(Z)]) - \mathbb{G}_P(g q^{k_n}) \|_2 \\
\leq \| \mathbb{E}_n^{(1)} - \mathbb{N}_P^{(1)} \|_{o,2} C_n = O_P(\frac{C_n \sqrt{k_n j_n} b_{1n} b_{2n} \log(n)}{\sqrt{n}})$$

uniformly in $P \in \mathbf{P}$, and hence the claim of the lemma follows.

Lemma S.4.7. Let $\{r_j\}_{j=1}^{j_n}$ be a set of functions of X, $r^{j_n}(x) \equiv (r_1(x), \dots, r_{j_n}(x))'$, and suppose $\sup_x \|r^{j_n}(x)\|_2 \lesssim b_{1n}$, $\sup_z \|q^{k_n}(z)\|_2 \lesssim b_{2n}$, and $E_P[q^{k_n}(Z)q^{k_n}(Z)']$ and $E_P[r^{j_n}(X)r^{j_n}(X)']$ have eigenvalues bounded uniformly in $P \in \mathbf{P}$, j_n , k_n . If $\{X_i, Z_i\}_{i=1}^n$ is i.i.d. with $(X, Z) \sim P \in \mathbf{P}$, then there is a $K < \infty$ such that for all $\delta \geq 0$

$$\sup_{P \in \mathbf{P}} P(\|\frac{1}{n} \sum_{i=1}^{n} q^{k_n}(Z_i) r^{j_n}(X_i)' - E_P[q^{k_n}(Z) r^{j_n}(X)']\|_{o,2} > \delta)$$

$$\leq (j_n + k_n) \exp\{-\frac{n\delta^2 K}{b_{1n}^2 \vee b_{2n}^2 + \delta b_{1n} b_{2n}}\}.$$

PROOF: We first define a $k_n \times j_n$ random matrix $\mathbb{M}_{i,n}$ satisfying $E_P[\mathbb{M}_{i,n}] = 0$ by

$$\mathbb{M}_{i,n} \equiv \frac{1}{n} \{ q^{k_n}(Z_i) r^{j_n}(X_i)' - E_P[q^{k_n}(Z) r^{j_n}(X)'] \}.$$

Since for any random matrix A we have $||E[A]||_o \leq E[||A||_o]$ by Jensen's inequality, $||A||_o^2 \leq \operatorname{trace}\{A'A\}$, $\sup_x ||r^{j_n}(x)||_2 \lesssim b_{1n}$, and $\sup_z ||q^{k_n}(z)||_2 \lesssim b_{2n}$ imply

$$\|\mathbb{M}_{i,n}\|_{o}^{2} \lesssim \|\frac{1}{n}q^{k_{n}}(Z_{i})r^{j_{n}}(X_{i})'\|_{o}^{2} + E_{P}[\|\frac{1}{n}q^{k_{n}}(Z)r^{j_{n}}(X)'\|_{o}^{2}]$$

$$\lesssim \frac{\sup_{z} \|q^{k_{n}}(z)\|_{2}^{2} \times \sup_{z} \|r^{j_{n}}(x)\|_{2}^{2}}{n^{2}} \lesssim \frac{b_{1n}^{2}b_{2n}^{2}}{n^{2}}. \quad (S.132)$$

Moreover, since the eigenvalues of $E_P[q^{k_n}(Z)q^{k_n}(Z)']$ are bounded uniformly in $P \in \mathbf{P}$ by assumption and $\sup_x ||r^{j_n}(x)||_2 \lesssim b_{1n}$ it additionally follows that

$$\sup_{P \in \mathbf{P}} \| \sum_{i=1}^{n} E_{P}[\mathbb{M}_{i,n}\mathbb{M}'_{i,n}] \|_{o} \le \sup_{P \in \mathbf{P}} \frac{2}{n} \| E_{P}[q^{k_{n}}(Z)q^{k_{n}}(Z)' \| r^{j_{n}}(X) \|_{2}^{2}] \|_{o} \lesssim \frac{b_{1n}^{2}}{n}.$$
 (S.133)

Identical arguments but relying on the eigenvalues of $E_P[r^{j_n}(X)r^{j_n}(X)']$ being bounded uniformly in $P \in \mathbf{P}$ and $\sup_x ||q^{k_n}(x)||_2 \lesssim b_{2n}$ by hypothesis further yield that

$$\sup_{P \in \mathbf{P}} \| \sum_{i=1}^{n} E_{P}[\mathbb{M}'_{i,n}\mathbb{M}_{i,n}] \|_{o} \lesssim \frac{b_{2n}^{2}}{n}.$$
 (S.134)

The claim of the lemma then follows from results (S.132), (S.133), and (S.134) allowing us to apply Theorem 1.6 in Tropp (2012) with $\sigma^2 \approx (b_{1n}^2 \vee b_{2n}^2)/n$ and $R \approx b_{1n}b_{2n}/n$.

Lemma S.4.8. If Assumptions 4.1(i)-(iii), 4.2(i)(ii) hold, and $j_n^3 k_n^2 \log(1 + j_n k_n) = o(n)$, then it follows that Assumption 3.11 holds with $R = \Theta$ for any sequence a_n satisfying $k_n^{1/p} (k_n^2 j_n^5 \log^3(1 + k_n j_n)/n)^{1/4} = o(a_n)$.

PROOF: Let $\mathcal{G}_n \equiv \{g: g(x) = 1\{y \leq c_j\} - p^{j_n}(w)'\beta \text{ for some } 1 \leq j \leq \mathcal{J} \text{ and } \|p^{j_n'}\beta\|_{\infty} \leq 2\}$ and $\tilde{\mathcal{F}}_n \equiv \{gq_k: g \in \mathcal{G}_n \text{ and } 1 \leq k \leq k_n\}$. Further let \mathbb{G}_P^* be a Gaussian process on $\tilde{\mathcal{F}}_n$ independent of $\{V_i\}_{i=1}^n$, satisfying $E[\mathbb{G}_P^*(f_1)] = 0$ and $E[\mathbb{G}_P^*(f_1)\mathbb{G}_P^*(f_2)] = \text{Cov}_P\{f_1, f_2\}$ for any $f_1, f_2 \in \tilde{\mathcal{F}}_n$, and for any $f \in \tilde{\mathcal{F}}_n$ define $\hat{\mathbb{G}}_n(f)$ to be given by

$$\hat{\mathbb{G}}_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ f(V_i) - \frac{1}{n} \sum_{j=1}^n f(V_j) \}$$

where $\{\omega_i\}_{i=1}^n$ are the same weights used in building $\hat{\mathbb{W}}_n$. Then note that when $R = \Theta$ and for $\mathbb{W}_P^*(\theta) \equiv (\mathbb{G}_P^*(\rho_1(\cdot,\theta)q^{k_n})', \dots \mathbb{G}_P^*(\rho_{\mathcal{J}}(\cdot,\theta)q^{k_n})')'$, we obtain

$$\sup_{\theta \in \Theta_n} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^{\star}(\theta)\|_p \lesssim k_n^{1/p} \sup_{f \in \tilde{\mathcal{F}}_n} |\hat{\mathbb{G}}_n(f) - \mathbb{G}_P^{\star}(f)|. \tag{S.135}$$

We will therefore establish the lemma by employing (S.135) and applying Theorem S.7.1(i) to the class $\tilde{\mathcal{F}}_n$. To this end, define $f^{d_n}(V)$ to be given by

$$f^{d_n}(V) \equiv g^{d_n}(V) - E_P[g^{d_n}(V)] \qquad g^{d_n}(V) \equiv q^{k_n}(Z) \otimes \begin{pmatrix} p^{j_n}(W) \\ 1\{Y \le c_1\} \\ \vdots \\ 1\{Y \le c_{\mathcal{J}}\} \end{pmatrix}$$
(S.136)

and note $d_n = k_n(j_n + \mathcal{J})$. Next observe that applying Lemma S.4.22 with $D_1 \equiv (p^{j_n}(W)', 1\{Y \leq c_1\}, \dots, 1\{Y \leq c_{\mathcal{J}}\})'$ and $D_2 = q^{k_n}(Z)$ allows us to conclude

$$\sup_{P \in \mathbf{P}} \overline{\operatorname{eig}} \{ E_P[g^{d_n}(V)g^{d_n}(V)'] \} \le \sup_{P \in \mathbf{P}} (\|\overline{\operatorname{eig}} \{D_1 D_1'\}\|_{P,\infty} \times \overline{\operatorname{eig}} \{ E_P[D_2 D_2'] \}) \lesssim j_n, \text{ (S.137)}$$

where the final inequality holds by Assumptions 4.1(ii) and 4.2(ii). Hence, since in addition $\overline{\text{eig}}\{E_P[g^{d_n}(V)]E[g^{d_n}(V)']\} \leq \overline{\text{eig}}\{E_P[g^{d_n}(V)g^{d_n}(V)']\}$, results (S.136) and (S.137) imply Assumption S.7.1(i) holds with $C_n \approx j_n$. Next note Assumption S.7.1(ii) is satisfied with $K_n \approx \sqrt{k_n j_n}$ by Assumptions 4.1(ii) and 4.2(i). By Assumptions 4.1(iii) it also follows that $\|\beta\|_2 \approx \sup_{P \in \mathbf{P}} \|p^{j_n'}\beta\|_{P,2} \leq \|p^{j_n'}\beta\|_{\infty}$. Hence, by definition of $\tilde{\mathcal{F}}_n$, there is a $C_0 < \infty$ such that any $f \in \tilde{\mathcal{F}}$ satisfies $f(V) - E_P[f(V)] = f^{d_n}(V)'\beta$ for some β in

$$\mathcal{B}_n \equiv \{ \beta \in \mathbf{R}^{d_n} : \beta = e_k \otimes \gamma \text{ for some } \gamma \in \mathbf{R}^{j_n + \mathcal{J}} \text{ with } \|\gamma\|_2 \le C_0 \},$$

where $e_k \in \mathbf{R}^{k_n}$ has its k^{th} coordinate equal to one and all other coordinates equal to zero. In particular, it follows that Assumption S.7.2(i) is immediate with $G_{n,P}$ equal to the zero function and $J_{1n} = 0$. Moreover, setting $C_n \equiv \{ \gamma \in \mathbf{R}^{j_n + \mathcal{J}} : ||\gamma||_2 \leq C_0 \}$, we can then conclude from the definition of \mathcal{B}_n and $N(\epsilon, C_n, ||\cdot||_2) \lesssim 1 \vee (C_0/\epsilon)^{j_n}$ that

$$\int_0^\infty \sqrt{\log(N(\epsilon, \mathcal{B}_n, \|\cdot\|_2))} d\epsilon$$

$$\lesssim \int_0^{C_0} \sqrt{\log(k_n) + \log(N(\epsilon, \mathcal{C}_n, \|\cdot\|_2))} d\epsilon \lesssim \sqrt{\log(k_n)} + \sqrt{j_n},$$

which verifies Assumption S.7.2(ii) is satisfied with $J_{2n} \simeq \sqrt{\log(k_n)} + \sqrt{j_n}$. Thus, applying Theorem S.7.1(i) with $K_n \simeq \sqrt{k_n, j_n}$, $C_n \simeq j_n$, $d_n \lesssim k_n j_n$, $J_{1n} = 0$, and $J_{2n} \simeq \sqrt{\log(k_n)} + \sqrt{j_n}$ implies that uniformly in $P \in \mathbf{P}$ we have

$$\sup_{f \in \hat{\mathcal{F}}_n} |\hat{\mathbb{G}}_n(f) - \mathbb{G}_P^*(f)| = O_P(\{\frac{k_n^2 j_n^5 \log^3 (1 + k_n j_n)}{n}\}^{1/4})$$
 (S.138)

provided that $j_n^3 k_n^2 \log(1 + j_n k_n) = o(n)$. Since the latter condition is satisfied by hypothesis, the claim of the lemma then follows from (S.135) and (S.138).

Lemma S.4.9. Define $\|\theta\|_{\mathbf{E}} = \sum_{j=1}^{\mathcal{J}} \sup_{P \in \mathbf{P}} \|F(c_j|\cdot)\|_{P,2}$ and for $\hat{V}_n(\theta, R| + \infty)$ as in

(34) let $\hat{V}_n(\theta, R|\ell_n) = \hat{V}_n(\theta, R|+\infty) \cap \{h : \|h/\sqrt{n}\|_{\mathbf{E}} \leq \ell_n\}$. If Assumptions 4.1, 4.2, and 4.3 hold, then for any $a_n = o(1)$ and $\ell_n = o(1)$ satisfying $k_n^4 j_n^5 \log^3(1 + k_n j_n)/n = o(a_n^4)$ and $k_n \sqrt{j_n} \log(n)/\sqrt{n} = o(\ell_n)$ it follows uniformly in $P \in \mathbf{P}_0$ that

$$\hat{U}_n(R|+\infty) = \inf_{\theta \in \hat{\Theta}_n^{\mathrm{r}}} \inf_{h \in \hat{V}_n(\theta, R|\ell_n)} \{ \sum_{j=1}^{\mathcal{J}} \|\hat{\mathbb{W}}_{j,n}(\theta) + \hat{\mathbb{D}}_{j,n}[h]\|_{\hat{\Sigma}_{j,n},2} \}^{1/2} + o_P(a_n).$$

PROOF: We establish the claim of the lemma by verifying the conditions of Lemma S.3.1. To this end, recall that in the proof of Theorem 4.1 we argued that Assumptions 3.2(i)(iii) and 3.7 hold with $B_n \simeq \sqrt{k_n}$ and $J_n \simeq \sqrt{j_n \log(1+j_n)}$. Moreover, Assumption 4.1(iii) implies that for any $(\{p^{j_n'}\beta_{j,h}\}_{j=1}^{\mathcal{J}}, \mu) = h \in \mathbf{B}_n$ we have

$$||h||_{\mathbf{E}} \lesssim \sum_{j=1}^{\mathcal{J}} ||\beta_{j,h}||_2 \lesssim \{\sum_{j=1}^{\mathcal{J}} ||\mathbb{D}_{j,P}[h]||_2^2\}^{1/2} = ||\mathbb{D}_P[h]||_2, \tag{S.139}$$

where the second inequality follows from $\mathbb{D}_{j,P}[h] = -E_P[q^{k_n}(Z)p^{j_n}(W)'\beta_{j,h}]$ and the smallest singular values of $E_P[q^{k_n}(Z)p^{j_n}(W)']$ being bounded away from zero uniformly in $P \in \mathbf{P}$ by Assumption 4.2(iii). Since $\nu_n \approx 1$ by Lemma S.4.1 and the derivative $\mathbb{D}_P(\theta)$ does not depend on θ , we conclude $\|h\|_{\mathbf{E}} \leq \nu_n \|\mathbb{D}_P[h]\|_2$ for all $h \in \mathbf{B}_n$ – i.e., in verifying the conditions of Lemma S.3.1 we may set $\mathcal{A}_n(P) = \Theta_n \cap R$. In order to verify condition (S.79) of Lemma S.3.1 we note that since $\|h\|_{\mathbf{E}} \approx \sum_{j=1}^{\mathcal{J}} \|\beta_{j,h}\|_2$ by Assumption 4.1(iii), the definitions of the operator norm $\|\cdot\|_{o,2}$, $\hat{\mathbb{D}}_{j,n}$, and $\mathbb{D}_{j,P}$ imply that

$$\sup_{h \in \mathbf{B}_n} \frac{\|\hat{\mathbb{D}}_n[h] - \mathbb{D}_P[h]\|_2}{\|h\|_{\mathbf{E}}} \lesssim \|\frac{1}{n} \sum_{i=1}^n q^{k_n}(Z_i) p^{j_n}(W_i)' - E_P[q^{k_n}(Z) p^{j_n}(W)']\|_{o,2} = o_P(1),$$

where the final equality holds uniformly in $P \in \mathbf{P}$ by applying Lemma S.4.7 with $b_{1n} = \sqrt{j_n}$, $b_{2n} = k_n$ (by Assumptions 4.1(ii) and 4.2(i)) and employing that $k_n \geq j_n$ and $k_n^2 \log(k_n)/n = o(1)$ by Assumptions 4.2(iii)(iv). Finally, we note that $j_n^5 k_n^4 \log^3(1 + j_n k_n)/n = o(a_n^4)$ by hypothesis, and employing Lemma S.4.8 with p = 2 yields that Assumption 3.11 holds for $R = \Theta$, and hence also for R as in (29). The only condition of Lemma S.3.1 that remains to be verified is that $\mathcal{S}_n(\mathbf{B}, \mathbf{E})\mathcal{R}_n = o(\ell_n)$. To this end, we observe that since $\hat{V}_n(\theta, R|\ell_n)$ is defined through the constraint $\|h\|_{\mathbf{E}} \leq \ell_n$ (instead of $\|\cdot\|_{\mathbf{B}} \leq \ell_n$), it suffices to verify $\mathcal{R}_n = o(\ell_n)$ – i.e. for the purposes of this lemma we may set $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{E}}$. However, since as argued $J_n \asymp \sqrt{j_n \log(1 + j_n)}$, $B_n = \sqrt{k_n}$, and $\nu_n \asymp 1$, we have $\mathcal{R}_n \asymp k_n \sqrt{j_n \log(1 + k_n) \log(1 + j_n)}/\sqrt{n}$, and the requirement $\mathcal{R}_n = o(\ell_n)$ is implied by $k_n \sqrt{j_n} \log(n)/\sqrt{n} = o(\ell_n)$. Thus, the claim of the lemma follow from Lemma S.3.1. \blacksquare

S.4.2 Proofs for Section A.2.1

PROOF OF THEOREM A.2.1: We establish the theorem by simply applying Theorem 3.1(ii) to both R as in (A.5) (to couple $I_n(R)$) and to $R = \Theta$ (to couple $I_n(\Theta)$). To this end, note that as discussed Assumption 3.1(ii)(iii) holds, while Assumption 3.1(i) is directly imposed in A.2.1(i). Since $q^{k_n}(Z)$ equals the vector $(1, \ldots, 1)' \in \mathbf{R}^{\mathcal{J}}$, it further follows Assumption 3.2(i) holds with $B_n = 1$, while Assumption 3.2(ii) is automatically satisfied. We further note that Assumption 3.2(iii) holds for $R = \Theta$ (and hence also for R as in (A.5)) with $J_n = C_0$ for some $C_0 < \infty$ by Assumption A.2.2(ii) and Lemma S.4.11. Also note Assumption 3.3(i) is satisfied for $R = \Theta$, and hence also for R as in (A.5), by Lemma S.4.12. Additionally, since Θ is convex by Assumption A.2.1(iii), the mean value theorem and Assumption A.2.2(ii) imply that

$$E_P[\|\rho(X,\theta_1) - \rho(X,\theta_2)\|_2^2] \le E_P[\sup_{\theta \in \Theta} \|\nabla_{\theta}\rho(X,\theta)\|_{o,2}^2] \|\theta_1 - \theta_2\|_2^2$$

for all $\theta_1, \theta_2 \in \Theta$, which verifies Assumption 3.3(ii) holds with $\kappa_{\rho} = 1$ and $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$. Lemma S.4.10 additionally verifies that Assumption 3.4 holds with $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$ and $\nu_n^{-1} = \eta$ for some $\eta > 0$ when $R = \Theta$ and hence also when R is as in (A.5). Furthermore, we note that in this problem $\mathcal{R}_n \approx n^{-1/2}$ because $\nu_n \approx 1$, $J_n = O(1)$, $k_n = \mathcal{J}$, and $B_n = 1$. To verify Assumption 3.5, note that in this application $\nabla m_{P,j}(\theta) = E_P[\nabla_{\theta}\rho_j(X,\theta)]$. Hence, Assumptions 3.5(i)(ii) hold with $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_2$ due to $E_P[\sup_{\theta \in \Theta} \|\nabla_{\theta}^2 \rho_j(X,\theta)\|_{o,2}]$ being bounded in $P \in \mathbf{P}$ by Assumption A.2.2(ii). Similarly, Assumption 3.5(iii) is satisfied due to $E_P[\sup_{\theta \in \Theta} \|\nabla_{\theta} \rho(X,\theta)\|_{o,2}]$ being bounded by Assumption A.2.2(ii). Finally, we note that since $\mathcal{R}_n \approx n^{-1/2}$ and $\kappa_{\rho} = 1$, Lemma S.4.11 verifies Assumption 3.6(i). Assumption 3.6(ii) is immediate since $E_P[\rho(X,\theta_0)] = 0$, while Assumption 3.7 holds by Assumption A.2.4. To conclude, simply note that the condition $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_{\rho}}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ is implied by $\ell_n \sqrt{\log(1/\ell_n)} = o(a_n)$ by Lemma S.4.11, and $K_m \mathcal{R}_n^2 = o(a_n/\sqrt{n})$ is implied by $n^{-1/2} = o(a_n)$.

PROOF OF THEOREM A.2.2: We first define a variable $\hat{E}_n(R|\ell_n)$ to be given by

$$\hat{E}_n(R|\ell_n) \equiv \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n(\hat{\theta}_n)[h]\|_{\hat{\Sigma}_n, 2}$$

and note Lemma S.4.15 implies $\hat{U}_n(R|+\infty) = \hat{E}_n(R|\ell_n) + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$ for any $\ell_n \downarrow 0$ satisfying the conditions of the theorem. Therefore, to establish the theorem it suffices to show that uniformly in $P \in \mathbf{P}_0$ we have

$$\hat{E}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n)$$

$$\hat{E}_n(R|\ell_n) - \hat{U}_n(\Theta| + \infty) \ge U_P^{\star}(R|\tilde{\ell}_n) - U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n).$$

with $\ell_n \simeq \tilde{\ell}_n$ and $\tilde{\ell}_n^{\rm u}$ satisfying the conditions of the theorem. To this end we rely on The-

orem 3.2 (for $\hat{E}_n(R|\ell_n)$) and Lemma S.3.7. Next note that in the proof of Theorem A.2.1 we established that Assumptions A.2.1, A.2.2, A.2.3, and A.2.4, imply Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, and 3.7 hold with $\mathcal{R}_n \times n^{-1/2}$, $\nu_n \times 1$, $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{E}} = \|\cdot\|_{\mathbf{L}} = \|\cdot\|_{2}$, $\kappa_{\rho} = 1$, and $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d_{\theta}}}$ for $R = \Theta$ and R as in (A.5). We thus avoid repeating the arguments, and verify only that Assumptions 3.8, 3.9, 3.10, 3.11, 3.12, and 3.13 hold for $R = \Theta$ and R as in (A.5).

Next note Lemma S.4.14 implies Assumptions 3.8, 3.9, and 3.10 are satisfied, while Lemma S.4.13 verifies Assumption 3.11 with $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d_{\theta}}}$ for $R = \Theta$, and hence also for R as in (A.5). Assumption 3.12(i) is immediate since $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_{\mathbf{B}} = \|\cdot\|_{2}$, while Assumptions 3.12(ii)(iii) are implied by Assumption A.2.5(i), $\|\hat{\theta}_n - \theta_0\|_2 = o_P(1)$ uniformly in $P \in \mathbf{P}_0$ (which we showed in establishing Theorem A.2.1), and $\mathcal{V}_n(P) \equiv \{\theta \in \Theta : \|\theta - \theta_0\|_2 \le \epsilon\}$ for some $\epsilon > 0$ by Lemma S.4.10. Assumption 3.13(i) is immediate since $\mathcal{S}_n(\mathbf{B}, \mathbf{E}) = 1$ and the choices of $\hat{\theta}_n$ and $\hat{\theta}_n^u$ correspond to setting $\tau_n = o(n^{-1/2})$. Similarly, Lemma S.4.11, $\mathcal{S}_n(\mathbf{L}, \mathbf{E}) = 1$, and $n^{-1/2} = o(\ell_n)$ imply that the condition $\ell_n^2 \sqrt{\log(1/\ell_n)} = o(a_n n^{-\frac{1}{2}})$ verifies Assumption 3.13(ii). Moreover, since $\ell_n = o(r_n)$ and $n^{-1/2} = o(r_n)$ Assumption 3.13(iii) holds. Hence, Theorem 3.2 implies

$$\hat{E}_n(R|\ell_n) \ge U_P^*(R|\tilde{\ell}_n) + o_P(a_n) \tag{S.140}$$

uniformly in $P \in \mathbf{P}_0$ for some $\ell_n \simeq \tilde{\ell}_n$. Similarly, since $\mathcal{R}_n^{\mathrm{u}} \simeq n^{-1/2}$, the conditions of Lemma S.3.7(ii) are immediate and hence by (S.140) there are $\ell_n \simeq \tilde{\ell}_n$ and $\ell_n^{\mathrm{u}} \simeq \tilde{\ell}_n^{\mathrm{u}}$ with

$$\hat{E}_n(R|\ell_n) - \hat{U}_n(\Theta| + \infty) \ge U_P^{\star}(R|\tilde{\ell}_n) - U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n). \tag{S.141}$$

The theorem therefore follows from (S.140), (S.141) and Lemma S.4.15.

Lemma S.4.10. If Assumptions A.2.1, A.2.2, A.2.3, and A.2.4(ii) hold, then Assumption 3.4 is satisfied with $R = \Theta$ and R as in (A.5), $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$, $\nu_n^{-1} = \eta$ for some $\eta > 0$, and $\mathcal{V}_n(P) \equiv \{\theta \in \Theta : \|\theta - \theta_0\|_2 \le \epsilon\}$ for some $\epsilon > 0$.

PROOF: To verify Assumption 3.4(ii), note Assumptions A.2.1(i), A.2.2(ii), A.2.4, and A.2.3(i) and Lemma S.4.11 allow us to apply Lemma S.1.1(i) with $\|\cdot\|_{\mathbf{A}} = \|\cdot\|_{2}$, $J_n = O(1)$ and $S_n(\epsilon) > 0$ to conclude $\hat{\theta}_n \in \mathcal{V}_n(P) \equiv \{\theta \in \Theta_n : \|\theta - \theta_0\|_2 \le \epsilon\}$ with probability tending to one uniformly in $P \in \mathbf{P}_0$ for any $\epsilon > 0$ and for both $R = \Theta$ and R as in (A.5). In order to verify Assumption 3.4(i), next note that Θ being convex and Assumption A.2.2(ii) imply that for some $C_0 < \infty$ we have

$$||E_P[\rho(X,\theta)] - E_P[\rho(X,\theta_0)] - E_P[\nabla_{\theta}\rho(X,\theta_0)](\theta - \theta_0)||_2 \le C_0||\theta - \theta_0||_2^2$$

for all $\theta \in \Theta$. Hence, since the smallest singular value of $E_P[\nabla_{\theta}\rho(X,\theta_0)]$ is bounded away

from zero uniformly in $P \in \mathbf{P}_0$ by Assumption A.2.3(ii), we obtain for some $C_1 < \infty$

$$\|\theta - \theta_0\|_2 \le C_1 \|E_P[\nabla_\theta \rho(X, \theta_0)](\theta - \theta_0)\|_2$$

$$\le C_1 \{\|E_P[\rho(X, \theta)] - E_P[\rho(X, \theta_0)]\|_2 + C_0 \|\theta - \theta_0\|_2^2\}$$
 (S.142)

for all $\theta \in \Theta$ and $P \in \mathbf{P}_0$. Therefore, provided $\epsilon > 0$ is set sufficiently small in defining $\mathcal{V}_n(P) \equiv \{\theta \in \Theta_n : \|\theta - \theta_0\|_2 \le \epsilon\}$, it follows that Assumption 3.4(i) holds with $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$ and $\nu_n^{-1} = \eta$ for some $\eta > 0$ due to (S.142) and Assumption A.2.4(ii).

Lemma S.4.11. Let $\mathcal{F} \equiv \{\rho_{\jmath}(\cdot,\theta) : \text{ for some } \theta \in \Theta \text{ and } 1 \leq \jmath \leq \mathcal{J}\}$. If Assumptions A.2.1(iii) and A.2.2 hold, then it follows that $\sup_{P \in \mathbf{P}} N_{[]}(\epsilon,\mathcal{F},\|\cdot\|_{P,2}) \lesssim 1 \vee \epsilon^{-d_{\theta}}$ and $\sup_{P \in \mathbf{P}} J_{[]}(\epsilon,\mathcal{F},\|\cdot\|_{P,2}) \lesssim \epsilon(1+\sqrt{\log(1\vee\epsilon^{-1})})$.

PROOF: Since Θ is convex by Assumption A.2.1(iii), the mean value theorem and Assumption A.2.2(i) imply for any $\theta_1, \theta_2 \in \Theta$ and $1 \leq j \leq \mathcal{J}$ that

$$|\rho_j(x,\theta_1) - \rho_j(x,\theta_2)| \le \sup_{\theta \in \Theta} \|\nabla_\theta \rho(x,\theta)\|_{o,2} \|\theta_1 - \theta_2\|_2.$$
 (S.143)

Setting $D(x) \equiv \sup_{\theta \in \Theta} \|\nabla_{\theta} \rho(x, \theta)\|_{o, 2}$, then note that Theorem 2.7.11 in van der Vaart and Wellner (1996) and the right hand side of (S.143) not depending on j imply

$$N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|_{P,2}) \le \mathcal{J} \times N(\frac{\epsilon}{2\|D\|_{P,2}}, \Theta, \|\cdot\|_2) \lesssim 1 \vee \epsilon^{-d_{\theta}}, \tag{S.144}$$

where we employed that $N(\epsilon, \Theta, \|\cdot\|_2) \lesssim 1 \vee \epsilon^{-d_{\theta}}$ due to Θ being bounded by Assumption A.2.1(iii) and $\sup_{P \in \mathbf{P}} \|D\|_{P,2} < \infty$ by Assumption A.2.2(ii).

For the second claim of the Lemma we employ the bound in (S.144) to obtain

$$\sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}, \|\cdot\|_{P,2}) \lesssim \int_0^{\epsilon} (1 + \log(1 \vee u^{-d_{\theta}}))^{1/2} du$$

$$= \epsilon \int_0^1 (1 + \log(1 \vee (\epsilon v)^{-d_{\theta}}))^{1/2} dv \lesssim \epsilon (1 + \sqrt{\log(1 \vee \epsilon^{-1})}),$$

where the first equality follows from the change of variables $v = u/\epsilon$ and the final inequality is implied by the inequality $1 \vee (ab) \leq (1 \vee a)(1 \vee b)$.

Lemma S.4.12. If Assumptions A.2.1(i)(iii) and A.2.2 hold, then it follows that Assumption 3.3(i) is satisfied with $R = \Theta$ and $a_n = \sqrt{\log(n)}/n^{\frac{1}{6+5d_\theta}}$.

PROOF: Let $\epsilon_n = \sqrt{\log(n)}/n^{\frac{1}{6+5d_{\theta}}}$ and set $\delta_n \equiv 1 \wedge (\epsilon_n^2 \sqrt{n})^{-\frac{2}{2+5d_{\theta}}}$, which note satisfies $1 \geq \delta_n = o(1)$. Further define $N_n \equiv N(\delta_n, \Theta, \|\cdot\|_2)$ and set $\{\theta_k\}_{k=1}^{N_n}$ to be the center of the N_n balls covering Θ . For notational simplicity, we also let

$$r_{n,P}(x) \equiv ((\rho(x,\theta_1) - E_P[\rho(X,\theta_1)])', \dots, (\rho(x,\theta_{N_n}) - E_P[\rho(X,\theta_{N_n})])')'$$

and note $r_{n,P}(x) \in \mathbf{R}^{\mathcal{I}N_n}$. For any $P \in \mathbf{P}$ and $\eta > 0$ further define $C_{n,P}(\eta)$ to equal

$$C_{n,P}(\eta) \equiv \frac{(\mathcal{J}N_n)E_P[\|r_{n,P}(X)\|_2^3]}{\eta^3 \epsilon_n^3 \sqrt{n}}.$$
 (S.145)

It then follows by Yurinskii's coupling (see, e.g., Theorem 10.10 in Pollard (2002)) that there exists a Gaussian vector $\mathbb{N}_{n,P} \in \mathbf{R}^{\mathcal{J}N_n}$ and universal constant K_0 such that

$$P(\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}r_{n,P}(X_i) - \mathbb{N}_{n,P}\|_2 > 3\eta\epsilon_n) \le K_0C_{n,P}(\eta)(1 + \frac{|\log(1/C_{n,P}(\eta))|}{\mathcal{J}N_n}). \quad (S.146)$$

Next note Assumption A.2.2(ii), Jensen's inequality, and the convexity of $u \mapsto |u|^{\frac{3}{2}}$ yield

$$\sup_{P \in \mathbf{P}} E_P[\|r_{n,P}(X)\|_2^3] \lesssim (\mathcal{J}N_n)^{\frac{3}{2}} \times \sup_{P \in \mathbf{P}} \frac{1}{\mathcal{J}N_n} \sum_{k=1}^{N_n} \sum_{j=1}^{\mathcal{J}} E_P[|\rho_j(X,\theta_k)|^3] \lesssim N_n^{\frac{3}{2}}. \quad (S.147)$$

In particular, since $N(\epsilon, \Theta, \|\cdot\|_2) \lesssim 1 \vee \epsilon^{-d_{\theta}}$, it follows from $\delta_n \leq 1$ that $N_n \lesssim \delta_n^{-d_{\theta}}$, and hence by (S.147) and the definition of $C_{n,P}(\eta)$ in (S.145) we obtain

$$\sup_{P \in \mathbf{P}} C_{n,P}(\eta) \lesssim \frac{N_n^{\frac{5}{2}}}{\eta^3 \epsilon_n^3 \sqrt{n}} \lesssim \frac{1}{\eta^3 \epsilon_n^3 (n \delta_n^{5d_\theta})^{\frac{1}{2}}}.$$
 (S.148)

Moreover, since the function $u \mapsto u(1+|\log(1/u)|/A)$ with $A \ge 1$ is increasing in u on the interval (0,1] and $\epsilon_n^3(n\delta_n^{5d\theta})^{\frac{1}{2}} \to \infty$, we obtain from results (S.146) and (S.148) that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}} P(\|\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (r_{n,P}(X_i) - \mathbb{N}_{n,P}\|_2 > 3\eta \epsilon_n)
\lesssim \limsup_{n \to \infty} \frac{1}{\eta^3 \epsilon_n^3 (n \delta_n^{5d_\theta})^{\frac{1}{2}}} (1 + \frac{|\log(\eta^3 \epsilon_n^3 (n \delta_n^{5d_\theta})^{\frac{1}{2}})|}{\mathcal{J} N_n}) = 0, \quad (S.149)$$

where the final result follows by direct calculation. Letting $\mathbb{S}_{n,P}$ denote the linear span of $r_{n,P}$ in L_P^2 we then employ $\mathbb{N}_{n,P}$ to define a Gaussian process $\mathbb{G}_P^{(1)}$ on $\mathbb{S}_{n,P}$ by setting

$$\mathbb{G}_P^{(1)}(\sum_{k=1}^{N_n} \lambda_k' \rho(\cdot, \theta_k)) \equiv (\lambda_1', \dots, \lambda_{N_n}') \mathbb{N}_{n,P}$$
 (S.150)

for any $\{\lambda_k\}_{k=1}^{N_n}$ with $\lambda_k \in \mathbf{R}^{\mathcal{J}}$. Letting $\operatorname{Proj}\{f|\mathbb{S}_{n,P}\}$ denote the projection of f onto $\mathbb{S}_{n,P}$ under $\|\cdot\|_{P,2}$, and assuming the probability space is suitably large to carry an isonormal process $\mathbb{G}_P^{(2)}$ on $\{(f-\int fdP) - \operatorname{Proj}\{f-\int fdP|\mathbb{S}_{n,P}\} : f \in \mathcal{F}\}$ that is independent of $\mathbb{G}_P^{(1)}$, we then define the isonormal process \mathbb{G}_P to be given by

$$\mathbb{G}_P(f) \equiv \mathbb{G}_P^{(1)}(\operatorname{Proj}\{f|\mathbb{S}_{n,P}\}) + \mathbb{G}_P^{(2)}(f - \operatorname{Proj}\{f|\mathbb{S}_{n,P}\}). \tag{S.151}$$

Next let $\Pi_n \theta$ denote the projection of any $\theta \in \Theta$ onto $\{\theta_k\}_{k=1}^{N_n}$ under $\|\cdot\|_2$ and define

$$\mathcal{G}_{n,P} \equiv \{ (\rho_{\jmath}(\cdot,\theta) - \rho_{\jmath}(\cdot,\Pi_n\theta)) - E_P[(\rho_{\jmath}(X,\theta) - \rho_{\jmath}(X,\Pi_n\theta))] : \theta \in \Theta, 1 \le \jmath \le \mathcal{J} \}.$$
 (S.152)

By the mean value theorem, Θ being convex by Assumption A.2.1(iii), and $\|\theta - \Pi_n \theta\|_2 \le \delta_n$ for every $\theta \in \Theta$ due to δ_n -balls around $\{\theta_k\}_{k=1}^{N_n}$ covering Θ , it follows that

$$\sup_{\theta \in \Theta} |(\rho_{\jmath}(x,\theta) - \rho_{\jmath}(x,\Pi_{n}\theta)) - E_{P}[(\rho_{\jmath}(X,\theta) - \rho_{\jmath}(X,\Pi_{n}\theta))]|$$

$$\leq \{\sup_{\theta \in \Theta} \|\nabla_{\theta}\rho(x,\theta)\|_{o,2} + \sup_{P \in \mathbf{P}} E_{P}[\sup_{\theta \in \Theta} \|\nabla_{\theta}\rho(X,\theta)\|_{o,2}]\} \times \delta_{n}.$$

Hence, setting $G(x) \equiv 1 \vee \{\sup_{\theta \in \Theta} \|\nabla_{\theta} \rho(x, \theta)\|_{o, 2} + \sup_{P \in \mathbf{P}} E_P[\sup_{\theta \in \Theta} \|\nabla_{\theta} \rho(X, \theta)\|_{o, 2}] \}$ it follows that $G\delta_n$ is an envelope for $\mathcal{G}_{n, P}$, which by Assumption A.2.2(ii) satisfies $\sup_{P \in \mathbf{P}} \|G\delta_n\|_{P, 2} \lesssim \delta_n$. Further note that if $[f_l, f_u]$ is a bracket containing a function f, then $[f_l - E_P[f_u(X)], f_u - E_P[f_l(X)]]$ contains $f - E_P[f(X)]$ and satisfies

$$||f_u - f_l - E_P[f_l(X) - f_u(X)]||_{P,2} \le 2||f_u - f_l||_{P,2}$$

by Jensen's inequality and the triangle inequality. Therefore, Lemma S.4.11 implies

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}) \lesssim N_n \times (1 \vee \epsilon^{-d_{\theta}}),$$

and hence Theorem 2.14.2 in van der Vaart and Wellner (1996) together with $\mathcal{G}_{n,P}$ having envelope $\delta_n G$ with $G \geq 1$, $\sup_{P \in \mathbf{P}} \|G\|_{P,2} < \infty$, and $N_n \lesssim \delta_n^{-d_{\theta}}$ yield

$$\sup_{P \in \mathbf{P}} E_{P}[\sup_{g \in \mathcal{G}_{n,P}} | \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g(X_{i}) - E_{P}[g(X)]) |]$$

$$\lesssim \sup_{P \in \mathbf{P}} \{ \delta_{n} || G ||_{P,2} \int_{0}^{1} (1 + \log N_{[]}(\epsilon \delta_{n} || G ||_{P,2}, \mathcal{G}_{n,P}, || \cdot ||_{P,2}))^{\frac{1}{2}} d\epsilon \}$$

$$\lesssim \delta_{n} \int_{0}^{1} (1 + \log(N_{n}) + \log(1 \vee (\epsilon \delta_{n})^{-d_{\theta}}))^{1/2} d\epsilon$$

$$\lesssim \delta_{n} (1 + \log(\delta_{n}^{-d_{\theta}}))^{1/2}.$$
(S.153)

Therefore, the definitions of δ_n and ϵ_n , result (S.153) and Markov's inequality imply

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}} P(\sup_{g \in \mathcal{G}_{n,P}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g(X_i) - E_P[g(X)]) \right| > \eta \epsilon_n)$$

$$\lesssim \limsup_{n \to \infty} \frac{\delta_n (1 + \log(\delta_n^{-d_\theta}))^{1/2}}{\eta \epsilon_n} = 0. \quad (S.154)$$

Similarly, since \mathbb{G}_P is Gaussian and $0 \in \mathcal{G}_{n,P}$, Corollary 2.2.8 in van der Vaart and Wellner

(1996) and packing numbers being bounded by bracketing numbers imply

$$\sup_{P \in \mathbf{P}} E_{P} \left[\sup_{g \in \mathcal{G}_{n,P}} |\mathbb{G}_{P}(g)| \right] \lesssim \sup_{P \in \mathbf{P}} \int_{0}^{\infty} (\log N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}))^{\frac{1}{2}} d\epsilon
\lesssim \sup_{P \in \mathbf{P}} \int_{0}^{2\delta_{n} \|G\|_{P,2}} (\log N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}))^{\frac{1}{2}} d\epsilon \lesssim \delta_{n} (1 + \log(\delta_{n}^{-d_{\theta}}))^{1/2}, \quad (S.155)$$

where in the second inequality we employed that the bracket $[-\delta_n G, \delta_n G]$ covers $\mathcal{G}_{n,P}$ due to $\delta_n G$ being an envelope for $\mathcal{G}_{n,P}$, and the final inequality follows from the change of variables $u = \epsilon/(2\delta_n ||G||_{P,2})$ and the same manipulations as in (S.153). Hence,

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}} P(\sup_{g \in \mathcal{G}_{n,P}} |\mathbb{G}_P(g)| > \eta \epsilon_n) \lesssim \limsup_{n \to \infty} \frac{\delta_n (1 + \log(\delta_n^{-d_\theta}))^{1/2}}{\eta \epsilon_n} = 0, \quad (S.156)$$

by result (S.155) and Markov's inequality. To conclude, for any $\theta \in \Theta$ set $\mathbb{W}_{P}(\theta)$ to be

$$\mathbb{W}_P(\theta) \equiv (\mathbb{G}_P(\rho_1(\cdot,\theta)), \dots, \mathbb{G}_P(\rho_{\mathcal{J}}(\cdot,\theta))'$$

and note that the definitions of \mathbb{G}_P in (S.150) and (S.151), and of $\mathcal{G}_{n,P}$ in (S.152), yield

$$\sup_{\theta \in \Theta} \|\mathbb{G}_{n}(\theta) - \mathbb{W}_{P}(\theta)\|_{2} \leq \|\frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_{n,P}(X_{i}) - \mathbb{N}_{n,P}\|_{2}
+ \sup_{g \in \mathcal{G}_{n,P}} \sqrt{\mathcal{J}} |\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g(X_{i}) - E_{P}[g(X)])| + \sup_{g \in \mathcal{G}_{n,P}} \sqrt{\mathcal{J}} |\mathbb{G}_{P}(g)|.$$

Thus the lemma follows from (S.149), (S.154), and (S.156).

Lemma S.4.13. If Assumptions A.2.1(i)(iii) and A.2.2 hold, then it follows that Assumption 3.11 is satisfied with $R = \Theta$ and $a_n = \log^{3/4}(n)/n^{\frac{1}{12+2d_{\theta}}}$.

PROOF: We establish the lemma by relying on Theorem S.7.1(i) in Section S.7. To this end set $\zeta_n = n^{-\frac{1}{2(6+d_\theta)}}$, $M_n = n^{\frac{1}{6+d_\theta}}$, and $N_n \equiv N(\zeta_n, \Theta, \|\cdot\|_2)$. By Assumption A.2.2(ii) the function $F(x) \equiv (1 + \sup_{\theta \in \Theta} \|\rho(x, \theta)\|_2)$ is integrable, and for any $\theta \in \Theta$ we let

$$\tilde{\rho}(x,\theta) \equiv (\rho_1(x,\theta)1\{F(x) \le M_n\}, \dots, \rho_{\mathcal{I}}(x,\theta)1\{F(x) \le M_n\})'.$$

Defining $d_n = \mathcal{J}N_n$ and $\{\theta_k\}_{k=1}^{N_n}$ to be the centers of the ζ_n -balls covering Θ we then let

$$f_n^{d_n}(X) \equiv (\tilde{\rho}(X, \theta_1)' - E_P[\tilde{\rho}(X, \theta_1)'], \dots, \tilde{\rho}(X, \theta_{N_n})' - E_P[\tilde{\rho}(X, \theta_{N_n})'])'.$$

Next note that since each entry of the matrix $f_n^{d_n}(X)f_n^{d_n}(X)'$ is almost surely bounded by $2M_n^2$ it follows that $||E_P[f_n^{d_n}(X)f_n^{d_n}(X)']||_{o,2} \leq 2d_nM_n^2$, and hence Assumption S.7.1 in Section S.7 holds with $C_n \approx d_nM_n^2$ and $K_n \approx M_n$. For every $\theta \in \Theta$ let $\Pi_n\theta$ denote its projection (under $\|\cdot\|_2$) onto $\{\theta_k\}_{k=1}^{N_n}$ and define the class $\mathcal{G}_{n,P} \equiv \{(\rho_j(\cdot,\theta) - \tilde{\rho}_j(\cdot,\Pi_n\theta)) - \tilde{\rho}_j(\cdot,\Pi_n\theta)\}$

$$E_P[\rho_j(X,\theta) - \tilde{\rho}_j(X,\Pi_n\theta)] : \theta \in \Theta \text{ and } 1 \leq j \leq \mathcal{J}\}.$$
 Further observe that

$$\sup_{g \in \mathcal{G}_{n,P}} |g(x)|
\leq \max_{1 \leq j \leq \mathcal{J}} \sup_{\theta \in \Theta} 2|\rho_{j}(x,\theta) - \rho_{j}(x,\Pi_{n}\theta)| + F(x)1\{F(x) > M_{n}\} + E_{P}[F(X)1\{F(X) > M_{n}\}]
\leq \sup_{\theta \in \Theta} 2\|\nabla_{\theta}\rho(x,\theta)\|_{o,2}\|\theta - \Pi_{n}\theta\|_{2} + F(x)1\{F(x) > M_{n}\} + E_{P}[F(X)1\{F(X) > M_{n}\}]$$
(S.157)

where in the second inequality we employed the mean value theorem and Θ being convex by Assumption A.2.1(iii). In particular, since the ζ_n -balls centered around $\{\theta_k\}_{k=1}^{N_n}$ cover Θ and $\zeta_n \leq 1$, result (S.157) implies that the function

$$G(x) \equiv 2 \sup_{\theta \in \Theta} \|\nabla_{\theta} \rho(x, \theta)\|_{o, 2} + F(x) + \sup_{P \in \mathbf{P}} E_{P}[F(X)]$$

is an envelope for $\mathcal{G}_{n,P}$, while Assumption A.2.2(ii) implies $\sup_{P\in\mathbf{P}} E_P[G^2(X)] < \infty$. Moreover, result (S.157) and Markov's, Jensen's, and Holder's inequalities yield that

$$\sup_{g \in \mathcal{G}_{n,P}} \|g\|_{P,2} \le \zeta_n \|G\|_{P,2} + 2\{E_P[F^3(X)]\}^{\frac{2}{3}} \{P(F(X) > M_n)\}^{\frac{1}{3}}$$

$$\le (\zeta_n + M_n^{-1/2} \times 2 \sup_{P \in \mathbf{P}} (E_P[F^3(X)])^{1/2}) \times \|G\|_{P,2}, \tag{S.158}$$

where in the final equality we employed that $||G||_{P,2} \ge 1$ because $F(X) \ge 1$. Thus, by result (S.158) and Assumption A.2.2(ii), we may set $\delta_n \equiv C(\zeta_n + M_n^{-1/2})$ and obtain $||g||_{P,2} \le \delta_n ||G||_{P,2}$ for all $g \in \mathcal{G}_{n,P}$ and $P \in \mathbf{P}$ provided C is chosen large enough. Next note that since Θ being bounded by Assumption A.2.1(iii) implies $N_n \lesssim \zeta_n^{-d_\theta}$, we obtain

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}) \lesssim N_n \times (1 \vee \epsilon)^{-d_{\theta}} \lesssim \zeta_n^{-d_{\theta}} \times (1 \vee \epsilon)^{-d_{\theta}}$$
 (S.159)

due to Lemma S.4.11. Hence, the change of variables $u = \epsilon/(\delta_n ||G||_{P,2})$ implies that

$$\sup_{P \in \mathbf{P}} \int_{0}^{\delta_{n} \|G\|_{P,2}} (1 + \log N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}))^{1/2} d\epsilon$$

$$\lesssim \sup_{P \in \mathbf{P}} \delta_{n} \|G\|_{P,2} \int_{0}^{1} (1 + \log(\zeta_{n}^{-d_{\theta}}) + \log(1 \vee (u\delta_{n} \|G\|_{P,2})^{-d_{\theta}}))^{1/2} du$$

$$\lesssim \delta_{n} (1 + \log(\zeta_{n}^{-1}))^{1/2} \tag{S.160}$$

where in the inequalities we employed result (S.159), $\zeta_n \lesssim \delta_n$, and $\sup_{P \in \mathbf{P}} ||G||_{P,2} < \infty$. In particular, results (S.159) and (S.160) together with Lemma S.7.3 imply that Assumption S.7.2(i) in Section S.7 is satisfied with $J_{1n} \lesssim \delta_n (1 + \log(\zeta_n^{-1}))^{1/2}$. Similarly, note that in this application, the set \mathcal{B}_n in Assumption S.7.2(ii) consists of $0 \in \mathbf{R}^{d_n}$ and the set of vectors in \mathbf{R}^{d_n} with one coordinate equal to one and all other coordinates equal

to zero. Thus, Assumption S.7.2(ii) holds with $J_{2n} = (\log(1+d_n))^{1/2} \lesssim (1+\log(\zeta_n^{-1}))^{1/2}$.

We have so far verified Assumptions S.7.1 and S.7.2 in Section S.7 hold with $d_n \lesssim \zeta_n^{-d_\theta}$, $K_n = M_n$, $C_n \lesssim M_n^2 \zeta_n^{-d_\theta}$, $J_{1n} \lesssim \delta_n (1 + \log(\zeta_n^{-1}))^{1/2}$, and $J_{2n} \lesssim (1 + \log(\zeta_n^{-1}))^{1/2}$. Since we had set $\zeta_n = n^{-1/(2(6+d_\theta))}$, $M_n = n^{1/(6+d_\theta)}$, and $\delta_n = C(\zeta_n + M_n^{-1/2})$ the requirement that $d_n \log(1 + d_n) K_n^2 = o(n)$ imposed by Theorem S.7.1(i) holds as well. Therefore, Assumption A.2.1(i) and Theorem S.7.1(i) finally enable us to conclude that there exists a process \mathbb{W}_p^* that is independent of the data $\{X_i\}_{i=1}^n$ and such that

$$\sup_{\theta \in \Theta} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^*(\theta)\|_2 = O_P(\log^{3/4}(n)n^{-1/(12+2d_\theta)})$$

uniformly in $P \in \mathbf{P}$, which conclude the proof of the lemma.

Lemma S.4.14. If Assumption A.2.1(ii), A.2.5(ii)-(vi), and A.2.6 hold, then it follows that Assumptions 3.8, 3.9, and 3.10 are satisfied.

PROOF: Recall that in this setting $\mathbf{G} = \mathbf{R}^{d_G}$ and $\|\cdot\|_{\mathbf{G}} = \|\cdot\|_{\infty}$. For ϵ and B^{ϵ} as in Assumption A.2.5, let $N_{\epsilon}(\theta_0) \equiv \{\theta \in \Theta : \|\theta - \theta_0\|_2 \le \epsilon\}$ noting that $N_{\epsilon}(\theta_0) \subseteq B^{\epsilon}$ and that $N_{\epsilon}(\theta_0)$ implicitly depends on P through θ_0 (which depends on P through (A.4)). For any $\theta_1, \theta_2 \in N_{\epsilon}(\theta_0)$, $N_{\epsilon}(\theta_0) \subseteq B^{\epsilon}$ and Proposition 7.3.3 in Luenberger (1969) imply

$$\|\Upsilon_{G}(\theta_{1}) - \Upsilon_{G}(\theta_{2}) - \nabla\Upsilon_{G}(\theta_{1})[\theta_{1} - \theta_{2}]\|_{\mathbf{G}} \leq \{ \sup_{\theta \in B^{\epsilon}} \max_{1 \leq j \leq d_{G}} \|\nabla^{2}\Upsilon_{G,j}(\theta)\|_{o,2} \} \frac{\|\theta_{1} - \theta_{2}\|_{2}^{2}}{2}.$$
(S.161)

Similarly, for any $\theta_1, \theta_2 \in N_{\epsilon}(\theta_0)$, Proposition 7.3.2 in Luenberger (1969) yields

$$\|\nabla \Upsilon_{G}(\theta_{1}) - \nabla \Upsilon_{G}(\theta_{2})\|_{o} = \sup_{\|h\|_{2}=1} \max_{1 \leq j \leq d_{G}} |(\nabla \Upsilon_{G,j}(\theta_{1}) - \nabla \Upsilon_{G,j}(\theta_{2}))[h]|$$

$$\leq \{ \sup_{\theta \in B^{\epsilon}} \max_{1 \leq j \leq d_{G}} \|\nabla^{2} \Upsilon_{G,j}(\theta)\|_{o,2} \} \|\theta_{1} - \theta_{2}\|_{2}.$$
 (S.162)

Since $\|\nabla^2 \Upsilon_{G,j}(\theta)\|_{o,2}$ is uniformly bounded on B^{ϵ} by Assumption A.2.5(v), it follows from results (S.161) and (S.162) that Assumptions 3.8(i)(ii) are satisfied with

$$K_g \equiv \sup_{\theta \in B^{\epsilon}} \max_{1 \le j \le d_G} \|\nabla^2 \Upsilon_{G,j}(\theta)\|_{o,2}.$$

Assumption A.2.5(iii) additionally implies $\sup_{\theta \in B^{\epsilon}} \|\nabla \Upsilon_G(\theta)\|_{o,2} < \infty$, and hence verifies Assumption 3.8(iii). By identical arguments, but recalling $\mathbf{F} = \mathbf{R}^{d_F}$ and $\|\cdot\|_{\mathbf{F}} = \|\cdot\|_2$, it follows Assumptions A.2.5(iii)-(iv) imply Assumptions 3.9(i)-(iii) hold with

$$K_f \equiv \sqrt{d_F} \sup_{\theta \in B^{\epsilon}} \max_{1 \le j \le d_F} \|\nabla^2 \Upsilon_{F,j}(\theta)\|_{o,2}. \tag{S.163}$$

To conclude, note that since Assumption A.2.5(vi) implies the range of $\nabla \Upsilon_F(\theta)$ equals \mathbf{R}^{d_F} for all $\theta \in B^{\epsilon}$, it follows that $\nabla \Upsilon_F(\theta)$ admits a right inverse. Moreover,

if Υ_F is affine, then $K_f = 0$ and hence Assumption 3.9(iv) holds. On the other hand, if Υ_F is nonlinear, then note $\nabla \Upsilon_F(\theta)^- = \nabla \Upsilon_F(\theta)'(\nabla \Upsilon_F(\theta) \nabla \Upsilon_F(\theta)')^{-1}$ and therefore $\|\nabla \Upsilon_F(\theta)^-\|_{o,2}$ is bounded for all $\theta \in B^\epsilon$ due to $\|\nabla \Upsilon_F(\theta)\|_{o,2}$ being bounded on B^ϵ by Assumption A.2.5(ii), and the smallest singular value of $\nabla \Upsilon_F(\theta)'$ being bounded away from zero on B^ϵ by Assumption A.2.6(ii). It follows Assumption 3.9(iv) holds as well. Since Assumption A.2.6 directly implies Assumption 3.10, the lemma follows.

Lemma S.4.15. Let Assumptions A.2.1, A.2.2, A.2.3, and A.2.4 hold, and set $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d_{\theta}}}$. For any ℓ_n with $n^{-1/2} = o(\ell_n)$, it follows uniformly in $P \in \mathbf{P}_0$ that

$$\hat{U}_n(R|+\infty) = \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n(\hat{\theta}_n)[h]\|_{\hat{\Sigma}_n, 2} + o_P(a_n).$$

PROOF: We establish the lemma by relying on Lemma S.3.1. To this end note that in the proof of Theorem A.2.1, Assumptions 3.1(i), 3.2(i)(iii), and 3.7 were verified and $\hat{\theta}_n$ was shown to be consistent for θ_0 uniformly in $P \in \mathbf{P}_0$ with $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$, $\mathcal{R}_n = n^{-1/2}$, and $\nu_n \approx 1$ for both R as in (A.5). Next, note Lemma S.4.13 verifies Assumption 3.11 holds with $a_n = \sqrt{\log(n)}/n^{\frac{1}{10+5d\theta}}$ for $R = \Theta$ and hence also for R as in (A.5). Moreover, the mean value theorem and Θ being convex imply that

$$\left|\frac{\partial}{\partial \theta_k} \rho_{\jmath}(x, \theta_1) - \frac{\partial}{\partial \theta_k} \rho_{\jmath}(x, \theta_2)\right| \le \max_{1 \le \jmath \le \mathcal{J}} \sup_{\theta \in \Theta} \|\nabla_{\theta}^2 \rho_{\jmath}(x, \theta)\|_{o, 2} \|\theta_1 - \theta_2\|_2 \tag{S.164}$$

for any $\theta_1, \theta_2 \in \Theta$, $1 \leq j \leq \mathcal{J}$, and $1 \leq k \leq d_{\theta}$. Thus, Assumption A.2.2(ii) implies there exists a $C_0 < \infty$ such that for all $P \in \mathbf{P}$ and $\theta_1, \theta_2 \in \Theta$ it follows that

$$||E_P[\nabla_{\theta}\rho(X,\theta_1) - \nabla_{\theta}\rho(X,\theta_2)]||_{o,2} \le C_0||\theta_1 - \theta_2||_2.$$

In particular, the function $\theta \mapsto E_P[\nabla_\theta \rho(X,\theta)]$ is uniformly continuous in θ and $P \in \mathbf{P}$, which implies by Assumption A.2.3(ii) that there is an $\epsilon_0 > 0$ such that the smallest singular value of $E_P[\nabla_\theta \rho(X,\theta)]$ is bounded away from zero on $\{\theta \in \Theta : \|\theta - \theta_0\|_2 \le \epsilon_0$ for some $P \in \mathbf{P}_0\}$ (where recall θ_0 implicitly depends on P through (A.4)). Since $\|\mathbb{D}_P(\theta)[h]\|_2 \equiv \|E_P[\nabla_\theta \rho(X,\theta)h]\|_2$, $\nu_n \asymp 1$, p = 2, and $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$, the Lemma S.3.1 requirement that $\|h\|_{\mathbf{E}} \le \nu_n \|\mathbb{D}_P(\theta)[h]\|_2$ for all $\theta \in \mathcal{A}_n(P)$, $P \in \mathbf{P}_0$, and $h \in \sqrt{n}\{\mathbf{B}_n \cap R - \theta\}$ holds with $\mathcal{A}_n(P) = (\theta_0)^{\epsilon_0}$ and $R = \Theta$ (and hence also for R as in (A.5)). Moreover, by uniform consistency (in $P \in \mathbf{P}_0$) of $\hat{\theta}_n$ it follows that $\hat{\theta}_n \in \mathcal{A}_n(P)$ with probability tending to one uniformly in $P \in \mathbf{P}_0$.

To conclude, define $\mathcal{F} \equiv \{\frac{\partial}{\partial \theta_k} \rho_J(\cdot, \theta) : \text{ for some } \theta \in \Theta, \ 1 \leq j \leq \mathcal{J}, \ 1 \leq k \leq d_\theta \}$ and let $F(x) \equiv \max_{1 \leq j \leq \mathcal{J}} \sup_{\theta \in \Theta} \|\nabla^2_{\theta} \rho_j(x, \theta)\|_{o,2}$. Then note that if ϵ -balls around $\{\theta_i\}_{i=1}^{N_{\epsilon}}$ cover Θ , then result (S.164) implies that the brackets $[\frac{\partial}{\partial \theta_k} \rho_j(\cdot, \theta_i) - \epsilon F, \frac{\partial}{\partial \theta_k} \rho_j(\cdot, \theta_i) + \epsilon F]$ cover \mathcal{F} . Writing these brackets as $\{[f_{l,k}, f_{u,k}]\}_{k=1}^{K_{\epsilon}}$ for conciseness, further note that $K_{\epsilon} = \mathcal{J} d_{\theta} N_{\epsilon} < \infty$ since $N_{\epsilon} < \infty$ due to Θ being compact, and $C_1 \equiv \sup_{P \in \mathbf{P}} \|F\|_{P,1} < \infty$

by Assumption A.2.2(ii). Moreover, by definition of $[f_{l,k}, f_{u,k}]$ it further follows that

$$E_P[f_{u,k}(X) - f_{l,k}(X)] = ||f_{u,k} - f_{l,k}||_{P,1} \le 2\epsilon C_1$$
(S.165)

for all $P \in \mathbf{P}$. Hence, employing the bound $f(x) - E_P[f(X)] \leq f_{u,k}(x) - E_P[f_{l,k}(X)]$ for $[f_{l,k}, f_{u,k}]$ the bracket containing f, we obtain from result (S.165) that

$$\sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(X_i) - E_P[f(X)] \right\} \\
\leq \max_{1 \leq k \leq K_{\epsilon}} \left| \frac{1}{n} \sum_{i=1}^{n} f_{u,k}(X_i) - E_P[f_{u,k}(X)] \right| + 2\epsilon C_1 = 2\epsilon C_1 + o_P(1), \quad (S.166)$$

where the equality holds uniformly in $P \in \mathbf{P}$ by Assumption A.2.2(ii), $K_{\epsilon} < \infty$, and Theorem 2.8.1 in van der Vaart and Wellner (1996). By identical arguments, we have

$$\inf_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(X_i) - E_P[f(X)] \right\} \\
\geq - \max_{1 \le k \le K_{\epsilon}} \left| \frac{1}{n} \sum_{i=1}^{n} f_{l,k}(X_i) - E_P[f_{l,k}(X)] \right| - 2\epsilon C_1 = -2\epsilon C_1 + o_P(1), \quad (S.167)$$

uniformly in $P \in \mathbf{P}$. We thus conclude from results (S.166) and (S.167) that \mathcal{F} is Glivenko-Cantelli uniformly in $P \in \mathbf{P}$. Since by Assumption A.2.5(i) there exists an $\epsilon > 0$ such that $\{\theta : \|\theta - \theta_0\|_2 \le \epsilon\} \subseteq \Theta$ for all $P \in \mathbf{P}_0$, we can conclude

$$\sup_{\theta: \|\theta - \theta_0\|_2 \le \epsilon} \sup_{h \in \mathbf{R}^{d_\theta}: \|\frac{h}{\sqrt{n}}\|_2 \ge \ell_n} \frac{\|\hat{\mathbb{D}}_n(\theta)[h] - \mathbb{D}_P(\theta)[h]\|_2}{\|h\|_2} \\
\le \sup_{\theta \in \Theta} \|\frac{1}{n} \sum_{i=1}^n \nabla_{\theta} \rho(X_i, \theta) - E_P[\nabla_{\theta} \rho(X, \theta)]\|_{o,2} = o_P(1) \quad (S.168)$$

uniformly in $P \in \mathbf{P}$, and where the equality follows from \mathcal{F} being Glivenko-Cantelli uniformly in $P \in \mathbf{P}$. Since $\nu_n \times 1$, result (S.168) verifies condition (S.79) in Lemma S.3.1. This concludes verifying the requirements of Lemma S.3.1 and hence the present Lemma follows for any ℓ_n satisfying $\mathcal{S}_n(\mathbf{B}, \mathbf{E})\mathcal{R}_n = o(\ell_n)$, which in this application is equivalent to $n^{-1/2} = o(\ell_n)$ due to $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{E}} = \|\cdot\|_2$ and $\mathcal{R}_n \times n^{-1/2}$.

S.4.3 Proofs for Section A.2.2

PROOF OF THEOREM A.2.3: We establish the theorem by simply verifying the conditions of Theorem 3.1(ii) for both R as corresponding to (A.13) and (A.14) (to couple $I_n(R)$) and to $R = \Theta$ (to couple $I_n(\Theta)$). To this end, note Assumption 3.1(i) is imposed in Assumption A.2.7(i), Assumption 3.2(i) holds with $B_n \times \sqrt{k_n}$ by Assumption

A.2.9(i), and Assumption 3.2(ii) is satisfied by Assumption A.2.9(ii). Further note that for $R = \Theta$, the class \mathcal{F}_n has bounded envelope F_n by Assumption A.2.7(iii) and $\|g\|_{\infty} \leq C_0$ for any $(g,\gamma) \in \Theta$. Hence, by Lemma S.4.17 it follows that Assumption 3.2(iii) holds with $J_n \approx \sqrt{j_n \log(1+j_n)}$ when $R = \Theta$, and therefore also for R as corresponding to (A.13) and (A.14). Next, we note Lemma S.4.18 and $j_n^2 k_n^3 \log^3(n)/n = o(1)$ by Assumption A.2.9(iv) imply Assumption 3.3(i) for both specifications of R under consideration and any a_n satisfying $a_n = O((\log(n))^{-1})$. To verify Assumption 3.3(ii), we observe that for any $(g_1, \gamma_1) \in \Theta_n$ and $(g_2, \gamma_2) \in \Theta_n$, Assumption A.2.7(iii) implies

$$E_{P}[((Q - g_{1}(S, Y) - W'\gamma_{1}) - (Q - g_{2}(S, Y) - W'\gamma_{2}))^{2}]$$

$$\lesssim \sup_{P \in \mathbf{P}} \|g_{1} - g_{2}\|_{P,2}^{2} + \|\gamma_{1} - \gamma_{2}\|_{2}^{2}. \quad (S.169)$$

Hence, since $\|\cdot\|_{\mathbf{E}} \equiv \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2} + \|\cdot\|_2$ it follows Assumption 3.3(ii) holds with $\kappa_{\rho} = 1$ and some $K_{\rho} < \infty$. Lemma S.4.16 additionally verifies that Assumption 3.4 holds with $\|\cdot\|_{\mathbf{E}} \equiv \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2} + \|\cdot\|_2$, $\mathcal{V}_n(P) = \Theta_n \cap R$, and $\nu_n^{-1} \asymp s_n$ for both $R = \Theta$ and R as corresponding to (A.13) and (A.14). Further note that in this application

$$\nabla m_P(\theta)[h] = -E_P[g_h(S, Y) + W'\gamma_h|Z]$$

for any $\theta \in \mathbf{B}$ and $(g_h, \gamma_h) = h \in \mathbf{B}$. By direct calculation it then follows Assumptions 3.5(i)(ii) hold with $K_m = 0$ for $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\mathbf{E}}$, and Assumption 3.5(iii) is satisfied for some $M < \infty$ by result (S.169) and Jensen's inequality. To verify Assumption 3.6(i), note that since as argued $\nu_n \approx 1/s_n$, p=2, $J_n \approx \sqrt{j_n \log(1+j_n)}$, and $B_n \approx \sqrt{k_n}$, it follows $\mathcal{R}_n \approx k_n \sqrt{j_n} \log(1+k_n)/s_n \sqrt{n}$ due to $j_n \leq k_n$ by Assumption A.2.9(iii). Therefore, since $\kappa_{\rho} = 1$, Lemma S.4.17 implies Assumption 3.6(i) demands $k_n \sqrt{j_n \log(1 + k_n)} \mathcal{R}_n (1 + k_n)$ $\sqrt{\log(1\vee(\sqrt{j_n}/\mathcal{R}_n))}$ = $o(a_n)$, which is satisfied with $a_n=1/\sqrt{\log(n)}$ by Assumption A.2.9(iv). In turn, Assumption 3.6(ii) holds with $a_n = (\log(n))^{-1/2}$ by Assumptions A.2.8(iv) and A.2.10(ii). Finally, we note that Assumption A.2.10 implies Assumption 3.7 due $B_n \lesssim \sqrt{k_n}$, p=2, $J_n \asymp \sqrt{j_n \log(1+j_n)}$, and $a_n=(\log(n))^{-1/2}$. To conclude, note that since $K_m = 0$, the condition $K_m \mathcal{R}_n^2 \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ is automatically satisfied, the requirement $\mathcal{R}_n = o(\ell_n)$ is equivalent to $k_n \sqrt{j_n} \log(1 + k_n) / s_n \sqrt{n} = o(\ell_n)$, and the condition $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ is implied by $k_n \sqrt{j_n \log(1+k_n)} \ell_n \sqrt{\log(\sqrt{j_n}/\ell_n)} = o((\log(n))^{-1/2})$. Thus, all the conditions of Theorem 3.1(ii) hold for both $R = \Theta$ and R corresponding to (A.13) and (A.14), and thus the claim of the theorem follows.

PROOF OF THEOREM A.2.4: We first define the variable $\hat{E}_n(R|\ell_n)$ to be given by

$$\hat{E}_n(R|\ell_n) \equiv \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n[h]\|_{\hat{\Sigma}_n, 2}$$

and note that for any sequence ℓ_n satisfying the conditions of the theorem, Lemma

S.4.21 implies $\hat{U}_n(R|+\infty) = \hat{E}_n(R|\ell_n) + o_P(a_n)$ uniformly in $P \in \mathbf{P}_0$. To establish the theorem, it therefore suffices to show that uniformly in $P \in \mathbf{P}_0$

$$\hat{E}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n)$$

$$\hat{E}_n(R|\ell_n) - \hat{U}_n(\Theta| + \infty) \ge U_P^{\star}(R|\tilde{\ell}_n) - U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n)$$

with $\ell_n \simeq \tilde{\ell}_n$ and $\tilde{\ell}_n^{\rm u}$ satisfying the requirements of the theorem. To this end we rely on Theorem 3.2 (for $\hat{E}_n(R|\ell_n)$) and Lemma S.3.7(ii). We note that in the proof of Theorem A.2.3 we established that Assumptions A.2.7, A.2.8, A.2.9, and A.2.10 imply Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 hold with $\mathcal{R}_n \simeq k_n \sqrt{j_n} \log(1+k_n)/s_n \sqrt{n}$, $B_n \simeq \sqrt{k_n}$, $\nu_n \simeq 1/s_n$, $\|\theta\|_{\mathbf{B}} = \|g\|_{1,\infty} \vee \|\gamma\|_2$ and $\|\theta\|_{\mathbf{L}} = \|\theta\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|g\|_{P,2} + \|\gamma\|_2$ for $\theta = (g, \gamma)$, $\kappa_\rho = 1$, and $a_n = (\log(n))^{-1/2}$ for $R = \Theta$ and R as corresponding to (A.13) and (A.14). We thus only verify Assumptions 3.8, 3.9, 3.10, 3.11, 3.12, and 3.13 for $R = \Theta$ and R as corresponding to (A.13) and (A.14).

Next note that Lemma S.4.20 implies Assumptions 3.8, 3.9, and 3.10 are satisfied with $K_g = 2$ and $K_f = 0$, while Lemma S.4.19 and Assumption A.2.11(iv) imply Assumption 3.11 holds with $R = \Theta$ (and hence for R corresponding to (A.13) and (A.14)) with $a_n = (\log(n))^{-1/2}$. Further note that since $\sup_{P \in \mathbf{P}} \|g\|_{P,2} + \|\gamma\|_2 \le 2(\|g\|_{1,\infty} \vee \|\gamma\|_2)$ for any $(g,\gamma) \in C_B^1(\Omega) \times \mathbf{R}^{d_w}$, it follows that Assumption 3.12(i) holds with M = 2. To verify Assumption 3.12(ii) note that by the definitions of $\|\cdot\|_{\mathbf{B}}$ and $\|\cdot\|_{\mathbf{E}}$ in this application and the eigenvalues of $E_P[p^{j_n}(S,Y)p^{j_n}(S,Y)']$ being bounded away from zero uniformly in $P \in \mathbf{P}$ by Assumption A.2.8(iii) we obtain

$$S_{n}(\mathbf{B}, \mathbf{E}) = \sup_{(\beta, \gamma)} \frac{\|p^{j_{n'}}\beta\|_{1,\infty} \vee \|\gamma\|_{2}}{\sup_{P \in \mathbf{P}} \|p^{j_{n'}}\beta\|_{P,2} + \|\gamma\|_{2}}$$

$$\leq 1 \vee \sup_{\beta} \frac{\|p^{j_{n'}}\beta\|_{1,\infty}}{\sup_{P \in \mathbf{P}} \|p^{j_{n'}}\beta\|_{P,2}} \lesssim 1 \vee \sup_{\beta} \frac{\|p^{j_{n'}}\beta\|_{1,\infty}}{\|\beta\|_{2}} \lesssim j_{n}^{3/2} \quad (S.170)$$

where the final equality follows from Assumptions A.2.8(i)(ii). In particular, note that result (S.170), $\mathcal{R}_n \asymp k_n \sqrt{j_n} \log(1+k_n)/s_n \sqrt{n}$, and Assumption A.2.9(iv) imply that $\mathcal{R}_n \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(1)$. Thus, since setting $\hat{\theta}_n$ and $\hat{\theta}_n^{\mathrm{u}}$ to be the minimizers of Q_n (respectively over $\Theta_n \cap R$ and Θ_n) corresponds to setting $\tau_n = 0$, Assumption A.2.11(ii) and $\mathcal{R}_n \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(1)$ implies Assumption 3.12(ii) holds. We also note Assumption 3.12(iii) is immediate since $\mathcal{V}_n(P) = \Theta_n \cap R$ by Lemma S.4.16. To conclude, note Assumption 3.13(i) holds since we showed $\mathcal{R}_n \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(1)$. Moreover, since $B_n \asymp \sqrt{k_n}$ and $K_f = K_m = 0$, Lemma S.4.17 implies Assumption 3.13(ii) holds for any $\ell_n, \ell_n^{\mathrm{u}}$ satisfying $k_n \sqrt{j_n} \log(1+k_n)(\ell_n \vee \ell_n^{\mathrm{u}})(1+\sqrt{\log(\sqrt{j_n}/(\ell_n \vee \ell_n^{\mathrm{u}})}) = o(a_n)$. Similarly, we obtain that Assumption 3.13(iii) is satisfied provided $\ell_n = o(r_n)$ (imposed in the theorem) and $k_n j_n^2 \log(1+k_n)/s_n \sqrt{n} = o(r_n)$ (implied by Assumption A.2.11(iii)), while the requirement $\mathcal{R}_n^{\mathrm{u}} = o(\ell_n^{\mathrm{u}})$ is implied by $k_n j_n^2 \log(1+k_n)/s_n \sqrt{n} = o(\ell_n^{\mathrm{u}})$. Hence, the conditions of Theorem 3.2 and Lemma S.3.7(ii) hold, and the theorem follows.

Lemma S.4.16. If Assumptions A.2.7(ii), A.2.8(iii), A.2.9(iii), and A.2.10(ii) hold, then Assumption 3.4 is satisfied with both $R = \Theta$ and R corresponding to (A.13) and (A.14), $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2} + \|\cdot\|_{2}$, $\mathcal{V}_{n}(P) = \Theta_{n} \cap R$, and $\nu_{n}^{-1} \asymp s_{n}$.

PROOF: By Assumption A.2.7(ii) there is a unique $\theta_0 \equiv (g_0, \gamma_0) \in \Theta \cap R$ for which (A.9) holds, and we let $\Pi_n \theta_0 = (g_n, \gamma_0) \in \Theta_n \cap R$ for $g_n = p^{j_n'}\beta_n$. To verify Assumption 3.4(i) is satisfied we set $\|\theta\|_{\mathbf{E}} \equiv \sup_{P \in \mathbf{P}} \|g\|_{P,2} + \|\gamma\|_2$ for any $(g, \gamma) = \theta \in \mathbf{B}$. Since the eigenvalues of $E_P[p^{j_n}(S, Y)p^{j_n}(S, Y)']$ are bounded uniformly in j_n and $P \in \mathbf{P}$ by Assumption A.2.8(iii) we can conclude for any $\theta = (p^{j_n'}\beta, \gamma)$ that

$$s_{n} \|\theta - \Pi_{n} \theta_{0}\|_{\mathbf{E}} \lesssim s_{n} \{ \|\beta - \beta_{n}\|_{2} + \|\gamma - \gamma_{0}\|_{2} \}$$

$$\lesssim \|E_{P}[(p^{j_{n}}(S, Y)'(\beta - \beta_{n}) + W'(\gamma - \gamma_{0}))q^{k_{n}}(Z)]\|_{\Sigma_{P}, 2}$$

$$= \|E_{P}[(\rho(X, \theta) - \rho(X, \Pi_{n} \theta_{0}))q^{k_{n}}(Z)]\|_{\Sigma_{P}, 2}$$
(S.171)

where the second inequality holds by Assumptions A.2.9(iii) and A.2.10(ii), while the final equality holds by definition of $\rho(X,\theta)$ (see (A.11)). Thus, we conclude from (S.171) that Assumption 3.4(i) holds with $\nu_n^{-1} \times s_n$ and $\mathcal{V}_n(P) = \Theta_n \cap R$. Finally, note Assumption 3.4(ii) is immediate since $\mathcal{V}_n(P) = \Theta_n \cap R$.

Lemma S.4.17. Define the class $\mathcal{F}_n \equiv \{f : f(v) = (q - g(s, y) - w'\gamma) \text{ for some } (g, \gamma) \in \Theta_n\}$ and suppose that Assumptions A.2.7(iii) and A.2.8(i)(iii) hold. Then, it follows that $\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim 1 \vee (\sqrt{j_n} K/\epsilon)^{j_n+d_w}$ for some $K < \infty$, and in addition $\sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim \epsilon \sqrt{j_n} (1 + \sqrt{\log(1 \vee (\sqrt{j_n}/\epsilon))})$.

PROOF: Define the classes $\mathcal{F}_{1n} \equiv \{f : f(v) = q - w'\gamma \text{ with } \|\gamma\|_2 \leq C_0\}$ and $\mathcal{F}_{2n} \equiv \{p^{j_n'}\beta : \|p^{j_n'}\beta\|_{1,\infty} \leq C_0\}$, and then note that by definition of \mathcal{F}_n we have

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \le \sup_{P \in \mathbf{P}} N_{[]}(\frac{\epsilon}{2}, \mathcal{F}_{1n}, \|\cdot\|_{P,2}) \times \sup_{P \in \mathbf{P}} N_{[]}(\frac{\epsilon}{2}, \mathcal{F}_{2n}, \|\cdot\|_{P,2}). \quad (S.172)$$

Next observe that since the support of W is bounded uniformly in $P \in \mathbf{P}$ by Assumption A.2.7(iii), the Cauchy-Schwarz inequality, the covering numbers of $\{\gamma \in \mathbf{R}^{d_w} : \|\gamma\|_2 \le C_0\}$ being bounded (up to a multiplicative constant) by $1 \vee \epsilon^{-d_w}$, and Theorem 2.7.11 in van der Vaart and Wellner (1996) allow us to conclude that

$$\sup_{P \in \mathbf{P}} N_{[]}(\frac{\epsilon}{2}, \mathcal{F}_{1n}, \|\cdot\|_{P,2}) \lesssim 1 \vee \epsilon^{-d_w}. \tag{S.173}$$

Similarly, for any $p^{j_n'}\beta_1, p^{j_n'}\beta_2 \in \mathcal{F}_{2n}$, the Cauchy-Schwarz inequality implies that

$$|p^{j_n}(s,y)'\beta_1 - p^{j_n}(s,y)'\beta_2| \le \sup_{(s,y)} ||p^{j_n}(s,y)||_2 ||\beta_1 - \beta_2||_2 \lesssim \sqrt{j_n} ||\beta_1 - \beta_2||_2,$$

where in the final inequality we employed Assumption A.2.8(i). Hence, Theorem 2.7.11

in van der Vaart and Wellner (1996), $\|\beta\|_2 \simeq \|p^{j_n\prime}\beta\|_{P,2}$ uniformly in $P \in \mathbf{P}$ by Assumption A.2.8(iii), and $\|p^{j_n\prime}\beta\|_{P,2} \leq \|p^{j_n\prime}\beta\|_{\infty} \leq C_0$ for any $p^{j_n\prime}\beta \in \Theta_n$ imply that

$$\sup_{P \in \mathbf{P}} N_{[]}(\frac{\epsilon}{2}, \mathcal{F}_{2n}, \|\cdot\|_{P,2}) \lesssim 1 \vee (\frac{K\sqrt{j_n}}{\epsilon})^{j_n}$$
 (S.174)

for some $K < \infty$. Thus, the first claim of the lemma follows from results (S.172), (S.173), and (S.174). For the second claim of the lemma, we employ the first claim of the lemma and the change of variables $v = u/\epsilon$ to obtain the bound

$$\sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim \epsilon + \int_0^{\epsilon} (\log(1 \vee (\frac{K\sqrt{j_n}}{u})^{j_n + d_w}))^{1/2} du$$

$$= \epsilon (1 + \sqrt{j_n + d_w} \int_0^1 (\log(1 \vee (\frac{K\sqrt{j_n}}{v\epsilon})))^{1/2} dv) \lesssim \sqrt{j_n} \epsilon (1 + \sqrt{\log(1 \vee (\sqrt{j_n}/\epsilon))}),$$

where we used that $(1 \vee ab) \leq (1 \vee a)(1 \vee b)$ whenever a and b are positive.

Lemma S.4.18. Let Assumptions A.2.7(i)(iii), A.2.8(i)(iii), and A.2.9(i) hold. If $a_n \downarrow 0$ and $k_n^3 j_n^2 \log^2(n)/n = o(a_n)$, then Assumption 3.3(i) holds with $R = \Theta$.

PROOF: We establish the claim of the lemma by relying on Lemma S.4.6. To this end, let $\tilde{j}_n = (1 + d_w) + j_n$, set $r^{j_n}(x) \equiv (q, w', p^{j_n}(x)')'$, and observe any $f \in \mathcal{F}_n$ can be written as $f = r^{j_n'}\delta$ for some $\delta \in \mathbf{R}^{\tilde{j}_n}$. Moreover, by Assumption A.2.8(iii) and definition of Θ_n , it follows that there exists an $M < \infty$ such that $\mathcal{F}_n \subseteq \{r^{\tilde{j}_n'}\delta : \|\delta\|_2 \leq M\}$, while Assumptions A.2.7(iii), A.2.8(i), and A.2.9(i) imply $\sup_x \|r^{j_n}(x)\|_2 \lesssim \sqrt{j_n}$ and $\sup_z \|q^{k_n}(z)\|_2 \leq \sqrt{k_n} \max_{1\leq k\leq k_n} \|q_k\|_\infty \lesssim k_n$. The claim of the lemma therefore follows from applying Lemma S.4.6 with $b_{1n} \approx \sqrt{j_n}$, $b_{2n} \approx k_n$, and $C_n = M$.

Lemma S.4.19. Suppose Assumptions A.2.7(i)(iii), A.2.8(i)(iii), A.2.9(i)(ii) hold and let $C_n \equiv \{\beta \in \mathbf{R}^{j_n} : \|p^{j_n'}\beta\|_{1,\infty} \leq C_0\}$ and $\mathcal{E}_n \equiv \int_0^\infty \sqrt{\log(N(\epsilon, C_n, \|\cdot\|_2))} d\epsilon$. If $j_n^2 k_n^2 \log(1 + k_n j_n) = o(n)$, then it follows that Assumption 3.11 holds with $R = \Theta$ for any sequence a_n satisfying $k_n^{1/p}(\sqrt{\log(k_n)} + \mathcal{E}_n)j_n^{3/4}k_n^{1/2}\log^{1/4}(1 + j_n k_n)/n^{1/4} = o(a_n)$.

PROOF: Recall that in this application $X \equiv (Q, S, Y, W')'$ and, when $R = \Theta$, we have $\mathcal{F}_n \equiv \{f : f(x) = (q - g(s, y) - w'\gamma) \text{ for some } (g, \gamma) \in \Theta_n\}$. Also define $\tilde{\mathcal{F}}_n \equiv \{fq_k : f \in \mathcal{F}_n \text{ and } 1 \leq k \leq k_n\}$, for $\{\omega_i\}_{i=1}^n$ the weights used in building $\hat{\mathbb{W}}_n$ set

$$\hat{\mathbb{G}}_n(f) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ f(V_i) - \frac{1}{n} \sum_{j=1}^n f(V_j) \}$$

for any $f \in \tilde{\mathcal{F}}_n$, and let \mathbb{G}_P^* denote an isonormal process on \mathcal{F}_n independent of $\{V_i\}_{i=1}^n$. Setting $\mathbb{W}_P^*(\theta) \equiv (\mathbb{G}_P^*(\rho(\cdot,\theta)q_1), \dots, \mathbb{G}_P^*(\rho(\cdot,\theta)q_{k_n}))'$, then note that

$$\sup_{\theta \in \Theta} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^{\star}(\theta)\|_p \le k_n^{1/p} \sup_{f \in \tilde{\mathcal{F}}_n} |\hat{\mathbb{G}}_n(f) - \mathbb{G}_P^{\star}(f)|. \tag{S.175}$$

We will establish the lemma by relying on (S.175) and applying Theorem S.7.1 to couple $\hat{\mathbb{W}}_n$ and \mathbb{G}_P^* on $\tilde{\mathcal{F}}_n$. To this end, define $d_n = k_n(j_n + d_w + 1)$ and let

$$f_n^{d_n}(V) \equiv g^{d_n}(V) - E_P[g^{d_n}(V)] \qquad g^{d_n}(V) \equiv q^{k_n}(Z) \otimes (p^{j_n}(S,Y)',Q,W')'.$$
 (S.176)

Next, we set $D_1 \equiv (Q, W', p^{j_n}(S, Y)')'$ and $D_2 = q^{k_n}(Z)$, and for $\overline{\text{eig}}\{D_1D_1'\}$ the largest eigenvalue of the matrix D_1D_1' , then note that we must have

$$\sup_{P \in \mathbf{P}} \|\overline{\operatorname{eig}}\{D_1 D_1'\}\|_{P,\infty} \le \sup_{P \in \mathbf{P}} \|\operatorname{trace}\{D_1 D_1'\}\|_{P,\infty} \lesssim j_n, \tag{S.177}$$

where the final inequality follows from Assumptions A.2.7(iii) and A.2.8(i). Hence, since $\overline{\operatorname{eig}}\{E_P[q^{k_n}(Z)q^{k_n}(Z)']\}$ is bounded uniformly in $P \in \mathbf{P}$ by Assumption A.2.9(ii), result (S.177) and Lemma S.4.22 imply $\overline{\operatorname{eig}}\{E_P[g^{d_n}(V)g^{d_n}(V)'] \lesssim j_n$. It thus follows from $\overline{\operatorname{eig}}\{E_P[g^{d_n}(V)]E_P[g^{d_n}(V)']\} \leq \overline{\operatorname{eig}}\{E_P[g^{d_n}(V)g^{d_n}(V)']\}$ and definition (S.176) that Assumption S.7.1(i) is satisfied with $C_n \asymp j_n$. Similarly, note that Assumptions A.2.7(iii), A.2.8(i), and A.2.9(i) imply Assumption S.7.1(ii) holds with $K_n \asymp \sqrt{j_n k_n}$. Moreover, Assumption S.7.2(i) is immediate with $G_{n,P}$ equal to the zero function and $J_{1n}=0$. Finally, note that any function $f \in \tilde{\mathcal{F}}_n$ has the structure

$$f(v) = q_k(z)(q - p^{j_n}(s, y)'\beta - w'\gamma) \text{ for some } (p^{j_n'}\beta, \gamma) \in \Theta_n.$$
 (S.178)

Therefore, for \mathcal{B}_n as defined in Assumption S.7.2(ii), $\mathcal{C}_n \equiv \{\beta \in \mathbf{R}^{j_n} : \|p^{j_n'}\beta\|_{1,\infty} \leq C_0\}$, and $\mathcal{G}_n \equiv \{\gamma \in \mathbf{R}^{d_w} : \|\gamma\|_2 \leq C_0\}$, we can conclude that

$$N(\epsilon, \mathcal{B}_n, \|\cdot\|_2) \le k_n \times N(\frac{\epsilon}{2}, \mathcal{G}_n, \|\cdot\|_2) \times N(\frac{\epsilon}{2}, \mathcal{C}_n, \|\cdot\|_2)$$

$$\lesssim k_n \times ((\frac{1}{\epsilon})^{d_w} \vee 1) \times N(\frac{\epsilon}{2}, \mathcal{C}_n, \|\cdot\|_2), \quad (S.179)$$

where in the second inequality we employed that $N(\epsilon, \mathcal{G}_n, \|\cdot\|_2) \lesssim (1/\epsilon)^{d_w} \vee 1$. Furthermore, note that Assumption A.2.8(iii) implies that $\|\beta\|_2 \approx \|p^{j_n'}\beta\|_{P,2}$ uniformly in j_n and $P \in \mathbf{P}$, and hence since $\|p^{j_n'}\beta\|_{P,2} \leq \|p^{j_n'}\beta\|_{1,\infty}$, the definition of Θ_n and (S.178) implies that the radius of \mathcal{B}_n under $\|\cdot\|_2$ is uniformly bounded in n. Thus, the bound in (S.179) yields that for some $M < \infty$ we must have

$$\int_{0}^{\infty} \sqrt{\log(N(\epsilon, \mathcal{B}_{n}, \|\cdot\|_{2}))} d\epsilon$$

$$\lesssim \int_{0}^{M} \sqrt{\log(k_{n})} d\epsilon + \int_{0}^{1} \sqrt{\log(1/\epsilon)} d\epsilon + \int_{0}^{M} \sqrt{\log(N(\epsilon/2, \mathcal{C}_{n}, \|\cdot\|_{2}))} d\epsilon$$

$$\lesssim \sqrt{\log(k_{n})} + \int_{0}^{\infty} \sqrt{\log(N(u, \mathcal{C}_{n}, \|\cdot\|_{2}))} du,$$

where the final inequality follows from $N(\epsilon, C_n, \|\cdot\|_2)$ being (weakly) larger than one for all ϵ and the change of variables $u = \epsilon/2$. Hence, Assumption S.7.2(ii) holds with

 $J_{2n} = \sqrt{\log(k_n)} + \mathcal{E}_n$, and as a result Theorem S.7.1 implies uniformly in $P \in \mathbf{P}$

$$\sup_{f \in \tilde{\mathcal{F}}_n} |\hat{\mathbb{G}}_n(f) - \mathbb{G}_P^{\star}(f)| = O_P((\sqrt{\log(k_n)} + \mathcal{E}_n) \{ \frac{j_n^3 k_n^2 \log(1 + j_n k_n)}{n} \}^{1/4}).$$
 (S.180)

The claim of the lemma therefore follows from (S.175) and (S.180).

Lemma S.4.20. If $\mathbf{B} = C_B^1(\Omega) \times \mathbf{R}^{d_w}$ and Υ_G , Υ_F , and Θ are as defined in (A.13), (A.14), and (A.15), then it follows that Assumptions 3.8, 3.9, and 3.10 are satisfied with $K_g = 2$, $K_f = 0$, and for any $\theta = (g, \gamma)$ and $h = (g_h, \gamma_h)$, $\nabla \Upsilon_G(\theta)[h]$ equals

$$\nabla \Upsilon_G(\theta)[h](s,y) = \frac{\partial}{\partial s} g_h(s,y) + g(s,y) \frac{\partial}{\partial y} g_h(s,y) + g_h(s,y) \frac{\partial}{\partial y} g(s,y).$$

PROOF: Recall that in this application $\mathbf{G} = C_B^0(\Omega)$ and $\|\theta\|_{\mathbf{B}} = \max\{\|g\|_{1,\infty}, \|\gamma\|_2\}$. Hence, for any $\theta_1 = (g_1, \gamma_1) \in \mathbf{B}$ and $\theta_2 = (g_2, \gamma_2) \in \mathbf{B}$ we obtain that

$$\|\Upsilon_{G}(\theta_{1}) - \Upsilon_{G}(\theta_{2}) - \nabla\Upsilon_{G}(\theta_{1})[\theta_{1} - \theta_{2}]\|_{\mathbf{G}}$$

$$\leq \sup_{(s,y)\in\Omega} |g_{1}(s,y) - g_{2}(s,y)| \times \sup_{(s,y)\in\Omega} |\frac{\partial}{\partial y}(g_{1}(s,y) - g_{2}(s,y))| \leq ||g_{1} - g_{2}||_{1,\infty}^{2},$$

which verifies Assumption 3.8(i) holds with $K_g = 2$. Similarly, we additionally conclude

$$\|\nabla \Upsilon_{G}(\theta_{1}) - \nabla \Upsilon_{G}(\theta_{2})\|_{o}$$

$$= \sup_{g_{h}:\|g_{h}\|_{1,\infty} \leq 1} \sup_{(s,y) \in \Omega} |(g_{1}(s,y) - g_{2}(s,y)) \frac{\partial}{\partial y} g_{h}(s,y) + g_{h}(s,y) \frac{\partial}{\partial y} (g_{1}(s,y) - g_{2}(s,y))|$$

$$\leq 2\|g_{1} - g_{2}\|_{1,\infty}, \tag{S.181}$$

which verifies Assumption 3.8(ii) holds with $K_g = 2$ as well. Moreover, note that since any $\theta = (g, \gamma) \in \Theta$ satisfies $||g||_{1,\infty} \leq C_0$, it follows that $||\tilde{g}||_{1,\infty} \leq C_0 + \epsilon$ for any $\tilde{g} \in \Theta^{\epsilon}$. Thus, by identical arguments to those in (S.181) we obtain

$$\|\nabla \Upsilon_G(\theta)\|_o \leq 2\|g\|_{1,\infty} \leq 2(C_0 + \epsilon),$$

which thus verifies Assumption 3.8(iii) holds with $M = 2(C_0 + \epsilon)$.

Next note $\Upsilon_F : \mathbf{B} \to \mathbf{F}$ is affine and continuous, and hence $\nabla \Upsilon_F(\theta)[h] = \Upsilon_F(h) - c_0$ for all $\theta, h \in \mathbf{B}$. Therefore, Assumptions 3.9(i)(ii) hold with $K_f = 0$, while

$$\sup_{g_h:||g_h||_{1,\infty}\leq 1}|g_h(s_0,y_0)|\leq 1$$

implies Assumption 3.9(iii) is satisfied with M=1. Since Υ_F being affine and $K_f=0$ further imply that Assumptions 3.9(iv) and 3.10 hold, the lemma follows.

Lemma S.4.21. Let $a_n = (\log(n))^{-1/2}$ and Assumptions A.2.7, A.2.8, A.2.9, A.2.10 hold. If ℓ_n satisfies $k_n j_n^2 \log(1 + k_n) / s_n \sqrt{n} = o(\ell_n)$, then uniformly in $P \in \mathbf{P}_0$:

$$\hat{U}_n(R|+\infty) = \inf_{h \in \hat{V}_n(\hat{\theta}_n, R|\ell_n)} \|\hat{\mathbb{W}}_n(\hat{\theta}_n) + \hat{\mathbb{D}}_n[h]\|_{\hat{\Sigma}_n, 2} + o_P(a_n).$$

PROOF: We establish the lemma by applying Lemma S.3.1. To this end, recall that in the proof of Theorem A.2.3, Assumptions 3.2(i)(iii) and 3.7 were verified to hold with $B_n \times \sqrt{k_n}$ and $J_n \times \sqrt{j_n \log(1+j_n)}$. Since the eigenvalues of $E_P[p^{j_n}(S,Y)p^{j_n}(S,Y)']$ are bounded uniformly in $P \in \mathbf{P}$ by Assumption A.2.8(iii) and $\|\theta\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|g\|_{P,2} + \|\gamma\|_2$ for any $\theta = (g, \gamma)$, it also follows that for any $h = (p^{j_n} \beta_h, \gamma_h)$ we have

$$||h||_{\mathbf{E}} = \sup_{P \in \mathbf{P}} ||p^{j_n'}\beta_h||_{P,2} + ||\gamma||_2 \lesssim ||\beta_h||_2 + ||\gamma_h||_2$$
$$\lesssim \frac{1}{s_n} ||E_P[q^{k_n}(Z)(p^{j_n}(S,Y)'\beta_h + W'\gamma_h)]||_2 = \frac{1}{s_n} ||\mathbb{D}_P[h]||_2,$$

where the second inequality holds by Assumption A.2.9(iii) and the final equality follows from the definition of $\mathbb{D}_P[h]$. Hence, since $\nu_n \approx 1/\mathrm{s}_n$ by Lemma S.4.16 and p=2, we conclude the Lemma S.3.1 requirement that $||h||_{\mathbf{E}} \leq \nu_n ||\mathbb{D}_P(\theta)[h]||_p$ for all $\theta \in \mathcal{A}_n(P)$ holds with $\mathcal{A}_n(P) = \Theta_n \cap R$. Next, define the $k_n \times (j_n + d_w)$ matrix

$$\mathbb{M}_{i,n} \equiv \frac{1}{n} \{ q^{k_n}(Z_i) (p^{j_n}(S_i, Y_i)' W_i') - E_P[q^{k_n}(Z) (p^{j_n}(S, Y)' W')] \}, \tag{S.182}$$

which satisfies $E_P[\mathbb{M}_{i,n}] = 0$. Since $\|(p^{j_n}\beta, \gamma)\|_{\mathbf{E}} \asymp \|\beta\|_2 + \|\gamma\|_2$ by Assumption A.2.8(iii), we then conclude from (S.182) that for some $C_1 < \infty$ we must have

$$\sup_{P \in \mathbf{P}} P(\sup_{h \in \mathbf{B}_n} \frac{\|\hat{\mathbb{D}}_n[h] - \mathbb{D}_P[h]\|_2}{\|h\|_{\mathbf{E}}} > \mathbf{s}_n) \le \sup_{P \in \mathbf{P}} P(\|\frac{1}{n} \sum_{i=1}^n \mathbb{M}_{i,n}\|_{o,2} > C_1 \mathbf{s}_n)$$

$$\le (j_n + d_w + k_n) \exp\{-\frac{n\mathbf{s}_n^2 C_2}{(k_n^2 \vee j_n) + \mathbf{s}_n k_n \sqrt{j_n}}\} = o(1), \quad (S.183)$$

where the final inequality follows by applying Lemma S.4.7 with $b_{1n} = \sqrt{j_n}$ (by Assumptions A.2.7(iii) and A.2.8(i)) and $b_{2n} = k_n$ (by Assumption A.2.9(i)), while the final equality results from $\log(k_n)k_n^2/s_n^2n = o(1)$ by Assumption A.2.9(iv) and $k_n \geq j_n$ by Assumption A.2.8(iii). Hence, $\nu_n \approx 1/s_n$ and (S.183) imply condition (S.79) in Lemma S.3.1 holds. Finally, we note that by Assumption A.2.11(iv), we may apply Lemma S.4.19 with p=2 to conclude that Assumption 3.11 holds with $R=\Theta$ (and hence for R as corresponding to (A.13) and (A.14)) with $a_n=(\log(n))^{-1/2}$. This concludes verifying the requirements of Lemma S.3.1 and therefore the present Lemma follows for any ℓ_n satisfying $\mathcal{S}_n(\mathbf{B}, \mathbf{E})\mathcal{R}_n = o(\ell_n)$, which in this application is equivalent to $k_n j_n^2 \log(1 + k_n)/s_n \sqrt{n} = o(\ell_n)$ due to $\mathcal{S}_n(\mathbf{B}, \mathbf{E}) \lesssim j_n^{3/2}$ and $\mathcal{R}_n \approx k_n \sqrt{j_n} \log(1 + k_n)/s_n \sqrt{n}$.

Lemma S.4.22. Let $D_1 \in \mathbf{R}^{d_1}$, $D_2 \in \mathbf{R}^{d_2}$ be distributed according to Q, and for any matrix A let $\overline{\operatorname{eig}}\{A\}$ denote its largest eigenvalue. Then it follows that

$$\overline{\operatorname{eig}}\{E_Q[(D_1 \otimes D_2)(D_1 \otimes D_2)']\} \leq \|\overline{\operatorname{eig}}\{D_1 D_1'\}\|_{Q,\infty} \times \overline{\operatorname{eig}}\{E_Q[D_2 D_2']\}.$$

PROOF: Let $\mathcal{A} \equiv \{\{a_i\}_{i=1}^{d_1} : a_i \in \mathbf{R}^{d_2} \text{ and } \sum_{i=1}^{d_1} \|a_i\|_2^2 \leq 1\}$, set $(D_1^{(1)}, \dots, D_1^{(d_1)}) = D_1 \in \mathbf{R}^{d_1}$, and then note that by direct calculation we obtain that

$$\overline{\operatorname{eig}} \{ E_{Q}[(D_{1} \otimes D_{2})(D_{1} \otimes D_{2})'] \}
= \sup_{\{a_{i}\}_{i=1}^{d_{1}} \in \mathcal{A}} (a'_{1}, \dots, a'_{d_{1}}) E_{Q}[(D_{1} \otimes D_{2})(D_{1} \otimes D_{2})'] (a'_{1}, \dots, a'_{d_{1}})'
= \sup_{\{a_{i}\}_{i=1}^{d_{1}} \in \mathcal{A}} E_{Q}[(\sum_{i=1}^{d_{1}} (a'_{i}D_{2})D_{1}^{(i)})^{2}]
\leq \|\overline{\operatorname{eig}} \{D_{1}D'_{1}\}\|_{Q,\infty} \sup_{\{a_{i}\}_{i=1}^{d_{1}} \in \mathcal{A}} \sum_{i=1}^{d_{1}} E_{Q}[(a'_{i}D_{2})^{2}].$$

However, since $\sum_{i=1}^{d_1} \|a_i\|_2^2 \leq 1$ for all $\{a_i\}_{i=1}^{d_1} \in \mathcal{A}$, we additionally have the inequality

$$\sup_{\{a_i\}_{i=1}^{d_1} \in \mathcal{A}} \sum_{i=1}^{d_1} E_Q[(a_i'D_2)^2] \le \sup_{\{a_i\}_{i=1}^{d_1} \in \mathcal{A}} \sum_{i=1}^{d_1} \overline{\operatorname{eig}} \{ E_Q[D_2D_2'] \} \|a_i\|_2^2 = \overline{\operatorname{eig}} \{ E_Q[D_2D_2'] \},$$

and therefore the claim of the lemma follows. ■

Lemma S.4.23. Let λ be the Lebesgue measure, $\{B_b^{(1)}\}_{b=1}^{j_{1n}}$ and $\{B_b^{(2)}\}_{b=1}^{j_{2n}}$ be B-splines on [0,1] of order $r \geq 3$ with no interior knot multiplicity, mesh ratio bounded in n, and $\|\cdot\|_{\lambda,2}$ normalized to have norm one. If $\{p_j\}_{j=1}^{j_n}$ is the tensor product of $\{B_b^{(1)}\}_{b=1}^{j_{1n}}$ and $\{B_b^{(2)}\}_{b=1}^{j_{2n}}$ and $\mathcal{C}_n \equiv \{\beta \in \mathbf{R}^{j_n} : \|p^{j_n'}\beta\|_{1,\infty} \leq C_0\}$, then it follows that

$$\int_0^\infty \sqrt{\log(N(\epsilon, \mathcal{C}_n, \|\cdot\|_2))} d\epsilon \lesssim \sqrt{j_{1n} \wedge j_{2n}} \log(j_n + 1).$$

PROOF: We rely heavily on Chapter 5 in DeVore and Lorentz (1993), and note that B_j corresponds to $N_j/\|N_j\|_{\lambda,2}$ in their notation. Throughout, for two sequences a_n and b_n we employ $a_n \approx b_n$ to mean that there exist constants \underline{c} and \overline{c} such that $\underline{c}a_n \leq b_n \leq \overline{c}a_n$ for all n. In what follows it will also prove convenient to index the elements of $\beta \in \mathbf{R}^{j_n}$ by β_{b_1,b_2} with $1 \leq b_1 \leq j_{1n}$ and $1 \leq b_2 \leq j_{2n}$. Then note that the mesh ratios corresponding to $\{B_b^{(1)}\}_{b=1}^{j_{1n}}$ and $\{B_b^{(2)}\}_{b=1}^{j_{2n}}$ being bounded uniformly in j_{1n} , j_{2n} and two applications

of Theorem 5.4.2 in DeVore and Lorentz (1993) imply that

$$||p^{j_n'}\beta||_{\infty} = \sup_{u_1 \in [0,1]} \sup_{u_2 \in [0,1]} |\sum_{b_2=1}^{j_{2n}} B_{b_2}^{(2)}(u_2) \sum_{b_1=1}^{j_{1n}} \beta_{b_1,b_2} B_{b_1}^{(1)}(u_1)|$$

$$\approx \sup_{u_1 \in [0,1]} \max_{1 \le b_2 \le j_{2n}} \sqrt{j_{2n}} |\sum_{b_1=1}^{j_{1n}} \beta_{b_1,b_2} B_{b_1}^{(1)}(u_1)| \approx \sqrt{j_{1n}j_{2n}} ||\beta||_{\infty} \quad (S.184)$$

uniformly in $\beta \in \mathbf{R}^{j_n}$. By similar arguments we also obtain uniformly in $\beta \in \mathbf{R}^{j_n}$ that

$$\sup_{u_{1}\in(0,1)} \sup_{u_{2}\in[0,1]} \left| \sum_{b_{2}=1}^{j_{2n}} B_{b_{2}}^{(2)}(u_{2}) \sum_{b_{1}=1}^{j_{1n}} \frac{\partial}{\partial u_{1}} \{\beta_{b_{1},b_{2}} B_{b_{1}}^{(1)}(u_{1})\} \right|
\approx \sup_{u_{1}\in(0,1)} \max_{1\leq b_{2}\leq j_{2n}} \sqrt{j_{2n}} \left| \sum_{b_{1}=1}^{j_{1n}} \frac{\partial}{\partial u_{1}} \{\beta_{b_{1},b_{2}} B_{b_{1}}^{(1)}(u_{1})\} \right|
\approx \max_{1\leq b_{2}\leq j_{2n}} \max_{2\leq b_{1}\leq j_{1n}} \sqrt{j_{2n}} j_{1n}^{3/2} \left| \beta_{b_{1},b_{2}} - \beta_{b_{1}-1,b_{2}} \right|,$$
(S.185)

where the second result follows by employing equation (3.11) and Theorem 5.4.2 in Chapter 5 of DeVore and Lorentz (1993) and the mesh ratio of $\{B_b^{(1)}\}_{b=1}^{j_{1n}}$ being bounded. Since by identical arguments we can also derive the symmetric (to (S.185)) relationship

$$\sup_{u_{1} \in [0,1]} \sup_{u_{2} \in (0,1)} \left| \sum_{b_{1}=1}^{j_{1n}} B_{b_{1}}^{(1)}(u_{1}) \sum_{b_{2}=1}^{j_{2n}} \frac{\partial}{\partial u_{2}} \{\beta_{b_{1},b_{2}} B_{b_{2}}^{(2)}(u_{2})\} \right| \\ \approx \max_{1 \leq b_{1} \leq j_{1n}} \max_{2 \leq b_{2} \leq j_{2n}} \sqrt{j_{1n}} j_{2n}^{3/2} |\beta_{b_{1},b_{2}} - \beta_{b_{1},b_{2}-1}|, \quad (S.186)$$

it follows from results (S.184), (S.185), and (S.186) that there is an $M_0 < \infty$ such that

$$\max_{1 \le b_1 \le j_{1n}} \max_{1 \le b_2 \le j_{2n}} |\beta_{b_1, b_2}| \le M_0 / \sqrt{j_n}$$

$$\max_{1 \le b_2 \le j_{2n}} \max_{2 \le b_1 \le j_{1n}} |\beta_{b_1, b_2} - \beta_{b_1 - 1, b_2}| \le M_0 / (j_{1n} \sqrt{j_n})$$

$$\max_{1 \le b_1 \le j_{1n}} \max_{2 \le b_2 \le j_{2n}} |\beta_{b_1, b_2} - \beta_{b_1, b_2 - 1}| \le M_0 / (j_{2n} \sqrt{j_n})$$
(S.187)

for all $\beta \in \mathcal{C}_n$. Hence, in order to establish the claim of the lemma, it suffices to bound the covering numbers for the set defined by (S.187).

We proceed by combining two bounds, one for "small" ϵ and one for "large" ϵ . First, assume without loss of generality $j_{1n} \geq j_{2n}$, let $c_n \equiv \lceil \log(j_{1n} + 1) \rceil$, and define the sets

$$\frac{\epsilon}{3\sqrt{j_n}}k_1 \le \beta_{b_1,b_2} \le \frac{\epsilon}{3\sqrt{j_n}}(k_1+1) \text{ for all } b_1 = mc_n + 1 \text{ with } 0 \le m \le \lceil j_{1n}/c_n \rceil - 1$$

$$\frac{\epsilon}{3c_n\sqrt{j_n}}k_2 \le \beta_{b_1,b_2} - \beta_{b_1-1,b_2} \le \frac{\epsilon}{3c_n\sqrt{j_n}}(k_2+1) \text{ otherwise}$$
(S.188)

where k_1, k_2 are non-zero integers – i.e. the sets (in \mathbf{R}^{j_n}) defined in (S.188) consist of "chains" along the b_1 dimension that reset every c_n integers. To compute the diameter of the sets in (S.188), then note that since all "chains" have the same structure

$$\sup \|\beta - \tilde{\beta}\|_{2}^{2} \text{ s.t. } \beta, \tilde{\beta} \text{ satisfying (S.188)}$$

$$\leq \sup j_{2n} \lceil \frac{j_{1n}}{c_{n}} \rceil \sum_{b_{1}=1}^{c_{n}} (\beta_{b_{1},j_{2n}} - \tilde{\beta}_{b_{1},j_{2n}})^{2} \text{ s.t. } \beta, \tilde{\beta} \text{ satisfying (S.188)}$$

$$\leq j_{2n} \lceil \frac{j_{1n}}{c_{n}} \rceil \sum_{b_{n}=1}^{c_{n}} \frac{\epsilon^{2}}{9j_{n}} \{1 + \frac{2(b_{1}-1)}{c_{n}}\}^{2}, \tag{S.189}$$

where the final inequality follows from (S.188). Since $\lceil j_{1n}/c_n \rceil c_n \leq (j_{1n}+c_n) \leq 2j_{1n}$ due to $\lceil j_{1n}/c_n \rceil \leq 1+j_{1n}/c_n$ and $c_n \leq j_{1n}$, it follows from $j_n = j_{1n}j_{2n}$ that every set in (S.188) is contained in a ball of radius ϵ . Moreover, by (S.187) the total number of sets with the structure in (S.188) needed to cover the set C_n is bounded by

$$\left(\left\lceil \frac{6M_0}{\epsilon} \right\rceil\right)^{j_{2n} \left\lceil \frac{j_{1n}}{c_n} \right\rceil} \left(\left\lceil \frac{6M_0 c_n}{\epsilon j_{1n}} \right\rceil\right)^{j_{2n} \left\lceil \frac{j_{1n}}{c_n} \right\rceil c_n}. \tag{S.190}$$

Next, we employ again the bound $\lceil j_{1n}/c_n \rceil c_n \leq 2j_{1n}$ and $\lceil a \rceil \leq 2a$ whenever $a \geq 1$, to obtain from (S.189) and (S.190) that whenever $\epsilon \leq 6M_0c_n/j_{1n}$ we have

$$N(\epsilon, \mathcal{C}_n, \|\cdot\|_2) \le \left(\frac{12M_0}{\epsilon}\right)^{\frac{2j_n}{c_n}} \left(\frac{12M_0c_n}{\epsilon j_{1n}}\right)^{2j_n}$$

$$= \left(\frac{12M_0c_n}{\epsilon j_{1n}} \left(\frac{j_{1n}}{c_n}\right)^{\frac{1}{c_n+1}}\right)^{\frac{2j_n(c_n+1)}{c_n}} \le \left(\frac{M_1\log(1+j_{1n})}{\epsilon j_{1n}}\right)^{4j_n}, \quad (S.191)$$

where the final equality holds for some $M_1 < \infty$ due to $(c_n+1)/c_n \le 2$ and $(j_{1n}/c_n)^{\frac{1}{c_n+1}} \le j_{1n}^{\frac{1}{\log(1+j_{1n})}} = O(1)$ because $c_n = \lceil \log(1+j_{1n}) \rceil$.

The bound in (S.191) is valid only for $\epsilon \leq 6M_0c_n/j_{1n}$. To obtain a bound for $\epsilon \geq 6M_0c_n/j_{1n}$, let $\{\mathbb{Z}_{b_1,b_2}\}_{b_1,b_2}$ be independent standard normal random variables. By Sudakov's inequality (see, e.g., Proposition A.2.5 in van der Vaart and Wellner (1996)), it then follows that for some $M_2 < \infty$ independent of n we have that

$$\sqrt{\log(N(\epsilon, C_n, \|\cdot\|_2))} \le \frac{M_2}{\epsilon} E[\sup_{\beta \in C_n} \sum_{b_1=1}^{j_{1n}} \sum_{b_2=1}^{j_{2n}} \beta_{b_1, b_2} \mathbb{Z}_{b_1, b_2}].$$
 (S.192)

Next, for notational convenience define $\Delta_{b_1}\beta_{b_1,b_2}=(\beta_{b_1,b_2}-\beta_{b_1-1,b_2})$ and $\Delta_{b_2}\beta_{b_1,b_2}=$

 $(\beta_{b_1,b_2} - \beta_{b_1,b_2-1})$, and then note that by (S.187) it follows that

$$\begin{split} \sup_{\beta \in \mathcal{C}_n} \sum_{b_1 = 1}^{j_{1n}} \sum_{b_2 = 1}^{j_{2n}} \beta_{b_1, b_2} \mathbb{Z}_{b_1, b_2} \\ &= \sup_{\beta \in \mathcal{C}_n} \sum_{b_2 = 1}^{j_{2n}} \sum_{b_1 = 2}^{j_{1n}} \Delta_{b_1} \beta_{b_1, b_2} \sum_{\tilde{b}_1 = b_1}^{j_{1n}} \mathbb{Z}_{\tilde{b}_1, b_2} + \sum_{b_2 = 2}^{j_{2n}} \Delta_{b_2} \beta_{1, b_2} \sum_{b_1 = 1}^{j_{1n}} \sum_{\tilde{b}_2 = b_2}^{j_{2n}} \mathbb{Z}_{b_1, \tilde{b}_2} + \beta_{1, 1} \sum_{b_1 = 1}^{j_{2n}} \sum_{b_2 = 1}^{j_{2n}} \mathbb{Z}_{b_1, b_2} \\ &\leq \sum_{b_2 = 1}^{j_{2n}} \sum_{b_1 = 2}^{j_{1n}} \frac{M_0}{j_{1n} \sqrt{j_n}} |\sum_{\tilde{b}_1 = b_1}^{j_{1n}} \mathbb{Z}_{\tilde{b}_1, b_2}| + \sum_{b_2 = 2}^{j_{2n}} \frac{M_0}{j_{2n} \sqrt{j_n}} |\sum_{b_1 = 1}^{j_{2n}} \sum_{\tilde{b}_2 = b_2}^{j_{2n}} \mathbb{Z}_{b_1, \tilde{b}_2}| + \frac{M_0}{\sqrt{j_n}} |\sum_{b_1 = 1}^{j_{1n}} \sum_{b_2 = 1}^{j_{2n}} \mathbb{Z}_{b_1, b_2}|. \end{split}$$

Hence, employing that if $\mathbb{W} \sim N(0, \sigma^2)$ then $E[|\mathbb{W}|] \lesssim \sigma$, we can conclude that

$$E[\sup_{\beta \in \mathcal{C}_n} \sum_{b_1=1}^{j_{1n}} \sum_{b_2=1}^{j_{2n}} \beta_{b_1,b_2} \mathbb{Z}_{b_1,b_2}] \lesssim \sum_{b_2=1}^{j_{2n}} \sum_{b_1=2}^{j_{1n}} \frac{\sqrt{j_{1n} - b_1}}{j_{1n}\sqrt{j_n}} + \sum_{b_2=2}^{j_{2n}} \frac{\sqrt{j_{1n}(j_{2n} - b_2)}}{j_{2n}\sqrt{j_n}} + 1$$

$$\leq \frac{j_n\sqrt{j_{1n}}}{j_{1n}\sqrt{j_n}} + \frac{j_{2n}\sqrt{j_{1n}j_{2n}}}{j_{2n}\sqrt{j_n}} + 1 \leq 3\sqrt{j_{2n}}$$

where in the final inequality we employed that $j_n = j_{1n}j_{2n}$. Hence, by (S.192) we have

$$\sqrt{\log(N(\epsilon, \mathcal{C}_n, \|\cdot\|_2))} \lesssim \frac{\sqrt{j_{2n}}}{\epsilon}.$$
 (S.193)

To conclude the proof, we combine the bounds in (S.191) and (S.193). In particular, setting $\delta_n \equiv 6M_0\lceil \log(j_{1n}+1)\rceil/j_{1n}$ and observing that $\|\beta\|_2 \approx \|p^{j_n\prime}\beta\|_{\lambda,2} \leq C_0$ for all $\beta \in \mathcal{C}_n$ allows us to conclude that for some $M_2 < \infty$ we must have

$$\int_{0}^{\infty} \sqrt{\log(N(\epsilon, C_{n}, \|\cdot\|_{2}))} d\epsilon \lesssim \int_{\delta_{n}}^{M_{2}} \frac{\sqrt{j_{2n}}}{\epsilon} + \sqrt{j_{n}} \int_{0}^{\delta_{n}} (\log(\frac{M_{1} \log(j_{1n})}{\epsilon j_{1n}}))^{1/2} d\epsilon$$
$$\lesssim \sqrt{j_{2n}} \log(1 + j_{1n}) + \frac{\sqrt{j_{n}} \log(1 + j_{1n})}{j_{1n}} \int_{0}^{1} (\log(\frac{1}{u}))^{1/2} du \lesssim \sqrt{j_{2n}} \log(1 + j_{1n})$$

where the second inequality follows from the change of variables $u = \epsilon/\delta_n$ and the final inequality employed that $j_n = j_{1n}j_{2n}$ and $j_{2n} \leq j_{1n}$. Substituting $j_{2n} = j_{1n} \wedge j_{2n}$ and employing $j_{1n} \leq j_n$ establishes the lemma.

S.4.4 Proofs for Section A.2.3

PROOF OF THEOREM A.2.5: We establish the result by applying Theorem 3.1 to both $R = \Theta$ and R corresponding to (A.26) and (A.27). To this end, note that Assumption 3.1(i) is directly imposed in Assumption A.2.12(i), Assumption 3.2(i) holds with $B_n = O(1)$ by Assumption A.2.15(ii), Assumption 3.2(ii) is directly imposed by Assumption A.2.15(iii), and Assumption 3.2(iii) holds with $F_n = 1$ and $J_n = O(1)$ by

Lemma S.4.27. Next, we apply Lemma S.4.28 with $\pi_{0n} = O(1)$ and $\pi_{1n} = O(k_n)$ (which is possible by Assumptions A.2.15(i)(ii)) to obtain that Assumption 3.3(i) holds for both $R = \Theta$ and R corresponding to (A.26) and (A.27) for any a_n satisfying $k_n^{1/p} \sqrt{j_n} \log(n) (n^{1/6} \vee k_n) / n^{1/3} = o(a_n)$ and in particular it holds for $a_n \asymp (\log(n))^{-1/2}$ by Assumption A.2.15(v). We also note Assumptions 3.3(ii), 3.4, and 3.5 hold with $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\infty}$, $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$, and $\kappa_{\rho} = 1/2$ by Lemmas S.4.25 and S.4.29. To verify Assumption 3.6, note $J_n = O(1)$, $B_n = O(1)$, and $\nu_n \asymp \sqrt{k_n} / s_n k_n^{1/p}$ by Lemma S.4.25 imply in this application we have $\mathcal{R}_n \asymp \sqrt{k_n \log(1+k_n)} / s_n \sqrt{n}$. Thus, Lemma S.4.27 and Assumption A.2.15(v) verify Assumption 3.6(i), while Assumption A.2.13(iii) implies Assumption 3.6(ii), and Assumption A.2.16 implies Assumption 3.7. Finally, since $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\infty}$ by Lemma S.4.29, Assumptions A.2.13(i)(ii) yield

$$\sup_{\beta} \frac{\|p^{j_n'}\beta\|_{\infty}}{\sup_{P \in \mathbf{P}} \|p^{j_n'}\beta\|_{P,2}} \lesssim \frac{\sqrt{j_n} \|\beta\|_2}{\|\beta\|_2} = \sqrt{j_n}.$$
 (S.194)

Therefore, the condition $K_m \ell_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$ is equivalent to $\ell_n^2 \sqrt{n j_n \log(n)} = o(1)$. Moreover, by Lemma S.4.27, Assumption A.2.13(ii), $\kappa_\rho = 1/2$, and $B_n = O(1)$ the condition $k_n^{1/p} \sqrt{\log(1+k_n)} B_n \times \sup_{P \in \mathbf{P}} J_{[]}(\ell_n^{\kappa_\rho}, \mathcal{F}_n, \|\cdot\|_{P,2}) = o(a_n)$ is implied by the restriction $k_n^{1/p} \sqrt{j_n \ell_n \log(1+k_n) \log(1/\ell_n)} = o((\log(n))^{-1/2})$. Thus, the first claim of the theorem follows from Theorem 3.1(i) applied to $I_n(R)$.

The second claim of the Theorem follows from applying Theorem 3.1(ii) to $I_n(\Theta)$. To this end, note that the only remaining condition to verify is that $\mathcal{R}_n^2 \times \mathcal{S}_n(\mathbf{L}, \mathbf{E}) = o(a_n n^{-1/2})$. Using that, as already argued, $\mathcal{R}_n \simeq \sqrt{k_n \log(1 + k_n)}/s_n \sqrt{n}$ and result (S.194) we note that a sufficient condition for this final requirement is that $k_n \log(1 + k_n) \sqrt{j_n \log(n)}/s_n^2 \sqrt{n} = o(1)$ and therefore the theorem follows.

PROOF OF THEOREM A.2.6: We proceed by relying on Theorem 3.2 and Lemma S.3.7(ii). To this end, we note that, for both $R = \Theta$ and R corresponding to (A.26) and (A.27), the proof of Theorem A.2.5 established that Assumptions 3.1(i), 3.2, 3.4, 3.3, 3.5, 3.6, and 3.7 are satisfied with $B_n = O(1)$, $J_n = O(1)$, $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$, $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\infty}$, $\nu_n \times \sqrt{k_n}/s_n k_n^{1/p}$, $\mathcal{R}_n \times \sqrt{k_n \log(1+k_n)}/s_n \sqrt{n}$, $\kappa_\rho = 1/2$, and $\mathcal{S}_n(\mathbf{L}, \mathbf{E}) \lesssim \sqrt{j_n}$.

Next, note that Assumptions 3.8, 3.9, and 3.10 are satisfied by Lemma S.4.30 with $K_g = 0$ and $K_f > 0$. To verify Assumption 3.11 we apply Lemma S.4.31 with $\pi_{0n} = O(1)$ and $\pi_{1n} \lesssim k_n$, which is possible by Assumptions A.2.15(i)(ii). In particular, Lemma S.4.31 evaluated at $d_n \approx (nk_n)^{3/13}$ and Assumption A.2.17(iii) yield

$$\sup_{\theta \in \Theta_n} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^*(\theta)\|_p = o_P((\log(n))^{-1/2})$$
 (S.195)

uniformly in $P \in \mathbf{P}$, which implies Assumption 3.11 is satisfied for both $R = \Theta$ and R corresponding to (A.26) and (A.27). Next note that Assumption 3.12(i) is immediate since $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}_0} \|\cdot\|_{P,2}$ and $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{2,\infty}$, while Assumption 3.12(iii) follows from

Lemma S.4.24 and $\mathcal{V}_n(P) \equiv \{\theta \in \Theta_n : \|\theta - \Pi_n \theta_0\|_{1,\infty} \leq \epsilon\}$ for some $\epsilon > 0$ by Lemma S.4.25. In order to verify Assumption 3.12(ii) and the rate conditions of Assumption 3.13, note that the eigenvalues of $E_P[p^{j_n}(D)p^{j_n}(D)']$ being bounded away from zero uniformly in $P \in \mathbf{P}$ by Assumption A.2.13(ii), Assumptions A.2.13(i) and A.2.17(iv) together with $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{2,\infty}$ and $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$ deliver the bound

$$S_n(\mathbf{B}, \mathbf{E}) = \sup_{\beta} \frac{\|p^{j_n \prime}\beta\|_{2,\infty}}{\sup_{P \in \mathbf{P}} \|p^{j_n \prime}\beta\|_{P,2}} \lesssim \frac{j_n^{5/2} \|\beta\|_2}{\|\beta\|_2} = j_n^{5/2}.$$
 (S.196)

Thus, we note Assumption 3.13(i) follows from Assumption A.2.17(v), result (S.196), and $\mathcal{R}_n \simeq \sqrt{k_n \log(1+k_n)}/s_n \sqrt{n}$ implying $\mathcal{R}_n \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(1)$. Furthermore, we note $\mathcal{R}_n \mathcal{S}_n(\mathbf{B}, \mathbf{E}) = o(1)$, Assumption A.2.17(ii), and the definition of Θ in (A.28) imply assumption 3.12(ii) is satisfied as well. Assumption 3.13(ii) similarly follows from Assumption A.2.17(v), result (S.196), $\mathcal{R}_n \simeq \sqrt{k_n \log(1+k_n)}/s_n \sqrt{n}$, $\kappa_\rho = 1/2$, and Lemma S.4.27. Finally, Assumption 3.13(iii) also follows by Assumption A.2.17(v), result (S.196), $\mathcal{R}_n \simeq \sqrt{k_n (\log(1+k_n))}/s_n \sqrt{n}$, and $K_g = 0$. We have thus verified the conditions of Theorem 3.2 for R corresponding to (A.26) and (A.27), and hence

$$\hat{U}_n(R|\ell_n) \ge U_P^{\star}(R|\tilde{\ell}_n) + o_P(a_n)$$

uniformly in $P \in \mathbf{P}_0$ for some $\tilde{\ell}_n \simeq \ell_n$. Similarly, under the additional conditions imposed on the second part of this theorem, Lemma S.3.7(ii) implies that for any $\tilde{\ell}_n^{\mathrm{u}}$ satisfying the conditions of Theorem A.2.5(ii) it follows that uniformly in $P \in \mathbf{P}_0$

$$\hat{U}_n(\Theta|+\infty) \le U_P^{\star}(\Theta|\tilde{\ell}_n^{\mathrm{u}}) + o_P(a_n),$$

which implies the second claim of the theorem also holds.

Lemma S.4.24. Let Assumptions A.2.12(i)(iii), A.2.13(ii)(iii), A.2.14(i), A.2.16, and A.2.15(i)(iii)(iv) hold, $k_n \log(1 + k_n)/n = o(1)$, and suppose that

$$Q_n(\hat{\theta}_n) \le \inf_{\theta \in \Theta_n \cap R} Q_n(\theta) + o(n^{-1/2}) \qquad Q_n(\hat{\theta}_n^{\mathrm{u}}) \le \inf_{\theta \in \Theta_n} Q_n(\theta) + o(n^{-1/2}).$$

Then: $\|\hat{\theta}_n - \Pi_n \theta_0\|_{1,\infty} \vee \|\hat{\theta}_n^{\mathrm{u}} - \Pi_n \theta_0\|_{1,\infty} = o_P(1)$ uniformly in $P \in \mathbf{P}_0$.

PROOF: We establish the result by verifying the conditions of Lemma S.1.1 with $\tau_n = o(n^{-1/2})$. First note that, for both $R = \Theta$ and R corresponding to (A.26), Assumption 3.1(i) is directly imposed in Assumption A.2.12(i), Assumption 3.2(ii) holds with $B_n = O(1)$ by Assumption A.2.15(i), Assumption 3.2(iii) holds with $F_n = 1$ and $J_n = O(1)$ by Lemma S.4.27, Assumption S.1.1 is implied by Assumption A.2.16, and Assumption S.1.2(i) follows from Assumption A.2.13(iii). Next, define the following neighborhood

$$\mathcal{V}_n(P) \equiv \{ \theta \in \Theta_n : \|\theta - \Pi_n \theta_0\|_{1,\infty} \le \epsilon \}$$
 (S.197)

for any $\epsilon > 0$ and $P \in \mathbf{P}_0$ (where recall θ_0 is implicitly a function of P through (A.24)). Further set $Q_P(\theta) \equiv \|E_P[\rho(X,\theta)q^{k_n}(Z)]\|_{\Sigma_{P,p}}$ and note that since for any $a \in \mathbf{R}^{k_n}$ we have $\|a\|_p \leq \|\Sigma_P^{-1}\|_{o,p} \|\Sigma_P a\|_p$ and $\|\Sigma_P^{-1}\|_{o,p}$ is bounded uniformly in k_n and $P \in \mathbf{P}$ by Assumption A.2.16(ii), we obtain from Lemma S.1.5 and Assumption A.2.13(iii) that

$$S_{n}(\epsilon) \equiv \inf_{P \in \mathbf{P}_{0}} \{ \inf_{\theta \in (\Theta_{n} \cap R) \setminus \mathcal{V}_{n}(P)} Q_{P}(\theta) - \inf_{\theta \in \Theta_{n} \cap R} Q_{P}(\theta) \}$$

$$\gtrsim \inf_{P \in \mathbf{P}_{0}} \inf_{\theta \in (\Theta_{n} \cap R) \setminus \mathcal{V}_{n}(P)} \frac{k_{n}^{1/p}}{\sqrt{k_{n}}} \|E_{P}[q^{k_{n}}(Z)\rho(X,\theta)]\|_{2} + O((n\log(n))^{-1/2}). \quad (S.198)$$

We further note that the eigenvalues of $E_P[q^{k_n}(Z)q^{k_n}(Z)']$ being bounded uniformly in k_n and $P \in \mathbf{P}$ together with Lemma S.2.5 and Assumption A.2.15(iv) yield

$$\sup_{P \in \mathbf{P}} \sup_{\theta \in \Theta} \|E_P[q^{k_n}(Z)(E_P[\rho(X,\theta)|Z] - q^{k_n}(Z)'\pi_n(\theta))]\|_2^2$$

$$\lesssim \sup_{P \in \mathbf{P}} \sup_{\theta \in \Theta} E_P[(E_P[\rho(X,\theta)|Z] - q^{k_n}(Z)'\pi_n(\theta))^2] = o(1). \quad (S.199)$$

Therefore, since the eigenvalues of $E_P[q^{k_n}(Z)q^{k_n}(Z)']$ are bounded away from zero by Assumption A.2.15(iii), we obtain from result (S.199) that

$$\inf_{P \in \mathbf{P}_0} \inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} ||E_P[q^{k_n}(Z)\rho(X,\theta)]||_2$$

$$\geq \inf_{P \in \mathbf{P}_0} \inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} ||E_P[q^{k_n}(Z)q^{k_n}(Z)'\pi_n(\theta)]||_2 + o(1)$$

$$\geq \inf_{P \in \mathbf{P}_0} \inf_{\theta \in (\Theta_n \cap R) \setminus \mathcal{V}_n(P)} (E_P[(q^{k_n}(Z)'\pi_n(\theta))^2])^{1/2} + o(1). \quad (S.200)$$

Also note that Assumption A.2.13(iii) implies that for n sufficiently large, we have the set inclusion $\Theta_n \cap R \setminus \mathcal{V}_n(P) \subseteq \{\theta \in \Theta : \|\theta - \Pi_n \theta_0\|_{1,\infty} \ge \epsilon\} \subseteq \{\theta \in \Theta : \|\theta - \theta_0\|_{1,\infty} \ge \epsilon/2\}$ holding for all $P \in \mathbf{P}_0$. Hence, (S.198), (S.200), and Assumption A.2.15(iv) yield

$$S_n(\epsilon) \gtrsim \inf_{P \in \mathbf{P}_0} \inf_{\theta \in \Theta: \|\theta - \theta_0\|_{1,\infty} \ge \epsilon} \frac{k_n^{1/p}}{\sqrt{k_n}} (E_P[(P(Y \le \theta(D)|Z) - \tau)^2])^{1/2} + o(\frac{k_n^{1/p}}{\sqrt{k_n}}). \quad (S.201)$$

Since $J_n \approx 1$ and $B_n \approx 1$, Assumption A.2.14(i), result (S.201) and $k_n \log(1 + k_n)/n = o(1)$ by hypothesis imply that $k_n^{1/p} \sqrt{\log(1 + k_n)} J_n B_n / \sqrt{n} = o(S_n(\epsilon))$ as required by lemma S.1.1. The preceding arguments apply for both $R = \Theta$ and R corresponding to (A.26) and (A.27), and therefore the claim of the lemma follows.

Lemma S.4.25. Let $k_n \log(1 + k_n)/n = o(1)$, and Assumptions A.2.12(i)(iii), A.2.14, A.2.13(ii)(iii), A.2.16, and A.2.15(i)(iii)(iv) hold. For both $R = \Theta$ and R corresponding to (A.26) and (A.27), it follows that Assumption 3.4 holds with $\|\cdot\|_{\mathbf{E}} \equiv \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$, $\nu_n \times \sqrt{k_n}/s_n k_n^{1/p}$, and $\mathcal{V}_n(P) \equiv \{\theta \in \Theta_n : \|\theta - \Pi_n \theta_0\|_{1,\infty} \le \epsilon\}$ for some $\epsilon > 0$.

PROOF: For either $R = \Theta$ or R corresponding to (A.26) and (A.27) set $\mathcal{V}_n(P) \equiv \{\theta \in \mathcal{V}_n(P) \in \mathcal{V}_n(P) \in \mathcal{V}_n(P) \in \mathcal{V}_n(P) \in \mathcal{V}_n(P) \}$

 $\Theta_n \cap R : \|\theta - \Pi_n \theta_0\|_{1,\infty} \leq \epsilon\}$ and note that Assumption 3.4(ii) is then satisfied by Lemma S.4.24. Further observe that since $\Pi_n \theta_0 \in \Theta_n$, there exists a β_{0n} such that $\Pi_n \theta_0 = p^{j_n \prime} \beta_{0n}$. For any $\theta = p^{j_n \prime} \beta \in \mathcal{V}_n(P)$, it then follows by Assumptions A.2.13(ii) and A.2.14(ii) that

$$\sup_{P \in \mathbf{P}} \|p^{j_n'}(\beta - \beta_{0n})\|_{P,2} \lesssim \|\beta - \beta_{0n}\|_{2}
\leq \frac{1}{s_n} \times \inf_{P \in \mathbf{P}_0} \inf_{\|\theta - \Pi_n \theta_0\|_{1,\infty} \leq \epsilon} \|E_P[f_{Y|DZ,P}(\theta(D)|D,Z)q^{k_n}(Z)p^{j_n}(D)'(\beta_n - \beta_{0n})]\|_{2}.$$

Hence, since $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$, the mean value theorem allows us to conclude that

$$\frac{\mathbf{s}_{n}k_{n}^{1/p}}{\sqrt{k_{n}}} \|p^{j_{n}'}(\beta - \beta_{0n})\|_{\mathbf{E}}$$

$$\lesssim \frac{k_{n}^{1/p}}{\sqrt{k_{n}}} \|E_{P}[(P(Y \leq p^{j_{n}}(D)'\beta|D, Z) - P(Y \leq p^{j_{n}}(D)'\beta_{0n}|D, Z))q^{k_{n}}(Z)]\|_{2}$$

$$\lesssim \|E_{P}[(P(Y \leq p^{j_{n}}(D)'\beta|D, Z) - P(Y \leq p^{j_{n}}(D)'\beta_{0n}|D, Z))q^{k_{n}}(Z)]\|_{\Sigma_{P}, p} \qquad (S.202)$$

for any $\theta = p^{j_n'}\beta \in \mathcal{V}_n(P)$ and $P \in \mathbf{P}_0$, and where the final inequality follows from Lemma S.1.5, $||a||_p \leq ||\Sigma_P^{-1}||_{o,p}||\Sigma_P a||_{o,p}$ for any $a \in \mathbf{R}^{k_n}$, and Assumption A.2.16(ii). Since the preceding arguments apply to both $R = \Theta$ and R corresponding to (A.26) and (A.27), result (S.202) verifies Assumption 3.4(i) is satisfied with $\nu_n \approx \sqrt{k_n}/s_n k_n^{1/p}$ for both choices of R and therefore the claim of the lemma follows.

Lemma S.4.26. Let Assumption A.2.12(iii) hold. If $f(y,d) = 1\{y \leq \theta(d)\} - \tau$ for some $\theta \in \Theta$ (Θ as in (A.28)) and $z \mapsto q(z)$ is differentiable with bounded level and derivative, then there exists a $K < \infty$ independent of f, such that for all $P \in \mathbf{P}$ we have

$$\varpi(fq, h, P) \le K \times \{ \|q\|_{\infty} \sqrt{h} + \|q\|_{1,\infty} h \}.$$

PROOF: First note that since $||f||_{\infty} \leq 1$, we can obtain by direct calculation and the definition of the integral modulus of continuity in (S.283) the upper bound

$$\varpi^2(fq, h, P) \le 2||q||_{\infty}^2 \varpi^2(f, h, P) + 2\varpi^2(q, h, P).$$
 (S.203)

For $\Omega_Z(P)$ the support of Z under P, the mean value theorem then implies that

$$\varpi^{2}(q, h, P) \equiv \sup_{\|s\|_{2} \le h} E_{P}[(q(Z+s) - q(Z))^{2} 1\{Z+s \in \Omega_{Z}(P)\}] \le \|q\|_{1,\infty}^{2} h^{2}. \quad (S.204)$$

Furthermore, for any $(s_y, s_d) \in \mathbf{R}^2$ and $d \in [0, 1]$ such that $d + s_d \in [0, 1]$, we also note

that the mean value theorem and Assumption A.2.12(iii) imply that

$$E_{P}[(1\{Y + s_{y} \leq \theta(D + s_{d})\} - 1\{Y \leq \theta(D)\})^{2} | D = d]$$

$$= |P(Y \leq \theta(D + s_{d}) - s_{y} | D = d) - P(Y \leq \theta(D) | D = d)|$$

$$\lesssim |\theta(d + s_{d}) - s_{y} - \theta(d)|. \tag{S.205}$$

Hence, by the law of iterated expectations, a second application of the mean value theorem, and employing that $\|\theta\|_{1,\infty} \leq C_0$ by definition of Θ , we can conclude

$$\varpi^{2}(f, h, P) \leq \sup_{\|(s_{y}, s_{d})\|_{2} \leq h} E_{P}[(1\{Y + s_{y} \leq \theta(D + s_{d})\} - 1\{Y \leq \theta(D)\})^{2}1\{D + s_{d} \in [0, 1]\}]
\lesssim \sup_{\|(s_{y}, s_{d})\|_{2} \leq h} E_{P}[|\theta(D + s_{d}) - s_{y} - \theta(D)|1\{D + s_{d} \in [0, 1]\}] \lesssim h. \quad (S.206)$$

The claim of the lemma then follows from (S.203), (S.204), and (S.206).

Lemma S.4.27. Define the class $\mathcal{F}_n \equiv \{f : f(v) = 1\{y \leq \theta(d)\} - \tau \text{ for some } \theta \in \Theta_n\}$ for Θ_n as in (A.29), and suppose that Assumption A.2.12(iii) and A.2.13(ii) hold. For $\zeta_{j_n} \geq (1 \wedge \sup_{d \in [0,1]} \|p^{j_n}(d)\|_2)$, it then follows that for all $\epsilon \leq 1$ and some $1 \leq K < \infty$

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \le \exp\{\frac{K}{\epsilon}\} \wedge (\frac{K\sqrt{\zeta_{j_n}}}{\epsilon})^{2j_n},$$

and $\sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \lesssim \sqrt{1 \wedge \epsilon} \wedge \sqrt{j_n(\log(\zeta_{j_n}) + \log(1 \vee \epsilon^{-1}))} (1 \wedge \epsilon).$

PROOF: We first note that if $\theta_l(d) \leq \theta(d) \leq \theta_u(d)$, then it immediately follows that

$$1\{y \le \theta_l(d)\} - \tau \le 1\{y \le \theta(d)\} - \tau \le 1\{y \le \theta_u(d)\},\tag{S.207}$$

which implies brackets for Θ_n readily yield brackets in \mathcal{F}_n . Moreover, by the mean value theorem and Assumption A.2.12(iii) we can in addition conclude that

$$E_{P}[(1\{Y \le \theta_{l}(D)\} - 1\{Y \le \theta_{u}(D)\})^{2}]$$

$$= E_{P}[P(Y \le \theta_{u}(D)|D) - P(Y \le \theta_{l}(D)|D)] \lesssim E_{P}[|\theta_{u}(D) - \theta_{l}(D)|]. \quad (S.208)$$

Hence, combining results (S.207) and (S.208) it follows that for some $M_0 < \infty$ we have

$$N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \le N_{[]}(\frac{\epsilon^2}{M_0}, \Theta_n, \|\cdot\|_{P,1}).$$
 (S.209)

On the other hand, since $\Theta_n \subseteq \Theta$, we also obtain by Corollary 2.7.2 in van der Vaart and Wellner (1996), $\|\cdot\|_{P,2} \leq \|\cdot\|_{\infty}$ and inequality (S.209) that

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \le N_{[]}(\frac{\epsilon^2}{M_0}, \Theta_n, \|\cdot\|_{\infty}) \le \exp\{\frac{M_1}{\epsilon}\}.$$
 (S.210)

In addition, the Cauchy-Schwarz inequality implies $\sup_{P\in\mathbf{P}} \|p^{j_n\prime}(\beta_1-\beta_2)\|_{P,1} \leq \zeta_{j_n}\|\beta_1-\beta_2\|_2$ for any $\beta_1,\beta_2\in\mathbf{R}^{j_n}$. Therefore, defining $\mathcal{B}_n\equiv\{\beta\in\mathbf{R}^{j_n}:p^{j_n\prime}\beta\in\Theta_n\}$ Theorem 2.7.11 in van der Vaart and Wellner (1996) allows us to conclude

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,1}) \le N(\frac{\epsilon^2}{2M_0\zeta_{j_n}}, \mathcal{B}_n, \|\cdot\|_2). \tag{S.211}$$

Further note that by Assumption A.2.13(ii), we have $||p^{j_n'}\beta||_{P,2} \approx ||\beta||_2$ uniformly in $P \in \mathbf{P}$ and n, and hence since $\sup_{P \in \mathbf{P}} ||p^{j_n'}\beta||_{P,2} \leq ||p^{j_n'}\beta||_{\infty} \leq C_0$, it follows

$$\sup_{P \in \mathbf{P}} N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,1}) \le \left(\frac{M_2 \sqrt{\zeta_{j_n}}}{\epsilon}\right)^{2j_n} \tag{S.212}$$

for some $M_2 < \infty$ due to result (S.211). The first claim of the lemma therefore follows from (S.210) and (S.212). Moreover, noting $N_{[]}(1, \mathcal{F}_n, \|\cdot\|_{\infty}) = 1$ we also obtain

$$\sup_{P \in \mathbf{P}} J_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \le \left(\int_0^{1/\epsilon} (1 + \frac{K}{u})^{1/2} du\right) \wedge \left(\int_0^{1/\epsilon} (1 + 2j_n \log(\frac{K\sqrt{\zeta_{j_n}}}{u}))^{1/2} du\right)$$

$$\lesssim \sqrt{1 \wedge \epsilon} \wedge \left\{\sqrt{j_n \log(\zeta_{j_n})} + \int_0^1 (j_n \log(\frac{1}{v(1 \wedge \epsilon)}))^{1/2} dv\right\} (1 \wedge \epsilon), \quad (S.213)$$

where the second inequality follows by the change of variables $v = u/(1 \wedge \epsilon)$. The claim of the lemma thus follows from (S.213) and direct calculation.

Lemma S.4.28. Let Assumptions A.2.12(i)(iii)(iv) and A.2.13(ii) hold, Θ_n be as in (A.29), set $\pi_{0n} \equiv \max_{1 \le k \le k_n} \|q_k\|_{\infty}$ and $\pi_{1n} \equiv \max_{1 \le k \le k_n} \|q_k\|_{1,\infty}$, and suppose $\log(k_n \vee \pi_{0n} \vee \sup_d \|p^{j_n}(d)\|_2) = O(\log(n))$. If $j_n/n = o(1)$, then Assumption 3.3(i) holds with $R = \Theta$ for any a_n with $(\pi_{0n}k_n^{1/p}\log(n)\sqrt{j_n}/\sqrt{n})(\sqrt{j_n} + \pi_{0n}n^{1/3} + \pi_{1n}n^{1/6}) = o(a_n)$.

PROOF: We establish the lemma by applying Theorem S.6.1. To this end, define the class $\tilde{\mathcal{F}}_n \equiv \{fq_k \text{ for some } f \in \mathcal{F}_n, 1 \leq k \leq k_n\}$ and let \mathbb{G}_P be a Gaussian process on $\tilde{\mathcal{F}}_n$ satisfying $E[\mathbb{G}_P(f_1)] = 0$ and $E[\mathbb{G}_P(f_1)\mathbb{G}_P(f_2)] = \text{Cov}_P\{f_1(V), f_2(V)\}$ for any $f_1, f_2 \in \tilde{\mathcal{F}}_n$. For any $\theta \in \Theta_n$, set $\mathbb{W}_P(\theta) \equiv (\mathbb{G}_P(\rho(\cdot, \theta)q_1), \dots, \mathbb{G}_P(\rho(\cdot, \theta)q_{k_n})'$ and note

$$\sup_{\theta \in \Theta_n} \|\mathbb{G}_n(\theta) - \mathbb{W}_P(\theta)\|_p \le \sup_{f \in \hat{\mathcal{F}}_n} k_n^{1/p} |\frac{1}{\sqrt{n}} \sum_{i=1}^n (f(V_i) - E_P[f(V)]) - \mathbb{G}_P(f)|. \quad (S.214)$$

We proceed by applying Theorem S.6.1 to the class $\tilde{\mathcal{F}}_n$ with $\delta_n \simeq \sqrt{j_n/n}$. Note that Assumptions S.6.1 and S.6.2 are directly imposed in Assumption A.2.12(iv), while Assumption S.6.3(i) is satisfied by Lemma S.4.26, and Assumption S.6.3(ii) holds with $K_n = \pi_{0n}$ since \mathcal{F} has envelope 1. Furthermore, for S_n as in (S.284), we have

$$S_n^2 \lesssim \sum_{i=0}^{\lceil \log_2 n \rceil} 2^i \left\{ \frac{\pi_{0n}^2}{2^{\frac{i}{3}}} + \frac{\pi_{1n}^2}{2^{\frac{2i}{3}}} \right\} \lesssim \pi_{0n}^2 n^{2/3} + \pi_{1n}^2 n^{1/3}$$

due to Lemma S.4.26. Also note that Lemmas S.1.3 and S.4.27 together imply

$$\sup_{P \in \mathbf{P}} \log(N_{[]}(\delta_n, \tilde{\mathcal{F}}_n, \|\cdot\|_{P,2})) \leq \sup_{P \in \mathbf{P}} \log(k_n N_{[]}(\frac{\delta_n}{\pi_{0n}}, \mathcal{F}_n, \|\cdot\|_{P,2})) \lesssim j_n \log(n),$$

where we employed that $\log(k_n \vee \pi_{0n} \vee \sup_d \|p^{j_n}(d)\|_2) = O(\log(n))$ and $\delta_n = \sqrt{j_n/n}$. Similarly, Lemmas S.1.3 and S.4.27 and the change of variables $v = u/\pi_{0n}$ yield

$$\sup_{P \in \mathbf{P}} J_{[]}(\delta_n, \tilde{\mathcal{F}}_n, \|\cdot\|_{P,2}) \leq \sup_{P \in \mathbf{P}} \int_0^{\delta_n} (1 + \log(k_n) + \log(N_{[]}(\frac{u}{\pi_{0n}}, \mathcal{F}_n, \|\cdot\|_{P,2})))^{1/2} du \\
\leq \delta_n \sqrt{\log(k_n)} + \sup_{P \in \mathbf{P}} J_{[]}(\frac{\delta_n}{\pi_{0n}}, \mathcal{F}_n, \|\cdot\|_{P,2}) \pi_{0n} \lesssim \frac{j_n \sqrt{\log(n)}}{\sqrt{n}},$$

where the final inequality follow from $\log(\pi_{0n}) = O(\log(n))$, $\log(k_n) = O(\log(n))$, and $\log(\sup_d \|p^{j_n}(d)\|_2) = O(\log(n))$. Hence, Theorem S.6.1 implies that

$$\sup_{f \in \tilde{\mathcal{F}}_n} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n (f(V_i) - E_P[f(V)]) - \mathbb{G}_P(f) \right| = O_P(\frac{\pi_{0n} \log(n) \sqrt{j_n}}{\sqrt{n}} \{ \sqrt{j_n} + \pi_{0n} n^{1/3} + \pi_{1n} n^{1/6} \})$$

uniformly in $P \in \mathbf{P}$, which together with (S.214) establishes the Lemma.

Lemma S.4.29. If Assumption A.2.12(iii) holds, then it follows that Assumptions 3.3(ii) and 3.5 are satisfied when $R = \Theta$ (Θ as in (A.28)) with $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$, $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\infty}$, $\kappa_{\rho} = 1/2$, $m_{P}(\theta)(Z) \equiv P(Y \leq \theta(D)|Z)$, and

$$\nabla m_P(\theta)[h](Z) \equiv E_P[f_{Y|DZ,P}(\theta(D)|D,Z)h(D)|Z]. \tag{S.215}$$

PROOF: For $\theta_1 \vee \theta_2$ and $\theta_1 \wedge \theta_2$ the pointwise minimum and maximum of θ_1 and θ_2 , note that the conditional density $f_{Y|DZ,P}$ being bounded in (D,Z) and $P \in \mathbf{P}$ by Assumption A.2.12(iii) together with the mean value theorem imply that

$$E_P[(\rho(X,\theta_1) - \rho(X,\theta_2))^2] = E_P[P(Y \le \theta_1(D) \lor \theta_2(D)|D) - P(Y \le \theta_1(D) \land \theta_2(D)|D)]$$

$$\lesssim E_P[\theta_1(D) \lor \theta_2(D) - \theta_1(D) \land \theta_2(D)] \le \sup_{P \in \mathbf{P}} \|\theta_1 - \theta_2\|_{P,2},$$

where in the final inequality we employed Jensen's inequality and that $\theta_1(d) \vee \theta_2(d) - \theta_1(d) \wedge \theta_2(d) = |\theta_1(d) - \theta_2(d)|$. It thus follows Assumption 3.3(ii) holds with $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$ and $\kappa_{\rho} = 1/2$. Moreover, Jensen's inequality and the mean value theorem imply for some $\bar{\theta}$ such that $\bar{\theta}(d)$ is a convex combination of $\theta_1(d)$ and $\theta_2(d)$ that

$$E_{P}[(P(Y \leq \theta_{1}(D)|Z) - P(Y \leq \theta_{2}(D)|Z) - \nabla m_{P}(\theta_{2})[\theta_{1} - \theta_{2}](Z))^{2}]$$

$$\leq E_{P}[(\{f_{Y|DZ,P}(\bar{\theta}(D)|D,Z) - f_{Y|DZ,P}(\theta_{2}(D)|D,Z)\}\{\theta_{1}(D) - \theta_{2}(D)\})^{2}]$$

$$\lesssim \|\theta_{1} - \theta_{2}\|_{\infty}^{2} \times \sup_{P \in \mathbf{P}} E_{P}[(\theta_{1}(D) - \theta_{2}(D))^{2}],$$

where the final inequality follows from $f_{Y|DZ,P}$ being Lipschitz uniformly in (D,Z) and $P \in \mathbf{P}$. Hence, we may conclude Assumption 3.5(i) is satisfied with $\|\cdot\|_{\mathbf{L}} = \|\cdot\|_{\infty}$ and $\|\cdot\|_{\mathbf{E}} = \sup_{P \in \mathbf{P}} \|\cdot\|_{P,2}$. Furthermore, once again employing Jensen's inequality and that $f_{Y|DZ,P}$ is Lipschitz uniformly in (D,Z) and $P \in \mathbf{P}$ yields

$$E_{P}[(E_{P}[\{f_{Y|DZ,P}(\theta_{1}(D)|D,Z) - f_{Y|DZ,P}(\theta_{2}(D)|D,Z)\}h(D)|Z])^{2}]$$

$$\lesssim \|\theta_{1} - \theta_{2}\|_{\infty}^{2} \times \sup_{P \in \mathbf{P}} \|h\|_{P,2}^{2} \quad (S.216)$$

which implies Assumption 3.5(ii) is also satisfied under the stated choices of $\|\cdot\|_{\mathbf{L}}$ and $\|\cdot\|_{\mathbf{E}}$. Finally, we note Assumption 3.5(iii) is immediate due to Jensen's inequality and $f_{Y|DZ,P}$ being bounded uniformly in (D,Z) and $P \in \mathbf{P}$.

Lemma S.4.30. If Assumption A.2.17(i) holds, $\mathbf{B} = C_B^2([0,1])$ and Υ_G , Υ_F , and Θ are as defined in (A.26), (A.27) (with $\lambda \neq 0$), and (A.28), then it follows that Assumptions 3.8, 3.9, and 3.10 are satisfied with $K_q = 0$, $\nabla \Upsilon_G(\theta)[h] = -\nabla^2 h$, and

$$\nabla \Upsilon_F(\theta)[h] = 2 \int_0^1 \theta(u)h(u)du - 2(\int_0^1 \theta(u)du)(\int_0^1 h(u)du).$$
 (S.217)

PROOF: Note that since Υ_G is linear and continuous, it immediately follows that Assumptions 3.8(i) and 3.8(ii) hold with $\nabla \Upsilon_G = \Upsilon_G$ and $K_g = 0$. It further follows from $\nabla \Upsilon_G = \Upsilon_G$ and the definitions of the operator norm $\|\cdot\|_o$ and $\|\cdot\|_{m,\infty}$ that

$$\|\nabla \Upsilon_G(\theta)\|_o = \sup_{\|h\|_{2,\infty} = 1} \|-\nabla^2 h\|_{\infty} \le 1,$$
 (S.218)

which implies Assumption 3.8(iii) holds with M=1. Moreover, by direct calculation

$$|\Upsilon_F(\theta_1) - \Upsilon_F(\theta_2) - \nabla \Upsilon_F(\theta_1)[\theta_1 - \theta_2]|$$

$$= |\int_0^1 (\theta_1(u) - \theta_2(u))^2 du - (\int_0^1 (\theta_1(u) - \theta_2(u)) du)^2| \le ||\theta_1 - \theta_2||_{2,\infty}^2, \quad (S.219)$$

which implies Υ_F is indeed Fréchet differentiable and its derivative is equal to $\nabla \Upsilon_F$ as defined in (S.217). In addition, by (S.217) and Jensen's inequality we have

$$\|\nabla \Upsilon_F(\theta_1) - \nabla \Upsilon_F(\theta_2)\|_o$$

$$= \sup_{\|h\|_{2,\infty} = 1} 2|\int_0^1 (\theta_1(u) - \theta_2(u))(h(u) - \int_0^1 h(\tilde{u})d\tilde{u})du| \le 2\|\theta_1 - \theta_2\|_{2,\infty}, \quad (S.220)$$

which together with (S.219) implies Assumptions 3.9(i) and 3.9(ii) hold with $K_f = 2$. Next, note that since $\lambda \neq 0$ it follows that $\mathbf{F}_n = \mathbf{R}$. For any $\theta \in \mathbf{B}_n$ such that $\Upsilon_F(\theta) \neq 0$, we then define $\nabla \Upsilon_F(\theta)^- : \mathbf{F}_n \to \mathbf{B}_n$ to be given (for any $c \in \mathbf{R}$) by

$$\nabla \Upsilon_F(\theta)^-[c](d) \equiv c \times \frac{\theta(d) - \int_0^1 \theta(u) du}{2\Upsilon_F(\theta)}, \tag{S.221}$$

and note that since $\theta \in \mathbf{B}_n$ and the constant function is in \mathbf{B}_n by Assumption A.2.17(i), it follows that $\nabla \Upsilon_F(\theta)^-[c] \in \mathbf{B}_n$. Moreover, by direct calculation we obtain

$$\nabla \Upsilon_F(\theta) \nabla \Upsilon_F(\theta)^-[c] = 2 \int_0^1 \theta(u) \{c \times \frac{\theta(u) - \int_0^1 \theta(\tilde{u}) d\tilde{u}}{2\Upsilon_F(\theta)} \} du = c \times \frac{2\Upsilon_F(\theta)}{2\Upsilon_F(\theta)} = c, \text{ (S.222)}$$

which verifies $\nabla \Upsilon_F(\theta)^-$ is indeed the right inverse of $\nabla \Upsilon_F(\theta)$. In addition note that

$$\|\nabla \Upsilon_F(\theta)^-\|_o = \sup_{|c|=1} \|c \times \frac{\theta - \int_0^1 \theta(u) du}{2\Upsilon_F(\theta)}\|_{2,\infty} \le \frac{\|\theta\|_{2,\infty}}{|\Upsilon_F(\theta)|},\tag{S.223}$$

and hence, since $\|\theta\|_{2,\infty} \leq C_0$ and $\Upsilon_F(\theta) = \lambda$ for any $\theta \in \Theta_{0n}^r$, it follows that we may select an $\epsilon > 0$ such that Assumption 3.9(iv) holds with $M = 4C_0/\lambda$.

Next, let θ_2 be the function given by $\theta_2(d) = d^2$ and note that by Assumption A.2.17(i) it follows that $\theta_2 \in \mathbf{B}_n$. For any $\theta \in \Theta_{0n}^r$ we may then set h to equal

$$h \equiv \frac{2\lambda}{C_0} \theta_2 - \frac{\nabla \Upsilon_F(\theta)[\theta_2]}{C_0} \theta, \tag{S.224}$$

which belongs to \mathbf{B}_n since $\theta_2, \theta \in \mathbf{B}_n$. Further observe $\nabla \Upsilon_F(\theta)[\theta] = 2\Upsilon_F(\theta) = 2\lambda$ due to $\theta \in R$, and hence by linearity of $\nabla \Upsilon_F(\theta)$ and (S.224) we can conclude that $h \in \mathbf{B}_n \cap \mathcal{N}(\nabla \Upsilon_F(\theta))$. In addition, it also follows from $\Upsilon_G = \nabla \Upsilon_G$ that

$$\Upsilon_G(\theta)(u) + \nabla \Upsilon_G(\theta)[h](u) = -\nabla^2 \theta(u) \left(1 - \frac{\nabla \Upsilon_F(\theta)[\theta_2]}{C_0}\right) - \frac{4\lambda}{C_0} \le -\frac{4\lambda}{C_0}, \quad (S.225)$$

where the inequality results from $-\nabla^2\theta(u) \leq 0$ due to $\theta \in \Theta_{0n}^r$, $\theta_2(d) = d^2$, and $|\nabla \Upsilon_F(\theta)[\theta_2]| \leq C_0$ because $\|\theta\|_{2,\infty} \leq C_0$ since $\theta \in \Theta_{0n}^r \subseteq \Theta_n$. By similar arguments and the triangle inequality we also have $\|h\|_{2,\infty} \leq 4\lambda/C_0 + C_0$ and hence by (S.225) we conclude Assumption 3.10 is satisfied.

Lemma S.4.31. Suppose Assumptions A.2.12(i)(iii)(iv) and A.2.13(ii) hold, Θ_n be as in (A.29), and let $\pi_{0n} \equiv \max_{1 \leq k \leq k_n} \|q_k\|_{\infty}$ and $\pi_{1n} \equiv \max_{1 \leq k \leq k_n} \|q_k\|_{1,\infty}$. For any sequence $d_n \uparrow \infty$ such that $d_n^4 \log(1+d_n) = o(n)$ and $\delta_n \asymp d_n^{-1/6} + \pi_{1n}/(\pi_{0n}d_n^{1/3})$ satisfies $\delta_n \log(1+k_n) = o(1)$ it follows that uniformly in $P \in \mathbf{P}$ we have

$$\sup_{\theta \in \Theta_n} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^*(\theta)\|_p$$

$$= O_P(\frac{k_n^{1/p} d_n^2 \pi_{0n} \sqrt{\log(1 + d_n)}}{\sqrt{n}} + \pi_{0n} k_n^{1/p} (\sqrt{\delta_n} + \sqrt{n} \exp\{-n\delta_n^3\})).$$

PROOF: We first define the class $\tilde{\mathcal{F}}_n \equiv \{fq_k \text{ for some } f \in \mathcal{F}_n \text{ and } 1 \leq k \leq k_n\}$, let \mathbb{G}_P^* be an isonormal Gaussian process on $\tilde{\mathcal{F}}_n$ independent of $\{V_i\}_{i=1}^n$, set $\mathbb{W}_P^*(\theta) \equiv (\mathbb{G}_P^*(\rho(\cdot,\theta)q_1),\ldots,\mathbb{G}_P^*(\rho(\cdot,\theta)q_{k_n}))'$, and for any $f \in \mathcal{F}_n$ define $\hat{\mathbb{G}}_n(f)$ to equal

$$\hat{\mathbb{G}}_n(f) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i (f(V_i) - \frac{1}{n} \sum_{j=1}^n f(V_j))$$

where $\{\omega_i\}_{i=1}^n$ are the same weights employed in $\hat{\mathbb{W}}_n$. These definitions then imply

$$\sup_{\theta \in \Theta_n} \|\hat{\mathbb{W}}_n(\theta) - \mathbb{W}_P^{\star}(\theta)\|_p \le \sup_{f \in \tilde{\mathcal{F}}_n} k_n^{1/p} |\hat{\mathbb{G}}_n(f) - \mathbb{G}_P^{\star}(f)|. \tag{S.226}$$

In what follows, we aim to establish the lemma by applying Theorem S.7.1 to the class $\tilde{\mathcal{F}}_n$ by relying on a Haar basis expansion as in Lemmas S.6.1 and S.6.2. Specifically, note that by Assumption A.2.12(iv) and Lemma S.6.1, there exists a sequence of partitions $\Delta_n(P) = \{\Delta_{d,n}(P) : d = 1, \ldots, d_n\}$ of the support of $V \equiv (Y, D, Z)$ such that $P(\Delta_{d,n}(P)) = 1/d_n$. For any $1 \le d \le d_n - 1$ we then set $\{f_{d,n,P}\}_{d=1}^{d_n-1}$ to be given by

$$f_{d,n,P}(V) \equiv \frac{(d_n 1\{V \in \Delta_{d,n}(P)\} - 1)}{\sqrt{d_n - 1}}$$
 (S.227)

and let $f_{n,P}^{d_n}(v) \equiv (f_{1,n,P}(v), \dots, f_{d_n-1,n,P}(v))'$. Then note that $E_P[f_{n,P}^{d_n}(V)] = 0$ and

$$E_{P}[f_{d,n,P}(V)f_{\tilde{d},n,P}(V)] = \begin{cases} 1 & \text{if } d = \tilde{d} \\ -\frac{1}{d_{n}-1} & \text{if } d \neq \tilde{d} \end{cases}$$
 (S.228)

By result (S.228) and direct calculation it follows Assumption S.7.1(i) holds with $C_n \approx 1$, while (S.227) implies Assumption S.7.1(ii) holds with $K_n \approx \sqrt{d_n}$. Also note that $\operatorname{Var}_P\{f_{n,P}^{d_n}(V)\} = E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']$ and therefore using that by (S.228) the smallest eigenvalue of $E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']$ is of order $1/d_n$, we obtain uniformly in $P \in \mathbf{P}$ that

$$\|\operatorname{Var}_{P}^{-1}\{f_{n,P}^{d_n}(V)\}\|_{o,2} \lesssim d_n.$$
 (S.229)

We next aim to verify that Assumption S.7.2 is satisfied by setting $\beta_{n,P}(f)$ to be

$$\beta_{n,P}(f) \equiv \begin{pmatrix} \frac{\sqrt{d_n - 1}}{d_n} (E_P[f(V)|V \in \Delta_{1,n}(P)] - E_P[f(V)|V \in \Delta_{d_n,n}(P)]) \\ \vdots \\ \frac{\sqrt{d_n - 1}}{d_n} (E_P[f(V)|V \in \Delta_{d_n - 1,n}(P)] - E_P[f(V)|V \in \Delta_{d_n,n}(P)]) \end{pmatrix}$$
(S.230)

for any $f \in \tilde{\mathcal{F}}_n$. Then observe that, by direct calculation, for any $f \in \tilde{\mathcal{F}}_n$ we have that

$$f_{n,P}^{d_n}(V)'\beta_{n,P}(f)$$

$$= \sum_{d=1}^{d_n-1} (E_P[f(V)|V \in \Delta_{d,n}(P)] - E_P[f(V)|V \in \Delta_{d,n}(P)])(1\{V \in \Delta_{d,n}(P)\} - 1/d_n)$$

$$= \sum_{d=1}^{d_n} E_P[f(V)|V \in \Delta_{d,n}(P)]1\{V \in \Delta_{d,n}(P)\} - E_P[f(V)], \tag{S.231}$$

where the final equality follows from $\{\Delta_{d,n}(P)\}_{d=1}^{d_n}$ being a partition of the support of V that satisfies $P(V \in \Delta_{d,n}(P)) = 1/d_n$. Defining $\mathcal{G}_{n,P} \equiv \{(f - \int f dP) - f_{n,P}^{d_{n'}}\beta_{n,P}(f) : f \in \tilde{\mathcal{F}}_n\}$, then observe that since \mathcal{F}_n has envelope 1, it follows the class $\tilde{\mathcal{F}}_n$ has envelope π_{0n} and hence by (S.231) and Jensen's inequality, the class $\mathcal{G}_{n,P}$ has envelope $G_{n,P} \equiv 2\pi_{0n}$. Moreover, by Lemmas S.6.2 and S.4.26 we can in addition conclude that

$$\sup_{P \in \mathbf{P}} \| (f - \int f dP) - f_{n,P}^{d_{n}\prime} \beta_{n,P}(f) \|_{P,2}^2 \lesssim \frac{\pi_{0n}^2}{d_n^{1/3}} + \frac{\pi_{1n}^2}{d_n^{2/3}},$$

and hence it follows that $||g||_{P,2} \le \delta_n ||G_{n,P}||_{P,2}$ for all $g \in \mathcal{G}_{n,P}$, $P \in \mathbf{P}$, and δ_n satisfying

$$\delta_n \simeq \frac{1}{d_n^{1/6}} + \frac{\pi_{1n}}{\pi_{0n} d_n^{1/3}}.$$

Next, note that if $f(V) \leq f(V) \leq \bar{f}(V)$ almost surely, then result (S.231) yields that

$$\underline{f}(V) - \sum_{d=1}^{d_n} E_P[\bar{f}(V)|V \in \Delta_{d,n}(P)] 1\{V \in \Delta_{d,n}(P)\} \le (f(V) - \int f dP) - f_n^{d_n}(V)' \beta_{n,P}(f)
\le \bar{f}(V) - \sum_{d=1}^{d_n} E_P[\underline{f}(V)|V \in \Delta_{d,n}(P)] 1\{V \in \Delta_{d,n}(P)\} \quad (S.232)$$

which implies brackets for $\tilde{\mathcal{F}}_n$ can be employed to obtain brackets for $\mathcal{G}_{n,P}$. Moreover, by the triangle and Jensen's inequality, the width of the brackets built in (S.232) is bounded by $2\|\bar{f} - \underline{f}\|_{P,2}$. Thus, Lemma S.1.3, and $\tilde{\mathcal{F}}_n$ having envelope π_{0n} yields

$$\sup_{P \in \mathbf{P}} \log(N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2})) \leq \sup_{P \in \mathbf{P}} \log(N_{[]}(\frac{\epsilon}{2}, \tilde{\mathcal{F}}_{n}, \|\cdot\|_{P,2}))$$

$$\leq \log(k_{n}) + \sup_{P \in \mathbf{P}} \log(N_{[]}(\frac{\epsilon}{2\pi_{0n}}, \mathcal{F}_{n}, \|\cdot\|_{P,2})) \lesssim \log(k_{n}) + \frac{\pi_{0n}}{\epsilon} 1\{\epsilon \leq 2\pi_{0n}\} \quad (S.233)$$

where the final inequality follows for any $\epsilon \leq 2\pi_{0n}$ by Lemma S.4.27, and for any $\epsilon > 2\pi_{0n}$ by observing that \mathcal{F}_n is contained in the brackets $[-\tau, 1-\tau]$ which has width 1, and hence $N_{[]}(1, \mathcal{F}_n, \|\cdot\|_{P,2}) = 1$. Recalling that $\mathcal{G}_{n,P}$ has envelope $G_{n,P} \equiv 2\pi_{0n}$, we can

then use result (S.233) to obtain the following upper bound

$$\sup_{P \in \mathbf{P}} J_{[]}(\delta_n \| G_{n,P} \|_{P,2}, \mathcal{G}_{n,P}, \| \cdot \|_{P,2}) \lesssim \int_0^{2\delta_n \pi_{0n}} \sqrt{1 + \log(k_n) + \frac{\pi_{0n}}{\epsilon} 1} \{ \epsilon \leq 2\pi_{0n} \} d\epsilon$$

$$\lesssim \delta_n \pi_{0n} \sqrt{\log(1 + k_n)} + \int_0^{2\delta_n \pi_{0n}} \sqrt{\frac{\pi_{0n}}{\epsilon}} d\epsilon \lesssim \pi_{0n} \sqrt{\delta_n}, \quad (S.234)$$

where in the final inequality we employed that $\delta_n \log(1 + k_n) = o(1)$ by hypothesis. Similarly, for $\eta_{n,P} \equiv 1 + \log N_{[]}(\delta_n || G_{n,P} ||_{P,2}, \mathcal{G}_{n,P}, || \cdot ||_{P,2})$ we can conclude that

$$\sqrt{n}E_{P}[G_{n,P}(V)\exp\{-\frac{n\delta_{n}^{2}\|G_{n,P}\|_{P,2}^{2}}{G_{n,P}^{2}(V)\eta_{n,P}}\}] \lesssim \sqrt{n}\pi_{0n}\exp\{-\frac{n\delta_{n}^{2}}{1+\log(k_{n})+\frac{1}{2\delta_{n}}}1\{\delta_{n}\leq 1\}}\}$$

$$\leq \sqrt{n}\pi_{0n}\exp\{-n\delta_{n}^{3}\} \tag{S.235}$$

where the second inequality holds for n sufficiently large due to $\delta_n \log(1 + k_n) = o(1)$. Together, results (S.234) and (S.235) verify Assumption S.7.2(i) is satisfied with $J_{1n} \approx \pi_{0n}(\sqrt{\delta_n} + \sqrt{n} \exp\{-n\delta_n^3\})$. Finally, let $\mathcal{B}_n \equiv \{\beta_{n,P}(f) : f \in \tilde{\mathcal{F}}_n, P \in \mathbf{P}\}$ and note (S.230), $P(\Delta_{i,n}(P)) = 1/d_n$, Jensen's inequality, and $||f||_{\infty} \leq \pi_{0n}$ for any $f \in \tilde{\mathcal{F}}_n$ imply $||\beta_{n,P}(f)||_2 \lesssim \pi_{0n}$ for all $f \in \tilde{\mathcal{F}}_n$ and $P \in \mathbf{P}$. It thus follows that \mathcal{B}_n is contained in a ball of radius $M\pi_{0n}$ for some $M < \infty$, which allows us to conclude

$$\int_0^\infty \sqrt{N(\epsilon, \mathcal{B}_n, \|\cdot\|_2)} d\epsilon \lesssim \int_0^{M\pi_{0n}} \sqrt{d_n \log(\frac{M\pi_{0n}}{\epsilon})} d\epsilon$$

$$= \sqrt{d_n} M\pi_{0n} \int_0^1 \sqrt{\log(\frac{1}{u})} du = O(\sqrt{d_n}\pi_{0n}), \quad (S.236)$$

where the first equality follows from the change of variables $u = \epsilon/M\pi_{0n}$. Result (S.236) verifies Assumption S.7.2(ii) is satisfied with $J_{2n} \simeq \sqrt{d_n}\pi_{0n}$. In summary, since $d_n^4 \log(1+d_n) = o(n)$ by hypothesis, it follows that the conditions of Theorem S.7.1(ii) hold with $C_n \simeq 1$, $K_n \simeq \sqrt{d_n}$, $\xi_n \simeq d_n$, $J_{1n} \simeq \pi_{0n}(\sqrt{\delta_n} + \sqrt{n}\exp\{-n\delta_n^3\})$, and $J_{2n} \simeq \sqrt{d_n}\pi_{0n}$. Therefore, Theorem S.7.1(ii) allows us to conclude, uniformly in $P \in \mathbf{P}$, that

$$\sup_{f \in \hat{\mathcal{F}}_n} |\hat{\mathbb{G}}_n(f) - \mathbb{G}_P^{\star}(f)| = O_P(\frac{\pi_{0n} d_n^2 \sqrt{\log(1 + d_n)}}{\sqrt{n}} + \pi_{0n}(\sqrt{\delta_n} + \sqrt{n} \exp\{-n\delta_n^3\})),$$

which together with (S.226) establishes the claim of the Lemma.

S.5 Local Parameter Space

This section contains analytical results concerning our approximation to the local parameter space. The main result of this section is the following theorem.

Theorem S.5.1. Let Assumptions 3.1(ii)(iii), 3.8, 3.9, and 3.10 hold, $\{\ell_n, \delta_n, r_n\}_{n=1}^{\infty}$

satisfy $\ell_n \downarrow 0$, $\delta_n 1\{K_f > 0\} \downarrow 0$, $r_n \geq 2(\ell_n + \delta_n) 1\{K_q > 0\}$, $r_n/\delta_n \downarrow 0$, and define

$$G_n(\theta) \equiv \{h \in \mathbf{B}_n : \Upsilon_G(\theta + \frac{h}{\sqrt{n}}) \le (\Upsilon_G(\theta) - K_g r_n \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \mathbf{1}_{\mathbf{G}}) \lor (-r_n \mathbf{1}_{\mathbf{G}}) \}$$

$$A_n(\theta) \equiv \{h \in \mathbf{B}_n : h \in G_n(\theta), \ \Upsilon_F(\theta + \frac{h}{\sqrt{n}}) = 0 \ and \ \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \le \ell_n \}$$

$$T_n(\theta) \equiv \{h \in \mathbf{B}_n : \Upsilon_F(\theta + \frac{h}{\sqrt{n}}) = 0, \ \Upsilon_G(\theta + \frac{h}{\sqrt{n}}) \le 0 \ and \ \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \le 2\ell_n \}.$$

(i) Then, there exist $M < \infty$, $\epsilon > 0$, and $n_0 < \infty$ such that for all $n > n_0$, $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^r$, and $\theta_1 \in (\Theta_{0n}^r)^\epsilon \cap R$ satisfying $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$ we have

$$\sup_{h_1 \in A_n(\theta_1)} \inf_{h_0 \in T_n(\theta_0)} \| \frac{h_1}{\sqrt{n}} - \frac{h_0}{\sqrt{n}} \|_{\mathbf{B}} \le M \times \ell_n(\ell_n + \delta_n) 1\{K_f > 0\}.$$
 (S.237)

(ii) If in addition Υ_G and Υ_F are affine, then for any $\theta_0, \theta_1 \in \mathbf{B}_n \cap R$ with $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$

$$\{h \in \mathbf{B}_n : h \in G_n(\theta_1) \text{ and } \Upsilon_F(\theta_1 + \frac{h}{\sqrt{n}}) = 0\}$$

$$\subseteq \{h \in \mathbf{B}_n : \Upsilon_G(\theta_0 + \frac{h}{\sqrt{n}}) \le 0 \text{ and } \Upsilon_F(\theta_0 + \frac{h}{\sqrt{n}}) = 0\}.$$

PROOF: We begin by establishing part (ii). First note that if Υ_G is affine, then $K_g = 0$ and since $r_n/\delta_n = o(1)$, Lemma S.5.1(ii) implies that for n sufficiently large

$$G_n(\theta_1) \subseteq \{ h \in \mathbf{B}_n : \Upsilon_G(\theta_0 + \frac{h}{\sqrt{n}}) \le 0 \}$$
 (S.238)

for any $\theta_0, \theta_1 \in \mathbf{B}_n$ with $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$. Moreover, if Υ_F is affine and continuous, then $\Upsilon_F(\theta) = L(\theta) + c_0$ for some continuous linear map $L : \mathbf{B} \to \mathbf{F}$ and $c_0 \in \mathbf{F}$. It follows that $\nabla \Upsilon_F(\theta)[h] = L(h)$, which does not depend on θ , and since any $\theta \in R$ must satisfy $L(\theta) = -c_0$ (since $\Upsilon_F(\theta) = 0$), we can conclude that $\{h : \Upsilon_F(\theta + h) = 0\} = \{h : L(h) = 0\}$ whenever $\theta \in R$. Therefore part (ii) follows from result (S.238) and $\theta_1, \theta_2 \in R$.

We next turn to the proof of part (i). Throughout, let $\tilde{\epsilon}$ be such that Assumptions 3.8 and 3.9 hold and set $\epsilon = \tilde{\epsilon}/2$. We break up the proof into four steps.

STEP 1: (Decompose h/\sqrt{n}). For any $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^r$, and $h \in \mathbf{B}_n$ set

$$h^{\perp_{\theta_0}} \equiv \nabla \Upsilon_F(\theta_0)^- \nabla \Upsilon_F(\theta_0)[h] \qquad h^{\mathcal{N}_{\theta_0}} \equiv h - h^{\perp_{\theta_0}}, \qquad (S.239)$$

where recall $\nabla \Upsilon_F(\theta_0)^- : \mathbf{F}_n \to \mathbf{B}_n$ denotes the right inverse of $\nabla \Upsilon_F(\theta_0) : \mathbf{B}_n \to \mathbf{F}_n$. Further note that $h^{\mathcal{N}_{\theta_0}} \in \mathcal{N}(\nabla \Upsilon_F(\theta_0))$ since $\nabla \Upsilon_F(\theta_0) \nabla \Upsilon_F(\theta_0)^- = I$ implies that

$$\nabla \Upsilon_F(\theta_0)[h^{\mathcal{N}_{\theta_0}}] = \nabla \Upsilon_F(\theta_0)[h] - \nabla \Upsilon_F(\theta_0)\nabla \Upsilon_F(\theta_0)^{-} \nabla \Upsilon_F(\theta_0)[h] = 0, \tag{S.240}$$

by definition of $h^{\perp_{\theta_0}}$ in (S.239). Next, observe that if $\theta_1 \in (\Theta_{0n}^r)^{\epsilon} \cap R$ and $h \in \mathbf{B}_n$ satisfies $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ and $\Upsilon_F(\theta_1 + h/\sqrt{n}) = 0$, then $\theta_1 + h/\sqrt{n} \in (\Theta_{0n}^r)^{\tilde{\epsilon}}$ for n sufficiently large, and hence by Assumption 3.9(i) and $\Upsilon_F(\theta_1) = 0$ due to $\theta_1 \in R$

$$\|\nabla \Upsilon_F(\theta_1)[\frac{h}{\sqrt{n}}]\|_{\mathbf{F}} = \|\Upsilon_F(\theta_1 + \frac{h}{\sqrt{n}}) - \Upsilon_F(\theta_1) - \nabla \Upsilon_F(\theta_1)[\frac{h}{\sqrt{n}}]\|_{\mathbf{F}} \le K_f \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}}^2.$$
 (S.241)

Hence, Assumption 3.9(ii), result (S.241), $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$, and $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ imply

$$\|\nabla \Upsilon_F(\theta_0)[\frac{h}{\sqrt{n}}]\|_{\mathbf{F}}$$

$$\leq \|\nabla \Upsilon_F(\theta_0) - \nabla \Upsilon_F(\theta_1)\|_o \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} + K_f \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}}^2 \leq K_f \ell_n (\delta_n + \ell_n). \quad (S.242)$$

Moreover, since $\nabla \Upsilon_F(\theta_0) : \mathbf{F}_n \to \mathbf{B}_n$ satisfies Assumption 3.9(iv), we also have that

$$K_f \|h^{\perp_{\theta_0}}\|_{\mathbf{B}} = K_f \|\nabla \Upsilon_F(\theta_0)^- \nabla \Upsilon(\theta_0)[h]\|_{\mathbf{B}}$$

$$\leq K_f \|\nabla \Upsilon_F(\theta_0)^-\|_o \|\nabla \Upsilon_F(\theta_0)[h]\|_{\mathbf{F}} \leq M_f \|\nabla \Upsilon_F(\theta_0)[h]\|_{\mathbf{F}} \quad (S.243)$$

for some $M_f < \infty$. Further note that if $K_f = 0$, then (S.239) and (S.242) imply $h^{\perp_{\theta_0}} = 0$. Thus, combining results (S.242) and (S.243) to handle the case $K_f > 0$ we conclude that for any $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^r$, $\theta_1 \in (\Theta_{0n}^r)^\epsilon \cap R$ satisfying $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$ and any $h \in \mathbf{B}_n$ such that $\Upsilon_F(\theta_1 + h/\sqrt{n}) = 0$ and $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ we must have

$$\|\frac{h^{\perp \theta_0}}{\sqrt{n}}\|_{\mathbf{B}} \le M_f \ell_n (\delta_n + \ell_n) 1\{K_f > 0\}.$$
 (S.244)

STEP 2: (Inequality Constraints). In what follows, it is convenient to define the set

$$S_n(\theta_0, \theta_1) \equiv \{ h \in \mathbf{B}_n : \Upsilon_G(\theta_0 + \frac{h}{\sqrt{n}}) \le 0, \ \Upsilon_F(\theta_1 + \frac{h}{\sqrt{n}}) = 0, \ \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} \le \ell_n \}.$$

Then note $r_n \ge 2(\ell_n + \delta_n) 1\{K_g > 0\}$, $r_n/\delta_n = o(1)$, and Lemma S.5.1(i) imply that

$$A_n(\theta_1) \subseteq S_n(\theta_0, \theta_1) \tag{S.245}$$

for n sufficiently large, all $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^{\mathrm{r}}$, and $\theta_1 \in (\Theta_{0n}^{\mathrm{r}})^{\epsilon}$ satisfying $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$. The proof will proceed by verifying (S.237) holds with $S_n(\theta_0, \theta_1)$ in place of $A_n(\theta_1)$. In particular, if $K_f = 0$, then $\Upsilon_F(\theta_0) = \Upsilon_F(\theta_1)$ due to $\theta_0, \theta_1 \in R$, and Assumptions 3.9(i)(ii) together with (S.245) imply $A_n(\theta_1) \subseteq S_n(\theta_0, \theta_1) \subseteq T_n(\theta_0)$. Hence, result (S.237) holds for the case $K_f = 0$.

For the rest of the proof we therefore assume $K_f > 0$. We further note that Lemma S.5.2 implies that for any $\eta_n \downarrow 0$, there is an $n_0 < \infty$ and $1 \le C < \infty$ (independent of η_n) such that for all $P \in \mathbf{P}_0$, $n > n_0$, and $\theta_0 \in \Theta_{0n}^r$ there exists a $h_{\theta_0,n} \in \mathbf{B}_n \cap$

 $\mathcal{N}(\nabla \Upsilon_F(\theta_0))$ such that for any $\tilde{h} \in \mathbf{B}_n$ for which there exists a $h \in S_n(\theta_0, \theta_1)$ satisfying $\|(\tilde{h} - h)/\sqrt{n}\|_{\mathbf{B}} \le \eta_n$ the following inequalities hold

$$\Upsilon_G(\theta_0 + \frac{h_{\theta_0,n}}{\sqrt{n}} + \frac{\tilde{h}}{\sqrt{n}}) \le 0 \qquad \|\frac{h_{\theta_0,n}}{\sqrt{n}}\|_{\mathbf{B}} \le C\eta_n. \tag{S.246}$$

<u>STEP 3:</u> (Equality Constraints). The results in this step allow us to address the challenge that $h \in S_n(\theta_0, \theta_1)$ satisfies $\Upsilon_F(\theta_1 + h/\sqrt{n}) = 0$ but not necessarily $\Upsilon_F(\theta_0 + h/\sqrt{n}) = 0$. To this end, let $\mathcal{R}(\nabla \Upsilon_F(\theta_0)^- \nabla \Upsilon_F(\theta_0))$ denote the range of the operator $\nabla \Upsilon_F(\theta_0)^- \nabla \Upsilon_F(\theta_0) : \mathbf{B}_n \to \mathbf{B}_n$ and define the vector subspaces

$$\mathbf{B}_{n}^{\mathcal{N}_{\theta_{0}}} \equiv \mathbf{B}_{n} \cap \mathcal{N}(\nabla \Upsilon_{F}(\theta_{0})) \qquad \mathbf{B}_{n}^{\perp_{\theta_{0}}} \equiv \mathcal{R}(\nabla \Upsilon_{F}(\theta_{0})^{-} \nabla \Upsilon_{F}(\theta_{0})). \tag{S.247}$$

Since $h^{\mathcal{N}_{\theta_0}} \in \mathbf{B}_n^{\mathcal{N}_{\theta_0}}$ by (S.240), the definitions in (S.239) and (S.247) imply that $\mathbf{B}_n = \mathbf{B}_n^{\mathcal{N}_{\theta_0}} + \mathbf{B}_n^{\perp_{\theta_0}}$. Furthermore, since $\nabla \Upsilon_F(\theta_0) \nabla \Upsilon_F(\theta_0)^- = I$, we also have

$$\nabla \Upsilon_F(\theta_0)^- \nabla \Upsilon_F(\theta_0)[h] = h \tag{S.248}$$

for any $h \in \mathbf{B}_n^{\perp \theta_0}$, and thus that $\mathbf{B}_n^{\mathcal{N}_{\theta_0}} \cap \mathbf{B}_n^{\perp \theta_0} = \{0\}$. Since $\mathbf{B}_n = \mathbf{B}_n^{\mathcal{N}_{\theta_0}} + \mathbf{B}_n^{\perp \theta_0}$, it then follows that $\mathbf{B}_n = \mathbf{B}_n^{\mathcal{N}_{\theta_0}} \oplus \mathbf{B}_n^{\perp \theta_0} - \text{i.e.}$ the decomposition in (S.239) is unique. Moreover, we observe that $\mathbf{B}_n^{\mathcal{N}_{\theta_0}} \cap \mathbf{B}_n^{\perp \theta_0} = \{0\}$ further implies the restricted map $\nabla \Upsilon_F(\theta_0) : \mathbf{B}_n^{\perp \theta_0} \to \mathbf{F}_n$ is in fact bijective, and by (S.248) its inverse is $\nabla \Upsilon_F(\theta_0)^- : \mathbf{F}_n \to \mathbf{B}_n^{\perp \theta_0}$.

Recall Υ_F is Fréchet differentiable on $(\Theta_{0n}^{\mathbf{r}})^{\tilde{\epsilon}}$ by Assumption 3.9(i). The preceding discussion and Assumption 3.9 imply we may apply Lemma S.5.4 with $\mathbf{A}_1 = \mathbf{B}_n^{\mathcal{N}_{\theta_0}}$, $\mathbf{A}_2 = \mathbf{B}_n^{\perp_{\theta_0}}$, and some $K_0 < \infty$ to obtain that for any $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^{\mathbf{r}}$ and $h^{\mathcal{N}_{\theta_0}} \in \mathbf{B}_n^{\mathcal{N}_{\theta_0}}$ satisfying $\|h^{\mathcal{N}_{\theta_0}}\|_{\mathbf{B}} \leq \{\tilde{\epsilon}/2 \wedge (2K_0)^{-2} \wedge 1\}^2$, there is a $h^{\star}(h^{\mathcal{N}_{\theta_0}}) \in \mathbf{B}_n^{\perp_{\theta_0}}$ such that

$$\Upsilon_F(\theta_0 + h^{\mathcal{N}_{\theta_0}} + h^*(h^{\mathcal{N}_{\theta_0}})) = 0 \qquad \|h^*(h^{\mathcal{N}_{\theta_0}})\|_{\mathbf{B}} \le 2K_0^2 \|h^{\mathcal{N}_{\theta_0}}\|_{\mathbf{B}}^2.$$
 (S.249)

Moreover, for any $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^{\mathrm{r}}$, $\theta_1 \in (\Theta_{0n}^{\mathrm{r}})^{\epsilon} \cap R$ with $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$, and any $h \in \mathbf{B}_n$ such that $\Upsilon_F(\theta_1 + h/\sqrt{n}) = 0$ and $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$, result (S.244), the decomposition in (S.239), $\delta_n \downarrow 0$ (since $K_f > 0$), and $\ell_n \downarrow 0$ imply that for n large

$$\|\frac{h^{N_{\theta_0}}}{\sqrt{n}}\|_{\mathbf{B}} \le \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} + \|\frac{h^{\perp_{\theta_0}}}{\sqrt{n}}\|_{\mathbf{B}} \le 2\ell_n.$$
 (S.250)

Thus, for $h_{\theta_{0,n}} \in \mathbf{B}_n^{\mathcal{N}_{\theta_0}}$ as in (S.246), $C \geq 1$, and results (S.249) and (S.250) imply that for n sufficiently large we must have for all $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^{\mathrm{r}}$, $\theta_1 \in \mathbf{B}_n \cap R$ with

 $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$ and $h \in \mathbf{B}_n$ satisfying $\Upsilon_F(\theta_1 + h/\sqrt{n}) = 0$ that

$$\Upsilon_{F}(\theta_{0} + \frac{h_{\theta_{0},n}}{\sqrt{n}} + \frac{h^{N_{\theta_{0}}}}{\sqrt{n}} + h^{*}(\frac{h_{\theta_{0},n}}{\sqrt{n}} + \frac{h^{N_{\theta_{0}}}}{\sqrt{n}})) = 0$$

$$\|h^{*}(\frac{h_{\theta_{0},n}}{\sqrt{n}} + \frac{h^{N_{\theta_{0}}}}{\sqrt{n}})\|_{\mathbf{B}} - 16K_{0}^{2}C^{2}(\ell_{n}^{2} + \eta_{n}^{2}) \leq 0.$$
(S.251)

Step 4: (Build Approximation). In order to employ Steps 2 and 3, we now set η_n to

$$\eta_n = 32(M_f + C^2 K_0^2)\ell_n(\ell_n + \delta_n).$$
(S.252)

In addition, for any $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^r$, $\theta_1 \in \mathbf{B}_n \cap R$ satisfying $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$, and any $h \in S_n(\theta_0, \theta_1)$, we let $h^{\mathcal{N}_{\theta_0}}$ be as in (S.239) and define

$$\frac{\hat{h}}{\sqrt{n}} \equiv \frac{h_{\theta_0,n}}{\sqrt{n}} + \frac{h^{\mathcal{N}_{\theta_0}}}{\sqrt{n}} + h^{\star} \left(\frac{h_{\theta_0,n}}{\sqrt{n}} + \frac{h^{\mathcal{N}_{\theta_0}}}{\sqrt{n}}\right). \tag{S.253}$$

From Steps 2 and 3 it then follows that for n sufficiently large (independent of $P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^r$, $\theta_1 \in \mathbf{B}_n \cap R$ with $\|\theta_0 - \theta_1\|_{\mathbf{B}} \leq \delta_n$ or $h \in S_n(\theta_0, \theta_1)$) we have

$$\Upsilon_F(\theta_0 + \frac{\hat{h}}{\sqrt{n}}) = 0. \tag{S.254}$$

Moreover, from results (S.251) and (S.252) we also obtain that for n sufficiently large

$$\|h^{\star}(\frac{h_{\theta_0,n}}{\sqrt{n}} + \frac{h^{\mathcal{N}_{\theta_0}}}{\sqrt{n}})\|_{\mathbf{B}} \le 16C^2 K_0^2(\ell_n^2 + \eta_n^2) \le \frac{\eta_n}{2} + 16C^2 K_0^2 \eta_n^2 \le \frac{3}{4}\eta_n.$$
 (S.255)

Thus, $h = h^{\mathcal{N}_{\theta_0}} + h^{\perp_{\theta_0}}$, (S.244), (S.252), (S.253) and (S.255) imply for large n that $\|(\hat{h} - h - h_{\theta_0,n})/\sqrt{n}\|_{\mathbf{B}} \le \eta_n$, and employing (S.246) with $\tilde{h} = (\hat{h} - h_{\theta_0,n})/\sqrt{n}$ yields

$$\Upsilon_G(\theta_0 + \frac{\hat{h}}{\sqrt{n}}) \le 0. \tag{S.256}$$

Since $||h_{\theta_0,n}/\sqrt{n}||_{\mathbf{B}} \leq C\eta_n$ by (S.246), results (S.244), (S.251), and $||h/\sqrt{n}||_{\mathbf{B}} \leq \ell_n$ for any $h \in S_n(\theta_0, \theta_1)$ imply by (S.252) and $\ell_n \downarrow 0$, $\delta_n \downarrow 0$ that

$$\|\frac{\hat{h}}{\sqrt{n}}\|_{\mathbf{B}} \leq \|\frac{h_{\theta_{0},n}}{\sqrt{n}}\|_{\mathbf{B}} + \|h^{\star}(\frac{h_{\theta_{0},n}}{\sqrt{n}} + \frac{h^{N_{\theta_{0}}}}{\sqrt{n}})\|_{\mathbf{B}} + \|\frac{h^{\perp_{\theta_{0}}}}{\sqrt{n}}\|_{\mathbf{B}} + \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}}$$

$$\leq C\eta_{n} + 16C^{2}K_{0}^{2}(\ell_{n}^{2} + \eta_{n}^{2}) + M_{f}\ell_{n}(\delta_{n} + \ell_{n}) + \ell_{n} \leq 2\ell_{n} \quad (S.257)$$

for n sufficiently large. Therefore, we conclude from (S.254), (S.256), and (S.257) that

 $\hat{h} \in T_n(\theta_0)$. Similarly, (S.244), (S.246), (S.251), and (S.252) yield for some $M < \infty$

$$\|\frac{\hat{h}}{\sqrt{n}} - \frac{h}{\sqrt{n}}\|_{\mathbf{B}} \leq \|\frac{h_{\theta_{0},n}}{\sqrt{n}}\|_{\mathbf{B}} + \|h^{\star}(\frac{h_{\theta_{0},n}}{\sqrt{n}} + \frac{h^{N_{\theta_{0}}}}{\sqrt{n}})\|_{\mathbf{B}} + \|\frac{h^{\perp_{\theta_{0}}}}{\sqrt{n}}\|_{\mathbf{B}}$$

$$\leq C\eta_{n} + 16C^{2}K_{0}^{2}(\ell_{n}^{2} + \eta_{n}^{2}) + M_{f}\ell_{n}(\ell_{n} + \delta_{n}) \leq M\ell_{n}(\ell_{n} + \delta_{n}),$$

which establishes the (S.237) for the case $K_f > 0$.

Lemma S.5.1. Let Assumptions 3.1(ii)(iii), 3.8 hold, and $\ell_n \downarrow 0$ be given. (i) Then, there are $n_0, M_g < \infty$ and $\epsilon > 0$ such that for all $n > n_0, P \in \mathbf{P}_0$, $\theta_0 \in \Theta_{0n}^r$, $\theta_1 \in (\Theta_{0n}^r)^{\epsilon}$:

$$\{h \in \mathbf{B}_n : \Upsilon_G(\theta_1 + \frac{h}{\sqrt{n}}) \le (\Upsilon_G(\theta_1) - K_g r \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \mathbf{1}_{\mathbf{G}}) \lor (-r \mathbf{1}_{\mathbf{G}}) \text{ and } \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \le \ell_n \}$$

$$\subseteq \{h \in \mathbf{B}_n : \Upsilon_G(\theta_0 + \frac{h}{\sqrt{n}}) \le 0 \text{ and } \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \le \ell_n \}$$

for any $r \geq \{M_g \|\theta_0 - \theta_1\|_{\mathbf{B}} + K_g \|\theta_0 - \theta_1\|_{\mathbf{B}}^2\} \vee 2\{\ell_n + \|\theta_0 - \theta_1\|_{\mathbf{B}}\} 1\{K_g > 0\}$. (ii) If in addition Υ_G is affine, then for any n, $\theta_0, \theta_1 \in \mathbf{B}_n$, and $r \geq M_g \|\theta_0 - \theta_1\|_{\mathbf{B}}$ we have

$$\{h \in \mathbf{B}_n : \Upsilon_G(\theta_1 + \frac{h}{\sqrt{n}}) \le \Upsilon_G(\theta_1) \lor (-r\mathbf{1}_{\mathbf{G}})\} \subseteq \{h \in \mathbf{B}_n : \Upsilon_G(\theta_0 + \frac{h}{\sqrt{n}}) \le 0\}.$$

Proof: Let $\tilde{\epsilon} > 0$ be such that Assumption 3.8 holds and set $M_g < \infty$ to satisfy

$$\|\nabla \Upsilon_G(\theta)\|_o \le M_a \tag{S.258}$$

for any $\theta \in (\Theta_{0n}^{\mathrm{r}})^{\tilde{\epsilon}}$, which is possible by Assumption 3.8(iii). Next, set $\epsilon = \tilde{\epsilon}/2$ and define $N(\delta) \equiv \{\theta \in \mathbf{B}_n : \overrightarrow{d}_H(\{\theta\}, \Theta_{0n}^{\mathrm{r}}, \|\cdot\|_{\mathbf{B}}) < \delta\}$ for any $\delta > 0$. Then note that for any $\theta_1 \in N(\epsilon)$ and $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ we have $\theta_1 + h/\sqrt{n} \in N(\tilde{\epsilon})$ for n sufficiently large. Therefore, Assumption 3.8(i) allows us to conclude that

$$\|\Upsilon_G(\theta_1 + \frac{h}{\sqrt{n}}) - \Upsilon_G(\theta_1) - \nabla \Upsilon_G(\theta_1) \left[\frac{h}{\sqrt{n}}\right]\|_{\mathbf{G}} \le K_g \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}}^2. \tag{S.259}$$

Similarly, Assumption 3.8(ii) implies that if $\theta_0 \in \Theta_{0n}^r$ and $\theta_1 \in N(\epsilon)$, then we have

$$\|\nabla \Upsilon_{G}(\theta_{0})[\frac{h}{\sqrt{n}}] - \nabla \Upsilon_{G}(\theta_{1})[\frac{h}{\sqrt{n}}]\|_{\mathbf{G}}$$

$$\leq \|\nabla \Upsilon_{G}(\theta_{0}) - \nabla \Upsilon_{G}(\theta_{1})\|_{o}\|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} \leq K_{g}\|\theta_{0} - \theta_{1}\|_{\mathbf{B}}\|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} \quad (S.260)$$

for any $h \in \mathbf{B}_n$. Hence, since $\Upsilon_G(\theta_0) \leq 0$ due to $\theta_0 \in \Theta_{0n}^{\mathrm{r}} \subseteq \Theta_n \cap R$ we obtain that

$$\Upsilon_{G}(\theta_{0} + \frac{h}{\sqrt{n}}) + \{\Upsilon_{G}(\theta_{1}) - \Upsilon_{G}(\theta_{1} + \frac{h}{\sqrt{n}})\}$$

$$\leq \{\Upsilon_{G}(\theta_{0} + \frac{h}{\sqrt{n}}) - \Upsilon_{G}(\theta_{0})\} + \{\Upsilon_{G}(\theta_{1}) - \Upsilon_{G}(\theta_{1} + \frac{h}{\sqrt{n}})\}$$

$$\leq K_{g} \|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} \{2\|\frac{h}{\sqrt{n}}\|_{\mathbf{B}} + \|\theta_{0} - \theta_{1}\|_{\mathbf{B}}\}\mathbf{1}_{\mathbf{G}}, \tag{S.261}$$

by (S.259), (S.260), and Lemma S.5.3. Also note for any $\theta_0 \in \Theta_{0n}^{\rm r}$, $\theta_1 \in N(\epsilon)$, and $h \in \mathbf{B}_n$ with $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ we have $\theta_0 + h/\sqrt{n} \in N(\tilde{\epsilon})$ and $\theta_1 + h/\sqrt{n} \in N(\tilde{\epsilon})$ for n sufficiently large. Thus, by Assumptions 3.8(i), result (S.258), and Lemma S.5.3

$$\Upsilon_{G}(\theta_{0} + \frac{h}{\sqrt{n}}) - \Upsilon_{G}(\theta_{1} + \frac{h}{\sqrt{n}}) \leq \nabla \Upsilon_{G}(\theta_{0} + \frac{h}{\sqrt{n}})[\theta_{0} - \theta_{1}] + K_{g}\|\theta_{0} - \theta_{1}\|_{\mathbf{B}}^{2} \mathbf{1}_{\mathbf{G}}$$

$$\leq \{M_{q}\|\theta_{0} - \theta_{1}\|_{\mathbf{B}} + K_{q}\|\theta_{0} - \theta_{1}\|_{\mathbf{B}}^{2}\} \mathbf{1}_{\mathbf{G}}. \tag{S.262}$$

Hence, (S.261) and (S.262) yield for $r \geq \{M_g \| \theta_0 - \theta_1 \|_{\mathbf{B}} + K_g \| \theta_0 - \theta_1 \|_{\mathbf{B}}^2 \} \vee 2\{\ell_n + \| \theta_0 - \theta_1 \|_{\mathbf{B}}^2 \} 1\{K_g > 0\}, \ \theta_0 \in \Theta_{0n}^r, \ \theta_1 \in N(\epsilon), \ \|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n, \ \text{and} \ n \ \text{large}$

$$\Upsilon_{G}(\theta_{0} + \frac{h}{\sqrt{n}}) \leq \Upsilon_{G}(\theta_{1} + \frac{h}{\sqrt{n}}) + (K_{g}r \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} - \Upsilon_{G}(\theta_{1})) \mathbf{1}_{\mathbf{G}} \wedge r \mathbf{1}_{\mathbf{G}}$$

$$= \Upsilon_{G}(\theta_{1} + \frac{h}{\sqrt{n}}) - (\Upsilon_{G}(\theta_{1}) - K_{g}r \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}}) \mathbf{1}_{\mathbf{G}} \vee (-r \mathbf{1}_{\mathbf{G}}) \qquad (S.263)$$

where the equality follows from $(-a)\vee(-b)=-(a\wedge b)$ by Theorem 8.6 in Aliprantis and Border (2006). Since $a_1 \leq a_2$ and $b_1 \leq b_2$ implies $a_1 \wedge b_1 \leq a_2 \wedge b_2$ in **G** by Corollary 8.7 in Aliprantis and Border (2006), (S.263) implies that for n sufficiently large and any $\theta_0 \in \Theta_{0n}^r$, $\theta_1 \in N(\epsilon)$ and $h \in \mathbf{B}_n$ satisfying $||h/\sqrt{n}||_{\mathbf{B}} \leq \ell_n$ and

$$\Upsilon_G(\theta_1 + \frac{h}{\sqrt{n}}) \le (\Upsilon_G(\theta_1) - K_g r \| \frac{h}{\sqrt{n}} \|_{\mathbf{B}} \mathbf{1}_{\mathbf{G}}) \lor (-r \mathbf{1}_{\mathbf{G}})$$

we must have $\Upsilon_G(\theta_0 + h/\sqrt{n}) \leq 0$, which verifies the first claim of the lemma. For the second claim, just note that if Υ_G is affine, then we may set $K_g = 0$ and $\epsilon = +\infty$ in Assumption 3.8, which leads to the desired simplification.

Lemma S.5.2. If Assumptions 3.1(ii)(iii), 3.8, 3.10(ii) hold, and $\eta_n \downarrow 0$, $\ell_n \downarrow 0$, then there is a n_0 (depending on η_n, ℓ_n) and a $C < \infty$ (independent of η_n, ℓ_n) such that for all $n > n_0$, $P \in \mathbf{P}_0$, and $\theta \in \Theta_{0n}^r$ there is $h_{\theta,n} \in \mathbf{B}_n \cap \mathcal{N}(\nabla \Upsilon_F(\theta))$ with

$$\Upsilon_G(\theta + \frac{h_{\theta,n}}{\sqrt{n}} + \frac{\tilde{h}}{\sqrt{n}}) \le 0 \qquad \qquad \|\frac{h_{\theta,n}}{\sqrt{n}}\|_{\mathbf{B}} \le C\eta_n$$
 (S.264)

for all $\tilde{h} \in \mathbf{B}_n$ for which there is a $h \in \mathbf{B}_n$ satisfying $\|(\tilde{h} - h)/\sqrt{n}\|_{\mathbf{B}} \le \eta_n$, $\|h/\sqrt{n}\|_{\mathbf{B}} \le \ell_n$, and the inequality $\Upsilon_G(\theta + h/\sqrt{n}) \le 0$.

PROOF: By Assumption 3.10(ii) there are $\epsilon > 0$ and $M_d < \infty$ such that for every $P \in \mathbf{P}_0$, n, and $\theta \in \Theta_{0n}^r$ there exists a $\bar{h}_{\theta,n} \in \mathbf{B}_n \cap \mathcal{N}(\nabla \Upsilon_F(\theta))$ satisfying

$$\Upsilon_G(\theta) + \nabla \Upsilon_G(\theta)[\bar{h}_{\theta,n}] \le -\epsilon \mathbf{1}_{\mathbf{G}} \qquad \|\bar{h}_{\theta,n}\|_{\mathbf{B}} \le M_d.$$
 (S.265)

Also note Assumption 3.8(iii) and $\ell_n = o(1)$ imply that there is an $M_g < \infty$ such that for n sufficiently large and any $h \in \mathbf{B}_n$ satisfying $\|h/\sqrt{n}\|_{\mathbf{B}} \le \ell_n$ we must have

$$\|\nabla \Upsilon_G(\theta + \frac{h}{\sqrt{n}})\|_o \le M_g. \tag{S.266}$$

Moreover, result (S.266), Assumption 3.8(i), Lemma S.5.3, and $\ell_n = o(1)$ imply that for n sufficiently large and any $h \in \mathbf{B}_n$ with $||h/\sqrt{n}||_{\mathbf{B}} \leq \ell_n$ we must have

$$\Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) \leq \Upsilon_{G}(\theta) + \nabla \Upsilon_{G}(\theta) \left[\frac{h}{\sqrt{n}}\right] + K_{g} \left\|\frac{h}{\sqrt{n}}\right\|_{\mathbf{B}}^{2} \mathbf{1}_{\mathbf{G}}$$

$$\leq \Upsilon_{G}(\theta) + \left\{ \|\nabla \Upsilon_{G}(\theta)\|_{o} \ell_{n} + K_{g} \ell_{n}^{2} \right\} \mathbf{1}_{\mathbf{G}} \leq \Upsilon_{G}(\theta) + 2M_{g} \ell_{n} \mathbf{1}_{\mathbf{G}}. \quad (S.267)$$

Hence, (S.265) and (S.267) imply for any $h \in \mathbf{B}_n$ with $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_n$ we must have

$$\Upsilon_G(\theta + \frac{h}{\sqrt{n}}) + \nabla \Upsilon_G(\theta)[\bar{h}_{\theta,n}] \le \{2M_g\ell_n - \epsilon\}\mathbf{1}_{\mathbf{G}}.$$
(S.268)

Next, we let $C_0 > 8M_g/\epsilon$ and aim to show (S.264) holds with $C = C_0M_d$ by setting

$$\frac{h_{\theta,n}}{\sqrt{n}} \equiv C_0 \eta_n \bar{h}_{\theta,n}. \tag{S.269}$$

To this end, we first note that if $\theta \in \Theta_{0n}^{r}$, $h \in \mathbf{B}_{n}$ satisfies $\|h/\sqrt{n}\|_{\mathbf{B}} \leq \ell_{n}$ and $\Upsilon_{G}(\theta + h/\sqrt{n}) \leq 0$, and $\tilde{h} \in \mathbf{B}_{n}$ is such that $\|(h - \tilde{h})/\sqrt{n}\|_{\mathbf{B}} \leq \eta_{n}$, then definition (S.269) implies that $\|(h_{\theta,n} + \tilde{h})/\sqrt{n}\|_{\mathbf{B}} = o(1)$. Therefore, Assumption 3.8(i), Lemma S.5.3, and $\|(\tilde{h} - h)/\sqrt{n}\|_{\mathbf{B}} \leq \eta_{n}$ together allow us to conclude that

$$\Upsilon_{G}(\theta + \frac{h_{\theta,n}}{\sqrt{n}} + \frac{\tilde{h}}{\sqrt{n}})$$

$$\leq \Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) + \nabla \Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) \left[\frac{h_{\theta,n}}{\sqrt{n}} + \frac{(\tilde{h} - h)}{\sqrt{n}} \right] + 2K_{g}(\|\frac{h_{\theta,n}}{\sqrt{n}}\|_{\mathbf{B}}^{2} + \eta_{n}^{2}) \mathbf{1}_{\mathbf{G}}$$

$$\leq \Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) + \nabla \Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) \left[\frac{h_{\theta,n}}{\sqrt{n}} \right] + \left\{ 2K_{g} \|\frac{h_{\theta,n}}{\sqrt{n}}\|_{\mathbf{B}}^{2} + 2M_{g}\eta_{n} \right\} \mathbf{1}_{\mathbf{G}}, \quad (S.270)$$

where the final result follows from result (S.266) and $2K_g\eta_n^2 \leq M_g\eta_n$ for n sufficiently

large. Similarly, Assumption 3.8(ii) and Lemma S.5.3 yield

$$\nabla \Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) \left[\frac{h_{\theta,n}}{\sqrt{n}} \right] \leq \nabla \Upsilon_{G}(\theta) \left[\frac{h_{\theta,n}}{\sqrt{n}} \right] + \|\nabla \Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) - \nabla \Upsilon_{G}(\theta)\|_{o} \|\frac{h_{\theta,n}}{\sqrt{n}}\|_{\mathbf{B}} \mathbf{1}_{\mathbf{G}}$$

$$\leq \nabla \Upsilon_{G}(\theta) \left[\frac{h_{\theta,n}}{\sqrt{n}} \right] + K_{g} \ell_{n} \|\frac{h_{\theta,n}}{\sqrt{n}}\|_{\mathbf{B}} \mathbf{1}_{\mathbf{G}}. \tag{S.271}$$

Hence, results (S.270) and (S.271), $||h_{\theta,n}/\sqrt{n}||_{\mathbf{B}} \leq C_0 M_d \eta_n$ due to $||\bar{h}_{\theta,n}||_{\mathbf{B}} \leq M_d$ by (S.265), and $\eta_n \downarrow 0$, $\ell_n \downarrow 0$, imply that for n sufficiently large we have

$$\Upsilon_G(\theta + \frac{h_{\theta,n}}{\sqrt{n}} + \frac{\tilde{h}}{\sqrt{n}}) \le \Upsilon_G(\theta + \frac{h}{\sqrt{n}}) + \nabla \Upsilon_G(\theta) \left[\frac{h_{\theta,n}}{\sqrt{n}}\right] + 4M_g \eta_n \mathbf{1}_{\mathbf{G}}.$$
 (S.272)

In addition, since $C_0\eta_n\downarrow 0$, we have $C_0\eta_n\leq 1$ eventually, and hence $\Upsilon_G(\theta+h/\sqrt{n})\leq 0$, $2M_g\ell_n\leq \epsilon/2$ for n sufficiently large due to $\ell_n\downarrow 0$, and result (S.268) imply that

$$\Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) + C_{0}\eta_{n}\nabla\Upsilon_{G}(\theta)[\bar{h}_{\theta,n}] \leq C_{0}\eta_{n}\{\Upsilon_{G}(\theta + \frac{h}{\sqrt{n}}) + \nabla\Upsilon_{G}(\theta)[\bar{h}_{\theta,n}]\}$$

$$\leq C_{0}\eta_{n}\{2M_{g}\ell_{n} - \epsilon\}\mathbf{1}_{\mathbf{G}} \leq -\frac{C_{0}\eta_{n}\epsilon}{2}\mathbf{1}_{\mathbf{G}}. \quad (S.273)$$

Thus, we can conclude from results (S.269), (S.272), (S.273), and $C_0 > 8M_g/\epsilon$ that

$$\Upsilon_G(\theta + \frac{h_{\theta,n}}{\sqrt{n}} + \frac{\tilde{h}}{\sqrt{n}}) \le \{4M_g - \frac{C_0\epsilon}{2}\}\eta_n \mathbf{1}_{\mathbf{G}} \le 0,$$

for n sufficiently large, which establishes the claim of the Lemma.

Lemma S.5.3. If **A** is an AM space with norm $\|\cdot\|_{\mathbf{A}}$ and unit $\mathbf{1}_{\mathbf{A}}$, and $a_1, a_2 \in \mathbf{A}$, then it follows that $a_1 \leq a_2 + C\mathbf{1}_{\mathbf{A}}$ for any $a_1, a_2 \in \mathbf{A}$ satisfying $\|a_1 - a_2\|_{\mathbf{A}} \leq C$.

PROOF: Since **A** is an AM space with unit $\mathbf{1}_{\mathbf{A}}$ we have that $||a_1 - a_2||_{\mathbf{A}} \leq C$ implies $|a_1 - a_2| \leq C \mathbf{1}_{\mathbf{A}}$, and hence the claim follows trivially from $a_1 - a_2 \leq |a_1 - a_2|$.

Lemma S.5.4. Let **A** and **C** be Banach spaces with norms $\|\cdot\|_{\mathbf{A}}$ and $\|\cdot\|_{\mathbf{C}}$, $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ and $F: \mathbf{A} \to \mathbf{C}$. Suppose $F(a_0) = 0$ and that there are $\epsilon_0 > 0$ and $K_0 < \infty$ such that:

- (i) $F: \mathbf{A} \to \mathbf{C}$ is Fréchet differentiable at all $a \in \mathcal{B}_{\epsilon_0}(a_0) \equiv \{a \in \mathbf{A} : ||a a_0||_{\mathbf{A}} \le \epsilon_0 \}$.
- (ii) $||F(a+h) F(a) \nabla F(a)[h]||_{\mathbf{C}} \le K_0 ||h||_{\mathbf{A}}^2$ for all $a, a+h \in \mathcal{B}_{\epsilon_0}(a_0)$.
- (iii) $\|\nabla F(a_1) \nabla F(a_2)\|_o \le K_0 \|a_1 a_2\|_{\mathbf{A}}$ for all $a_1, a_2 \in \mathcal{B}_{\epsilon_0}(a_0)$.
- (iv) $\nabla F(a_0) : \mathbf{A} \to \mathbf{C}$ has $\|\nabla F(a_0)\|_o \leq K_0$.
- (v) $\nabla F(a_0): \mathbf{A}_2 \to \mathbf{C}$ is bijective and $\|\nabla F(a_0)^{-1}\|_o \leq K_0$.

Then, for all $h_1 \in \mathbf{A}_1$ with $||h_1||_{\mathbf{A}} \leq (\epsilon_0/2 \wedge (4K_0^2)^{-1} \wedge 1)^2$ there is a unique $h_2^{\star}(h_1) \in \mathbf{A}_2$ with $F(a_0 + h_1 + h_2^{\star}(h_1)) = 0$. In addition, $h_2^{\star}(h_1)$ satisfies $||h_2^{\star}(h_1)||_{\mathbf{A}} \leq 4K_0^2 ||h_1||_{\mathbf{A}}$ for arbitrary \mathbf{A}_1 , and $||h_2^{\star}(h_1)||_{\mathbf{A}} \leq 2K_0^2 ||h_1||_{\mathbf{A}}^2$ when $\mathbf{A}_1 = \mathcal{N}(\nabla F(a_0))$.

PROOF: We closely follow the arguments in the proof of Theorems 4.B in Zeidler (1985). First, we define $g: \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{C}$ pointwise for any $h_1 \in \mathbf{A}_1$ and $h_2 \in \mathbf{A}_2$ by

$$g(h_1, h_2) \equiv \nabla F(a_0)[h_2] - F(a_0 + h_1 + h_2). \tag{S.274}$$

Since $\nabla F(a_0): \mathbf{A}_2 \to \mathbf{C}$ is bijective by hypothesis, $F(a_0 + h_1 + h_2) = 0$ if and only if

$$h_2 = \nabla F(a_0)^{-1}[g(h_1, h_2)].$$
 (S.275)

Letting $T_{h_1}: \mathbf{A}_2 \to \mathbf{A}_2$ be given by $T_{h_1}(h_2) \equiv \nabla F(a_0)^{-1}[g(h_1, h_2)]$, we see from (S.275) that the desired $h_2^*(h_1)$ must be a fixed point of T_{h_1} . Next, define the set

$$M_0 \equiv \{h_2 \in \mathbf{A}_2 : ||h_2||_{\mathbf{A}} \le \delta_0\}$$

for $\delta_0 \equiv (\epsilon_0/2) \wedge (4K_0^2)^{-1} \wedge 1$, and consider an arbitrary $h_1 \in \mathbf{A}_1$ with $||h_1||_{\mathbf{A}} \leq \delta_0^2$. Notice that then $a_0 + h_1 + h_2 \in \mathcal{B}_{\epsilon_0}(a_0)$ for any $h_2 \in M_0$ and hence g is differentiable with respect to h_2 with derivative $\nabla_2 g(h_1, h_2) \equiv \nabla F(a_0) - \nabla F(a_0 + h_1 + h_2)$. Thus, if $h_2, \tilde{h}_2 \in M_0$, then Proposition 7.3.2 in Luenberger (1969) implies that

$$||g(h_1, h_2) - g(h_1, \tilde{h}_2)||_{\mathbf{C}} \leq \sup_{0 < \tau < 1} ||\nabla_2 g(h_1, h_2 + \tau(\tilde{h}_2 - h_2))||_o ||h_2 - \tilde{h}_2||_{\mathbf{A}}$$

$$\leq \frac{1}{2K_0} ||h_2 - \tilde{h}_2||_{\mathbf{A}}, \tag{S.276}$$

where the final inequality follows by Condition (iii) and $\delta_0^2 \leq \delta_0 \leq (4K_0^2)^{-1}$. Moreover,

$$\|\nabla F(a_0)[h_2] - \nabla F(a_0 + h_1)[h_2]\|_{\mathbf{C}}$$

$$\leq \|\nabla F(a_0) - \nabla F(a_0 + h_1)\|_o \|h_2\|_{\mathbf{A}} \leq K_0 \|h_1\|_{\mathbf{A}} \|h_2\|_{\mathbf{A}} \leq \frac{\|h_2\|_{\mathbf{A}}}{4K_0} \quad (S.277)$$

by Condition (iii) and $||h_1||_{\mathbf{A}} \leq \delta_0 \leq (4K_0^2)^{-1}$. Similarly, for any $h_2 \in M_0$ we have

$$||F(a_0 + h_1 + h_2) - F(a_0 + h_1) - \nabla F(a_0 + h_1)[h_2]||_{\mathbf{C}} \le K_0 ||h_2||_{\mathbf{A}}^2 \le \frac{||h_2||_{\mathbf{A}}}{4K_0} \quad (S.278)$$

due to $a_0 + h_1 \in \mathcal{B}_{\epsilon_0}(a_0)$ and Condition (ii). Moreover, since $F(a_0) = 0$ by hypothesis, Conditions (ii) and (iv), $||h_1||_{\mathbf{A}} \leq \delta_0^2$, and $\delta_0 \leq (4K_0^2)^{-1}$ yield that

$$||F(a_0+h_1)||_{\mathbf{C}} = ||F(a_0+h_1)-F(a_0)||_{\mathbf{C}} \le K_0||h_1||_{\mathbf{A}}^2 + ||\nabla F(a_0)||_o||h_1||_{\mathbf{A}} \le \frac{\delta_0}{2K_0}.$$
 (S.279)

Hence, by (S.274) and (S.277)-(S.279) we obtain for any $h_2 \in M_0$ and h_1 with $||h_1||_{\mathbf{A}} \leq \delta_0^2$

$$||g(h_1, h_2)||_{\mathbf{C}} \le \frac{||h_2||_{\mathbf{A}}}{2K_0} + \frac{\delta_0}{2K_0} \le \frac{\delta_0}{K_0}.$$
 (S.280)

Thus, since $\|\nabla F(a_0)^{-1}\|_o \leq K_0$ by Condition (v), result (S.280) implies $T_{h_1}: M_0 \to M_0$, and (S.276) yields $\|T_{h_1}(h_2) - T_{h_1}(\tilde{h}_2)\|_{\mathbf{A}} \leq 2^{-1}\|h_2 - \tilde{h}_2\|_{\mathbf{A}}$ for any $h_2, \tilde{h}_2 \in M_0$. By Theorem 1.1.1.A in Zeidler (1985) we then conclude T_{h_1} has a unique fixed point $h_2^{\star}(h_1) \in M_0$, and the first claim of the lemma follows from (S.274) and (S.275).

Next, we note that since $h_2^{\star}(h_1)$ is a fixed point of T_{h_1} , we can conclude that

$$||h_2^{\star}(h_1)||_{\mathbf{A}} = ||T_{h_1}(h_2^{\star}(h_1))||_{\mathbf{A}} \le ||T_{h_1}(h_2^{\star}(h_1)) - T_{h_1}(0)||_{\mathbf{A}} + ||T_{h_1}(0)||_{\mathbf{A}}.$$
 (S.281)

Thus, since $||T_{h_1}(h_2^{\star}(h_1)) - T_{h_1}(0)||_{\mathbf{A}} \le 2^{-1}||h_2^{\star}(h_1)||_{\mathbf{A}}$ by (S.276) and $||\nabla F(a_0)^{-1}||_o \le K_0$, it follows from result (S.281) and $T_{h_1}(0) \equiv -\nabla F(a_0)^{-1}F(a_0 + h_1)$ that

$$\frac{1}{2} \|h_2^{\star}(h_1)\|_{\mathbf{A}} \leq \|T_{h_1}(0)\|_{\mathbf{A}} \leq K_0 \|F(a_0 + h_1)\|_{\mathbf{C}}$$

$$\leq K_0 \{K_0 \|h_1\|_{\mathbf{A}}^2 + \|\nabla F(a_0)\|_o \|h_1\|_{\mathbf{A}}\} \leq 2K_0^2 \|h_1\|_{\mathbf{A}}, \quad (S.282)$$

where in the second inequality we employed $\|\nabla F(a_0)^{-1}\|_o \leq K_0$, in the third inequality we used (S.279), and in the final inequality we exploited $\|h_1\|_{\mathbf{A}} \leq 1$. While the estimate in (S.282) applies for generic \mathbf{A}_1 , we note that if in addition $\mathbf{A}_1 = \mathcal{N}(\nabla F(a_0))$, then

$$\frac{1}{2} \|h_2^{\star}(h_1)\|_{\mathbf{A}} \leq \|T_{h_1}(0)\|_{\mathbf{A}} \leq K_0 \|F(a_0 + h_1)\|_{\mathbf{C}} \leq K_0^2 \|h_1\|_{\mathbf{A}}^2 ,$$

due to $F(a_0) = 0$ and $\nabla F(a_0)[h_1] = 0$, and thus the final claim of the lemma follows.

S.6 Coupling via Koltchinskii (1994)

In this section we develop uniform coupling results for empirical processes that help verify Assumption 3.3(i) in specific applications. Our analysis is based on the Hungarian construction of Massart (1989) and Koltchinskii (1994), and we state the results in a notation that abstracts from the rest of the paper due to their potential independent interest. Thus, in what follows we consider $V \in \mathbf{R}^d$ to be a generic random variable distributed according to $P \in \mathbf{P}$, denote its support under P by $\Omega(P) \subset \mathbf{R}^d$, and let λ denote the Lebesgue measure on \mathbf{R}^d . For any function f we further set

$$\mathbb{G}_n(f) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (f(V_i) - E_P[f(V)]).$$

The rates obtained through a Hungarian construction crucially depend on the ability of the functions indexing the empirical process to be approximated by a suitable Haar basis. Here, we follow Koltchinskii (1994) and control the relevant approximation errors through primitive conditions stated in terms of the integral modulus of continuity. For a measure P and a function $f: \mathbf{R}^d \to \mathbf{R}$, the integral modulus of continuity of f is the

function $\varpi(f,\cdot,P): \mathbf{R}_+ \to \mathbf{R}_+$ defined for every $h \in \mathbf{R}_+$ as

$$\varpi(f, h, P) \equiv \sup_{\|s\| \le h} \left(\int_{\Omega(P)} (f(v+s) - f(v))^2 1\{v+s \in \Omega(P)\} dP(v) \right)^{\frac{1}{2}}.$$
 (S.283)

Intuitively, the integral modulus of continuity quantifies the "smoothness" of a function f by examining the difference between f and its own translation. For example, it is straightforward to verify that $\varpi(f,h,P)\lesssim h$ whenever f is Lipschitz. In contrast indicator functions such as $f(v)=1\{v\leq t\}$ typically satisfy $\varpi(f,h,P)\lesssim h^{1/2}$.

We impose the following assumptions to establish the uniform coupling results.

Assumption S.6.1. (i) For all $P \in \mathbf{P}$, $P \ll \lambda$ and $\Omega(P) \subset \mathbf{R}^d$ is compact; (ii) The densities $dP/d\lambda$ satisfy $\sup_{P \in \mathbf{P}} \sup_{v \in \Omega(P)} \frac{dP}{d\lambda}(v) < \infty$ and $\inf_{P \in \mathbf{P}} \inf_{v \in \Omega(P)} \frac{dP}{d\lambda}(v) > 0$.

Assumption S.6.2. (i) For each $P \in \mathbf{P}$ there is a continuously differentiable bijection $T_P : [0,1]^d \to \Omega(P)$; (ii) The Jacobian JT_P and its determinant $|JT_P|$ satisfy $\inf_{P \in \mathbf{P}} \inf_{v \in [0,1]^d} |JT_P(v)| > 0$ and $\sup_{P \in \mathbf{P}} \sup_{v \in [0,1]^d} |JT_P(v)||_o < \infty$.

Assumption S.6.3. The classes \mathcal{F}_n satisfy: (i) $\sup_{P \in \mathbf{P}} \sup_{f \in \mathcal{F}_n} \varpi(f, h, P) \leq \varphi_n(h)$ for some $\varphi_n : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying $\varphi_n(Ch) \leq C^{\kappa} \varphi_n(h)$ for all $n, C \geq 1$, and some $\kappa > 0$; and (ii) $\sup_{f \in \mathcal{F}_n} \|f\|_{\infty} \leq K_n$ for some $K_n > 0$.

In Assumption S.6.1 we impose that $V \sim P$ be continuously distributed for all $P \in \mathbf{P}$, with uniformly (in P) bounded supports and densities bounded from above and away from zero. Assumption S.6.2 requires that the support of V under each P be "smooth" in the sense that it be a differentiable transformation of the unit square. Together, Assumptions S.6.1 and S.6.2 enable us to construct partitions of $\Omega(P)$ such that the diameter of each set in the partition is controlled uniformly in P; see Lemma S.6.1. As a result, the approximation error by the Haar basis implied by each partition can be controlled uniformly by the integral modulus of continuity; see Lemma S.6.2. Together with Assumption S.6.3, which imposes conditions on the integral modulus of continuity of \mathcal{F}_n uniformly in P, we can obtain a uniform coupling result through the analysis in Koltchinskii (1994). We note that the homogeneity condition on φ_n in Assumption S.6.3(i) is not necessary, but we impose it to simplify the bound.

The next theorem establishes a coupling for the empirical process \mathbb{G}_n .

Theorem S.6.1. Let Assumptions S.6.1-S.6.3 hold, $\{V_i\}_{i=1}^n$ be i.i.d. with $V_i \sim P \in \mathbf{P}$ and for any $\delta_n \downarrow 0$ let $N_n \equiv \sup_{P \in \mathbf{P}} N_{[]}(\delta_n, \mathcal{F}_n, \|\cdot\|_{P,2}), J_n \equiv \sup_{P \in \mathbf{P}} J_{[]}(\delta_n, \mathcal{F}_n, \|\cdot\|_{P,2}),$

$$S_n \equiv \left(\sum_{i=0}^{\lceil \log_2 n \rceil} 2^i \varphi_n^2 (2^{-\frac{i}{d}})\right)^{\frac{1}{2}}.$$
 (S.284)

If $N_n \uparrow \infty$, there is a Gaussian \mathbb{G}_P (possible depending on n) so that uniformly in $P \in \mathbf{P}$

$$\|\mathbb{G}_n - \mathbb{G}_P\|_{\mathcal{F}_n} = O_P(\frac{K_n \log(nN_n)}{\sqrt{n}} + \frac{K_n \sqrt{\log(nN_n)\log(n)}S_n}{\sqrt{n}} + J_n(1 + \frac{J_n K_n}{\delta_n^2 \sqrt{n}})).$$
 (S.285)

Theorem S.6.1 is a mild modification of the results in Koltchinskii (1994). The proof of Theorem S.6.1 relies on a coupling of the empirical process on a sequence of grids of cardinality N_n , and employs the equicontinuity of \mathbb{G}_n and \mathbb{G}_P to obtain a coupling on the entire class \mathcal{F}_n . The conclusion of Theorem S.6.1 applies to any choice of grid accuracy δ_n . In order to obtain the best rate, δ_n must be chosen to balance the terms in (S.285) and thus depends on the metric entropy of \mathcal{F}_n through the terms N_n and J_n .

Below, we include the proof of Theorem S.6.1 and auxiliary results.

PROOF OF THEOREM S.6.1: Let $\{\Delta_i(P)\}$ be the partitions of $\Omega(P)$ in Lemma S.6.1 and $\mathcal{B}_{P,i}$ the σ -algebra generated by $\Delta_i(P)$. By Lemma S.6.2 and Assumption S.6.3,

$$\sup_{P \in \mathbf{P}} \sup_{f \in \mathcal{F}_n} \left(\sum_{i=0}^{\lceil \log_2 n \rceil} 2^i E_P[(f(V) - E_P[f(V) | \mathcal{B}_{P,i}])^2])^{\frac{1}{2}} \right)$$

$$\leq C_1 \left(\sum_{i=0}^{\lceil \log_2 n \rceil} 2^i \varphi_n^2 (2^{-\frac{i}{d}})^{\frac{1}{2}} \equiv C_1 S_n \quad (S.286) \right)$$

for some constant $C_1 > 0$ and for S_n as defined in (S.284). Next, let $\mathcal{F}_{P,n,\delta_n} \subseteq \mathcal{F}_n$ denote a finite δ_n -net of \mathcal{F}_n with respect to $\|\cdot\|_{P,2}$. Since $N(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \leq N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2})$, it follows from the definition of N_n that we may choose $\mathcal{F}_{P,n,\delta_n}$ so that

$$\sup_{P \in \mathbf{P}} \operatorname{card}(\mathcal{F}_{P,n,\delta_n}) \le \sup_{P \in \mathbf{P}} N_{[]}(\delta_n, \mathcal{F}_n, \| \cdot \|_{P,2}) \equiv N_n.$$
 (S.287)

By Theorem 3.5 in Koltchinskii (1994), (S.286) and (S.287), it follows that for each $n \ge 1$ there exists an isonormal process \mathbb{G}_P , such that for all $\eta_1 > 0$, $\eta_2 > 0$

$$\sup_{P \in \mathbf{P}} P(\frac{\sqrt{n}}{K_n} \| \mathbb{G}_n - \mathbb{G}_P \|_{\mathcal{F}_{P,n,\delta_n}} \ge \eta_1 + \sqrt{\eta_1} \sqrt{\eta_2} (C_1 S_n + 1))$$

$$\lesssim N_n \exp\{-C_2 \eta_1\} + n \exp\{-C_2 \eta_2\}, \quad (S.288)$$

for some $C_2 > 0$. Since $N_n \uparrow \infty$, (S.288) implies for any $\varepsilon > 0$ there are $C_3 > 0$, $C_4 > 0$ sufficiently large, such that setting $\eta_1 \equiv C_3 \log(N_n)$ and $\eta_2 \equiv C_3 \log(n)$ yields

$$\sup_{P \in \mathbf{P}} P(\|\mathbb{G}_n - \mathbb{G}_P\|_{\mathcal{F}_{P,n,\delta_n}} \ge C_4 K_n \times \frac{\log(nN_n) + \sqrt{\log(N_n)\log(n)}S_n}{\sqrt{n}}) < \varepsilon. \quad (S.289)$$

Next, note that by definition of $\mathcal{F}_{P,n,\delta_n}$, there exists a $\Gamma_{n,P}: \mathcal{F}_n \to \mathcal{F}_{P,n,\delta_n}$ such that $\sup_{P\in\mathbf{P}} \sup_{f\in\mathcal{F}_n} \|f-\Gamma_{n,P}(f)\|_{P,2} \leq \delta_n$. Let $D(\epsilon,\mathcal{F}_n,\|\cdot\|_{P,2})$ denote the ϵ -packing number

for \mathcal{F}_n under $\|\cdot\|_{P,2}$, and note $D(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2}) \leq N_{[]}(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2})$. Therefore, by Corollary 2.2.8 in van der Vaart and Wellner (1996) we can conclude that

$$\sup_{P \in \mathbf{P}} E_P[\|\mathbb{G}_P - \mathbb{G}_P \circ \Gamma_{n,P}\|_{\mathcal{F}_n}]$$

$$\lesssim \sup_{P \in \mathbf{P}} \int_0^{\delta_n} \sqrt{\log D(\epsilon, \mathcal{F}_n, \|\cdot\|_{P,2})} d\epsilon \leq \sup_{P \in \mathbf{P}} J_{[]}(\delta_n, \mathcal{F}_n, \|\cdot\|_{P,2}) \equiv J_n. \quad (S.290)$$

Similarly, employing Lemma 3.4.2 in van der Vaart and Wellner (1996) yields that

$$\sup_{P \in \mathbf{P}} E_P[\|\mathbb{G}_n - \mathbb{G}_n \circ \Gamma_{n,P}\|_{\mathcal{F}_n}]$$

$$\lesssim \sup_{P \in \mathbf{P}} J_{[]}(\delta_n, \mathcal{F}_n, \|\cdot\|_{P,2})(1 + \sup_{P \in \mathbf{P}} \frac{J_{[]}(\delta_n, \mathcal{F}_n, \|\cdot\|_{P,2})K_n}{\delta_n^2 \sqrt{n}}) \equiv J_n(1 + \frac{J_n K_n}{\delta_n^2 \sqrt{n}}). \quad (S.291)$$

Therefore, combining (S.289), (S.290), and (S.291) together with the decomposition

$$\|\mathbb{G}_n - \mathbb{G}_P\|_{\mathcal{F}_n} \le \|\mathbb{G}_n - \mathbb{G}_P\|_{\mathcal{F}_{P,n,\delta_n}} + \|\mathbb{G}_n - \mathbb{G}_n \circ \Gamma_{n,P}\|_{\mathcal{F}_n} + \|\mathbb{G}_P - \mathbb{G}_P \circ \Gamma_{n,P}\|_{\mathcal{F}_n},$$

establishes the claim of the theorem by Markov's inequality.

Lemma S.6.1. Let \mathcal{B}_P denote the completion of the Borel σ -algebra on $\Omega(P)$ with respect to P. If Assumptions S.6.1 and S.6.2 hold, then for each $P \in \mathbf{P}$ there exists a sequence $\{\Delta_i(P)\}$ of partitions of $(\Omega(P), \mathcal{B}_P, P)$ such that:

- (i) $\Delta_i(P) = {\Delta_{i,k}(P) : k = 0, \dots, 2^i 1}, \ \Delta_{i,k}(P) \in \mathcal{B}_P, \ and \ \Delta_{0,0}(P) = \Omega(P).$
- (ii) $\Delta_{i,k}(P) = \Delta_{i+1,2k}(P) \cup \Delta_{i+1,2k+1}(P)$ and $\Delta_{i+1,2k}(P) \cap \Delta_{i+1,2k+1}(P) = \emptyset$ for any integers $k = 0, \dots 2^i 1$ and $i \ge 0$.
- (iii) $P(\Delta_{i+1,2k}(P)) = P(\Delta_{i+1,2k+1}(P)) = 2^{-i-1}$ for $k = 0, \dots 2^i 1$, $i \ge 0$.
- (iv) $\sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^i 1} \sup_{v, v' \in \Delta_{i, k}(P)} ||v v'||_2 = O(2^{-\frac{i}{d}}).$
- (v) \mathcal{B}_P equals the completion with respect to P of the σ -algebra generated by $\bigcup_i \Delta_i(P)$.

PROOF: Let \mathcal{A} denote the Borel σ -algebra on $[0,1]^d$, and for any $A \in \mathcal{A}$ define

$$Q_P(A) \equiv P(T_P(A)),\tag{S.292}$$

where $T_P(A) \in \mathcal{B}_P$ due to T_P^{-1} being measurable. Moreover, $Q_P([0,1]^d) = 1$ due to T_P being surjective, and Q_P is σ -additive due to T_P being injective. Hence, we conclude Q_P defined by (S.292) is a probability measure on $([0,1]^d, \mathcal{A})$. In addition, for λ the Lebesgue measure, we obtain from Theorem 3.7.1 in Bogachev (2007) that

$$Q_P(A) = P(T_P(A)) = \int_{T_P(A)} \frac{dP}{d\lambda}(v)d\lambda(v) = \int_A \frac{dP}{d\lambda}(T_P(a))|JT_P(a)|d\lambda(a), \quad (S.293)$$

where $|JT_P(a)|$ denotes the Jacobian of T_P at any point $a \in [0,1]^d$. Hence, Q_P has density with respect to Lebesgue measure given by $g_P(a) \equiv \frac{dP}{d\lambda}(T_P(a))|JT_P(a)|$ for any

 $a \in [0,1]^d$. Next, let $a = (a_1, \ldots, a_d)' \in [0,1]^d$ and define for any $t \in [0,1]$

$$G_{l,P}(t|A) \equiv \frac{Q_P(a \in A : a_l \le t)}{Q_P(A)},\tag{S.294}$$

for any $A \in \mathcal{A}$ with $Q_P(A) > 0$ and $1 \le l \le d$. Also let $m(i) \equiv i - \lfloor \frac{i-1}{d} \rfloor \times d$ (i.e. m(i) equals i modulo d), set $\tilde{\Delta}_{0,0}(P) = [0,1]^d$, and inductively define the partitions of $[0,1]^d$

$$\tilde{\Delta}_{i+1,2k}(P) \equiv \{ a \in \tilde{\Delta}_{i,k}(P) : G_{\mathrm{m}(i+1),P}(a_{\mathrm{m}(i+1)}|\tilde{\Delta}_{i,k}(P)) \le \frac{1}{2} \}$$

$$\tilde{\Delta}_{i+1,2k+1}(P) \equiv \tilde{\Delta}_{i,k}(P) \setminus \tilde{\Delta}_{i+1,2k}(P)$$
(S.295)

for $0 \leq k \leq 2^i - 1$. For $\operatorname{cl}(\tilde{\Delta}_{i,k}(P))$ the closure of $\tilde{\Delta}_{i,k}(P)$, we then note that by construction each $\tilde{\Delta}_{i,k}(P)$ is a hyper-rectangle in $[0,1]^d$ – i.e. it is of the general form

$$\operatorname{cl}(\tilde{\Delta}_{i,k}(P)) = \prod_{i=1}^{d} [l_{i,k,j}(P), u_{i,k,j}(P)].$$

Moreover, since g_P is positive on $[0,1]^d$ by Assumptions S.6.1(ii) and S.6.2(ii), it follows that for any $i \geq 0$, $0 \leq k \leq 2^i - 1$ and $1 \leq j \leq d$, we have

$$l_{i+1,2k,j}(P) = l_{i,k,j}(P)$$

$$u_{i+1,2k,j}(P) = \begin{cases} u_{i,k,j}(P) & \text{if } j \neq m(i+1) \\ \text{solves } G_{m(i+1),P}(u_{i+1,2k,j}(P)|\tilde{\Delta}_{i,k}(P)) = \frac{1}{2} & \text{if } j = m(i+1) \end{cases}$$
(S.296)

Similarly, since $\tilde{\Delta}_{i+1,2k+1}(P) = \tilde{\Delta}_{i,k}(P) \setminus \tilde{\Delta}_{i+1,2k}(P)$, it additionally follows that

$$u_{i+1,2k+1,j}(P) = u_{i,k,j}(P) \qquad l_{i+1,2k+1,j}(P) = \begin{cases} l_{i,k,j}(P) \text{ if } j \neq m(i+1) \\ u_{i+1,2k,j}(P) \text{ if } j = m(i+1) \end{cases}$$
(S.297)

Since $Q_P(\operatorname{cl}(\tilde{\Delta}_{i+1,2k}(P))) = Q_P(\tilde{\Delta}_{i+1,2k}(P))$ by $Q_P \ll \lambda$, (S.294) and (S.296) yield

$$\begin{split} Q_{P}(\tilde{\Delta}_{i+1,2k}(P)) &= Q_{P}(a \in \tilde{\Delta}_{i,k}(P) : a_{\mathrm{m}(i+1)} \leq u_{i+1,2k,\mathrm{m}(i+1)}(P)) \\ &= G_{\mathrm{m}(i+1),P}(u_{i+1,2k,\mathrm{m}(i+1)}(P) | \tilde{\Delta}_{i,k}(P)) Q_{P}(\tilde{\Delta}_{i,k}(P)) \\ &= \frac{1}{2} Q_{P}(\tilde{\Delta}_{i,k}(P)). \end{split}$$

Therefore, since $\tilde{\Delta}_{i,k}(P) = \tilde{\Delta}_{i+1,2k}(P) \cup \tilde{\Delta}_{i+1,2k+1}(P)$, it follows $Q_P(\tilde{\Delta}_{i+1,2k+1}(P)) = \frac{1}{2}Q_P(\tilde{\Delta}_{i,k}(P))$ for $0 \le k \le 2^i - 1$ as well. In particular, $Q_P(\tilde{\Delta}_{0,0}(P)) = 1$ implies that

$$Q_P(\tilde{\Delta}_{i,k}(P)) = \frac{1}{2^i} \tag{S.298}$$

for any integers $i \geq 1$ and $0 \leq k \leq 2^i - 1$. Moreover, we note that result (S.293) and

Assumptions S.6.1(ii) and S.6.2(ii) together imply that the density g_P of Q_P satisfies

$$0 < \inf_{P \in \mathbf{P}} \inf_{a \in [0,1]^d} g_P(a) < \sup_{P \in \mathbf{P}} \sup_{a \in [0,1]^d} g_P(a) < \infty, \tag{S.299}$$

and therefore $Q_P(A) \simeq \lambda(A)$ uniformly in $A \in \mathcal{A}$ and $P \in \mathbf{P}$. Hence, since by (S.296) $u_{i+1,2k,j}(P) = u_{i,k,j}(P)$ and $l_{i+1,2k,j}(P) = l_{i,k,j}(P)$ for all $j \neq m(i+1)$, we obtain

$$\frac{(u_{i+1,2k,m(i+1)}(P) - l_{i+1,2k,m(i+1)}(P))}{(u_{i,k,m(i+1)}(P) - l_{i,k,m(i+1)}(P))} = \frac{\prod_{j=1}^{d} (u_{i+1,2k,j}(P) - l_{i+1,2k,j}(P))}{\prod_{j=1}^{d} (u_{i,k,j}(P) - l_{i,k,j}(P))}
= \frac{\lambda(\tilde{\Delta}_{i+1,2k}(P))}{\lambda(\tilde{\Delta}_{i,k}(P))} \approx \frac{Q_P(\tilde{\Delta}_{i+1,2k}(P))}{Q_P(\tilde{\Delta}_{i,k}(P))} = \frac{1}{2} \quad (S.300)$$

uniformly in $P \in \mathbf{P}$, $i \ge 0$, and $0 \le k \le 2^i - 1$ by results (S.298) and (S.299). Moreover, by identical arguments but using (S.297) instead of (S.296) we conclude

$$\frac{(u_{i+1,2k+1,m(i+1)}(P) - l_{i+1,2k+1,m(i+1)}(P))}{(u_{i,k,m(i+1)}(P) - l_{i,k,m(i+1)}(P))} \approx \frac{1}{2}$$
(S.301)

also uniformly in $P \in \mathbf{P}$, $i \geq 0$ and $0 \leq k \leq 2^i - 1$. Thus, since $(u_{i+1,2k,j}(P) - l_{i+1,2k,j}(P)) = (u_{i+1,2k+1,j}(P) - l_{i+1,2k+1,j}(P)) = (u_{i,k,j}(P) - l_{i,k,j}(P))$ for all $j \neq m(i+1)$, and $u_{0,0,j}(P) - l_{0,0,j}(P) = 1$ for all $1 \leq j \leq d$ we obtain from $m(i) = i - \lfloor \frac{i-1}{d} \rfloor \times d$, results (S.300) and (S.301), and proceeding inductively that

$$(u_{i,k,j}(P) - l_{i,k,j}(P)) \approx 2^{-\frac{i}{d}},$$
 (S.302)

uniformly in $P \in \mathbf{P}$, $i \geq 0$, $0 \leq k \leq 2^i - 1$, and $1 \leq j \leq d$. Thus, result (S.302) yields

$$\sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^{i} - 1} \sup_{a, a' \in \tilde{\Delta}_{i,k}(P)} \|a - a'\|$$

$$\leq \sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^{i} - 1} \max_{1 \le j \le d} \sqrt{d} \times (u_{i,j,k}(P) - l_{i,j,k}(P)) = O(2^{-\frac{i}{d}}). \quad (S.303)$$

We next obtain the desired sequence of partitions $\{\Delta_i(P)\}$ of $(\Omega(P), \mathcal{B}_P, P)$ by constructing them from the partition $\{\tilde{\Delta}_{i,k}(P)\}$ of $[0,1]^d$. To this end, set

$$\Delta_{i,k}(P) \equiv T_P(\tilde{\Delta}_{i,k}(P))$$

for all $i \ge 0$ and $0 \le k \le 2^i - 1$. Note that $\{\Delta_i(P)\}$ satisfies conditions (i) and (ii) due to T_P^{-1} being a measurable map, T_P being bijective, and result (S.295). In addition, $\{\Delta_i(P)\}$ satisfies condition (iii) since by definition (S.292) and result (S.298) we have

$$P(\Delta_{i,k}(P)) = P(T_P(\tilde{\Delta}_{i,k}(P))) = Q_P(\tilde{\Delta}_{i,k}(P)) = 2^{-i},$$

for all $0 \le k \le 2^i - 1$. Moreover, by Assumption S.6.2(ii), $\sup_{P \in \mathbf{P}} \sup_{a \in [0,1]^d} ||JT_P(a)||_o < \infty$, and hence by the mean value theorem we can conclude that

$$\sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^{i} - 1} \sup_{v, v' \in \Delta_{i,k}(P)} \|v - v'\| = \sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^{i} - 1} \sup_{a, a' \in \tilde{\Delta}_{i,k}(P)} \|T_{P}(a) - T_{P}(a')\|$$

$$\lesssim \sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^{i} - 1} \sup_{a, a' \in \tilde{\Delta}_{i,k}(P)} \|a - a'\| = O(2^{-\frac{i}{d}}),$$

by result (S.303), which verifies that $\{\Delta_i(P)\}$ satisfies condition (iv). Also note that to verify $\{\Delta_i(P)\}$ satisfies condition (v) it suffices to show that $\bigcup_{i\geq 0} \Delta_i(P)$ generates the Borel σ -algebra on $\Omega(P)$. To this end, we first aim to show that

$$\mathcal{A} = \sigma(\bigcup_{i \ge 0} \tilde{\Delta}_i(P)), \tag{S.304}$$

where for a collection of sets C, $\sigma(C)$ denotes the σ -algebra generated by C. For any closed set $A \in A$, then define $D_i(P)$ to be given by

$$D_i(P) \equiv \bigcup_{k: \tilde{\Delta}_{i,k}(P) \cap A \neq \emptyset} \tilde{\Delta}_{i,k}(P).$$

Notice that since $\{\tilde{\Delta}_i(P)\}$ is a partition of $[0,1]^d$, $A \subseteq D_i(P)$ for all $i \ge 0$ and hence $A \subseteq \bigcap_{i \ge 0} D_i(P)$. Moreover, if $a_0 \in A^c$, then A^c being open and (S.303) imply $a_0 \notin D_i(P)$ for i sufficiently large. Hence, $A^c \cap (\bigcap_{i \ge 0} D_i(P)) = \emptyset$ and therefore $A = \bigcap_{i \ge 0} D_i(P)$. It follows that if A is closed, then $A \in \sigma(\bigcup_{i \ge 0} \tilde{\Delta}_i(P))$, which implies $A \subseteq \sigma(\bigcup_{i \ge 0} \tilde{\Delta}_i(P))$. On the other hand, since $\tilde{\Delta}_{i,k}(P)$ is Borel for all $i \ge 0$ and $0 \le k \le 2^i - 1$, we also have $\sigma(\bigcup_{i \ge 0} \tilde{\Delta}_i(P)) \subseteq A$, and hence (S.304) follows. To conclude, we then note that

$$\sigma(\bigcup_{i\geq 0} \Delta_i(P)) = \sigma(\bigcup_{i\geq 0} T_P(\tilde{\Delta}_i(P))) = T_P(\sigma(\bigcup_{i\geq 0} \tilde{\Delta}_i(P))) = T_P(\mathcal{A}), \tag{S.305}$$

by Corollary 1.2.9 in Bogachev (2007). However, T_P and T_P^{-1} being continuous implies $T_P(\mathcal{A})$ equals the Borel σ -algebra in $\Omega(P)$, and therefore (S.305) implies $\{\Delta_i(P)\}$ satisfies condition (v) establishing the lemma.

Lemma S.6.2. Let $\{\Delta_i(P)\}$ be as in Lemma S.6.1, and $\mathcal{B}_{P,i}$ denote the σ -algebra generated by $\Delta_i(P)$. If Assumptions S.6.1 and S.6.2 hold, then there are $K_0 > 0$, $K_1 \ge 1$ such that for all $P \in \mathbf{P}$ and any f satisfying $f \in L_P^2$ for all $P \in \mathbf{P}$:

$$E_P[(f(V) - E_P[f(V)|\mathcal{B}_{P,i}])^2] \le K_0 \times \varpi^2(f, K_1 \times 2^{-\frac{i}{d}}, P).$$

PROOF: Since $\Delta_i(P)$ is a partition of $\Omega(P)$ and $P(\Delta_{i,k}(P)) = 2^{-i}$ for all $i \geq 0$ and

 $0 \le k \le 2^i - 1$, we may express $E_P[f(V)|\mathcal{B}_{P,i}]$ as an element of L_P^2 by

$$E_P[f(V)|\mathcal{B}_{P,i}] = 2^i \sum_{k=0}^{2^i - 1} 1\{V \in \Delta_{i,k}(P)\} \int_{\Delta_{i,k}(P)} f(v) dP(v).$$

Hence, employing that $P(\Delta_{i,k}(P)) = 2^{-i}$ for all $i \geq 0$ and $0 \leq k \leq 2^i - 1$ together with $\Delta_i(P)$ being a partition of $\Omega(P)$, and applying Holder's inequality to the term $(f(v) - f(\tilde{v}))1\{v \in \Omega(P)\}1\{\tilde{v} \in \Delta_{i,k}(P)\}$ we can conclude that

$$\begin{split} E_{P}[(f(V) - E_{P}[f(V) | \mathcal{B}_{P,i}])^{2}] \\ &= \sum_{k=0}^{2^{i}-1} \int_{\Delta_{i,k}(P)} (f(v) - 2^{i} \int_{\Delta_{i,k}(P)} f(\tilde{v}) dP(\tilde{v}))^{2} dP(v) \\ &= \sum_{k=0}^{2^{i}-1} 2^{2i} \int_{\Delta_{i,k}(P)} (\int_{\Delta_{i,k}(P)} (f(v) - f(\tilde{v})) 1\{v \in \Omega(P)\} dP(\tilde{v}))^{2} dP(v) \\ &\leq \sum_{k=0}^{2^{i}-1} 2^{2i} P(\Delta_{i,k}(P)) \int_{\Delta_{i,k}(P)} \int_{\Delta_{i,k}(P)} (f(v) - f(\tilde{v}))^{2} 1\{v \in \Omega(P)\} dP(\tilde{v}) dP(v) \\ &= \sum_{k=0}^{2^{i}-1} 2^{i} \int_{\Delta_{i,k}(P)} \int_{\Delta_{i,k}(P)} (f(v) - f(\tilde{v}))^{2} 1\{v \in \Omega(P)\} dP(\tilde{v}) dP(v). \end{split}$$

Let $D_i \equiv \sup_{P \in \mathbf{P}} \max_{0 \le k \le 2^i - 1} \operatorname{diam} \{\Delta_{i,k}(P)\}$, where $\operatorname{diam} \{\Delta_{i,k}(P)\}$ is the diameter of $\Delta_{i,k}(P)$. Further note that by Lemma S.6.1(iv), $D_i = O(2^{-\frac{i}{d}})$ and hence we have $\lambda(\{s \in \mathbf{R}^d : \|s\| \le D_i\}) \le M_1 2^{-i}$ for some $M_1 > 0$ and λ the Lebesgue measure. Noting that $\sup_{P \in \mathbf{P}} \sup_{v \in \Omega(P)} \frac{dP}{d\lambda}(v) < \infty$ by Assumption S.6.1(ii), and doing the change of variables $s = v - \tilde{v}$ we then obtain for some constant $M_0 > 0$ that

$$E_{P}[(f(V) - E_{P}[f(V)|\mathcal{B}_{P,i}])^{2}]$$

$$\leq M_{0} \sum_{k=0}^{2^{i}-1} 2^{i} \int_{\Delta_{i,k}(P)} \int_{\|s\| \leq D_{i}} (f(\tilde{v}+s) - f(\tilde{v}))^{2} 1\{s + \tilde{v} \in \Omega(P)\} d\lambda(s) dP(\tilde{v})$$

$$\leq M_{0} M_{1} \sup_{\|s\| \leq D_{i}} \sum_{k=0}^{2^{i}-1} \int_{\Delta_{i,k}(P)} (f(\tilde{v}+s) - f(\tilde{v}))^{2} 1\{\tilde{v}+s \in \Omega(P)\} dP(\tilde{v}). \quad (S.306)$$

Hence, since $\{\Delta_{i,k}(P): k=0...2^i-1\}$ is a partition of $\Omega(P)$, $\varpi(f,h,P)$ is decreasing in h, and $D_i \leq K_1 2^{-\frac{i}{d}}$ for some $K_1 \geq 1$ by Lemma S.6.1(iv), we obtain

$$E_P[(f(V) - E_P[f(V)|\mathcal{B}_{P,i}])^2] \le M_0 M_1 \times \varpi^2(f, K_1 \times 2^{-\frac{i}{d}}, P)$$
 (S.307)

by (S.306). Setting $K_0 \equiv M_0 \times M_1$ in (S.307) then establishes the lemma.

S.7 Uniform Bootstrap Coupling

We next provide uniform coupling results for the multiplier bootstrap that allow us to verify Assumption 3.11 in a variety of problems. The results in this appendix may be of independent interest, as they extend the validity of the multiplier bootstrap to suitable non-Donsker classes \mathcal{F}_n . For this reason, as in Section S.6, we state the results in a notation that abstracts from the rest of the paper. Hence, here $V \in \mathbf{R}^d$ should be interpreted as a generic random variable whose distribution is given by $P \in \mathbf{P}$. For $\{\omega_i\}_{i=1}^n$ i.i.d. standard normal random variables independent of $\{V_i\}_{i=1}^n$ we also set

$$\hat{\mathbb{G}}_n(f) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \{ f(V_i) - \frac{1}{n} \sum_{j=1}^n f(V_j) \}.$$

Our coupling results rely on a series approximation to the elements of \mathcal{F}_n . To this end, we will assume that for each $P \in \mathbf{P}$ there is a basis $\{f_{d,n,P}\}_{d=1}^{d_n}$, with d_n possibly diverging to infinity, that provides a suitable approximation to every $f \in \mathcal{F}_n$. Formally, for $f_{n,P}^{d_n}(v) \equiv (f_{1,n,P}(v), \dots, f_{d_n,n,P}(v))'$, we impose the following:

Assumption S.7.1. For each $P \in \mathbf{P}$ there is an array of functions $\{f_{d,n,P}\}_{d=1}^{d_n} \subset L_P^2$ such that: (i) The eigenvalues of $E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']$ are bounded by $1 \leq C_n$ uniformly in $P \in \mathbf{P}$; (ii) $\sup_{P \in \mathbf{P}} \max_{1 \leq d \leq d_n} ||f_{d,n,P}||_{\infty} \leq K_n$ with $1 \leq K_n$ finite.

Assumption S.7.2. For every $f \in \mathcal{F}_n$ and $P \in \mathbf{P}$ there is a $\beta_{n,P}(f) \in \mathbf{R}^{d_n}$ such that: (i) The class $\mathcal{G}_{n,P} \equiv \{(f - \int f dP) - f_{n,P}^{d_{n'}} \beta_{n,P}(f) : f \in \mathcal{F}_n\}$ has envelope $G_{n,P}$ which satisfies $\|g\|_{P,2} \leq \delta_n \|G_{n,P}\|_{P,2}$ for all $P \in \mathbf{P}$, $g \in \mathcal{G}_{n,P}$, and some $\delta_n > 0$ with

$$J_{1n} \equiv \sup_{P \in \mathbf{P}} \{ J_{[]}(\delta_n \| G_{n,P} \|_{P,2}, \mathcal{G}_{n,P}, \| \cdot \|_{P,2}) + \sqrt{n} E_P[G_{n,P}(V) \exp\{-\frac{n\delta_n^2 \| G_{n,P} \|_{P,2}^2}{G_{n,P}^2(V) \eta_{n,P}}\}] \}$$

finite and $\eta_{n,P} \equiv 1 + \log N_{[]}(\delta_n \|G_{n,P}\|_{P,2}, \mathcal{G}_{n,P}, \|\cdot\|_{P,2});$ (ii) The set $\mathcal{B}_n \equiv \{\beta_{n,P}(f) : f \in \mathcal{F}_n, P \in \mathbf{P}\} \cup \{0\} \text{ satisfies } J_{2n} \equiv \int_0^\infty \sqrt{\log(N(\epsilon, \mathcal{B}_n, \|\cdot\|_2))} d\epsilon < \infty.$

Assumption S.7.1 imposes our regularity conditions on the approximating functions $\{f_{d,n,P}\}_{d=1}^{d_n}$. We emphasize that the functions $\{f_{d,n,P}\}_{d=1}^{d_n}$ need not be known: They are only employed in the theoretical construction of the coupling. In certain applications, such as when \mathcal{F}_n is finite dimensional, a basis $\{f_{d,n,P}\}_{d=1}^{d_n}$ may be naturally available. The approximating requirements on $\{f_{d,n,P}\}_{d=1}^{d_n}$ are imposed in Assumption S.7.2. In particular, Assumption S.7.2(i) requires that the remainder of the approximation of \mathcal{F}_n by $\{f_{d,n,P}\}_{d=1}^{d_n}$ not be "too large." Intuitively, Assumption S.7.2(i) controls the "bias" in a series approximation of \mathcal{F}_n by linear combinations of $\{f_{d,n,P}\}_{d=1}^{d_n}$. Assumption S.7.2(ii) in turn controls the "variance" of the series approximation by demanding that the class of approximating functions have a finite entropy.

We next show Assumptions S.7.1 and S.7.2 suffice for coupling $\hat{\mathbb{G}}_n$.

Theorem S.7.1. Let Assumptions S.7.1, S.7.2 hold, $\{(\omega_i, V_i)\}_{i=1}^n$ be i.i.d. with $V_i \sim P \in \mathbf{P}$, $\omega_i \sim N(0,1)$, ω_i and V_i independent, and $d_n \log(1+d_n)K_n^2C_n = o(n)$. Then: (i) There is a linear Gaussian \mathbb{G}_P^* (possibly depending on n) independent of $\{V_i\}_{i=1}^n$ with

$$\|\hat{\mathbb{G}}_n - \mathbb{G}_P^*\|_{\mathcal{F}_n} = O_P(J_{2n}\{\frac{K_n^2 C_n d_n \log(1 + d_n)}{n}\}^{1/4} + J_{1n})$$

uniformly in $P \in \mathbf{P}$ with $E[\mathbb{G}_P^{\star}(f)] = 0$ and $E[(\mathbb{G}_P^{\star}(f)\mathbb{G}_P^{\star}(g))] = \operatorname{Cov}_P\{f(V), g(V)\}$ for any $f, g \in \mathcal{F}_n$. (ii) If in addition $\sup_{P \in \mathbf{P}} \|(\operatorname{Var}_P\{f_{n,P}^{d_n}(V)\})^{-1}\|_{o,2} \leq \xi_n < \infty$ and $\xi_n \sqrt{d_n \log(1 + d_n)C_n} K_n / \sqrt{n} = o(1)$, then uniformly in $P \in \mathbf{P}$

$$\|\hat{\mathbb{G}}_n - \mathbb{G}_P^*\|_{\mathcal{F}_n} = O_P(\frac{J_{2n}K_n\sqrt{\xi_nC_nd_n\log(1+d_n)}}{\sqrt{n}} + J_{1n}).$$

Theorem S.7.1(i) derives a rate of convergence for the coupled process, while Theorem S.7.1(ii) improves on the rate under the additional requirement that $\operatorname{Var}_P\{f_{n,P}^{d_n}(V)\}$ be bounded away from singularity. The rates of both Theorems S.7.1(i) and S.7.1(ii) depend on the selected sequence d_n , which should be chosen optimally. Heuristically, the proof of Theorem S.7.1 proceeds in two steps. First, we construct a multivariate normal random variable $\mathbb{G}_P^{\star}(f_{n,P}^{d_n}) \in \mathbf{R}^{d_n}$ that is coupled with $\hat{\mathbb{G}}_n(f_{n,P}^{d_n}) \in \mathbf{R}^{d_n}$, and then employ the linearity of $\hat{\mathbb{G}}_n$ to obtain a suitable coupling on the subspace $\mathbb{S}_{n,P} \equiv \overline{\operatorname{span}}\{f_{n,P}^{d_n}\}$. Second, we employ Assumption S.7.2(i) to show that a successful coupling on $\mathbb{S}_{n,P}$ leads to the desired construction since \mathcal{F}_n is well approximated by $\{f_{d,n,P}\}_{d=1}^{d_n}$.

Below, we include the proof of Theorem S.7.1 and auxiliary results.

PROOF OF THEOREM S.7.1: We first couple $\hat{\mathbb{G}}_n$ on a finite dimensional subspace and then show that such a result suffices for coupling $\hat{\mathbb{G}}_n$ and \mathbb{G}_P^{\star} on \mathcal{F}_n . To this end, let $\mathbb{S}_{n,P} \equiv \overline{\operatorname{span}}\{f_{n,P}^{d_n}\}$ and note that Assumption S.7.2(ii) and Lemma S.7.1 imply that there exists a linear Gaussian process $\mathbb{G}_p^{(1)}$ on $\mathbb{S}_{n,P}$ and a sequence $R_n = o(1)$ such that

$$\sup_{\beta \in \mathcal{B}_n} |\hat{\mathbb{G}}_n(f_{n,P}^{d_{n'}}\beta) - \mathbb{G}_P^{(1)}(f_{n,P}^{d_{n'}}\beta)| = O_P(J_{2n}R_n)$$
 (S.308)

uniformly in $P \in \mathbf{P}$, $E[\mathbb{G}_P^{(1)}(f_{n,P}^{d_{n'}}\beta)] = 0$, and also $E[(\mathbb{G}_P^{(1)}(f_{n,P}^{d_{n'}}\beta_1))(\mathbb{G}_P^{(1)}(f_{n,P}^{d_{n'}}\beta_2))] = \operatorname{Cov}_P(f_{n,P}^{d_n}(V)'\beta_1, f_{n,P}^{d_n}(V)'\beta_2)$. To establish part (i) of the theorem we will set $R_n = (d_n \log(1+d_n)C_nK_n^2/n)^{1/4}$ and employ Lemma S.7.1(i), while to establish part (ii) we will set $R_n = (\xi_n d_n \log(1+d_n)C_nK_n^2/n)^{1/2}$ and employ Lemma S.7.1(ii) instead.

Next note that since $\hat{\mathbb{G}}_n(f-c) = \hat{\mathbb{G}}_n(f)$ for any $c \in \mathbf{R}$ and $f \in L_P^2$, we may assume without loss of generality that $E_P[f(V)] = 0$ for all $f \in \mathcal{F}_n$. For any closed linear subspace $\mathbb{A}_{n,P} \subseteq L_P^2$ let $\operatorname{Proj}\{f|\mathbb{A}_{n,P}\}$ denote the $\|\cdot\|_{P,2}$ projection of f onto $\mathbb{A}_{n,P}$ and $\mathbb{A}_{n,P}^{\perp} \equiv \{f \in L_P^2 : f = g - \operatorname{Proj}\{g|\mathbb{A}_{n,P}\}\$ for some $g \in L_P^2\}$. Assuming the underlying

probability space is suitably enlarged to carry a linear isonormal Gaussian process $\mathbb{G}_P^{(2)}$ on $\{\operatorname{Proj}\{f|\mathbb{S}_{n,P}^{\perp}\}: f \in \mathcal{F}_n \cup \mathcal{G}_{n,P}\}$ independent of $\mathbb{G}_P^{(1)}$ and $\{V_i\}_{i=1}^n$, we then set

$$\mathbb{G}_P^{\star}(f) \equiv \mathbb{G}_P^{(1)}(\operatorname{Proj}\{f|\mathbb{S}_{n,P}\}) + \mathbb{G}_P^{(2)}(\operatorname{Proj}\{f|\mathbb{S}_{n,P}^{\perp}\}),$$

which is linear in f by linearity of $f \mapsto \operatorname{Proj}\{f|\mathbb{S}_{n,P}\}$, $\mathbb{G}_P^{(1)}$, and $\mathbb{G}_P^{(2)}$, and satisfies $E[\mathbb{G}_P^{\star}(f)] = 0$ and $E[\mathbb{G}_P^{\star}(f)\mathbb{G}_P^{\star}(g)] = \operatorname{Cov}_P\{f(V), g(V)\}$. Moreover, since \mathbb{G}_P^{\star} is sub-Gaussian with respect to $\|\cdot\|_{P,2}$, it follows from Corollary 2.2.8 in van der Vaart and Wellner (1996), $N(\delta_n\|G_{n,P}\|_{P,2}, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}) = 1$ due to $\|g\|_{P,2} \leq \delta_n\|G_{n,P}\|_{P,2}$ for all $g \in \mathcal{G}_{n,P}$ and $P \in \mathbf{P}$, bracketing numbers being larger than covering numbers, Jensen's inequality, and the definition of J_{1n} in Assumption S.7.2(i) that

$$E_{P}[\sup_{g \in \mathcal{G}_{n,P}} | \mathbb{G}_{P}^{\star}(g) |] \lesssim \delta_{n} \|G_{n,P}\|_{P,2} + \int_{0}^{\infty} \sqrt{\log(N(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}))} d\epsilon$$

$$\leq \delta_{n} \|G_{n,P}\|_{P,2} + \int_{0}^{\delta_{n} \|G_{n,P}\|_{P,2}} \sqrt{1 + \log(N_{[]}(\epsilon, \mathcal{G}_{n,P}, \|\cdot\|_{P,2}))} d\epsilon \lesssim J_{1n}. \quad (S.309)$$

To obtain an analogous bound for $\hat{\mathbb{G}}_n$, note $\sup_{g \in \mathcal{G}_{n,P}} \|g\|_{P,2} \le \delta_n \|G_{n,P}\|_{P,2}$ by Assumption S.7.2(i) and $|E_P[g(V)]| \le \|g\|_{P,2}$ by Jensen's inequality imply that

$$\sup_{g \in \mathcal{G}_{n,P}} |\hat{\mathbb{G}}_n(g)| \le \sup_{g \in \mathcal{G}_{n,P}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i g(V_i) \right| + \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i \right| \times \left\{ \sup_{g \in \mathcal{G}_{n,P}} \left| \frac{1}{n} \sum_{i=1}^n g(V_i) - E_P[g(V)] \right| + \delta_n \|G_{n,P}\|_{P,2} \right\}.$$
 (S.310)

Next, define the class $\tilde{\mathcal{G}}_{n,P} \equiv \{(\omega,v) \mapsto \omega g(v) : g \in \mathcal{G}_{n,P}\}$, and with some abuse of notation let P index the joint distribution of (V,ω) . Further note that if $\{[g_{i,l,P},g_{i,u,P}]\}_i$ is a bracket for $\mathcal{G}_{n,P}$, then the functions $\{[\tilde{g}_{i,l,P},\tilde{g}_{i,u,P}]\}$ given by

$$\tilde{g}_{i,l,P}(\omega, v) \equiv \max\{\omega, 0\} g_{i,l,P}(v) + \min\{\omega, 0\} g_{i,u,P}(v)$$

$$\tilde{g}_{i,u,P}(\omega, v) \equiv \min\{\omega, 0\} g_{i,l,P}(v) + \max\{\omega, 0\} g_{i,u,P}(v)$$

form a bracket for $\tilde{\mathcal{G}}_{n,P}$. Moreover, since $E[\omega^2] = 1$ and ω and V are independent, it follows that $\|\tilde{g}_{i,u,P} - \tilde{g}_{i,l,P}\|_{P,2} = \|g_{i,u,P} - g_{i,l,P}\|_{P,2}$. Setting $\tilde{G}_{n,P}(\omega,v) \equiv |\omega|G_{n,P}(v)$, then note that $\tilde{G}_{n,P}$ is an envelope for $\tilde{\mathcal{G}}_{n,P}$, which satisfies $\|\tilde{G}_{n,P}\|_{P,2} = \|G_{n,P}\|_{P,2}$. For $\eta_{n,P} \equiv 1 + \log N_{[]}(\delta_n \|G_{n,P}\|_{P,2}, \mathcal{G}_{n,P}, \|\cdot\|_{P,2})$ (as in Assumption S.7.2(i)) we then obtain

by Theorem 2.14.2 in van der Vaart and Wellner (1996) that

$$E_{P}[\sup_{g \in \mathcal{G}_{n,P}} | \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_{i} g(V_{i}) |] \lesssim J_{[]}(\delta_{n} \| G_{n,P} \|_{P,2}, \mathcal{G}_{n,P}, \| \cdot \|_{P,2})$$

$$+ \sqrt{n} E_{P}[|\omega| G_{n,P}(V) 1\{ |\omega| \frac{G_{n,P}(V)}{\| G_{n,P} \|_{P,2}} > \frac{\sqrt{n} \delta_{n}}{\sqrt{\eta_{n,P}}} \}]. \quad (S.311)$$

Moreover, since ω follows a standard normal distribution, we have $E[|\omega|1\{|\omega| > a\}] \lesssim \exp\{-a^2/2\}$ for any $a \geq 0$. Therefore, the independence of ω and V implies

$$E_{P}[|\omega|G_{n,P}(V)1\{|\omega|\frac{G_{n,P}(V)}{\|G_{n,P}\|_{P,2}} > \frac{\sqrt{n}\delta_{n}}{\sqrt{\eta_{n,P}}}\}] \lesssim E_{P}[G_{n,P}(V)\exp\{-\frac{n\delta_{n}^{2}\|G_{n,P}\|_{P,2}^{2}}{2G_{n,P}^{2}(V)\eta_{n,P}}\}]$$

which together with result (S.311) and the definition of J_{1n} in Assumption S.7.2(i) yields

$$E_P\left[\sup_{g \in \mathcal{G}_{n,P}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \omega_i g(V_i) \right| \right] \lesssim J_{1n}. \tag{S.312}$$

Moreover, by Lemmas 2.3.1 and 2.9.1 in van der Vaart and Wellner (1996) we have

$$E_{P}[\sup_{g \in \mathcal{G}_{n,P}} | \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(V_{i}) - E_{P}[g(V)]|] + \delta_{n} \|G_{n,P}\|_{P,2}$$

$$\lesssim E_{P}[\sup_{g \in \mathcal{G}_{n,P}} | \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_{i} g(V_{i})|] + \delta_{n} \|G_{n,P}\|_{P,2} \lesssim J_{1n}, \quad (S.313)$$

where the final inequality follows from (S.312) and the definition of J_{1n} . Thus, (S.310), (S.312), and (S.313) together with Markov's inequality imply that uniformly in $P \in \mathbf{P}$

$$\|\hat{\mathbb{G}}_n\|_{\mathcal{G}_{n,P}} = O_P(J_{1n}). \tag{S.314}$$

Next, we use the linearity of the processes $f \mapsto \hat{\mathbb{G}}_n(f)$ and $f \mapsto \mathbb{G}_P^*(f)$ to obtain that

$$\|\hat{\mathbb{G}}_{n} - \mathbb{G}_{P}^{\star}\|_{\mathcal{F}_{n}} \leq \sup_{f \in \mathcal{F}_{n}} |\hat{\mathbb{G}}_{n}(f_{n,P}^{d_{n}\prime}\beta_{n,P}(f)) - \mathbb{G}_{P}^{\star}(f_{n,P}^{d_{n}\prime}(\beta_{n,P}(f)))| + \|\hat{\mathbb{G}}_{n} - \mathbb{G}_{P}^{\star}\|_{\mathcal{G}_{n,P}}$$

$$\leq \sup_{\beta \in \mathcal{B}_{n}} |\hat{\mathbb{G}}_{n}(f_{n,P}^{d_{n}\prime}\beta) - \mathbb{G}_{P}^{\star}(f_{n,P}^{d_{n}\prime}\beta)| + O_{P}(J_{1n}) = O_{P}(J_{2n}R_{n} + J_{1n}),$$

where the second inequality holds uniformly in $P \in \mathbf{P}$ by (S.309) and Markov's inequality, result (S.314), and set inclusion, while the equality holds uniformly in $P \in \mathbf{P}$ by result (S.308). The first claim of the theorem then follows by using Lemma S.7.1(i) to set $R_n = (d_n \log(1 + d_n)C_nK_n^2/n)^{1/4}$ in (S.308), and the second part of the theorem follows from using Lemma S.7.1(ii) to set $R_n = (\xi_n d_n \log(1 + d_n)C_nK_n^2/n)^{1/2}$ instead.

Lemma S.7.1. Let $\{(\omega_i, V_i)\}_{i=1}^n$ be i.i.d. with $V_i \sim P \in \mathbf{P}$, $\omega_i \sim N(0,1)$, and ω_i and V_i independent. Suppose Assumption S.7.1 holds, $d_n \log(1+d_n)K_n^2C_n = o(n)$, and

 $\mathcal{B}_n \subset \mathbf{R}^{d_n}$ satisfies $0 \in \mathcal{B}_n$ and $J_{2n} \equiv \int_0^\infty \sqrt{\log(N(\epsilon, \mathcal{B}_n, \|\cdot\|_2))} d\epsilon < \infty$. Then: (i) There is a linear Gaussian process \mathbb{G}_P^{\star} on $\mathbb{S}_{n,P} \equiv \overline{\operatorname{span}}\{f_{n,P}^{d_n}\}$ independent of $\{V_i\}_{i=1}^n$ with

$$\sup_{\beta \in \mathcal{B}_n} |\hat{\mathbb{G}}_n(f_{n,P}^{d_{n'}}\beta) - \mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)| = O_P(J_{2n}\{\frac{d_n \log(1 + d_n)C_nK_n^2}{n}\}^{1/4})$$

uniformly in $P \in \mathbf{P}$ and satisfying $E[\mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)] = 0$ and $E[\mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta_1)\mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta_2)] = \operatorname{Cov}_P\{f_{n,P}^{d_n}(V)'\beta_1, f_{n,P}^{d_n}(V)'\beta_2\}.$ (ii) If in addition $\sup_{P \in \mathbf{P}} \|\operatorname{Var}_P^{-1}\{f_{n,P}^{d_n}(V)\}\|_{o,2} \leq \xi_n < \infty$ and $\xi_n \sqrt{d_n \log(1 + d_n)C_n}K_n/\sqrt{n} = o(1)$, then uniformly in $P \in \mathbf{P}$

$$\sup_{\beta \in \mathcal{B}_n} |\hat{\mathbb{G}}_n(f_{n,P}^{d_{n'}}\beta) - \mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)| = O_P(\frac{J_{2n}\sqrt{\xi_n d_n \log(1 + d_n)C_n}K_n}{\sqrt{n}}).$$

PROOF: First note that $\hat{\mathbb{G}}_n(f-c) = \hat{\mathbb{G}}_n(f)$ for any $c \in \mathbf{R}$ and $f \in L_P^2$. We may therefore assume without loss of generality that $E_P[f_{n,P}^{d_n}(V)] = 0$, and for every $P \in \mathbf{P}$ we let $\Sigma_n(P) \equiv \operatorname{Var}_P\{f_{n,P}^{d_n}(V)\} = E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']$ and define

$$\hat{\Sigma}_n(P) \equiv \frac{1}{n} \sum_{i=1}^n (f_{n,P}^{d_n}(V_i) - \frac{1}{n} \sum_{j=1}^n f_{n,P}^{d_n}(V_j)) (f_{n,P}^{d_n}(V_i) - \frac{1}{n} \sum_{j=1}^n f_{n,P}^{d_n}(V_j))'.$$

For a sequence R_n with $R_n = o(1)$, and any constant M > 0 and $P \in \mathbf{P}$ define the event

$$A_{n,P}(M) \equiv \{ \|\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P)\|_{o,2} \le MR_n \}.$$
 (S.315)

Further note that by Lemma S.7.2 it follows we may select $R_n = o(1)$ such that we have

$$\liminf_{M \uparrow \infty} \liminf_{n \to \infty} \inf_{P \in \mathbf{P}} P(\{V_i\}_{i=1}^n \in A_{n,P}(M)) = 1.$$
 (S.316)

In particular, to establish part (i) we will set $R_n = (d_n \log(1+d_n)C_nK_n^2/n)^{1/4}$ and employ Lemma S.7.2(i), while to establish part (ii) we will set $R_n = (\xi_n d_n \log(1+d_n)C_nK_n^2/n)^{1/2}$ and employ Lemma S.7.2(ii) instead.

Next, let $\mathcal{N}_{d_n} \in \mathbf{R}^{d_n}$ follow a standard normal distribution and be independent of $\{(\omega_i, V_i)\}_{i=1}^n$ (defined on the same suitably enlarged probability space). Further let $\{\hat{\nu}_d\}_{d=1}^{d_n}$ denote eigenvectors of $\hat{\Sigma}_n(P)$, $\{\hat{\lambda}_d\}_{d=1}^{d_n}$ represent the corresponding (possibly zero) eigenvalues and define the random variable $\mathbb{Z}_{n,P} \in \mathbf{R}^{d_n}$ to be given by

$$\mathbb{Z}_{n,P} \equiv \sum_{d:\hat{\lambda}_d \neq 0} \hat{\nu}_d \frac{(\hat{\nu}_d' \hat{\mathbb{G}}_n(f_{n,P}^{d_n}))}{\hat{\lambda}_d^{1/2}} + \sum_{d:\hat{\lambda}_d = 0} \hat{\nu}_d(\hat{\nu}_d' \mathcal{N}_{d_n}). \tag{S.317}$$

Then note that since $\hat{\mathbb{G}}_n(f_{n,P}^{d_n}) \sim N(0,\hat{\Sigma}_n(P))$ conditional on $\{V_i\}_{i=1}^n$, and \mathcal{N}_{d_n} is inde-

pendent of $\{(\omega_i, V_i)\}_{i=1}^n$, $\mathbb{Z}_{n,P}$ is Gaussian conditional on $\{V_i\}_{i=1}^n$. Furthermore,

$$E[\mathbb{Z}_{n,P}\mathbb{Z}'_{n,P}|\{V_i\}_{i=1}^n] = \sum_{d=1}^{d_n} \hat{\nu}_d \hat{\nu}'_d = I_{d_n}$$

by direct calculation for I_{dn} the $d_n \times d_n$ identity matrix. Hence, $\mathbb{Z}_{n,P} \sim N(0,I_{dn})$ conditional on $\{V_i\}_{i=1}^n$ almost surely in $\{V_i\}_{i=1}^n$ and is thus independent of $\{V_i\}_{i=1}^n$. Moreover, we also note that by Theorem 3.6.1 in Bogachev (1998) and $\hat{\mathbb{G}}_n(f_{n,P}^{d_n}) \sim N(0,\hat{\Sigma}_n(P))$ conditional on $\{V_i\}_{i=1}^n$, it follows that $\hat{\mathbb{G}}_n(f_{n,P}^{d_n})$ belongs to the range of $\hat{\Sigma}_n(P): \mathbf{R}^{d_n} \to \mathbf{R}^{d_n}$ almost surely in $\{(\omega_i,V_i)\}_{i=1}^n$. Therefore, since $\{\hat{\nu}_d: \hat{\lambda}_d \neq 0\}_{d=1}^{d_n}$ spans the range of $\hat{\Sigma}_n(P)$, we conclude from (S.317) that for any $\beta \in \mathbf{R}^{d_n}$

$$\beta' \hat{\Sigma}_n^{1/2}(P) \mathbb{Z}_{n,P} = \beta' \sum_{d: \hat{\lambda}_d \neq 0} \hat{\nu}_d(\hat{\nu}_d' \hat{\mathbb{G}}_n(f_{n,P}^{d_n})) = \hat{\mathbb{G}}_n(\beta' f_{n,P}^{d_n}).$$

Analogously, we define for any $\beta \in \mathbf{R}^{d_n}$ the linear Gaussian process \mathbb{G}_P^{\star} on $\mathbb{S}_{n,P}$ by

$$\mathbb{G}_P^{\star}(\beta' f_{n,P}^{d_n}) \equiv \beta' \Sigma_n^{1/2}(P) \mathbb{Z}_{n,P},$$

which by construction is independent of $\{V_i\}_{i=1}^n$ and satisfies $E[\mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)] = 0$ and $E[\mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta_1)\mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta_2)] = \text{Cov}_P\{f_{n,P}^{d_n}(V)'\beta_1, f_{n,P}^{d_n}(V)'\beta_2\}$. Setting

$$\bar{\mathbb{G}}_{P}(\beta) \equiv (\beta'(\hat{\Sigma}_{n}^{1/2}(P) - \Sigma_{n}^{1/2}(P))\mathbb{Z}_{n,P})1\{A_{n,P}(M)\},\tag{S.318}$$

where $1\{A_{n,P}(M)\}$ denotes the indicator for the event $\{V_i\}_{i=1}^n \in A_{n,P}(M)$, then note

$$\sup_{\beta \in \mathcal{B}_n} |\hat{\mathbb{G}}_n(f_{n,P}^{d_{n'}}\beta) - \mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)|1\{A_{n,P}(M)\} = \sup_{\beta \in \mathcal{B}_n} |\bar{\mathbb{G}}_P(\beta)|. \tag{S.319}$$

Moreover, we note that conditional on $\{V_i\}_{i=1}^n$, $\bar{\mathbb{G}}_P$ is sub-Gaussian under the semimetric $\rho_n(\tilde{\beta}, \beta) \equiv \|(\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P))(\tilde{\beta} - \beta)\|_2$. Since $\|\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P)\|_{o,2} \leq MR_n$ whenever $1\{A_{n,P}(M)\} = 1$ we obtain, whenever $\{V_i\}_{i=1}^n \in A_{n,P}(M)$, that

$$\int_{0}^{\infty} \sqrt{\log(N(\epsilon, \mathcal{B}_{n}, \rho_{n}))} d\epsilon \leq \int_{0}^{\infty} \sqrt{\log(N(\epsilon/MR_{n}, \mathcal{B}_{n}, \|\cdot\|_{2}))} d\epsilon$$

$$= MR_{n} \int_{0}^{\infty} \sqrt{\log(N(u, \mathcal{B}_{n}, \|\cdot\|_{2}))} du, \tag{S.320}$$

where the equality follows from the change of variables $\epsilon = MR_n u$. Therefore, since $0 \in \mathcal{B}_n$, Corollary 2.2.8 in van der Vaart and Wellner (1996) and (S.320) imply

$$E[\sup_{\beta \in \mathcal{B}_n} |\bar{\mathbb{G}}_P(\beta)| |\{V_i\}_{i=1}^n] \lesssim MR_n \int_0^\infty \sqrt{\log(N(u, \mathcal{B}_n, \|\cdot\|_2))} du \equiv MR_n J_{2n}. \quad (S.321)$$

Next, note (S.318), (S.319), and (S.321) together with Markov's inequality imply that

$$P(\sup_{\beta \in \mathcal{B}_n} |\hat{\mathbb{G}}_n(f_{n,P}^{d_{n'}}\beta) - \mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)| > M^2 R_n J_{2n}; \ A_{n,P}(M))$$

$$\leq P(\sup_{\beta \in \mathcal{B}_n} |\bar{\mathbb{G}}_P(\beta)| > M^2 R_n J_{2n}) \lesssim \frac{1}{M} \quad (S.322)$$

for all $P \in \mathbf{P}$. Therefore, combining results (S.316) and (S.322), we can finally conclude

$$\limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}} P(\sup_{\beta \in \mathcal{B}_n} |\hat{\mathbb{G}}_n(f_{n,P}^{d_{n'}}\beta) - \mathbb{G}_P^{\star}(f_{n,P}^{d_{n'}}\beta)| > M^2 R_n J_{2n})$$

$$\lesssim \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}} \{ \frac{1}{M} + P(\{V_i\}_{i=1}^n \notin A_{n,P}(M)) \} = 0.$$

The first claim of the lemma then follows by employing Lemma S.7.2(i) to set $R_n = (d_n \log(1 + d_n) C_n K_n^2/n)^{1/4}$ in (S.315), while the second claim follows by employing Lemma S.7.2(ii) to set $R_n = (\xi_n d_n \log(1 + d_n) C_n K_n^2/n)^{1/2}$.

Lemma S.7.2. Let $\{V_i\}_{i=1}^n$ be i.i.d. with $V \sim P \in \mathbf{P}$, suppose Assumption S.7.1 holds, define $\Sigma_n(P) \equiv \operatorname{Var}_P\{f_{n,P}^{d_n}(V)\}$ and its sample analogue $\hat{\Sigma}_n(P)$ to equal

$$\hat{\Sigma}_n(P) \equiv \frac{1}{n} \sum_{i=1}^n (f_{n,P}^{d_n}(V_i) - \frac{1}{n} \sum_{j=1}^n f_{n,P}^{d_n}(V_j)) (f_{n,P}^{d_n}(V_i) - \frac{1}{n} \sum_{j=1}^n f_{n,P}^{d_n}(V_j))',$$

and assume $d_n \log(1+d_n) K_n^2 C_n = o(n)$. (i) Then, it follows that uniformly in $P \in \mathbf{P}$

$$\|\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P)\|_{o,2} = O_P(\{\frac{d_n \log(1 + d_n)C_n K_n^2}{n}\}^{1/4}).$$

(ii) If in addition $\sup_{P\in\mathbf{P}} \|\Sigma_n^{-1}(P)\|_{0,2} \leq \xi_n < \infty$ and $\xi_n \sqrt{d_n \log(1+d_n)C_n} K_n/\sqrt{n} = o(1)$, then we can also conclude uniformly in $P\in\mathbf{P}$ that

$$\|\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P)\|_{o,2} = O_P(\frac{\sqrt{\xi_n d_n \log(1 + d_n)C_n} K_n}{\sqrt{n}}).$$

PROOF: First note that we may without loss of generality assume that $E_P[f_{n,P}^{d_n}(V)] = 0$. Next note that Assumption S.7.1(ii) implies that for all $P \in \mathbf{P}$ we must have

$$\|\frac{1}{n}\{f_{n,P}^{d_n}(V_i)f_{n,P}^{d_n}(V_i)' - E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']\}\|_{o,2} \le \frac{2d_nK_n^2}{n}$$
(S.323)

almost surely for all $P \in \mathbf{P}$ since each entry of the matrix $f_{n,P}^{d_n}(V_i)f_{n,P}^{d_n}(V_i)'$ is bounded by K_n^2 . Similarly, employing $\|f_{n,P}^{d_n}(V_i)f_{n,P}^{d_n}(V_i)'\|_{o,2} \leq d_n K_n^2$ almost surely we obtain

$$\|\frac{1}{n}E_P[\{f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)' - E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']\}^2]\|_{o,2} \le \frac{2d_nK_n^2C_n}{n}.$$
 (S.324)

Thus, employing results (S.323) and (S.324), together with $d_n \log(1 + d_n) K_n^2 C_n = o(n)$, we obtain by Theorem 6.1(ii) in Tropp (2012) that for all $P \in \mathbf{P}$

$$P(\|\frac{1}{n}\sum_{i=1}^{n}f_{n,P}^{d_{n}}(V_{i})f_{n,P}^{d_{n}}(V_{i})' - E_{P}[f_{n,P}^{d_{n}}(V)f_{n,P}^{d_{n}}(V)']\|_{o,2} > \frac{M\sqrt{d_{n}\log(1+d_{n})C_{n}}K_{n}}{\sqrt{n}})$$

$$\leq d_{n}\exp\{-\frac{M^{2}(d_{n}\log(1+d_{n})K_{n}^{2}C_{n})}{2n}\frac{n}{MBd_{n}K_{n}^{2}C_{n}}\} \quad (S.325)$$

for some $B < \infty$. Hence, we can conclude from (S.325) that uniformly in $P \in \mathbf{P}$

$$\|\frac{1}{n}\sum_{i=1}^{n}f_{n,P}^{d_{n}}(V_{i})f_{n,P}^{d_{n}}(V_{i})' - E_{P}[f_{n,P}^{d_{n}}(V)f_{n,P}^{d_{n}}(V)']\|_{o,2} = O_{P}(\frac{\sqrt{d_{n}\log(1+d_{n})C_{n}}K_{n}}{\sqrt{n}}).$$
(S.326)

Recalling that we had without loss of generality set $E_P[f_{n,P}^{d_n}(V)] = 0$, next note that $E_P[f_{n,P}^{d_n}(V)] \leq ||E_P[f_{n,P}^{d_n}(V)f_{n,P}^{d_n}(V)']||_o \leq C_n$, Markov's inequality, and Lemmas 2.2.9 and 2.2.10 in van der Vaart and Wellner (1996) imply, uniformly in $P \in \mathbf{P}$, that

$$\|\frac{1}{n}\sum_{i=1}^{n}f_{n,P}^{d_{n}}(V_{i})\|_{2} \leq \sqrt{d_{n}}\max_{1\leq d\leq d_{n}}\left|\frac{1}{n}\sum_{i=1}^{n}f_{d,n,P}(V_{i})\right|$$

$$=O_{P}\left(\frac{K_{n}\log(1+d_{n})\sqrt{d_{n}}}{n}+\frac{\sqrt{C_{n}d_{n}\log(1+d_{n})}}{\sqrt{n}}\right). \quad (S.327)$$

Therefore, since for any $a, b \in \mathbf{R}^{d_n}$ we have $||ab'||_{o,2} \leq ||a||_2 ||b||_2$, results (S.326) and (S.327) together with the triangle inequality yield, uniformly in $P \in \mathbf{P}$, that

$$\|\hat{\Sigma}_n(P) - \Sigma_n(P)\|_{o,2} = O_P(\frac{\sqrt{d_n \log(1 + d_n)C_n}K_n}{\sqrt{n}}).$$
 (S.328)

Finally, since $\hat{\Sigma}_n(P) \geq 0$ and $\Sigma_n(P) \geq 0$, Theorem X.1.1 in Bhatia (1997) implies that

$$\|\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P)\|_{o,2} \le \|\hat{\Sigma}_n(P) - \Sigma_n(P)\|_{o,2}^{1/2}$$
(S.329)

almost surely, and hence the first claim the lemma follows from (S.328) and (S.329).

For the second claim, let $\underline{\operatorname{eig}}\{A\}$ denote the smallest eigenvalue of any Hermitian matrix A. Since $\|\Sigma_n^{-1}(P)\|_{o,2} = 1/\underline{\operatorname{eig}}\{\Sigma_n(P)\}$, $\sup_{P\in\mathbf{P}}\|\Sigma_n^{-1}(P)\|_{o,2} \leq \xi_n$, result (S.328), Corollary III.2.6 in Bhatia (1997), and $\xi_n\sqrt{d_n\log(1+d_n)C_n}K_n/\sqrt{n} = o(1)$ imply

$$\lim_{n \to \infty} \inf_{P \in \mathbf{P}} P(\underline{\operatorname{eig}}\{\hat{\Sigma}_n(P)\} > \frac{1}{2\xi_n})$$

$$\geq \lim_{n \to \infty} \inf_{P \in \mathbf{P}} P(\underline{\operatorname{eig}}\{\Sigma_n(P)\} > \frac{1}{2\xi_n} + \|\hat{\Sigma}_n(P) - \Sigma_n(P)\|_{o,2}) = 1.$$

Therefore, Applying Theorem X.3.8 in Bhatia (1997) we can then conclude that

$$\limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}} P(\|\hat{\Sigma}_n^{1/2}(P) - \Sigma_n^{1/2}(P)\|_{o,2}) > M \frac{\sqrt{\xi_n d_n \log(1 + d_n) C_n} K_n}{\sqrt{n}})$$

$$\leq \limsup_{M \uparrow \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}} P(\|\hat{\Sigma}_n(P) - \Sigma_n(P)\|_{o,2}) > M \frac{\sqrt{d_n \log(1 + d_n) C_n} K_n}{\sqrt{n}}) = 0,$$

where the final equality follows from result (S.328). \blacksquare

Lemma S.7.3. For any positive random variable U with $E[U^2] < \infty$ and finite constant A > 0 it follows that $E[U \exp\{-A/U^2\}] \le E[U] \exp\{-A/E[U^2]\} + E[U^2]/\sqrt{2A}$.

PROOF: First note $u \mapsto u \exp\{-A/u^2\}$ is convex on $u \in (0, \sqrt{2A}]$. Therefore Jensen's inequality, $u \mapsto u \exp\{-A^2/u^2\}$ being increasing in $u \in (0, \infty)$, $E[1\{0 < U < \sqrt{2A}\}U] \le E[U]$ due to U being positive a.s., and $\exp\{-A/U^2\} \le 1$ due to A > 0, imply

$$\begin{split} E[U\exp\{-\frac{A}{U^2}\}] &= E[1\{0 < U \le \sqrt{2A}\}U\exp\{-\frac{A}{U^2}\}] + E[1\{U > \sqrt{2A}\}U\exp\{-\frac{A}{U^2}\}] \\ &\le E[U]\exp\{-\frac{A}{E[U^2]}\} + E[1\{U > \sqrt{2A}\}U]. \end{split}$$

The claim of the lemma therefore follows from $E[1\{U > \sqrt{2A}\}U] \le E[U^2]/\sqrt{2A}$ by the Cauchy Schwarz inequality and Markov's inequality.

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