Power of CLR and Related Tests

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Introduction

- Standard linear IV setting, one endogenous variable, homoskedasticity, $y = x\beta + u$, $x = Z\pi + v$
- Weak instrument robust inference, AR, LM and CLR tests for $H_0 : \beta = \beta_0$
- Conditional Wald (CW) tests, not unbiased
- Power comparisons in general done for two different designs, varying β whilst
  - keeping Σ, variance of $(u_i, v_i)'$, fixed
  - keeping Ω, variance of $(r_i, v_i)'$ fixed, $y = Z\pi_y + r$
- Comparison of CLR and CW done for fixed-Ω design, CLR found to be superior.

Literature

- Andrews, Moreira and Stock (Ecta, 2006), introduced fixed-Ω design
- Andrews, Moreira and Stock (JoE, 2007), Ω compared CLR and CW tests
- Mills, Moreira and Vilela (JoE, 2014), Ω compared CLR and CW_0 tests
- Andrews, Marmer and Yu, (QE, 2019), vary β_0 instead of β, but that is same as fixed-Σ design.

Main Issues

Let

$$\Sigma = \begin{bmatrix} \sigma_u^2 & \rho_{uv}\sigma_u\sigma_v \\ \rho_{uv}\sigma_u\sigma_v & \sigma_v^2 \end{bmatrix}; \quad \Omega = \begin{bmatrix} \sigma_r^2 & \rho_\Omega\sigma_r\sigma_v \\ \rho_\Omega\sigma_r\sigma_v & \sigma_v^2 \end{bmatrix}$$

- ρ_{uv} is measure for endogeneity, ρ_Ω is not
- Fixed-Ω power curve considers quite specific ρ_{uv}, β combinations.
  - Very little space devoted to low/moderate endogeneity and/or ρ_{uv} and β having same sign when testing $H_0 : \beta = 0$.
  - Can investigate power for these cases better with fixed-Σ design
  - Find in simulations that *CW* tests have more power than *CLR* test when ρ_{uv} is small/moderate and/or when ρ_{uv} and β have the same sign

Main Issues

- Fixed-Ω design: cannot set wlog $\beta_0 = 0$ and $\sigma_r^2 = \sigma_v^2 = 1$, can do latter conditional on former.
- Fixed-Σ design: can set wlog $\beta_0 = 0$ and $\sigma_u^2 = \sigma_v^2 = 1$

AR and LR Tests

The LR test statistic is criterion difference given by

$$\begin{aligned} LR &= \frac{u_0' P_Z u_0}{\widehat{\sigma}_0^2} - \frac{\hat{u}_L' P_Z \hat{u}_L}{\widehat{\sigma}_L^2} \\ &= AR - B\left(\hat{\beta}_L\right) \end{aligned}$$

where $\hat{\beta}_L$ is the LIML estimator and $B\left(\hat{\beta}_L\right)$ the Basmann test for overidentifying restrictions, $\hat{u}_L = y - x\hat{\beta}_L = Wb_L$, $b_L = \left(1 \quad -\hat{\beta}_L\right)'$, $\widehat{\sigma}_L^2 = b_L' \widehat{\Omega} b_L$.

$$LR \underset{H_0}{\overset{d}{\to}} \chi_1^2$$

under strong instrument asymptotics, but not with weak or uninformative instruments. For CLR use conditional critical values.

LR Statistic

Proposition

LR statistic is identical to $W_0\left(\widehat{\beta}_L\right) = t_0(\widehat{\beta}_L)^2$

$$LR = W_0\left(\widehat{\beta}_L\right) = \frac{\left(\widehat{\beta}_L - \beta_0\right)^2 (x'P_Z x - n\widehat{\kappa}\widehat{\omega}_{22})}{\widehat{\sigma}_0^2}$$

$$\widehat{\beta}_L = \frac{x'P_Z y - n\widehat{\kappa}\widehat{\omega}_{12}}{x'P_Z x - n\widehat{\kappa}\widehat{\omega}_{22}},$$

$$\widehat{\kappa} = \min \operatorname{eval}\left(\widehat{\Omega}^{-1}\left(n^{-1}W'P_Z W\right)\right).$$

$W_0\left(\widehat{\beta}_L\right)$ proposed by Mills, Moreira and Vilela (JoE, 2014).

Fixed-Ω Design

DGP for fixed-Ω design is given by

$$y = x\beta + r - \beta v$$
$$x = Z\pi + v$$

as then reduced form is $y = Z\pi\beta + r$.
We cannot set $\beta_0 = 0$ and $\sigma_r^2 = \sigma_v^2 = 1$ wlog.

Ω and $\Sigma(\beta)$

For fixed-Ω design with

$$\Omega = \begin{bmatrix} 1 & \rho_\Omega \\ \rho_\Omega & 1 \end{bmatrix},$$

we have that

$$\Sigma(\beta) = \begin{bmatrix} 1 - 2\beta\rho_\Omega + \beta^2 & \rho_\Omega - \beta \\ \rho_\Omega - \beta & 1 \end{bmatrix},$$

or

$$\frac{\sigma_u^2(\beta)}{\beta^2} \to 1 \text{ when } |\beta| \to \infty$$
$$\rho_{uv}(\beta) \to -1 \text{ when } \beta \to \infty$$
$$\rho_{uv}(\beta) \to 1 \text{ when } \beta \to -\infty$$

which are the combinations where the power of the *CLR* test goes to 1.

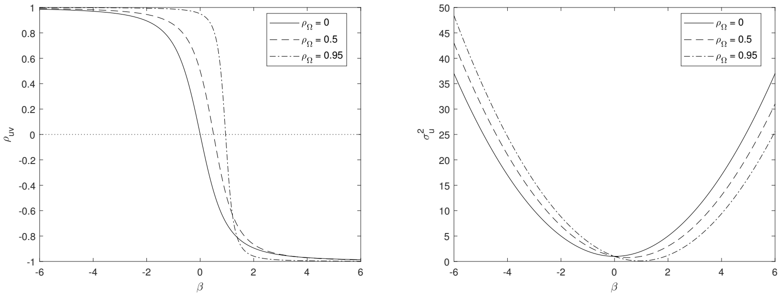

Figure: Values of ρ_{uv} and σ_u^2 as a function of β when holding $\Omega = \begin{bmatrix} 1 & \rho_\Omega \\ \rho_\Omega & 1 \end{bmatrix}$ constant.

Note that ρ_{uv} and β have here only the same sign for $0 < \beta < \rho_\Omega$. For $\rho_\Omega = 0.5$, ρ_{uv} only moderate, say $-0.5 \leq \rho_{uv} \leq 0.5$ for $0 \geq \beta \geq 1$.

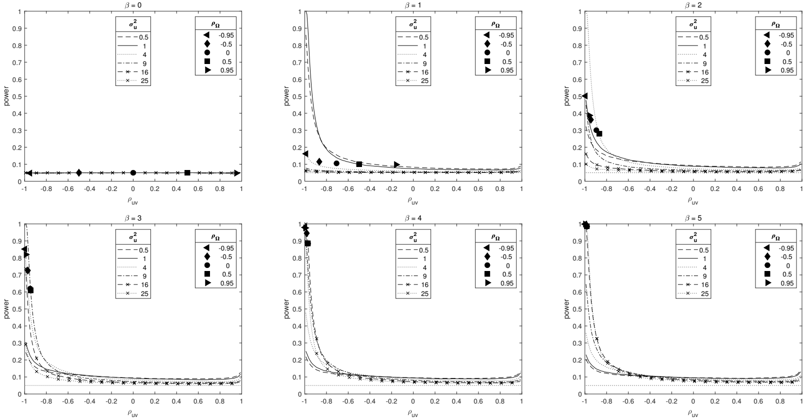

Figure: Weak instruments asymptotic power of *CLR* test, $k_z = 5$, $\lambda = 1$.

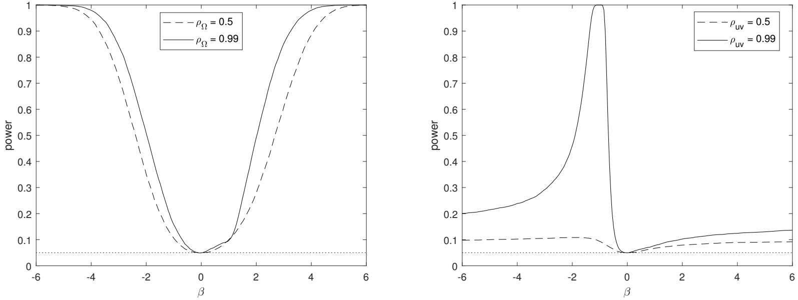

Figure: Weak instruments asymptotic power curves of the *CLR* test for fixed Ω design, left panel, and fixed Σ design, right panel. $k_z = 5$, $\lambda = 1$.

Power of *CLR* and *CW* Tests

First replicate Andrews, Moreira and Stock (JoE, 2007):

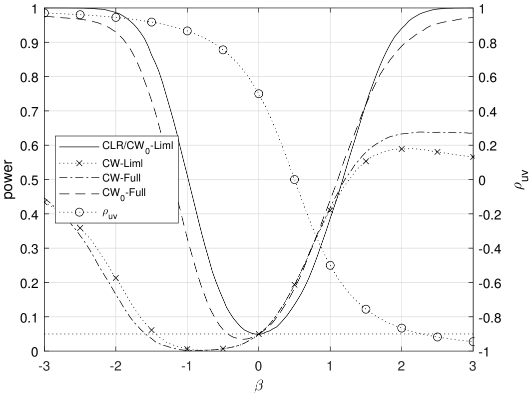

Figure: Asymptotic power of tests, fixed Ω design, $\rho_\Omega = 0.5$, $k_z = 5$, $\lambda / k_z = 1$.

Power of CLR and CW Tests

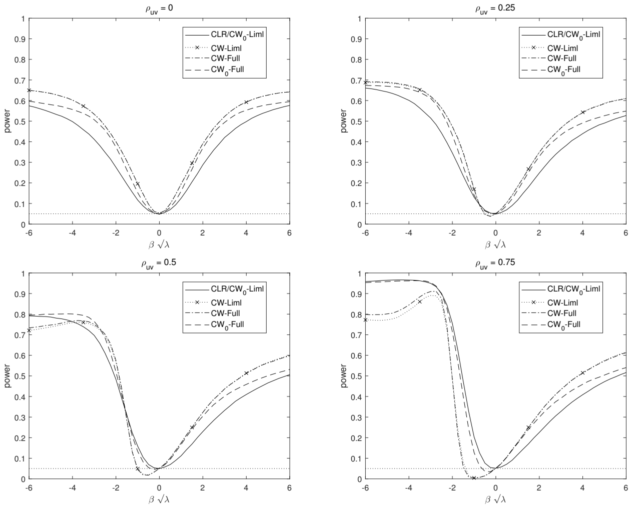

Figure: Asymptotic power of tests, fixed Σ design, $k_z = 5$ and $\lambda / k_z = 2$, for different values of ρ_{uv}.

Conclusions

- Fixed-Ω design does not appear a good design to assess relative properties of tests.
- Cannot set $\beta_0 = 0$ and $\sigma_r^2 = \sigma_v^2 = 1$ wlog in fixed-Ω design
- Argument often used for keeping Ω fixed is that it can be estimated consistently and hence treated as known, but keeping it fixed changes the design which has not before been specified.
- Can better control endogeneity features in fixed-Σ design
- Fixed-Σ designs shows more power for CW tests in low/moderate endogeneity settings, and settings with signs of β and ρ_{uv} the same.
- Behaviour of test-based construction of confidence intervals from fixed-Σ design.