Costly Multi-Unit Search[*]

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Abstract

We formulate and solve a costly multi-unit search problem for the optimal selling of a stock of goods. Our showcase application is an inventory liquidation problem with fixed holding costs, such as warehousing, salaries or floor planning. A seller faces a stream of buyers periodically arriving with random capped demands. At each decision point, he decides *how to price* each unit and also whether to *stop searching or not*. We set this as a dynamic programming problem and solve it inductively by characterizing optimal search rules and reservation prices.

We show that combining multiple units with a fixed per period search cost might translate into non-monotone selling costs and reservation prices. This lack of monotonicity naturally leads to *discontinuities* of the pricing strategy. In particular, the seller optimally employs strategies such as *bundling*, and more sophisticated ones that endogenously combine *purchase premiums*, when inventory is large, with *clearance sales* and *discounts*, when inventory is low.

Our model extends search theory by explicitly accounting for the effects of fixed costs on optimal multi-unit pricing strategies, pushing it into a richer class of problems and offering solutions that extend beyond optimal stopping rules.

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1 Introduction

Strategies for the pricing and selling of goods are of central importance to firms and, more generally, to economists. In presence of transaction frictions, search theory provides rigorous decision theoretic tools to analyze these problems. Yet, in an attempt to offer clear insights, most search models usually simplify environments by assuming that a competitive and organized market exists, that agents trade single units or that they face no search cost. In this paper, we relax all of these assumptions by formulating a costly search model for the optimal selling of a large stock of goods. In doing so, we set up and solve a dynamic programming exercise that combines the extensive margin of search with the intensive margin of selling that commonly arise in most liquidation problems.

Our motivating example is an inventory liquidation problem with fixed holding costs (e.g., warehousing, salaries and floor planning), but real world cases that combine search frictions, costly search and multi-unit liquidation without an accesible centralized market abound.[1] For instance, in finance, the inter-dealers over-the-counter markets are better modeled as fully or semi-decentralized (Weill, 2020). In this case, when dealers search for counterparts they typically face the opportunity cost of giving up investments in other markets (Rocheteau and Weill, 2011). These opportunity costs may very well be interpreted as a fixed per-period participation cost, especially when investment decisions are sequential. Do these fixed selling costs matter? In the canonical McCall (1970) single unit search model, if search is costly, then sales surely accelerate as the seller reduces his reservation price. But, how specifically would a seller act differently if he were to hold a large stock of units it is not obvious. How would he try to accelerate sales? What specific optimal pricing strategies would he employ? These are precisely the questions this paper aims to answer. A large class of other applications relate to trade models. For example, most of the monetary theory literature that allows for multi-unit trades only accounts for agent intensive margin decisions (Lagos and Wright, 2005; Molico, 2006; Lagos et al., 2017) without taking into account dynamic participation decisions. Other examples include real assets where traders must incur costs to search for trading partners (Gavazza, 2011). For instance, car dealerships that sell off inventory while facing high floor planning costs, real estate brokers holding multiple properties and paying a fixed mortgage, store liquidation with sales agents paid with a fixed daily wage, ticket sales or even consultants that hire out their time piecemeal and face a flow fixed cost due to

[1] Every year, and for many diverse reasons (e.g., store closing, bankruptcy, etc.), retailers liquidate billions of dollars of inventory. In 2013, Barnes and Noble planed to close a third of its retail stores (20 stores a year) over the next decade (*Wall Street Journal, January 28*). Furthermore, in 1992-2011, more than 15% of public retailers entered bankruptcy, and 3.4% were liquidated (Gaur et al., 2014).

renting their space work.

We consider a seller liquidating multiple indivisible units of a good that lack an organized competitive market; instead, trade is *fully decentralized.* This means that if the seller decides to search, he then randomly meets buyers who sequentially arrive with *capped demands* wishing to buy only up to a *cap* or maximum number of units. Then, in each trading opportunity, either the buyer's cap limits the trade size, or the seller declines the trade, or the seller partially exercises the buyer's offer. But searching for counterparts is costly for the seller. In particular, we assume that the seller faces a *fixed per-period search cost* that can only be escaped once inventory is liquidated, or obviously once he decides to stop searching. Without loss of generality, our analysis assumes that once he stops, any remaining inventory is sold at a zero salvage price. We inductively solve for the seller optimal *search rule* by simultaneously specifying a *search strategy* (when to start the search and whether to stop or not) and a *pricing strategy*, which is related to *reservation prices* (how to price each unit). Unlike McCall (1970), with multiple units the search rule adjust as liquidation proceeds reflecting the endogenous time-varying option value of inventory. By solving this dynamic programming exercise we aim to understand how this liquidation sale optimally proceeds, extending search theory to a richer class of problems with solutions beyond optimal stopping rules.

As in any search theoretical problem, how the seller balances each unit *optionality* (option value) is critical. But crucially, as we combine multiple units with an escapable fixed search cost, this optionality is twofold. Not only does each additional unit allow the seller to sell more in each trade opportunity, providing a valuable *selling optionality*; it also allows him to split each period fixed search cost among more units, lowering average selling costs and improving the *search optionality.* That is, more units of inventory allows the seller to sell more and also, on average, to search more cheaply.

We find that, even though each additional unit only helps, the selling optionality falls in the size of the inventory (due to capped demands) and the search optionality rises simply because the seller might liquidate units facing a lower per unit search cost. These optionalities give rise to new trade offs that in turn might translate into a non-monotone endogenous *marginal selling cost.* We show that this lack of monotonicity of the marginal selling cost naturally leads to *discontinuities* of the optimal pricing strategy, as the conventional marginal analysis fails to solve the seller liquidation problem. In particular, we show that the seller optimally employs a new set of selling strategies, such as *bundling* (when discontinuities are the largest), and more sophisticated ones that endogenously combine *purchase premiums* at the beginning of the liquidation, with *clearance sales* and *volume discounts* at the end. These kinds of strategies are commonly used by firms to try to speed up sales during liquidations (Craig and Raman, 2016; Avittathur and Biswas, 2017), and our model is able to make sense of them. Indeed, the use of discounts to reduce retail inventory increased from 8% in 1970 to close to 30% at the turn of the century (Fisher and Raman, 2010).

Summary of Results: The shape of the demand distribution plays a critical role in determining the seller's optimal search rule; in particular, whether buyers demand single or multiple units, and whether capped demands are random or not. We find that when buyers arrive willing to buy multiple units, the seller stops searching only if the inventory falls below a *shut-down threshold* (Lemma 1). Instead, when sales are carried through in single units, if the seller ever starts searching then he only stops once inventory has been fully sold off. No seller abandons his inventory if sales exclusively happen in single units. Furthermore, as in this case an additional unit of inventory obviously translates into an extra search period, whose cost is never split among units, optionality gains are exclusively driven by decreasingly better future expected sales. This *diminishing selling optionality* in turn has one key economic implication: it translates into an increasing marginal selling cost and so into a pricing strategy that sets higher reservation prices to sell additional units. That is, the seller employs a pricing strategy that continuously sells more units only for higher prices.

With multiple unit caps, we still obtain diminishing returns to optionality and increasing reservation prices but only when the fixed cost is low enough (Proposition 1). Furthermore, in this case reservation prices continuously rise through the liquidation process as each unit optionality increases. In the more general case, when capped demands and a more significant per-period search cost coexist, our model solution dictates the use of more sophisticated discontinuous strategies, such as optimal bundling or strategies that combine clearance sales, bundling, and volume discounts.

In the special case with *non-random demands*, the pricing strategy discontinuity is the largest and optimal liquidation translates into *optimal bundle pricing*. More specifically, the seller designs only two bundles, but unlike most bundling menus, sets a higher per-unit reservation price for the larger one (Proposition 2). In fact, in the extreme case of uncapped demands, when the seller never faces selling restrictions, the value rises at an increasing rate purely due to the increasing search optionality (i.e., each additional unit allows him split the per-period search cost). To wit, selling costs are decreasing and so sales are all-or-nothing with reservation price equal to the average selling cost. Otherwise, with *random demands*, the optimal strategy combines selling additional units at a premium — when inventory is large — with sales in the form of bundling and volume discounts, when inventory is low (Proposition 3). That is, reservation prices are lower at initial stages of a liquidation process, and they progressively rise as time transpires. At some point, however, which we precisely identify in our paper, the seller tries to speed up sales by lowering reservation prices and offering discounts (within each trade period) to buyers with larger demands.

Literature Review: Our theory relates to the literature on optimal trading behavior with frictions, when trade is fully or semi-decentralized (i.e., intermittent, and sometimes costly, access to a centralized market). This infrequent access to trade opportunities is a key characteristic in over-the-counter markets. Abundant literature, pioneered by Duffie et al. (2005, 2007), use search theory to model trading behavior in the presence of frictions. However, since they usually impose tight restrictions on asset holdings — that investors can hold either one or zero unit of the asset — they do not account for how trading behavior might depend on the asset holdings size. Neither do they address agents dynamic participation decisions, since they usually ignore search costs. Unlike them, we account for both multi-units and costly search.

Costly access to trade opportunities in over-the-counter markets has been usually modeled as a one-time entry decision problem, exclusively faced by buyers. Afonso (2011) and Rocheteau and Weill (2011) consider buyers entry decision in a single and indivisible unit trade protocol. Unlike them, we consider a fixed per-period search cost for the seller that in turn translates into a dynamic participation decision. Atkeson et al. (2015) accounts for entry-exit of banks in a specific derivatives market application.

Despite the ubiquity of multi-unit search and multi-unit trade, the search literature in over-the-counter markets has been largely concerned with single-unit search or trades.[2] Among the exceptions, Lagos and Rocheteau (2009) assume unrestricted asset holdings in an equilibrium search model for semi-centralized over-the-counter markets. Unlike them, our capped demands translate into trading restrictions, a key difference between infrequent access to a centralized market (semi-decentralized trade), and frictions without a formal organized market (fully decentralized trade). More recent work that account for unrestricted holdings, but fully decentralized search markets include Afonso and Lagos (2015) and Üslü (2019). In contrast to this work, our model accounts for selling restrictions (caps) and for a per-period search cost to access trade opportunities, which in turn yields a dynamic asset optionality. These extra two ingredients are at

[2]Even the small branch of the search theoretical literature that investigates the optimal stopping rule in multi-product search restricts trades to happen exclusively in single units (Burdett and Malueg, 1981; Carlson and McAfee, 1984; Zhou, 2014); (Gatti, 1999) assumes that consumers search for prices to maximize a general indirect utility function. They all derive, in different environments, a "reservation sum" property which is the multi-product equivalent of a reservation price rule. Unlike them, our optimal search rule does not have a direct equivalence to the single unit search case.

the core of our analysis. We venture that accounting for costly search in these models might also yield a non-monotonicity of the marginal selling cost, and thus might push agents optimal strategies towards non-linear pricing (e.g., bundling, volume discounts and clearance sales.)

The seminal work by Stigler (1962) offers an optimal sample size application that accounts for costly and simultaneous search. In a static environment, a consumer samples prices by choosing a *search intensity* (i.e., how many searches to make). Upon observing prices, he buys at the lowest sampled price. Instead, ours is a dynamic and a purely sequential search problem where selling costs are obtained endogenously. This means, due to our multiple unit ingredient, that the seller has to optimally specify a genuinely *dynamic search intensity* (i.e., our pricing strategy) that evolves as time transpires. Morgan (1983) and Morgan and Manning (1985) endogenize Stigler's sample size in a dynamic environment, thereby combining the intensive and extensive margins of search. Analogous to them, we also combine these two margins, but in a multiunit and purely sequential search environment. That is, each period our seller is confronted with a Stigler multiple-unit optimization, albeit with a constraint that arise due to random caps. In addition, he faces a fixed per-period search cost. In this way, we like to think our problem as the cross product of Stigler (1962) and McCall (1970) wage search model.

In a recent work, Carrasco and Smith (2017) extend search theory to multiple units. In their sequential model, due to rising endogenous *holding costs* (i.e., the opportunity cost of delaying optionality of inframarginal units), the seller *searches at the margin* and sets higher reservation prices to sell additional units. In fact, the seller's trading behavior is fully summarized by the opportunity cost of selling the marginal unit. Our model builds on Carrasco and Smith (2017), but unlike them we assume that the seller also faces a fixed search cost per unit of time, as arises when one has hired out the liquidation task to another agent. We find that such a small change is economically relevant, and that it might significantly change the seller's optimal liquidation strategy if the search cost is large enough. In particular, we show that a sufficiently large fixed cost translates into a non-monotone endogenous marginal selling cost that precludes us to just using first order conditions when solving the seller problem. Crucially then, as reservation prices are guided by total revenue considerations, a fixed search cost leads to a discontinuous pricing strategy giving rise to bundling, volume discounts and sales. These predictions for sales behavior specifically emerge because of the fixed cost assumption, and they radically differ from Carrasco and Smith (2017). In fact, their main message is one of penalizing quantity (i.e., sell more only for higher prices), but in our model we have the opposite message as volume discounts emerge. In addition, as we combine an extensive and an intensive margin throughout the selling process, the inductive logic becomes far more challenging. For we have to simultaneously keep track of the endogenous marginal and average selling costs, as the former it is no longer a sufficient statistic to predict the seller's actions.

We are also related to the literature on inventory management in markets with search frictions. Although there are many theoretical contributions on this topic, most of them focus on how intermediaries with larger and heterogeneous inventory holdings facilitate trade (Johri and Leach, 2002; Shevchenko, 2004; Smith, 2004). As a result, the single-unit trade restriction is commonly used. In contrast, we focus on the optimal liquidation and pricing strategies that explicitly account for multiple-unit trade and fixed search costs. These kind of liquidations naturally arise when inventory is endogenous and its size evolve in time. Recently, Li et al. (2019) solve for the equilibrium pricing and optimal inventory management decision problem for intermediaries in a directed search environment. Buyers and sellers face a fixed search cost, but they hold and trade single units. Intermediaries hold multiple units but face convex holding costs, which naturally yields diminishing returns to inventory holdings for intermediaries. The characterization of the inventory distribution if intermediaries were to face an escapable flow fixed cost without the single unit trade restriction is still an open question. We believe that our paper might shed light on what kind of optimal inventory-based pricing strategies might arise in this case. In particular, that bundling and volume discounts should arise as optimal policies.

2 The Model

Time is discrete and runs forever with discount factor $\beta \in [0, 1)$. In any given period, the seller holds an inventory of indivisible units of size $n \in \mathbb{N}$ that wishes to sell off. The inventory lacks a formal organized market, but the seller can *search* for counterparties and sequentially sell part of it to buyers that he randomly meets. Crucially, since inventory is costly to hold (e.g., warehousing, salaries and floor planning), search is also costly. Specifically, we assume a *fixed per-period search cost* $c > 0$. It can be avoided only once the seller stops search or if the whole inventory is liquidated.

Each period, the seller meets a new buyer with a *capped demand* (ℓ, p) with probability $\alpha_\ell \geqslant 0$; this means, a buyer who wishes at most ℓ units at a *price* p each. In order to ease the notation, prices and *caps* are assumed to be independent random variables. While caps obey $\ell \in \{1, 2, \ldots, m\}$ and $\alpha_m > 0$, we assume prices have CDF $G(p)$ with densities $g(p) > 0$ on $(0, \infty)$.

Conditional on the buyer's offer (ℓ, p), the seller selects how many units of inventory to sell from a *constrained set* $i \in \{0, 1, \ldots, \min\{\ell, n\}\}$, earns cash flow $p \cdot i$ and keeps a post trade inventory of $n - i$ units. He then decides whether to stop the search or to continue in order to sell the remaining units of inventory. Without loss of generality, we assume that once he stops searching any remaining inventory is sold at a price normalized to zero. Since the seller faces the same constrained set in meetings with buyers with larger demands $\ell \geqslant n$, the *relevant meeting chances* are summarized by the vector α^n, where $\alpha^n_\ell = \alpha_\ell$ for $\ell \leqslant n-1$ and $\alpha^n_n \equiv 1 - \sum_{j=1}^{n-1} \alpha_j$.

3 When to Stop Searching?

To characterize the seller optimal search rule, we first solve for the optimal search strategy (or stopping rule). More specifically, we characterize the cases when it is better to stop searching and abandon any unsold inventory. As we will show, a search strategy like this only arises due to our multiple unit sales assumption. For our analysis, the inventory size $n \geqslant 1$ is the only relevant *state variable*.

The seller maximizes the present value of cash inflows, which yields the *option value of holding inventory* V_n. More formally, holding values obey the *Bellman* recursion:

$$V_n \equiv \max\left\{-c + \beta \textstyle\sum_{\ell=1}^{n} \alpha^n_\ell \mathbb{E}\left[\max_{0 \leqslant i \leqslant \ell} \left(P \cdot i + V_{n-i}\right)\right], 0\right\}. \tag{1}$$

The max functions in (1) account for both of the seller decisions; when to stop searching and how much to sell in each meeting. Of course, the value is zero when $n = 0$ or when the seller stops searching. Otherwise, values V_n are endogenous objects that we compute inductively, for there is no last selling period to proceed by backward induction. Instead, there is always a last unit to sell whose value solves the standard McCall (1970) Bellman equation $V_1 = \max\{-c + \beta \mathbb{E}(\max(P, V_1)), 0\}$. Since there is a unique solution for $V_1 \geqslant 0$, then (1) is well defined for all n and values can be computed by inductive logic.

Exploiting (1), we see that a policy that combines no-selling and no-stopping only yields the discounted costs $-c/(1-\beta)$. Since by stopping the seller earns zero value, an optimizing strategy yields $V_n \geqslant 0$. Of course, the seller might increase his optionality by holding larger inventory and searching more, but he can also save himself the cost c if he searches less or if, in an extreme case, he stops and abandons unsold units.

Value monotonicity is obvious since more units only help (left panel of Figure 1). We now show that the optimal search strategy dictates to stop searching only if inventory falls below a *shut-down threshold* n_0 that uniquely solves $V_{n_0+1} > V_{n_0} = 0$. In other

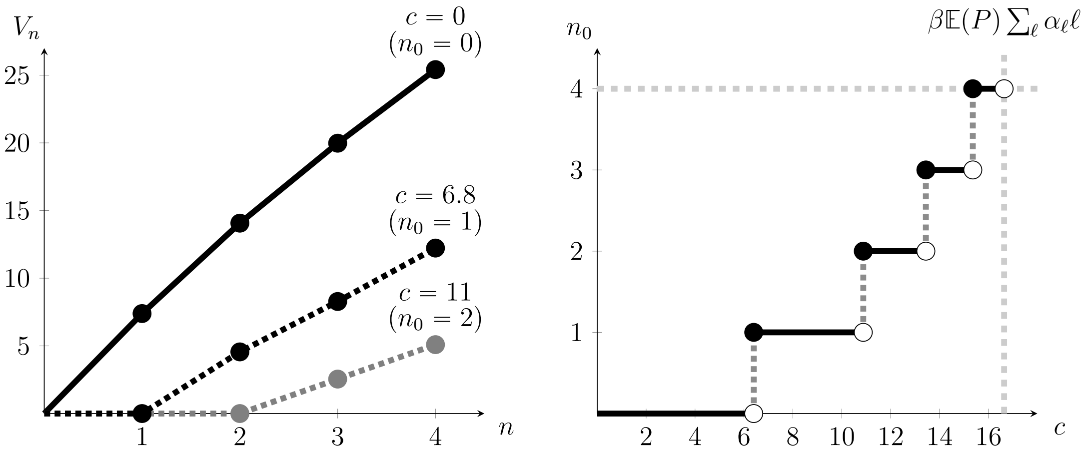

Figure 1: **Optimal Search Strategy.** Left: value rises in inventory and the seller stops search if $n \leqslant n_0$, by Lemma 1. Right: the shutdown threshold n_0 is an increasing step function of the search cost; if $c \geqslant \beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell$ the seller never searches. We use $\beta = 0.8$, $P \sim \Gamma(4,2)$ and $m = 5$ with $\alpha_1 = \alpha_2 = 0.3$, $\alpha_3 = \alpha_4 = 0.1$ and $\alpha_5 = 0.2$.

words, whenever $n > n_0$ the seller is better off incurring the cost c to search for trade opportunities which in turn yields $V_n > 0$. However, if $n \leqslant n_0$, the seller stops searching, saving himself the search cost but earning zero lifetime payoffs.

Lemma 1 (Optimal Search Strategy) *Start searching only if $c < \beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell$ and continue searching as long as $n > n_0$, where n_0 uniquely solves:*

$$\beta\mathbb{E}(P)\sum_{\ell=1}^{\infty}\min(\ell, n_0+1)\alpha_\ell > c \geqslant \beta\mathbb{E}(P)\sum_{\ell=1}^{\infty}\min(\ell, n_0)\alpha_\ell. \tag{2}$$

In our proof we show that n_0 always exists. Furthermore, as depicted on the right panel of Figure 1, the shutdown threshold n_0 is an increasing step function of the search cost and obeys $0 \leqslant n_0 \leqslant m-1$, by (2). For if $n_0 \geqslant m$, then the seller should have never started to search.[3] More notably, Lemma (1) reflects the stationarity of the search strategy resembling Stigler (1962) simultaneous search decision rule. Crucially, and unlike Stigler's model, here search is dynamic and purely sequential (without recall), so exercising an option also requires to specify a dynamic selling intensity. Interestingly though, at this stage we do not require a full characterization of the pricing strategy.[4]

[3] To see this, note that if this were the case, $V_{n_0} = 0$ yields $V_{n_0+1} = 0$, by (1). Intuitively, an extra unit of inventory provides no extra value neither through continuation values nor through better sales opportunities. The same logic yields $V_{n_0+2} = V_{n_0+3} = \cdots = 0$.

[4] As n_0 solves $V_{n_0+1} > V_{n_0} = 0$ and inventory only falls as time transpires, to compute n_0 we only require the pricing strategy at n_0+1. In this case, the seller is aware that after any trade all continuation values are zero and so his pricing strategy is binary; he either fully exploits a buyer demand or he does not sell at all. To wit, he sets reservation prices that only depend on the value of V_{n_0+1}. Specifically,

Intuitively, and analogous to Stigler's fixed sample size search, in order to continue searching the marginal benefits of searching must exceed its marginal costs.[5] Here, the marginal cost is just the flow fixed cost c of search, analogous to the sample cost in Stigler's model. In Stigler (1962) job search application, since search is simultaneous, the returns from search are given by the marginal wage rate increase from one additional search. However, in our setting, given our purely sequential search technology, the marginal benefit is well summarized by the discounted value of one-period ahead *expected maximum sales* $\beta\mathbb{E}(P)\sum_\ell \min(\ell, n)\alpha_\ell$. This precisely accounts for the magnitude of the returns to search as a function of inventory. As the expected maximum sales rise in the size of the inventory, the stopping decision also adjusts as inventory evolves. In particular, the marginal benefit-marginal cost comparison dictates that it is optimal to search and never stop searching if $c < \beta\mathbb{E}(P)$ (i.e., when the expected maximum sales exceeds the search cost even when $n = 1$), and to never search if $c \geqslant \beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell$ (i.e., if the expected maximum sales never exceeds the cost of search), as deduced from (2). Otherwise, if $\beta\mathbb{E}(P) \leqslant c < \beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell$ it is optimal to start searching, but eventually stop if inventory is low enough.

Exploiting Lemma 1, we also deduce that for a given search cost value $c > 0$, a strategy that combines searching with eventually stopping and abandoning part of the inventory exclusively arises if we allow multiple unit sales. For if buyers arrive with single unit demands, the decision to stop searching is independent of the size of the inventory. The expected maximum sales in this case always equals $\beta\mathbb{E}(P)$, and so the seller either always or never searches, but never stops before having sold all units. To obtain this difference between policies, our multiple unit sales assumption is critical.

How the value n_0 and the expected numbers of search periods vary as the model parameters adjust only depend on how the expected maximum sales change. Exploiting (2), we deduce that the shutdown threshold falls in $\beta\mathbb{E}(P)$, rises in c, and falls with stochastic increases in ℓ or with mean-preserving spreads in ℓ.[6] Intuitively, stochastically better prices or demands improve inventory optionality, and so the seller searches more by delaying the stopping decision; a higher search cost or impatience anticipates it. As for an increased price risk, despite the fact it improves inventory optionality, it does not affect the stopping decision as long as the expected price remains unchanged (mean preserving spreads). This is unlike increments in demand risk that delays stopping.

he sells $j \geqslant 1$ units only if $p \geqslant V_{n_0+1}/j$. Imposing $V_{n_0+1} > 0$ yields (2).

[5] Morgan and Manning (1985) refers as *regular* to the search rules that allow a further single observation when it is expected to increase the searcher's utility; these rules obey (2) in our model.

[6] As $\sum_{\ell=1}^{\infty} \min(\ell, n+1)\alpha_\ell > \sum_{\ell=1}^{\infty} \min(\ell, n)\alpha_\ell$, then n_0 falls in $\beta\mathbb{E}(P)$ and rises in c. Our last claim follows by standard stochastic ranking theorems, since $\mathbb{E}(\min(\ell, n))$ is increasing and concave.

4 How to Price Units?

Now that we have characterized when it is optimal for the seller to stop searching, we pursue a more ambitious goal. The task is to characterize the optimal pricing strategy by deducing *how to price* each unit of inventory, and thus *how much* should the seller sell in each trade opportunity. This ultimately completes the characterization of the optimal search rule. As in any sequential search problem, the pricing strategy is related to and well summarized by *optimal reservation prices.* Since these reservation prices will adjust depending on how inventory size evolves, we let $\mathcal{R}_{i,n}$ be the *per unit reservation price to sell i units when inventory is n.*

Furthermore, as in each period the seller chooses to search by incurring a fixed cost, reservation prices are guided by total revenue considerations instead of pure marginalism. This will naturally prevent us from just using first order conditions when solving the seller optimization problem. For this reason, we have to simultaneously keep track of both the marginal and average cost of selling each unit. Both of these functions are endogenous objects that can be derived from the shape of the value function, exploiting an inductive logic. More specifically, the *marginal cost* of selling is given by *value differences* $\Delta V_n \equiv V_n - V_{n-1}$, while the *average selling cost* is determined by *average value differences* or *average value loss* of selling j units, defined here as $\phi_{n,j} \equiv (V_n - V_{n-j})/j$.

For a clear analysis and exposition, we now divide our analysis in *three cases.* We will show that in each of the following cases the seller employs a different optimal pricing strategy. First, we explore the case when the search cost is low enough. In this case, the marginal selling cost is increasing and thus selling happens at the margin, as in Carrasco and Smith (2017). Second, when demands are non-random and all buyers arrive willing to buy the same maximum amount of units. Finally, in the third case, we study the case when demands are random; that is, when some buyers might be willing to buy more units than others. Our main finding is that, except in the first case, a fixed search cost translates into a non-monotone endogenous marginal selling cost that leads to discontinuous selling strategies; specifically to bundling and volume discounts.

A. Low Search Cost: Here we show that a low enough search cost yields diminishing returns to optionality extending the findings in Carrasco and Smith (2017) to a costly search environment.

Proposition 1 *Value differences ΔV_n are positive and decreasing if the search cost is low enough; specifically, whenever $c \leqslant \bar{c}$ where $\bar{c}$ uniquely solves $V_2 - V_1 = V_1$.*

The argument to find the value of $\bar{c}$ is purely inductive. If $c \leqslant \bar{c}$, then $V_2 - V_1 \leqslant V_1$

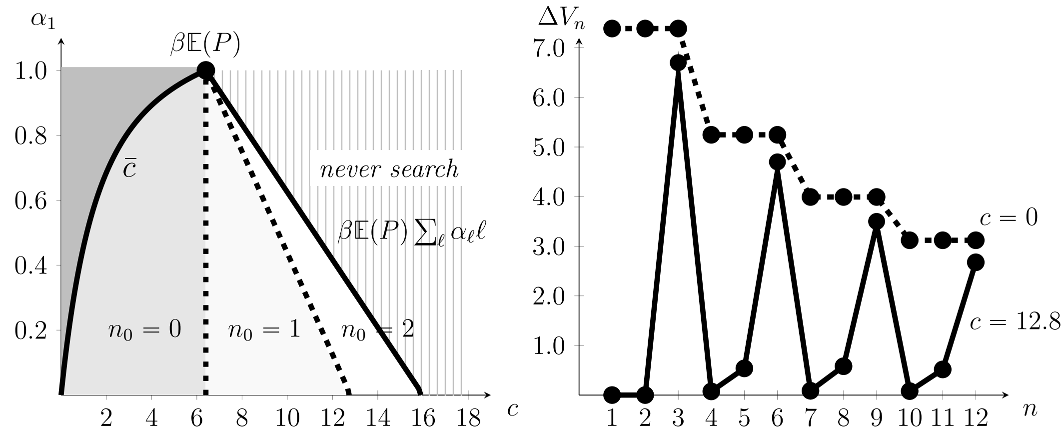

Figure 2: **Only Low Search Cost leads to Diminishing Returns.** Assume $m = 3$, $\beta = 0.8$ and $P \sim \Gamma(4, 2)$. Left: diminishing returns only arise if $c \leqslant \bar{c}$ (dark gray region), by Proposition 1. For $\alpha_1 \in (0, 1)$, if $\bar{c} < c < \beta\mathbb{E}(P)$ then $n_0 = 0$ but as c rises, search stops at $n_0 = \{1, 2\}$; if $c \geqslant \beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell$ search never happens, by Lemma 1. Right: If $\alpha_1 = 0$ then $\bar{c} = \beta\mathbb{E}(P)$ so any $c > 0$ invalidates the diminishing returns.

which in turn yields $V_3 - V_2 \leqslant V_2 - V_1$, so on and so forth. This condition ultimately yields a decreasing value increment ΔV_n.

A direct consequence of Proposition 1 is that the reservation price for selling i units when inventory is n equals $\mathcal{R}_{i,n} = \Delta V_{n-i+1}$, as in Carrasco and Smith (2017). There is, of course, an inverse relationship between returns to optionality and reservation prices. Intuitively, since the diminishing returns to optionality means that an additional unit of inventory is progressively less valuable, they unambiguously translate into an increasing marginal and average selling cost. To wit, the seller sells more units only for higher prices and those prices rise through the entire liquidation process.

For more insights, in the proof of Proposition 1 we precisely identify that the value $\bar{c}$ that uniquely solves $V_2 - V_1 = V_1$ obeys:

$$\bar{c} = \beta\alpha_1 \int_{\frac{\bar{c}(1-\alpha_1)}{\alpha_1(1-\beta)}}^{\infty} [1 - G(p)]dp. \tag{3}$$

As shown in Figure (2), the value of $\bar{c}$ rises in α_1 and there are diminishing returns to optionality only if $c \leqslant \bar{c}$.[7] This extends the results in Carrasco and Smith (2017) to a costly search environment, as long as the search cost is low enough. Exploiting (3) we deduce that our multiple unit sales assumption is critical to offer new insights about the optimal search rule when the search cost is higher. For, when sales are carried

[7]Furthermore, since $\bar{c} = \alpha_1\beta\mathbb{E}\left(\max\left(P - \bar{c}(1-\alpha_1)/\alpha_1(1-\beta), 0\right)\right)$, as shown in the proof of Proposition 1, $\bar{c}$ rises with stochastic improvements in P and mean preserving price dispersion.

in multiple units ($\alpha_1 = 0$), since $\bar{c} = 0$ then any search cost invalidates their results. "Search at the margin" is not longer optimal. Instead, with single units sales ($\alpha_1 = 1$) we have $\bar{c} = \beta\mathbb{E}(P)$, which exactly coincides with cost value that precludes search at all, by Lemma 1. That is, whenever the seller decides to search, he does it at the margin and sells each additional unit only at a premium, as in Carrasco and Smith (2017). Intuitively, in this case the search cost is analogous to paying a storage cost exclusively on the marginal unit (whereas all the other units are free to store) meaning the seller treats each unit as if it were the last one, which further diminish its optionality.

B. Non-Random Demands: We now assume that buyers arrive with deterministic demands; that is, they always demand up to $m \geqslant 2$ units, and so $\alpha_m = 1$. Obviously, as in this case $\alpha_1 = 0$ any search cost precludes the diminishing returns property since $\bar{c} = 0$, by (3). Hence, the marginal selling cost is no longer increasing and so the seller will employ a different optimal pricing strategy. He will not sell at the margin anymore. We restrict to $c < \beta\mathbb{E}(P)m$ as otherwise the seller never search, by Lemma 1.

In this case, the seller needs to carefully balance the two kinds of optionalities provided by each unit of inventory. Obviously, as each extra unit improves the seller's ability to sell more in each trade opportunity, it provides him a valuable "selling optionality". However, with a fixed search cost, as it also allows him to split it among more sold units, it lowers the average selling costs and improves the "search optionality". Crucially, these optionalities adjust as the size of the inventory evolves. Due to caps, the selling optionality falls in the size of the inventory as shown in Carrasco and Smith (2017). We argue that the search optionality rises in the size of the inventory whenever it allows the seller to split the search cost among more units, lowering the per unit search cost. The fact that these two optionalities vary differently in the inventory size yields a non-monotone selling cost, which naturally translates into a discontinuous pricing strategy.

For clear insights, let us first partition the inventory state space into smaller *search domains* $\Omega_m(k) \equiv \{j \in \mathbb{N} : m(k-1) + 1 \leqslant j \leqslant mk\}$ for $k \in \mathbb{N}$. As we now formally show, compared to our previous low search cost case, the shape of the value function is radically different; it rises at increasing rate, but only in each search domain.

Proposition 2 (Non-random demands) *Value differences are increasing in each search domain, but they otherwise fall and so $\Delta V_{mk} > \Delta V_{mk+1}$ for all $k \in \mathbb{N}$. Additionally, average value differences $\phi_{mk,m}$ fall in k.*

Our results in Proposition 2 are illustrated in Figure 3. We provide an intuition of our result here. For this, let $n_0 = 0$ and suppose for now that $n \in \Omega_m(k)$, but $n \leqslant mk-1$. That is, inventory lies inside a search domain but is not a multiple of the demand size m.

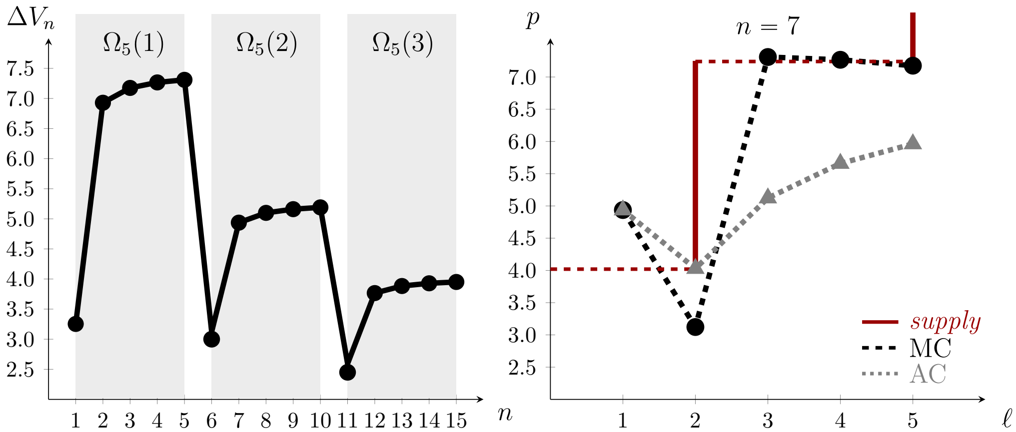

Figure 3: **Non-Random Demands yield Optimal Bundling.** Let $m = 5$, $c = 3.2$, $\beta = 0.8$ and $P \sim \Gamma(4, 2)$. Left: value differences rise only on each search domain, by Proposition 2. Right: the seller optimally employs a bundling strategy, as neither the marginal cost (MC) nor the average cost (AC) are monotone. If $n = 7$, he bundles 2 and 5 units and sells them if $p \geqslant \mathcal{R}_{2,7} = \phi_{7,2}$ and $p \geqslant \mathcal{R}_{5,7} = \phi_{5,3}$, respectively.

In this case, the seller necessarily requires at least k search periods to fully liquidate his inventory. For, in a best case scenario, he is only going to be able to sell m units in each encounter (and he might even find optimal not to sell in a search period). Then, given this fixed number of required search periods, the demand caps do not harm the selling optionality of an additional unit, for the search surplus of none of the units of inventory is delayed. Even more, the seller realizes that having one extra unit of inventory does not add search periods since $(n + 1) \in \Omega_m(k)$; that is, he also requires at least k search periods to sell $n + 1$ units. Altogether, one extra unit improves both the selling and the search optionalitites. To wit, additional units in a search domain are always more valuable and thus the marginal cost of selling is decreasing, as depicted in Figure 3.

Instead, suppose now that the size of the inventory is a product of the demand size; that is, $n = mk$ and $n \in \Omega_m(k)$. In this case, if the seller were to hold an additional unit, then the exact opposite happens as both optionalities fall. First, notice that since $n + 1 \in \Omega_m(k + 1)$, an extra unit of inventory necessarily requires an additional search period for the seller to fully liquidate inventory. Then, as the additional unit has to fully bear the search cost, its search optionality is severely harmed. In addition, since the surplus of the additional unit is delayed due to caps, the selling optionality is also reduced. This unambiguously yields a lower value increment $\Delta V_{mk} > \Delta V_{mk+1}$, as depicted on the left panel of Figure 3.

Our next corollary shows how the shape of the value function described in Proposition 2 — with a non-monotone marginal benefit of inventory — determines a discontinuous optimal pricing strategy. As shown in Figure 3, the shape of the value function yields non-monotone selling costs, which in turn gives rise to a pricing strategy where the discontinuity is the largest: *optimal bundle pricing.*

Corollary 1 (Bundling) *For low inventory, optimal selling is all-or-nothing. Specifically, if $n \leqslant m$, then the seller bundles the inventory and fully sells it for a per unit price that equals the average cost $\mathcal{R}_{n,n} = \phi_{n,n}$. For larger inventory, it is optimal to design and offer at most two bundles. Specifically, if $n \in \Omega_m(k)$ and $k \geqslant 2$, the offered bundle sizes are $j = n - m(k-1)$ and $m \geqslant j$, with reservation prices equal to the average cost $\mathcal{R}_{j,n} = \phi_{n,j}$, and to the incremental average cost $\mathcal{R}_{m,n} = \phi_{m(k-1),m-j}$, respectively.*

This follows immediately from Proposition (2). Unlike Carrasco and Smith (2017), the seller now prices each unit of inventory at its average, rather than marginal, cost. For low inventory $n \leqslant m$ the seller faces increasing search optionality gains of inventory (because the search cost is split among more units), and caps are not binding. He then faces a decreasing marginal selling cost, and thus bundles all of his inventory, willing to completely unload it if the price exceeds it average cost $\mathcal{R}_{n,n} = \phi_{n,n}$. Otherwise, for larger inventory, the seller faces a non monotone marginal selling cost, as shown on the right panel of Figure 3. To wit, bundle pricing arise as an optimal pricing strategy.

Ultimately, the seller designs only two bundles, but unlike most bundling menus, sets a higher per-unit price for the larger one.[8] By doing so, he designs a (more expensive) bundle for high valuation buyers, and an additional one (cheaper) for low valuation ones. The per-unit price of the small bundle with j units is equal to the average selling cost $\phi_{n,j}$, while the per-unit price of the larger one equals the average selling cost of the additional $m - j$ units; which is $\phi_{m(k-1),m-j}$. This is a direct consequence of the fact that the seller faces *diseconomies of scale* (an increasing average selling cost, as depicted on the right panel of Figure 3). In the particular case of $n \in \{m, 2m, 3m, \dots\}$ the seller always bundles and sells m units only for prices higher than the reservation price $\mathcal{R}_{m,n} = \phi_{n,m}$. Furthermore, in this case the price of the bundle rises through the liquidation process, which follows immediately from the fact that $\phi_{m(k+1),m} < \phi_{mk,m}$, by Proposition 2. In the special case of *uncapped demands* ($m \to \infty$) the seller employs an all-or-nothing pricing strategy with reservation price $\mathcal{R}_{n,n} = \phi_{n,n}$, by Corollary 1.[9]

Observe also that the seller adjusts reservation prices depending on the size of his

[8]By our first result in Proposition 2 we obtain $\mathcal{R}_{j,n} < \phi_{mk,m}$ and $\mathcal{R}_{m,n} > \phi_{m(k-1),m}$, while by the second we have $\phi_{mk,m} < \phi_{m(k-1),m}$.

[9]Exploiting (1), in this case the reservation prices solve $(1-\beta)\mathcal{R}_{n,n} = -c/n + \beta\mathbb{E}(\max(P - \mathcal{R}_{n,n}, 0)$. Since caps never bind, the search cost is analogous to a storage cost that is divided in equal parts among units.

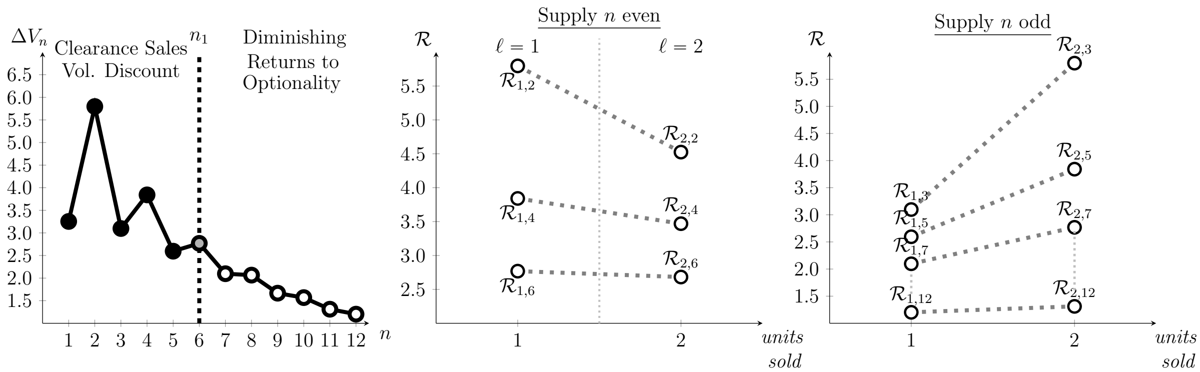

Figure 4: **Random Demands yield Clearance Sales and Volume Discounts.** Let $m = 2$, $\alpha_2 = 0.75$, $c = 3.2$, $P \sim \Gamma(4, 2)$ and $\beta = 0.8$. At left, value differences fall only if $n \geqslant n_1$, by Proposition 3. We plot optimal reservation prices for $n \leqslant n_1$ in the other panels. The seller optimally combines premiums when inventory is odd (right) with volume discounts offered to buyers with large demands ($\ell = 2$) when it is even (middle), by Corollary 2.

post trade-inventory. This in turn relates to a *dynamic search intensity*, which is in stark contrast to the findings in the optimal search literature Morgan and Manning (1985); Morgan (1983).[10] In particular, for infinite horizon problems without recall like ours, Morgan (1983) show that it is optimal to search with constant intensity until stopping (Proposition 1). Crucially, and unlike their time invariant "psychic cost" of sampling, our selling costs are endogenous and fully determined by previous period sales. It is then genuinely dynamic reflecting the endogenous time-varying optionality of each unit of inventory. Then, after each trade, the seller adjusts his selling intensity by revising the reservation prices of each bundle and thus his selling intensity, simply because optionality has changed.

C. Random Demands $\alpha_m < 1$: We now explore the case when buyers arrive with different capped demands. In order to illustrate our results and offer clear insights, we restrict to the $m = 2$ case. That is, buyers demanding a maximum of one unit arrive with probability $\alpha_1 \geqslant 0$, and a maximum of two units with complementary chance $\alpha_2 = 1 - \alpha_1 \geqslant 0$. For the $m > 2$ case we rely on numerical simulations to offer more general insights. Obviously, we assume $\bar{c} < c < \beta \mathbb{E}(P) \sum_\ell \alpha_\ell \ell$ as otherwise the seller never searches or $n_0 = 0$ and with decreasing value differences, as in our first case.

We first show that in this case the shape of the value function, and thus the pricing

[10] In this literature, search intensity relates to the number of simultaneous observations drawn from a known distribution. There are, of course, alternative ways to model search intensity in sequential environments like ours. In particular, our pricing strategy is directly related to this intensive margin since a drop in the reservation price is analogous (yet, not exactly the same) to an increase in the search intensity.

strategy, radically adjust depending on the size of the inventory. In particular, we show that there exists a unique *inventory threshold for diminishing returns* $n_1 \geqslant 0$, after which value differences are decreasing. That is, $\Delta V_n > \Delta V_{n+1}$ for $n \geqslant n_1$, but $\Delta V_{n_1} > \Delta V_{n_1-1}$. Then, the optimal pricing strategy for large enough inventory naturally follows, since marginal selling costs rise. For low inventory values, we show that value differences are highly non monotone, and that it *zigzags* as inventory increases.

Proposition 3 (Random demands) *For $c > \bar{c}$, there exist a unique inventory threshold $n_1 \geqslant 2$ such that value differences are decreasing when $n \geqslant n_1$. Instead, when $n \leqslant n_1$ value differences zigzag; that is, $\Delta V_n > \max\{\Delta V_{n-1}, \Delta V_{n+1}\}$ for all even numbers $n \geqslant 2$.*

We now show how this specific shape of the value function determines the optimal pricing strategy, which is radically different than the ones in our previous cases. As shown in Figure 4 and formalized in Corollary 2, the optimal pricing strategy is again discontinuous. In particular, it combines the diminishing returns insights — and thus *purchase premiums* when inventory is large — with *clearance sales* in the form of *bundling with volume discounts* when inventory is low. We omit the proof as it is an immediate implication of Proposition 3.

Corollary 2 (Discounts and Premiums) *For low inventory $n \leqslant n_1$: if n is even then it is optimal to offer volume discounts and set $\mathcal{R}_{1,n} > \mathcal{R}_{2,n}$, where $\mathcal{R}_{i,n} = \phi_{n,i}$. If n is odd, it is optimal to charge purchase premiums and set $\mathcal{R}_{2,n} > \mathcal{R}_{1,n}$ where $\mathcal{R}_{i,n} = \Delta V_{n-i+1}$. For larger inventory $n \geqslant n_1 + 1$, it is also optimal to charge purchase premiums and set reservation prices equal to marginal costs; i.e., $\mathcal{R}_{i,n} = \Delta V_{n-i+1}$.*

The fact that value differences fall when inventory is large enough induces the seller to sell each additional unit at a premium if $n \geqslant n_1 + 1$, as in Carrasco and Smith (2017). Otherwise, if the seller holds low enough inventory, then the pricing strategy adjusts and reflects the possibility of increasing value differences, as shown in the left panel of Figure 4. In particular, when $n - 1$ is odd, one additional unit of inventory not only provides a better selling optionality to the seller — improving his ability to sell more — but also allows him to split the search cost among more units. Due to the increasing optionality, when inventory is n (and thus, is even) the seller faces a decreasing marginal and average selling cost. Obviously then, in this case the seller will try to sell as many units as possible. This in turn translates into volume discounts offered to buyers that arrive with larger demands. Instead, when n is odd, since the seller faces an increasing selling cost (decreasing optionality) purchase premiums are optimal.

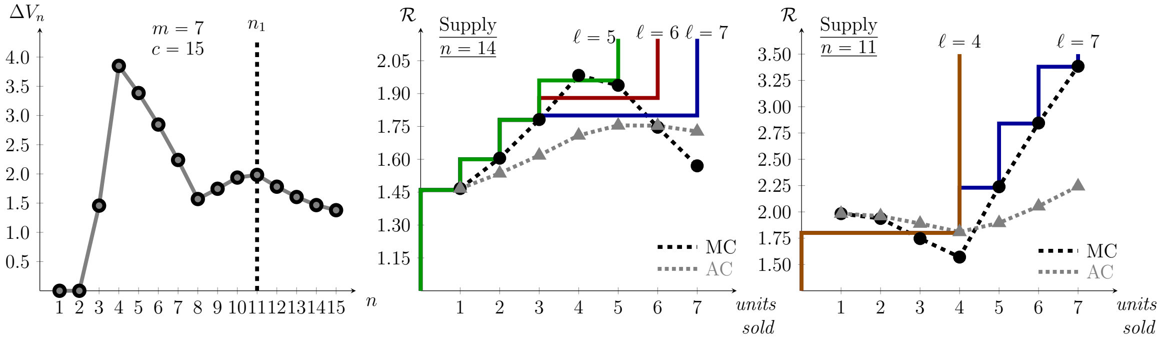

Figure 5: **Optimal Discounts and Premiums in a More General Case.** We posit $m = 7$, $\alpha_\ell = 1/7$, $\beta = 0.8$ and $P \sim \Gamma(4, 2)$. At left, value differences fall only if $n \geqslant 1$. The other two panels show the supply (reservation prices) for different capped demand values (solid lines) for $n = 14$ (middle) and $n = 11$ (right). Volume discounts arise whenever the marginal cost falls, but stop as soon as it rises.

Altogether, reservation prices are lower at initial stages of a liquidation process and they progressively rise as time transpires. However, as soon as inventory falls below the inventory threshold for diminishing returns n_1, the seller speeds up sales by lowering reservation prices and offering volume discounts, as depicted in Figure 4.

A More General Case: With random demands and $m > 2$ it is hard to analytically characterize the value function and the pricing strategy, but we rely on numerical exercises to offer key economic insights. To illustrate our findings, we explore the example of equally likely capped demands with $m = 7$ (Figure 5). First, we still observe that when inventory is large enough and $n \geqslant n_1$, value differences are decreasing and so the diminishing property of inventory optionality is obtained. We deduce that if $n \geqslant n_1 + m - 1$ the seller faces an increasing marginal selling cost and thus sets increasing reservation prices $\mathcal{R}_{n,i} = \Delta V_{n-i+1}$ to sell i units. Otherwise, for low enough inventory $n \leqslant n_1$ there might be multiple regions where value differences are increasing, as shown in the left panel of Figure 5. This means the marginal selling cost is again non-monotone. However, given the lack of regularities, its shape translates into a more sophisticated combination of volume discounts and purchase premiums.

In this more general case, and unlike in our previous $m = 2$ case, the seller optimally offers volume discounts but *not only to buyers arriving with large demands*. For instance, when $n = 11$ (right panel of Figure 5), it is optimal to offer volume discounts to buyers arriving with smaller demands that wish to buy no more than four units. The reservation prices in this case are equal to the average selling costs. In addition, we see that there might be *limits to volume discounts*. For instance, in the same case when $n = 11$ the seller is willing to sell the first four units at progressively lower prices if $\ell \leqslant 4$, but charges a purchase premium again for more units. In fact, all these additional units are sold at their marginal cost. Finally, we also see that in some cases the seller offers volume discounts exclusively to buyers with large demands. For instance, when $n = 14$ (middle panel of Figure 5) he sells the first four units at progressively higher unitary prices if $\ell \leqslant 4$, but volume discounts star for $\ell \geqslant 5$, when the marginal cost of selling the next unit is smaller than the cost of the previous one.

5 Concluding Remarks

Despite the ubiquity of multi-unit search and multi-unit trade, most sequential search problems in economics simplify the trade environment by assuming a single indivisible unit. Furthermore, models that account for multiple units have been usually combined with semi-decentralized trading, assuming that a competitive and organized market exist, and thus focusing exclusively in the intensive margin of trade; i.e., how much to sell given a competitive equilibrium price. In this paper, we relax all of these assumptions by formulating a costly search model for the optimal selling of a stock of goods. In doing so, we explore and solve a dynamic programming exercise that combines the extensive margin of search with the intensive margin of selling that commonly arise in most selling problems when inventory is endogenous and its size evolve in time.

We inductively solve for the seller's optimal search rule by simultaneously specifying a search strategy — i.e., when to start searching and whether to stop or not — and a pricing strategy that specifies how to price each unit of inventory, and thus how much to sell in each trade opportunity. Our theory explicitly accounts for the effect of escapable fixed costs on optimal liquidation strategies, pushing search theory into a richer class of problems that are not just optimal stopping exercises. Indeed, and to the best of our knowledge, our is the first search theoretical model that simultaneously makes sense of bundling, purchase premiums and volume discounts as part of the optimal search policy.

As in any search problem, understanding each unit optionality is critical. This is especially important in our case, for by combining multiple units with an escapable search cost, the optionality is twofold. First, and more obvious, each additional unit allows the seller to sell more in each trade opportunity, providing a valuable selling optionality. Second, it also allows him to split each period search cost among more units, thus lowering average selling costs and improving search optionality. We find that the selling optionality falls with the size of the inventory (due to capped demands), and that the search optionality rises because the seller might liquidate units facing a lower per unit search cost. These optionalities give rise to new trade offs that yield a non-monotone endogenous marginal selling cost. Ultimately this translates into new selling strategies, such as bundling, and more sophisticated ones that endogenously combine purchase premiums with sales and volume discounts.

Among the most relevant model extensions, one might think that the form of the cost matters. That is, if the seller were to face a per unit cost, rather than a fixed search cost. It is hard to offer general insights here, as the results and the model tractability obviously depend on the specific shape of the cost function that is assumed. However, with a linear holding cost it is easy to verify that the inventory unambiguously have diminishing returns to optionality, due to capped demands (a convex holding cost would of course further reinforce the diminishing returns). Hence, the seller then always faces an increasing marginal cost and only sells more units for higher prices. However, unlike the fixed cost case, the value function is no longer monotone, which in turn this yields an optimal inventory size. This means that if the seller ever abandons inventory he does it at the beginning of the liquidation, and not at the end as it happens in our fixed cost case. In fact, the seller should even be willing to pay in order to get rid of part of his inventory when this exceeds the optimal size.

Our model can also easily be further extended to account for price-quantity Nash bargaining and continuous time. Hence, our results all naturally extend to this more general trade protocol usually employed, for instance, in over-the-counter markets.[11] A more general demand schedule that allows buyers to trade off price and quantity is also an interesting extension, but it significantly reduces the model tractability.

We are currently extending the analysis to account for multiple units with costly but ordered search (e.g., submarkets), as in Weitzman (1979). In this case, the seller's optimal policy not only would need to specify how intensively to act upon each trade opportunity, but it would also need to specify a search order. In this case, a seller would be able to direct his search based on his knowledge about each submarket. Naturally, a seller's strategy, both about how many buyers to search for and which buyers to search for, will change as its inventory level changes.

[11]When bargaining powers are δ for the seller and $1-\delta$ buyers, as the bargained price is the weighted average of reservation values, we just have to account that inventory readjusting happens with chance δ. Ultimately, this means that the effective fixed cost and discount are $c/(1-(1-\delta)\beta)$ and $\beta\delta/(1-(1-\delta)\beta)$, respectively. The continuous time version is similar, except that $\beta = \mathbb{E}(e^{tT})$ and $c = k(1 - \mathbb{E}(e^{tT})/r$, where r is the discount rate, T the random time for the first meeting, and k is the continuous time flow cost. In the tractable Poisson arrival version with meeting rate $\rho > 0$, we have $\beta = \rho/(r + \rho)$ and $c = k/(r + \rho)$. Furthermore, for the price-quantity Nash bargaining version, we just adjust the meeting rate to $\rho\delta$ where δ is the seller bargaining power.

6 Appendix

Proof of Lemma 1: If $n_0 \geqslant 1$ then $V_{n_0-1} = V_{n_0-2} = \cdots = V_1 = 0$, due to value monotonicity. Then, substituting zero continuation values in (1) yields $V_{n_0} = \max\{-c + \beta\mathbb{E}(P)\sum_{\ell=1}^{n_0}\alpha_\ell^{n_0}\ell, 0\}$. By the same logic, imposing $V_{n_0} = 0$ we have $V_{n_0+1} = \max\{-c + \beta\sum_{\ell=1}^{n_0+1}\alpha_\ell^{n_0+1}\mathbb{E}(\max(V_{n_0+1}, P\ell)), 0\}$. Then, the shutdown threshold obeys $n_0 \geqslant 1$ if $V_{n_0+1} > 0 = V_{n_0}$, which yields (2) by fixed point reasoning. When $m = \alpha_1 = 1$ then (2) yields $n_0 = 0$. For $m \geqslant 2$, as both sides of (2) are increasing in n_0, a unique solution $m - 1 \geqslant n_0 \geqslant 1$ exists if $\beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell > c \geqslant \beta\mathbb{E}(P)$. Shutdown never happens ($n_0 = 0$) when $V_1 > 0$ which is iff $c < \beta\mathbb{E}(P)$. Finally, since $V_n \leqslant -c/(1-\beta) + \beta\mathbb{E}(P)\sum_{\ell=1}^{\infty}\ell\alpha_\ell/(1-\beta)$, the seller shuts down regardless of his holdings if $c \geqslant \beta\mathbb{E}(P)\sum_\ell \alpha_\ell \ell$.□

Proof of Proposition 1: We aim to prove when $\Delta V_{n+1} < \Delta V_n$ for all $n \geqslant 1$. We assume that $V_n > 0$ for all $n \in \mathbb{N}$ and check this at the end. Let us first stochastically increase the demand schedule α^n to $\bar{\alpha}^n$, by incrementing every demand for $n-1$ units by one. In this case, $\bar{\alpha}_\ell^n = \alpha_\ell^n$ for $\ell = 1, 2, \ldots, n-2$ but $\bar{\alpha}_{n-1}^n = 0$ and $\bar{\alpha}_n^n = 1 - (\alpha_1 + \cdots + \alpha_{n-2})$. We consider the higher value $V_{n+1} < \bar{V}_{n+1}$ when all probability weight shifts from demand for n units to $n+1$ units. To show $\Delta V_{n+1} < \Delta V_n$, it suffices that $\bar{V}_{n+1} - V_n \leqslant \Delta V_n$. We now show this second inequality. Let $\mathbb{1}^n \equiv (1, 1, \ldots, 1) \in \mathbb{R}^n$, the vector $V^n = (V_1, \ldots, V_n)$ and define:

$$F_n(\cdot, v|\alpha^n) \equiv -c + \beta\sum_{\ell=1}^n \alpha_\ell^n \mathbb{E}\left(\max(v, \max_{1\leqslant i\leqslant \ell}(Pi + V_{n-i} - V_{n-1}))\right) \tag{4}$$

By (1), $v = \Delta V_n$ solves

$$v = F_n(V^{n-1} - V_{n-1}\mathbb{1}^{n-1}, v|\alpha^n) - (1-\beta)V_{n-1} \tag{5}$$

Likewise, $v = \bar{V}_{n+1} - V_n$ solves

$$v = F_{n+1}(V^n - V_n\mathbb{1}^n, v|\bar{\alpha}^{n+1}) - (1-\beta)V_n \tag{6}$$

Then $\bar{V}_{n+1} - V_n \leqslant \Delta V_n$ if the (unique) fixed point of (6) is at most the (unique) fixed point of (5). Next, let the *grap* $\mathcal{G}_n(v)$ be the right side of (6) less the right side of (5):

$$\mathcal{G}_n(v) = F_{n+1}(V^n - V_n\mathbb{1}^n, v|\bar{\alpha}^{n+1}) - F_n(V^{n-1} - V_{n-1}\mathbb{1}^{n-1}, v|\alpha^n) - (1-\beta)\Delta V_n \tag{7}$$

Next, consider the *upper envelope* of $\ell+1$ linear functions ($\ell = 1, 2, \ldots, n$):

$$\mathcal{U}_n(v,p,\ell) \equiv \max(v, \max_{1\leqslant i\leqslant \ell}(pi + V_{n-i} - V_{n-1})) \tag{8}$$

Since $\bar{\alpha}_{n+1}^{n+1} = \alpha_n^n$, the gap $\mathcal{G}_n(v)$ in (7) is a weighted sum, with identical probability weights, on the differences of analogous terms:

$$\mathcal{G}_n(v) = \beta\sum_{\ell=1}^{n-1}\alpha_\ell\mathbb{E}[\Delta\mathcal{U}_{n+1}(v,P,\ell)] + \beta\alpha_n^n\mathbb{E}[\mathcal{U}_{n+1}(v,P,n+1) - \mathcal{U}_n(v,P,n)] - (1-\beta)\Delta V_n$$

Each difference $\Delta\mathcal{U}_{n+1}(v,p,\ell)$ has the form $\max(v,m_{n+1}) - \max(v,m_n)$ where $m_n = \max_{1\leqslant i\leqslant\ell}(pi - \sum_{j=1}^{i-1}\Delta V_{n-j})$. Now, suppose $\Delta V_n \leqslant \Delta V_{n-1} \leqslant \ldots \leqslant \Delta V_1$. Then we have $\max(v,m_{n+1}) - \max(v,m_n) > 0$. Since such a function falls in v, so does $\Delta\mathcal{U}_{n+1}(v,p,\ell)$. By similar logic, the middle term in $\mathcal{G}_n(v)$ can be written as a sum of such terms falling in v. So $\mathcal{G}_n(v) \leqslant \mathcal{G}_n(0)$. We now find a surplus representation for the gap $\mathcal{G}_n(0)$.

Selling i units is optimal if the price exceeds the average cost of selling any m' less units, and is at most the average cost of selling any m more units:

$$(V_{n-i+m'} - V_{n-i})/m' \leqslant p \leqslant (V_{n-i} - V_{n-i-m})/m \quad \text{for } 1 \leqslant m' \leqslant i, 1 \leqslant m \leqslant \ell - i$$

By recursive assumption, optimality reduces to the discrete first order condition with $m = m' = 1$. So selling $i \geqslant 2$ units is optimal iff $\Delta V_{n+1-i} \leqslant p \leqslant \Delta V_{n-i}$. So as a function of p, and for $0 \leqslant v \leqslant \Delta V_{n-1}$, the upper envelope in (8), kinks upward at $p = v$ as the seller chooses $i = 1$, and then at every ΔV_{n+1-i} for $i = 2, \ldots, \ell$, as the seller chooses to sell $i+1$. Let $\Phi_{n,j} \equiv \mathbb{E}(\max(P - \phi_{n,j}, 0)$ so that $\Phi_{n,1} = \mathbb{E}(\max(P - \Delta V_n, 0)$, and so:

$$\mathbb{E}[\,\mathcal{U}_n(v,P,\ell)] = \mathbb{E}(\max(v,P)) + \textstyle\sum_{j=n+1-\ell}^{n-1}\Phi_{j,1} \tag{9}$$

Rewrite V_n using (8) and (9):

$$(1-\beta)V_n = -f + \beta\sum_{\ell=1}^{n}\alpha_\ell^n(\mathbb{E}[\mathcal{U}_n(\Delta V_n, P, \ell)] - \Delta V_n) = -f + \beta\sum_{\ell=1}^{n}\alpha_\ell^n\sum_{j=n+1-\ell}^{n}\Phi_{j,1} \tag{10}$$

Then:

$$(1-\beta)\Delta V_n \quad = \quad \beta\sum_{\ell=1}^{n-1}\alpha_\ell\sum_{j=n-\ell}^{n-1}[\Phi_{j+1,1} - \phi_{j,1}] + \beta\alpha_n^n\Phi_{n,1} \tag{11}$$

Next, by (9):

$$\mathbb{E}[\Delta\mathcal{U}_{n+1}(0,P,\ell)] = \textstyle\sum_{j=n+2-\ell}^{n}\Phi_{j,1} - \sum_{j=n+1-\ell}^{n-1}\Phi_{j,1} = \sum_{j=n+1-\ell}^{n-1}[\Phi_{j+1,1}-\Phi_{j,1}] \tag{12}$$

$$\mathbb{E}[\mathcal{U}_{n+1}(0,P,n+1)-\mathcal{U}_n(0,P,n)] = \textstyle\sum_{j=1}^{n}\Phi_{j,1} - \sum_{j=1}^{n-1}\Phi_{j,1} = \Phi_{n,1} \tag{13}$$

Using (11)-(13), yields:

$$\mathcal{G}_n(0) = \beta\sum_{\ell=1}^{n-1}\alpha_\ell[\Phi_{n-\ell,1}-\Phi_{n+1-\ell,1}] = \beta\sum_{j=1}^{n-1}\alpha_{n-j}\left(\Phi_{j,1}-\Phi_{j+1,1}\right) \leqslant 0$$

Altogether, we have shown that if $\Delta V_n \leqslant \Delta V_{n-1} \leqslant \ldots \leqslant \Delta V_1$, then $\mathcal{G}_n(v) \leqslant \mathcal{G}_n(0)$ and so $\bar{V}_{n+1} - V_n \leqslant \Delta V_n$ and since $V_{n+1} < \bar{V}_{n+1}$ we have $\Delta V_{n+1} < \Delta V_n$.

Finally, value increments to be decreasing it suffices that $\Delta V_2 \leqslant \Delta V_1 = V_1$. First, observe that if $c \geqslant \beta\mathbb{E}(P)(2-\alpha_1)$ then $V_2 = V_1 = 0$, by (2), in which case value increments are obviously not decreasing. By the same logic, if $\beta\mathbb{E}(P) \leqslant c < \beta\mathbb{E}(P)(2-\alpha_1)$, then $n_0 = 1$ and $\Delta V_2 > \Delta V_1 = 0$. To wit, $V_1 > 0$ and so $V_n > 0$. We now restrict to $c < \beta\mathbb{E}(P)$, in which case $n_0 = 0$ by Lemma (1). The value functions for $n = 1, 2$:

$$\begin{aligned} V_1 &= -c + \beta\mathbb{E}(\max(V_1, P)) && (14)\\ V_2 &= -c + \beta\alpha_1\mathbb{E}(\max(V_2, P+V_1)) + \beta(1-\alpha_1)\mathbb{E}(\max(V_2, P+V_1, 2P)) && (15)\end{aligned}$$

Subtract V_1 from (15), so that $\Delta V_2 = V_2 - V_1$ obeys $\Delta V_2 = Q(\Delta V_2)$, where:

$$Q(v) = -c + \beta\alpha_1\mathbb{E}(\max(v,P)) + \beta(1-\alpha_1)\mathbb{E}(\max(v,P,2P-V_1)) - (1-\beta)V_1$$

As $Q(v)$ is increasing with $Q'(v) \in (0,1)$ and at $v = V_1$ we obtain $Q(V_1) = V_1 + c(1-\alpha_1) - V_1\alpha_1(1-\beta)$, we have that $V_2 - V_1 \leqslant V_1$ iff $Q(V_1) \leqslant V_1$, which is iff $V_1 \geqslant c(1-\alpha_1)/\alpha_1(1-\beta)$. Exploiting (14) we deduce that this is iff:

$$c \geqslant \alpha_1\beta\mathbb{E}\left(\max\left(P - \frac{f(1-\alpha_1)}{\alpha_1(1-\beta)}, 0\right)\right) \tag{16}$$

Call $\bar{c} > 0$ to the c value that solves (16) with equality. As the left side rises linearly c but the right side falls in c, there a unique value of $\bar{c}$. To wit, $V_2 - V_1 \leqslant V_1$ iff $c \leqslant \bar{c}$. Integrating by parts (16) yields (3).□

Proof of Proposition 2: Let $\alpha_m = 1$ and $c < m\beta\mathbb{E}(P)$. We argue inductively throughout the proof. We first show that for $k \geqslant 1$, if (♣) $\Delta V_{mk} \geqslant \Delta V_{mk-1} \geqslant \Delta V_{mk-2} \geqslant \cdots \geqslant \Delta V_{m(k-1)+1}$ then (♠) $\Delta V_{mk} \geqslant \Delta V_{mk+1}$ but $\Delta V_{mk+2} \geqslant \Delta V_{mk+1}$. In this case, we deduce that when $n = mk$ an all-or-nothing pricing strategy is optimal. Then, substracting V_{mk-1} from (1) we obtain that ΔV_{mk} solves $v = J_{mk}(v)$, where

$$J_{mk}(v) = -c + \beta\mathbb{E}(\max(v, P, mP - [V_{mk-1} - V_{m(k-1)}]) - (1-\beta)V_{mk-1}$$

By the same logic, as when $n = mk+1$ it is optimal to sell nothing, one or m units, we obtain that ΔV_{mk+1} solves $v = H_{mk+1}(v)$. As ♣ dictates $\Delta V_{mk} \geqslant \Delta V_{m(k-1)+1}$, then $H_{mk+1}(v) \leqslant H_{mk}(v)$ and so $\Delta V_{mk} \geqslant \Delta V_{mk+1}$. To show that $\Delta V_{mk+2} \geqslant \Delta V_{mk+1}$ we compute the value of the feasible pricing strategy when $n = mk+2$ that uses reservation prices as if holdings were $mk+1$ selling one extra unit when the strategy dictates to sell just one unit; let $\tilde{V}_{mk+2}$ be the value of that strategy. Next, let $\Phi_{n,j} = \mathbb{E}(\max(P - \phi_{n,j}, 0)$ and guess $\phi_{mk+1,1} \leqslant \phi_{mk,m-1}$. We obtain

$$(1-\beta)\tilde{V}_{mk+2} = (1-\beta)V_{mk+1} + \beta(\Phi_{mk+1,1} - \Phi_{mk,m-1}) \tag{17}$$

As $(1-\beta)V_{mk} = -c + m\beta\Phi_{mk,m}$ and $(1-\beta)V_{mk+1} = -c + \beta\Phi_{mk+1,1} + (m-1)\beta\Phi_{mk,m-1}$ we obtain $(1-\beta)\Delta V_{mk+1} = \beta(\Phi_{mk+1,1} - \Phi_{mk,m-1}) + m\beta(\Phi_{mk,m-1} - \Phi_{mk,m})$. Since ♣ implies $\phi_{mk,m-1} \geqslant \phi_{mk,m}$ we obtain $\Phi_{mk,m-1} \leqslant \Phi_{mk,m}$ and so $(1-\beta)\Delta V_{mk+1} \leqslant \beta(\Phi_{mk+1,1} - \Phi_{mk,m-1})$, which in turn yields $V_{mk+2} - V_{mk+1} \geqslant \tilde{V}_{mk+2} - V_{mk+1} \geqslant \Delta V_{mk+1}$, by (17); then, it follows that $\Delta V_{mk+2} \geqslant \Delta V_{mk+1}$

We now show that if (♥) $\Delta V_{mk+i} \geqslant \Delta V_{mk+i-1} \geqslant \Delta V_{mk+i-2} \geqslant \cdots \geqslant \Delta V_{mk+1}$ then $\Delta V_{mk+i+1} \geqslant \Delta V_{mk+i}$, for $2 \leqslant i \leqslant m-1$. We use our previous logic and argue that $V_{mk+i+1} \geqslant \tilde{V}_{mk+i+1}$, where the latter is the value of employing the reservations prices as if holdings were $mk+i$ and sell one extra unit when the strategy dictates to sell just i units. More generally, we assume $\phi_{mk+i,i} \leqslant \phi_{mk,m-i}$. Then,

$$(1-\beta)\tilde{V}_{mk+i+1} = (1-\beta)V_{mk+i} + \beta(\Phi_{mk+i,i} - \Phi_{mk,m-i}) \tag{18}$$

As $(1-\beta)V_{mk+i} = -c + i\beta\Phi_{mk+i,i} + (m-i)\beta\Phi_{mk,m-i}$, we obtain $(1-\beta)\Delta V_{mk+i} = \beta(\Phi_{mk+i,i} - \Phi_{mk,m-i}) + (i-1)\beta(\Phi_{mk+i,i} - \Phi_{mk+i-1,i-1}) + (m-i+1)\beta(\Phi_{mk,m-i} - \Phi_{mk,m-i+1})$. Since ♥ implies $\phi_{mk+i,i} \geqslant \phi_{mk+i-1,i-1}$ we obtain $\Phi_{mk+i,i} \leqslant \Phi_{mk+i-1,i-1}$ and as ♣ implies $\Phi_{mk,m-i} \geqslant \Phi_{mk,m-i+1}$ we obtain $\Phi_{mk,m-i} \leqslant \Phi_{mk,m-i+1}$. Altogether, $(1-\beta)\Delta V_{mk+i} \leqslant \beta(\Phi_{mk+i,i} - \Phi_{mk,m-i})$, which in turn yields $V_{mk+i+1} - V_{mk+i} \geqslant \tilde{V}_{mk+i+1} - V_{mk+i} \geqslant \Delta V_{mk+i}$ and so $\Delta V_{mk+i+1} \geqslant \Delta V_{mk+i}$, by (18).

We now show that $\phi_{m(k+1),m} \leqslant \phi_{mk,m}$. As for $n = mk$ and $n = mk+1$ it is optimal to employ an all-or-nothing pricing strategy, we deduce that $\phi_{mk,mk}$ solves $v = T_k(v)$ where $T_k(v) = \beta\mathbb{E}(\max(P - v, 0)) - [c + (1-\beta)V_{m(k-1)}]/m$. Since $T_{k+1}(v) \leqslant T_k(v)$

we obtain $\phi_{m(k+1),m} \leqslant \phi_{mk,m}$, by fixed point reasoning. We use this to verify that $\phi_{mk+i,i} \leqslant \phi_{mk,m-i}$ for $1 \leqslant i \leqslant m-1$. As $\phi_{mk+i,i} \leqslant \phi_{m(k+1),m}$ and $\phi_{m(k+1),m} \leqslant \phi_{mk,m}$ we obtain $\phi_{mk+i,i} \leqslant \phi_{mk,m}$; as $\phi_{mk,m} \leqslant \phi_{mk,m-i}$ we get $\phi_{mk+i,i} \leqslant \phi_{mk,m-i}$.

Finally, we show $2 \leqslant n \leqslant m-1$ that ($\blacklozenge$) $\Delta V_n \geqslant \Delta V_{n-1} \geqslant \cdots \geqslant \Delta V_2 \geqslant V_1$ then $\Delta V_{n+1} \geqslant \Delta V_n$. In this case, since it is optimal to employ an all-or-nothing pricing strategy, we get $(1-\beta)V_n = -c + \beta n \Phi_{n,n}$ and so $(1-\beta)\Delta V_n = \beta \Phi_{n,n} + \beta(n-1)[\Phi_{n,n} - \Phi_{n-1,n-1}] \leqslant \beta \Phi_{n,n}$. Given that at $n+1$ it is feasible to sell one extra unit using the reservation prices as if holdings were n, we obtain $(1-\beta)\tilde{V}_{n+1} = (1-\beta)V_n + \beta \Phi_{n,n}$. Hence, $\Delta V_{n+1} \geqslant \beta \Phi_{n,n}/(1-\beta) \geqslant \Delta V_n$.

Altogether, $\Delta V_2 \geqslant V_1$ implies $\blacklozenge$ for $n = 3$. As $\blacklozenge$ for n implies $\blacklozenge$ for $n+1$, inductively we obtain $\clubsuit$. The latter implies $\spadesuit$, which in turn implies ($\heartsuit$) for $i = 2$. As $\heartsuit$ for i implies $\heartsuit$ for $i+1$ we obtain $\Delta V_{m(k+1)} \geqslant \Delta V_{m(k+1)-1} \geqslant \cdots \geqslant \Delta V_{mk+1}$. As when $c > 0$ we obtain $\Delta V_2 \geqslant V_1$, by Proposition (1), this completes the proof. For the uncapped demand case ($m \to \infty$), as $(1-\beta)V_1 = -c + \beta \Phi_{1,1}$, and given that ΔV_2 solves $v = -c + \beta \mathbb{E}(\max(v, P, 2P - V_1)) - (1-\beta)V_1$, whose right hand side equals $V_1 + c > V_1$ when $v = V_1$ we obtain $\Delta V_2 > V_1$. That is, $\Delta V_{n+1} > \Delta V_n$ for all $n \in \mathbb{N}$.$\square$

Proof of Corollary 1: For low inventory $n \leqslant m$, as ΔV_{n-i+1} falls in $i \in \{1, \ldots, n\}$, by Proposition 2, the marginal cost of selling is decreasing. To wit, the seller optimally sells all of his inventory as long as the price exceeds the average selling cost; hence $\mathcal{R}_{n,n} = \phi_{n,n}$. Next, suppose $n \in \Omega_m(k)$ and $k \geqslant 2$. If $m \leqslant mk-1$ then ΔV_{n-i+1} falls in $i \in \{1, \ldots, n - m(k-1)\}$ and $\Delta V_{m(k-1)} > \Delta V_{m(k-1)+1}$, but ΔV_{n-i+1} falls again in $i \in \{n - m(k-1) + 1, \ldots, m\}$, by Proposition 2. To wit, the marginal cost of selling falls for the first $n - m(k-1)$ units sold, rises for the $n - m(k-1) + 1$ unit, and falls again if more units are sold. Then, the offered bundles size are $j = n - m(k-1)$ and m, with reservation prices equal to the average cost $\mathcal{R}_{j,n} = \phi_{n,j}$ and to the incremental average cost $\mathcal{R}_{m,n} = \phi_{m(k-1),m-j} \geqslant \mathcal{R}_{j,n}$, respectively. Finally, if $n = mk$ then as ΔV_{n-i+1} falls in $i \in \{1, \ldots, m\}$ by Propositon 2, the seller optimally sells all of his inventory as long as the price exceeds the average selling cost; hence $\mathcal{R}_{m,n} = \phi_{n,m}$.$\square$

Proof of Proposition 3: If $\bar{c} < c < \beta \mathbb{E}(P)(1 + \alpha_2)$ then $n_0 = 0$ or $n_0 = 1$, by Lemma 1. Let $n \geqslant 2$. We first show that ($\heartsuit$) if $\Delta V_n > \Delta V_{n-1}$ then $\Delta V_{n+1} < \Delta V_n$. Subtracting V_{n-1} from (1), ΔV_n solves $v = H_n(v)$:

$$H_n(v) = -c + \beta \alpha_1 \mathbb{E}(\max(v, P)) + \beta \alpha_2 \mathbb{E}(\max(v, P, 2P - \Delta V_{n-1})) - (1-\beta)V_{n-1} \quad (19)$$

If $\Delta V_n > \Delta V_{n-1}$, as $V_n \geqslant V_{n-1}$ then $H_{n+1}(v) < H_n(v)$ and so $\Delta V_{n+1} < \Delta V_n$. Furthermore, as $H_n(v)$ is increasing, then $\Delta V_{n+2} > \Delta V_{n+1}$ iff $H_{n+2}(\Delta V_{n+1}) > \Delta V_{n+1}$. We now find the conditions that guarantees that $\Delta V_{n+2} > \Delta V_{n+1}$. Exploiting (1), we deduce

$$(1-\beta)V_{n+1} = -c+\beta\alpha_1\mathbb{E}(\max(\Delta V_{n+1}, P)) + \beta\alpha_2\mathbb{E}(\max(\Delta V_{n+1}, P, 2P-\Delta V_n)) - \beta\Delta V_{n+1}$$

which in turn yields that $H_{n+2}(\Delta V_{n+1}) > \Delta V_{n+1}$ iff:

$$(1-\beta)\Delta V_{n+1} < \beta\alpha_2[\mathbb{E}(\max(\Delta V_{n+1}, P, 2P-\Delta V_{n+1})) - \mathbb{E}(\max(\Delta V_{n+1}, P, 2P-\Delta V_n))] \quad (20)$$

Write $\mathbb{E}(\max(\Delta V_{n+1}, P, 2P-\Delta V_{n+1})) = \Delta V_{n+1} + 2\Phi_{n+1,1}$. Since $\Delta V_{n+1} < \Delta V_n$, we have that $\mathbb{E}(\max(\Delta V_{n+1}, P, 2P-\Delta V_n)) = \Delta V_{n+1} + \Phi_{n,1} + \Phi_{n+1,1}$. By (20), $H_{n+2}(\Delta V_{n+1}) > \Delta V_{n+1}$ iff $(1-\beta)\Delta V_{n+1} < \beta\alpha_2[\Phi_{n+1,1} - \Phi_{n,1}]$. Next, let $J_n(v) \equiv [\mathbb{E}(\max(P-v, 0)) - \Phi_{n,1}]$; as $J_n(v)$ is decreasing there is a unique solution $\psi(\Delta V_n)$ for $(1-\beta)v = \beta\alpha_2 J_n(v)$. To wit, $H_{n+2}(\Delta V_{n+1}) > \Delta V_{n+1}$ iff $\Delta V_{n+1} < \psi(\Delta V_n)$. On the other hand, as $\Delta V_{n+1} < \Delta V_n$ we have that ΔV_{n+1} solves $(1-\beta)v = K_n(v) = -c-(1-\beta)V_n + \beta(\mathbb{E}(\max(P-v, 0)) + \alpha_2\Phi_{n,1})$, by (19). Furthermore, since $(1-\beta)V_n = -c+\beta\alpha_1\Phi_{n,1} + 2\beta\alpha_2\Phi_{n,2}$ then $K_n(v) = \beta[\mathbb{E}(\max(P-v, 0)) - \Phi_{n,1} - 2\alpha_2(\Phi_{n,2} - \Phi_{n,1})]$. As $J_n(v)$ and $K_n(v)$ are decreasing, by fixed point logic we deduce that $\Delta V_{n+1} < \psi(\Delta V_n)$ iff $K_n(\psi(\Delta V_n)) < \beta\alpha_2 J_n(\psi(\Delta V_n))$, which is iff $\alpha_1 J_n(\psi(\Delta V_n)) < 2\alpha_2(\Phi_{n,2} - \Phi_{n,1})$. Altogether, ($\diamondsuit$) if $\Delta V_n > \Delta V_{n-1}$ then $\Delta V_{n+1} < \Delta V_n$ and also $\Delta V_{n+2} > \Delta V_{n+1}$ iff $\alpha_1 J_n(\psi(\Delta V_n)) < 2\alpha_2(\Phi_{n,2} - \Phi_{n,1})$.

We now show that if the value difference falls twice if falls a third time and thus continuously fall; that is ($\bigstar$) if $\Delta V_n < \Delta V_{n-1} < \Delta V_{n-2}$ then $\Delta V_{n+1} < \Delta V_n$. By (19), ΔV_n and ΔV_{n+1} solve $v = H_n(v)$ and $v = H_{n+1}(v)$, respectively. We now compute the difference $\Delta H_{n+1} = H_{n+1}(v) - H_n(v) = \beta\alpha_2[\mathbb{E}(\max(v, P, 2P-\Delta V_n)) - \mathbb{E}(\max(v, P, 2P-\Delta V_{n-1}))] - (1-\beta)\Delta V_n$. As $\Delta V_n < \Delta V_{n-1}$, the difference $\mathbb{E}(\max(v, P, 2P-\Delta V_n)) - \mathbb{E}(\max(v, P, 2P-\Delta V_{n-1}))$ falls in v and so $\Delta H_{n+1} \leqslant \beta\alpha_2[\mathbb{E}(\max(P, 2P-\Delta V_n)) - \mathbb{E}(\max(P, 2P-\Delta V_{n-1}))] - (1-\beta)\Delta V_n = \beta\alpha_2[\Phi_{n,1} - \Phi_{n-1,1}] - (1-\beta)\Delta V_n$. Now, as $\Delta V_n < \Delta V_{n-1} < \Delta V_{n-2}$ we write $(1-\beta)\Delta V_n = \beta\alpha_1[\Phi_{n,1} - \Phi_{n-1,1}] + \beta\alpha_2[\Phi_{n,1} - \Phi_{n-2,1}]$, by (19) and since $(1-\beta)V_{n-1} = -c + \beta\Phi_{n-1,1} + \beta\alpha_2\Phi_{n-2,1}$, by (1). That is, $\Delta H_{n+1} < \beta\alpha_1[\Phi_{n-1,1} - \Phi_{n,1}] + \beta\alpha_1[\Phi_{n-2,1} - \Phi_{n-1,1}] < 0$, which yields $\Delta V_{n+1} < \Delta V_n$.

Now, assume $\Delta V_n > \Delta V_{n-1}$. Suppose $\alpha_1 J_n(\psi(\Delta V_n)) \geqslant 2\alpha_2(\Phi_{n,2} - \Phi_{n,1})$. Then, by $\diamondsuit$ we have $\Delta V_{n+1} < \Delta V_n$ and $\Delta V_{n+2} \leqslant \Delta V_{n+1}$ and so $\Delta V_{n+1+j} < \Delta V_{n+j}$ for all $j \geqslant 0$, by $\bigstar$. To wit, $n_1 = n$. Otherwise, if $\alpha_1 J_n(\psi(\Delta V_n)) < 2\alpha_2(\Phi_{n,2} - \Phi_{n,1})$, then $\Delta V_{n+1} < \Delta V_n$ and $\Delta V_{n+2} > \Delta V_{n+1}$ and $\Delta V_{n+3} \leqslant \Delta V_{n+2}$, by $\bigstar$. That is, $n_1 = n+2$. Finally, since $\bar{c} < c < \beta\mathbb{E}(P)(1+\alpha_2)$, then $\Delta V_2 > \Delta V_1$ and so $\Delta V_n > \Delta V_{n-1}$ and $\Delta V_{n+1} < \Delta V_n$ for all even numbers $2 \leqslant n \leqslant n_1$. The existence of n_1 is guaranteed by the boundedness of V_n, while uniqueness, by $\bigstar$.$\square$

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