Delay and Learning in Coordination*

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Abstract

A continuum of agents with heterogenous private information have an option to delay a fixed size investment that pays off only when the aggregate investment exceeds some imperfectly known level. The delay option enables the agents to observe a binary signal depending on whether the mass of early actions surpasses a threshold. A critical difference from previous studies is the non-linearity of this information process. In equilibrium, the benefit of this information is balanced by the cost of delay. The results depend on the discount rate and on the critical mass for the signal. If the discount rate (or the period length) is small, there is less coordination than in the one period (static) case, and coordination takes place in the first period. The opposite is true when the discount rate is large. However, in that case, when the heterogeneity of agents (measured by the variance of their private signal) is sufficiently small, there is no stable equilibrium in monotone strategies. This property is indicative of the difficulty that agents may have in the coordination of actions with strategic complementarities in a multi-period context. We further discuss extensions to an arbitrary number of periods and some implications in macroeconomics, finance and political stability.

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1 Introduction

This paper analyzes, in a theoretical model with a large number of agents, the coordination problem of making once a fixed size "investment," with the option to delay, that pays off only if the total mass of investment exceeds some imperfectly known level, called the fundamental. The delay option enables the agents to observe a binary signal depending on whether the mass of early actions surpasses a threshold. The process of learning from the observation of others generates a strong non-linearity in the precision of the information that is provided by this public signal, *i.e.*, in the signal-to-noise ratio.

To put the framework of the paper in context, one should expect that the quality of the information revealed by others' actions increases with their level of activity. (Trivially, if there is no activity, *ex ante*, there is nothing to learn.) (Vives, 1993) was the first one to analyze this property in a simple model with a signal-to-noise ratio that increases with the level of endogenous activity: the publicly observed signal is the sum of the endogenous activity, which depends on the fundamental parameter, and of the exogenous noise with a fixed amplitude.

A stylized model of a business cycle that exhibits a similar mechanism has been presented by Van Nieuwerburgh and Veldkamp (2006). The fundamental, which determines the payoff of an individual's action/investment, changes evolves between two different states according to a Markov process. The recovery from a through in the business cycle is slow because, due to the low level of activity, agents are slow to recognize a good state of the fundamental. On the contrary, at the top of the cycle, a switch of the fundamental to the low state is recognized immediately because of the high signal-to-noise ratio and the crash is sudden.

The activity that is subject to observation can also be the absence of investment. In the model of Caplin and Leahy (1994*a*), where agents have an option to suspend their activity, a high level of activity reveals no information for some time because agents with a negative information on the fundamental are herding. Eventually, a sufficient mass of agents accumulate a string of negative signals have a private belief that is sufficiently strong to overcome the public belief. Their withdrawal from activity breaks the herd and may precipitate a crash.

In the model of Chamley and Gale (1994), the individual signal is the option to make a fixed size investment. For delaying agents the observation of a high level of economic activity is good news because the number of agents is an increasing function of the fundamental and therefore of the expected payoff. In equilibrium, the cost of delay, which is an increasing function of the expected payoff must be equal to the value of information through delay. When optimism rises, the propensity to delay decreases, investment rises, and, by intertemporal arbitrage, the value of the information that is obtained from observation also increases. In this model, as in the previous one, the interaction between actions and information generates a strategic substitution between agents in their willingness to invest. More investment by others increases the incentive for delays and reduces one's willingness to invest.

The model in (Chamley, 2004) is similar to the previous one with the difference that the mass of players is fixed. All agents have an option to invest and they are differentiated by a private signal which is good or bad. In that setting, no delay by all agents is an equilibrium, in addition to the previous one, since it conveys no information and delay is costly. Therefore, information externalities, in the absence of payoff externalities, may generate both strategic substitutability and complementarity.

Strategic complementarities that are due to payoff externalities have been the subject of vast number of analyses. From a theoretical point of view, the main issue is the multiplicity of equilibria. A "selection mechanism" seems to be indispensable for a policy analysis. In this respect, a method that has been very popular for the last thirty years is the removal of the assumption of common knowledge in a global game, (Carlsson and Van Damme, 1993), where individual observations are subject to independent idiosyncratic noise. So far, this method has been applicable only in static games, with one period.[1] In most multi-period models with strategic complementarity and learning, the observation in the first period truncates the public distribution of the fundamental, removing an essential assumption in the global game method, after which there are multiple equilibria (Angeletos, Hellwig and Pavan, 2007).

In the multi-period model of strategic complementarity with learning of Chamley (1999), there is a unique equilibrium, which is rationalizable. The fundamental evolves over time, randomly and by small steps, and agents observe at the end of each period the aggregate activity. There is no delay and agents take a zero-one action. In the equilibrium, which is rationalizable and therefore unique, the aggregate activity goes through random cycles and in most periods the mass of acting agents is either small or large, and the observation of the aggregate provides a weak signal. There is a strong non-linearity in the information mechanism which is similar to the one in (Vives, 1993). That non-linearity arises because of a feature that is found in many observation settings. In a boom

[1] Of course, one can present a multi-period model as a sequence of one period model with a new public information in each period.

or a trough, the aggregate activity is driven by the agents in the tail of the distribution of private signals. For example, set the aggregate activity to be in the form of $X(\hat{s}, \theta) = F(\theta - \hat{s}) + \epsilon$, where θ is the fundamental, F the cdf of the private signals, and ϵ a small noise so that the observation of Y does not provide perfect information on θ. In equilibrium, the agents with a signal greater than $\hat{s}$ take action 1, "invest". The aggregate activity provides a signal on θ. In a boom, $\hat{\theta}$ is large. The investing agents are in the upper tail of the distribution, and–this is the essence of the mechanism–the value of the fundamental has a small impact on the mass of investing agents. In this case, the aggregate variable X provide little information on the fundamental.

The previous mechanism arises in model where agents have an imperfect signal on the fundamental and where, in equilibrium, the margin between acting and non-acting agents is in the tail of that distribution of the private signals. Such a property is likely to hold in most models of strategic complementarity. Because of the payoff externality, in an equilibrium, the mass of acting agent is either large or small. Payoff complementarity, when sufficiently large, generates strategic complementarity and a non-linear response to the actions of others. (In a simple canonical model with common knowledge, the response is either 1 or 0.) When there is learning from others, this non-linear response in actions generates a strong non-linearity in the information that is obtained through the observation of others. Such a non-linearity is a key mechanism in the present paper.

The non-linearity of the information from the observation of others is suppressed in the model of Dasgupta (2007) because, using the previous notation, the public signal after the first period is $G(F(\theta - \hat{s}))$, where G is set explicitly to neutralize the non-linearity by choosing it to be the inverse function of F. Such an inverse function multiplies the small variations of Y when the margin for action is in the tail of the distribution. The impact of the function $G \equiv F^{-1}$, is to transform the activity in the first period, no matter how small it can be, into a linear signal on θ. An argument for this choice of the function G is that the model makes also the assumption that the observation of aggregate activity in the first period is nearly perfect. In this case, the transformation of the actual activity through the function G, may not matter, asymptotically.

In this paper, the signal that transforms the aggregate action into information has the property that it is equal to 0 if that aggregate is less than some threshold. This embodies the realistic property that a small level of endogenous activity, which carries information, is drowned by noise and measurement errors. We think that this non-linearity of the information process is critical in models of delay and payoff strategic complementarity. In sharp contrast to the linear model of information, our study demonstrates the inability of delay option in promoting coordination, when the discount rate is sufficiently small. This finding is largely consistent with experimental evidence provided by Jin, Zhou and Brandenburger (2021).[2]

The properties of the model are also related to (Angeletos, Hellwig and Pavan, 2006). In their model, there is no delay. In each period, a new game is played with learning from past periods' outcomes about the fundamental and additional private signals. Agents cannot observe the level of investment as long as coordination has not been successful, in which case the game ends. In the model here, agents observe a zero-one signal Y that is equal to one if the mass of investment is greater than the exogenous parameter γ. The outcome of the game in the first period truncates the distribution of the fundamental in the public information and this leads to the possibility of multiple equilibria after the first period. The model that is presented here is a special case that model in which agent get no new private information after the first period and, most importantly, play only one game with delay over multiple periods. A more detailed comparison is made in the paper, but one can already point out two cases. First, when the discount rate is small, all the uncertainty is resolved after the first period, with the success or the failure of coordination. When the discount rate is sufficiently large and the degree of heterogeneity in private information sufficiently small (a case where there are multiple equilibria in (Angeletos, Hellwig and Pavan, 2007)), there is no stable equilibrium with a non-random monotone strategy in the game with delays.

Let us sketch the argument of the paper. Following related works, let us assume a continuum of agents, of mass one, each with an option to make a fixed size investment, at a fixed cost c, ($0 < c < 1$), in one of two periods. Coordination is successful when, at the end of the game, the mass of investment, X, is greater than $1 - \theta$, where θ is the "fundamental". When coordination is successful, each agent who has "invested" receives a gross payoff of 1. The payoff of investment in the second period is reduced by a discount factor. In a one period setting, the multiplicity of equilibria that arises under common knowledge when $0 < \theta < 1$, is resolved when the common knowledge assumption is replaced by the imperfect information of individuals about the fundamental θ. As in these previous works, we will assume that this imperfect information takes the form of individual signals $s_i = \theta + \sigma\epsilon_i$, where ϵ_i is a conditionally independent noise.

Assume that the strategy in the first period is monotone: for some $\hat{s}_1$, agents with a signal $s_i \geq \hat{s}_1$ invest without delay. The opportunity cost of delay is the expected payoff multiplied by the rate of discount. The benefit of delay arises from avoiding to pay the investment cost in the

[2] Jin, Zhou and Brandenburger (2021) study the option of delay in a complete-information coordination game with no delay cost. They provide experimental evidence to show that the delay option favors coordination on the reversible action (no invest) but impairs coordination on the irreversible action, and a theory (with agents who hold other-regarding preferences) built on iterated weak dominance to explain this finding. The experimental evidence is largely consistent with our theory by taking the delay cost vanishingly small.

first period when coordination eventually fails. For the marginal agent with signal $\hat{s}_1$, the two are equal. For an agent with a higher value of s, the opportunity cost is higher since he is more confident that the fundamental θ has a high value, and that coordination will succeed.

The sub-game in period 2 depends on the observation of Y at the end of the first period. Assume that if $Y = 0$, which is bad news about the fundamental, there is no more investment (a property that will be proven in the analysis). When $Y = 1$ more agents invest in period 2. There are two possibilities.

When the parameter γ is higher than γ^*, the observation $Y = 1$ takes place only if aggregate investment in the first period is high. $Y = 1$ is also a signal of a high value of the fundamental (that is necessary to generate that investment). Since both investment and fundamental are high, the first period investment is sufficient to generate coordination. The same property holds when for given γ, the discount rate is sufficiently low. The cost of delay for information is low and given the *ex ante* low propensity to invest in the first period, the realization $Y = 1$ the property of the model occurs under the same condition as when γ is sufficiently large. All remaining players, having learned that $Y = 1$ implies coordination, invest in the second period, but they are like free riders on the early investors. Coordination is based only on the first period investors, who have a signal higher than $\hat{s}_1$, a value that is higher than the strategy of the one period game. Because the set of signals for investment in the first period is smaller than in the static game, and agents in the second period are irrelevant for achieving a successful coordination, that success requires a higher value of the fundamental than in the static game. The option for delays makes the success of coordination less likely.

The second case takes place when $\gamma < \gamma^*$, or, equivalently, when the discount rate is higher than some threshold. Here, a successful coordination requires the contribution of some investors in the second period, only after $Y = 1$ (as in the previous case), with a signal greater than $\hat{s}_2$ (with, obviously, $\hat{s}_2 < \hat{s}_1$). That value is lower than the cutoff point in the one period model. Consequently, the minimum value of the fundamentals with coordination is lower than in the one period model. The option for delay facilitates coordination.

Since the financial market moves fast and speculative attacks take place in a relatively short time, the discount rate can be very low in these applications.That strong impact of the option for delay can be illustrated by the application of the model to the stylized description of a speculative attack that has been used by Morris and Shin. Assume that the cost of speculation is 10 percent of the gross payoff after a successful attack. If the crisis takes two weeks, a long time for a crisis, and the annual interest rate is 24 percent, a very high value (see the case of Sweden in 1992), $\gamma^* = 0.09$. Suppose now that agents notice the bank run when at least 10 percent of them run to the bank.

The case $\gamma > \gamma^*$ applies. We will see that in this example, the bank should keep a level of reserves that is no more than 10 percent of the deposits (instead of the 90 percent ratio of the static case).

Applying our theory to the investment game, the dynamic model also provides a novel channel through which a policy of low interest rate can work against stimulating investment and economic recovery. In a low interest rate environment, the discount rate is low and therefore the investors have a higher incentive of waiting for more information. As discussed, the inaction in the early period makes the "good news" $Y = 1$ more difficult to be generated and, therefore, makes the coordination on investing more difficult to achieve.

Outline The remainder of the paper is organized as follows. Section 2 presents our model and we solve the unique equilibrium for any given parameter choice in Section 2.2. In Section 3, we discuss how the learning opportunity and the delay cost affects the impact of the delay option on facilitating coordination. Equilibrium selection and policy implications are discussed in Section 4. The model is extended to more than two periods in Section 5.

2 Baseline Model

2.1 Model Setup

We first consider a two-period setting ($t = 1, 2$) in which agents can delay their choices at $t = 1$, and make more informed decisions after learning from others' actions in the first period.

Players and Actions There is a unit mass of risk neutral agents, indexed by $i \in [0, 1]$. At the beginning of the first period, each agent is endowed with one option to make a fixed size irreversible investment, which is normalized to one, in one of the two periods. If an agent chooses to invest, we write agent i's time of investment as $t_i \in \{1, 2\}$.

Payoffs By a standard abuse of notation, the mass of agents who invest is the same as the mass of investment. Formally, we write the mass of investment at $t \in \{1, 2\}$ as $X_t = \int_{i \in [0,1]} \mathbb{1}\{t_i = t\} \mathrm{d}i$. The payoff of investment depends on the total investment $X := X_1 + X_2$ at the end of period 2 and on a "fundamental" variable, θ. To model the strategic complementarity, the expected payoff is an increasing function of the total investment X, and, by convention, an increasing function of the fundamental θ. These properties are satisfied by the canonical representation that we take. If there is no uncertainty on θ or the actions of agents, the payoff of an agent i who invests in period t_i is

$$U(t_i, \theta, X) = \delta^{t_i - 1}(Z - c) \text{ with } Z = \begin{cases} 1, & \text{if } X \geq 1 - \theta, \\ 0, & \text{if } X < 1 - \theta, \end{cases} \tag{1}$$

where c the fixed cost of investment, and $\delta < 1$ is the discount factor between periods, which generates a cost of delay.

This canonical representation could apply to an economy with agents make investments in an economy with strength θ. In other applications, the action can be labelled as "speculative attack against a currency" whereas $1 - \theta$ represents the level of reserves of a central bank which has to devalue if the speculative attack exceeds these reserves (Morris and Shin, 1999) or the reverse a bank holds ; or "run on a bank" whereas $1 - \theta$ measures the bank's reserve holding (Rochet and Vives, 2004); or "attack a political regime" whereas θ represents the weakness of the political regime (Chamley, 1999).

When $X \geq 1 - \theta$, we say that coordination is successful. The investment return $Z = 1$ is realized only when the coordination is successful, that is, when sufficiently many other investors choose to invest and/or economic condition (i.e., fundamental θ) is sufficiently strong. The investment opportunity is profitable as the cost of investment $c \in (0, 1)$. Delayed investment is costly since if the coordination is successful, the payoff from investing earlier is strictly higher –i.e., $1 - c > \delta(1 - c)$. (This cost of delay could also be represented by a cost of investment increasing with time.)

Exogenous Information As in many studies since Carlsson and Van Damme (1993), we assume that agents have imperfect information on θ. Here, they have a common prior of θ, which is diffuse (uniformly distributed) over the real line.[3] In addition, each agent i is endowed at the beginning of the first period with a private noisy signal s_i on the fundamental θ:

$$s_i = \theta + \sigma \epsilon_i,$$

where the noise terms ϵ_i have conditionally independent standard normal distribution $N(0, 1)$, and σ scales the noise.

It may be useful to recall that a high signal s_i is a "good" signal in the sense that it shifts the subjective distribution of θ to the right and may be an indication that a successful coordination is more likely.

[3]One can interpret this prior distribution as normal $\theta \sim N(\theta_0, \sigma_0^2)$, in which $\sigma_0 \to \infty$.

With the diffused prior, an agent with private signal s_i has a subjective posterior distribution on θ with a mean s_i and a cumulative distribution function $F(\theta - s_i)$, where F is the CDF of the Gaussian distribution with mean zero and variance σ^2. Most of the results of the paper do not depend on σ. That property is not surprising. The model belongs to the class of global games, where the equilibrium properties depend on the ratio between the precision of the individual signals, on the fundamental, and the precision of the prior distribution. Here, because of the diffuse prior, that ratio is equal to zero for any value of σ. If convenient, the reader may assume that F is the CDF of the standardized distribution. The density associated to $F(\cdot)$ will be denoted by $f(\cdot)$. Given the diffuse prior on θ, for any given θ, the mass of agents who have a private signal lower than s_i is $F(s_i - \theta)$. By symmetry of the normal distribution, $F(s_i - \theta) = 1 - F(\theta - s_i)$. This property will simplify the technical analysis.

Endogenous Information Agents who delay and do not invest in the first period get some imperfect information on the aggregate investment in the first period, X_1 in the form of a public signal, Y, that depends on X_1. Vives (1993) has emphasized that the precision of such a signal may be poor when the activity X_1 is small. For example, when Y is the sum of X_1 and a noise, (the case considered by Vives), small values of X_1 are dwarfed by the noise. We incorporate this property by assuming that if X_1 is smaller than some parameter γ, there is no signal, that means, $Y = 0$. When $X_1 > \gamma$, then, by assumption, $Y = 1$. That signal Y is imprecise in both cases when X_1 is in a low range, which can be very low, and in the high range. The second property is not important here, essentially because investment is irreversible.[4] Following this discussion, it is assumed that agents observe at the end of the first period a public signal $Y \in \{0, 1\}$ which is equal to one if and only if the mass of investment in the first period exceeds some threshold level $\gamma \in (0, 1)$:

$$Y = \begin{cases} 0, \text{ if } X_1 < \gamma, \\ 1, \text{ if } X_1 \geq \gamma. \end{cases} \tag{2}$$

This step function can also be motivated by discrete policies, *e.g.*, the raising of the Central Bank's discount rate to defend the currency, the minimum size of a demonstration to create a news event. Aside from this public signal Y and private signal s_i, we assume agents cannot receive any other information regarding the fundamental θ or the history X_1.

Monotone Strategies Throughout the paper, we will restrict our attention to monotone strategies and use the *Perfect Bayesian Equilibrium in monotone strategies* as our solution concept. Recall

[4]In the model of Caplin and Leahy (1994*b*), the poor information when investment is high generates, for fixed fundamental, a sudden crash.

that a high value of the fundamental θ facilitates a successful coordination and a positive payoff of investment. Hence, a high value of the private signal s_i is good news for investment. Accordingly, we focus on monotone strategies that are defined by q triplet $(\hat{s}_1, \hat{s}_2^0, \hat{s}_2^1)$. An agent who plays this strategy invests at $t = 1$ if $s_i \geq \hat{s}_1$, and delays if $s_i < \hat{s}_1$. The values of s_2^0 and $\hat{s}_2^1$ define the thresholds for investment (by delaying agents) in period 2 after observing $Y = 0$ or 1, respectively. That is, this agent invests after observing Y at $t = 2$ if and only if $s_i \geq \hat{s}_2^Y$ for $Y = 0, 1$. As investments are irreversible, $s_2^Y \leq \hat{s}_1$.

2.2 Preliminaries

The static model with no delay option is a special case with the discount factor $\delta = 0$. It will provide a benchmark to assess whether the option of delay facilitates coordination.

Static Benchmark

If agents play a monotone strategy and invest with a signal greater than $\hat{s}$, the aggregate investment is

$$X(\theta, \hat{s}) = \mathbb{P}[s_i \geq \hat{s} | \theta] = F(\theta - \hat{s}). \tag{3}$$

From the definition of the payoff in (1), coordination succeeds if and only if $F(\theta - \hat{s}) \geq 1 - \theta$, or equivalently, $\theta \geq F(\hat{s} - \theta)$. Define the function $\Theta(s)$ by the solution of the equation

$$\Theta(s) = F(s - \Theta(s)). \tag{4}$$

The solution $\Theta(s)$ is unique for any s and the function $\Theta(s)$ is strictly increasing in s. Coordination is successful if and only if $\theta \geq \Theta(\hat{s})$.

We have seen that an agent with marginal signal $\hat{s}$ has a subjective probability distribution on θ with the CDF $F(\theta - \hat{s})$. Therefore, if other agents are playing the monotone strategy $\hat{s}$, i.e., investing if and only if $s_i \geq \hat{s}$, then for agent $\hat{s}$, the probability of successful coordination ($\theta \geq \Theta(\hat{s})$) is equal to

$$\mathbb{P}[\theta \geq \Theta(\hat{s}) | \hat{s}] = F(\hat{s} - \Theta(\hat{s})) = 1 - F(\Theta(\hat{s}) - \hat{s}) = \Theta(\hat{s}).$$

Lemma 1. *If agents play the strategy $\hat{s}$ in the one-period model, for an agent with signal $\hat{s}$, the probability of a successful coordination is equal to $\Theta(\hat{s}) = \mathbb{P}[\theta \geq \Theta(\hat{s}) | \hat{s}]$, solution of (4).*

By Lemma 1, when agents follow the strategy $\hat{s}$, the function $\Theta(\hat{s})$ has a *double* interpretation: (1) it measures both the minimum value of θ for which coordination succeeds, and (2) for the marginal agent with signal $\hat{s}$, the probability that coordination succeeds (or $\theta \geq \Theta(\hat{s})$). That double interpretation will play an important role later.

The equilibrium is then characterized by the arbitrage condition for the marginal agent with signal $\hat{s}$. For this agent, the expected payoff from investing is 0, the same as that from not investing;

$$\mathbb{P}[\theta \geq \Theta(\hat{s})|\hat{s}] \times 1 - c = \Theta(\hat{s}) - c = 0.$$

The next lemma summarizes the unique equilibrium (s^*, θ^*) for the static game. It restates a well-known result in the global game literature, to serve as a comparison benchmark.[5]

Lemma 2 (Static Benchmark)**.** *In the benchmark case with one period, the equilibrium strategy is unique, to invest if $s > s^*$ with $s^* = \Theta^{-1}(c)$, and $\Theta(\cdot)$ defined in (4). In this equilibrium, coordination is successful if and only if $\theta \geq \theta^* = c$.*

It is well known that the benchmark model belongs to class of models that have a strongly rationalizable equilibrium, and that the strategy s^* is the unique strategy when there is no monotone restriction on strategies. The purpose of the discussion before Lemma 1 was only to characterize the equilibrium under the assumptions of Section 2.1.

We will see later that the coordination outcomes are different in the one-period setting and in the two-period setting with a discount factor equal to 0.

A property of the variable Y

Before the analysis of the equilibrium with two periods, we need to investigate the properties of the signal Y that affects the agents' choices. The information content of Y depends on the play of the agents at $t = 1$. Recall that $\hat{s}_1$ denotes the strategy in the first period. $Y = 1$ when the aggregate investment in period 1 is greater than γ, which is equivalent to

$$X_1(\theta, \hat{s}_1) = \mathbb{P}[s_i \geq \hat{s}_1|\theta] = F(\theta - \hat{s}_1) \geq \gamma,$$

or, equivalently, $\theta \geq \Theta_\gamma(\hat{s}_1)$ where $\Theta_\gamma(\cdot)$ is defined as

$$\Theta_\gamma(\hat{s}) := \hat{s} + F^{-1}(\gamma). \tag{5}$$

[5]See, for example, among others, Morris and Shin (2003). A standard sufficient condition of the uniqueness of the equilibrium is that the ratio between the variances of the private signal and of the prior should be lower than some bound. Here, that ratio is zero and the case is technically equivalent to a vanishingly small variance of the private signal in the standard global game framework.

The observation $Y = 1$ reveals that $\theta \geq \Theta_\gamma(\hat{s}_1)$: the distribution of the fundamental θ is truncated on its left-hand side, below $\Theta_\gamma(\hat{s}_1)$. Because of this, the posterior belief about θ is shifted to the right, and agents become more optimistic on θ, and therefore on a successful coordination: $Y = 1$ is good news, whereas $Y = 0$, which truncates the right tail of the distribution and induces a shift to the left, is bad news.[6]

In addition, for the marginal agent with private signal $\hat{s}_1$, the probability that $Y = 1$ (i.e., $\theta \geq \Theta_\gamma(\hat{s}_1)$), is equal to

$$\mathbb{P}\left[Y = 1|\hat{s}_1\right] = \mathbb{P}[\theta \geq \Theta_\gamma(\hat{s}_1)|\hat{s}_1] = F(\hat{s}_1 - \Theta_\gamma(\hat{s}_1)),$$

which by the definition in (5), is equal to $1 - \gamma$.[7]

Lemma 3. *If all agents take the monotone strategy $\hat{s}_1$ at $t = 1$, the public signal $Y = 1$ ($Y = 0$) is generated when $\theta \geq \Theta_\gamma(\hat{s}_1)$ ($\theta < \Theta_\gamma(\hat{s}_1)$), where $\Theta_\gamma(\hat{s}_1)$ is defined in (5) and is an increasing function of $\hat{s}_1$ and γ. In addition, for the marginal agent with private signal $\hat{s}_1$, the probability of $Y = 1$ is $1 - \gamma$.*

Remark 1. *If $\hat{s}_1$ is higher, the realization $Y = 1$ requires a higher value of the fundamental θ. Therefore, $Y = 1$ is less likely ex-ante but it generates a higher jump of optimism. Likewise if γ is higher for a fixed $\hat{s}_1$.*

Thus far, we understand that $Y = 0$ is a bad signal, which makes agents less optimistic about the coordination success. In view of this, it is intuitive that the observation $Y = 0$ has a negative impact on investment in period 2. The next lemma shows that, after $Y = 0$, in an equilibrium, there is no investment.

The game stops with no investment after the bad news $Y = 0$

Any delaying agent cannot invest at $t = 2$ both after the observations $Y = 1$ and $Y = 0$. Such a strategy, which is equivalent to delaying with a commitment to invest in period 2, without regard to the realization of Y, yields a payoff that is equal to the discounted value of the first period investment. Because of the discounting, it is strictly dominated by no delay. Therefore, in an equilibrium, at least one of the two values, $\hat{s}_2^1$ and $\hat{s}_2^0$ must be equal to $\hat{s}_1$.

[6] It is worth noting that, as $\theta \in (-\infty, +\infty)$, for any possible equilibrium threshold $\hat{s}_1$, both signals $Y = 1$ and $Y = 0$ will be on path. Therefore, in any possible equilibrium, agents can always apply the Bayes rule to determine their posterior beliefs about θ.

[7] This is often referred to as the *Laplacian belief* of the marginal agent about the aggregate action of others in the global game literature (see, for example, Morris and Shin (2003)), which says that, if all others play the monotone strategy $\hat{s}_1$, for the marginal agent with $\hat{s}_1$ believes, the aggregate action X_1 is uniformly distribution over $[0, 1]$.

Lemma 4. *In any equilibrium with monotone strategy $\hat{s}_1$ in the first period, and $\hat{s}_2^Y$ in the second period, after the observation of Y,* $\max\{\hat{s}_2^0, \hat{s}_2^1\} = \hat{s}_1$.

That argument is the same as in Chamley and Gale (1994) for an environment without payoff externalities. In that model, the game stops when the news is not sufficiently good. In the present model of dynamic coordination with payoff externalities, one cannot rule out *a priori* that the observation $Y = 0$ could be a coordination device for more investment (i.e., $\hat{s}_2^0 < \hat{s}_1$). This case is ruled out by the next Lemma.

Lemma 5. *In any monotone equilibrium, $\hat{s}_2^0 = \hat{s}_1$: there is no investment after the bad news $Y = 0$.*

The proof, in the appendix, proceeds by contradiction in two steps. First, the observation $Y = 0$ shifts the distribution of the fundamental θ to the left and agents are less optimistic about coordination success in the subgame following $Y = 0$ than they would be in a static game with the initial beliefs. Thefore, they are less willing to invest: $\hat{s}_2^0 > s^*$, and $\Theta(\hat{s}_2^0) > \Theta(s^*) = c$.

Second, by Lemma 4, $\hat{s}_2^0 < \hat{s}_2^1 = \hat{s}_1$, which, by the first step of the argument, means that $\hat{s}_2^1 > s^*$. But then if $Y = 1$ had occurred, the distribution of θ would have shifted to right,[8] compared to the one-period model, and agents with a signal in the neighborhood of $\hat{s}_2^1$ would invest, a contradiction of Lemma 4.

Following Lemma 5, we only need to consider monotone equilibrium with $\hat{s}_2^1 \leq \hat{s}_2^0 = \hat{s}_1$. For that reason, we will abuse the notation and use $\hat{s}_2$ to stand for $\hat{s}_2^1$. Next, we proceed to solve for the equilibrium strategy $(\hat{s}_1, \hat{s}_2)$.

So far, the analysis has been independent of the discount factor, δ. It is intuitive that this parameter will now play a critical role because affects the trade-off between the payoff of no delay and the discounted payoff of delay. For the interpretation of some properties, it will also be useful to use the discount rate, ρ, which is related to the discount factor by $\delta = 1/(1 + \rho)$.

2.3 A high discount factor, or high γ

The impact of the option for delay will be stronger when the discount rate is low because delaying will be less costly. The game goes on in period 2 only after $Y = 1$. If the mass of investment is higher than γ, it must be that θ takes a high value. But, in that case, could the mass of investment

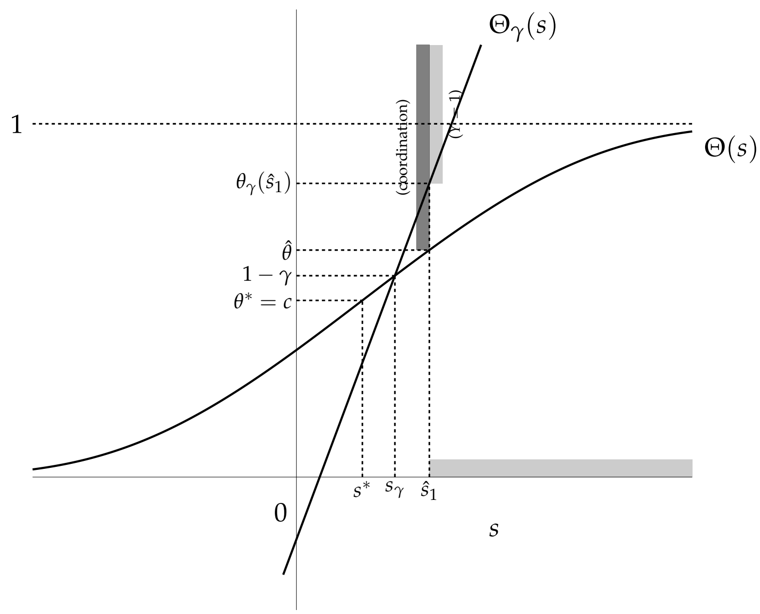
[8] More precisely, since $Y = 1$ implies $\theta \geq \Theta_\gamma(\hat{s}_1) > \Theta(\hat{s}_2^0) > c$, this public signal will generate sufficient optimism and induce agent with $\hat{s}_1^- := \lim_{\varepsilon \downarrow 0} \hat{s}_1 - \varepsilon$ to invest. See the proof in the appendix for details.

in the first period be sufficient for coordination, even if no investment takes place in period 2? It will turn out that this property is true if the discount rate is below some value that depends on the parameters of the model. We begin with this case. Recall that $Y = 1$ is equivalent to $\theta > \Theta_\gamma(\hat{s}_1)$ and that coordination is achieved with the first period investment if $\theta > \Theta(\hat{s}_1)$. Therefore $Y = 1$ implies coordination if and only if

$$\Theta_\gamma(\hat{s}_1) > \Theta(\hat{s}_1). \tag{6}$$

Define s_γ such that $\Theta_\gamma(s_\gamma) = \Theta(s_\gamma)$. Using Lemmata 1 and 3,

Figure 1: A high discount factor



For the equilibrium strategy $\hat{s}_1$, $Y = 1$ reveals that investment in the first period is sufficient for coordination. If $Y = 0$, there is no more investment in period 2 although coordination will be achieved at the end of the game if θ is in the segment $(\hat{\theta}, \Theta_\gamma(\hat{s}_1))$.

$$\Theta_\gamma(s_\gamma) = \Theta(s_\gamma) = 1 - \gamma. \tag{7}$$

Because the functions $\Theta_\gamma(\cdot)$ and $\Theta(\cdot)$ are increasing, and the second function has derivative strictly less than one, while the first function has a constant slope of 1 (see (5)), the equilibrium can be represented in Figure 1, where (6) is equivalent to

$$\hat{s}_1 > s_\gamma. \tag{8}$$

Since $Y = 1$ predicts coordination success, the net discounted payoff of investment in the second period following $Y = 1$ is equal to $\delta(1 - c)$. Therefore, the arbitrage equation for the marginal agent with signal $\hat{s}_1$ is

$$\mathbb{P}\left[\theta \geq \Theta(\hat{s}_1)|\hat{s}_1\right] - c = \delta \mathbb{P}\left[Y = 1|\hat{s}_1\right](1-c). \tag{9}$$

Another interpretation of this indifference condition is that

$$\rho(\mathbb{P}\left[\theta \geq \Theta(\hat{s}_1)|\hat{s}_1\right] - c) = \mathbb{P}(Y = 0|\hat{s}_1)\left(c - \mathbb{P}\left[\theta < \Theta(\hat{s}_1)|Y = 0, \hat{s}_1\right]\right).$$

The left-hand side of the above equation is equal to the cost of delay: it is equivalent to a return, for one period, on the net value of the investment. The right-hand side measures the value of delay. It is equal to the expected value of undoing the investment after the bad news $Y = 0$.

Using Lemmata 1 and 3, (9) is equivalent to

$$\Theta(\hat{s}_1) = c + \delta(1-\gamma)(1-c). \tag{10}$$

Using this equation and (7), the inequality (8) is equivalent to

$$1 - c - \gamma < \delta(1-c)(1-\gamma). \tag{11}$$

This inequality holds when for given c and γ, the discount rate satisfies the inequality

$$\delta > \delta^* \equiv \frac{1-c-\gamma}{(1-c)(1-\gamma)}. \tag{12}$$

We will also discuss the properties of the model with respect to the mass γ for the signal $Y = 1$. In this respect, the inequality (11) is equivalent to

$$\gamma > \gamma^* \equiv \frac{(1-\delta)(1-c)}{1-\delta(1-c)}. \tag{13}$$

After establishing the necessary conditions for an equilibrium, we now show that the arbitrage equation (9) is also sufficient for the equilibrium. Let $D(s_i)$ the payoff difference between no delay and delay for an agent with a signal s_i when other agents play $\hat{s}_1$. It is proven in the appendix that any agent with signal $s < \hat{s}_1$ ($s > \hat{s}_1$) strictly prefers (not) to delay, that is , $D(s_i) < 0$ ($D(s_i) > 0$). Therefore, $\hat{s}_1$ solved in equation (10) and $\hat{s}_2 = -\infty$ can constitute a monotone equilibrium. In the appendix, we further prove the uniqueness of this equilibrium; that is, there exists no equilibrium with $\hat{s}_1 \leq s_\gamma$ under the condition $\delta > \delta^*$.

Proposition 1. *If $\gamma \geq 1 - c$, or if $\delta > \delta^*$ given $\gamma < 1 - c$, there exists a unique equilibrium $(\hat{s}_1, \hat{s}_2^1 = \infty)$, with*

$$\Theta(\hat{s}_1) = c + \delta(1-\gamma)(1-c).$$

Coordination is successful if and only if $\theta \geq \Theta(\hat{s}_1) > c$. In equilibrium, the outcome of coordination is achieved by the investment in the first period.

In equilibrium, the game proceeds as follows: if $Y = 0$ at the end of the first period, the game stops; if $Y = 1$, all the delayers invest but they do not contribute to the success of coordination, as it is achieved in the first period. Under the conditions of the proposition, $Y = 1$ only if the fundamental θ takes a high value, and after observing $Y = 1$, agents learn that investment in the first period is already sufficient for a successful coordination. This case should be expected when γ takes a high value.

Note that if $\gamma \geq 1 - c$, then $\gamma > \gamma^*$ holds regardless of discount factor δ as $\delta^* \leq 0$. A simple argument explains why the result of Proposition 1 holds when $\gamma > 1 - c$, whatever the value of the discount factor. Recall that in the benchmark case with one period, coordination is successful when $\theta > \theta^* = c$, and coordination is successful when $X \geq 1 - \theta$. That means that if $X \geq 1 - c$, coordination is successful in the one period setting. Now introduce two periods with $\gamma > 1 - c$. $Y = 1$ implies that $X_1 > 1 - c$, which is sufficient for a successful coordination. Investment in the second period is irrelevant for coordination. Hence, by definition of $\Theta(\cdot)$, the payoff of investment in the first period for the marginal agent $\hat{s}_1$ is equal to

$$\mathbb{P}\left[\theta \geq \Theta(\hat{s}_1)|\hat{s}_1\right] - c = \Theta(\hat{s}_1) - c > 0.$$

That payoff is equal to the payoff of delay, which is strictly positive, whatever the discount factor, because $Y = 1$ occurs with a positive probability and in that event the payoff of investment is equal to $1 - c$. The previous inequality shows that there is less coordination than in the static case.

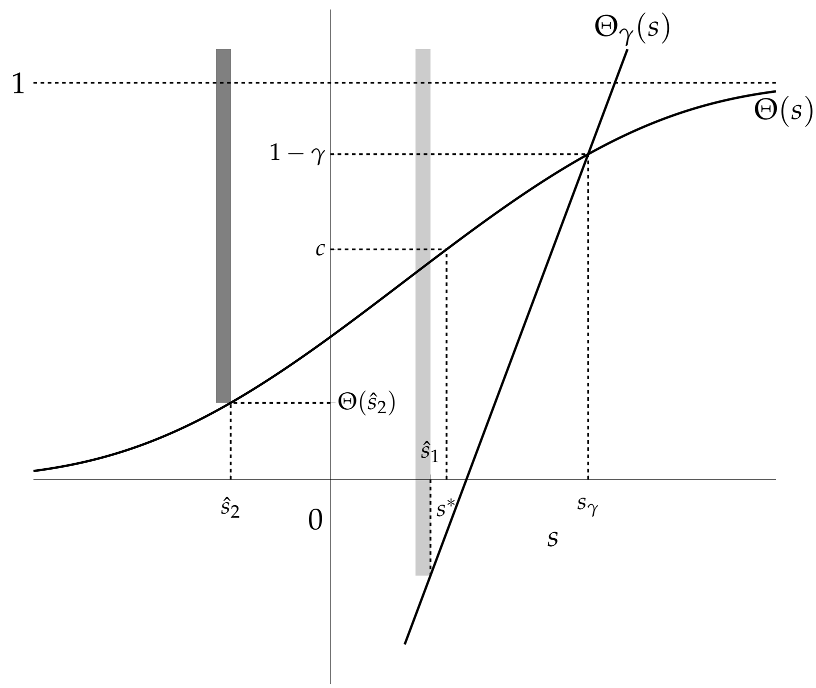
When $\gamma < 1 - c$, the previous mechanism requires that the incentive for delay should be sufficiently strong and here, that requires a discount factor greater than the value δ^*. Of course, the lower the value of γ, the higher the incentive must be for the effect to hold, as can be verified in the expression (8) of δ^*. Note that in this case, if agents follow the "static" strategy s^*, the observation $Y = 1$ does not guarantee that coordination is achieved in the first period. That is the case represented in Figure 1, where $\Theta_\gamma(s^*) < c$.

Comparing with the result in Lemma 2, there is *less* coordination than in the static case: $\Theta(\hat{s}_1) > \theta^* = c$. This has a simple interpretation. The coordination outcome is only generated by investment in the first period, as in the one-period setting. But with two periods, some agents who would invest in the one-period setting, are delaying. That makes a successful coordination less likely. The delaying agents will invest after $Y = 1$ but they are useless for the success of coordination.

2.4 A low discount factor

Let us now assume that $0 < \delta < \delta^*$ (defined in Proposition 1). If there is an equilibrium, following the previous discussion, the observation $Y = 1$ takes place also for some values of θ for which the first-period investment is not sufficient for coordination. We have the case of Figure 2 with the first period strategy $\hat{s}_1 < s_\gamma$. This case is represented in Figure 2.

Figure 2: Low discount factor



After the observation $Y = 1$, the support of the θ is truncated to $(\Theta_\gamma(\hat{s}_1), \infty)$, (green segment in the figure). The mass of first-period may be sufficient for a successful coordination, if $\theta > \Theta(\hat{s}_1)$. But it may not be if the realization of θ is not sufficiently high, i.e., $\theta \in (\Theta_\gamma(\hat{s}_1), \Theta(\hat{s}_1)]$. In this case, because the realization $Y = 1$ has shifted the distribution of the fundamental to higher values, agents did not invest at $t = 1$ become more optimistic after observing $Y = 1$. Because of this learning effect, more agents are willing to invest. Of course, this additional investment has been expected by agents who do not delay in the first period when they choose the strategy $\hat{s}_1$, and the coordination success is determined by these additional investments.

The equilibrium $(\hat{s}_1, \hat{s}_2)$ is characterized by the two necessary arbitrage conditions for the marginal agents in each of the two periods. Coordination is achieved if $\theta > \Theta(\hat{s}_2)$. In the second period (after $Y = 1$), the marginal agent with signal $\hat{s}_2$ who learns $\theta > \Theta_\gamma(\hat{s}_1)$ from the observation of $Y = 1$ is indifferent between investing and not investing at $t = 2$:

$$\mathbb{P}(\theta > \Theta(\hat{s}_2)|\theta > \Theta_\gamma(\hat{s}_1), \hat{s}_2) - c = 0, \tag{14}$$

which is equivalent to

$$\frac{\Theta(\hat{s}_2)}{F(\hat{s}_2 - \Theta_\gamma(\hat{s}_1))} = c. \tag{15}$$

In the first period, a marginal agent, with signal $\hat{s}_1$, is indifferent between no delay and delay with a commitment to invest if and only if $Y = 1$:

$$\mathbb{P}(\theta > \Theta(\hat{s}_2)|\hat{s}_1) - c = \delta(\mathbb{P}(\theta > \Theta(\hat{s}_2)|\hat{s}_1) - c\mathbb{P}(\theta > \Theta_\gamma(\hat{s}_1)|\hat{s}_1)), \tag{16}$$

which, by Lemma 3, is equivalent to

$$(1-\delta)\mathbb{P}(\theta > \Theta(\hat{s}_2)|\hat{s}_1) = c(1 - \delta(1-\gamma)). \tag{17}$$

In this arbitrage equation, the reduction of the gross payoff of investment for the marginal agent $\hat{s}_1$ is equal to the difference in investment cost, which is paid in period 2 only if $Y = 1$. This equation is equivalent to

$$F(\hat{s}_1 - \Theta(\hat{s}_2)) = c\frac{1-\delta(1-\gamma)}{1-\delta}. \tag{18}$$

Recalling $\hat{\theta} = F(\hat{s}_2 - \hat{\theta})$,

$$\hat{s}_2 = \hat{\theta} + \sigma\Phi^{-1}(\hat{\theta}), \tag{19}$$

where $\hat{\theta} = \Theta(\hat{s}_2)$, to simplify the notation. Equations (15) and (18) can be rewritten

$$\hat{\theta} = c\Phi(\frac{\hat{s}_2 - \hat{s}_1}{\sigma} - \Phi^{-1}(\gamma)), \qquad \hat{s}_1 = \hat{\theta} + \sigma\Phi^{-1}(c\frac{1-\delta(1-\gamma)}{1-\delta}). \tag{20}$$

$$\hat{\theta} = c\Phi\Big(\Phi^{-1}(\hat{\theta}) + a\Big), \quad \text{with} \quad a = -\Phi^{-1}(\gamma) - \Phi^{-1}(c\frac{1-\delta(1-\gamma)}{1-\delta}). \tag{21}$$

The condition $\delta < \delta^*$ is equivalent to $a > 0$. It is shown in the appendix that this equation has a unique solution in $\hat{\theta} = \Theta(\hat{s}_2)$.

Proposition 2. *If $\gamma < 1 - c$ and $\delta \in (0, \delta^*)$, there is a unique monotone equilibrium characterized by two values $\hat{s}_1$ and $\hat{s}_2 < \hat{s}_1$ that solve (15) and (18). In equilibrium, an agent with signal $s_i > \hat{s}_1$ does not delay, an agent with $s_i \in (\hat{s}_2, \hat{s}_1)$ invest in period 2 after the observation $Y = 1$, and an agent with signal $s_i < \hat{s}_2$ never invests.*

In contrast to Proposition 1, because of the smaller discount factor there is less incentive to delay. When $Y = 1$, agents in the second period know that the worst values of the fundamental (below $\Theta_\gamma(\hat{s}_1)$), are ruled out and some investment takes place in the second period. In the first period, the probability of more action in the second period stimulates investment. One can verify in (15) that $\Theta(\hat{s}_2) < c$. Therefore, coordination is more likely to be succeed than in the static case. This property will be discussed later.

2.5 A special/marginal case

Next, we will end this section by discussing a special case in which $\delta = \delta^*$, which connects the two different cases with $\delta > \delta^*$ and $\delta < \delta^*$ and gives a complete picture of the equilibrium characterization. As discussed above, this case only arises when $\gamma < 1 - c$.

Recall that when $\delta > \delta^*$, coordination success is guaranteed following $Y = 1$ but it can also occur after $Y = 0$; and when $\delta < \delta^*$, coordination success occurs only after $Y = 1$ but that is not guaranteed. As shown in the following proposition, in the special case where $\delta = \delta^*$, there is a continuum of equilibria and in any possible equilibrium, coordination success is guaranteed and only occurs after $Y = 1$.

Proposition 3. *If $\gamma = \gamma^*$ (or $\delta = \delta^*$), any $\hat{s}_1 \in [\sigma\Phi^{-1}(1-\gamma^*), s_{\gamma^*}]$ and $\hat{s}_2 = -\infty$ can constitute an equilibrium. In any equilibrium, coordination is successful if and only if $\theta > \Theta_\gamma(\hat{s}_1)$.*

To understand this result, suppose that $Y = 1$ perfectly predicts the coordination success; that is, $\theta \geq \Theta_\gamma(\hat{s}_1)$ is not only the condition for $Y = 1$ but it also presents the condition for coordination success. We already know that this happens when $\hat{s}_1 = s_\gamma$ and accordingly, $\Theta(\hat{s}_1) = \Theta_\gamma(\hat{s}_1)$. However, when $\delta = \delta^*$, it is also possible under other first period strategy $\hat{s}_1$. To see this, the payoff difference of the marginal agent with $\hat{s}_1$ is

$$\mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|\hat{s}_1) - c - \delta^*\mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|\hat{s}_1)(1-c) = (1-\gamma) - c - \delta^*(1-\gamma)(1-c) = 0. \quad (22)$$

Interestingly, regardless of the value of $\hat{s}_1$, when $\delta = \delta^*$, the marginal agent is always indifferent. To understand the underlying reason for the multiplicity of equilibrium, first note that, if coordination always succeeds after $Y = 1$, it must be that all delaying agents will invest after $Y = 1$, i.e., $\hat{s}_2 = -\infty$. More importantly, it is possible that coordination is not successful purely driven by first-period investment (i.e., $\hat{s}_1 < s_\gamma$ and $\Theta(\hat{s}_1) > \Theta_\gamma(\hat{s}_1)$), but it is induced by the subsequent investment at $t = 2$ after $Y = 1$, which happens as long as $\Theta_\gamma(\hat{s}_1) \geq \Theta(\hat{s}_2 = -\infty) = 0$. In other words, the fact that all delaying agents will invest at $t = 2$ following $Y = 1$ makes $Y = 1$ a perfect predictor of coordination success, which, in turn, justifies the choice of $\hat{s}_2 = -\infty$.[9] For that reason, it is an equilibrium as long as $\hat{s}_1$ satisfies $\Theta(\hat{s}_1) \geq \Theta_\gamma(\hat{s}_1) \geq 0$.

[9] One may wonder why this cannot happen in the case where $\delta < \delta^*$. As in the subgame following $Y = 1$, agents learn that $\theta \geq \Theta_\gamma(\hat{s}_1)$. If $\Theta_\gamma(\hat{s}_1) \geq 0$, in that subgame, it is an equilibrium where all delaying agent choose to invest, i.e., $\hat{s}_2 = -\infty$. However, that cannot constitute an equilibrium for the entire two-period game because the marginal agent with any $\hat{s}_1$ always strictly prefers to act earlier. This can be seen by replacing δ^* by any $\delta < \delta^*$ in (22).

3 Does the delay option facilitate coordination?

Facilitating coordination means expanding the range of the fundamental's values for which coordination takes place. Here, coordination takes place if the fundamental θ is in an open interval with a lower-bound $\hat{\theta}$ that is determined by the equilibrium. The delay option facilitates coordination $\hat{\theta}$ is lower than the lower-bound θ^* in the one-period setting. Whether more coordination is good or bad depends on the interpretation of the model and will be seen in the next section. We have already seen that the answer to the question in this section depends on the discount factor, δ, and on the minimum mass, γ for the public signal $Y = 1$.

Proposition 4. *There is less coordination with two periods than with one period, i.e.,* $\hat{\theta} > \theta^*$*, when* $\delta(1-c)(1-\gamma) > 1 - c - \gamma$*. When this inequality is reversed, there is more coordination:* $\hat{\theta} < \theta^*$*.*

The inequality in the proposition depends on the relative positions of the critical mass parameter, γ, and on the discount factor. We consider them in turn.

3.1 Coordination and the critical mass for $Y = 1$

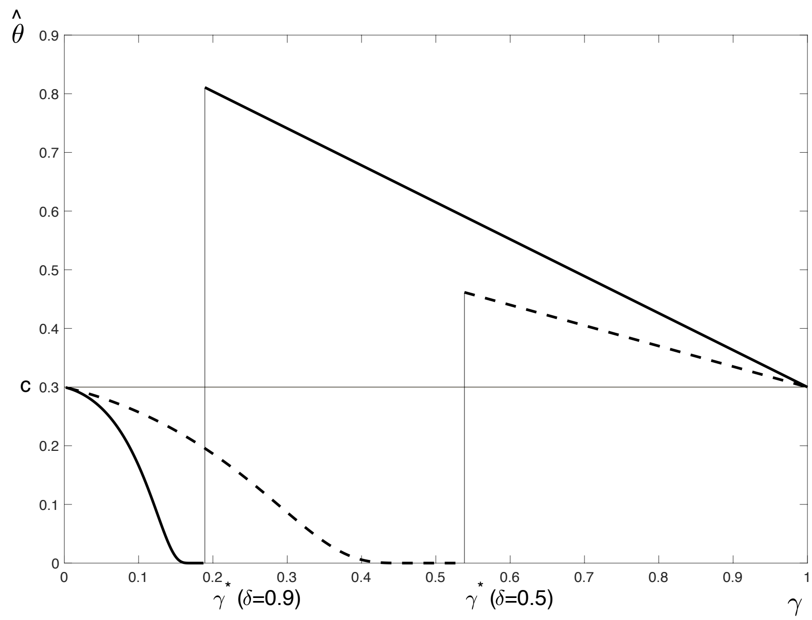
The expression of the lower-bound $\hat{\theta}$ a function of the critical mass γ is relevant when this parameter is affected by policy or some interpretations of the model (see below). An example is illustrated by Figure 3, for two values of the discount factor, $\delta = 0.5$ and $\delta = 0.9$, respectively.

The lower-bound for coordination is a decreasing function of γ in both cases when γ is greater and smaller than γ^*, with a sharp discontinuity at γ^*. To see this, assume first that $\gamma > \gamma^*$. When γ increases, the probability of the good news $Y = 1$ for the marginal agent with $\hat{s}_1$ decreases, as it is equal to $1 - \gamma$ (Lemma 3). The content of good news ($Y = 1$) does not change: it is that the payoff of investment in the second period is one. The value of delay decreases as the likelihood of receiving good news ($Y = 1$) decreases. When γ tends to one, the incentive for delay becomes vanishingly small. At the limit, the model is the same as the one-period model and $\hat{\theta} = c$.

For the case $\gamma < \gamma^*$, assume first that γ is close to 0. The probability of $Y = 1$ is close to 1, which reveals very little information. There is little incentive to delay. At the limit, all agents invest in the first period and the model is equivalent to the one-period model with the same lower-bound for coordination, c.

When γ increases, the observation of $Y = 1$ becomes more informative. In Figure 2, the line that represents the function $\Theta_\gamma(\cdot)$ is shifted upwards. The value $\Theta_\gamma(\hat{s}_1)$, which is extremely low

Figure 3: Lower-bound of θ for coordination as a function of γ



when γ is near zero, increases. When γ increases, more values are truncated away in the distribution of θ. That stimulates investment in period 2 and lowers $\hat{s}_2$. Therefore, $\Theta(\hat{s}_2)$ decreases and there is more coordination.

These properties are summarized in the following proposition, which is proven in the Appendix.

Proposition 5. *For given parameters of the model, the lower-bound $\hat{\theta}$ for coordination is a decreasing function of γ. When γ tends to γ^* with $\gamma < \gamma^*$, the lower-bound for coordination tends to 0, which is the lower-bound in the first-best when the objective is to achieve coordination whenever possible.*

In the proposition, when $\gamma \to \gamma^*-$, the realization $Y = 1$ reveals, asymptotically that the values $\theta < 0$, under which no coordination is possible, have a vanishingly small probability. The threshold for investment in the second period, $\hat{s}_2$ tends to $-\infty$. Asymptotically, all remaining agents invest in the second period. The outcome of coordination, success or failure, is, asymptotically the same as in the first-best in the sense of the proposition. However, the investment allocation does not converge to the first-best allocation when γ tends to γ^*- because that a significant amount of agents choose to delay.[10] In addition, investing agents in the first period take the

[10]When $\gamma \to \gamma^*-$, the equilibrium threshold $\hat{s}_1$ is determined by the arbitrage condition (18) which, asymptotically is equivalent to $F(\hat{s}_1) = c/(1 - \delta(1 - c))$.

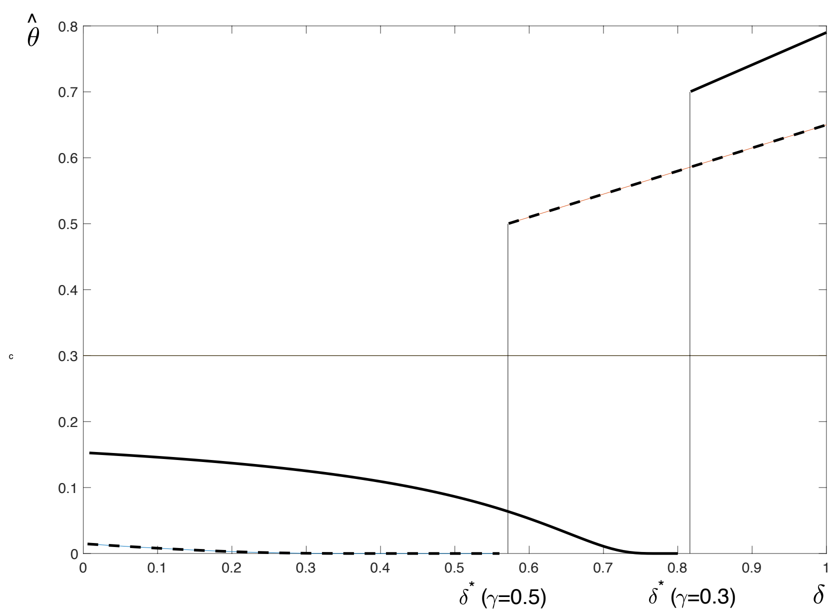
risk that $Y = 0$, which asymptotically, means $\theta < 0$.

Figure 4 also illustrates the non-continuous variation of $\hat{\theta}$ around $\gamma = \gamma^*$. The fundamental cutoff $\hat{\theta}$ "jumps" from 0 (when $\gamma = \gamma^*-$) to $\Theta(s_{\gamma^*})$ (when $\gamma = \gamma^*+$). It is worth noting that, as shown in Proposition 3, any point within the range of $[0, \Theta(s_{\gamma^*})]$ represents an equilibrium for the special case of $\gamma = \gamma^*$.

3.2 Coordination and the discount factor

The lower-bound $\hat{\theta}$ for a successful coordination is represented as a function of the discount factor δ in Figure 4.

Figure 4: Lower-bound of θ for coordination as a function of δ



Proposition 6. *For given γ and c with $\gamma < 1 - c$, the lower-bound $\hat{\theta}$ for coordination is a decreasing function of δ if $\delta < \delta^*$ and an increasing function of δ if $\delta > \delta^*$. When δ tends to δ^*-, the lower-bound for coordination tends to 0, which is the lower-bound in the first-best when the objective is to achieve coordination whenever possible.*

When the discount rate greater than δ^*, there is less coordination than in the static case (Proposition 1). Furthermore, the lower-bound $\hat{\theta}$ is increasing with the discount factor δ. The argument can be put in terms of the discount rate $\rho = 1/\delta - 1$. A lower discount rate, by reducing the cost of delay and increasing the incentive for delay, reduces the level of investment in the first period and therefore the probability that time-1 investment triggers the realization $Y = 1$, which in this case ($\delta > \delta^*$), is a necessary condition for a successful coordination. (Recall that whether coordination is successful or not is determined only by the mass of investment in the first period.) Therefore, when the threshold for investment, $\hat{s}_1$, increases, a successful coordination requires a higher value of the fundamental.

When $\delta < \delta^*$, the threshold value of $\hat{\theta}$ for coordination success is equal to $\Theta(\hat{s}_2)$, which depends on the strategy of the delaying agents in the second period. Conditional on $Y = 1$, these delaying agents are more optimistic than in the belief before the game starts, which is the same belief of the one-period setting. Therefore, they have more invest more than in the one-period setting. $\Theta(\hat{s}_2) < c$. The option for delay *facilitates* coordination as it expands the range of values of θ for which coordination is successful.

It is interesting to observe what happens when the discount factor tends to 0. At the limit, the future has a value of approximately zero. However, not all of them invest, obviously. At the end of the first period, the good news, $Y = 1$, arrives with a strictly positive probability. When $Y = 1$, beliefs are upgraded for the agents who did not invest in the first period and some of them will invest in the second period. *That* investment is taken into account by the agents in the first period when they optimize. They do not weigh the future payoffs for their optimization, but they take into account the future actions for the payoff from coordination. This property can be observed in Figure 4, in which the limit of lower-bound $\hat{\theta}$ is strictly below $\theta^* = c$, when $\delta \to 0$. Therefore, the static model is not the limit of a two-period model with vanishingly small discount factor.

The figure shows the remarkable property that if $\delta \to \delta^*-$, then $\hat{\theta} \to 0$, which means that coordination takes place if and only if it would take place in the first-best. This is identical to the case with $\gamma \to \gamma^*-$.

Recall that in the region of $\delta > \delta^*$, $\hat{\theta}$ is determined by the cutoff $\hat{s}_1$ in the first period when agents play as if they ignored the play in the second period. On the right of δ^*, agents in the first period do take into account the possibility of more investment in the second period. As shown in Figure 4, the threshold $\hat{\theta}$, above which coordination is successful, is greater than θ^*, indicating that delay option *impairs* coordination.

To summarize, in the case $\delta > \delta^*$, with a strong delay incentive, the delaying agents wait for very good news: the signal $Y = 1$ reveals that it is perfectly safe to invest; while, in the case $\delta < \delta^*$, with a weak incentive to delay, the delaying agents wait to avoid the bad news $Y = 0$.

4 Discussions

4.1 Equilibrium Selection and Public Information

It is well known that introducing public information into the global game homogenizes agents' beliefs, and is likely to result in multiplicity of equilibrium.[11] In particular, Hellwig (2002) and Morris and Shin (2003) show that the uniqueness of equilibrium persists only when the private signal is significantly precise, relative to the public signal. Basak and Zhou (2020) find that, in a static game, when agents receive public signal that truncates their beliefs about the fundamental θ, unique equilibrium occurs only when the private signal is sufficiently imprecise. In our dynamic model, we find generically unique monotone equilibrium as shown in Proposition 1 and 2. Interestingly, the fundamental threshold $\hat{\theta}$ above which coordination succeeds is invariant to σ, the dispersion of agents' private signals.

Corollary 1. *The generically unique equilibrium in Proposition 1 and 2 has the following properties.*

1. *when $\gamma > \gamma^*$, the lower-bound for coordination, $\Theta(\hat{s}_1)$, does not depend on σ;*
2. *when $\gamma < \gamma^*$, the lower-bound for coordination, $\Theta(\hat{s}_2)$ does not depend on σ, while the lower-bound of the distribution of θ after the realization $Y = 1$, $\Theta_\gamma(\hat{s}_1)$, is a decreasing function of σ, which tends to $\Theta(\hat{s}_2)$ if σ tends to zero and is negative if and only if $\sigma > \sigma^*$ for some σ^*.*

In the case with $\gamma > \gamma^*$ (or, equivalently, $\delta > \delta^*$), recall that

$$\hat{s}_1 = \hat{\theta} + \sigma\Phi^{-1}(\hat{\theta}), \quad \text{with} \quad \hat{\theta} = \Theta(\hat{s}_1) = c + \delta(1-\gamma)(1-c). \tag{23}$$

Consider that the individual signals are less precise, that is, σ increases. In that case, for any given θ, the mass of investing agents must stay constant. Therefore, the distribution of individual signals is more dispersed but with the linear relation 23). That is why in Proposition 1, the lower-bound for of θ for coordination, $\Theta(\hat{s}_1)$ does not depend on σ.

In the case with $\gamma < \gamma^*$ (or, equivalently, $\delta < \delta^*$), the invariant property is based on equation (21), which is independent of σ. Based on (20), the threshold $\Theta_\gamma(\hat{s}_1)$ can be written as

$$\Theta_\gamma(\hat{s}_1) = \Theta(\hat{s}_2) - \sigma b. \tag{24}$$

Recall that b is a strictly positive number under $\gamma < \gamma^*$ (see (21)). As $\hat{\theta} = \Theta(\hat{s}_2) \in [0, c)$ is independent of σ, it is easy to check that $\Theta_\gamma(\hat{s}_1)$ obtains a positive value only when σ is sufficiently small, that is, private signals are sufficiently precise.

[11] See, for example, Hellwig (2002) and Angeletos, Hellwig and Pavan (2007).

Let us now consider the case in which both $\gamma < \gamma^*$ and $\sigma \leq \sigma^*$ holds. Based on the unique equilibrium $(\hat{s}_1, \hat{s}_2)$ solved in Proposition 2, we know that the corresponding $\Theta_\gamma(\hat{s}_1) \geq 0$ according to Corollary 1. As it is publicly known that $\theta_\gamma \geq 0$, all agents invest following $Y = 1$ at $t = 2$ is obviously a *sub-game* equilibrium. However, given $\delta < \delta^*$, all delaying agents invest after $Y = 1$ (i.e., $\hat{s}_2 = -\infty$) cannot constitute an equilibrium for *the entire game* since, if that is the case, the marginal agents with $\hat{s}_1$ always strictly prefers to invest earlier. See footnote 9 for the details. Next, we will show that when the binary public information Y is not an endogenous signal, but rather an exogenous one purely depends on θ, this can constitute an equilibrium.

A Comparison with Exogenous Information

Next, to see why the endogenous learning is a critical feature of the model, let us consider an alternative information generating process with $Y' = 0, 1$: the value of Y' is equal to 1 if and only if $\theta \geq \theta_Y$ but this value of θ_Y is set as fixed, which does not depend on the strategy of agents in the first period.

In this new setup, there exists an equilibrium that is characterized by the next result.

Proposition 7. *For any $\theta_Y \in [0, \frac{c}{1-\delta(1-c)}]$, there exists one monotone equilibrium such that all remaining players invest at $t = 2$ after $Y' = 1$ (i.e., $\hat{s}_2 = -\infty$), no one invests after $Y = 0$, and, in the first period, the investment set is $s_i \in (-\infty, \hat{s}_1^Y)$ with*

$$F(\hat{s}_1^Y - \theta_Y) = \frac{c}{1 - \delta(1-c)}. \tag{25}$$

Proposition 7 presents one equilibrium in which all delaying agents will coordinate and invest following $Y' = 1$. The condition $\theta_Y \geq 0$ guarantees that successful coordination after all agents invest at $t = 2$ following $Y' = 1$ and the condition $\theta_Y \leq \frac{c}{1-\delta(1-c)}$ ensures that, in equilibrium, coordination never succeeds beyond $Y' = 1$.

$$\mathbb{P}(\theta \geq \theta_Y | \hat{s}_1) - c - \delta \mathbb{P}(\theta \geq \theta_Y | \hat{s}_1)(1-c) = 0$$

The equilibrium condition (25) is derived from this arbitrage condition, which ensures the existence of such an equilibrium.

Recall that, under $\gamma < \gamma^*$ and $\sigma < \sigma^*$, the unique equilibrium $(\hat{s}_1, \hat{s}_2)$ solved in Proposition 2 features $\Theta_\gamma(\hat{s}_1) \geq 0$. If we set $\theta_Y = \Theta_\gamma(\hat{s}_1)$, then, based on Proposition 7, all delaying agents investing following $Y' = 1$ is in fact an equilibrium.[12] Obviously, the equilibrium presented in

[12]Note that when $\gamma < \gamma^*$, $\Theta_\gamma(\hat{s}_1) < \Theta(\hat{s}_2) < c$. Therefore, $\theta_Y = \Theta_\gamma(\hat{s}_1)$ must satisfy the condition $\theta_Y \in [0, \frac{1}{1-\delta(1-c)}]$.

Proposition 2 will remain as an equilibrium given that $\theta_Y = \Theta_\gamma(\hat{s}_1)$. This clearly demonstrates that endogenous learning is an essential feature accounting for the uniqueness of equilibrium. To understand this, note that when the public signal Y is endogenously generated, the investment strategy $\hat{s}_1$ at $t = 1$ determines the meaning of $Y = 1$ at $t = 2$ (or $\Theta_\gamma(\hat{s}_1)$), while, at the same time, the marginal agent with $\hat{s}_1$ must be indifferent between investing and waiting for $Y = 1$ (with the endogenous meaning).

However, if we only consider the sub-game starting from $t = 2$, as $\Theta_\gamma(\hat{s}_2) \geq 0$, investing for any signal ($\hat{s}_2 = -\infty$) is always a sub-game equilibrium. This raises the issue of the stability of the equilibrium with strategy $\hat{s}_2$.

Stability in period 2

Here, let us define stability in the following way. Assume that all agents play the strategy s_2 in the second period. Then one can define the reaction function $R(s_2)$ which satisfied the arbitrage condition for a deviating agent. This reaction function is determined by

$$\frac{F(R(s_2) - \Theta(s_2))}{F(R(s_2) - \Theta_\gamma(\hat{s}_1))} = c. \tag{26}$$

Obviously $\hat{s}_2$ is a fixed point of R: $R(\hat{s}_2) = \hat{s}_2$. An equilibrium is unstable when $R'(\hat{s}_2) > 1$. When there are multiple fixed points (equilibria), by geometry, stable and unstable equilibria alternate. The equilibrium where all remaining agents invest is (locally stable) because agents with a signal greater than $\bar{s}$ invest, when $\bar{s}$ is sufficiently small, the reaction is to invest whatever the signal, $R(\bar{s}) = -\infty$.

The following proposition demonstrates that under $\gamma < \gamma^*$, the equilibrium strategy of $\hat{s}_2$ in Proposition 2 is not stable if the private signals are sufficiently precise.

Proposition 8. *For any $\gamma < \gamma^*$ and $\delta < \delta^*$, there always exists $\sigma_0 > 0$ such that the equilibrium strategy $\hat{s}_2$ given in Proposition 2 cannot constitute an stable (sub-game) equilibrium.*

In the "real world" with, say, small exogenous shocks to the technologies (whose realization can be common knowledge before the decision in period 2), or some uncertainty about the rationality of other agents, or their information, or whatever, under the conditions where $\hat{s}_2$ is unstable, the realization $Y = 1$ would provide a much stronger signal (than the previous expectation, which is not a commitment) to coordinate on the equilibrium where remaining agents invest.

4.2 Application to the Financial Market

To understand how the dynamic setting and the resulting equilibrium selection can make a difference, let us consider the example of coordination in financial market, in which market participants and, especially, financial institutions react quickly. That implies, the period length of waiting for the public information or τ, is short, thereby inducing a high discount factor δ. The present model highlights that short periods, or fast reactions, can stabilize a financial institution that faces multiple equilibria.[13]

When parameter values are plugged into a static model, the implication for policy is not plausible. Consider that agents are choosing whether or not to attack a financial institution (instead of invest). Attacking is irreversible and it succeeds if and only if the bank's reserve $1-\theta < X$.

The payoff from coordination corresponds to the rate of a devaluation, say, more than 10 percent. The cost is the interest rate of the speculators during the crisis. If the crisis takes two weeks, a long time for a crisis, and the annual interest rate (as the borrowing cost) is 24 percent(example of Sweden**), the cost of speculation is 10 percent of the potential gain. For the present model, with a payoff normalized at 1, $c = 0.1$. As predicted by the static coordination, for any value of $\theta > \theta^* = c = 0.1$, agents will coordinate on attacking and, as a result, the bank fails. In other terms, the bank, in order to defend any attack, has to keep 90 percent of its deposits in cash.

The present model shows that when agents have the choice of not running and waiting for the observation that there is a run (with $Y = 1$), the bank may keep a much smaller amount of reserves to be safe. Suppose that agents notice the bank run when at least 10 percent of them run on the bank early, i.e., $\gamma = 0.1$, and the discount factor $\delta = 0.99$. Taking the same numerical values as in the above example of the static game ($c = 0.1$), we have $\gamma^* = 0.0826 < \gamma$. (The value of γ^* is not very sensitive to the plausible values of the discount rate δ as δ is very close to 1). The amount of reserves that the bank should keep is given by equation by Proposition 1: $\theta^* = c + \delta(1-\gamma)(1-c) = 0.911$. Therefore, a reserve of only less than 10 percent of the deposits can make a bank safe (instead of 90 percent in the static case).

In what follows, we discuss the policy implication of our theory. In our discussion, we consider a policy maker who either only cares about the ex-ante probability of successful coordination, or only cares about the welfare of agents but only focuses on the abovementioned limiting case. In either case, the policy maker judges this economy by the fundamental cutoff θ^*. Moreover, the policy maker cannot observe the true realization of θ, and, thus, the adopted policy (no matter it

[13] It is worth noting that the following discussion does not rely on the assumption $\delta \to 0$. As shown in Proposition 1 and **??**, the fundamental cutoff θ^* does not rely on parameter σ.

is about the interest rate r or the information threshold γ) cannot signal the true fundamental θ.

4.3 Inaction and Interest Rate r

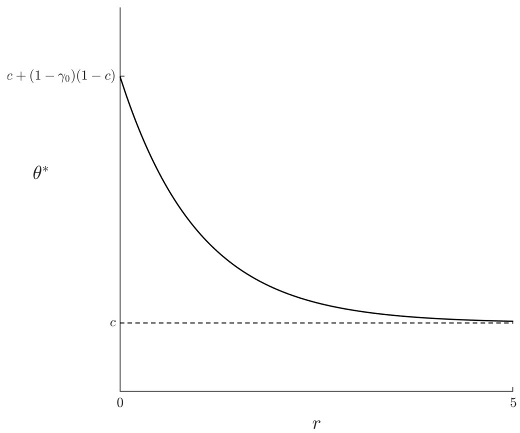
Consider an economy experiencing a recession and the policy maker wants to stimulate investments for the economic recovery. However, the investment game features strategic complementarity; that is, whether investment will be successful or not depend on how many other investor opt in. Naturally, during the economic turmoil, the investors are uncertain about the fundamental of the economy, which also matter for the investment return. Another important feature that is highlighted in our model is that investors can always choose to wait for more information before pledging their capital. This economic environment can be perfectly mapped to our model. In reality, we often observe policy makers to reduce the interest rate r as a way to promote investments and boost recovery. To fulfill this goal, the nominal interest rate was even reduced to zero after the great recession and in recent years. How does such monetary policy affect the incentive of waiting and inaction, as well as the coordination on investment?

Our model is well suited to address these questions. Recall that, given any period length of waiting τ, the discount factor in our model $\delta = e^{-r\tau}$ increases when the interest rate r decreases. When $r \to 0$, δ converges to 1. Recall that when $\gamma \geq 1 - c$, the condition $\gamma > \gamma^*$ holds true regardless of δ (or r). Therefore, the equilibrium is specified in Proposition 1, and delay option always hurts coordination on investment. Given $\gamma \geq 1 - c$ and taking everything else the same, any decrease in the interest rate r create extra incentive for waiting, thereby increasing s_1^* and making coordination success less likely to occur. See Figure 5 (a) for a graphical illustration of $\theta^*(r)$.

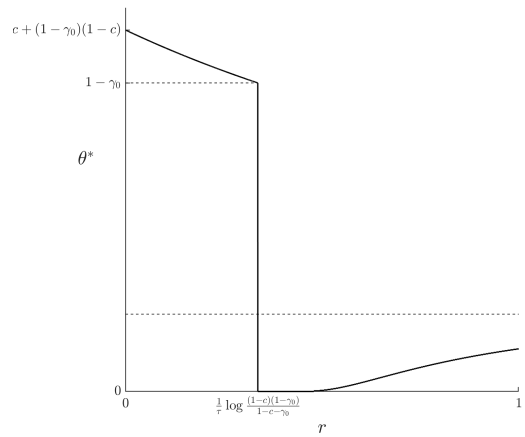
Now consider the cases in which $\gamma < 1 - c$. Given this condition, whether or not the delay option can be helpful in promoting coordination depends on whether $\delta < \delta_0$, which, in turn, depends on whether $r > r_0 := -\frac{1}{\tau} \ln \delta_0$. Suppose the interest rate is reduced significantly from above r_0 to a value below r_0. This essentially makes $\delta > \delta_0$, and based on Proposition **??**, this change in r makes the delay option counterproductive in promoting coordination on investment, and, therefore, such a policy is ineffective in stimulating the economy.

Interestingly, no matter $\gamma \geq 1 - c$ or not, when r decreases from a relative low value and becomes sufficiently close to 0, such a policy results in more inaction (or waiting) at $t = 1$, thereby reducing the chance of having the positive news $Y = 1$ and impairing aggregate investment. In fact, the worst case scenario arises at the zero lower bound when $r \to 0$, where the fundamental cutoff θ^* reaches its maximum. Figure 5(b) clearly demonstrates how the change in r affect the fundamental cutoff θ^* when $\gamma < 1 - c$.

Figure 5: The equilibrium θ^* and the interest rate r



(a) Cases with $\gamma \geq 1 - c$



(b) Cases with $\gamma < 1 - c$

Note: We adopt standard normal distribution for F, $\tau = 1$, $c = 0.4$, $\gamma = 0.3$ for sub-figure (a), and $\gamma = 0.7$ for sub-figure (b).

Admittedly, lowering the interest rate may directly reduce the cost of investment c and this, in turn, helps boost investment. However, we are interested in how changes in interest rate would affect the incentive of waiting in our dynamic coordination model, and, in this way, changes the investment outcome. Notably, our model identifies a clean channel through which a lower interest rate may promote more inaction in the economic recovery, which makes such a monetary policy ineffective in stimulating the economic recovery. We believe this channel is important to be taken into account given that the macroeconomy always features strong complementarity, a dynamic structure with delay options and a low interest rate environment.

4.4 Waiting for the fall of the regime

Theoretical models of coordination have been related to the sustainability of an oppressive regime facing demonstrations. Lohmann (1994) has taken the example of the Leipzig demonstrations that preceded the fall of the Berlin Wall. Kuran applies the concept of strategic complementarity to the the sudden, and apparently unexpected, change of public opinion (Kuran (1987), Kuran (1995)). Similar references were in Chamley (1999) and Angeletos, Hellwig and Pavan (2007). In the latter model, agents play a multi-period game where they do not observe the mass of investment (or "demonstration") as long as the regime survives, that is, as long as coordination has not been achieved. If we assume that agents do not get additional private information after the first period, that model is reduced to the present model in which $Y = 1$ if coordination succeeds, which means that γ is endogenous. The arbitrage equation (9) is replaced by the following

$$\Theta(\hat{s}_1) - c = \delta(1 - c)\Theta(\hat{s}_1), \tag{27}$$

and the lower-bound for coordination is given by

$$\Theta(\hat{s}_1) = \frac{c}{1 - \delta(1 - c)}. \tag{28}$$

This value is greater than the value in Proposition 1 if and only if $\gamma > \gamma^*$, which is a condition for that proposition. The mechanism is simple. The expected payoff of delay, on the right-hand side of (27) is greater than in the case of Proposition **??**, where coordination may take place even if $Y = 0$. The higher value of the expected payoff in delaying implies, by arbitrage, a higher expected payoff for the marginal agent who invests in the first period. There is less "attack" of the regime in the present case.

If the objective function is to keep the regime in place, the policy implication is straightforward: one should repress any manifestation however small, or at least the news about such manifestation, and also, which is no less important, any news about the repression of such manifestation. Such news would be equivalent to the trigger of a variable $Y = 1$ in the present model.

4.5 Accessibility of Information γ

Consider the cases in which the policy maker can influence the accessibility of information, which captures the minimum size of a demonstration to generate a news event and is governed by the parameter γ. When γ is close to 0, then the agents can easily easily observe $Y = 1$ as long as a small amount of (irreversible) actions occurs. Therefore, information is easily accessible. Otherwise, if γ is relatively large and close to 1, then by design, observing past actions become very difficult and, therefore, information is not accessible. Moreover, we consider the policy maker does not know θ so her policy that influences the accessibility of news event would not signal any information about θ.[14]

[14] An alternative and possibly more plausible assumption is that, although the policy maker understands θ much better than the agents, any policy or institution concerning the accessibility of information is designed ex-ante before θ is realized. Such policy and institution, e.g., enacted by law, cannot be modified easily and thus lasts for long periods of time.

In some applications, for example, coordination on investment, the policy maker wants to promote the taking of the irreversible action. Therefore, she would influence accessibility of information, trying to reduce the fundamental cutoff θ^*. In this case, she would make the good news $Y = 1$ very easy to be generated; that is, it appears as long as a very small number of agents invest early (or $\gamma < \gamma^*$). It is worth noting that when the good news is easily accessible, it may not predict the ultimate success. However, only with high accessibility of information, the delay option together with the news Y can operate to make coordination success on investing more likely.

In other application (e.g., bank runs), the policy maker would prevent the agents from taking the irreversible actions (or running on a bank). Recall that, in such applications, the adverse outcome occurs (or bank fails) as long as $1 - \theta > 1 - \theta^*$. Therefore, the policy maker wants to influence γ so as to increase θ^*. Based on our theory, this requires the "bad" news $Y = 1$ occurs only if a significant share of agents have attacked (or ran on a bank). Such a bad news, only it is generated, perfectly predicts the policy maker's unfavorable outcome. [15]

This finding is largely close to what we observe in reality. Any little progress in a collective investment project (e.g., a fundraising event) will be reported, although such small progress cannot predict the success of the project. In contrast, the vulnerability of financial institutions or occurrence of existent attacks will not be revealed publicly until the failure of these institutions are doomed. By building a dynamic coordination model, out study provides some rationale for such design of information revelation. In more detail, from an ex-ante perspective, such design help increases the incidence of investment success as well as strengthen the resilience of financial institutions.

5 Extension: Dynamic Model with $T \geq 3$ Periods

A standard context for a coordination game is a bank run (Diamond-Dybvig) or a central bank (Obstfeld). With regard to these applications, we believe that these crises do not take place in a unique period, on which, and this is an important coordination requirement, agents coordinate. Multiple periods is a necessary assumption for the analysis of such crises (Chamley, 2003). From this perspective, our model, building on the heterogeneous private information setting which follows the global game approach, presents an interesting observation of dynamic coordination.

[15] It is worth pointing out that since the dynamic model is reduced to a static one in the limiting cases with $\gamma \to 0$ and $\gamma \to 1$. In those cases, the information $Y = 1$ is produced with probability 1 and 0, and, thus, is useless. As such, the optimal choice for the policy maker, if we take the model seriously, is $\gamma \to \gamma^*-$ in the former case where the policy maker wants to minimize θ^*, and $\gamma \to \gamma^*+$ if the policy maker wants to maximize θ^*.

In this section, we extend our two-period model to a full dynamic model. To map our model to the fast-moving financial markets, we primarily focus on case of the high discount factor $\delta > \delta^*$.

The coordination game lasts for some fixed period of time, which is normalized from 0 to 1. We consider the case in which agents can learn a binary signal over a fixed time intervals $\frac{1}{T}$, where T is a positive integral. T is greater if the market moves faster and thus information arrives more frequently. As in the benchmark model with $T = 2$, agents get private information s_i about θ at $t = 0$. They learn the binary public signal y_t at *time* $\frac{t}{T}$ for *period* $t = 1, \ldots, T-1$, in which

$$Y_t = \begin{cases} 1 & \text{if} \ \ \sum_{u=1}^{t-1} X_u \geq \gamma \\ 0 & \text{if} \ \ \sum_{u=1}^{t-1} X_u < \gamma \end{cases}$$

Recall that X_t stands for the new investment at period t. In other words, in this dynamic setting, the binary signal Y_t switches from 0 to 1 at the beginning of the period t if the cumulative investments reaches γ by last period. One can interpret this information generating process as following: there is some news media that monitoring the evolution of the dynamic investment (or bank run), and a news event will be triggered as soon as the cumulative action reaches some critical threshold. Given this information generating process, since the investment is irreversible, if $Y_t = 1$, then $Y_{t'} = 1$ for all $t' > t$.

As there is no investment or speculative attack at a time without new information arrival, agents can invest at $t = 0$, or wait until period t and invest at that time after observing Y_t ($t = 1, 2, \ldots, T-1$). It is worth noting that the arrival of information and decision making of agents end at period $T - 1$ (or time $\frac{T-1}{T}$), whereas the coordination outcome is determined afterwards at period T (or time 1). Coordination is successful if and only if $\sum_{u=0}^{T-1} X_u + \theta \geq 1$. Clearly, the 2-period benchmark model is a special case of this general model with $T = 2$.

This setup helps us to better understand how the coordinating behavior changes with the multiple delay options and future information arrivals. In particular, it enables us to study the impact of speed of information flows ($\frac{1}{T}$). The speed of information flow governs how fast agents can can respond to others' actions. In this way, it also determines the discount rate across periods, i.e, $\delta = e^{-r\frac{1}{T}}$. With a higher speed of learning (a greater T), the discount factor becomes closer to 1 so that delay becomes less costly. In the limiting case where $T \to \infty$, the model converges to a continuous time setting in which the observation of cumulative past actions is instantaneous.

5.1 The case of $T = 3$

Before we dive into the full dynamic model, let us consider a simple example with $T = 3$. The possible history for this example is that (1) $Y_1 = 1, Y_2 = 1$; (2) $Y_1 = 0, Y_2 = 0$ and (3) $Y_1 = 0, Y_2 = 1$. Therefore, we can write the monotone strategy as $(\hat{s}_0, \hat{s}_1^0, \hat{s}_1^1, \hat{s}_2^0, \hat{s}_2^1)$. That is, (1) agents with $s_i \geq \hat{s}_0$ invest at $t = 0$; (2) following $Y_1 = 1$ (then $Y_2 = Y_1 = 1$), agents with $s_i \geq \hat{s}_1^1$ will invest at $t = 1$, and no one invests at $t = 2$; and (3) following $Y_1 = 0$, the agent with $s_i \geq \hat{s}_1^0$ will invest at $t = 1$, then, if $Y_2 = 1$, the agent with $s_i \geq \hat{s}_2^1$ will invest at $t = 2$, and, if $Y_2 = 0$, the agent will invest at $t = 2$ iff $s_i \geq \hat{s}_2^0$.

It is important to note that no one will invests after $Y_2 = Y_1 = 1$ at $t = 2$, since if so, he should have invested at $t = 1$ after $Y_1 = 1$. However, it is theoretically possible that some agents who refuse to invest at $t = 1$ after $Y_1 = 0$, would switch to investing at $t = 2$ after observing $Y_2 = 1$. By definition, $\hat{s}_1^{Y_1} \leq \hat{s}_0$ and $\hat{s}_2^{Y_2} \leq \hat{s}_1^0$.

Our goal is to show that $\hat{s}_2^0 = \hat{s}_1^0 = \hat{s}_0$; that is, no one would invest after $Y_1 = 0$ at $t = 1$, and no one invests after $Y_2 = 0$ at $t = 2$. If this holds true, then as no one invests after $Y_1 = 0$, $Y_1 = 0$ implies $Y_2 = 0$. Moreover, since $Y_1 = 1$ always implies $Y_2 = 1$, in any possible monotone equilibrium, Y_2 does not provide any additional information on top of Y_1. As such, we only need to solve for $\hat{s}_0$ and $\hat{s}_1^1$ since $\hat{s}_1^1 \leq \hat{s}_0 = \hat{s}_1^0 = \hat{s}_2^{Y_2}$. In other words, the game degenerates to a 2-period model as we have studied.

Lemma 6. *In the case with $T = 3$, under $\gamma > \gamma^*$ (or, equivalently, $\delta > \delta^*$), the game degenerates to a two period model with no one waits and then invests at $t = 3$. The unique cutoff equilibrium is characterized in Proposition 1.*

Lemma 6 demonstrates that under the condition $\delta > \delta^*$, one agents will invest after $Y_1 = 0$, so that Y_2 will not provide any additional information and no one will wait to the last period, and then invest. As such, the 3-period model degenerates to our benchmark model with two periods.

5.2 Full Dynamic Model

Next, we extend the model to a full dynamic model with T periods with $T \geq 3$. We begin with a formal description of the dynamic model, in particular, the definition of history and monotone strategies.

Full History Fix any T and consider the dynamic game in its strategic form. The possible histories can be written as $h_{T-1} = (Y_0, Y_1, \ldots, Y_{T-1})$. Recall that, if $Y_t = 1$, then $Y_{t'} = 1$ for all $t' > t$ regardless of what agents do after period t. Therefore, for any history h_{T-1} with $Y_{T-1} = 1$, we can define $t_1(h_{T-1}) := \min\{t | Y_t = 1\}$ as the first time when Y_t switches from 0 to 1. Therefore, all possible histories can be characterized as $t_1(h_{T-1}) \in \{1, 2, \ldots, T-1\}$ and $(Y_t = 0)_{t=1}^{T-1}$.

Monotone Strategy Given that, we can simply characterize the monotone strategy as $(\hat{s}_0, (\hat{s}_t^{Y_t})_{t=1}^{T-1})$ with $\hat{s}_t^{Y_t} \leq \hat{s}_{t-1}^{Y_{t-1}=0}$ for all $t \in \{1, 2, \ldots, T-1\}$. For history h_{T-1} with $Y_{T-1} = 0$, the agent playing monotone strategy $(\hat{s}_0, (\hat{s}_t^{Y_t})_{t\geq 1})$ invests at period 0 if and only if $s_i \geq \hat{s}_0$; and invests at period t if and only if $s_i \geq \hat{s}_t^{Y_t=0}$. For any other history h_t that admits $t_1(h_t) \leq T-1$, the agents playing monotone strategy $(\hat{s}_0, (\hat{s}_t^{Y_t})_{t\geq 1})$ invests at period 0 iff $s_i \geq \hat{s}_0$; invests at period $t \in [1, t_1(h_t))$ iff $s_i \geq \hat{s}_t^{Y_t=0}$; and invests at $t = t_1(h_{T-1})$ iff $s_i \geq \hat{s}_t^{Y_t=1}$. Recall that, for any $t > t_1(h_{T-1})$, by design, $Y_t = 1$ regardless of the agents' play. Therefore, there is no unnecessary delay so that no one invests after $t_1(h_{T-1})$.

Based on the insight gained from Lemma 6, we can show that in a dynamic model with multiple periods, T, under the binary signal Y we consider, agents, if waiting at $t = 0$, invest only at the first period $t = 1$ after $Y_1 = 1$; otherwise, the delaying agents never invest. As such, the game reduces to our benchmark model. Under the condition that market moves fast and the the waiting period is sufficiently short, coordinating on the irreversible action (e.g., run on a bank) becomes more difficult than that in the static model.

Proposition 9. *In any monotone equilibrium with $T \geq \frac{r}{\ln \frac{(1-c)(1-\gamma)}{1-c-\gamma}}$, no one will wait and then invest at period $t = 1$ when $Y_1 = 0$, and no one will wait and then invest at period $t \geq 1$ regardless of Y_t.*

6 Conclusion

Coordination usually happens in a dynamic fashion and agents always have the option of delay for more information. In this paper, we construct a dynamic model of coordination with an option for delay. A binary public signal arises depending on the history of action. Interestingly, the good news — sufficiently many other agents have already pledged to this action — is more likely to be generated ex-ante only when it is less informative ex-post. (Both the availability and informativeness depend on $\Theta_\gamma(s_1^*)$, and, in turn, on equilibrium strategy $\hat{s}_1$.) We solve for the unique equilibrium in monotone strategies and observe that the option of delay may not facilitate coordination. When agents are inclined to wait for the good news instead of acting earlier, this makes the good news difficult to be produced, which, in turn, hinders successful coordination. This always happen if the delay cost is sufficiently low, either because of a small waiting period or a low interest rate.

Theoretically, this study offers a simple and tractable dynamic model, which features option of delay (dynamic structure), privately informed agents with public learning about history (information environment), and strategic complementarity (payoff interaction). When the public information is generated by the past actions, our model proves the uniqueness of equilibrium holds true generically. Compared with static models, we construct some numerical example to demonstrate that this equilibrium selection provides a more plausible description of coordination problems in reality. With regard to policy implications, our dynamic model identifies a channel through which lowering the interest rate may encourage the inaction (or waiting) and slow the economic recovery.

Throughout the paper, we take the information generating process of the public signal Y exogenous. It can be thought of as the agent naturally have such learning opportunities. We briefly discuss the case in which a policy maker can influence the accessibility of information (or γ). It would be interesting to study how to provide additional information to facilitate coordination (given the exogenous source of public information).[16] In addition, we restrict our attention to monotone strategies throughout the paper, and the uniqueness of equilibrium relies on that restriction. The standard iterated elimination idea in global game literature cannot be applied to the dynamic setting with endogenous timing because of the lack of complementarity in the sense that if more agents choose to attack early, the incentive of attacking early might be reduced when the delay option is available. It will be interesting, although challenging, to study under what conditions, the equilibrium monotone strategies is uniquely rationalizable. We believe these are promising areas for future studies.

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[16]In a recent study, Basak and Zhou (2021) study the optimal information provision policy in a different dynamic coordination setup where agents who choose to wait may miss the opportunity of successful attack. They do not consider some exogenous source of public information as in the present paper but assume that the policy maker have perfect control of information flow overtime. The optimal policy found in Basak and Zhou (2021) requires the policy maker to have access to the fundamental θ, and thus, is different from the exogenous public signal Y considered in the present paper.

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Appendix

Proof of Lemma 5. Suppose there exists an equilibrium strategy $(\hat{s}_1, \hat{s}_2^0, \hat{s}_2^1)$ in which $\hat{s}_2^0 < \hat{s}_1$. Then, based on Lemma 4, $\hat{s}_2^1 = \hat{s}_1$ must holds, meaning that no one will choose to invest after $Y = 1$ but some agents with $s_i \in (\hat{s}_2^0, \hat{s}_1)$ would wait and then invest after $Y = 0$, which is equivalent to $\theta < \Theta_\gamma(\hat{s}_1)$. Successful coordination requires that $\theta > \Theta(\hat{s}_2^0)$. Therefore $\Theta(\hat{s}_2^0) < \Theta_\gamma(\hat{s}_1)$. Another condition that must hold is $\Theta_\gamma(\hat{s}_1) < \Theta(\hat{s}_1)$. Otherwise, if $\Theta(\hat{s}_1) \leq \Theta_\gamma(\hat{s}_1)$, then $Y = 1$ reveals that the mass of investment in the first period is sufficient for a successful coordination and investment is a dominating strategy in the second period, thus contradicting our assumption $\hat{s}_2^1 = \hat{s}_1$ To summarize,

$$\Theta(\hat{s}_2^0) < \Theta_\gamma(\hat{s}_1) < \Theta(\hat{s}_1). \tag{A.1}$$

We now prove the Lemma by contraction in the two steps that were described heuristically in the text. First, agents in the one-period subgame, after the bad news $Y = 0$ invest less than in the static game. In the subgame with strategy $\hat{s}_2^0$, the agent with signal $\hat{s}_2^0$ is indifferent between investing and not investing, i.e.,

$$\mathbb{P}\left(\theta \in [\Theta(\hat{s}_2^0), \Theta_\gamma(\hat{s}_1)) \middle| \theta < \Theta_\gamma(\hat{s}_1), \hat{s}_2^0\right) = \frac{F(\Theta_\gamma(\hat{s}_1) - \hat{s}_2^0) - F(\Theta(\hat{s}_2^0) - \hat{s}_2^0)}{F(\Theta_\gamma(\hat{s}_1) - \hat{s}_2^0)} = c.$$

Therefore, we have

$$F(\Theta(\hat{s}_2^0) - \hat{s}_2^0) = (1 - c)F(\Theta_\gamma(\hat{s}_1) - \hat{s}_2^0).$$

By definition of $\Theta(\cdot)$ (see (4)), we have[17]

$$\Theta(\hat{s}_2^0) = 1 - F(\Theta(\hat{s}_2^0) - \hat{s}_2^0) = 1 - (1 - c)F(\Theta_\gamma(\hat{s}_1) - \hat{s}_2^0) > c.$$

Since $c = \Theta(s^*)$, and $\Theta(\cdot)$ is an increasing function, this inequality shows that $s^* < s_1^0$: the investment set is smaller than in the one-period game. Therefore,

$$c < \Theta(\hat{s}_2^0) < \Theta_\gamma(\hat{s}_1) < \Theta(\hat{s}_1). \tag{A.2}$$

In the second step, we consider an agent with $s_i = \hat{s}_1^- := \lim_{\varepsilon \downarrow 0} \hat{s}_1 - \varepsilon$. To have such an equilibrium, in the sub-game starting from $Y = 1$, this agent does not invest. However, the ex-ante payoff from investing after $Y = 1$ for agent with $\hat{s}_1$ is

$$\mathbb{P}(\theta \geq \Theta(\hat{s}_1)|\hat{s}_1) \cdot 1 - \mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|\hat{s}_1) \cdot c = \Theta(\hat{s}_1) - (1 - \gamma)c.$$

This payoff is strictly positive as $\Theta(\hat{s}_1) > c$ (see (A.1)). By continuity, this implies that the agent with $s_i = \hat{s}_1^-$ would strictly prefer to investing following $Y = 1$, which contradicts with the condition that $\hat{s}_2^1 = \hat{s}_1$. Thus, there does not exist any equilibrium with $\hat{s}_2^0 < \hat{s}_2^1 = \hat{s}_1$. □

[17] The inequality is strict because, obviously, $\hat{s}_2^0 > -\infty$ and $\Theta_\gamma(\hat{s}_1) < +\infty$. As such, $\hat{s}_2^0 > s^*$ must hold.

Proof of Proposition 1. We have shown the indifferent condition of $\hat{s}_1$ in the main text. Now, we proceed to show that for any $s_i > \hat{s}_1$ (or $s_i < \hat{s}_1$), the agent would prefer invest at $t = 1$ (or delay the investment). Denote the expected payoff difference as

$$D(s, \theta_\gamma, \hat{\theta}) := \mathbb{P}(\theta \geq \hat{\theta}|s) - c - \delta(1-c)\mathbb{P}(\theta \geq \theta_\gamma|s) = \Phi(\frac{s-\hat{\theta}}{\sigma}) - c - \delta(1-c)\Phi(\frac{s-\theta_\gamma}{\sigma}),$$

in which, $\hat{\theta} = \Theta(\hat{s}_1)$ and $\theta_\gamma = \Theta_\gamma(\hat{s}_1)$ for any given $\hat{s}_1$. Recall that $\hat{\theta} < \theta_\gamma$. Next, observe that $D(\hat{s}_1, \theta_\gamma, \hat{\theta}) = 0$ and

$$\lim_{s\to\infty} D(s, \theta_\gamma, \hat{\theta}) = (1-\delta)(1-c) > 0, \quad \lim_{s\to-\infty} D(s, \theta_\gamma, \hat{\theta}) = -c < 0$$

Furthermore,

$$D_s(s, \theta_\gamma, \hat{\theta}) = \frac{1}{\sigma}\left(\phi(\frac{s-\hat{\theta}}{\sigma}) - \delta(1-c)\phi(\frac{s-\theta_\gamma}{\sigma})\right) = \frac{1}{\sigma}\phi(\frac{s-\theta_\gamma}{\sigma})\left(e^{\frac{1}{2\sigma^2}(\hat{\theta}-\theta_\gamma)(2s-\hat{\theta}-\theta_\gamma)} - \delta(1-c)\right)$$

Therefore, $D_s(s, \theta_\gamma, \hat{\theta}) = 0$ when $s = s^D \equiv \frac{\sigma^2}{\hat{\theta}-\theta_\gamma}\ln(\delta(1-c)) + \frac{\theta_\gamma+\hat{\theta}}{2}$. Moreover, $D_s(s, \theta_\gamma, \hat{\theta}) > 0$ if and only if $s < s^D(\theta_\gamma, \hat{\theta})$. Therefore, we know that D increases with s from $s = -\infty$, crossing 0 at $\hat{s}_1$, reaches its maximum at $s^D(\theta_\gamma, \hat{\theta})$, then decreases with s, and ultimately converging to $(1-\delta)(1-c)$. As such, it is clear that for any $s < \hat{s}_1$ ($s > \hat{s}_1$), $D(s, \theta_\gamma, \hat{\theta}) < 0$ ($D(s, \theta_\gamma, \hat{\theta}) > 0$).

Next, we prove that there does not exist any equilibrium with $\hat{s}_1 \leq s_\gamma$, so that the one characterized by (10) is the unique monotone equilibrium. Consider any $\hat{s}_1 \leq s_\gamma$ as an candidate equilibrium time-1 investment strategy and suppose the equilibrium fundamental cutoff is $\hat{\theta}$. Under the condition $\hat{s}_1 \leq s_\gamma$, $\Theta(\hat{s}_1) \geq \Theta_\gamma(\hat{s}_1)$. As there must be no investment after $Y = 0$, coordination cannot be successful for any $\theta < \Theta_\gamma(\hat{s}_1)$. Therefore, regardless of the equilibrium $\hat{s}_2$, we know that the equilibrium fundamental cutoff is such that $\hat{\theta} \geq \Theta_\gamma(\hat{s}_1)$. In this case, the expected payoff difference for the marginal agent is

$$\begin{aligned}&\mathbb{P}(\theta \geq \hat{\theta}|\hat{s}_1) - c - \delta\left(\mathbb{P}(\theta \geq \hat{\theta}|\hat{s}_1) \times 1 - \mathbb{P}(Y=1|\hat{s}_1)c\right)\\ \leq&\mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|\hat{s}_1) - c - \delta\mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|\hat{s}_1)(1-c) = H(\gamma) < 0,\end{aligned}$$

which implies that, for any $\hat{s}_1 \leq s_\gamma$, the marginal agent always prefers to wait. This completes the proof. □

Proof of Proposition 2. The equilibrium is solution of the system of equations (15), (18), which is repeated here

$$\frac{\Theta(\hat{s}_2)}{F(\hat{s}_2 - \Theta_\gamma(\hat{s}_1))} = c, \qquad F(\hat{s}_1 - \Theta(\hat{s}_2)) = c\frac{1-\delta(1-\gamma)}{1-\delta}.$$

Substituting in the first equation $\Theta_\gamma(\hat{s}_1) = \hat{s}_1 + F^{-1}(\gamma)$, eliminating $\hat{s}_1$ that is given in the second equation, and using $F(\hat{s}_2 - \Theta(\hat{s}_2)) = \Theta(\hat{s}_2)$, the system is reduced to

$$G(x;a) = c, \qquad x = \Theta(\hat{s}_2), \tag{A.3}$$

with

$$G(x;a) = \frac{x}{F(F^{-1}(x)+a)}, \qquad a = -F^{-1}(c\frac{1-\delta(1-\gamma)}{1-\delta}) - F^{-1}(\gamma).$$

Using the symmetry property of F, $\gamma < \gamma^*$ is equivalent to $a > 0$.

Next, we prove the existence and uniqueness of $x = \Theta(\hat{s}_2)$ as a solution to (A.3). Recall that F presents a normal CDF with mean 0 and variance σ^2. Therefore, as normal distribution is log-concave, it is easy to show that $G(x;a)$ is strictly increasing in $x \in (0,1)$. For the limiting cases, we have $Lim_{x\to 1} G(x;a) = 1$, and

$$\lim_{x\to 0} G(x;a) = \lim_{x\to 0} \frac{x}{\Phi\left(\Phi^{-1}(x)+a\right)} = \lim_{x\to 0} \frac{1}{\phi(\Phi^{-1}(x)+a)\frac{1}{\phi(\Phi^{-1}(x))}} = \lim_{x\to 0} e^{\frac{1}{2\sigma^2}\left(2a\Phi^{-1}(x)+a^2\right)} = 0.$$

Therefore, for any $c \in (0,1)$, the equation (A.3) has a unique solution.[18] Accordingly, there is a unique solution for $\Theta(\hat{s}_2) = x$ and, since $\Theta(\cdot)$ is monotone increasing, a unique solution for $\hat{s}_2$. The unique solution for $\hat{s}_1$ is given by (18).

As the function G is independent of σ, the standard deviation in the cdf F, so is the equilibrium cutoff $\Theta(\hat{s}_2)$. However, the cutoff strategy does depend on σ as $\hat{s}_2 = \hat{x} + \sigma\Phi^{-1}(\hat{x})$.

As $G(x;a)$ is strictly decreasing in a and a is strictly decreasing in γ (or δ), the solution lower bound of coordination $\Theta(\hat{s}_2)$ and, accordingly, the information cutoff $\hat{s}_2$ is decreasing in γ for $\gamma < \gamma^*$ (or δ for $\delta < \delta^*$).

Next, we go beyond the indifference conditions to prove that $(\hat{s}_1, \hat{s}_1)$ is indeed an equilibrium. We further verify the sufficient conditions.

[18] The proposition is true for a class of distributions that is wider than the normal distribution. It is sufficient that $G(0) = 0$, and for that, the left tail of the distribution of F cannot be "thick", which is equivalent to a sufficiently strong concavity of $Log(F)$. If, for example, $F(x)$ is asymptotically equivalent to $x - 1/x$ when $x \to -\infty$ (asymptotically Log-linear), $G(x)$ tends to a strictly positive limit when $x \to -\infty$, and if that limit is strictly greater than c, the proof does not apply.

Sufficient Condition for $\hat{s}_1$ For agent i with s_i, given other agents' strategy $(\hat{s}_1, \hat{s}_2)$, the expected payoff difference between investing early and waiting and then investing (after $Y = 1$) is

$$\begin{aligned}&\mathbb{P}(\theta \geq \Theta(\hat{s}_2)|s_i) - c - \delta\left[\mathbb{P}(\theta \geq \Theta(\hat{s}_2)|s_i) - \mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|s_i)c\right]\\=&(1-\delta)\Phi(\frac{s_i - \Theta(\hat{s}_2)}{\sigma}) + \delta c \Phi(\frac{s_i - \Theta_\gamma(\hat{s}_1)}{\sigma}) - c\end{aligned}$$

Let us define a function

$$\beta(s_i, \hat{\theta}, \theta_\gamma) \equiv (1-\delta)\Phi(\frac{s_i - \hat{\theta}}{\sigma}) + \delta c \Phi(\frac{s_i - \theta_\gamma}{\sigma}) - c$$

Recall that the equilibrium $(\hat{s}_1, \hat{s}_2)$ satisfy (1) $\Theta(\hat{s}_2) = \hat{\theta} > \Theta_\gamma(\hat{s}_1) = \theta_\gamma$, and (2) $\beta(\hat{s}_1, \hat{\theta}, \theta_\gamma) = 0$. It is easy to check that the payoff difference $\beta(s_i, \hat{\theta}, \theta_\gamma)$ is strictly increasing in s_i. Further, we know that

$$\lim_{s_i \to -\infty} \beta(s_i, \hat{\theta}, \theta_\gamma) = -c \qquad \text{and} \qquad \lim_{s_i \to +\infty} \beta(s_i, \hat{\theta}, \theta_\gamma) = (1-\delta)(1-c) > 0.$$

Therefore, this payoff difference is strictly positive (negative) for $s_i > \hat{s}_1$ ($s_i < \hat{s}_1$), thereby completing the proof that agents with $s_i > \hat{s}_1$ strictly prefer to investing early.

Sufficient Condition for $\hat{s}_2$ Next, consider the expected payoff from waiting and then investing after $Y = 1$:

$$\delta\left[\mathbb{P}(\theta \geq \Theta(\hat{s}_2)|s_i) - \mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|s_i)c\right] = \delta\Phi(\frac{s_i - \Theta_\gamma(\hat{s}_1)}{\sigma})\left[\frac{\Phi(\frac{s_i - \Theta(\hat{s}_2)}{\sigma})}{\Phi(\frac{s_i - \Theta_\gamma(\hat{s}_1)}{\sigma})} - c\right]. \tag{A.4}$$

An agent would strictly prefer to taking this strategy if the following payoff is strictly positive. Based on the log-concavity of Φ and the fact that $\Theta(\hat{s}_2) > \Theta_\gamma(\hat{s}_1)$, it is easy to check that

$$\frac{\Phi(\frac{s_i - \Theta(\hat{s}_2)}{\sigma})}{\Phi(\frac{s_i - \Theta_\gamma(\hat{s}_1)}{\sigma})} - c$$

is strictly increasing in s_i. Recall that when $s_i = \hat{s}_2$, the above expression obtains a value of 0. Therefore, this expression and accordingly the expected payoff in (A.4) is strictly positive (negative) if and only if $s_i > \hat{s}_2$ ($s_i < \hat{s}_2$). This confirms that agents with signal $s_i < \hat{s}_2$ strictly prefer not to invest.

Note that under the condition $\gamma < \gamma^*$ (or $\delta < \delta^*$), there does not exist equilibrium with $\hat{s}_1 > s_\gamma$ and $\hat{s}_2 = -\infty$. This is because, under this parameter condition, the indifference condition cannot hold for the marginal agent with $\hat{s}_1$ for any $\hat{s}_1 > s_\gamma$, which can be seen from condition (11). This complete the proof.

□

Proof of Proposition 3. We first prove that, in any possible equilibrium, the lower-bound of coordination $\hat{\theta} = \Theta_\gamma(\hat{s}_1)$. Or equivalently, $Y = 1$ perfectly predicts the coordination success, which requires $\Theta(\hat{s}_2) \leq \Theta_\gamma(\hat{s}_1) \leq \Theta(\hat{s}_1)$.

To show this, suppose the fundamental cutoff $\hat{\theta} > \Theta_{\gamma^*}(\hat{s}_1)$ for some equilibrium $\hat{s}_1$. Then, for the marginal agent with $\hat{s}_1$, the expected payoff difference between investing early and waiting and then investing after $Y = 1$ is strictly negative when $\gamma = \gamma^*$, regardless of the value of $\hat{s}_1$; that is,

$$
\begin{aligned}
&\mathbb{P}(\theta \geq \hat{\theta}|\hat{s}_1) - c - \delta\left(\mathbb{P}(\theta \geq \hat{\theta}|\hat{s}_1) - \mathbb{P}(\theta \geq \Theta_{\gamma^*}(\hat{s}_1)|\hat{s}_1)c\right) \\
<&(1-\delta)\mathbb{P}(\theta \geq \Theta_{\gamma^*}(\hat{s}_1)|\hat{s}_1) - c + \delta\mathbb{P}(\theta \geq \Theta_{\gamma^*}(\hat{s}_1)|\hat{s}_1)c \\
=&(1-\delta)(1-\gamma^*) - c + \delta(1-\gamma^*)c = 0.
\end{aligned}
$$

Therefore, no such equilibrium can exist. Similarly, suppose $\hat{\theta} < \Theta_{\gamma^*}(\hat{s}_1)$, this expected payoff difference for the marginal agent is always positive regardless of $\hat{s}_1$; that is,

$$
\begin{aligned}
&\mathbb{P}(\theta \geq \hat{\theta}|\hat{s}_1) - c - \delta\mathbb{P}(\theta \geq \Theta_{\gamma^*}(\hat{s}_1)|\hat{s}_1)\,(1-c) \\
>&\mathbb{P}(\theta \geq \Theta_{\gamma^*}(\hat{s}_1)|\hat{s}_1) - c - \delta\mathbb{P}(\theta \geq \Theta_{\gamma^*}(\hat{s}_1)|\hat{s}_1)\,(1-c) \\
=&(1-\gamma^*) - c - \delta(1-\gamma^*)(1-c) = 0.
\end{aligned}
$$

Therefore, $\hat{\theta}$ must be equal to $\Theta_{\gamma^*}(\hat{s}_1)$ in equilibrium. It is then easy to check that, under $\gamma = \gamma^*$, when $\hat{\theta} = \Theta_{\gamma^*}(\hat{s}_1)$, the expected payoff difference is always 0 regardless of $\hat{s}_1$.

Next, as $\hat{\theta} = \Theta_{\gamma^*}(\hat{s}_1)$, $Y = 1$ predicts coordination success and thus, $\hat{s}_2 = -\infty$. We need to find the time-1 cutoff $\hat{s}_1$ which makes the condition $\Theta(\hat{s}_2) = 0 \leq \Theta_{\gamma^*}(\hat{s}_1) \leq \Theta(\hat{s}_1)$ holds true. (Only under this condition, $\hat{\theta} = \Theta_{\gamma^*}(\hat{s}_1)$.) Recall that $\hat{s}_1 \leq s_{\gamma^*}$ is the sufficient and necessary condition for $\Theta_{\gamma^*}(\hat{s}_1) \leq \Theta(\hat{s}_1)$ (see (6) and (8)). In addition, as $\Theta_{\gamma^*}(\hat{s}_1) = \hat{s}_1 + \sigma\Phi^{-1}(\gamma^*)$, $\Theta(\hat{s}_1 = -\infty) \leq \Theta_{\gamma^*}(\hat{s}_1)$ requires $\hat{s}_1 \geq \sigma\Phi^{-1}(1-\gamma^*)$. Therefore, s_1^* can be any number between $\sigma F^{-1}(1-\gamma^*)$ and s_{γ^*} and $s_2^* = -\infty$. In equilibrium, $\theta^* = \Theta_\gamma(s_1^*)$.

We then check the sufficient conditions to confirm that any $\hat{s}_1 \in [\sigma\Phi^{-1}(1-\gamma), s_{\gamma^*}]$ and $\hat{s}_2 = -\infty$ can constitute an equilibrium. Given that all other agents take the strategy of any $\hat{s}_1 \in [\sigma\Phi^{-1}(1-\gamma^*), s_{\gamma^*}]$, and $\hat{s}_2 = -\infty$, it is easy to check that the expected payoff difference between investing early and delaying and then only investing after $Y = 1$

$$
\mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|s_i) - c - \delta\mathbb{P}(\theta \geq \Theta_\gamma(\hat{s}_1)|s_i)(1-c) = (1-\delta(1-c))\Phi(\frac{s_i - \Theta_\gamma(\hat{s}_1)}{\sigma}) - c
$$

is strictly increasing in s_i, and, as we have shown, the difference obtains 0 when $s_i = \hat{s}_1$. As such, for any $s_i > \hat{s}_1$, the agent strictly prefers to investing at $t = 1$; and for $s_i < \hat{s}_1$, the agent strictly prefers to waiting and then investing (after $Y = 1$). This completes the proof. □

Proof of Proposition 8. Following (26),

$$R'(\hat{s}_2) = \frac{\Theta'(\hat{s}_2)\phi(\frac{R(\hat{s}_2)-\Theta(\hat{s}_2)}{\sigma})}{\phi(\frac{R(\hat{s}_2)-\Theta(\hat{s}_2)}{\sigma}) - c\phi(\frac{R(\hat{s}_2)-\Theta_\gamma(\hat{s}_1)}{\sigma})}$$

By definition of $\Theta(\cdot)$, we know (recall that $R(\hat{s}_2) = \hat{s}_2$)

$$\frac{R(\hat{s}_2) - \Theta(\hat{s}_2)}{\sigma} = \Phi^{-1}(\Theta(\hat{s}_2)) \text{ and } \Phi'(\hat{s}_2) = \frac{\phi(\Phi^{-1}(\Theta(\hat{s}_2))}{\sigma + \phi(\Phi^{-1}(\Theta(\hat{s}_2)))} \in (0,1)$$

Therefore,

$$\begin{aligned} R'(\hat{s}_2) &= \Theta'(\hat{s}_2)\frac{\phi(\Phi^{-1}(\Theta(\hat{s}_2)))}{\phi(\Phi^{-1}(\Theta(\hat{s}_2))) - c\phi(\frac{\hat{s}_2-\Theta(\hat{s}_2)+\Theta(\hat{s}_2)-\Theta_\gamma(\hat{s}_1)}{\sigma})} \\ &= \frac{\phi(\Phi^{-1}(\Theta(\hat{s}_2))}{\sigma + \phi(\Phi^{-1}(\Theta(\hat{s}_2)))}\frac{\phi(\Phi^{-1}(\Theta(\hat{s}_2)))}{\phi(\Phi^{-1}(\Theta(\hat{s}_2))) - c\phi(\Phi^{-1}(\Theta(\hat{s}_2)) + \frac{\Theta(\hat{s}_2)-\Theta_\gamma(\hat{s}_1)}{\sigma})} \\ &= \frac{\phi(\Phi^{-1}(\Theta(\hat{s}_2))}{\sigma + \phi(\Phi^{-1}(\Theta(\hat{s}_2)))}\frac{\phi(\Phi^{-1}(\Theta(\hat{s}_2)))}{\phi(\Phi^{-1}(\Theta(\hat{s}_2))) - c\phi(\Phi^{-1}(\Theta(\hat{s}_2)) + b)}. \end{aligned}$$

Recall that we defined $b = \frac{\Theta(\hat{s}_2)-\Theta_\gamma(\hat{s}_1)}{\sigma} = -\Phi^{-1}\left(c\frac{1-\delta(1-\gamma)}{1-\delta}\right) - \Phi^{-1}(\gamma) > 0$. Obviously, $R'(\hat{s}_2)$ decreases with σ. When $\sigma \to 0$, $R'(\hat{s}_2) > 1$ and when $\sigma \to +\infty$, $R'(\hat{s}_2) \to 0$. By continuity, we can find $\sigma^{*'} > 0$ such that when $\sigma < \sigma^{*'}$, $R'(\hat{s}_2) > 1$. Let $\sigma_0 = \min\{\sigma^{*'}, \sigma^*\}$. By definition, for any $\sigma < \sigma_0$, as $\Theta(\hat{s}_1) \geq 0$, all investing at $t = 2$ is a sub-game equilibrium, which the equilibrium $\hat{s}_2 > -\infty$ cannot constitute a stable sub-game equilibrium. □

Proof of Lemma 6. First of all, consider any monotone equilibrium with $\hat{s}_1^1 < \hat{s}_0$. In this case, following the option value argument (Lemma xxx), we know that $\hat{s}_1^0 = \hat{s}_0$. (That is, it is not possible that an agent with $s_i = \hat{s}_0 - \epsilon$ will invest at $t = 1$ regardless of Y_1 since delay is costly.) Therefore, if $\hat{s}_1^1 < \hat{s}_0$, no one invests following $Y_1 = 0$ at $t = 1$ ($\hat{s}_1^0 = \hat{s}_0$), and, because of that, it is certain that $Y_2 = 0$ and no one will wait and then invest at $t = 2$. That said, a necessary condition for having investment after $Y_t = 0$ ($t = 1, 2$) is $\hat{s}_1^1 = \hat{s}_0$ and $\hat{s}_1^0 < \hat{s}_0$; that is, no one invests at $t = 1$ following $Y_1 = 1$ but some one chooses to invest at $t = 1$ following $Y_1 = 0$.

Therefore, to show that $\hat{s}_2^0 = \hat{s}_1^0 = \hat{s}_0$, it is sufficient to show that no monotone equilibrium with $\hat{s}_1^1 = \hat{s}_0$ and $\hat{s}_1^0 < \hat{s}_0$ exists. Next, we prove that such equilibrium cannot exist under the condition $\gamma > \gamma^*$.

First, the condition $\hat{s}_1^1 = \hat{s}_0$ and $\hat{s}_1^0 < \hat{s}_0$ implies that an agent with $s_i = \hat{s}_0 - \epsilon$ will not invest at $t = 0$ or invest at $t = 1$ after $Y_1 = 1$; rather, he would invest at $t = 1$ after $Y_1 = 0$. Therefore, the expected payoff from investing at $t = 1$ following $Y_1 = 1$ cannot be strictly positive for agent with

$s_i = \hat{s}_1$, i.e.,

$$\mathbb{P}(\theta \geq \Theta(\hat{s}_0)|Y_1 = 1, \hat{s}_1) - c = \mathbb{P}(\theta \geq \Theta(\hat{s}_0)|\theta \geq \Theta_\gamma(\hat{s}_0), \hat{s}_1) - c \leq 0.$$

That implies $\Theta(\hat{s}_0) \leq (1-\gamma)c$.

Recall that $\Theta(s) > \Theta_\gamma(s)$ if and only if $\Theta(s) < 1-\gamma$. Therefore, $\Theta_\gamma(\hat{s}_0) < \Theta(\hat{s}_0)$ must hold. Moreover, since $\hat{s}_1^0 \leq \hat{s}_0$, we must have $\Theta_\gamma(\hat{s}_1^0) < \Theta(\hat{s}_1^0)$.

Next, we prove that, under the condition $\hat{s}_1^0 < \hat{s}_0$, in the last period ($t = 2$), it is not possible to have $\hat{s}_2^0 < \hat{s}_1^0$. Suppose that $\hat{s}_2^0 < \hat{s}_1^0$. Then, by the option value argument, we have $\hat{s}_2^1 = \hat{s}_1^0$. $Y_2 = 0$ implies that $\theta < \Theta_\gamma(\hat{s}_1^0) < (1-\gamma)c$. Following the same argument as in Lemma XXX, we cannot find such $\hat{s}_2^0$. That is because, since $\Theta(\hat{s}_1^0) > \Theta_\gamma(\hat{s}_1^0)$, the threshold $\hat{s}_2^0$ has to satisfy

$$\mathbb{P}(\theta \geq \Theta(\hat{s}_2^0)|\theta < \Theta_\gamma(s_1^0), \hat{s}_2^0) - c \geq 0,$$

which implies that

$$\Theta(\hat{s}_2^0) \geq 1 - (1-c)F(\Theta_\gamma(\hat{s}_0) - \hat{s}_2^0) \geq c > \Theta(\hat{s}_0).$$

Therefore, the possible monotone equilibrium with $\hat{s}_1^0 < \hat{s}_0$ and $\hat{s}_1^1 = \hat{s}_0$ must feature $\hat{s}_2^0 = \hat{s}_0$; that is, no one invests following $Y_2 = 0$.

Then, we show that the necessary condition for $\hat{s}_1^0 < \hat{s}_0$ is that (1) $\hat{s}_2^1 < \hat{s}_1^0$ and (2) the success following $Y_1 = 0$ is determined by $\theta \geq \Theta(\hat{s}_2^1)$. To see this, consider the case in which the success range is $[\Theta(\hat{s}_1^0), \Theta_\gamma(\hat{s}_0))$ (i.e., coordination success is achieved purely by investment at $t = 2$ following $Y_1 = 0$). If that is the case, following the same reasoning for $\hat{s}_2^0$ as well as the proof for 2-period model, we need to have $\Theta(\hat{s}_1^0) > c$, which contradicts with $\hat{s}_1^0 \leq \hat{s}_0$.

In the last step, we show that it is not possible to construct a monotone equilibrium in which (1) $\hat{s}_2^1 < \hat{s}_1^0 < \hat{s}_0$ and (2) the success range is $[\Theta(\hat{s}_0), +\infty)$ following $Y_1 = 1$ and $[\Theta(\hat{s}_2^1), \Theta_\gamma(\hat{s}_0))$ following $Y_1 = 0$.

Note that, in such an equilibrium, the marginal agent with $\hat{s}_1^0$ is indifferent between investing at $t = 1$ after $Y_1 = 0$ and investing at $t = 2$ following $Y_2 = 1$. That is,

$$\mathbb{P}\left(\theta \in [\Theta(\hat{s}_2^1), \Theta_\gamma(\hat{s}_0))|\hat{s}_1^0, \theta < \Theta_\gamma(\hat{s}_0)\right) - c - \delta\left[\mathbb{P}(\theta \in [\Theta(\hat{s}_2^1), \Theta_\gamma(\hat{s}_0)) - c\mathbb{P}(\theta \in [\Theta_\gamma(\hat{s}_1^0), \Theta_\gamma(\hat{s}_0))\right] = 0$$

Multiplying the above equation by $\mathbb{P}(\theta < \Theta_\gamma(\hat{s}_0)|\hat{s}_1^0) = F(\Theta_\gamma(\hat{s}_0) - \hat{s}_1^0)$ and reorganizing, we have

$$(1-\delta)(1-c)F(\Theta_\gamma(\hat{s}_0) - \hat{s}_1^0) - (1-\delta)F(\Theta(\hat{s}_2^1) - \hat{s}_1^0) - \delta c F(\Theta_\gamma(\hat{s}_1^0) - \hat{s}_1^0) = 0 \qquad \text{(A.5)}$$

The LHS of the above equation is less than (since $\Theta(\hat{s}_2^1) \geq \Theta_\gamma(\hat{s}_1^0)$)

$$\begin{aligned}
&(1-\delta)(1-c)F(\Theta_\gamma(\hat{s}_0) - \hat{s}_1^0) - (1-\delta)F(\Theta(\hat{s}_2^1) - \hat{s}_1^0) - \delta c F(\Theta_\gamma(\hat{s}_1^0) - \hat{s}_1^0) \\
<&(1-\delta)(1-c) - (1-\delta)F(\Theta_\gamma(\hat{s}_1^0) - \hat{s}_1^0) - \delta c\gamma \\
=&(1-\delta)(1-c) - (1-\delta+\delta c)\gamma = (1-\delta)(1-c)(\gamma^* - \gamma)
\end{aligned}$$

Therefore, as long as $\gamma \geq \gamma^*$, then the equation (A.5) cannot hold.

Thus, under the condition that $\gamma \geq \gamma^*$ (or equivalently, $\delta \geq \delta^* = \frac{1-c-\gamma}{(1-c)(1-\gamma)}$), any monotone equilibrium cannot have $\hat{s}_1^0 < \hat{s}_0$ and $\hat{s}_1^1 = \hat{s}_0$. As such, all monotone equilibrium must satisfy that $\hat{s}_1^0 = \hat{s}_0$ and $\hat{s}_1^1 \leq \hat{s}_0$. As $\hat{s}_1^0 = \hat{s}_0$ (that is, no one invests after $Y_1 = 0$), the history ($Y_1 = 0, Y_2 = 1$) is not on path in any possible monotone equilibrium, and therefore, $\hat{s}_2^1$ is irrelevant for the equilibrium outcome and Y_2 does not provide any relevant information and therefore, no one will wait for that information and $X_2 = 0$. For convenience, we write $\hat{s}_2^{Y_2} = \hat{s}_0$ for $Y_2 = 0, 1$. As such, the 3-period model degenerates to the 2-period model we have studied before. □

Proof of Proposition 9. Essentially, we want to prove that, in any monotone equilibrium, $\hat{s}_{T-1}^{Y_{T-1}=0} = \hat{s}_{T-2}^{Y_{T-2}=0} = \ldots = \hat{s}_1^{Y_1=0} = \hat{s}_0$. Under this, as long as $Y_1 = 0$, all subsequent $Y_t = 0$ for all $t \geq 2$. The agent may only wait and then invest at $t = 1$ after $Y_1 = 1$ (i.e., $s_1^1 \leq \hat{s}_0$).

The proof follows the one for $T = 3$ and the condition $T \geq \frac{r}{\ln \frac{1}{\delta^*}}$ ensures that $\delta = e^{-\frac{r}{T}} \geq \delta^*$.

First of all, if $\hat{s}_1^1 < \hat{s}_0$, then $\hat{s}_1^0 = \hat{s}_0$. Given that, conditional on $Y_1 = 0$, $Y_2 = 0$ and therefore, as no one invests following $Y_1 = 0$, no one waits and then invests following $Y_2 = 0$. This argument can go forward to show that $\hat{s}_{T-1}^{Y_{T-1}=0} = \hat{s}_{T-2}^{Y_{T-2}=0} = \ldots = \hat{s}_1^{Y_1=0} = \hat{s}_0$. As such, in order to have investments following $Y_t = 0$ for $t \geq 1$ in an equilibrium, we must need $\hat{s}_1^1 = \hat{s}_0$. That implies, again,

$$\Theta(\hat{s}_0) \leq (1-\gamma)c.$$

Next, since, by definition, $\hat{s}_{T-2}^0 \leq \hat{s}_0$, $\Theta_\gamma(\hat{s}_{T-2}^0) \leq \Theta(\hat{s}_0) \leq (1-\gamma)c$. For the last period, if $Y_{T-1} = 0$, or equivalently, $\theta < \Theta_\gamma(\hat{s}_{T-2}^0) \leq (1-\gamma)c$, following the same argument as for the last period in the example of $T = 3$, in any monotone equilibrium, we must have $\hat{s}_{T-1}^0 = \hat{s}_{T-2}^0$.

As we have shown in the case with $T = 3$, in order to have $\hat{s}_{T-2}^0 < \hat{s}_{T-3}^0$, we must need $s_{T-1}^1 < \hat{s}_{T-2}^0$. However, that is impossible to since under $\delta \geq \delta^*$, the indifference condition for $\hat{s}_{T-2}^0$ (between investing at $t = T-2$ following $Y_{T-2} = 0$ and waiting and then investing at $t = T-1$ after $Y_{T-1} = 1, Y_{T-2} = 0$) cannot hold true. As such, in any monotone equilibrium, $\hat{s}_{T-2}^0 = \hat{s}_{T-3}^0$, and therefore, $Y_{T-2} = 0$ implies $Y_{T-1} = 0$ (no one invests at $t = T-2$). For that reason, Y_{T-1} is redundant since $Y_{T-1} = Y_{T-2}$ regardless of $Y_{T-2} = 0$ or 1, and $\hat{s}_{T-1}^1$ is irrelevant.

Given that, in any monotone equilibrium, $\hat{s}_{T-1}^{Y_{T-1}} = \hat{s}_{T-2}^{0}$ for $Y_{T-1} = 0, 1$, $T-2$ is essentially the last period for the dynamic game. Applying the same arguments inductively, we can prove that $\hat{s}_{T-1}^{Y_{T-1}=0} = \hat{s}_{T-2}^{Y_{T-2}=0} = \ldots = \hat{s}_{1}^{Y_1=0} = \hat{s}_0$. □