Pecuniary Externalities in Competitive Economies with Limited Pledgeability*

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Abstract

We analyze the efficiency properties of competitive economies with strategic default and limited pledgeability. We show that laissez-faire equilibria can be constrained suboptimal: under certain conditions, imposing *tighter* borrowing constraints (relative to the laissez-faire regime) can make everybody in the economy better off. The inefficiency is due to the interaction between debt pricing and the default option, which generates a pecuniary externality. We also show that a Pigouvian subsidy on net financial positions may induce borrowers to internalize this externality and increase welfare.

Keywords: Limited pledgeability; debt constraints; constrained inefficiency; macroprudential interventions.

JEL codes: E00; E10; F00.

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1 Introduction

Understanding whether competitive economies with financial frictions are vulnerable to potential inefficiencies or market failures is an important question in macroeconomics with many relevant implications. In particular, it helps us understand whether and when policy interventions are warranted. However, asserting that equilibria might be inefficient from a second-best point of view turns out to be more nuanced than it may appear. There are broadly two strands of the literature that provide different answers and implications. On the one hand, a large and growing body of research has emphasized the presence of pecuniary externalities as a fundamental source of inefficiency, especially in settings where contractual arrangements are subject to limited commitment and/or informational asymmetries.[1] There, the frictions take the form of borrowing constraints that depend on market prices of goods or assets. Private agents fail to take into account the general equilibrium effects of their individual decisions on market prices, and that failure could lead, for instance, to excessive borrowing in equilibrium. On the other hand, standard general equilibrium models with self-enforcing debt constraints have found it generally harder to show that competitive equilibria are constrained suboptimal. In the well-known class of single-commodity models, which is widely used in applications, where debt constraints are microfounded by the threat of financial autarky, the competitive equilibria are indeed constrained efficient.[2] This is despite the fact that the debt constraints depend on market prices. In addition to the nontrivial problem of establishing inefficiency, how the resulting externalities are related to the precise nature of the underlying financial frictions is less obvious than commonly understood. For example, little theoretical work has explored whether different debt enforcement mechanisms will lead to different types of inefficiency.

In this paper, we revisit these issues in the context of a standard dynamic general equilibrium model with microfounded borrowing constraints. More precisely, we study endowment economies in which agents cannot commit to honor their liabilities and debt repayment is sustained because a part of the private resources is pledgeable, and/or due to exclusion from credit markets upon default. Pledgeable resources represent output contraction in the case

[1] A nonexhaustive list includes, among others, Kiyotaki and Moore (1997), Gromb and Vayanos (2002), Golosov and Tsyvinski (2007), Lorenzoni (2008), Farhi et al. (2009), Bianchi (2011), Bianchi and Mendoza (2011), and Dávila and Korinek (2018).

[2] See e.g. Kehoe and Levine (1993, 2001), Kocherlakota (1996), Alvarez and Jermann (2000, 2001), Bloise and Reichlin (2011).

of sovereign default, or recourse and seized collateral in the case of consumer and corporate default. Exclusion from credit reflects the adverse effects on debtors' reputation in financial markets. Agents can smooth their consumption by trading one-period ahead contingent claims (Arrow securities), but their borrowing is subject to endogenous borrowing constraints induced by the default punishment. Following Alvarez and Jermann (2000), we consider laissez-faire equilibria where debt limits are *not too tight*, i.e., they are set at the largest possible levels so that repayment is always individually rational. This set up serves well our purposes as it encompasses economies where debt repudiation leads to deadweight losses and exclusion from the credit market as well as economies with collateral constraints.[3]

Our main result is to show that in economies with limited pledgeability, laissez-faire equilibria might be constrained inefficient, in the sense that restricting the amount of credit private agents can obtain may lead to Pareto improvement. More precisely, we consider policy interventions where a regulator imposes tighter debt constraints than the not-too-tight constraints. We interpret such interventions as a parsimonious representation of regulatory or prudential policies that aim to constrain leverage in the financial markets. We show that, under certain conditions, the policy intervention can increase the ex-ante welfare of all agents in the economy.

Intuitively, though all agents are fully rational and forward looking, they fail to internalize how changes in the severity of credit restrictions in the future feedback on equilibrium prices and, most crucially, what is the effect of changes in market prices on the default option. In particular, tightening the debt constraints from some period τ onward might increase bond prices, or equivalently, lower the implied interest rates. In the setting where defaulters are subject to endowment losses and exclusion from credit (à la Bulow and Rogoff 1989 and Hellwig and Lorenzoni 2009), this tightening might reduce the value of the default in periods $t < \tau$, since it is now more costly to smooth consumption over time by saving

[3]There are many variations of models with collateralized or reputation debt, and the literature is too vast to be summarized here. The reputation mechanism strand of the literature was significantly spurred by the early contributions of Eaton and Gersovitz (1981) and Bulow and Rogoff (1989) and embedded to general equilibrium by Kehoe and Levine (1993), Zhang (1997), Alvarez and Jermann (2000), Kehoe and Levine (2001) and Hellwig and Lorenzoni (2009). The seminal papers of Geanakoplos (1997), Kiyotaki and Moore (1997) and Geanakoplos and Zame (2002) brought together collateral with rigorous general equilibrium theory. Recent contributions include, among others, Kubler and Schmedders (2003), Azariadis and Kaas (2007), Fostel and Geanakoplos (2008), Chien and Lustig (2010) and Gottardi and Kubler (2015). The papers of Hellwig and Lorenzoni (2009) and Chien and Lustig (2010) provide the closest settings to our setup.

only. As a consequence, the not-too-tight debt limits increase at periods $t < \tau$ and this opens the possibility for Pareto improvement: the benefits from the relaxed debt constraints at periods $t < \tau$ may compensate for the costs of facing tighter constraints in subsequent periods. In a setting with collateral constraints (à la Chien and Lustig 2010), the argument is similar. There, lower interest rates raise the value of pledgeable income and increase trade opportunities in the periods that precede the tightening of the collateral constraints. Again, this opens the possibility for Pareto improvement. This is the essence of the mechanism we explore in this paper.

Our analysis exploits an intuitive and powerful characterization of not-too-tight debt limits in economies with limited pledgeability: debt limits are always decomposed into a component that equals the present value of pledgeable resources and a credit bubble component that is interpreted as the amount of credit agents can rollover indefinitely.[4] This characterization serves well our purposes as it simplifies substantially the computation of the laissez-faire equilibria, ruling out complications related to the fixed-point determination of the debt limits. Importantly, it allows us to map the set of laissez-faire equilibria in the environment with reputation debt to the set of equilibria in the environment with collateralized debt and vice versa. This equivalent mapping between different equilibrium concepts offers a useful benchmark upon which we can carry out our policy interventions.

To provide more clarity on the underlying mechanism we concentrate to a simple economy with two agents facing uncertainty only at the initial period. Once uncertainty is resolved, the economy is a deterministic one where endowments switch from a high value to a low value between periods. We further assume that pledgeable resources are time-invariant and identical for both agents. Within this setting we restrict attention to symmetric Markov laissez-faire equilibria where, by an appeal to our characterization result, debt limits are bubble-free and equal to the present value of pledgeable resources. The policy intervention takes the form of tightening debt limits by a fraction ε from some period t onward, and analyzing the feedback effect of such a distortion on equilibrium prices and the default option.

It is worth remarking two important features of our policy experiment. First, the intervention in financial markets is not equivalent to modifying pledgeable resources, which

[4]Though this decomposition can be seen as the analogue of Hellwig and Lorenzoni (2009)'s characterization result in an augmented set up with output losses, the result can not be derived by a simple adaptation of their argument. It rather builds on novel insights that have no analogue in the absence of output losses.

remain fixed. The reallocation is induced by tightening the borrowing limits with respect to their level endogenously determined in equilibrium. Second, the coincidence between the set of equilibria in economies with reputation debt and in economies with collateralized debt breaks down in the post-intervention economy, where the debt limits are no longer not-too-tight. We do have, therefore, to conduct our analysis and compute the new equilibrium variables in each setup separately. Interestingly enough, though the source of inefficiency is common in both environments, Pareto improving equilibria might feature very different qualitative properties. We show that delaying the intervention in financial markets in the economy with collateral constraints can lead to equilibria that are close to the first-best outcome.

The fact that private agents fail to internalize the pecuniary externality at the competitive equilibrium with limited pledegability implies that there is room for government intervention by means of macroprudential controls on financial markets in the lines of Jeanne and Korinek (2010, 2019) and Farhi and Werning (2016). We show that the externality discussed above can be tackled by means of corrective Pigouvian subsidies on net financial positions supported by lump-sum taxes. In particular, we show that a planner who has flexibility in the choice of the subsidy rate can improve welfare without intervening in each individual decision made by each agent. The distortion created by the subsidy leads to a wedge in marginal rates of substitution between the high income and the low income agent. When compared to the laissez-faire equilibrium, the wedge generates higher prices and looser debt limits that can reduce the extent of market failure. An interesting observation is that the equilibrium coincidence between the reputation debt model and the collateral debt model is not distorted by this type of intervention, and this permits to study in a unified way whether macroprudential controls can be welfare improving.

Related Literature. The idea that economies with limited commitment are prone to market failures dates back to Kehoe and Levine (1993). When there is more than one commodity and default cannot exclude agents from trading in spot markets, constrained efficiency might fail because private contracts cannot internalize their effect on relative prices and the default option. The logic there is conceptually the same as in incomplete markets economies where a redistribution of asset holdings, through the induced price changes, affects the spanning properties of the limited assets (Geanakoplos and Polemarchakis 1986). In the single good model studied here, however, there are no spot markets, and as a result this mechanism is absent. Moreover, Alvarez and Jermann (2000, 2001) show that competitive equilibria are constrained efficient when the default option is autarky. We instead show that constrained inefficiency obtains in economies with a single commodity when debt enforcement relies on the limited pledgeability of private resources and/or a weak form of exclusion (i.e., one-sided exclusion) from financial markets. Changes in the severity of credit restrictions induce price changes in bond markets. These price changes, in turn, affect the value of default and, therefore, the extent of risk sharing, potentially improving efficiency. This source of inefficiency is not present in Alvarez-Jermann's framework since the value of default does not respond to changes in bond prices.

Our work is related to a well-developed literature studying the emergence of pecuniary externalities in production economies with collateral constraints. Gromb and Vayanos (2002) show that both distributive and collateral externalities can emerge due to market segmentation. Lorenzoni (2008) shows that financial distress might lead to fire sales whose effects on asset prices are not internalised by highly leveraged investors. Dávila and Korinek (2018) characterize pecuniary externalities in dynamic settings that are subject to reduced-form, price-dependent collateral constraints. They distinguish between distributive and collateral externalities and show that each of these two types can be quantified as a function of intuitive sufficient statistics. In all these works, because of capital accumulation, the reallocation of resources is induced by a change in the level of investment. A planner can overcome the market failure by reducing aggregate investment ex ante and, therefore, the size of the asset sales in bad states. In contrast, in our pure exchange setup this channel is absent as aggregate resources are fixed and only their distribution can vary. The reallocation of resources is solely induced by the tightening of the endogenously determined debt constraints. This relates to the work of Guerrieri and Lorenzoni (2017) who study the effects of unexpected credit contractions in Bewley-type economies with incomplete markets and exogeneous borrowing limits.

Gottardi and Kubler (2015) provide an antecedent to our paper by analyzing constrained suboptimality in a collateral economy à la Chien and Lustig (2010). Our analysis differs from theirs in two important aspects. First, they assume that the intervention is unexpectedly announced at the initial period after all trades have taken place. We instead assume that the intervention is fully anticipated by private agents. Second, their policy experiment exploits an equivalence between equilibria in the economy with collateral constraints and equilibria of an auxiliary economy with financial intermediaries where agents can only take long positions on contingent trees. They show that Pareto improvement obtains in the auxiliary economy, however, they do not show whether the established equivalence is preserved post intervention, so that the equilibrium with financial intermediaries is mapped back to the equilibrium with tighter collateral constraints. The online appendix of this paper offers a more detailed discussion on how our analysis is distinct from Gottardi and Kubler (2015).

Finally, our work is related to a complementary strand of literature that focuses on macroprudential controls that take the form of Pigouvian taxes or subsidies to reduce pecuniar externalities. Park (2014) studies optimal taxation in an Alvarez and Jermann (2000) production economy. There, individuals do not take into account that their labor and saving decisions affect aggregate labor and capital supply and wages, and thus the value of autarky. Jeanne and Korinek (2010, 2019) and Dávila and Korinek (2018) provide a welfare rational for the taxation of capital flows to mitigate the financial amplification effects of fire sales in economies with collateral constraints. In Farhi and Werning (2016) the focus is on demand externalities that are associated with the presence of nominal price rigidities. Though such externalities are qualitatively different from the pecuniary externalities that we study here, Korinek and Simsek (2016) argue that the two types of externalities interact and may mutually reinforce each other. We show that, in an exchange set up, Pigouvian corrective subsidies on net financial deliveries can be welfare improving because they induce a wedge in marginal rates of substitutions that results in inflating bond prices and relaxing credit conditions.

The plan of the paper is as follows. Section 2 describes the baseline model environment. Section 3 provides a characterization of not-too-tight debt limits in two environments with microfounded borrowing constraints. Section 4 shows that laissez-faire equilibria can be Pareto inferior to equilibria with tighter debt constraints. Section 5 shows that corrective Pigouvian subsidies can mitigate the extent of market failure. Section 6 concludes. All proofs are in the Appendix. The Online Supplement provides additional derivations and discussion.

2 General Model

2.1 Fundamentals

Consider an infinite-horizon endowment economy with a single nonstorable consumption good at each date. Time and uncertainty are both discrete. We use an event tree Σ to describe the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \ldots\}$ there is a finite set $S^t \subseteq \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. The notation $s^{t+1} \succ s^t$ specifies that s^{t+1} is a successor of s^t. The event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$.[5] The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : \ s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree starting at event s^t is then given by:

$$\Sigma(s^t) := \bigcup_{\tau \geqslant 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : \ s^\tau \succeq s^t\}$.

There is a finite set I of household types, each consisting of a unit measure of identical, infinitely lived agents who consume the single perishable good. Preferences over (nonnegative) consumption processes $c = (c(s^t))_{s^t \succeq s^0}$ are represented by the lifetime expected and discounted utility:

$$U(c) := \sum_{t \geqslant 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t)),$$

where $\beta \in (0, 1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of s^t, and $u : [0, \infty) \to \mathbb{R}$ is a utility function that is strictly increasing, stricly concave, continuous on $[0, \infty)$, differentiable on $(0, \infty)$, and satisfies Inada's condition $\lim_{\varepsilon \to 0}[u(\varepsilon) - u(0)]/\varepsilon = \infty$. To simplify further the exposition we assume that u is bounded. This restriction ensures that the lifetime utility U is continuous (for the product topology) and the demand set is nonempty.[6] Given an event s^t, we denote by $U(c|s^t)$ the lifetime continuation utility conditional on s^t, as defined by:

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geqslant 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau})),$$

[5] Formally, σ is a mapping from $\Sigma \setminus \{s^0\}$ to Σ such that $\sigma(S^{t+1}) = S^t$ for every $t \geqslant 0$. We pose $\sigma^{(1)} := \sigma$ and $\sigma^{(\tau+1)} := \sigma \circ \sigma^{(\tau)}$ for every $\tau \geqslant 1$.

[6] The analysis can be extended, and our results continue to hold even when u is unbounded. In particular, when u belongs to the class of constant relative risk aversion utility functions $u(c) = c^{1-\alpha}/(1-\alpha)$ with $\alpha > 0$. In fact, in the simple economy we consider to illustrate our main result we assume a logarithmic period utility function, i.e., $\alpha = 1$. A general treatment of unbounded utility functions requires some additional technical assumptions on endowment processes together with a suitable modification of the utility function u outside a specific interval such that the equilibrium outcomes remain unaffected. For a detailed discussion, see Martins-da-Rocha and Santos (2019).

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t.

Agents' endowments are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \succeq s^0}$ the process of positive endowments $y^i(s^t) > 0$ of a representative agent of type i. For notational convenience, we assume that agents' preferences and beliefs are homogeneous as it simplifies the exposition of the theoretical results. We remark that all of our arguments remain valid when agents have heterogeneous preferences and beliefs, and the only necessary change is to replace (u, β, π) with (u^i, β^i, π^i). In fact, we explicitly consider a setting with heterogeneous beliefs in the main Sections 4 and 5.

2.2 Debt-Constrained Asset Markets

At any event s^t, agents can issue and trade state-contingent one-period bonds, each one promising to pay one unit of the consumption good contingent on the realization of a successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ denote the price, at event s^t, of the s^{t+1}-contingent bond (the inverse of q is the interest rate between s^t and s^{t+1}). Agent i's bond holdings are $a^i = (a^i(s^t))_{s^t \succeq s^0}$, where $a^i(s^t) \leqslant 0$ is a liability and $a^i(s^t) \geqslant 0$ is a claim. Each agent's debt is observable and subject to certain (state-contingent, nonnegative and finite) debt limits $D^i = (D^i(s^t))_{s^t \succeq s^0}$.

Given an initial bond holding $a^i(s^0)$ and debt limits D^i, we denote by $B^i(D^i, a^i(s^0)|s^0)$ the budget set of an agent who never defaults. It consists of all pairs (c^i, a^i) of consumption and bond holdings satisfying the following budget flows and debt constraints: for all $s^t \succeq s^0$,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leqslant y^i(s^t) + a^i(s^t), \tag{2.1}$$

and

$$a^i(s^{t+1}) \geqslant -D^i(s^{t+1}), \quad \forall s^{t+1} \succ s^t. \tag{2.2}$$

We naturally restrict attention to allocations where the initial asset holdings clear the market, i.e., $\sum_{i \in I} a^i(s^0) = 0$, and satisfy the debt constraints, i.e., $a^i(s^0) \geqslant -D^i(s^0)$ for each i. Similarly, let $B^i(D^i, b|s^\tau)$ be the set of all plans (c^i, a^i) satisfying restrictions (2.1) and (2.2) at every successor node $s^t \succeq s^\tau$ with initial claim $a^i(s^\tau) = b$. Let $V^i(D^i, b|s^t)$ be the contingent value function defined by:

$$V^i(D^i, b|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B^i(D^i, b|s^t)\}.$$

When $b = a^i(s^t)$, this will be the equilibrium value, i.e., the payoff to each agent i along the equilibrium path following any event s^t.

Definition 2.1. Given initial asset holdings $(a^i(s^0))_{i\in I}$ satisfying $\sum_{i\in I} a^i(s^0) = 0$, an *equilibrium* $(q, (c^i, a^i, D^i)_{i\in I})$ is a collection of state-contingent bond prices q, a consumption allocation $(c^i)_{i\in I}$, a bond holdings allocation $(a^i)_{i\in I}$, and a family of nonnegative and finite debt limits $(D^i)_{i\in I}$ satisfying:

(a) each agent i, taking prices and the debt limits as given, chooses a plan (c^i, a^i) that is optimal among budget feasible plans in $B^i(D^i, a^i(s^0)|s^0)$;

(b) markets clear: $\sum_{i\in I} c^i = \sum_{i\in I} y^i$ and $\sum_{i\in I} a^i = 0$.

In the above definition, debt limits are arbitrary. Our main object of interest is the endogenous determination of the debt limits, which are a critical determinant of equilibrium allocations and equilibrium payoffs.

2.3 Debt Limits

The limits represent the maximal amount of debt that borrowers can issue. In general equilibrium, they also represent the maximal amount of liquidity (or storage of value) that savers have access to. For reasons and microfoundations that will soon be provided, we specify that debt limits satisfy the following general decomposition property:

$$D^i(s^t) = \ell^i(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \forall s^t \succ s^0, \tag{2.3}$$

where the first term $\ell^i(s^t) \in [0, y^i(s^t)]$ is exogenously given and the second-term is the maximum amount the agent can get from rolling over debt. If the debt level $D^i(s^t)$ is self-enforceable, $\ell^i(s^t)$ represents the amount agent i is willing to deliver beyond the resources he can get from rolling over debt to next period. We use the terms *pledgeable endowment* and *endowment loss* interchangeably when referring to the process $\ell^i = (\ell^i(s^t))_{s^t\succeq s^0}$.

It is straightforward that a process of debt limits D^i satisfies property (2.3) if, and only if, it can be decomposed into a *fundamental* and a *bubble* component:

$$D^i(s^t) = \underbrace{\mathrm{PV}(\ell^i|s^t)}_{\text{fundamental}} + \underbrace{M^i(s^t)}_{\text{bubble}}, \quad \text{for all } s^t \succeq s^0. \tag{2.4}$$

Here, the fundamental component is simply the present value of pledgeable income, defined as:

$$\mathrm{PV}(\ell^i|s^t) := \frac{1}{p(s^t)} \sum_{s^\tau\succeq s^t} p(s^\tau)\ell^i(s^\tau), ^7$$

[7] Similarly, the *wealth* of an agent at event s^t is defined as the present value $\mathrm{PV}(y^i|s^t)$ of her endowments.

where $p(s^t)$ is the date-0 price of consumption at event s^t.[8] The bubble component of D^i is a nonnegative process satisfying the following exact rollover property:

$$M^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) M^i(s^{t+1}), \quad \forall s^t \succeq s^0.$$

We now have the following equilibrium definition:

Definition 2.2. Given pledgeable endowment processes $(\ell^i)_{i \in I}$, we call an *equilibrium with limited pledgeability* any equilibrium $(q, (c^i, a^i, D^i)_{i \in I})$ such that the debt limits D^i of each agent i satisfy the condition (2.3), or equivalently the condition (2.4).

Our setting nests several important benchmarks. When the whole endowment is pledgeable, i.e., $\ell^i = y^i$ for any $i \in I$, the debt limits coincide with the natural debt limits, i.e., $D^i = \mathrm{PV}(y^i)$.[9] When no endowment is pledgeable, i.e., $\ell^i = 0$ for each agent i, then our setting collapses to Hellwig and Lorenzoni (2009), where debt is necessarily rolled over as a credit bubble. In Martins-da Rocha et al. (2021), we provide an example of an equilibrium where the fundamental and bubble components co-exist, i.e., $\mathrm{PV}(\ell^i) > 0$ and $M^i > 0$. We also provide conditions on primitives sufficient to guarantee the existence of an equilibrium with limited pledgeability.

When can we rule out bubbly equilibria where the credit bubble component M^i is positive? The following proposition shows that this is the case when the pledgeable resources constitute a *nonnegligible* fraction of aggregate resources:

Proposition 2.1. *If pledgeable resources are a nonnegligible fraction of aggregate resources, in the sense that there exists $\varepsilon > 0$ such that:*

$$\sum_{i \in I} \ell^i(s^t) \geqslant \varepsilon \sum_{i \in I} y^i(s^t), \quad \forall s^t \succeq s^0,$$

then in any equilibrium with limited pledgeability, the bubble component is necessarily zero. As a consequence, $D^i = \mathrm{PV}(\ell^i)$ for every agent i.

Proof. See Appendix A.1. □

[8] Formally, $p(s^t)$ is defined recursively by $p(s^0) = 1$ and $p(s^{t+1}) = q(s^{t+1})p(s^t)$ for all $s^{t+1} \succ s^t$.

[9] Proposition 2.1 below implies that when $\ell^i = y^i$ for each agent i, the bubble component is necessarily zero in equilibrium, and hence $D^i = \mathrm{PV}(y^i)$.

3 Microfoundations for Debt Limits

In equilibrium with limited pledgeability debt limits satisfy condition (2.3), or equivalently condition (2.4). It turns out that this decomposition property arises naturally in environments with limited commitment. Formally, consider an environment where agents cannot commit to their financial contracts and may opt for default. We denote by $V^i_{\text{def}}(s^t)$ agent i's value of the default option at event s^t. Following Alvarez and Jermann (2000), we impose that the debt limits reflect the fact that repayment is always individually rational. Specifically, we say that debt limits D^i are *self-enforcing* if debtors prefer to repay even the maximum debt allowed, i.e.,

$$V^i(D^i, -D^i(s^t)|s^t) \geqslant V^i_{\text{def}}(s^t), \quad \text{for all } s^t \succeq s^0. \tag{3.1}$$

We say that D^i are *not too tight* if (3.1) always holds with equality, i.e., borrowers are indifferent between repaying and defaulting:

$$V^i(D^i, -D^i(s^t)|s^t) = V^i_{\text{def}}(s^t), \quad \text{for all } s^t \succeq s^0. \tag{3.2}$$

Given future debt limits $(D^i(s^\tau))_{s^\tau \succ s^t}$, the level $D^i(s^t)$ satisfying (3.2) is interpreted as the largest self-enforcing debt limit contingent to event s^t. We say that D^i are *too tight* if they are self enforcing and (3.1) holds with strict inequality at some event $s^t \succ s^0$.

Definition 3.1. Given a family of default value functions $(V^i_{\text{def}})_{i\in I}$, we call a *self-enforcing equilibrium* any equilibrium $(q, (c^i, a^i, D^i)_{i\in I})$ such that the debt limits D^i of each agent i satisfy condition (3.1). When the debt limits satisfy the not-too-tight condition (3.2), we use the term *not-too-tight equilibrium*. Similarly, when the debt limits are too tight, we use the term *too-tight equilibrium*.

It is reasonable to expect that in a competitive market, competition among lenders should naturally lead them to offer as much credit as possible, without violating borrowers' incentive to repay. Hence, we will also use the term *laissez-faire equilibrium* as a synonym for not-too-tight equilibrium.

The value of default is the key object that determines the debt limits. We analyze two well-established frameworks: a *reputation debt* environment in which default entails restricted market participation and a loss of endowments (Bulow and Rogoff 1989 and Hellwig and Lorenzoni 2009), and a *collateralized debt* environment in which the only consequence of default is the seizure of a collateral asset (Chien and Lustig 2010 and Gottardi and Kubler 2015).

3.1 Reputation Debt

In this section, we consider a framework à la Bulow and Rogoff (1989) where all assets are seized upon default and debtors loose access to credit while retaining the ability to save (by purchasing other people's debt). In addition, default causes a (dead weight) endowment loss: if agent i defaults at s^τ, then her endowments will reduce to $y^i(s^t) - \ell^i(s^t)$ for all successor events $s^t \succeq s^\tau$, with $\ell^i(s^t) \in [0, y^i(s^t)]$ exogenously given.[10] As a consequence, the default utility for any agent i at any event s^t is given by:

$$V^i_{\text{def}}(s^t) = V^i_{\ell^i}(0,0|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B^i_{\ell^i}(0,0|s^t)\}, \tag{3.3}$$

where $B^i_{\ell^i}(0,0|s^t)$ is the budget set of any agent i who has zero liabilities, cannot borrow, and is endowed with $y^i - \ell^i$ resources. The condition (3.2) then reads as follows:

$$V^i(D^i, -D^i(s^t)|s^t) = V^i_{\ell^i}(0,0|s^t), \quad \text{for all } s^t. \tag{3.4}$$

The following result shows that not-too-tight debt limits can be decomposed into a fundamental component and a credit bubble component that captures the possibility of rolling over a fraction of debt indefinitely. It provides our first microfoundation for the decomposition property (2.4).

Theorem 3.1. *In the reputation debt framework, where the value of default is given by (3.3), any process of not-too-tight debt limits D^i can be decomposed as the sum of the present value of the endowment loss process ℓ^i and a bubble component M^i, i.e.,*

$$D^i = \text{PV}(\ell^i) + M^i,$$

where M^i is a nonnegative exact rollover process.

Proof. See Appendix A.2. □

Intuitively, the bubble component reflects the fact that credit beyond the fundamental component is sustainable only if agents can rollover their debt. A crucial and nontrivial

[10]In the context of consumer credit, the endowment loss is a parsimonious way to capture recourse and other legal consequences of default (see e.g. Chatterjee et al. 2007; Livshits et al. 2007; Livshits 2015). In the context of sovereign debt, the endowment loss parsimoniously captures the negative effects of default on domestic production (see e.g. Eaton and Gersovitz 1981; Bulow and Rogoff 1989; Cole and Kehoe 2000; Aguiar and Gopinath 2006; Arellano 2008).

step to prove the result is to show that the process $\mathrm{PV}(\ell^i)$ is a lower bound to any sequence of not-too-tight debt limits. A second step, based on a translation invariance of the flow budget constraints, then shows that the process $\mathrm{PV}(\ell^i)$ is itself not too tight. The result then follows from the well-known fact that the difference between two processes of not-too-tight debt limits necessarily satisfies the exact rollover property (see, for instance, Martins-da-Rocha and Santos 2019).

Besides providing a microfoundation for our specification of debt limits in Section 2, Theorem 3.1 is also useful for the computation of equilibria. It eliminates the usual complications related to the fixed-point process of determining not-too-tight debt limits, where the value of default depends on prices (as defaulting agents can still save), which in turn depend on equilibrium allocations and hence the debt limits. The usefulness will become clear when we conduct our policy intervention experiments in Section 4.

3.2 Collateralized Debt

In this section, we shift our attention to an environment where all borrowing and lending is fully secured by collateral. As in Chien and Lustig (2010) and Gottardi and Kubler (2015), we assume that agents back their promises by means of trading a long-lived asset (or Lucas tree). In contrast to the economy studied in the previous section, debt repudiation does not induce any form of exclusion from financial markets. Upon default, debtors lose their collateralizable assets which are handed over to creditors, but they still maintain access to financial markets. Within this framework, our aim is to provide another microfoundation for the specification of debt limits in Section 2. Furthermore, we will use Theorem 3.1 to establish an interesting and nontrivial equivalence mapping between the two settings. This equivalence unravels an interesting link between credit limits and asset prices.

Consider an economy where each agent i receives an endowment of $e^i(s^t) \geqslant 0$ units of the consumption good at event s^t. At the initial period, each agent i is also endowed with an exogenous share $\alpha^i(s^{-1}) \geqslant 0$ of a Lucas tree. The tree is an infinitely lived physical asset that pays a dividend of $\delta(s^t) \geqslant 0$ units of the consumption good at event s^t. Agent i's total endowment is therefore $y^i(s^t) := e^i(s^t) + \alpha^i(s^{-1})\delta(s^t)$ at event s^t. The tree exists in unit supply and its shares can be traded at the ex-dividend price $P(s^t)$, determined in equilibrium. We denote by $\alpha^i(s^t) \geqslant 0$ the post-trade tree holding of agent i at event s^t. Agents can also trade one-period-ahead contingent bonds at any event s^t. Let $b^i(s^{t+1}) \in \mathbb{R}$ denote the position on the bond paying at event s^{t+1}, whose price, expressed in units of

s^t-consumption, is $q(s^{t+1})$. For each agent i, given an initial financial claim $b^i(s^0)$, the initial financial wealth is given by $a^i(s^0) := b^i(s^0) + \alpha^i(s^{-1})[P(s^0) + \delta(s^0)]$.

Since the tree holdings can be seized by creditors, it is intuitive to assume that debt limits are imposed on the net asset position.[11] Formally, we let $\widetilde{B}^i(\widetilde{D}^i, a^i(s^0)|s^0)$ denote the budget set consisting of all triples (c^i, α^i, b^i) of consumption processes $c^i = (c^i(s^t))_{s^t \succeq s^0}$, non-negative tree holdings $\alpha^i = (\alpha^i(s^t))_{s^t \succ s^0}$, and contingent claims $b^i = (b^i(s^t))_{s^t \succ s^0}$ satisfying the following flow budget constraints and debt constraints: for all $s^t \succeq s^0$,[12]

$$c^i(s^t) + P(s^t)\alpha^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) b^i(s^{t+1}) \leqslant \\ e^i(s^t) + b^i(s^t) + \alpha^i(\sigma(s^t))\big[\delta(s^t) + P(s^t)\big], \quad (3.5)$$

and

$$\forall s^{t+1} \succ s^t, \quad b^i(s^{t+1}) + \alpha^i(s^t)\big[\delta(s^{t+1}) + P(s^{t+1})\big] \geqslant -\widetilde{D}^i(s^{t+1}). \quad (3.6)$$

Since we have more markets than in the environment described in Section 2, we need to modify Definition 2.1 as follows.

Definition 3.2. Given initial financial claims $(b^i(s^0))_{i\in I}$ satisfying $\sum_{i\in I} b^i(s^0) = 0$, and initial shares $(\alpha^i(s^{-1}))_{i\in I}$ satisfying $\sum_{i\in I} \alpha^i(s^{-1}) = 1$, an *equilibrium* $(q, P, (c^i, \alpha^i, b^i, \widetilde{D}^i)_{i\in I})$ is a collection of state-contingent bond prices q, tree prices P, a consumption allocation $(c^i)_{i\in I}$, an allocation of tree holdings $(\alpha^i)_{i\in I}$, an allocation of contingent claims $(b^i)_{i\in I}$ and finite debt limits $(\widetilde{D}^i)_{i\in I}$ such that:

(a) each agent i, taking prices and the debt limits as given, chooses a plan (c^i, α^i, b^i) that is optimal among budget feasible plans in $\widetilde{B}^i(\widetilde{D}^i, a^i(s^0)|s^0)$;[13]

(b) all markets clear:

$$\sum_{i\in I} c^i = \sum_{i\in I} y^i, \quad \sum_{i\in I} b^i = 0 \quad \text{and} \quad \sum_{i\in I} \alpha^i = 1.$$

For every event $s^\tau \succ s^0$ and every beginning-of-period net financial wealth $x \in \mathbb{R}$, we let $\widetilde{B}^i(\widetilde{D}^i, x|s^\tau)$ be the set of triples (c^i, α^i, b^i) satisfying the flow budget constraint (3.5) and

[11] This follows Kocherlakota (2008).

[12] To keep notational consistency, we extend the domain of the predecessor function σ to the whole tree Σ by posing $\sigma(s^0) := s^{-1}$.

[13] Recall that the initial financial wealth is given by $a^i(s^0) = b^i(s^0) + \alpha^i(s^{-1})[P(s^0) + \delta(s^0)]$.

the debt constraints (3.6) for all successor events $s^t \succeq s^\tau$, together with the initial wealth condition:

$$x = b^i(s^\tau) + \alpha^i(\sigma(s^\tau))[\delta(s^\tau) + P(s^\tau)].$$

The continuation value conditional on no default is then given by:

$$\widetilde{V}^i(\widetilde{D}^i, x|s^\tau) := \sup\{U(c^i|s^\tau) \ : \ (c^i, \alpha^i, b^i) \in \widetilde{B}^i(D^i, x|s^\tau)\}.$$

At any contingency, debtors have the option to renege on their contracts and file for bankruptcy. In this case, all tree holdings and current period dividends are seized and transferred to lenders to redeem their debt. The part $e^i(s^t)$ of total endowment $y^i(s^t) = e^i(s^t)+\alpha^i(s^{-1})\delta(s^t)$ cannot be seized, and defaulters still maintain access to financial markets. We refer to the process $(\alpha^i(s^{-1})\delta(s^t))_{s^t \succeq s^0}$ as the *collateralizable income*. The residual $e^i(s^t)$ constitutes the non-pledgeable component of the total endowment $y^i(s^t)$, since it cannot be sold in advance to finance consumption or savings at any date before the endowment is received.

This specification of the default punishment leads to the following value of default:

$$\widetilde{V}^i_{\text{def}}(s^\tau) := \widetilde{V}^i(\widetilde{D}^i, 0|s^\tau), \tag{3.7}$$

and the condition (3.2) for debt limits $\widetilde{D}^i$ to be not too tight becomes:

$$\widetilde{V}^i(\widetilde{D}^i, -\widetilde{D}^i(s^t)|s^t) = \widetilde{V}^i(\widetilde{D}^i, 0|s^t) \quad \text{for all } s^t \succeq s^0. \tag{3.8}$$

Comparing condition (3.8) and its counterpart (3.4) reveals that it is simpler to solve for the not-too-tight debt limits in the collateral model than in the reputation model, as the former condition does not depend on future debt limits. In fact, we have the following immediate result: when the value of default is given by (3.7), any process of not-too-tight debt limits must be equal to zero.[14] This in turn implies that the not-too-tight debt constraints (3.6) are equivalent to the collateral constraints:

$$\forall s^{t+1} \succ s^t, \quad b^i(s^{t+1}) \geqslant -\alpha^i(s^t)\big[P(s^{t+1}) + \delta(s^{t+1})\big]. \tag{3.9}$$

[14] In the reputation debt framework, the debt limit $D^i(s^t)$ is not too tight when $V^i(D^i, -D^i(s^t)|s^t) = V^i_{\ell^i}(0, 0|s^t)$. Since future debt limits only appear in the LHS of the equation, the determination of $D^i(s^t)$ depends on the value of the future debt limits $(D^i(s^\tau))_{s^\tau \succ s^t}$. Since we also require future debt limits to be not too tight, this involves a fixed point in the space of debt limits processes. In the collateralized debt environment, however, future debt limits appear both in the RHS and LHS of (3.8). Since the mapping $x \mapsto \widetilde{V}^i(\widetilde{D}^i, x|s^t)$ is strictly increasing, we deduce that $\widetilde{D}^i(s^t) = 0$ is the only possible solution of (3.8). Observe that this property is valid even if future debt limits were too tight. This will be crucial in the analysis conducted in Section 4.4.

To connect the above constraint to the decomposition of debt limits in (2.4), we recall the following standard asset-pricing result:

$$\delta + P = \mathrm{PV}(\delta) + M, \tag{3.10}$$

where M is a nonnegative exact rollover process.[15] Asset pricing equation (3.10) implies that tree holdings are indeterminate since what matters for consumption smoothing purposes is the net financial position

$$\theta^i(s^t) := b^i(s^t) + \alpha^i(\sigma(s^t))\big[P(s^t) + \delta(s^t)\big].$$

Indeed, given Equation (3.10), the flow budget constraint (3.5) can be written as:

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\theta^i(s^{t+1}) \leqslant e^i(s^t) + \theta^i(s^t).$$

Therefore, adjusting contingent claims b^i if necessary, we can assume without any loss of generality that agents do not trade their equity shares i.e., $\alpha^i(s^t) = \alpha^i(s^{-1})$. The flow budget constraint (3.5) then becomes:

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})b^i(s^{t+1}) \leqslant y^i(s^t) + b^i(s^t) \tag{3.11}$$

while the debt constraint is stated as:

$$b^i(s^{t+1}) \geqslant -\alpha^i(s^{-1})P(s^{t+1}) = -\big[\mathrm{PV}(\alpha^i(s^{-1})\delta|s^{t+1}) + \alpha^i(s^{-1})M(s^{t+1})\big]. \tag{3.12}$$

Fix an arbitrary decomposition of the process $M = \sum_{i \in I} M^i$ where each M^i is a nonnegative exact roll-over process. Consider the allocation $(a^i)_{i \in I}$ given by:

$$a^i(s^t) := b^i(s^t) + \alpha^i(s^{-1})M(s^t) - M^i(s^t).$$

We can check that the flow budget constraint (3.11) is equivalent to:

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a^i(s^{t+1}) \leqslant y^i(s^t) + a^i(s^t)$$

and the debt constraint (3.12) is equivalent to:

$$a^i(s^{t+1}) \geqslant -\big[\mathrm{PV}(\alpha^i(s^{-1})\delta|s^{t+1}) + M^i(s^{t+1})\big].$$

[15] We refer to Appendix A.3 for a detailed derivation of this equation.

The above implies that agents' borrowing capacity in the economy with collateralized debt is decomposed into a fundamental and a bubble component exactly the same way it is decomposed in the economy with reputation debt. We can now present the following equivalence theorem.

Theorem 3.2. *A consumption allocation is the outcome of a laissez-faire equilibrium in the collateralized debt framework, where the tree's dividend process is δ and the tree's initial holdings are $(\alpha^i(s^{-1}))_{i\in I}$, if, and only if, it is the outcome of a laissez-faire equilibrium in the reputation debt framework, where the endowment losses are $(\alpha^i(s^{-1})\delta)_{i\in I}$.*

Proof. See Appendix A.4. □

Our equivalence result has implications for the effects of vanishing pledgeable income on borrowing capacity and intertemporal trade. Assume, as in Chien and Lustig (2010), that endowments are bounded and that collateralizable income represents a constant fraction of the endowment, i.e., there exists $\delta \geqslant 0$ such that for all s^t, $\delta(s^t) = \delta \geqslant 0$. When $\delta > 0$, Proposition 2.1 implies that the present value of pledgeable resources is finite and assets are priced at their fundamental value, so prices are bubble-free. One may think that when $\delta = 0$ (i.e., assets pay no dividends), asset prices must equal zero, so autarky is the only equilibrium outcome. But such a claim presupposes that the aggregate wealth is still finite, or equivalently, that the implied interest rates remain positive (higher than the growth rate) when passing to the limit. However, as documented by Hellwig and Lorenzoni (2009), when $\delta = 0$, equilibrium interest rates can be sufficiently low (equal to zero in the absence of growth) so that the economy's aggregate wealth is infinite. The implication for the collateral equilibrium, is that, even if the trees pay no dividend, assets may be priced as a speculative bubble. Indeed, it is sufficient to appeal to Theorem 3.2 and translate the bubbly equilibrium of Hellwig and Lorenzoni (2009) in the environment of Chien and Lustig (2010). The intuition for this discrepancy relies on the dual role of collateral as a source of liquidity. As dividends become negligible (i.e., δ approaches zero), the value of the asset increases to compensate for the decreased investment value. In the limit, the value of the collateral asset is still positive, reflecting purely a bubble, even though there is no collateral in the market anymore.

4 Tightening Debt Constraints

The common belief in models where financial frictions are due to limited commitment is that borrowing should be subject to not-too-tight debt limits. As we mentioned before, this choice is justified on the grounds that competition among lenders will eventually permit borrowers to issue the largest amount of debt compatible with repayment incentives. This view is further reinforced by the misguided intuition that not-too-tight debt limits allow for maximum risk-sharing. Though this is trivially true in a partial equilibrium framework where prices are fixed, this intuition is questionable in general equilibrium settings where both prices and debt limits are determined endogenously.

For a given specification of the default option, an obvious way to justify the choice of imposing not-too-tight debt limits is to show that any other choice of debt limits consistent with repayment incentives cannot lead to Pareto improvement. Equivalently, the test amounts to explore whether it is possible that equilibria with too-tight debt limits (see Definition 3.1) can Pareto dominate laissez-faire equilibria. We adopt the term *debt-constrained efficiency*, defined formally below, for this optimality property.

Definition 4.1. Fix a feasible allocation $(a^i(s^0))_{i\in I}$ of initial financial claims and a self-enforcing equilibrium $(q,(c^i,a^i,D^i)_{i\in I})$.[16] This equilibrium is said to be *debt-constrained efficient* if there does not exist another self-enforcing equilibrium $(\tilde{q},(\tilde{c}^i,\tilde{a}^i,\widetilde{D}^i)_{i\in I})$ with, possibly, a different feasible allocation $(\tilde{a}^i(s^0))$ of initial financial claims such that the consumption allocation $(\tilde{c}^i)_{i\in I}$ Pareto dominates $(c^i)_{i\in I}$.[17]

We hereafter explore whether laissez-faire equilibria in economies with limited commitment, like those analysed in the previous sections, are debt-constrained efficient and whether policy interventions are warranted. In doing so, it is useful to first revisit a well-known benchmark where debt-constrained efficiency is unambiguous: the Alvarez and Jermann (2000) model, where default induces complete financial autarky and non-negligible deadweight losses. There, we have the following result:

Theorem 4.1. *Assume that for each agent i, the value of the default option is:*

$$V^i_{\mathrm{def}}(s^t) = U^i(y^i - \ell^i|s^t), \quad \textit{for all } s^t \succ s^0.$$

[16] In the sense that debt limits D^i satisfy the condition (3.1) with (possibly) strict inequality in some contingencies. Recall that laissez-faire equilibria are a particular case of self-enforcing equilibria.

[17] In the sense that $U(\tilde{c}^i) > U(c^i)$ for each agent $i \in I$.

If the endowment losses $(\ell^i)_{i\in I}$ are a non-negligible fraction of aggregate resources (see Proposition 2.1), then any laissez-faire equilibrium is debt-constrained efficient.

Proof. See Appendix A.5.[18] □

This section's main contribution is to overturn this efficiency result in economies where agents can still save upon default. In particular, we show that, in the environments of Sections 3.1 and 3.2, a policy intervention that tightens the debt limits can Pareto improve upon the laissez-faire allocation. Specifically, we allow for a credit agency or the government to impose too-tight debt limits and show that, under certain conditions, in the new equilibrium all agents are better off with respect to the equilibrium that is subject to not-too-tight debt limits. We interpret such interventions as a parsimonious representation of regulatory or prudential policies that aim to constrain leverage in the financial markets.

Intuitively, when do we expect laissez-faire allocations to be debt-constrained inefficient? When the value of default depends on market prices, there is a pecuniary externality that is not internalized by agents in a competitive environment. In particular, we will show that a reduction of the borrowing capacity from a period τ onward reduces the credit volume and increases bond prices, or equivalently, lowers the implied interest rates. This impact on prices has a negative feedback effect on the value of the default option at periods $t < \tau$, since it is now more costly to smooth consumption over time by saving only. This implies that the not-too-tight debt limits at periods $t < \tau$ must be looser compared to their level before the intervention. Pareto improvement can be obtained when the benefits from the relaxed credit conditions at periods $t < \tau$ compensate the costs of the tighter credit conditions at subsequent periods.

To illustrate the intuition above in the simplest possible manner, we consider an economy with two agents facing uncertainty only at the initial period. The economy is thereafter a deterministic one where every other period agents' endowments switch from a high value to a low value. Within this setting we perform the following exercise. We first construct a Markov laissez-faire equilibrium $(q,(c^i,a^i,D^i)_{i\in I})$ where after the realization of uncertainty the economy settles in a cyclical and symmetric steady-state equilibrium where debt limits are not too tight. We then construct another equilibrium $(\tilde{q},(\tilde{c}^i,\tilde{a}^i,\widetilde{D}^i)_{i\in i})$, supported by the same allocation of initial financial claims but with debt constraints that are tighter

[18]In fact, the proof shows that this efficiency result extends to any laissez-faire equilibrium where the present value of initial endowments is finite and the value of the default punishment is independent of equilibrium prices.

than necessary. We then show that the consumption allocation $(\tilde{c}^i)_{i\in I}$ Pareto dominates the consumption allocation $(c^i)_{i\in I}$ of the laissez-faire equilibrium. Throughout our analysis we make use of our decomposition result (Theorem 3.1) and equivalence result (Theorem 3.2).

4.1 Primitives of the Example

There are two agents $I = \{a, b\}$ who enter the market with an identical endowment $y_0 > 0$ and no financial claims (i.e., $a^a(s^0) = a^b(s^0) = 0$). There is uncertainty only at the initial period $t = 0$, described by two possible states $z^a \neq z^b$. After the realization of the state z^i, the economy becomes deterministic where agents endowments' switch between a high value y_{H} and a low value y_{L} with $y_{\text{H}} > y_{\text{L}}$. The realization of state z^i means that it is agent i who starts with the high endowment at $t = 1$. The beliefs are heterogeneous, with each agent assigning a probability $\pi_{\text{H}} < 1/2$ ($\pi_{\text{L}} := 1 - \pi_{\text{H}}$, respectively) of getting the high (low, respectively) endowment at $t = 1$.

Since there is uncertainty only at the initial period, we simplify notation by writing a generic process $(x(s^t))_{s^t \succeq s^0}$ as follows: $x(s^0) = x_0$ and

$$x(s^t) = x_t(z), \quad \text{if } s^t \succeq (s^0, z) \text{ with } z \in \{z^a, z^b\}.$$[19]

The representation of the event tree is as in Figure 4.1.

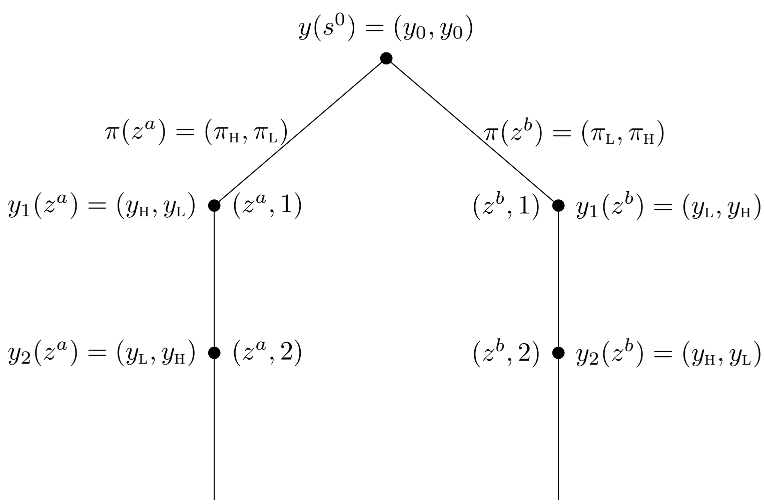


Figure 4.1: Event tree and endowments

[19] The event-tree Σ can be formally defined as follows: $S^0 := \{s^0\}$ and for every $t \geqslant 1$, $S^t = \{(z^a, t), (z^b, t)\}$. The binary relation $\succ$ is defined as follows: $(z, 1) \succ s^0$ and $(z, \tau) \succ (\zeta, t)$ when $z = \zeta$ and $\tau > t$.

For future reference, we point out that the *symmetric first-best allocation* of this economy obtains when both agents consume their endowment at $t = 0$ and, conditional on the realization of state z^i, agent i consumes $\underline{c}^{\text{fb}}$ while agent $j \neq i$ consumes $\overline{c}^{\text{fb}}$ at every period $t \geqslant 1$. The consumption levels $\overline{c}^{\text{fb}}$ and $\underline{c}^{\text{fb}}$ solve the following system of equations:

$$\pi_{\text{H}} u'(\underline{c}^{\text{fb}}) = \pi_{\text{L}} u'(\overline{c}^{\text{fb}}) \quad \text{and} \quad \overline{c}^{\text{fb}} + \underline{c}^{\text{fb}} = y_{\text{H}} + y_{\text{L}}. \tag{4.1}$$

Observe that $\overline{c}^{\text{fb}} > \underline{c}^{\text{fb}}$: since both agents believe that reaching the low endowment state at $t = 1$ has a higher likelihood $\pi_{\text{L}} > \pi_{\text{H}}$, they will trade to implement the larger consumption level $\overline{c}^{\text{fb}}$ contingent to this event.

All figures in the paper are obtained assuming:

$$u = \ln, \quad y_0 = 1, \quad y_{\text{L}} = 2, \quad y_{\text{H}} = 2.5, \quad \beta = 0.9 \quad \text{and} \quad \pi_{\text{H}} = 0.35.$$

Setting specific values for the primitives serves well the purpose to provide a graphical illustration of our policy interventions. It also helps to verify straightaway the validity of the first-order optimality conditions which is an essential part of the construction of Pareto improving equilibria. It should be clear that, given our assumptions (continuity), the whole analysis is valid for an open set of parameters values. For the sake of exposition we abstract from presenting the technical details of the robustness of our results.

4.2 Laissez-faire Equilibrium

Suppose that pledgeable endowment is time-invariant and identical for both agents, i.e., $\ell^i(s^t) = \ell$ for all $i \in I$ and $s^t \in \Sigma$. Within this framework, we restrict attention to symmertic Markov equilibria with limited pledgeability and recall that equilibrium debt limits are equal to the present value of pledgeable endowment.[20]

We first notice that the first-best allocation can be implemented as an equilibrium when the level ℓ of pledgeable endowment is larger than the following threshold:

$$\ell^{\text{fb}} := \frac{(y_{\text{H}} - \underline{c}^{\text{fb}}) - \beta(\underline{c}^{\text{fb}} - y_{\text{L}})}{1 + \beta}. \tag{4.2}$$

Consider next the following lower level of pledgeable endowment:

$$\ell^\star := \frac{1 - \beta}{1 + \beta} \times \frac{y_{\text{H}} - y_{\text{L}}}{2}. \tag{4.3}$$

[20] When $\ell > 0$, it follows from Proposition 2.1 that debt limits in an equilibrium with limited pledgeability cannot display bubbles.

We assume the parameters such that $\ell^\star < y_{\mathrm{L}}$.[21] The following claim shows that ℓ^* supports an equilibrium with the following characteristics: at period $t = 0$, both agents borrow against their high-income state and save contingent to their low-income state. After the resolution of the uncertainty at period $t = 1$, the economy settles in a cyclical steady-state where the low-income agent borrows up to the not-too-tight debt limit, the high-income agent saves, and consumption is constant and equal to $c^{\mathrm{lf}} := (y_{\mathrm{L}} + y_{\mathrm{L}})/2$ for every $t \geqslant 1$.

Claim 4.1. *Let $\ell^\star$ be specified as in (4.3) and denote:*

$$q(0) := \beta \quad \text{and} \quad d(0) := \frac{\ell^\star}{1 - q(0)}.$$

There exists an equilibrium with limited pleadgebility $(q, (c^i, a^i, D^i)_{i \in I})$ where for each $z \in \{z^a, z^b\}$ and every $i \in I$:

(i) Debt limits equal $D^i_t(z) = d(0)$, for $t \geqslant 1$;

(ii) Consumption is risk-free: $c^i_0 = y_0$ and $c^i_t(z^a) = c^i_t(z^b) = c^{\mathrm{lf}}$, for $t \geqslant 1$;

(iii) Net asset positions are $a^i_t(z) = -d(0)$ (i.e., the debt limit binds) if $y^i_t(z) = y_{\mathrm{H}}$ and $a^i_t(z) = d(0)$ if $y^i_t(z) = y_{\mathrm{L}}$, for $t \geqslant 1$;

(iv) Prices are given by $q_1(z) = \beta \pi_{\mathrm{L}} u'(c^{\mathrm{lf}})/u'(y_0)$ and $q_{t+1}(z) = q(0)$, for $t \geqslant 1$;

(v) The continuation utility at period $t = 1$ is:

$$U^{\mathrm{lf}}_1 = \frac{u(c^{\mathrm{lf}})}{1 - \beta},$$

and the expected utility at period $t = 0$ is:

$$U^{\mathrm{lf}}_0 = u(y_0) + \beta U^{\mathrm{lf}}_1.$$

We omit the straightforward proof of the claim and conclude by noting that Theorem 3.1 and Theorem 3.2 imply that the equilibrium described above can be supported as a *laissez-faire* equilibrium where debt limits are not-too-tight as it is the case in the model with reputation debt and the model with collateralized debt. Indeed, the collateral equilibrium is obtained when the dividend process δ of the Lucas tree is constant and equal to $2\ell^\star$, and the initial tree holdings are symmetric, i.e., $\alpha^i(s^{-1}) = 1/2$ for each i. Non-pledgeable endowment is then given by $e^i(s^t) := y^i(s^t) - \ell^\star$.[22]

[21] This is the case when β is sufficiently close to 1 or $y_{\mathrm{H}} - y_{\mathrm{L}}$ is sufficiently small.

[22] Recall that the values of primitive parameters are such that $\ell^\star < y_{\mathrm{L}}$.

4.3 Tightening Debt Limits in the Reputation Debt Environment

We now proceed to show that a policy intervention that tightens the debt limits can potentially Pareto improve upon the laissez-faire allocation, starting with the reputation debt framework (Section 4.4 will analyze the collateral debt framework). Formally, for each tightening parameter $\varepsilon \in [0,1]$, we will show that there is an equilibrium

$$(q^\varepsilon, (c^{i,\varepsilon}, a^{i,\varepsilon}, D^{i,\varepsilon})_{i\in I}),$$

where, for every state $z \in \{z^a, z^b\}$, the debt limits $D_1^{i,\varepsilon}(z)$ satisfy the not-too-tight condition (3.2), but the debt limits in subsequent periods are too-tight and equal to:

$$D_t^{i,\varepsilon}(z) = (1-\varepsilon)\,\mathrm{PV}_t^\varepsilon(\ell^\star|z), \text{ for } t \geqslant 2.^{23} \tag{4.4}$$

Our aim is to show that for some values of ε, the new consumption allocation $(c^{i,\varepsilon})_{i\in I}$ Pareto dominates the laissez-faire consumption allocation $(c^i)_{i\in I}$ in Claim 4.1. To facilitate the exposition of this policy experiment we split the argument in several steps.

4.3.1 Steady-State Phase

The first step amounts to show that the debt limits in (4.4) support a cyclical steady-state from period $t = 2$ onward. For each tightening parameter ε, denote

$$q(\varepsilon) := \beta\frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{\mathrm{H}}(\varepsilon))} \quad \text{and} \quad d(\varepsilon) := (1-\varepsilon)\frac{\ell^\star}{1-q(\varepsilon)}, \tag{4.5}$$

where:

$$c_{\mathrm{H}}(\varepsilon) := y_{\mathrm{H}} - (1+q(\varepsilon))d(\varepsilon) \quad \text{and} \quad c_{\mathrm{L}}(\varepsilon) := y_{\mathrm{L}} + (1+q(\varepsilon))d(\varepsilon).$$

We have the following result:

Claim 4.2. *Contingent to any state $z \in \{z^a, z^b\}$, the economy reaches a steady-state phase at $t = 2$ with the following characteristics: for all $t \geqslant 2$,*

(i) Debt limits $D_t^{i,\varepsilon}(z) = d(\varepsilon)$ are too tight.

(ii) The consumption allocation is $c_t^{i,\varepsilon}(z) = c_{\mathrm{H}}(\varepsilon)$ if $y_t^i(z) = y_{\mathrm{H}}$ and $c_t^{i,\varepsilon}(z) = c_{\mathrm{L}}(\varepsilon)$ if $y_t^i(z) = y_{\mathrm{L}}$;

[23] The notation $\mathrm{PV}_t^\varepsilon(\ell^\star|z)$ represents the present value $\mathrm{PV}(\ell^\star|s^t)$ computed with the price process q^ε and conditional to time-t event $s^t = (z,t)$.

(iii) Net asset positions are $a_t^{i,\varepsilon}(z) = -d(\varepsilon)$ (i.e., the debt limit binds) if $y_t^i(z) = y_{\rm H}$ and $a_t^{i,\varepsilon}(z) = d(\varepsilon)$ if $y_t^i(z) = y_{\rm L}$;

(iv) Prices are given by $q_{t+1}^{\varepsilon}(z) = q(\varepsilon)$;

In words, agents borrow the amount $d(\varepsilon)$ when their income is low and save the amount $d(\varepsilon)$ when their income is high. It is shown below (see Figures 4.2(a) and 4.2(b)) that the higher is the tightening coefficient ε, the tighter are the debt limits (i.e., the function $\varepsilon \mapsto d(\varepsilon)$ is decreasing), and the higher is the steady-state price $q(\varepsilon)$ (or, equivalently, the lower is the steady-state interest rate). In the limit, when ε tends to 1, the interest rate is zero (i.e., $\lim_{\varepsilon\to 1} q(\varepsilon) = 1$) and debt limits form a bubble, i.e., $D_t^{i,1}(z) = d(1)$ where $d(1)$ is determined by the equation: $u'(y_{\rm H} - 2d(1)) = \beta u'(y_{\rm L} + 2d(1))$.[24]

By construction, the steady-state variables satisfy market clearing. To be part of an equilibrium they should also be optimal. This requires that the following inequality holds true:[25]

$$q(\varepsilon) \geqslant \beta \frac{u'(c_{\rm H}(\varepsilon))}{u'(c_{\rm L}(\varepsilon))}. \tag{4.6}$$

The claim that $D_t^{i,\varepsilon}(z)$ is too tight relies on the following observation. From the decomposition result (Theorem 3.1) we infer that the equilibrium described in Claim 4.2 is in fact a laissez-faire equilibrium of another economy where the endowment loss upon default is $(1-\varepsilon)\ell^\star$. Indeed, for every $z \in \{z^a, z^b\}$ and $s^t = (z,t)$ with $t > 0$ we have:

$$d(\varepsilon) = \mathrm{PV}_t^{\varepsilon}((1-\varepsilon)\ell^\star | z).$$

So $D^{i,\varepsilon}(s^t) = d(\varepsilon)$ satisfies the not-too-tight condition:

$$V^i(D^{i,\varepsilon}, -D^{i,\varepsilon}(s^t)|s^t) = V^i_{(1-\varepsilon)\ell^\star}(0,0|s^t).$$

Since in the actual economy the endowment loss equals $\ell^\star$, we deduce that:

$$V^i(D^{i,\varepsilon}, -D^{i,\varepsilon}(s^t)|s^t) = V^i_{(1-\varepsilon)\ell^\star}(0,0|s^t) > V^i_{\ell^\star}(0,0|s^t) = V^i_{\rm def}(s^t),$$

which proves the claim.

[24] We notice that under the chosen values for the primitives we have that $\beta u'(y_{\rm L})/u'(y_{\rm H}) = 1.125 > 1$ which ensures the existence of a pure bubbly equilibrium when there is no endowment loss.

[25] The Online Supplement shows that inequality (4.6) is satisfied for the values of the primitives we consider.

4.3.2 Transition Phase

The second step is to determine the equilibrium variables for the transition periods $t = 0$ and $t = 1$. This is nontrivial since we have to compute the not-too-tight debt limits $D_1^{i,\varepsilon}(z)$ without being able to appeal to our decomposition result (Theorem 3.1).[26] We address this issue in the next step (Section 4.3.3). For the moment, fix a parameter $d_1 \in [0, y_{\mathrm{H}})$ representing the debt issued at period $t = 0$, and look for an equilibrium where for every $i \in I$:

$$c_0^{i,\varepsilon} = y_0, \quad a_1^{i,\varepsilon}(z^i) = -d_1, \quad a_1^{i,\varepsilon}(z^j) = d_1 \quad \text{and} \quad D_1^{i,\varepsilon}(z^i) = d_1.$$

That is, at the initial period, both agents borrow against next period's high-income state, and save contingent to the low-income state.

Since at period $t = 2$ the economy settles in the cyclical steady-state described in Claim 4.2, bond holdings at the end of period $t = 1$ should be equal to:

$$a_2^{i,\varepsilon}(z) = \begin{cases} d(\varepsilon) & \text{if } y_2^i(z) = y_{\mathrm{L}}, \\ -d(\varepsilon) & \text{if } y_2^i(z) = y_{\mathrm{H}}. \end{cases}$$

This in turn implies that the corresponding consumption levels at $t = 1$ are given by:

$$c_1^{i,\varepsilon}(z^i) = y_{\mathrm{H}} - d_1 - q_2^\varepsilon(z) d(\varepsilon) =: c_{1,\mathrm{H}}(\varepsilon, d_1)$$

and

$$c_1^{i,\varepsilon}(z^j) = y_{\mathrm{L}} + d_1 + q_2^\varepsilon(z) d(\varepsilon) =: c_{1,\mathrm{L}}(\varepsilon, d_1),$$

where the bond prices $q_2^\varepsilon(z)$ at period $t = 1$ are determined by the first order conditions:

$$q_2^\varepsilon(z) = \beta \pi_{\mathrm{L}} \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{1,\mathrm{H}}(\varepsilon, d_1))} =: q_2(\varepsilon, d_1), \quad \text{for } z \in \{z^a, z^b\}.$$

Similarly, the bond prices at period $t = 0$ are determined by the following first order conditions:

$$q_1^\varepsilon(z) = \beta \pi_{\mathrm{L}} \frac{u'(c_{1,\mathrm{L}}(\varepsilon, d_1))}{u'(y_0)} =: q_1(\varepsilon, d_1), \quad \text{for } z \in \{z^a, z^b\}.$$

Optimality requires that:

$$\frac{u'(c_{1,\mathrm{L}}(\varepsilon, d_1))}{u'(c_{1,\mathrm{H}}(\varepsilon, d_1))} \geqslant \max\left\{ \frac{\pi_{\mathrm{H}}}{\pi_{\mathrm{L}}}, \frac{u'(c_{\mathrm{H}}(\varepsilon))}{u'(c_{\mathrm{L}}(\varepsilon))} \right\}.^{27} \tag{4.7}$$

[26] Note that we only need to specify the debt limit $D_1^{i,\varepsilon}(z^i)$ as the debt limit $D_1^{i,\varepsilon}(z^j)$ that is contingent to the low income state will be nonbinding.

[27] The inequality obtains from the first order conditions of the borrowing decisions at $t = 0$ and $t = 1$

4.3.3 Determination of d_1

We next identify the level of d_1 (that is not too tight) given that the debt limits at all successor periods $t \geqslant 2$ are set to be too tight (see Claim 4.2). Let us denote by $d_1(\varepsilon)$ this level and remark that we cannot appeal to Theorem 3.1 to claim that $d_1(\varepsilon) = \mathrm{PV}_1^{\varepsilon}(\ell^\star|z)$. This would be the case if future debt limits were also not-too-tight (i.e., they are also equal to the present value of pledgeable income), which we have ruled out by construction. Therefore, the determination of $d_1(\varepsilon)$ requires that we do compute the value functions associated to equilibrium and out-of-equilibrium paths. For this purpose, we introduce the following notations.

Let $U_{1,\mathrm{H}}(\varepsilon, d_1)$ denote the value function

$$U_{1,\mathrm{H}}(\varepsilon, d_1) := V^i(D^{i,\varepsilon}, -d_1|(z^i, 1))$$

that corresponds to the largest continuation utility when the debt of the high-income agent at period $t = 1$ equals to d_1. Provided that (4.7) holds true, it follows that:

$$U_{1,\mathrm{H}}(\varepsilon, d_1) = u(c_{1,\mathrm{H}}(\varepsilon, d_1)) + \beta U_{\mathrm{L}}(\varepsilon) \quad \text{where} \quad U_{\mathrm{L}}(\varepsilon) := \frac{u(c_{\mathrm{L}}(\varepsilon)) + \beta u(c_{\mathrm{H}}(\varepsilon))}{1-\beta^2}.$$

Let also $W_{1,\mathrm{H}}(\varepsilon, d_1)$ denote the default option of the high-income agent at $t = 1$:

$$W_{1,\mathrm{H}}(\varepsilon, d_1) := V^i_{\ell^\star}(0, 0|(z^i, 1)).$$

We preliminary notice that the default option depends indirectly on the debt level d_1 as the later affects the bond prices $q_2(\varepsilon, d_1)$. In the Online Supplement, we derive the following solution for $W_{1,H}$:

$$W_{1,\mathrm{H}}(\varepsilon, d_1) = u(\bar{c}_1(\varepsilon, d_1)) + \beta u(\bar{c}_2(\varepsilon, d_1)) + \beta^2 \left[\frac{u(\bar{c}_{\mathrm{H}}(\varepsilon)) + \beta u(\bar{c}_{\mathrm{L}}(\varepsilon))}{1-\beta^2}\right],$$

where the consumption levels satisfy the flow budget constraint at $t = 1$:

$$\bar{c}_1(\varepsilon, d_1) + q_2(\varepsilon, d_1)\bar{\theta}_2(\varepsilon, d_1) = y_{\mathrm{H}} - \ell^\star \quad \text{with} \quad \bar{\theta}_2(\varepsilon, d_1) \geqslant 0,$$

the flow budget constraint at $t = 2$:

$$\bar{c}_2(\varepsilon, d_1) = y_{\mathrm{L}} - \ell^\star + \bar{\theta}_2(\varepsilon, d_1),$$

respectively, i.e., $q_1(\varepsilon, d_1) \geqslant \beta\pi_{\mathrm{H}}\frac{u'(c_{1,\mathrm{H}}(\varepsilon,d_1))}{u'(y_0)}$ and $q_2(\varepsilon, d_1) \geqslant \beta\frac{u'(c_{\mathrm{H}}(\varepsilon))}{u'(c_{1,\mathrm{L}}(\varepsilon,d_1))}$. The Online Supplement shows that inequality (4.7) is satisfied for the values of the primitives we consider.

the flow budget constraint at any date $t \geqslant 3$ where agents are receiving high-income:

$$\bar{c}_{\mathrm{H}}(\varepsilon) + q(\varepsilon)\bar{\theta}(\varepsilon) = y_{\mathrm{H}} - \ell^\star \quad \text{with} \quad \bar{\theta}(\varepsilon) \geqslant 0,$$

and the flow budget constraint at any date $t \geqslant 3$ where agents are receiving low-income

$$\bar{c}_{\mathrm{L}}(\varepsilon) = y_{\mathrm{L}} - \ell^\star + \bar{\theta}(\varepsilon).$$

There are no savings when income is low. The saving choices $\bar{\theta}_2(\varepsilon, d_1)$ and $\bar{\theta}(\varepsilon)$ when income is high are determined by the first order conditions

$$q_2(\varepsilon, d_1) = \beta \frac{u'(\bar{c}_2(\varepsilon, d_1))}{u'(\bar{c}_1(\varepsilon, d_1))} \quad \text{and} \quad q(\varepsilon) = \beta \frac{u'(\bar{c}_{\mathrm{L}}(\varepsilon))}{u'(\bar{c}_{\mathrm{H}}(\varepsilon))}, \tag{4.8}$$

where the prices $q_2(\varepsilon, d_1)$ and $q(\varepsilon)$ are determined in the transition and steady-state phases. We finally notice that the above (out-of-equilibrium) consumption and saving choices are optimal provided that they satisfy[28]

$$\frac{u'(\bar{c}_{\mathrm{L}}(\varepsilon))}{u'(\bar{c}_{\mathrm{H}}(\varepsilon))} \geqslant \frac{u'(\bar{c}_{\mathrm{H}}(\varepsilon))}{u'(\bar{c}_2(\varepsilon, d_1))} \quad \text{and} \quad \bar{c}_{\mathrm{H}}(\varepsilon) \geqslant \bar{c}_{\mathrm{L}}(\varepsilon). \tag{4.9}$$

Figure 4.2(a) plots the debt level $d_1(\varepsilon)$ obtained as the solution to the following not-too-tight condition

$$U_{1,\mathrm{H}}(\varepsilon, d_1) = W_{1,\mathrm{H}}(\varepsilon, d_1). \tag{4.10}$$

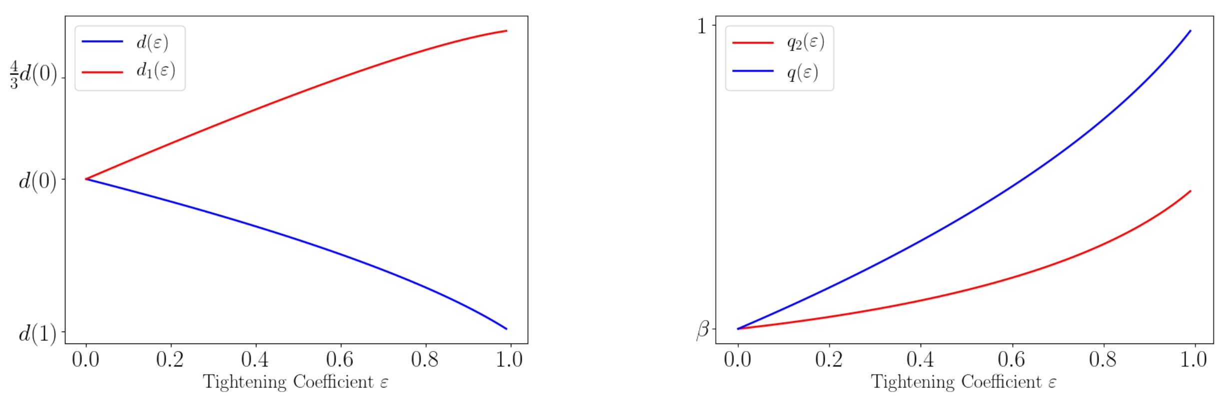
For comparison, we also plot the equilibrium too-tight debt level $d(\varepsilon)$ as defined in (4.5). We see that $\varepsilon \mapsto d_1(\varepsilon)$ is an increasing function while $\varepsilon \mapsto d(\varepsilon)$ is a decreasing function. With the determination of $d_1(\varepsilon)$ well understood, we can simplify the notation for the equilibrium variables along the transition as follows:

$$c_{1,\mathrm{H}}(\varepsilon) := c_{1,\mathrm{H}}(\varepsilon, d_1(\varepsilon)), \quad c_{1,\mathrm{L}}(\varepsilon) := c_{1,\mathrm{L}}(\varepsilon, d_1(\varepsilon)) \quad \text{and} \quad q_t(\varepsilon) := q_t(\varepsilon, d_1(\varepsilon)), \quad \text{for } t \in \{1, 2\}.$$

To understand why the policy intervention might be Pareto improving it is useful to disentangle the effects it has on the not-too-tight debt limit $D_1^i(z^i)$. Before the intervention ($\varepsilon = 0$), the economy is at the laissez-faire equilibrium where $D_1^i(z^i)$ equals to $d(0)$. If we ignore the impact on prices, the deterioration of future credit conditions has a first-order

[28]The inequalities correspond to the first order conditions of the saving decisions at $t = 2$ and $t \geqslant 3$ respectively, i.e., $q(\varepsilon) \geqslant \beta \frac{u'(\bar{c}_{\mathrm{H}}(\varepsilon))}{u'(\bar{c}_2(\varepsilon, d_1))}$ and $q(\varepsilon) \geqslant \beta \frac{u'(\bar{c}_{\mathrm{H}}(\varepsilon))}{u'(\bar{c}_{\mathrm{L}}(\varepsilon))}$. The Online Supplement shows that both inequalities are satisfied for the values of the primitives we consider.

effect: restricting borrowing in the future (i.e., $d(\varepsilon)$ falls below $d(0)$ as ε increases) reduces the value of honoring the debt $d(0)$ at period $t = 1$ while it leaves the default option unaffected. This implies that $D_1^i(z^i)$ has to decrease below $d(0)$ for the not-too-tight condition (3.1) to be satisfied. Taking into account the feedback on prices produces a second-order effect: as ε increases both the current bond price $q_2(\varepsilon)$ and the steady-state price $q(\varepsilon)$ increase, as shown in Figure 4.2(b). Thus the intervention makes the option of default less appealing by reducing the interest rate on saving. The impact on the value for honoring the debt $d(0)$ is, however, ambiguous since along the equilibrium path the agents both save and borrow. Figure 4.3(a) shows that the value $U_{1,\mathrm{H}}(\varepsilon, d(0))$ of repaying the debt level $d(0)$ is strictly above the value $W_{1,\mathrm{H}}(\varepsilon, d(0))$ of the default option, so $D_1^i(z^i)$ has to increase above $d(0)$ for the not-too-tight condition (3.1) to be satisfied. The overall effect of policy intervention on the level of $D_1^i(z^i)$ is reflected in the value of $d_1(\varepsilon)$, the new not-too-tight debt level. As shown in Figure 4.2(a), $d_1(\varepsilon) > d(0)$, so the second-order effect outweighs the first-order effect.



(a) Debt limits

(b) Prices

Figure 4.2: Equilibrium debt limits and prices as functions of the tightening coefficient ε.

The following claim summarizes the construction of the equilibrium with too-tight debt constraints.

Claim 4.3. *The consumption allocations, bond holdings and debt limits of the steady-state and transition phases support a competitive equilibrium with not-too-tight reputation debt at* $t = 1$ *and too-tight reputation debt at every subsequent date* $t \geqslant 2$.

4.3.4 Pareto Improvement

We now numerically show that the equilibrium described in Claim 4.3 Pareto dominates the laissez-faire equilibrium. To identify the overall impact on expected utility, we introduce the following notations. Let $U_{1,\mathrm{H}}(\varepsilon)$ and $U_{1,\mathrm{L}}(\varepsilon)$ be the continuation utilities contingent to high and low income at $t = 1$ when the debt limit is $d_1(\varepsilon)$. That is,

$$U_{1,\mathrm{H}}(\varepsilon) = u(c_{1,\mathrm{H}}(\varepsilon)) + \beta U_{\mathrm{L}}(\varepsilon) \quad \text{and} \quad U_{1,\mathrm{L}}(\varepsilon) = u(c_{1,\mathrm{L}}(\varepsilon)) + \beta U_{\mathrm{H}}(\varepsilon),$$

where

$$U_{\mathrm{H}}(\varepsilon) := \frac{u(c_{\mathrm{H}}(\varepsilon)) + \beta u(c_{\mathrm{L}}(\varepsilon))}{1-\beta^2} \quad \text{and} \quad U_{\mathrm{L}}(\varepsilon) := \frac{u(c_{\mathrm{L}}(\varepsilon)) + \beta u(c_{\mathrm{H}}(\varepsilon))}{1-\beta^2}.$$

are the steady-state continuation utilities. Time-0 utility $U_0(\varepsilon)$ is then given by

$$U_0(\varepsilon) = u(y_0) + \beta[\pi_{\mathrm{H}} U_{1,\mathrm{H}}(\varepsilon) + \pi_{\mathrm{L}} U_{1,\mathrm{L}}(\varepsilon)].$$

Since the equilibrium is symmetric, we have

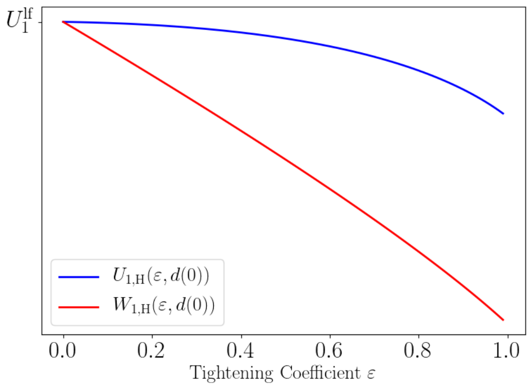
$$U^i(c^{i,\varepsilon}|s^0) = U_0(\varepsilon)$$

for each agent $i \in I$. It is straightforward to verify that if $\varepsilon = 0$, then we recover the laissez-faire equilibrium with not-too-tight debt limits, that is

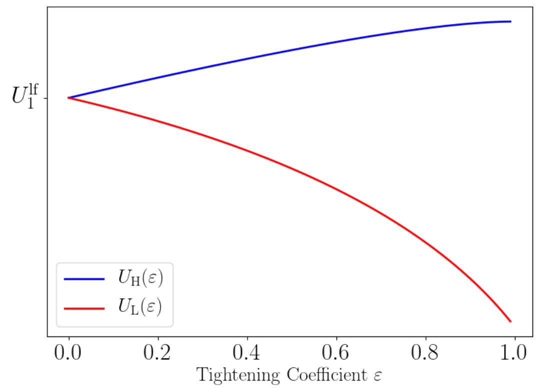
$$(q^0, (c^{i,0}, a^{i,0}, D^{i,0})_{i\in I}) = (q, (c^i, a^i, D^i)_{i\in I})$$

and we deduce that $U_0(0) = U^i(c^i|s^0)$. Therefore, to show that the consumption allocation $(c^{i,\varepsilon})_{i\in I}$ Pareto dominates the consumption allocation $(c^i)_{i\in I}$, it is sufficient to show that $U_0(\varepsilon) > U_0(0)$ for some value of ε.

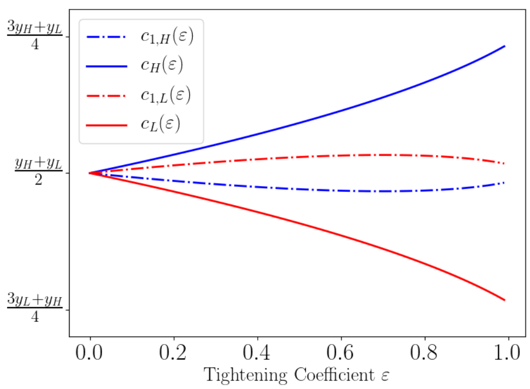
Figure 4.3(b) shows that the steady-state utility of the high income (low income, respectively) agent increases (decreases, respectively) with ε. We can also see from Figure 4.3(c) that the period $t = 1$ consumption of the high income (low income, respectively) agent decreases (increases, respectively). The above impacts the ex-ante (i.e., at $t = 0$) utility in two ways. There is a negative effect due to the decrease of period $t = 1$ continuation utility of the high income agent, and a positive effect due to the increase of period $t = 1$ continuation utility of the low income agent. This is illustrated in Figure 4.3(d). Since both agents assign a higher probability on low income state than on high income state, i.e., $\pi_{\mathrm{L}} > \pi_{\mathrm{H}}$, it is possible that, for some values of ε, the positive effect might offset the negative effect, so that the ex-ante utility increases. Figure 4.3(e) confirms this conjecture: for the values of the primitives we consider the benefit due to the increased borrowing capacity at $t = 0$ can outweigh the cost of reduced borrowing opportunities at each $t \geqslant 1$. In summary, our numerical analysis has shown that for some values of ε the intervention of tightening the debt limits can *increase* the ex-ante utility for both agents.



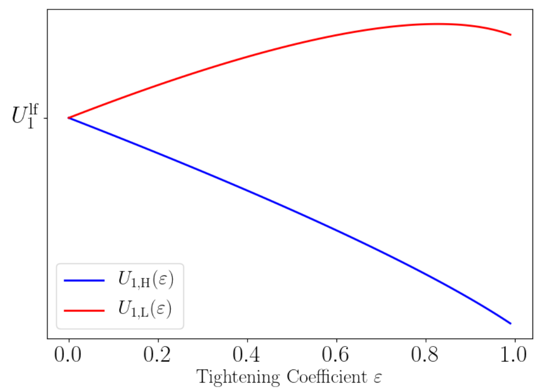
(a) Default and repayment values.



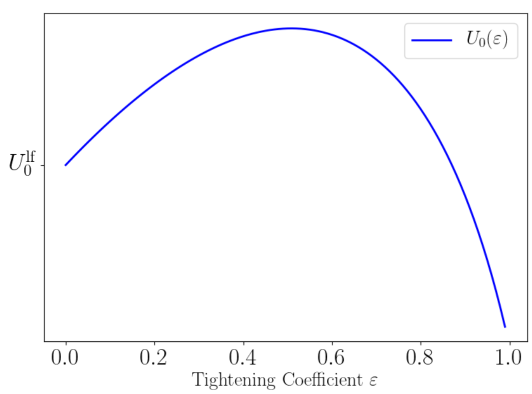
(b) Steady-state utility.



(c) Transitory vs. steady-state consumption.



(d) Utility at $t = 1$.



(e) Expected lifetime utility at $t = 0$.

Figure 4.3: Consumption and utilities as functions of tightening coefficient ε.

4.4 Tightening Debt Limits in the Collateral Debt Environment

We now show that tightening debt limits can also increase welfare in the environment of Section 3.2 where debt is collateralized. We mentioned before that the laissez-faire equilirium described in Claim 4.1 can be supported as an equilibrium with collateralized debt when the dividend of the Lucas tree is constant and equal to $2\ell^\star$ and the initial tree holdings are symmetric, i.e., $\alpha^i(s^{-1}) = 1/2$ for each agent i. Non-pledgeable endowment is then given by $e^i(s^t) := y^i(s^t) - \ell^\star$.

We recall from Section 3.2 that debt limits are self-enforcing at event $s^t = (z,t)$ when

$$\widetilde{V}^i_t(\widetilde{D}^i, -\widetilde{D}^i_t(z)|z) \geqslant \widetilde{V}^i_t(\widetilde{D}^i, 0|z).$$[29]

The above condition is satisfied with equality if, and only if, $\widetilde{D}^i_t(z) = 0$, whereas a strict inequality obtains if, and only if, $\widetilde{D}^i_t(z) < 0$. Equivalently, in the collateral environment, the debt limit is too tight at some contingency if, and only if, it forces mandatory saving in net terms.

Our objective is to construct a collateral equilibrium with too tight debt limits at some events that Pareto dominates the laissez-faire equilibrium. To do this, we fix a sequence $(\eta_t)_{t\geqslant 1}$ of tightening parameters $\eta_t \in [0,1]$ and set

$$\widetilde{D}^i_t(z) = -\eta_t[P_t(z) + \ell^\star], \tag{4.11}$$

where $P_t(z)$ is the price of the tree at event $s^t = (z,t)$. When $\eta_t = 0$, the debt limit $\widetilde{D}^i_t(z) = 0$ is not too tight, whereas when $\eta_t > 0$, the debt limit $\widetilde{D}^i_t(z) < 0$ is too tight. The borrowing (collateral) constraints (3.6) take now the following form: for all $z \in \{z^a, z^b\}$

$$b^i_t(z) + \alpha^i_{t-1}(z)[P_t(z) + \ell^\star] \geqslant \eta_t[P_t(z) + \ell^\star], \tag{4.12}$$

where $0 \leqslant \eta_t \leqslant 1$ is interpreted as a margin requirement imposed by a regulatory agency or the government that requires agents to keep at least (the market value of) a fraction η_t of the physical asset in their balance-sheet.

An important observation is that, even under the possibly too-tight debt limits (4.11), the asset pricing equation (3.10) remains valid so we do have that $P_t(z) + \ell^\star = \mathrm{PV}_t(\ell^\star|z)$. This permits, without any loss of generality, to focus attention to the case where there is no trade in the equity market, i.e., $\alpha^i(s^t) = 1/2$ for each i and all s^t. In particular, as

[29] We replace the notation $\widetilde{V}^i(\widetilde{D}^i, x|s^t)$ by the simpler $\widetilde{V}^i_t(\widetilde{D}^i, x|z)$ when $s^t = (z,t)$.

argued in Section 3.2, we can show that an equilibrium $(q, P, (c^i, \alpha^i, b^i, \widetilde{D}^i)_{i\in I})$ with self-enforcing (possibly too-tight) collateral constraints (4.12) is equivalent to an equilibrium $(q, (c^i, a^i, D^i)_{i\in I})$ where the debt limits are given by $D^i_t(z) := (1-\eta_t)\,\mathrm{PV}_t(\ell^\star|z)$. In the rest of section, we compute such equilibria by considering a nonzero sequence of tightening coefficients, i.e., $(\eta_t)_{t\geqslant 1} \neq 0$.

We perform two policy experiments that give rise to equilibria with different characteristics. We first look the case where the margin requirements (or, equivalently, the too-tight collateral constraints) are imposed from period $t=1$ onward, similar to what we did in the economy with reputation debt. We then look the case where the intervention takes place from period $t=2$ onward. Interestingly enough, the analysis reveals that delaying the tightening of the collateral constraints one period ahead generates higher welfare gains. We show that this is a general property in our example: the later in the future the intervention takes place, the higher are the welfare gains. In the limit, if we delay the intervention for a sufficiently long time, we can get as close as we desire to the first-best regime.

4.4.1 Tightening Collateral Constraints at $t \geqslant 1$

Assume that $\eta_1 = 0$ and $\eta_t = \varepsilon > 0$ for every $t \geqslant 2$. We construct an equilibrium $(q^\varepsilon, (c^{i,\varepsilon}, a^{i,\varepsilon}, D^{i,\varepsilon})_{i\in I})$ where the debt limits satisfy

$$D^{i,\varepsilon}_1(z) = \mathrm{PV}^\varepsilon_1(\ell^\star|z) \quad \text{and} \quad D^{i,\varepsilon}_t(z) = (1-\varepsilon)\,\mathrm{PV}^\varepsilon_t(\ell^\star|z), \quad \text{for all } t \geqslant 2.$$

As argued above, such an equilibrium can be implemented as an equilibrium with not-too-tight collateral constraints at $t=0$ and too-tight collateral constraints at every $t \geqslant 1$.

The characteristics of the equilibrium are as follows: the economy reaches at period $t=3$ a cyclical steady-state $(q(\varepsilon), c_{\mathrm{H}}(\varepsilon), c_{\mathrm{L}}(\varepsilon), d(\varepsilon))$ similar to the one obtained in the model with reputation debt (i.e., Claim 4.2 applies for $t \geqslant 3$). In the transition periods $t \in \{1,2\}$, consumption, asset holdings and debt limits are symmetric, i.e., for any $z \in \{z^a, z^b\}$,

$$c^{i,\varepsilon}_t(z) = \begin{cases} c_{t,\mathrm{H}}(\varepsilon), & \text{if } y^i_t(z) = y_{\mathrm{H}}, \\ c_{t,\mathrm{L}}(\varepsilon), & \text{if } y^i_t(z) = y_{\mathrm{L}}; \end{cases} \quad \text{and} \quad a^{i,\varepsilon}_t(z) = \begin{cases} -d_t(\varepsilon), & \text{if } y^i_t(z) = y_{\mathrm{H}}, \\ d_t(\varepsilon), & \text{if } y^i_t(z) = y_{\mathrm{L}}; \end{cases}$$

together with $q^\varepsilon_{t+1}(z) =: q_{t+1}(\varepsilon)$ and $D^{i,\varepsilon}(z) =: D_t(\varepsilon)$ where:

$$D_1(\varepsilon) = \ell^\star\left[1 + q_2(\varepsilon)\left(1 + q_3(\varepsilon)\frac{1}{1-q(\varepsilon)}\right)\right] \quad \text{and} \quad D_2(\varepsilon) = (1-\varepsilon)\ell^\star\left[1 + q_3(\varepsilon)\frac{1}{1-q(\varepsilon)}\right].$$

At $t = 0$, both agents consume their endowment $c_0^{i,\varepsilon} = y_0$, with asset prices given by:

$$q_1^\varepsilon(z) = \beta\pi_{\text{L}} \frac{u'(c_{1,\text{L}}(\varepsilon))}{u'(y_0)}, \quad \text{for} \quad z \in \{z^a, z^b\}.$$

An important feature of the cyclical steady state described in Claim 4.2 is that the interest rates tend to zero when ε converges to 1 (i.e., $\lim_{\varepsilon\to 1} q(\varepsilon) = 1$). This property has implications for the not-too-tight debt limit $D_1(\varepsilon)$ and the too-tight debt limit $D_2(\varepsilon)$. In particular, Figure 4.4(a) shows that $D_1(\varepsilon)$ explodes to infinite while $D_2(\varepsilon)$ decreases as ε increases. These features have in turn implications for the determination of the equilibrium consumption, asset positions and prices over the transition period. Specifically, they give rise to three threshold values $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < 1$ over which equilibria differ. We delegate the detailed equilibrium derivations to the Online Supplement of this paper and present the main characteristics hereafter. A graphical illustration is given in Figure 4.4.

- For $\varepsilon \in [0, \varepsilon_1]$, both agents borrow up to the debt limit against their high income at periods $t \in \{1, 2\}$, i.e., $d_1(\varepsilon) = D_1(\varepsilon)$ and $d_2(\varepsilon) = D_2(\varepsilon)$. For $\varepsilon \in (0, \varepsilon_1)$, we have $c_{1,\text{L}}(\varepsilon) < c_{2,\text{H}}(\varepsilon)$ and $c_{1,\text{H}}(\varepsilon) > c_{2,\text{L}}(\varepsilon)$. At the threshold value ε_1, the debt limit $D_1(\varepsilon_1)$ is large enough so that agents' consumption levels at $t = 1$ and $t = 2$ are equalized, i.e., $c_{1,\text{H}}(\varepsilon_1) = c_{2,\text{L}}(\varepsilon_1)$ and $c_{1,\text{L}}(\varepsilon_1) = c_{2,\text{H}}(\varepsilon_1)$.

- For $\varepsilon \in (\varepsilon_1, \varepsilon_2]$, both agents borrow up to the debt limit against their high income state at period $t = 1$, i.e., $d_1(\varepsilon) = D_1(\varepsilon)$. But now $D_1(\varepsilon)$ is sufficiently large so that the low-income agent at period $t = 1$ does not need to borrow up to the debt limit i.e., $d_2(\varepsilon) < D_2(\varepsilon)$ to achieve perfect consumption smoothing between $t = 1$ and $t = 2$. We then have $c_{1,\text{H}}(\varepsilon) = c_{2,\text{L}}(\varepsilon)$ and $c_{1,\text{L}}(\varepsilon) = c_{2,\text{H}}(\varepsilon)$, which implies that $q_2(\varepsilon) = \beta$. As ε increases, $D_1(\varepsilon)$ becomes so large that the high-income agent at period $t = 1$ finds it optimal to borrow, i.e., $d_2(\varepsilon)$ becomes negative. The threshold value ε_2 is determined by the binding constraint $d_2(\varepsilon_2) = -D_2(\varepsilon_2)$.

- For $\varepsilon \in (\varepsilon_2, \varepsilon_3]$, both agents borrow up to the debt limit against their high income state at $t = 1$, i.e., $d_1(\varepsilon) = D_1(\varepsilon)$. However, perfect consumption smoothing between periods $t = 1$ and $t = 2$ is not anymore feasible since the debt constraint of the high-income agent at period $t = 1$ is binding, i.e., the agent continues to borrow up to $d_2(\varepsilon) = -D_2(\varepsilon)$. The consumption $c_{1,\text{L}}(\varepsilon)$ of the low-income agent continues to increase with ε while the consumption $c_{1,\text{H}}(\varepsilon)$ of the high-income agent continues to decrease. At the threshold level ε_3, the consumption levels $c_{1,\mathrm{L}}(\varepsilon_3)$ and $c_{1,\mathrm{H}}(\varepsilon_3)$ equal the first best values $\overline{c}^{\mathrm{fb}}$ and $\underline{c}^{\mathrm{fb}}$, so we have $\pi_{\mathrm{L}} u'(c_{1,\mathrm{L}}(\varepsilon_3)) = \pi_{\mathrm{H}} u'(c_{1,\mathrm{H}}(\varepsilon_3))$.

- Finally, for $\varepsilon \in (\varepsilon_3, 1]$, the debt limit level $D_1(\varepsilon)$ is so large that the debt constraint at $t = 0$ is not binding, i.e., $d_1(\varepsilon) < D_1(\varepsilon)$. The first best allocation is implemented at period $t = 1$. The high-income agent continues to borrow up to the debt limit contingent to low income, i.e., $d_2(\varepsilon) = -D_2(\varepsilon)$.

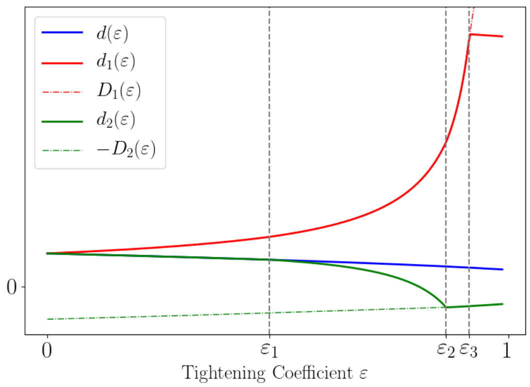
Figure 4.4(a) and Figure 4.4(b) plot the debt levels and equilibrium prices as functions of the tightening parameter ε. As in the model with reputation debt, the tightening of debt constraints at every $t \geqslant 1$ leads to lower interest rates in the cyclical steady-state ($\varepsilon \mapsto q(\varepsilon)$ is increasing). This has a positive feedback effect on the equity price at $t = 1$, which in turn relaxes the collateral constraints at $t = 0$: the price $\mathrm{PV}_1^{\varepsilon}(\ell^\star|z)$ of the asset is increasing with ε and tends to infinite when ε converges to 1.

In terms of utility values, the tightening of debt constraints increases (decreases) the steady-state continuation utility $U_{\mathrm{H}}(\varepsilon)$ ($U_{\mathrm{L}}(\varepsilon)$) of the high income (low income) agent. Figures 4.4(c) and 4.4(d) plot the consumption levels at dates $t = 1$ and $t = 2$. The consumption $(c_{1,\mathrm{L}}(\varepsilon), c_{2,\mathrm{H}}(\varepsilon))$ of the agent having low income at period $t = 1$ increases with ε. As shown in Figure 4.4(e), this increase in consumption more than compensates for the lower steady-state utility value $U_{\mathrm{L}}(\varepsilon)$, so the period-1 continuation utility $U_{1,\mathrm{L}}(\varepsilon)$ increases with ε. Symmetrically, the consumption $(c_{1,\mathrm{H}}(\varepsilon), c_{2,\mathrm{L}}(\varepsilon))$ of the agent having high income at $t = 1$ decreases with ε and this outweighs the increase in the steady-state utility $U_{\mathrm{H}}(\varepsilon)$, so the period-1 continuation utility $U_{1,\mathrm{H}}(\varepsilon)$ decreases with ε. The overall effect on ex-ante utility $U_0(\varepsilon)$ is driven by the trade off of period-1 continuation utility values. Since both agents assign a higher probability to low income state than to high income state, i.e., $\pi_{\mathrm{L}} > \pi_{\mathrm{H}}$ we get Pareto improvement. This is illustrated in Figure 4.4(f).

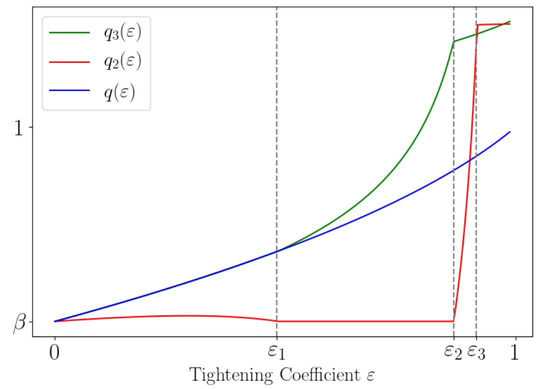
When ε is close enough to 1 (formally, $\varepsilon \geqslant \varepsilon_3$), interest rates are so low and the value of collateral is so large that the debt constraints at $t = 0$ are not anymore binding. Therefore, there is no gain (in terms of period $t = 1$ and period $t = 2$ consumption levels) from restricting trade in the future and the ex-ante expected utility decreases with ε.

4.4.2 Tightening Collateral Constraints at $t \geqslant 2$

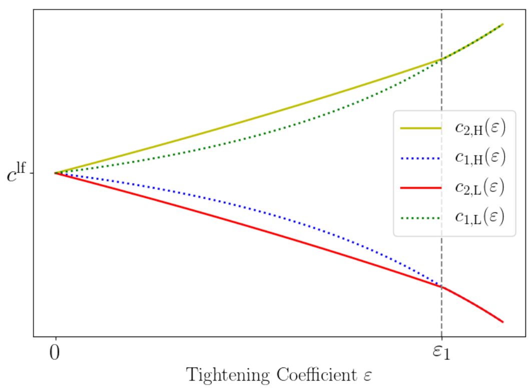
Consider now the case where the collateral constraints are tightened at all dates $t \geqslant 2$, but at $t = 0$ and $t = 1$ constraints are not-too-tight. Formally, we assume that $\eta_1 = \eta_2 = 0$



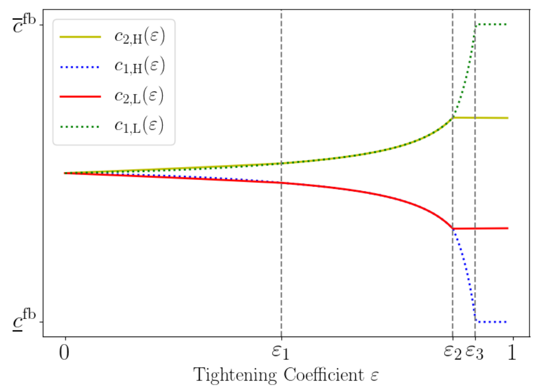
(a) Debt limits



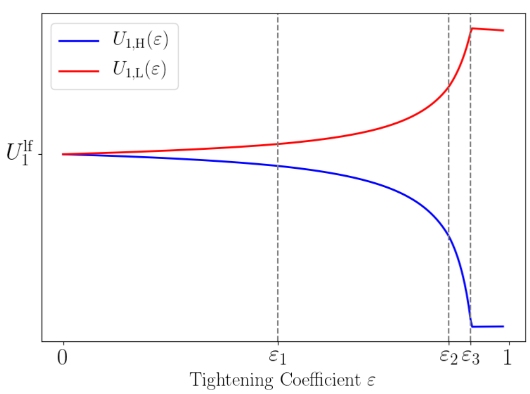
(b) Prices



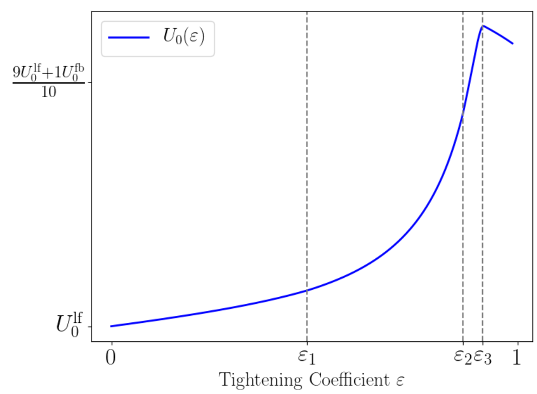
(c) Consumption for low values of ε



(d) Consumption for high values of ε



(e) Time 1 continuation utility



(f) Time 0 expected utility

Figure 4.4: Equilibrium debt limits, prices, consumption and utility as functions of the tightening coefficient ε.

and for every $t \geqslant 3$, $\eta_t = \varepsilon$ for some $\varepsilon > 0$. As in the previous section, depending on the value of ε, we exhibit an equilibrium $(q^{\varepsilon}, (c^{i,\varepsilon}, a^{i,\varepsilon}, D^{i,\varepsilon})_{i\in I})$ that has different characteristics before the economy reaches the cyclical steady-state at $t = 3$. The difference is that we now only have a single threshold value $\varepsilon_1 \in (0, 1)$ where the equilibrium variables differ over the transition phase. We hereafter discuss the main equilibrium characteristics and delegate the detailed derivations to the Online Supplement of this paper.

Agents now can borrow at period $t = 1$ up to

$$D_2(\varepsilon) = \ell^{\star}\left[1 + q_3(\varepsilon)\frac{1}{1 - q(\varepsilon)}\right].$$

Figure 4.5(a) shows that $D_2(\varepsilon)$ is increasing in ε and explodes to infinite as ε converges to 1 (and $q(\varepsilon) \mapsto 1$), exactly as it happens with $D_1(\varepsilon)$. Figures 4.5(b) and 4.5(c) plot the equilibrium bond prices and consumption levels for the transition period as functions of the tightening parameter ε.

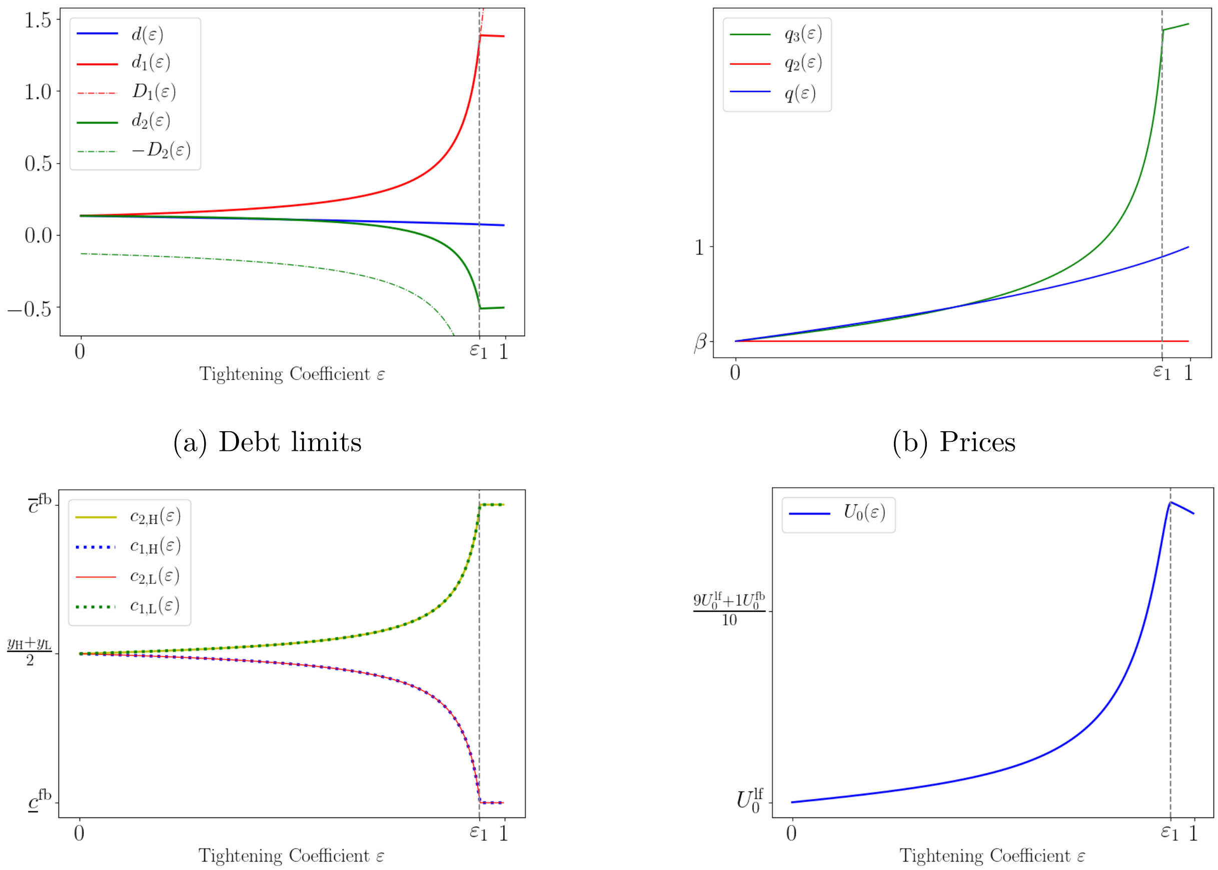
- For $\varepsilon \in [0, \varepsilon_1]$, both agents borrow at $t = 0$ up to the debt limit contingent to period-1 high income state. However, the low income agent does not exhaust all borrowing opportunities at period $t = 1$, that is, the debt constraint is non-binding: $-D_2(\varepsilon) < d_2(\varepsilon) < D_2(\varepsilon)$. In doing so, agents perfectly smooth consumption between $t = 1$ and $t = 2$ before reaching the cyclical steady-state at $t = 3$: $c_{1,\text{L}}(\varepsilon) = c_{2,\text{H}}(\varepsilon)$ and $c_{1,\text{H}}(\varepsilon) = c_{2,\text{L}}(\varepsilon)$.

- For $\varepsilon \in (\varepsilon_1, 1]$, the debt limit $D_1(\varepsilon)$ is so large that both the period-0 and period-1 debt constraints are non-binding. In this case, the first best consumption levels can be implemented at periods $t = 1$ and $t = 2$: $c_{1,\text{L}}(\varepsilon) = c_{2,\text{H}}(\varepsilon) = \overline{c}^{\text{fb}}$ and $c_{1,\text{H}}(\varepsilon) = c_{2,\text{L}}(\varepsilon) = \underline{c}^{\text{fb}}$.

Figure 4.5(d) plots the ex-ante expected utility and shows that the tightening of debt constraints can lead to Pareto improvement. A comparison with Figure (4.4(f)) also reveals that, the tightening of debt constraints one period ahead, generates higher utility gains. This is because the first-best consumption levels are achieved not only at $t = 1$, but also at $t = 2$.

4.4.3 Tightening Collateral Constraints in The Long Run

The analysis in Section 4.4.2 suggests that the later the government decides to intervene in financial markets, by means of tightening the debt constraints, the larger are the utility



(a) Debt limits

(b) Prices

(c) Consumption

(d) Expected lifetime utility at $t = 0$

Figure 4.5: Equilibrium debt limits, prices, consumption and utility as functions of the tightening coefficient ε. Panel (d) shows that tightening debt limits can lead to a Pareto improvement of the laissez-faire equilibrium.

gains. We provide more insight on this issue by showing that late interventions can support equlbria that are as much close as to the first-best outcome. To formalize this property, given a date $T \geqslant 2$ and $\varepsilon > 0$, we consider an equilibrium where the tightening of debt constraints is given by

$$\eta_1 = \eta_2 = \ldots = \eta_T = 0 \quad \text{and} \quad \eta_t = \varepsilon, \quad \forall t \geqslant T+1.$$

When $T \geqslant 2$ is an even date, choosing ε close enough to one, we can verify (arguing as in the previous section) that there exists a competitive equilibrium where, at any $t \leqslant T-1$, the debt constraints are not binding and consumption equals the first-best level, while a steady-state is reached at period $T+1$.[30]

That is, for every $t \leqslant T$, we have

$$c_t^{i,\varepsilon}(z) = \begin{cases} \overline{c}^{\text{fb}}, & \text{if } y_1^i(z) = y_{\text{L}}, \\ \underline{c}_{\text{fb}}, & \text{if } y_1^i(z) = y_{\text{H}}, \end{cases}$$

and $q_t^{\varepsilon}(z) = \beta$. For $t \geqslant T+1$, we have

$$c_t^{i,\varepsilon}(z) = \begin{cases} c_{\text{L}}(\varepsilon), & \text{if } y_t^i(z) = y_{\text{L}}, \\ c_{\text{H}}(\varepsilon), & \text{if } y_t^i(z) = y_{\text{H}}, \end{cases}$$

and $q_{t+1}^{\varepsilon}(z) = q(\varepsilon)$.

The bond price $q_{T+1}^{\varepsilon}(z)$ is determined by the first-order condition of the saving decision of the agent with the high income at date T. Observe that since T is even, the high-income agent at date T had low income at $t=1$ so her current consumption level is $\overline{c}^{\text{fb}}$. This implies that

$$q_{T+1}^{\varepsilon}(z) = \beta \frac{u'(c_L(\varepsilon))}{u'(\overline{c}^{\text{fb}})}.$$

Debt limits at every $t \leqslant T$ satisfy

$$\frac{D_t^{\varepsilon}}{\ell^\star} = 1 + \beta\left[1 + \beta\left[\ldots + \left[1 + q_{T+1}^{\varepsilon}\frac{1}{1-q(\varepsilon)}\right]\right]\right].$$

We notice that

$$\lim_{\varepsilon \to 1} c_{\text{L}}(\varepsilon) = y_{\text{L}} + 2d(1),$$

[30] Since $\eta_{T+1} = \varepsilon$, the debt constraint $a_{T+1}^{i,\varepsilon}(z) \geqslant (1-\varepsilon)\,\text{PV}(\ell^\star|(z, T+1))$ imposed at date T is too-tight.

where $d(1)$ is the unique positive value satisfying

$$1 = \beta \frac{u'(y_{\mathrm{L}} + 2d(1))}{u'(y_{\mathrm{H}} - 2d(1)}.$$

Since $\lim_{\varepsilon \to 1} q(\varepsilon) = 1$, we get that for every $t \leqslant T$,

$$\lim_{\varepsilon \to 1} D_t^{\varepsilon} = \infty.$$

This verifies the claim that, choosing ε close enough to 1, debt constraints do not bind at every $t \leqslant T - 1$. The debt constraint binds at T and the variables defined above form a competitive equilibrium if, and only if,

$$\frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(\overline{c}^{\mathrm{fb}})} \geqslant \frac{u'(c_{\mathrm{H}}(\varepsilon))}{u'(\underline{c}^{\mathrm{fb}})}. \tag{4.13}$$

The above condition is always satisfied since $c_{\mathrm{L}}(\varepsilon) \leqslant c_{\mathrm{H}}(\varepsilon)$ and $\underline{c}^{\mathrm{fb}} \leqslant \overline{c}^{\mathrm{fb}}$. The period-0 expected utility is then given by

$$U(c^{i,\varepsilon}|s^0) = u(y_0) + \pi_{\mathrm{H}}\big[u(\underline{c}^{\mathrm{fb}})\big(\beta + \ldots + \beta^T\big) + \beta^{T+1} U_{\mathrm{H}}(\varepsilon)\big] + \\ \pi_{\mathrm{L}}\big[u(\overline{c}^{\mathrm{fb}})\big(\beta + \ldots + \beta^T\big) + \beta^{T+1} U_{\mathrm{L}}(\varepsilon)\big].$$

In particular, we have

$$\sup_{\varepsilon \in [0,1]} U(c^{i,\varepsilon}|s^0) \geqslant \underline{U}_0(T) := U(c^{i,1}|s^0).$$

Since

$$\lim_{T \to \infty} \underline{U}_0(T) = u(y_0) + \pi_{\mathrm{H}} \frac{u(\underline{c}^{\mathrm{fb}})}{1 - \beta} + \pi_{\mathrm{L}} \frac{u(\overline{c}^{\mathrm{fb}})}{1 - \beta},$$

we can get as close as desired to the first-best utility level by choosing T large enough and ε close enough to 1. This property is illustrated by Figure (4.6) where we plot the function $T \mapsto \underline{U}_0(T)$.

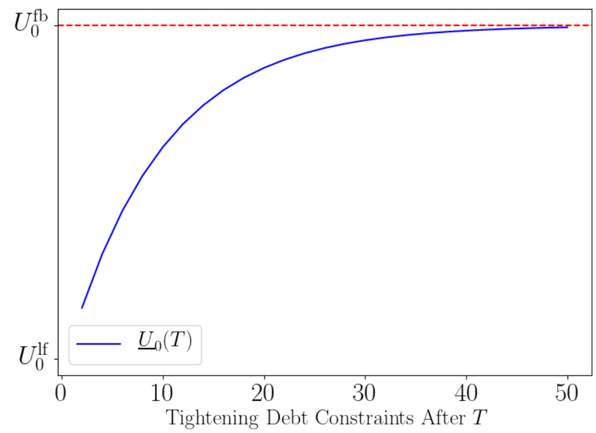


Figure 4.6: Expected lifetime utility at $t = 0$ (for tightening parameter ε close enough to 1) as a function of tightening period T.

5 Pigouvian Subsidies

We have illustrated in the previous section that private agents fail to internalize how their financial decisions affect the debt limits via prices. This gives room for a government intervention by means of macroprudential controls on financial markets in the lines of Jeanne and Korinek (2010, 2019) and Farhi and Werning (2016). We explore below the effects of such a policy experiment by considering corrective Pigouvian subsidies on net deliveries financed by lump-sum taxes.

Formally, each agent i maximizes $U^i(c|s^0)$ among all plans (c, a) satisfying, for every event s^t, the flow budget constraint

$$c(s^t) + T^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1}) \leqslant y^i(s^t) + a(s^t) + \kappa \left[-a(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1}) \right]^+ \quad (5.1)$$

while we keep unchanged the debt constraints

$$a(s^{t+1}) \geqslant -D^i(s^{t+1}), \quad \forall s^{t+1} \succ s^t. \quad (5.2)$$

When maximizing her utility, agent i takes as given not only the price process $(q(s^t))_{s^t \succ s^0}$ and the process $(D^i(s^t))_{s^t \succ s^0}$ of debt limits, but also the subsidy coefficient $\kappa \in [0, 1]$ and the process $(T^i(s^t))_{s^t \succeq s^0}$ of lump-sum taxes. The subsidy applies only when the net financial position $-a(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1})$ is positive. If the agent starts with some debt, i.e., $a(s^t) < 0$, the net financial position is positive when the agent repays at least a part of her debt out of her endowment, or equivalently, when not all of the current debt is rolled over. If, instead, the agent starts with some positive claim, i.e., $a(s^t) \geqslant 0$, the net financial position is positive when the agent saves more than the value of his initial financial claim.

The following equilibrium concept is the analogue of Definition 2.2 in the current environment.

Definition 5.1. Given pledgeable endowment processes $(\ell^i)_{i \in I}$, a competitive equilibrium with limited pledgeability and Pigouvian subsidy rate $\kappa \in [0, 1]$ is a family $(q, (c^i, a^i, D^i, T^i)_{i \in I})$ such that

(a) for each i, the plan (c^i, a^i) maximizes $U^i(c|s^0)$ among all plans (c, a) satisfying the flow budget constraints (5.1) and the debt constraints (5.2);

(b) for each i, there exists a nonnegative exact rollover process M^i such that the debt limits satisfy $D^i = \mathrm{PV}(\ell^i) + M^i$;

(c) subsidies are financed by lump-sum taxes along the equilibrium path:

$$T^i(s^t) = \kappa \left[-a^i(s^t) + \sum_{s^{t+1} \succ s^t} q^\kappa(s^{t+1}) a^i(s^{t+1}) \right]^+ ; \tag{5.3}$$

(d) markets clear.

Two observations are worth remarking. First, we only require that the tax revenue $T^i(s^t)$ offsets the subsidy along the equilibrium path. Second, we notice that the microfoundations for limited pledgeability, discussed in Section 3, remain valid in the current environment. This follows from our assumption that the subsidy only applies on the net financial position. The equivalence between the reputation debt model and the collateral debt model is preserved, and this permits to study in a unified way whether macroprudential controls can be welfare improving.

5.1 A Simple Equilibrium Characterization

To better understand how corrective subsidies can improve welfare, we here present a characterization of equilibria with limited pledgeability and Pigouvian subsidies on net deliveries. Consider a pair of subsidy rate and lump-sum taxes $(\kappa, (T^i)_{i\in I})$ such that $\kappa \in [0,1]$ and

$$T^i(s^t) \leqslant \kappa y^i(s^t) \tag{5.4}$$

for every $i \in I$ and every $s^t \in \Sigma$. Let (c,a) be a post-tax/subsidy plan that satisfies the flow budget constraints (5.1) with equality and the debt constraints (5.2). Denote by $\tilde{c}$ the pre-tax/subsidy consumption process defined by:

$$\tilde{c}(s^t) := y^i(s^t) + a(s^t) - \sum_{s^{t+1}\succ s^t} q(s^{t+1})a(s^{t+1}).$$

Observe that:

$$c(s^t) = F^i(\tilde{c}(s^t), s^t) \quad \text{where} \quad F^i(x, s^t) := x - T^i(s^t) + \kappa[y^i(s^t) - x]^+.$$

Equivalently, F^i can also be written as follows:

$$F^i(x, s^t) = \begin{cases} x - T^i(s^t) & \text{if } x > y^i(s^t), \\ (1-\kappa)x + \kappa y^i(s^t) - T^i(s^t) & \text{elsewhere.} \end{cases}$$

By construction, the function $F^i(\cdot, s^t)$ is well-defined on the whole domain $[0,\infty)$ with non-negative values. Let $\tilde{u}^i(\cdot, s^t)$ be the period utility function defined by:

$$\tilde{u}^i(x, s^t) := u(F^i(x, s^t)).$$

Denote the corresponding continuation utility by:

$$\widetilde{U}^i(\tilde{c}|s^t) := \tilde{u}^i(\tilde{c}(s^t), s^t) + \sum_{\tau\geqslant 1} \beta^\tau \sum_{s^{t+\tau}\succ s^t} \pi(s^{t+\tau}|s^t)\tilde{u}^i(\tilde{c}(s^{t+\tau}), s^{t+\tau}).$$

We can see that a post-tax/subsidy plan (c,a) satisfies the flow budget constraints (5.1) with equality if, and only if, the pre-tax/subsidy plan $(\tilde{c}, a)$ satisfies with equality the standard flow budget constraint

$$\tilde{c}(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1})a(s^{t+1}) = y^i(s^t) + a(s^t). \tag{5.5}$$

This implies that (c^i, a^i) maximizes the utility U^i among all plans (c,a) satisfying the post-tax/subsidy flow budget constraints (5.1) and the debt constraints (5.2) if, and only if, $(\tilde{c}^i, a^i)$

maximizes the utility $\widetilde{U}^i$ among all plans $(\tilde{c}, a)$ satisfying the pre-tax/subsidy flow budget constraints (5.5) and the debt constraints (5.2). Moreover, if

$$T^i(s^t) = \kappa\left[-a^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a^i(s^{t+1})\right]^+$$

then the pre- and post-tax/subsidy consumption plans coincide, $c^i = \tilde{c}^i$.[31] It then follows that

$$\tilde{u}'(\tilde{c}^i(s^t), s^t) = \begin{cases} u'(c^i(s^t)) & \text{if } c^i(s^t) > y^i(s^t), \\ (1-\kappa)u'(c^i(s^t)) & \text{if } c^i(s^t) < y^i(s^t). \end{cases}$$

This allows us to establish the following characterization.

Proposition 5.1. *Fix a collection $(q, (c^i, a^i, D^i)_{i \in I})$ satisfying market clearing such that*

$$c^i(s^t) \neq y^i(s^t), \quad \textit{for all } i \in I \textit{ and all } s^t \succ s^0.$$

Let

$$\chi^i(s^t) := \begin{cases} 1-\kappa & \textit{if } c^i(s^t) < y^i(s^t), \\ 1 & \textit{if } c^i(s^t) > y^i(s^t). \end{cases}$$

The collection $(q, (c^i, a^i, D^i)_{i \in I})$ is a competitive equilibrium with limited pledgeability and Pigouvian subsidy $\kappa \in [0, 1]$ if, and only if, for each agent i:

- *the post-tax/subsidy plan (c^i, a^i) satisfies the pre-tax/subsidy flow budget constraints with equality and the debt constraints;*
- *debt limits take the following form $D^i = \mathrm{PV}(\ell^i) + M^i$ where M^i is a non-negative exact rollover process;*
- *the following Euler equations are satisfied: for every event s^t and each successor event $s^{t+1} \succ s^t$, we have*

$$q(s^{t+1}) = \max_{i \in I} \frac{\chi^i(s^{t+1})}{\chi^i(s^t)} \left[\frac{\beta\pi(s^{t+1}|s^t)u'(c^i(s^{t+1}))}{u'(c^i(s^t))}\right]; \tag{5.6}$$

- *the following standard transversality condition holds*

$$\liminf_{t \to \infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^i(s^t)) = 0;$$

[31] Observe that condition (5.4) is satisfied since $T^i(s^t) = \kappa[y^i(s^t) - \tilde{c}^i(s^t)]^+$.

- *the lump-sum taxes satisfy*

$$T^i(s^t) := \kappa \left[-a^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \right]^+ .$$

We remark that the subsidy rate affects only the term $\chi^i(s^{t+1})/\chi^i(s^t)$ of the Euler Equation (5.6). However, this term plays a crucial role in determining the remaining equilibrium variables as we illustrate in subsequent sections. The proof of Proposition 5.1 is straightforward once we observe that an equilibrium with Pigouvian subsidies is nothing more than a standard equilibrium with limited pledgeability but with a different period utility function (the function $u(\cdot)$ is replaced by $\tilde{u}^i(\cdot, s^t)$). This also reveals why the microfoundations for limited pleadgeability, discussed in Section 3, remain valid.

5.2 Example

We consider again the example of Section 4.1. For any possible value of the subsidy rate $\kappa \in [0, 1]$, we look for an equilibrium with limited pleadgeability and subsidies on net deliveries having the following characteristics: at period $t = 0$, both agents borrow against their high-income state and save contingent to their low-income state. After the resolution of the uncertainty at period $t = 1$, the economy settles in a cyclical steady-state where the low-income agent borrows up to the not-too-tight debt limit and the high-income agent saves. To describe the equilibrium variables, we denote by $q(\kappa)$ the solution of the following equation:

$$q(\kappa) := \frac{\beta}{1-\kappa} \times \frac{u'(c_{\mathrm{L}}(\kappa))}{u'(c_{\mathrm{H}}(\kappa))} \tag{5.7}$$

where the consumption levels are defined by

$$c_{\mathrm{H}}(\kappa) := y_{\mathrm{H}} - (1 + q(\kappa))d(\kappa) \quad \text{and} \quad c_{\mathrm{L}}(\kappa) := y_{\mathrm{L}} + (1 + q(\kappa))d(\kappa) \tag{5.8}$$

and the level of debt is given by

$$d(\kappa) := \frac{\ell^\star}{1 - q(\kappa)} \tag{5.9}$$

where ℓ^* be specified as in (4.3). We claim that the above quantities support a competitive equilibrium provided that the subsidy rate is such that

$$\frac{u'(c_{\mathrm{L}}(\kappa))}{u'(c_{\mathrm{H}}(\kappa))} \geqslant 1 - \kappa.^{32} \tag{5.10}$$

[32] The inequality corresponds to the sufficient optimality condition for $t \geqslant 1$.

Claim 5.1. *Let $q(\kappa)$, $c_{\text{H}}(\kappa)$, $c_{\text{L}}(\kappa)$ and $d(\kappa)$ be specified as in (5.7), (5.8), (5.9) and assume that the rate κ satisfies condition (5.10). There exists an equilibrium $(q^\kappa, (c^{i,\kappa}, a^{i,\kappa}, D^{i,\kappa}, T^i)_{i\in I})$ with limited pledgeability and subsidity rate κ where for each $z \in \{z^a, z^b\}$:*

(i) Debt limits are $D_t^{i,\kappa}(z) = d(\kappa)$;

(ii) The consumption allocation is $c_0^{i,\kappa} = y_0$, $c_t^{i,\kappa}(z) = c_{\text{H}}(\kappa)$ if $y_t^i(z) = y_{\text{H}}$ and $c_t^{i,\kappa}(z) = c_{\text{L}}(\varepsilon)$ if $y_t^i(z) = y_{\text{L}}$, for $t \geqslant 1$;

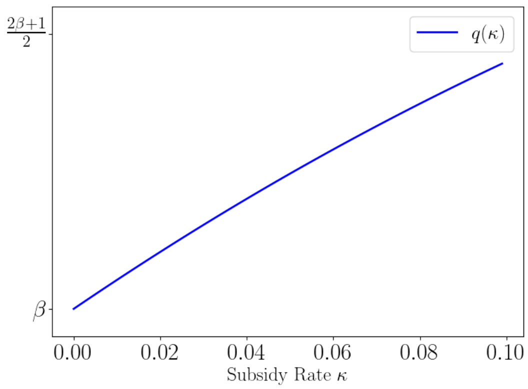
(iii) Net asset positions are $a_t^{i,\kappa}(z) = -d(\kappa)$ (i.e., the debt limit binds) if $y_t^i(z) = y_{\text{H}}$ and $a_t^{i,\kappa}(z) = d(\kappa)$ if $y_t^i(z) = y_{\text{L}}$, for $t \geqslant 1$;

(iv) Prices are given by:

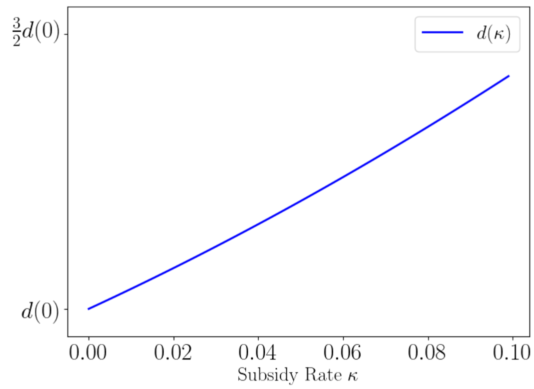
$$q_1^\kappa(z) = \pi_{\text{L}} \frac{\beta}{1-\kappa} \times \frac{u'(c_{\text{L}}(\kappa))}{u'(y_0)} \quad \text{and} \quad q_{t+1}^\kappa(z) = q(\kappa);$$

(v) Lump sum taxes are $T_t^{i,\kappa}(z) = 0$ if $y_t^i(z) = y_{\text{H}}$ and $T_t^{i,\kappa}(z) = \kappa d(\kappa)$ if $y_t^i(z) = y_{\text{L}}$ for $t \geqslant 1$.

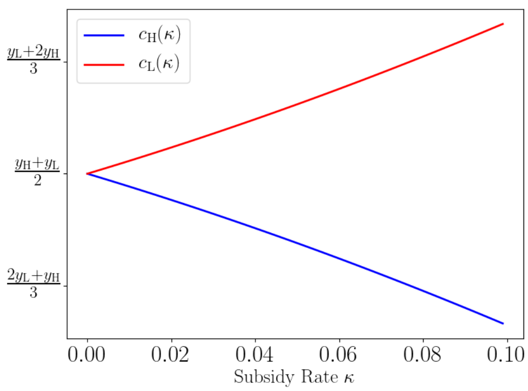
We delegate to the Online Supplement the straightforward proof of the claim. We here show numerically that there are values of κ such that the equilibrium described in Claim 5.1 Pareto dominates the laissez-faire equilibrium described in Claim 4.1. To this purpose Figures 5.1(a), 5.1(b) and 5.1(c) plot the steady state bond prices, debt levels and consumption allocations as a function of the subsidy rate κ. We also show in Figure 5.1(d) that the sufficient condition (5.10) is satisfied for the values of primitives we consider.



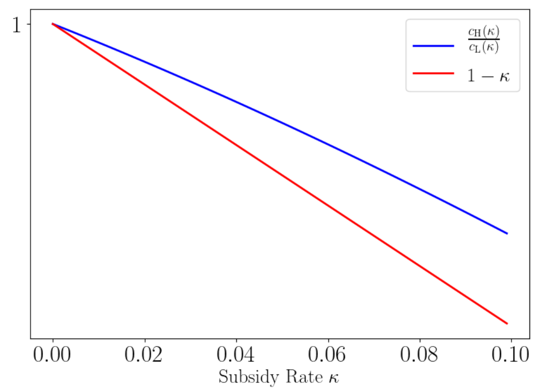
(a) Steady-state bond price



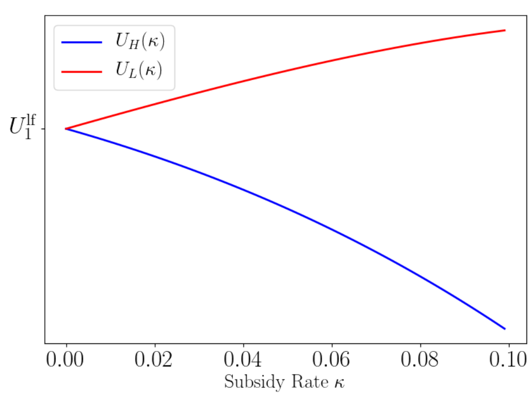
(b) Steady-state debt



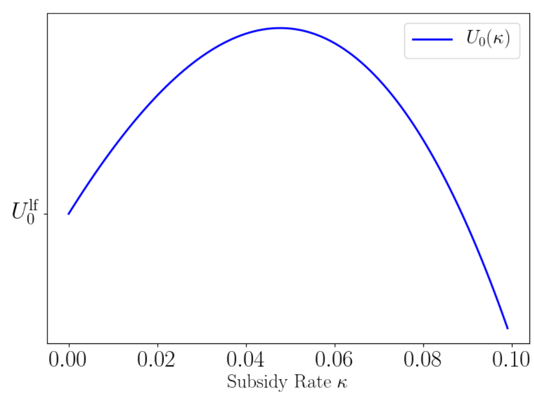
(c) Steady-state consumption



(d) FOC for borrowing



(e) Time 1 utility



(f) Time 0 utility

Figure 5.1: Equilibrium variables and utilities

Given our specifications of the model, an equilibrium with subsidies can be seen as a standard equilibrium (without subsidies) where agents' time preference coefficient β is replaced by the higher $\beta(\kappa) := \beta/(1-\kappa)$ when agents' current income is high (and, consequently, future income is low). In other words, agents are more patient when their current income is high than when their current income is low. That is, the distortion created by the subsidies leads to a wedge in marginal rates of substitution between the high income and the low income agents. When compared to the laissez-faire equilibrium, this wedge allows for higher prices (Figure 5.1(a)), looser debt limits (Figure 5.1(b)), higher consumption when income is low and lower consumption when income is high (Figure 5.1(c)).

Let $U_H(\kappa)$ and $U_L(\kappa)$ be the continuation utilities when the agents' income is high and low respectively. Observe that

$$U_{\mathrm{H}}(\kappa) = \frac{u(c_{\mathrm{H}}(\kappa)) + \beta u(c_{\mathrm{L}}(\kappa))}{1-\beta^2} \quad \text{and} \quad U_{\mathrm{L}}(\kappa) = \frac{u(c_{\mathrm{L}}(\kappa)) + \beta u(c_{\mathrm{H}}(\kappa))}{1-\beta^2}.$$

Let also

$$U_0(\kappa) = u(y_0) + \beta[\pi_{\mathrm{H}} U_{\mathrm{H}}(\kappa) + \pi_{\mathrm{L}} U_{\mathrm{L}}(\kappa)].$$

Since the equilibrium is symmetric, for each agent i, the period-0 utility satisfies

$$U^i(c^{i,\kappa}|s^0) = U_0(\kappa).$$

It is straightforward to verify that, for $\kappa = 0$, we recover the laisser-faire equilibrium in Claim 4.1, that is

$$(q^0, (c^{i,0}, a^{i,0}, D^{i,0})_{i\in I}) = (q, (c^i, a^i, D^i)_{i\in I})$$

and we deduce that $U_0(0) = U^i(c^i|s^0) = U_0^{\mathrm{lf}}$.[33]

To show that the consumption allocation $(c^{i,\kappa})_{i\in I}$ Pareto dominates the consumption allocation $(c^i)_{i\in I}$, it is sufficient to show that $U_0(\kappa) > U_0(0)$ for some values of κ. Figure 5.1(c) shows that consumption contingent to low (high) income at $t = 1$ increases (decreases) with κ. Figure 5.1(e) then shows that the continuation utility $U_{\mathrm{L}}(\kappa)$ ($U_{\mathrm{L}}(\kappa)$) contingent to low (high) income increases (decreases) with κ. Since agents believe that it is more likely that income is low at period $t = 1$ ($\pi_{\mathrm{L}} > \pi_{\mathrm{H}}$), in expectation, the increase of $U_{\mathrm{L}}(\kappa)$ more than compensates the loss of $U_{\mathrm{H}}(\kappa)$ as shown in Figure 5.1(f). This proves our claim.

[33]Recall from Claim 4.1 that

$$U_0^{\mathrm{lf}} := u(y_0) + \beta\frac{u((y_{\mathrm{H}} + y_{\mathrm{L}})/2)}{1-\beta}.$$

6 Conclusion

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A Appendix: Omitted Proofs

A.1 Proof of Proposition 2.1

Let $(q, (c^i, a^i, D^i)_{i \in I})$ be an equilibrium with limited pledgeability. Since pledgeable income is nonnegligible, we must have

$$\sum_{i \in I} \mathrm{PV}(y^i | s^0) \leqslant \frac{1}{\varepsilon} \sum_{i \in I} \mathrm{PV}(\ell^i | s^0).$$

By the decomposition property (2.4), we have that $\mathrm{PV}(\ell^i | s^0) < \infty$ for each agent i, so we deduce that the aggregate wealth of the economy $\sum_{i \in I} \mathrm{PV}(y^i | s^0)$ must be finite. Since consumption markets clear, we obtain that the present value of optimal consumption is finite for all agents. In addition, due to the Inada's condition, the optimal consumption is strictly positive.[34] Lemma A.1 in Martins-da-Rocha and Vailakis (2017) then implies that

[34] See the Supplemental Material of Martins-da-Rocha and Santos (2019) for a detailed proof.

the following market transversality condition holds true:[35]

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0. \tag{A.1}$$

The decomposition property (2.4) implies that, for each i, there exists a nonnegative discounted martingale process M^i such that $D^i = \mathrm{PV}(\ell^i) + M^i$. Condition A.1 can then be rewritten as follows:

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)a^i(s^t) = -p(s^0)M^i(s^0).$$

Since bond markets clear, we deduce that $\sum_{i\in I} M^i(s^0) = 0$, proving the desired result: $M^i = 0$ for each i.

A.2 Proof of Theorem 3.1

The proof of Theorem 3.1 exploits two intermediate results. The first and crucial observation, that has no analogue in the absence of output contraction, is to show that the present value of foregone endowment imposes a lower bound on not-too-tight debt limits. A direct implication of this property is that the process $\mathrm{PV}(\ell^i)$ is finite. This is summarized in the following lemma.

Lemma A.1. *Not-too-tight debt limits are at least as large as the present value of endowment losses, i.e., for each agent i, $D^i(s^t) \geqslant \mathrm{PV}(\ell^i|s^t)$ at any event s^t.*

A natural approach to prove this result is to show that $D^i(s^t) \geqslant \ell^i(s^t) + \widetilde{D}^i(s^t)$, where $\widetilde{D}^i(s^t) := \sum_{s^{t+1}\succ s^t} q(s^{t+1})D^i(s^{t+1})$ is the present value of next period's debt limits, and then use a standard iteration argument. Because, in equilibrium, debt limits are not too tight, this is equivalent to proving that agent i does not have an incentive to default when her net asset position is $\ell^i(s^t) + \widetilde{D}^i(s^t)$, i.e.,

$$V^i(D^i, -\ell^i(s^t) - \widetilde{D}^i(s^t)|s^t) \geqslant V^i_{\ell^i}(0, 0|s^t). \tag{A.2}$$

By definition, the value function $V^i_{\ell^i}$ satisfies:

$$V^i_{\ell^i}(0, 0|s^t) \geqslant u(y^i(s^t) - \ell^i(s^t)) + \beta \sum_{s^{t+1}\succ s^t} \pi(s^{t+1}|s^t)V^i_{\ell^i}(0, 0|s^{t+1}). \tag{A.3}$$

[35]The market transversality condition differs from the individual transversality condition. Indeed, due to the lack of commitment, agent i's debt limits may bind, in which case we do not necessarily have that $p(s^t) = \beta^t\pi(s^t)u'(c^i(s^t))/u'(c^i(s^0))$.

If we had an equality in (A.3), then inequality (A.2) would be straightforward. Indeed, consuming $y^i(s^t)-\ell^i(s^t)$ and borrowing up to each debt limit $D^i(s^{t+1})$ at event s^t leads to the right-hand side continuation utility in (A.3) and satisfies the solvency constraint at event s^t in the budget set defining the left-hand side of (A.2). Unfortunately, in our environment where agents can save upon default condition (A.3) may not hold as an equality.[36] Overcoming this problem is the technical challenge in the proof of Lemma A.1. The formal argument is presented below.

The second observation is that the process $\mathrm{PV}(\ell^i)$ of present values of endowment losses, when it is finite, is itself not too tight. The following lemma provides the formal statement. The proof follows from a simple translation invariance of the flow budget constraints.

Lemma A.2. *If* $\mathrm{PV}(\ell^i|s^0)$ *is finite, then the process* $\mathrm{PV}(\ell^i)$ *is not too tight, i.e.,*

$$V^i(\mathrm{PV}(\ell^i), -\mathrm{PV}(\ell^i|s^t)|s^t) = V^i_{\ell^i}(0,0|s^t), \quad \forall s^t \succeq s^0.$$

Equipped with Lemma A.1 and Lemma A.2, we can now provide a simple proof of Theorem 3.1.

Proof of Theorem 3.1. Fix a process D^i of not-too-tight debt limits. Lemma A.1 implies that $\mathrm{PV}(\ell^i|s^0)$ is finite. From Lemma A.2 we also deduce that the process $\underline{D}^i := \mathrm{PV}(\ell^i)$ is not too tight. Martins-da-Rocha and Santos (2019) show that the difference between two processes of not-too-tight debt limits must be an exact rollover process. Therefore, there exists a process M^i satisfying the exact rollover property such that $D^i = \underline{D}^i + M^i$. By Lemma A.1, $D^i \geqslant \underline{D}^i$, in which case the process M^i must be nonnegative. $\square$

A.2.1 Proof of Lemma A.1

Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i. Let D be a process of not-too-tight limits. We first show that there exists a nonnegative process $\underline{D}$ satisfying

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0. \tag{A.4}$$

[36] In the simpler environment where, upon default, saving is not possible (as it is the case in Alvarez and Jermann 2000) condition (A.3) always hold as an equality.

Indeed, let Φ be the mapping $B \in \mathbb{R}^\Sigma \longmapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leqslant B \leqslant \bar{D}$ where

$$\bar{D}(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \succeq s^0.$$

The mapping Φ is continuous (for the product topology), and we have $\Phi[0, \bar{D}] \subseteq [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that Φ admits a fixed point $\underline{D}$ in $[0, \bar{D}]$.

Claim A.1. *The process $\underline{D}$ is tighter than the process D, i.e., $\underline{D} \leqslant D$.*

Proof of Claim A.1. Fix a node s^t. Since $V_\ell(0, 0|s^t) = V(D, -D(s^t)|s^t)$ and $V(D, \cdot|s^t)$ is strictly increasing, it is sufficient to show that $V(D, -\underline{D}(s^t)|s^t) \geqslant V_\ell(0, 0|s^t)$. Denote by $(c, \tilde{a})$ the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t.[37] We let $\widehat{D}$ be the process defined by $\widehat{D}(s^t) := \min\{D(s^t), \underline{D}(s^t)\}$ for all s^t. Observe that

$$\begin{aligned} y(s^t) - \underline{D}(s^t) &= y(s^t) - \ell(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \widehat{D}(s^{t+1}) \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) [\tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})] \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) \end{aligned}$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})$. Since $\widehat{D} \leqslant D$, we have $a(s^{t+1}) \geqslant -D(s^{t+1})$. At any successor event $s^{t+1} \succ s^t$, we have

$$\begin{aligned} y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\ &\geqslant y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\ &\geqslant y(s^{t+1}) - \ell(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) \widehat{D}(s^{t+2}) \\ &\geqslant c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) [\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\ &\geqslant c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) a(s^{t+2}) \end{aligned}$$

[37] That is, the process $\tilde{a}$ supports consumption c such that $U(c|s^t) := V_\ell(0, 0|s^t)$.

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.[38] Observe that $a(s^{t+2}) \geqslant -D(s^{t+2})$ as $\widehat{D} \leqslant D$).

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that (c, a) belongs to the budget set $B(D, -\underline{D}(s^t)|s^t)$. It follows that

$$V(D, -\underline{D}(s^t)|s^t) \geqslant U(c|s^t) = V_\ell(0, 0|s^t)$$

implying the desired result: $\underline{D}(s^t) \leqslant D(s^t)$. □

It follows from Claim A.1 that $\underline{D}$ satisfies

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \tag{A.5}$$

Applying equation (A.5) recursively, we get

$$\begin{aligned} p(s^t)\underline{D}(s^t) = p(s^t)\ell(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} p(s^{t+1})\ell(s^{t+1}) + \dots \\ \dots + \sum_{s^T \in S^T(s^t)} p(s^T)\ell(s^T) + \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})\underline{D}(s^{T+1}) \end{aligned}$$

for any $T > t$. Since $\underline{D}$ is nonnegative, it follows that

$$p(s^t)\underline{D}(s^t) \geqslant \sum_{\tau=t}^{T} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)\ell(s^\tau).$$

Passing to the limit when T goes to infinity, we get that $\mathrm{PV}(\ell|s^t)$ is finite for any event s^t (in particular for s^0). Recalling that $D \geqslant \underline{D}$, we also get that $D(s^t) \geqslant \mathrm{PV}(\ell|s^t)$.

A.2.2 Proof of Lemma A.2

Denote by $(c, \tilde{a})$ the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t. We pose $\underline{D} := \mathrm{PV}(\ell)$ and observe that

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}).$$

It is easy to show that (c, a) is optimal in the budget set $B(\underline{D}, -\underline{D}(s^0)|s^t)$ where $a := \tilde{a} - \underline{D}$. We then deduce that $V^i(\underline{D}, -\underline{D}(s^t)|s^t) = V_\ell(0, 0|s^t)$, so proving the claim.

[38] To get the second weak inequality, we use equation (A.4).

A.3 Derivation of the Asset Price Equation (3.10)

Fix an event $s^t \succeq s^0$. Market clearing implies that there exists at least one agent $i \in I$ who is holding a positive amount $\alpha^i(s^t) > 0$ of the tree shares. Fix $\varepsilon \in \mathbb{R}$ such that $\varepsilon \geqslant -\alpha^i(s^t)$. The following changes in contingent claims and equity's holding are admissible

$$\tilde{\alpha}^i(s^t) := \alpha^i(s^t) + \varepsilon \quad \text{and} \quad \tilde{b}^i(s^{t+1}) := b^i(s^{t+1}) - \varepsilon[P(s^{t+1}) + \delta(s^{t+1})].$$

Since the agent's welfare cannot improve after these changes, we must have

$$P(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})[P(s^{t+1}) + \delta(s^{t+1})]. \tag{A.6}$$

Given this recursive equation, it follows that $\mathrm{PV}(\delta|s^0)$ is finite. Moreover, for every event s^t, the following limit

$$M(s^t) = \lim_{\tau \to \infty} \frac{1}{p(s^t)} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) P(s^\tau)$$

is well-defined, so we obtain equation (3.10).

Remark A.1. The proof that the equity's price satisfies the asset-pricing recursive equation (A.6) relies on a standard no-arbitrage argument. This differs from the proof that endowment losses have finite present value (Lemma A.1) where the recursive equation is obtained by means of fixed-point of a suitably defined operator on debt limits.

A.4 Proof of Theorem 3.2

Consider first the collateralized debt model where δ denotes the process of dividends of the long-lived tree. Let $(q, P, (c^i, \alpha^i, b^i, 0)_{i \in I})$ be a laissez-faire equilibrium.[39] Denote by M the bubble component of the tree's price and choose an arbitrary decomposition $M = \sum_{i \in I} M^i$ where each M^i is a nonnegative exact roll-over process. Then, from the decomposition of debt limits in Theorem 3.1, it is straightforward to see that the collection $(q, (c^i, a^i, D^i)_{i \in I})$ is a laissez-faire equilibrium in the reputation debt model, where endowment losses are given by:

$$\ell^i(s^t) := \alpha^i(s^{-1})\delta(s^t), \quad \text{for every } s^t,$$

debt limits are given by:

$$D^i := \mathrm{PV}(\ell^i) + M^i \tag{A.7}$$

[39] Recall that the debt limits $\widetilde{D}^i$ are not too tight if, and only if, they are equal to zero.

and contingent claims are given by

$$a^i(s^t) := b^i(s^t) + \alpha^i(\sigma(s^t))[P(s^t) + \delta(s^t)] - D^i(s^t). \tag{A.8}$$

Reciprocally, let $(q, (c^i, a^i, D^i)_{i\in I})$ be a laissez-faire equilibrium in the reputation debt model where endowment losses $(\ell^i)_{i\in I}$ satisfy $\ell^i(s^t) = \alpha^i(s^{-1})\delta(s^t)$ for each event s^t. Recall that $D^i = \mathrm{PV}(\ell^i) + M^i$ where M^i is agent i's credit bubble. Fix any family $(\alpha^i)_{i\in I}$ of equity shares satisfying market clearing.[40] Then $(q, P, (c^i, \alpha^i, b^i, 0)_{i\in I})$ constitutes an equilibrium with collateralized debt where δ is the dividend process, equity is given by

$$P := \mathrm{PV}(\delta) - \delta + M, \quad \text{where} \quad M = \sum_{i\in I} M^i, \tag{A.9}$$

and bond holdings are defined by

$$b^i(s^t) := a^i(s^t) + D^i(s^t) - \alpha^i(\sigma(s^t))[P(s^t) + \delta(s^t)]. \tag{A.10}$$

A.5 Proof of Theorem 4.1

Let $(q, (c^i, a^i, D^i)_{i\in I})$ be an equilibrium with not-too-tight debt constraints. Since endowment losses are nonnegligible fraction of aggregate resources, initial endowment have finite present value (see Proposition 2.1). We can then apply Bloise and Reichlin (2011) to deduce that it is not possible to Pareto dominate $(c^i)_{i\in I}$ by another feasible consumption allocation $(\tilde{c}^i)_{i\in I}$ that satisfies the participation constraints

$$U^i(\tilde{c}^i|s^t) \geqslant U^i(y^i - \ell^i|s^t)$$

for every agent i and every event $s^t \succeq s^0$. This is sufficient to get the desired result. Indeed, assume by way of contradiction, that there exists another equilibrium $(\tilde{q}, (\tilde{c}^i, \tilde{a}^i, \widetilde{D}^i)_{i\in I})$ with self-enforcing debt constraints such that the consumption allocation $(\tilde{c}^i)_{i\in I}$ Pareto dominates $(c^i)_{i\in I}$. Since $(\tilde{c}^i, \tilde{a}^i)$ is optimal in the budget set $B^i(\widetilde{D}^i, a^i(s^0)|s^0)$, it follows from the Principle of Optimality that $(\tilde{c}^i, \tilde{a}^i)$ is optimal in $B^i(\widetilde{D}^i, \tilde{a}^i(s^t)|s^t)$ for any event s^t. Since $\tilde{a}^i(s^t) \geqslant -\widetilde{D}^i(s^t)$, we deduce that

$$U^i(\tilde{c}^i|s^t) = V^i(\widetilde{D}^i, \tilde{a}^i(s^t)|s^t) \geqslant V^i(\widetilde{D}^i, -\widetilde{D}^i(s^t)|s^t).$$

Since the debt limits $\widetilde{D}^i$ are self-enforcing, we deduce that $U^i(\tilde{c}^i|s^t) \geqslant U^i(y^i - \ell^i|s^t)$: a contradiction.

[40] In the sense that $\sum_{j\in I} \alpha^i(s^t) = 1$ for all $i \in I$ and all $s^t \succeq s^0$.

B Appendix: Omitted Derivations

B.1 Detailed Derivations for Section 4.4.1

For $\varepsilon \in [0, \varepsilon_1]$, both agents borrow up to the debt limit contingent to the high income at periods $t \in \{1, 2\}$. In this case, the cyclical steady-state is reached at period $t = 2$. Formally, we have $c_{2,\mathrm{L}}(\varepsilon) = c_{\mathrm{L}}(\varepsilon)$, $c_{2,\mathrm{H}}(\varepsilon) = c_{\mathrm{H}}(\varepsilon)$, $d_2(\varepsilon) = D_2(\varepsilon) = d(\varepsilon)$, $q_3(\varepsilon) = q(\varepsilon)$ with the remaining equilibrium variables be determined by

- the period-1 first-order condition:
$$q_2(\varepsilon) = \beta \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{1,\mathrm{H}}(\varepsilon))},$$
- the binding debt limit:
$$d_1(\varepsilon) = D_1(\varepsilon),$$
- and the binding budget constraints:
$$c_{1,\mathrm{H}}(\varepsilon) = y_{\mathrm{H}} - D_1(\varepsilon) - q_2(\varepsilon)d(\varepsilon) \quad \text{and} \quad c_{1,\mathrm{L}}(\varepsilon) = y_{\mathrm{L}} + D_1(\varepsilon) + q_2(\varepsilon)d(\varepsilon).$$

The variables defined above form an equilibrium if, and only if, the following first-order conditions for the borrowing decisions at $t = 0$ and $t = 2$ are satisfied

$$\frac{u'(c_{1,\mathrm{L}}(\varepsilon))}{u'(c_{1,\mathrm{H}}(\varepsilon))} \geqslant \frac{\pi_{\mathrm{H}}}{\pi_{\mathrm{L}}} \quad \text{and} \quad \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{1,\mathrm{H}}(\varepsilon))} \geqslant \frac{u'(c_{\mathrm{H}}(\varepsilon))}{u'(c_{1,\mathrm{L}}(\varepsilon))}. \tag{B.1}$$

The threshold value ε_1 is determined as the value of ε that equates consumption at periods $t = 1$ and $t = 2$, i.e., it corresponds to the solution of

$$c_{1,\mathrm{H}}(\varepsilon) = c_{2,\mathrm{L}}(\varepsilon) \quad \text{or} \quad c_{1,\mathrm{L}}(\varepsilon) = c_{2,\mathrm{H}}(\varepsilon).$$

For $\varepsilon \in (\varepsilon_1, \varepsilon_2]$, it is not anymore optimal to borrow up to the debt limit contingent to high income at period $t = 2$, i.e., $d_2(\varepsilon) < D_2(\varepsilon)$. This is because the debt at $t = 1$ is so large that we get perfect consumption smoothing between date $t = 1$ and $t = 2$,

$$c_{1,\mathrm{H}}(\varepsilon) = c_{2,\mathrm{L}}(\varepsilon) \quad \text{and} \quad c_{1,\mathrm{L}}(\varepsilon) = c_{2,\mathrm{H}}(\varepsilon).$$

This implies that $q_2(\varepsilon) = \beta$ with the remaining equilibrium variables be determined by:

- the binding debt limit:

$$d_1(\varepsilon) = D_1(\varepsilon),$$

- the period-2 first-order condition :

$$q_3(\varepsilon) = \beta \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{2,\mathrm{H}}(\varepsilon))},$$

- the two equations associated to perfect smoothing:

$$\underbrace{y_{\mathrm{H}} - D_1(\varepsilon) - \beta d_2(\varepsilon)}_{c_{1,\mathrm{H}}} = \underbrace{y_{\mathrm{L}} + d_2(\varepsilon) + q_3(\varepsilon) d(\varepsilon)}_{c_{2,\mathrm{L}}}$$

  and

$$\underbrace{y_{\mathrm{L}} + D_1(\varepsilon) + \beta d_2(\varepsilon)}_{c_{1,\mathrm{L}}} = \underbrace{y_{\mathrm{H}} - d_2(\varepsilon) - q_3(\varepsilon) d(\varepsilon)}_{c_{2,\mathrm{H}}}$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decisions at $t = 0$, $t = 2$ and the debt constraint at $t = 1$ are satisfied:

$$\frac{u'(c_{1,\mathrm{L}}(\varepsilon))}{u'(c_{1,\mathrm{H}}(\varepsilon))} \geqslant \frac{\pi_{\mathrm{H}}}{\pi_{\mathrm{L}}}, \quad \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{2,\mathrm{H}}(\varepsilon))} \geqslant \frac{u'(c_{\mathrm{H}}(\varepsilon))}{u'(c_{2,\mathrm{L}}(\varepsilon))} \quad \text{and} \quad d_2(\varepsilon) \geqslant -D_2(\varepsilon). \tag{B.2}$$

When ε get close to ε_2, the debt $d_1(\varepsilon)$ contingent to high income at $t = 1$ is so large that the high-income agent borrows against her low income at $t = 2$, i.e., $d_2(\varepsilon) < 0$. The threshold value ε_2 is determined by the binding constraint $d_2(\varepsilon) = -D_2(\varepsilon)$.

For $\varepsilon \in (\varepsilon_2, \varepsilon_3]$, agents borrow up to the debt limit contingent to high income at $t = 1$, i.e., $d_1(\varepsilon) = D_1(\varepsilon)$ but they do not anymore perfectly smooth consumption between dates $t = 1$ and $t = 2$. This is because the debt constraint binds at $t = 2$: agents borrow up to the debt limit contingent to low income, i.e., $d_2(\varepsilon) = -D_2(\varepsilon)$. The remaining equilibrium variables are determined by:

- the first-order conditions associated to the saving decisions at $t = 1$ and $t = 2$:

$$q_2(\varepsilon) = \beta \frac{u'(c_{2,\mathrm{H}}(\varepsilon))}{u'(c_{1,\mathrm{L}}(\varepsilon))} \quad \text{and} \quad q_3(\varepsilon) = \beta \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{2,\mathrm{H}}(\varepsilon))}$$

- the period-1 binding flow budget constraints:

$$c_{1,\mathrm{H}}(\varepsilon) = y_{\mathrm{H}} - D_1(\varepsilon) + q_2(\varepsilon) D_2(\varepsilon) \quad \text{and} \quad c_{1,\mathrm{L}}(\varepsilon) = y_{\mathrm{L}} + D_1(\varepsilon) - q_2(\varepsilon) D_2(\varepsilon)$$

- and the period-2 binding flow budget constraints:

$$c_{2,\text{H}}(\varepsilon) = y_{\text{H}} + D_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon) \quad \text{and} \quad c_{2,\text{L}}(\varepsilon) = y_{\text{L}} - D_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon).$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 0$, $t = 1$ and $t = 2$ are satisfied:

$$\frac{u'(c_{1,\text{L}}(\varepsilon))}{u'(c_{1,\text{H}}(\varepsilon))} \geqslant \frac{\pi_{\text{H}}}{\pi_{\text{L}}}, \quad \frac{u'(c_{2,\text{H}}(\varepsilon))}{u'(c_{1,\text{L}}(\varepsilon))} \geqslant \frac{u'(c_{2,\text{L}}(\varepsilon))}{u'(c_{1,\text{H}}(\varepsilon))} \quad \text{and} \quad \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))} \geqslant \frac{u'(c_{\text{H}}(\varepsilon))}{u'(c_{2,\text{L}}(\varepsilon))}. \tag{B.3}$$

The threshold level ε_3 is attained when the first-order condition for borrowing at $t = 0$ binds, i.e., $\pi_{\text{L}} u'(c_{1,\text{L}}(\varepsilon)) = \pi_{\text{H}} u'(c_{1,\text{H}}(\varepsilon))$.

Finally, for $\varepsilon \in (\varepsilon_3, 1]$, the debt limit level $D_1(\varepsilon)$ is so large that we can implement the first best consumption at $t = 1$:

$$c_{1,\text{H}}(\varepsilon) = \underline{c}^{\text{fb}} \quad \text{and} \quad c_{1,\text{L}}(\varepsilon) = \overline{c}^{\text{fb}}.$$

The debt level $d_1(\varepsilon)$ (which turns out to be strictly lower than the debt limit $D_1(\varepsilon)$) and the remaining equilibrium variables are determined by:

- the period-1 binding flow budget constraints:

$$\underline{c}^{\text{fb}} = y_{\text{H}} - d_1(\varepsilon) + q_2(\varepsilon)D_2(\varepsilon) \quad \text{and} \quad \overline{c}^{\text{fb}} = y_{\text{L}} + d_1(\varepsilon) - q_2(\varepsilon)D_2(\varepsilon)$$

- the period-2 binding flow budget constraints:

$$c_{2,\text{H}}(\varepsilon) = y_{\text{H}} + D_2(\varepsilon) - q_3(\varepsilon)d(\varepsilon) \quad \text{and} \quad c_{2,\text{L}}(\varepsilon) = y_{\text{L}} - D_2(\varepsilon) + q_3(\varepsilon)d(\varepsilon)$$

- and the first-order conditions associated to the saving decisions at $t = 2$ and $t = 3$:

$$q_2(\varepsilon) = \beta \frac{u'(c_{2,\text{H}}(\varepsilon))}{u'(c_{1,\text{L}}(\varepsilon))} \quad \text{and} \quad q_3(\varepsilon) = \beta \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))}.$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 1$ and $t = 2$ are satisfied:

$$\frac{u'(c_{2,\text{H}}(\varepsilon))}{u'(c_{1,\text{L}}(\varepsilon))} \geqslant \frac{u'(c_{2,\text{L}}(\varepsilon))}{u'(c_{1,\text{H}}(\varepsilon))} \quad \text{and} \quad \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))} \geqslant \frac{u'(c_{\text{H}}(\varepsilon))}{u'(c_{2,\text{L}}(\varepsilon))}. \tag{B.4}$$

B.2 Detailed Derivations for Section 4.4.2

For $\varepsilon \in [0, \varepsilon_1]$, both agents borrow at $t = 0$ up to the debt limit contingent to period-1 high income state, but they do not exhaust all borrowing opportunities at period $t = 1$ when income is low, that is, the debt constraint is non-binding. In doing so they perfectly smooth consumption between $t = 1$ and $t = 2$ before reaching the cyclical steady-state at $t = 3$. Formally, we have $d_1(\varepsilon) = D_1(\varepsilon)$, $d_2(\varepsilon) \in (-D_2(\varepsilon), D_2(\varepsilon))$ and $d_t(\varepsilon) = d(\varepsilon)$ for every $t \geqslant 3$. Since debt constraints at $t = 1$ do not bind, we have

$$c_{1,\mathrm{H}}(\varepsilon) = c_{2,\mathrm{L}}(\varepsilon), \quad c_{1,\mathrm{L}}(\varepsilon) = c_{2,\mathrm{H}}(\varepsilon) \quad \text{and} \quad q_2(\varepsilon) = \beta.$$

The remaining equilibrium variables are determined by the two equations associated to perfect smoothing

$$\underbrace{y_{\mathrm{H}} - D_1(\varepsilon) - \beta d_2(\varepsilon)}_{c_{1,\mathrm{H}}} = \underbrace{y_{\mathrm{L}} + d_2(\varepsilon) + q_3(\varepsilon) d(\varepsilon)}_{c_{2,\mathrm{L}}}$$

and

$$\underbrace{y_{\mathrm{L}} + D_1(\varepsilon) + \beta d_2(\varepsilon)}_{c_{1,\mathrm{L}}} = \underbrace{y_{\mathrm{H}} - d_2(\varepsilon) - q_3(\varepsilon) d(\varepsilon)}_{c_{2,\mathrm{H}}}$$

together with the first-order condition associated to the saving decision at $t = 2$

$$q_3(\varepsilon) = \beta \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{2,\mathrm{H}})}.$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 0$ and $t = 2$ are satisfied:

$$\frac{u'(c_{1,\mathrm{L}}(\varepsilon))}{u'(c_{1,\mathrm{H}}(\varepsilon))} \geqslant \frac{\pi_{\mathrm{H}}}{\pi_{\mathrm{L}}}, \quad \frac{u'(c_{\mathrm{L}}(\varepsilon))}{u'(c_{2,\mathrm{H}}(\varepsilon))} \geqslant \frac{u'(c_{\mathrm{H}}(\varepsilon))}{u'(c_{2,\mathrm{L}}(\varepsilon))} \quad \text{and} \quad d_2(\varepsilon) \geqslant -D_2(\varepsilon). \tag{B.5}$$

The threshold level ε_1 is attained when the first-order condition for borrowing at $t = 0$ binds, i.e., $\pi_{\mathrm{L}} u'(c_{1,\mathrm{L}}(\varepsilon)) = \pi_{\mathrm{H}} u'(c_{1,\mathrm{H}}(\varepsilon))$.

For $\varepsilon \in (\varepsilon_1, 1]$, the debt limit $D_1(\varepsilon)$ is so large so both agents borrow less than the debt limit contingent to period-1 high income state, i.e., both the period-0 and period-1 debt constraints are non-binding. Not only we support perfect consumption smoothing, but also we implement the first best allocation at $t = 1$ and $t = 2$:

$$c_{1,\mathrm{H}}(\varepsilon) = c_{2,\mathrm{L}}(\varepsilon) = \underline{c}^{\mathrm{fb}} \quad \text{and} \quad c_{1,\mathrm{L}}(\varepsilon) = c_{2,\mathrm{H}}(\varepsilon) = \overline{c}^{\mathrm{fb}}.$$

This implies that $q_2(\varepsilon) = \beta$. The debt levels $d_1(\varepsilon)$ and $d_2(\varepsilon)$ together with the price $q_3(\varepsilon)$ are determined by the two equations associated to perfect smoothing

$$\underbrace{y_{\text{H}} - d_1(\varepsilon) - \beta d_2(\varepsilon)}_{c_{1,\text{H}}} = \underbrace{y_{\text{L}} + d_2(\varepsilon) + q_3(\varepsilon) d(\varepsilon)}_{c_{2,\text{L}}}$$

and

$$\underbrace{y_{\text{L}} + d_1(\varepsilon) + \beta d_2(\varepsilon)}_{c_{1,\text{L}}} = \underbrace{y_{\text{H}} - d_2(\varepsilon) - q_3(\varepsilon) d(\varepsilon)}_{c_{2,\text{H}}}$$

and the first-order condition associated to the saving decisions at $t = 2$

$$q_3(\varepsilon) = \beta \frac{u'(c_{\text{L}}(\varepsilon))}{u'(c_{2,\text{H}}(\varepsilon))} = \beta \frac{u'(c_{\text{L}}(\varepsilon))}{u'(\overline{c}^{\text{fb}})}.$$

The variables defined above form an equilibrium if, and only if, the first-order conditions for the borrowing decision at $t = 2$ is satisfied:

$$\frac{u'(c_{\text{L}}(\varepsilon))}{u'(\overline{c}^{\text{fb}})} \geqslant \frac{u'(c_{\text{H}}(\varepsilon))}{u'(\underline{c}^{\text{fb}})} \quad \text{and} \quad d_2(\varepsilon) \geqslant -D_2(\varepsilon). \tag{B.6}$$