A Smooth Shadow-Rate Dynamic Nelson-Siegel Model for Yields at the Zero Lower Bound

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Introduction

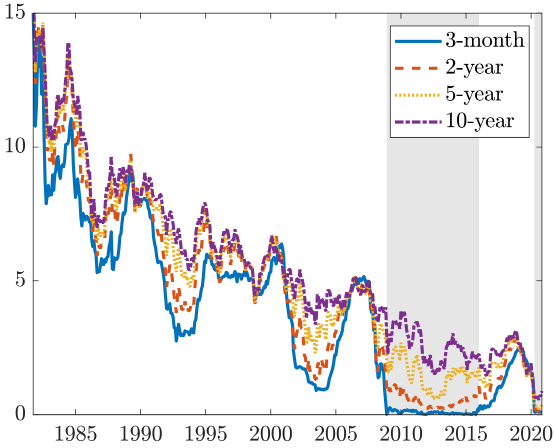

Figure: U.S. government bond yields

Introduction

Problem: Traditional term structure models ignore the zero lower bound (ZLB) and are not able to capture asymmetric yield behaviour.

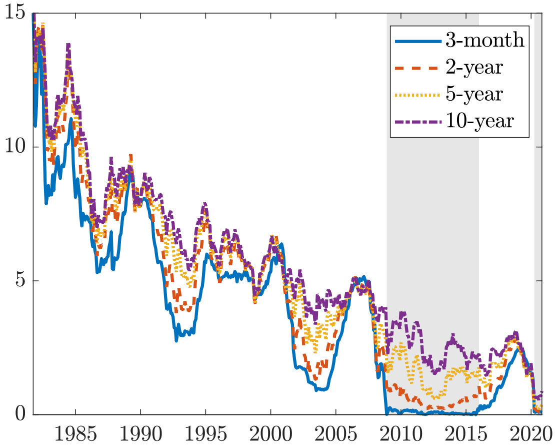

Figure: U.S. government bond yields

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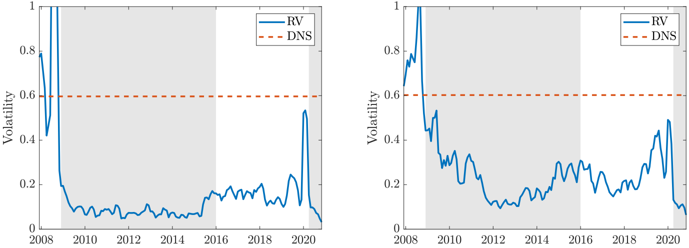
- Traditional models ignore the prolonged low volatility of yields close to the ZLB (Christensen and Rudebusch, 2015, 2016).



(a) Three-month yield (b) Two-year yield

Figure: Three-month ahead realized yield volatility and model-implied conditional yield volatility series of the Dynamic Nelson-Siegel (DNS) model

(i) Problems at the ZLB

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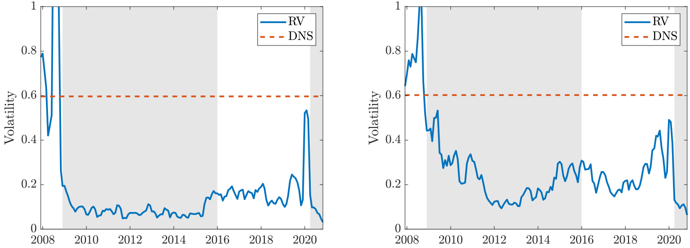
$\Longrightarrow$ Generate yield forecasts that revert too quickly to their long-term means.



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(ii) Problems at the ZLB

- At the same time, traditional models generate implausible negative interest rate forecasts (Christensen and Rudebusch, 2016; Bauer and Rudebusch, 2016).

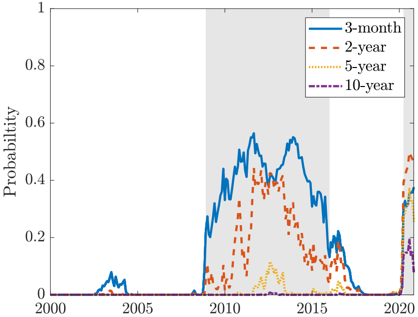


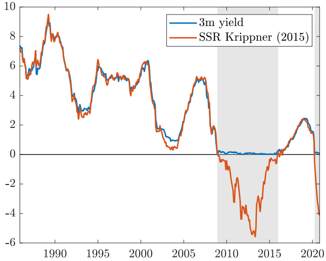
Figure: Conditional probabilities of negative three-month ahead U.S. interest rates from DNS model based on simulation

Solution: Shadow-rate models

- Shadow-rate concept introduced by Black (1995), where

$$r_t = \max(r_{LB}, s_t),$$

with r_t being the observed short rate, s_t the shadow short rate and r_{LB} the lower bound value.



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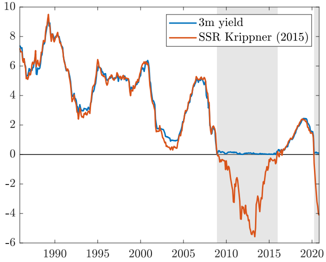

- Under this nonlinear framework, approximate closed-form bond price formulas are derived for shadow-rate affine term structure models (Krippner, 2012; Wu and Xia, 2016)

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   $\Longrightarrow$ We illustrate this by allowing for time-varying factor loadings (Koopman et al., 2010).

Contribution to literature

1. Our work extends literature on shadow-rate models, which has mostly focused on less tractable affine term structure model class (Krippner, 2012; Christensen and Rudebusch, 2015, 2016; Wu and Xia, 2016; Bauer and Rudebusch, 2016).

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	Structural form (arbitrage-free)	Reduced form (model flexibility)
Traditional model (ignores ZLB)	$\surd$	$\surd$
Shadow-rate model (respects ZLB)	$\surd$	**This paper**

Nelson-Siegel curve

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$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

where

- ◇ $y_t(\tau)$: yield at time t with time to maturity τ,
- ◇ λ: factor loading parameter,
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  - $\beta_{1t}, \beta_{2t}, \beta_{3t}$: latent time-varying factors with interpretation of level, slope and curvature, respectively.
- *Extension:* Koopman et al. (2010) allow for time-varying factor loadings by considering $\beta_{4t} = \lambda_t$ as an additional latent factor.

Dynamic Nelson-Siegel model

- Let $\boldsymbol{y}_t^o = (y_t^o(\tau_1), \ldots, y_t^o(\tau_N))'$ denote the collection of N observed yields at time t.
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- Then, the complete Dynamic Nelson-Siegel (DNS) model can be represented as the linear state-space model

$$\begin{aligned} \boldsymbol{y}_t^o &= \boldsymbol{\Lambda}(\lambda)\boldsymbol{\beta}_t + \boldsymbol{\varepsilon}_t, & \boldsymbol{\varepsilon}_t &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon), \\ \boldsymbol{\beta}_t &= \boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, & \boldsymbol{\eta}_t &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\eta). \end{aligned}$$

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- Estimation can proceed via maximum likelihood estimation in combination with the Kalman filter.

Lower bound restriction

- The DNS model, however, does not restrict yields to be non-negative and does not allow for the asymmetry in the yield curve at the ZLB.

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- This approach assumes that yields are either behaving in a traditional way above r_{LB} or are flat and equal to r_{LB}.

  $\Longrightarrow$ Hence, the model is non-smooth and results in a kink at r_{LB} that separates yields into two possible states

Smooth lower bound restriction

- To allow for a smooth transition between these two states, we consider a smooth approximation of the max function such that

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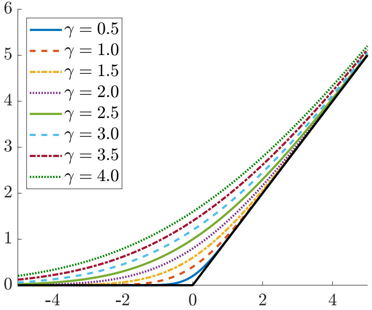

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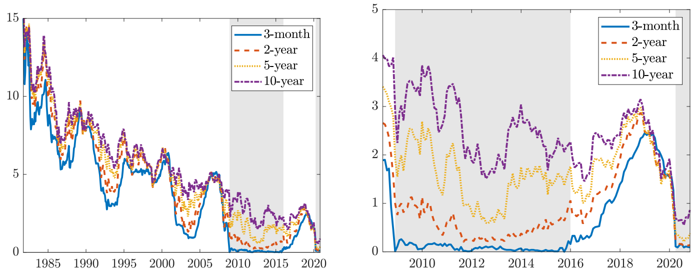
- Estimation again proceeds via maximum likelihood estimation, but now in combination with the Extended Kalman filter to deal with nonlinearity.

U.S. government bond yield curve data

- We consider end-of-the month U.S. Treasury zero-coupon bond yields for eight maturities ranging from three months to ten years.
- This data can be obtained from the H.15 series of the Federal Reserve Board.
- Sample: September 1981 - October 2020 (470 observations)



(a) Total period (b) Post-GFC period

Figure: Time series of U.S. government bond yields with shaded ZLB periods

Estimation output

	Log-likelihood	$\#\Theta$	AIC	BIC
DNS	2615.7	27	-11.0	-10.8
DNS-TVL	3007.6	38	-12.6	-12.3
B-DNS	2614.1	27	-11.0	-10.8
SB-DNS	3080.6	28	-13.0	-12.7
SB-DNS-TVL	3251.0	39	-13.7	-13.3
AFNS	2245.1	27	-9.4	-9.2
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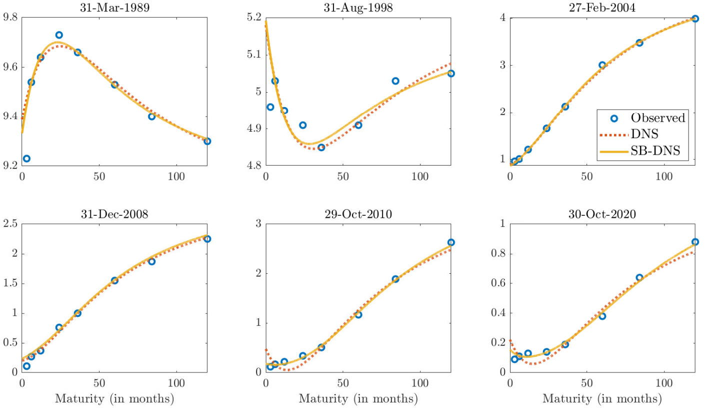
- Time-varying loadings improves the log-likelihood and criteria even further.
- Arbitrage-free models have lower log-likelihood, but imposition of shadow-rate framework still improves fit.

Illustration of yield curve fits

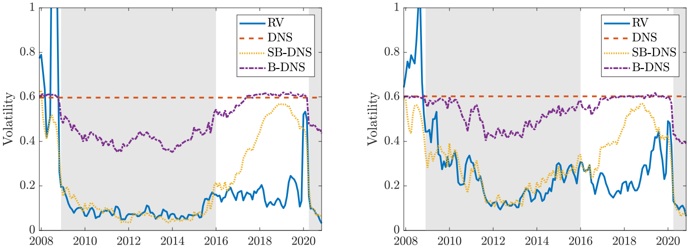
Both the DNS and SB-DNS model accurately fit different yield curve shapes. Yet, the SB-DNS model seems to be more flexible for short- and long-term yields.



▸▸ Overall in-sample fit

Volatility compression at the ZLB

- The SB-DNS model replicates the decrease in volatility and closely follows RV series after financial crisis 2007-2008.
- The B-DNS models is only partly able to do so.



(a) Three-month yield (b) Two-year yield

Figure: Three-month ahead realized and model-implied conditional volatility of yields

Forecasting exercise

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- Consider four forecast horizons: one-month ahead ($h = 1$), six-month ahead ($h = 6$), one-year ahead ($h = 12$) and two-year ahead ($h = 24$).
- Include random walk forecasts, which are known to be a hard-to-beat benchmark for yields (Duffee, 2002).

Out-of-sample performance ($h = 6$)

Table: Relative RMSFEs compared to SB-DNS model

	Maturities (in months)							
	3	**6**	**12**	**24**	**36**	**60**	**84**	**120**
RW	1.06	1.02	0.97	0.93	0.93**	0.93**	0.94**	0.95*
DNS	1.11*	1.04	1.02	1.03	1.05	1.06	1.06	1.08**
DNS-TVL	1.19**	1.09	1.07	1.07	1.10*	1.11**	1.11**	1.14**
B-DNS	1.13*	1.06	1.02	1.02	1.03	1.04	1.04	1.07
SB-DNS-TVL	1.13*	1.06	1.05	1.05	1.06	1.05	1.04	1.05
AFNS	1.97***	1.81***	1.69***	1.50***	1.34***	1.12**	1.07*	1.00
B-AFNS	1.16**	1.15**	1.10*	1.02	0.97**	0.91***	0.91***	0.91***

Notes: Green cell indicates that SB-DNS has lower RMSFE. The asterisks $*$,$**$, and $***$ indicate significance at the 10%, 5% and 1% level, respectively, based on Diebold-Mariano test.

- The SB-DNS model outperforms all other DNS-variants for all maturities, although outperformance is generally insignificant.
- The SB-DNS model significantly outperforms B-AFNS model for short-term yields, but is significantly outperformed for long-term yields.

Cumulative SSE plots ($h = 6$)

Out-of-sample performance ($h = 24$)

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RW	1.13*	1.12*	1.10	1.04	0.98	0.90	0.88*	0.87*
DNS	1.18*	1.17*	1.18*	1.20**	1.20**	1.20**	1.22**	1.27**
DNS-TVL	1.14*	1.12	1.14*	1.19**	1.21**	1.23***	1.25***	1.31***
B-DNS	1.11	1.10	1.09	1.10	1.09	1.09	1.10	1.15
SB-DNS-TVL	1.09	1.08	1.10	1.13*	1.13*	1.13*	1.13	1.16*
AFNS	1.55***	1.51***	1.52***	1.49***	1.38***	1.16**	1.05	0.94
B-AFNS	1.04	1.03	1.03	1.02	0.97	0.90**	0.88**	0.87***

Notes: Green cell indicates that SB-DNS has lower RMSFE. The asterisks $*, **$, and $***$ indicate significance at the 10%, 5% and 1% level, respectively, based on Diebold-Mariano test.

- The SB-DNS model significantly outperforms AFNS and DNS(-TVL) for most maturities.
- But RW and B-AFNS still perform better for long-term yields.

Cumulative SSE plots ($h = 24$)

Concluding remarks

- We proposed a smooth shadow-rate version of the DNS model to analyze and forecast the term structure of interest rates during the ZLB period.

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  - ◇ The SB-DNS model outperforms the baseline DNS model in terms of fitting and forecasting the yield curve, particularly during the ZLB period, and is competitive with the more rigorous B-AFNS model.

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  - The DNS model lacks in generating plausible future yield curve behaviour, which is resolved with our smooth shadow-rate adaption that can in turn be used to shape future policy expectations at the ZLB.

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In-sample fit across all maturities

Overall RMSE	Pre-ZLB period (Dec 1981 - Oct 2008)	ZLB period (Nov 2008 - Dec 2015)	Total period (Dec 1981 - Oct 2020)
DNS	9.0	7.3	8.2
DNS-TVL	**7.1**	4.3	**6.3**
B-DNS	9.0	7.2	8.2
SB-DNS	8.4	4.5	7.5
SB-DNS-TVL	7.6	**4.1**	6.8
AFNS	8.6	6.6	7.8
B-AFNS	8.4	4.8	7.5

- The SB-DNS model has lower RMSE than the DNS model for all periods with the largest improvement for the ZLB period.
- Time-varying loading models improve the in-sample fit even further.
- Arbitrage-free models also better overall in-sample fit than the DNS model, particularly the B-AFNS model.

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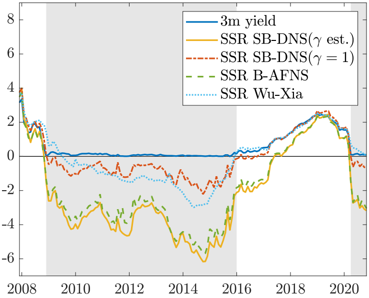
Shadow short rate estimates



- Some advocate shadow short rates (SSR) to be a useful measure of the stance of unconventional monetary policy (Bullard, 2012; Wu and Xia, 2016)

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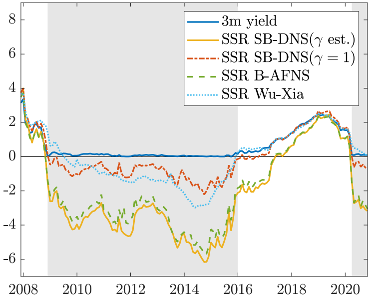

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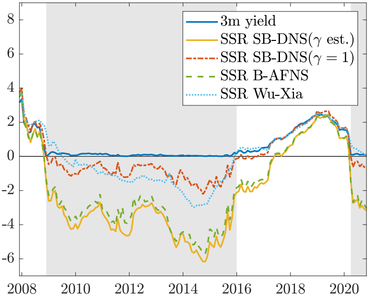

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- SSR of SB-DNS model with $\gamma = 1$ close to the one of Wu and Xia (2016).

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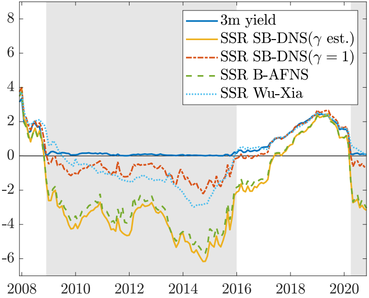
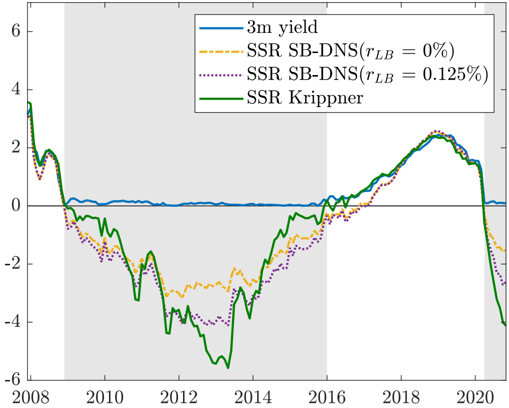


- Some advocate shadow short rates (SSR) to be a useful measure of the stance of unconventional monetary policy (Bullard, 2012; Wu and Xia, 2016)
  - ! Use with caution as estimates are sensitive to model and estimation choices.
- SSR of SB-DNS model with $\gamma = 1$ close to the one of Wu and Xia (2016).
- SSR of SB-DNS model with estimated γ generates similar SSR as the B-AFNS model.

SSR comparison with Krippner | Sensitivity of SSRs | Lift-off horizon estimates

SSR estimates based on two-factor models

- Krippner (2015) argues that two-factor term structure models produce more robust and economically meaningful estimates.
- SSR of two-factor SB-DNS model with $\gamma = 1$ closely follows the one from Krippner (2015).



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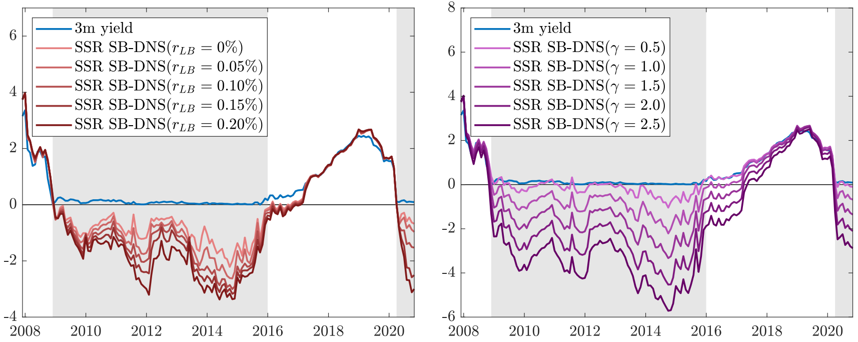
Sensitivity of SSR estimates



(a) Lower bound analysis with $\gamma = 1$

(b) Smoothness analysis with $r_{LB} = 0\%$

Figure: Robustness analysis of SSR estimates towards lower bound value and smoothness parameter

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Policy insights at the ZLB

- Liftoff horizon starts to increase after financial crisis, but decreases almost linearly from 2013 onwards.
- Closely follows realized liftoff, although with a six-months delay.
- Liftoff horizon again large at start of corona virus pandemic

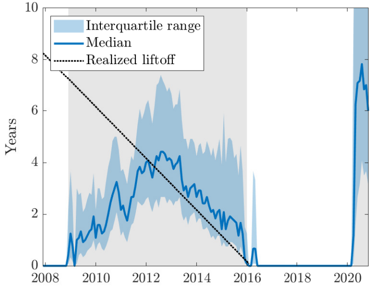

Figure: Liftoff horizon estimates from the SB-DNS model (including realized liftoff horizon)

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Out-of-sample performance ($h = 6$)

Outperformance of SB-DNS compared to DNS model mostly stems from ZLB period, in which DNS and AFNS perform particularly poor.

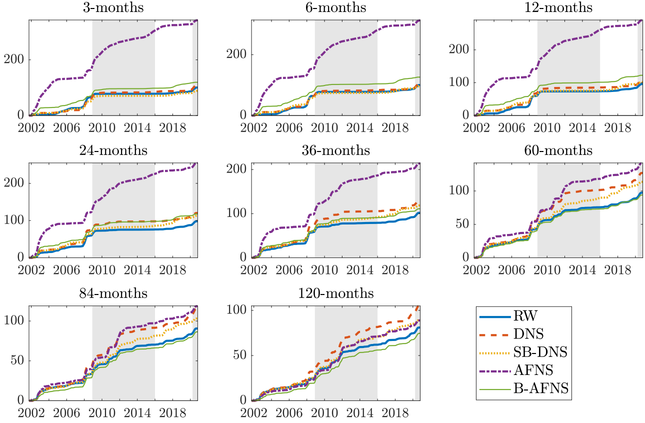

Figure: Cumulative sum of squared forecast errors for six-month ahead forecasts

Out-of-sample performance ($h = 24$)

Clear out-performance of the SB-DNS model for short- and medium-term maturities.

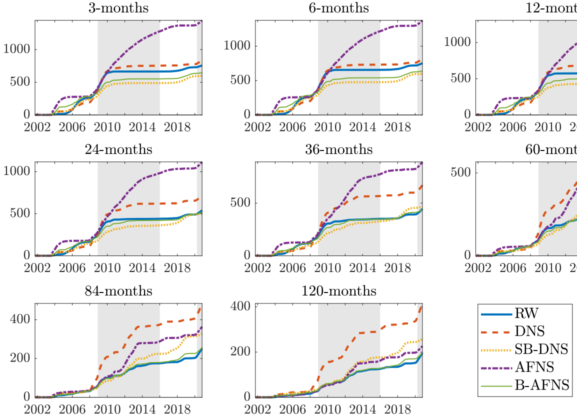

Figure: Cumulative sum of squared forecast errors for two-year ahead forecasts