Perfect Altruism Breeds Time Consistency*

Antoine Billot† and Xiangyu Qu‡

Abstract

Economic analysis and policy making, in particular about intergenerational equity, mainly centre on the determination of parameters of social lifetime utility. This paper focuses on the general process that makes these policies socially acceptable to individuals through their own discount factors and instantaneous utilities. We show that *perfect altruism* via an adapted form of unanimity is the key condition helping to characterize a time-consistent society in the presence of individuals who are heterogeneous in discount factors and instantaneous utilities. In addition, different intensity levels of altruism are proven to provide different forms of aggregated social discounting and instantaneous utility, these forms giving rise to several lifetime utilities, from the standard exponential discounted function to the quasi-hyperbolic and the k-hyperbolic functions. Moreover, by demonstrating that the degree of social present bias can be regulated by the choice of the number of periods involving altruism through unanimity, novel insights might emerge and potentially overturn some recommendations for economic policy.

1 Introduction

The roots of most social economic decisions, ranging from fiscal policy (Barro [1974]) to environmental policy (Nordhaus [2007]), are linked to a practice of setting social discount factor and instantaneous utility. The standard approach is paternalistic, such as Ramsey's near-one discount factor and utilitarian social instantaneous utility. This method largely neglects individual preferences, which violates the spirit of democracy. In particular, in the presence of social heterogeneity, Zuber [2011] and Jackson and Yariv [2014] prove that any nonpaternalistic society is incompatible

*We thank Itzhak Gilboa, Marcus Pivato, Norio Takeoka, Jean-Marc Tallon and David Schmeidler for helpful comments and discussions.

†LEMMA, Université Panthéon-Assas (Paris 2): billot@u-paris2.fr. Billot conducts this research as part of the project Labex MMEDII (ANR 11-LABX-0033-01) for support.

‡CNRS, Centre d'Economie de la Sorbonne: xiangyuqu@gmail.com

with Pareto principle of unanimity, which, claimed by Buchanan and Tullock [1962], is ethically superior to all alternative principles. Without the support by any reasonable existing principle, the legitimacy of the paternalism is widely questioned (Marglin [1963] and Feldstein [1964]).

In this paper, we argue that, in the context of preference heterogeneity, the Pareto principle of unanimity is far less compelling than preference homogeneity counterpart. We, therefore, suggest some novel principles, which are superior to Pareto condition in our opinion. We show that under our conditions, social preferences are consistent with certain aggregation of individual preferences, in particular, social discount factor and instantaneous utility takes certain form of individual discount factors and instantaneous utilities, respectively. Unlike standard approach, ours is nonpaternalistic and is immune to the question of legitimacy.

The *exponential time discounted utility* (EDU) model due to Ramsey [1928] and Samuelson [1937] has long been recognized as the canonical model of this representative agent. Although Marglin [1963] and Feldstein [1964] have highlighted the difficulty of deriving a social lifetime EDU by aggregating a society of heterogeneous individuals, this form is still widely used to evaluate various policies because of its simplicity and, more importantly, preferences homogeneity assumption. The issue of the social aggregation, however, has recently introduced novel challenges at the academic frontier between theoretical considerations and policy debates. One of these challenges is invalidity of homogeneous discounting assumption, which has effect in the context of climate change. Optimal climate policy is related to the social value of the future and, therefore, depends critically on the discounting factor (Nordhaus [2007]). Surveys by Frederick, Loewenstein, and O'Donoghue [2002] and more recent by Cohen, Ericson, Laibson, and White [2020], individual discount rates differ dramatically across different studies and estimations. As Weitzman [2001] and Drupp, Freeman, Groom, and Nesje [2018] demonstrate, there is no convergence toward a unique rate of impatience even among experts. Furthermore, there is long traditional beliefs in economics that individuals differ in tastes and, therefore, are heterogenous in instantaneous utilities. This entails both a conceptual and a theoretical difficulty in justifying this difference through a consistent preference aggregation process. Neglecting no aspect of individual heterogeneity is unequivocally a target of our paper.

Another challenge is descriptive and has arisen in the context of political power rotation. It is well known that political turnover leads to time inconsistency, which descriptively falsifies the social EDU assumption (Harstad [2020]). Although the potential implications of social time inconsistency have been frequently noted, few studies have formalized the mechanisms under which preference aggregation may lead to, for instance, quasi-hyperbolic discounting. This constitutes another target of our contribution.

More generally, the approach of this paper seeks to contribute to the literature on time preferences in several ways. We jointly characterize the social discount factor and instantaneous utility across two settings: a general *time-separable utility* (TSU) setting where individuals have TSUs and an EDU setting where individuals are fitted with EDUs. We identify the conditions to quantify social entities through parameters by aggregating individual entities in a nondictatorial fashion, namely, when every individual lifetime utility influences the formation of social utility. Specifically, we advocate the 'utilitarian' idea that society should take a weighted average of individual discount factors, which stands in stark contrast to the argument that society should value the future more than individuals.[1]

The economic tradition advises justifying the transition from individual to social entities by means of an aggregation rule and imposing that this rule satisfies the Pareto principle. As noted by Zuber [2011] and Jackson and Yariv [2014], this clearly contexts to the literature devoted to preference aggregation.[2] However, it is well known that a possible aggregation result becomes impossible when individuals are too heterogeneous. In terms of time preferences, the standard Pareto condition (PC) is not sufficient to withstand the effect of the heterogeneity of individual discount factors when individual instantaneous utilities are supposed to be heterogeneous and consequently to provide an axiomatic justification of social time preferences.

Intuitively, preference unanimity can result from the fact that conflicts over individual instantaneous utilities and conflicts over individual discounting factors cancel out in a TSU. We therefore suggest an alternative condition, the so-called impartial Pareto condition (IPC), which states that if all individuals rank one consumption stream higher than another, even when individual discount factors are impartially and arbitrarily permuted, then society should endorse this ranking. This new condition is significant: if both society and individuals have TSUs, then adopting the IPC yields a social discount factor and a social instantaneous utility that are equal to a weighted average of individual discount factors and individual utilities, respectively.

The EDU model is generally regarded as a cornerstone for policy studies, thus demanding principles supporting its theoretical feasibility. We thus consider principles from the perspective of a society whose preferences are represented by an EDU and show first that, given individual preference heterogeneity, social lifetime utility may be dictatorial even under the IPC. We find that a *perfectly altruistic* society, that is, a society who is only altruistic towards the next generation,

[1] Weitzman [2001] and Drupp, Freeman, Groom, and Nesje [2018] contemplate some individuals who express support for a near-one discount factor. A society can, therefore, place a high weight on higher discounting factors and maintain intergenerational ethical concerns for long-run projects. Moreover, this weighted average method is flexible enough to accommodate the demand for mild discounting in the case of short-run projects.

[2] In both cases, however, they are more interested in a dictatorial society than in utilitarianism.

as initiated by Phelps and Pollak [1968], is compatible with a social EDU. More precisely, the IPC must be accordingly adjusted to *perfectly altruistic impartial* Pareto condition, which compare consumption streams that only differ in the same two periods. We therefore show that a social TSU satisfying this *restricted* IPC and a condition of stationarity must be a *separatedly aggregated* EDU: social entities are identified as weighted means of associated individual entities. In particular, social discount factor lies between the maximum and minimum of individual discount factors. One prominent insight of this result is that perfect altruism, which drives time consistency, is shown to simply correspond to altruism between any two generations and not necessarily between two successive generations nor between the current and the next generations, as prescribed by Barro [1974]. Initiated by Ramsey's ethical critique in support of near-one discounting factor, many studies have thus proposed that the planner should place a higher discount factor above the current generation's private discount factor towards the future (Bernheim [1989], Farhi and Werning [2007], Caplin and Leahy [2004], Feng and Ke [2018]). However, this approach is built on the frameworks where a dynastic individual and the planner discount future differently. Therefore, it introduces conceptual and theoretical difficulty of justifying this planner through a preference aggregation process inside this society. Recently, Nesje [2021] demonstrates that this critique can also be derived when a society would have the same preferences for the future as dynasties.

Finally, as Weitzman [2001] noted, in certain empirical situations, many individuals that can be assimilated to a society behave in a time-inconsistent way. Hence, we propose to study the extent to which deviating from perfect altruism would affect social discounting and, consequently, the time consistency of social lifetime utilities. The building block of this analysis is, perhaps surprisingly, that if a social TSU respects a *quasi-altruistic impartial* Pareto condition corresponding to the occurrence of imperfect altruism and a stationarity-like condition, then the social discount factor is that of the quasi-hyperbolic discounting model (Phelps and Pollak [1968], Laibson [1997]). More interestingly, we find that when the number of periods involving the IPC increases, society is more present biased. This suggests that the degree of social present bias can be regulated by controlling the number of periods involving the IPC. We do not search for an abstract specification of the optimal number of periods involving unanimity. Rather, in empirical situations, a society can be assumed to be nondogmatic, as in Millner [2020], which means that he must choose the very principle, i.e., the appropriate number of periods involving the IPC, in accordance with the problem at hand. In technology policy, for instance, Harstad [2020] stresses that time inconsistency and strategic investments are important for policies addressing externalities. Thus, once an optimal degree of time inconsistency is determined, the society can select the associated number of periods involving the IPC to match this inconsistency.

The remainder of this paper proceeds as follows. Section 2 sets up the benchmark model. Section 3 motivates and formally states the IPC. Section 4 presents the separate aggregation results when individual utilities are TSUs, while Section 5 considers a society composed of EDU individuals. Then, the characterization results of social time consistency and time inconsistency are both presented. Section 6 reviews the related literature and Section 7 concludes the paper. All proofs are contained in the Appendix.

2 The Model

We consider a finite society $\mathcal{I}$ consisting of n individuals. Each individual i is assumed to live infinitely and to consume in discrete periods $t \in \mathbb{N} = \{1, 2, \ldots\}$. Let $\mathcal{L}$ be a consumption space, formally a set of lotteries over *finite* outcomes X, i.e. $\mathcal{L} = \Delta(X)$. Each period t consumption z_t belongs to $\mathcal{L}$, and a *consumption stream* is denoted by $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$. For any $z \in \mathcal{L}$, the *constant* consumption stream $(z, z, \ldots)$ is denoted by $\bar{\mathbf{z}}$. For any $x \in \mathcal{L}$ and $\mathbf{z} \in \mathcal{L}^\infty$, we denote consumption streams $(x, \mathbf{z})$ by $(x, z_1, z_2, \ldots)$. The consumption streams $(x, y, \mathbf{z})$ and so on are denoted similarly. More generally, for any $t \in \mathbb{N}$ and any $\mathbf{x}, \mathbf{z} \in \mathcal{L}^\infty$, the stream $\mathbf{x}_t\mathbf{z}$ denotes $(x_1, \ldots, x_t, z_1, \ldots)$.

Individual preferences over alternative consumption streams are represented by a lifetime utility function $U_i : \mathcal{L}^\infty \to \mathbb{R}$. We assume that such preferences are represented by *time-separable utility* (TSU).[3] Namely, for each $t \in \mathbb{N}$ and each $i \in \mathcal{I}$, there exists an individual time weight or *discount function* d_{it} and a nonconstant and continuous *instantaneous expected utility* denoted by $u_i : \mathcal{L} \to \mathbb{R}$ such that a consumption stream $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$ is evaluated as follows:

$$U_i(\mathbf{z}) = \sum_{t=1}^{\infty} d_{it} u_i(z_t), \tag{1}$$

where $0 < d_{it} < 1$ is (strictly) decreasing in t.[4] In a TSU model, d_{it} depends on time but not on consumption. Wlog, we normalize $d_{i1} = 1$, for all i. The assumption that $0 < d_{it} < 1$ decreases in t reflects individual devalue the future utility flows based on the time distance from present. Positive discount factors reflect individual desirabilities of future consumption. The triple

[3] In fact, TSU, whether individual or social, is implicitly assumed to depend only on relative time and flow variables but not on absolute time, i.e., it is *time invariant* in the sense of Halevy [2015]. History-dependent lifetime utility would then be an example violating time separability. Also, TSU assumes that discount function is independent of consumption streams, which rule out, for instance, *costly empathy* model of Noor and Takeoka [2021].

[4] Since d_{it} is strictly decreasing in t, the series $\sum_{i=1}^{\infty} d_{it}$ converges. That outcome set X is finite implies each u_i is bounded. Hence, $U_i(\mathbf{z})$ is finite for all i and $\mathbf{z}$.

(U_i, u_i, d_i) fully characterizes individual i's TSU.

The most important case of a TSU, which we will further discuss below, is the *exponential time discounted utility* (EDU). When preferences are TSU and satisfy Koopmans [1960]'s axioms, they can be represented by an EDU. Namely, for $i \in \mathcal{I}$, there exists a *constant* discount factor $\delta_i \in (0,1)$ and a nonconstant and continuous instantaneous utility function $u_i : \mathcal{L} \to \mathbb{R}$ such that a consumption stream $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$ is evaluated by i as follows:

$$U_i(\mathbf{z}) = \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(z_t). \tag{2}$$

Hence, the triple (U_i, u_i, δ_i) fully characterizes individual i's EDU.

We have three requirement for a society. First, we assume that social preferences over streams of consumption are also represented by a TSU. That is, there exists a continuous social instantaneous expected utility u and social discount factor $0 < d_t < 1$ such that the social lifetime utility function $U : \mathcal{L}^\infty \to \mathbb{R}$ is defined by:

$$U(\mathbf{z}) = \sum_{t=1}^{\infty} d_t u(z_t). \tag{3}$$

Once again, we normalize $d_1 = 1$ and d_t strictly decreases in t. The TSU representation is the most general model of preferences satisfying time separability. This model is commonly used for both normative applications (prescribing optimal policy) and positive applications (describing and predicting behavior). The TSU model includes the hyperbolic discounting model where $d_t = (1+\gamma t)^{-\frac{\alpha}{\gamma}}$ and $\alpha > \gamma$, the quasi-hyperbolic discounting model where $d_t = \beta\delta^{t-1}$, for $t > 1$ and many others. Similarly, we use the triplet (U, u, d) and (U, u, δ) to represent TSU and EDU society, respectively.

Second, we require that a society should abide by a separate aggregation rule, in which the social discount function and instantaneous utility are aggregated separately.

Definition 1. A TSU society (U, u, d) admits a *separate* aggregation rule, if there exist functions f and g such that

$$d = f(d_1, \ldots, d_n) \quad \text{and} \quad u = g(u_1, \ldots, u_n).$$

In particular, we say a TSU society admits a *linearly* separate aggregation rule, if there exists nonnegative $\{\gamma_i\}_{i\in\mathcal{I}}$ and $\{\alpha_i\}_{i\in\mathcal{I}}$ with $\sum_i \gamma_i = \sum_i \alpha_i = 1$ such that

$$d = \sum_i \gamma_i d_i \quad \text{and} \quad u = \sum_i \alpha_i u_i.$$

One could imagine a social TSU in which the social discount function depends on the individual discount functions as well as on the individual instantaneous utilities, and similarly for the social instantaneous utility function. Since a critical part of TSU theory for decision making is that discount function and instantaneous utility function are defined independently. Therefore, separate aggregation is a natural rule to social lifetime utility. Notice that a TSU society assumption rules out some well-known aggregation rules. Examples include utilitarian aggregation rule, namely weighted average of individual lifetime utilities and, Rawlsian aggregation rule, namely the minimum of individual lifetime utilities. However, separate aggregation rule is flexible enough to allow a society demonstrating many important properties, such as dynamic consistency, decreasing impatience and so on. For instance, if all individuals are EDU, i.e. $d_{it} = \delta_i^{t-1}$ for all i and t, then a discount function aggregation $d_t = (\sum_i \gamma_i \delta_i)^{t-1}$ makes a TSU society to be EDU, which satisfies dynamic consistency.

Third, we assume the existence of a *minimum agreement over consumption*: there are $x^*, x_* \in X$ such that $u_i(x^*) > u_i(x_*)$, for all $i \in \mathcal{I}$. Therefore, we normalize $u_i(x^*) = 1$ and $u_i(x_*) = 0$ for all i.

3 Pareto Dilemma and Impartiality

The standard PC was long widely accepted and considered an indisputable benchmark principle for preference aggregation. However, as suggested by Zuber [2011] and Jackson and Yariv [2015], among many others, the PC is a source of dilemmas in the dynamic preference aggregation setting. Indeed, when individual and social preferences are assumed to be represented by an EDU, the PC implies society to be dictatorial. Therefore, to better motivate the necessity of resorting to an alternative PC, we first demonstrate that even in our framework, where the lifetime utility of both individuals and society is assumed to be a TSU, the PC and nondictatorship are mutually exclusive.

3.1 Pareto Dilemma

When individual preferences and social preferences are supposed to be represented by a TSU, the PC can be written in the following way.

Pareto Condition (PC). For any $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_i(\mathbf{z}) \geq U_i(\hat{\mathbf{z}})$, for all $i \in \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

The PC means that if every individual prefers one consumption stream to another, then so does society. Unfortunately, although Buchanan and Tullock [1962] claim that it is ethically superior to all alternative principles, the PC is basically inconsistent with a nondictatorship requirement, which is commonly regarded as the minimum imperative for democracy.

In our context, society displays a degree of heterogeneity through individual instantaneous utilities and time preferences. A society is said to be *regular* if (i) there are $i, j \in \mathcal{I}$ such that $d_{it} \neq d_{jt}$, for some t, and (ii) individual instantaneous utilities are linearly independent.[5] Finally, a society is said to be *dictatorial* if there exists $i \in \mathcal{I}$ such that social and individual i's preferences coincide.

Proposition 1. *Assume a regular society. Then, the* PC *holds if and only if the society is dictatorial.*

Proposition 1 basically means that when individuals are heterogeneous, the Pareto principle of unanimity is equivalent to the existence of a dictator. Although our setting is similar to that of Jackson and Yariv [2015], their result[6] regarding the equivalence between Pareto condition and dictatorship is different from ours. Despite some technical details, they assume all lifetime utilities to be EDUs. In contrast, Proposition 1 introduces considerably more flexibility in generalizing Jackson and Yariv [2015]'s negative result to TSUs, a larger class of lifetime utilities.

3.2 Fictitious Individuals and Impartial Pareto Condition

Before presenting our suggestion, we propose a simple example to question the legitimacy of the PC. Consider a household consisting of two individuals, Ana, who is characterized by (u_a, d_{at}), and Bob, who is characterized by (u_b, d_{bt}). If this household is not dictatorial, then wlog, there exists a $\lambda \in (0, 1)$ such that[7]

$$u = \lambda u_a + (1 - \lambda) u_b.$$

This household wants to decide whether to have a child. If they do not have a child, then their consumption stream is constant, written as, $(x, x, \cdots)$. If they have a child, then their first-period consumption y stands for consumption with *baby* child and the second-period consumption z stands for consumption with *adult* child. From the third period, consumption is constant, which equals to x. Therefore, the consumption stream with a child is $(y, z, x, x, \cdots)$. Their instantaneous utility and relative discounting function are presented in the following table:[8]

[5] Recall that $u_1, \ldots, u_n$ are linearly independent if $\sum_i \lambda_i u_i = 0$ implies $\lambda_i = 0$, for all $i \in \mathcal{I}$.

[6] See Theorem 2 of Jackson and Yariv [2015].

[7] Since u_i and u are expected utility, restricted to constant consumption streams, Harsanyi Aggregation Theorem requires that social instantaneous utility u is a weighted sum of individual instantaneous utilities.

[8] Since instantaneous utility after the second period is always null, values of the discount function after the second period do not affect the calculation.

$\mathcal{L}$	x	y	z
u_a	0	$\frac{0.98}{\lambda}$	$-\frac{1}{\lambda}$
u_b	0	$-\frac{0.95}{1-\lambda}$	$\frac{9}{1-\lambda}$
u	0	0.03	8

t	1	2
d_a	1	0.99
d_b	1	0.1
d	1	d_2

Ana enjoys the time with baby child but worries about the future of adult child. As a result, she evaluates y positive and z negative. However, Bob finds it boring and expensive to care for baby child but enjoys family happiness once the child grows up. As a result, he evaluates y negative and z positive. Furthermore, Ana is highly patient and has a high discount value for second-period consumption. In contrast, Bob is very impatient and has a low discount value for second-period consumption. By simple calculation, we have:

$$\begin{aligned} U_a(y, z, x, x, \cdots) &= \frac{0.98}{\lambda} - \frac{1}{\lambda} \times 0.99 < 0 = U_a(x, x, \cdots), \\ U_b(y, z, x, x, \cdots) &= -\frac{0.95}{1-\lambda} + \frac{9}{1-\lambda} \times 0.1 < 0 = U_b(x, x, \cdots). \end{aligned}$$

It is straightforward to see that both Ana and Bob prefer not to have a child. However, for any positive household discount function d, we have

$$U(y, z, x, x, \cdots) = 0.03 + 8d_2 > 0 = U(x, x, \cdots).$$

Therefore, *regardless of the discount function*, this household should have a child. This contradiction between the decision by the PC and that from the household lifetime utlity with utilitarian instantaneous utility reveals that the current unanimity is *spurious*[9]. This situation explains that unanimity as formalized through the PC violates the household interest and, hence, can hardly be adopted as a righteous principle. Intuitively, to avoid such spurious unanimity, both Ana and Bob could introduce some empathetic considerations. Ana (resp. Bob) should replace her (his) discount function with Bob's (Ana's) and reconsider the choices accordingly. If unanimity remains even while exchanging discount functions, then it is no longer spurious but rather *impartial* insofar as each individual discount function is unbiasedly favored, and therefore, unanimity can be considered righteous.

A convincing PC should be rooted in a mutual acceptance of diverging opinions. This acceptance can be translated through a *speculative* experimentation. In terms of time preferences, a preference can be considered irresistibly unanimous only if, when all individuals replace their own discount function with any others' discount functions, this permutation never involves any

[9]This notion first appears in Mongin [1995] under the uncertain environment.

preference reversal. The absence of preference reversal reveals that such a speculative unanimity is robust to any individual discount function. Therefore, a collection of all the individual discount functions could serve as a common ground for irresistible unanimity.

Formally, any individual i can be represented by a pair (d_i, u_i). If, in this pair, d_i is replaced by discount function d_j of another individual j, then we can have a fictitious pair (d_j, u_i) corresponding to a fictitious individual ji. Since this fictitious individual ji composed of j's discount function and i's instantaneous utility corresponds to a non-actual individual, he is basically *fictitious*. Note that only fictitious individuals $ii \in \mathcal{I} \times \mathcal{I}$, for all $i \in \mathcal{I}$, are the *actual* individuals in society. Here, fiction is an introspective experiment involving the association of a discount function and an instantaneous utility that are not jointly observable in actual society. Namely, there is no actual individual corresponding to the fictitious individual ji when $j \neq i$. Then, call *fictitious society* the product set, for convenience, $\mathcal{I} \times \mathcal{I}$. Assume that the preferences of any fictitious individual $ij \in \mathcal{I} \times \mathcal{I}$ over streams of consumption are represented by a lifetime utility U_{ij} such that for all $\mathbf{z} \in \mathcal{L}^\infty$:

$$U_{ij}(\mathbf{z}) = \sum_{t=1}^{\infty} d_{it} u_j(z_t).$$

The form U_{ij} expresses how individual j evaluates alternative consumption streams if she replaces her own discount factor with that of individual i.

Let us now introduce a modified PC that takes all fictitious individuals into account.

Impartial Pareto Condition (IPC). For any $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\hat{\mathbf{z}})$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

This modified PC means that, for each pair of consumption streams, if all fictitious individuals unanimously prefer one stream to the other, then so does society. In some ways, the IPC recalls the impartial observer principle of Harsanyi [1953]. A simple way to understand Harsanyi's intuition about impartiality is the following: to help choose among social alternatives, each individual is assumed to imagine herself as an impartial observer who does not know which discount function and instantaneous utility she will be allocated. Consequently, the impartial observer considers not only an actual preference over the social outcomes but also fictitious preferences whose identity in $\mathcal{I} \times \mathcal{I}$ she will assume. When an impartial observer imagines herself being fictitious individual ij, she adopts i's discount function and j's instantaneous utility to form preferences. Hence, the so-called *acceptance principle* in Harsanyi's and IPC in our framework play an equivalent role and can be similarly interpreted. In addition, there is another layer of meaning that IPC brings in: the idea that individuals should infer the knowledge of others by observing their parameters. The heterogeneity of individual discount functions explains that the information or the knowledge about *ideal* discount function held by individuals are different. With this perspective, a society should encourage individuals to learn from each other and to suppress the role of actual individuals. This is, in a way, like the role of *nondogmatism* of Millner [2020].

When comparing the IPC to the PC, one critical difference arises. Under the IPC, society builds unanimous preferences from all possible fictitious preferences, not just actual preferences. Each individual is required to reevaluate every stream based on other individual discount functions to ensure unanimity to be fully compelling. Impartial introspection, considering the discount factor of anyone else as a possible introspective experience for oneself, can effectively eliminate a spurious unanimity, for instance, one induced by a double disagreement of instantaneous utilities and time preferences.[10]

One might call for *individualism* principle and argue that a society should base all accounts of choices on the actual individual decisions, not the fictitious ones. It is worth to emphasize that IPC is primarily a theory of society, an attempt to understand the forces which should determine the social decisions. Therefore, IPC is considered as a "ought" proposition, but not a "is" proposition in the sense of David Hume. Later, we will suggest an alternative principle which can be considered as "is" proposition.

4 Separate Aggregation of Time Preferences

In this section, we consider the situations that the social time discount function and the social instantaneous utility are aggregated separately. The next theorem clarifies the connection between IPC and separate aggregation.

Theorem 1. *A society satisfies the* IPC *if and only if it admits a linearly separate aggregation rule.*

Theorem 1 states that if a social lifetime utility satisfies the IPC, social functions (utility and discount) take the form of a convex combination of individual functions. In contrast to impossibility results such as Proposition 1 and that of Jackson and Yariv [2014], the IPC weakens the PC in a way that avoids spurious unanimity. Hence, it naturally gives rise to a possibility. To see how the IPC works, note that it requires unanimity with respect to the *fictitious society*, which implies that there exists a nonnegative λ_{ij} such that, for a consumption stream $\mathbf{z}$,

$$U(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}(\mathbf{z}).$$

[10] This type of spurious unanimity is first discovered by Gilboa, Samet, and Schmeidler [2004] in the context of uncertainty.

Let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, it can be shown that linearly separate aggregation holds.

One might argue that, in practice, the social decision to protect the environment only rests on individualistic valuation, but not the fictitious valuation. It is plausible to obtain a separate aggregation by a Pareto-like condition which is solely based on individual choice. Two consumption streams are said to be *common-value* streams if either both streams are constants or every individuals have the same instantaneous utilities over the possible outcomes. Formally, take two consumption streams $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, if, (i) $\mathbf{z}$ and $\mathbf{z}'$ are constant streams; or (ii) $u_i(z) = u_j(z)$ for all $z \in \text{conv}(\{z_t : t \in \mathbb{N}\} \cup \{z'_t : t \in \mathbb{N}\})$ and all $i, j \in \mathcal{I}$, then $\mathbf{z}$ and $\mathbf{z}'$ are common-value streams. Due to MAC assumption, the non-constant common-value streams always exist.

Common-value Pareto Condition (CV-PC). For any pair of common-value streams $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_i(\mathbf{z}) \geq U_i(\hat{\mathbf{z}})$, for all $i \in \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

CV-PC states that for every pair of common-value streams, if all individuals prefer the former one to the latter one, then so is the society. In contrast, this condition strengthens classic PC by focusing on the pairs of streams that are either constant or identical in utility values over set of stream consumptions. Notice that IPC implies CV-PC, simply because the preferences of fictitious individual ij over common-value streams coincide with actual individual i. The next result shows that CV-PC, a weaker Pareto condition, along with MAC assumption also imply linearly separate aggregation.

Theorem 2. *A social lifetime utility U satisfies the* CV-PC *if and only if society admits a linearly separate aggregation rule.*

This theorem demonstrates that linearly separate aggregation is obtained if a society follows CV-PC. Although both IPC and CV-PC could equivalently imply the separate aggregation rule, particular stress will be placed on the IPC in the rest of analysis because it is, in our opinion, normatively sounding. However, all of our following results based on IPC can be equivalently replaced by CV-PC.

The separate aggregation result does not specify the properties of social discount function. Now we focus on two important types of discount functions: the *constant-impatience discount function* and the *present-bias discount function*, translating decreasing impatience. (We henceforth use the terms 'decreasing impatience' and 'present bias' interchangeably since it is widely agreed that the first serves as a testable implication of the second.) The key property of the first class of discount functions is that social choices are *time consistent*, which is not only normatively plausible but also widely applicable due to its tractability. This motivates the following question: is there a way of characterizing a constant-impatience social discount function? In addition, at least since Thaler [1981], the finding that decision-makers become present biased as the time delay increases is a canonically descriptive result. Therefore, one might wonder what a social criterion driving such a present bias looks like.[11] Consequently, we attempt to characterize whether this criterion, while generating a social discount function, defines decreasingly impatient behavior based on a reasonable definition of present bias. Such a criterion is expected, allowing not only the identification of discount functions but also the clarification of the key behavioral principles behind the collective decision-making process.

Let $(x_t, \bar{\mathbf{x}}_{*-t})$ denote a consumption stream with $z_t = x$ and $z_s = x_*$ for $s \neq t$.

Definition 2. A lifetime utility $U : \mathcal{L} \to \mathbb{R}$ is *present biased* (resp. *constant impatient*) if, for any $t > s$, any $k \geq 1$, $U(x_t, \bar{\mathbf{x}}_{*-t}) = U(y_s, \bar{\mathbf{x}}_{*-s})$ implies $U(x_{t+k}, \bar{\mathbf{x}}_{*-(t+k)}) \geq U(y_{s+k}, \bar{\mathbf{x}}_{*-(s+k)})$ (resp. $U(x_{t+k}, \bar{\mathbf{x}}_{*-(t+k)}) = U(y_{s+k}, \bar{\mathbf{x}}_{*-(s+k)})$).

A lifetime utility is present biased if, once closer consumption x at time s and further consumption y at time t are indifferent, then further consumption y is preferred when both consumption streams are shifted further by time k. Intuitively, if such time shifting does not change preferences, then this lifetime utility is said to be constant impatient.

Next, a present-biased U can also be characterized by its discount function. A *discount factor* measured at date t, i.e., $\delta(t)$, of a TSU with a discounting function d_t is defined as

$$\delta(t) = \frac{d_{t+1}}{d_t}.$$

The following lemma expresses that if a lifetime utility is present biased, then its discount factor is increasing. Similarly, if a lifetime utility is constant impatient, then its discount factor is constant.[12] This result is stated without proof because of its triviality.[13]

Lemma 1. *Suppose that a lifetime utility U is a TSU characterized by (u, d_t). Then, U is present biased if and only if its discount factor δ is increasing. Moreover, U is constant impatient if and only if its discount factor δ is constant.*

Since we exclusively consider two types of lifetime utility functions, a natural question to ask is the following: is a society composed of constant-impatient or present-biased individuals and

[11] For the relation between social decreasing impatience and social present bias, see Jackson and Yariv [2015].

[12] In the case of an EDU, $d_{t+1}/d_t = \delta^t/\delta^{t-1} = \delta$.

[13] Recently, Chambers, Echenique, and Miller [2021] in parallel develop a similar definition of decreasing impatience and characterize it in different ways. The focus of this paper is different from theirs.

with lifetime preferences governed by the IPC necessarily present biased? The next proposition provides a positive answer to this question.

Proposition 2. *Assume that a social lifetime utility U admits a linearly separable aggregation. If each individual is either constant impatient or present biased, then a nondictatorial social utility U is necessarily present biased.*

Proposition 2 can be viewed as a result displaying the relation between nondictatorship and present bias, while a society is composed of constant-impatient or present-biased individuals and abides by a linearly separate aggregation rule. This shows that if the domain of individual lifetime utilities is restricted to the only constant-impatient or present-biased TSUs, the IPC implies that society is also present biased. In contrast to Jackson and Yariv [2015], this result thereby establishes that a present-biased society does not necessarily rely on the assumption of constant-impatient individuals. Moreover, a present-biased society does not imply individuals to be either constant impatient or present biased. A simple example could be easily constructed with a first individual being present biased and a second individual increasingly impatient. If society assigns a small enough weight to the latter, society can still be present biased.

5 Altruism and Social Impatience

In this section, all individual preferences are assumed to be represented by an EDU lifetime utility $(U_i, u_i, \delta_i)_{i\in\mathcal{I}}$ and social preferences are represented by a TSU lifetime utility (U, u, d_t). [14]

5.1 Perfect Altruism and Constant Social Discounting

Although separate aggregation is compatible with the IPC, there also remains the question of the extent to which a society can be collectively dynamically consistent. Since "*the simplicity and elegance of this* (EDU) *formulation is irresistible*", as claimed by Frederick, Loewenstein, and O'Donoghue [2002], it is of vital importance to suggest a principle that would characterize a society admitting an EDU representation for its preferences. In this subsection, we show that an appropriately modified IPC would imply a time-consistent society that has instantaneous utility and

[14] Substantial empirical evidence supports that individuals do not behave as EDU maximizers when making decisions involving tradeoffs over time. Indeed, as noted by Frederick, Loewenstein, and O'Donoghue [2002], one generally assumes other kinds of behaviors that are more realistic, such as hyperbolic discounting. However, since our setting essentially revolves around common goods, individual preferences might very well differ from those concerning private goods. Furthermore, it is not clear why hyperbolic discounting behavior for private consumption should inform the assumption of discounting for common goods in the time horizon.

discount factor defined as the convex combination of individual utilities and of individual discount factors, respectively.

Consider first the stationarity axiom, which is required to ensure constant discounting, as shown by Koopmans [1960].

Stationarity. A lifetime utility function U is *stationary* if, for all $x \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) \geq U(\mathbf{z}') \text{ if and only if } U(x, \mathbf{z}) \geq U(x, \mathbf{z}').$$

Stationarity means that the ranking between two streams remains unchanged when common consumption is inserted in the first period for both streams. A decision-maker who obeys this axiom should be insensitive to what consumption is inserted. Recursively, it requires that the evaluation of two consumption streams does not change if all dates are shifted according to the same time constant.

However, the IPC and stationarity are not sufficient to characterize constant discounting for a nondictatorial social lifetime utility.[15] Therefore, a further weakened PC is needed to derive constant social discounting. We say two consumption streams $\mathbf{z}$ and $\mathbf{z}'$ are *diperiodic* if $z_t = z'_t$ for $t > 2$. That is, a pair of diperiodic consumption streams can only differ in consumption at the first two periods.

Perfectly Altruistic Impartial Pareto Condition (PAI-PC). For any diperiodic consumption streams $\mathbf{z}$ and $\mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z})$.

PAI-PC states that while comparing two consumption streams that differ only in their first two consecutive periods, if all true individuals and all fictitious individuals prefer the first to the second stream, then so does society.

Phelps and Pollak [1968] recall that the intuition whereby individual preferences may be linked by a kind of 'generational' commitment already exists in Ramsey [1928], who assumes that each generation's preferences for its own consumption relative to the next generation's preferences do not differ from preferences for any future generation's consumption relative to that of the next generation. This commitment is equivalent to a stationarity postulate: the present generation's preferences over consumption streams are supposed to be invariant to changes in their timing. Phelps and Pollak [1968] suggest calling this *perfect altruism*. Later, Barro [1974]'s analysis of

[15] Note that the IPC is equivalent to the PC when individual instantaneous utilities are identical. Jackson and Yariv [2015] demonstrate that there does not exist a nondictatorial social lifetime EDU if the PC and stationarity are imposed on a society where individuals have heterogeneous discount factors.

debt neutrality is based on a similar assumption: individuals are motivated by a special form of intergenerational altruism (here called *dynastic altruism*) such that individuals have an altruistic concern for their children, who in turn also have altruistic feelings for their own children, and so forth.[16]

While assimilating a period to a generation length, the restriction imposed by PAI-PC for time consistency is precisely to avoid imperfect altruism between generations, since imperfect altruism leads to a violation of stationarity. By considering only the first two consumptions, the lifetime utility of each individual i is then defined as a discounted utility of i and his immediate descendant. As a result, social lifetime utility in the first two periods also corresponds to a discounted utility of the current generation and the next generation. Stationarity further implies that social lifetime utility would be evaluated recursively as a discounted sum of all future utilities in which the discount factor is constant.

We can now state one of our main results. If social preferences are represented by a TSU that satisfies stationarity and respects PAI-PC, then the social lifetime utility is an EDU. Furthermore, social instantaneous utility and the social discount factor are equal to a weighted average of individual instantaneous utilities and a weighted average of individual discount factors, respectively.

Theorem 3. *The* PAI-PC *and* stationarity *are satisfied if and only if social lifetime utility is a EDU, in which u is a convex combination of $\{u_i\}_{i\in\mathcal{I}}$ and social discount function is exponential, i.e. $d_t = \delta^{t-1}$ for all $t \in \mathbb{N}$, with δ being a convex combination of $\{\delta_i\}_{i\in\mathcal{I}}$.*

Theorem 3 means that to be time consistent, the social lifetime utility must be an EDU function; hence, society should respect both stationarity and PAI-PC. In fact, in this situation, the social discount factor can only rest between the minimum and the maximum of individual discount factors, and the social instantaneous utility is a weighted sum of individual instantaneous utilities. Thus, the exact value of the social discount factor and the exact form of the social utility function would depend on the choice of weights. Note that the weights for discount factors can differ from those affecting utilities. This means that society can believe in individual i's judgment about time and place high weight (or even full weight) on her discount factor but be more concerned about individual j's welfare and, consequently, place more weight (or even full weight) on his instantaneous utility. In other words, society can locally arbitrate between a discount factor and individual welfare and generalize this arbitrage across individuals.

Since PAI-PC restricts stream comparisons to streams that only differ in the first two periods,

[16]Through this recursive relation, all generations of a single family (i.e., a *dynasty*) are linked together by a chain of private intergenerational transfers, countervailing any attempt by the government to redistribute resources across them.

it is conceivable to strengthen this condition and, thus, to remove stationarity. For example, stream restrictions can be relaxed to streams that differ in any two arbitrary successive periods: i.e., for any $t \in \mathbb{N}$, any $x, y, x', y' \in \mathcal{L}$, and any $\mathbf{z} \in \mathcal{L}^\infty$, if $U_{ij}(x_t, y_{t+1}, \mathbf{z}_{-(t,t+1)}) \geq U_{ij}(x'_t, y'_{t+1}, \mathbf{z}_{-(t,t+1)})$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(x_t, y_{t+1}, \mathbf{z}_{-(t,t+1)}) \geq U(x'_t, y'_{t+1}, \mathbf{z}_{-(t,t+1)})$. In view of a recursive evaluation of welfare for every pair of successive generations, a natural question is whether this version of the IPC along with recursive evaluation would imply stationarity of the social lifetime utility U. In order words, in this situation, is stationarity redundant?

The following example proves that stationarity is not useless. Consider a society of 2 individuals $\{1, 2\}$. Suppose that individuals have identical instantaneous utilities but that their discount factors differ, i.e., $\delta_1 \neq \delta_2$. Suppose hence that society has the same instantaneous utility as individuals and adopts the following discount function:

$$d(t) = \frac{1}{t}\delta_1 + (1 - \frac{1}{t})\delta_2.$$

Clearly, this society does not have a constant discounting factor. Therefore, the associated social lifetime utility U violates stationarity. However, it is clear that U satisfies PAI-PC. In fact, with stationarity, the above alternative PC turns is equivalent to PAI-PC.

Now, another possibility to modify PAI-PC is to relax the requirement for the two considered periods to be successive. Namely, altruism would no longer be restricted to only the next generation and rather jumps to a later generation. It can be the case, for instance, that individuals do not care about their children but only about their grandchildren. Is this *postponed altruism* also *perfect* in the sense of time consistency? Surprisingly, as proved below in Proposition 3, the answer is positive. Let us first adapt the IPC to capture the idea of postponed altruism. Let $k \geq 2$, we say two consumption streams $\mathbf{z}, \mathbf{z}'$ are *k-diperiodic* if $z_t = z'_t$ for $t \in \mathbb{N} \setminus \{1, k\}$.

k-PAI-PC Let $k \in \mathbb{N}$. For any k-diperiodic streams $\mathbf{z}, \mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$ for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z}')$.

This condition, k-PAI-PC, requires impartial unanimity to apply only if the compared streams differ for the current generation and the k-th generation. Along with stationarity, we can then prove that it also implies a time-consistent society. Furthermore, social lifetime utility and social discount factors are weighted averages of individual utilities and factors.

Proposition 3. *The* k-PAI-PC *and* stationarity *are satisfied if and only if social lifetime utility is a EDU, in which u is a convex combination of $\{u_i\}_{i \in \mathcal{I}}$ and $d_t = \delta^{t-1}$ for all $t \in \mathbb{N}$, with δ being a convex combination of $\{\delta_i\}_{i \in \mathcal{I}}$.*

Proposition 3 means that if a stationary social lifetime utility evaluates individual welfare such that society is concerned only about the utilities of the current generation and the k-th generation, then the lifetime utility of this society is an EDU. Relative to Theorem 3, where PAI-PC is assumed along with stationarity, Proposition 3 leads to the same utilitarian characterization while assuming k-PAI-PC and stationarity. Without delving into the formal proof, to be convinced of this, it is sufficient to consider a situation where individual utilities are identical. Therefore, k-PAI-PC implies that the value of the social discount function at time k is a weighted average of $\{\delta_i^{k-1}\}_{i\in\mathcal{I}}$. Since this average is between $(\max_{i\in\mathcal{I}} \delta_i)^{k-1}$ and $(\min_{i\in\mathcal{I}} \delta_i)^{k-1}$, there should exist a $\delta \in [\min_{i\in\mathcal{I}} \delta_i, \max_{i\in\mathcal{I}} \delta_i]$ such that δ^{k-1} corresponds exactly to that weighted average value. Stationarity further implies that this society admits a lifetime utility that has a constant discounting factor.

Proposition 3 turns out to have surprisingly striking implications. To be time consistent, society only needs to consider the utilities of any two apart generations that are not necessarily successive. This amounts to the fact that a society affected by a remote generation can be regarded as a society affected by the next generation. This in a sense redresses the prevalence of the belief that a perfect altruistic society cares only about the utility of immediate children and not about distant descendants.

5.2 Quasi-hyperbolic Social Discounting

Although time consistency is appealing in economic theory, little of it can be seen in economic policy. This can be either explained by the fact that society lacks the power to commit or by the fact that commitment benefits are overwhelmed by commitment costs. Consequently, a demand for social time consistency must be seen as special rather than universal. The quasi-hyperbolic discounting model of Phelps and Pollak [1968] and Laibson [1997] has long served as a standard norm for economic analysis when time inconsistency arises. We present its representative form.

Definition 3. A lifetime utility $U : \mathcal{L}^\infty \to \mathbb{R}$ admits a *quasi-hyperbolic discounting* form if there exists a continuous function u on $\mathcal{L}$ and parameters $\beta \in (0, 1]$ and $\delta \in (0, 1)$ such that for $z \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = u(z_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(z_t). \tag{4}$$

Of particular interest is the question of the principles that society should respect for social lifetime utility to admit a quasi-hyperbolic discounting form. Such a social lifetime utility being time inconsistent, we already know that it violates stationarity. Hence, a weaker stationarity-like condition is required.

Quasi-stationarity. A lifetime utility U is *quasi-stationary* if, for all $x, y \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$,

$$U(x, \mathbf{z}) \geq U(x, \mathbf{z}') \text{ if and only if } U(x, y, \mathbf{z}) \geq U(x, y, \mathbf{z}').$$

Quasi-stationarity means that the social evaluation of consumption streams relative to the next period is invariant to changes in future periods. Thus, this condition admits the possibility that society could assign to the current consumption a place of relative importance out of proportion to all future consumptions.

It is clear that stationarity implies quasi-stationarity (but not vice versa). Therefore, it is intuitive that quasi-stationarity and PAI-PC are compatible with quasi-hyperbolic social discounting. However, in this situation, β and δ in (4) are indeterminate. From the above analysis, quasi-stationarity and PAI-PC imply the product of β and δ to be a weighted average of individual discount factors. As a result, society can freely choose, for instance, either β or δ to be a weighted average. Such indeterminacy contradicts the democratic intuition that every individual should have a say in every social issue. To avoid this indeterminacy, a stronger condition than PAI-PC is required. We say that two consumption streams $\mathbf{z}, \mathbf{z}'$ are triperiodic if $z_t = z'_t$ for $t > 3$.

Quasi-Altruism Impartial Pareto Condition (QAI-PC). For any pair of triperiodic consumption streams $\mathbf{z}, \mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z})$.

QAI-PC states that if two consumption streams only differ in the first three periods, then the fact that all individuals, true and fictitious, rank the these two streams in the same way would imply that society also adopts this ranking. In this situation, society cares directly about the two next generations, which, in the spirit of Phelps and Pollak [1968], reflects *imperfect* altruism. However, this imperfectness is not only compatible with quasi-hyperbolic social discounting but also resolves the indeterminacy of β and δ.

Theorem 4. *The* QAI-PC *and* quasi-stationarity *are satisfied if and only if there exist nonnegative* $\{\alpha_i\}_{i\in\mathcal{I}}$ *and* $\{\lambda_i\}_{i\in\mathcal{I}}$ *with* $\sum_i \alpha_i = \sum_i \lambda_i = 1$ *such that social lifetime utility admits a quasi-hyperbolic discounting form as defined in (4), with*

$$u = \sum_i \alpha_i u_i \quad \textit{and} \quad \delta = \frac{\sum_i \lambda_i \delta_i^2}{\sum_i \lambda_i \delta_i} \quad \textit{and} \quad \beta = \frac{(\sum_i \lambda_i \delta_i)^2}{\sum_i \lambda_i \delta_i^2}.$$

Furthermore, $\delta \in [\min_{i\in\mathcal{I}} \delta_i, \max_{i\in\mathcal{I}} \delta_i]$ *and* $\beta \in [\frac{\min_i \delta_i}{\max_i \delta_i}, 1]$.

Theorem 4 shows that if a society respects QAI-PC and quasi-stationarity, then the social lifetime utility has a quasi-hyperbolic discounting form. Social instantaneous utility is a weighted average of individual utility, while the social discount factor δ is a proportion of second-moment to first-moment individual discount factors, and social present bias is a proportion of the square of first-moment to second-moment. This result proves that a deviation from perfect altruism incurs present bias. There is a clear tradeoff between present bias and impatience towards future generations. If society would be less present biased, then it would have to place more weight on a particular individual. As β is close to one, society tends to dictate the discount factor, i.e., $\delta \approx \delta_i$, for some individual i. Moreover, the range of β is determined by the degree of discount factor heterogeneity. If individuals are more diverse in terms of impatience, society would be more present biased. This sheds light on the source of present bias, namely, individual discount factor heterogeneity.

Similar to the discussion in Subsection 5.1, QAI-PC can also be relaxed, allowing the compared streams to be different in three periods that are not necessarily consecutive. Note that these arbitrary three periods have to include the current period to reflect the different arbitrages between current and future generations. This difference is scaled by β.

5.3 Delayed Social Stationarity

Constant and quasi-hyperbolic social discounting are compatible with PAI-PC and QAI-PC, respectively. This naturally gives rise to the following question: what would social discounting be if we were to force this extension to a more general level, in which altruistic concern is spread up to the kth generations? We already observed that QAI-PC triggers dynamic inconsistency. Then, it would not be surprising that spreading altruism will also lead to such inconsistency. A deeper issue is that this *imperfect* altruism may result in more inconsistency as k grows larger. In fact, ignoring the degree of dynamic inconsistency might harm, for example, the sustainability of society. Therefore, inconsistency regulation should be a critical concern for a society in making decisions. In what follows, after the formal definition of a generalization of QAI-PC, we then explore how the intensity of social inconsistency can be characterized through this condition. We say that two consumption streams $\mathbf{z}$ and $\mathbf{z}'$ are *k-periodic* if $z_t = z'_t$ for $t > k$.

k-Imperfect Altruism Impartial Pareto Condition (k-IAI-PC): Let $k \in \mathbb{N}$. For any pair of k-periodic consumption streams $\mathbf{z}, \mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$ for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z}')$.

Consider now a subclass of TSUs satisfying k-IAI-PC along with a stationarity-like condition.

Definition 4. A lifetime utility $U : \mathcal{L}^\infty \to \mathbb{R}$ admits a k-*hyperbolic* form if there exists $0 < \beta_1 \leq \ldots \leq \beta_k \leq 1$ and $\delta \in (0,1)$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = u(z_1) + \beta_1 \delta u(z_2) + \beta_1 \beta_2 \delta^2 u(z_3) + \cdots + \prod_{\ell=1}^{k} \beta_\ell \sum_{t=\ell+1}^{\infty} \delta^{t-1} u(z_t). \tag{5}$$

This formulation assumes a declining discount factor until period k but a constant discount factor thereafter. The parameters β_ℓ can be thought of as a measure of the 'horizon $(\ell - 1)$' *bias*. Each β_ℓ can also represent the size of the perceived distance between periods $(\ell - 1)$ and ℓ. This definition includes the case of an EDU for $\beta_1 = \cdots = \beta_k = 1$ and the classic quasi-hyperbolic utility for $k = 1$. Note that k-hyperbolic utilities are a subclass of 'semi-hyperbolic' utilities, as proposed in Montiel Olea and Strzalecki [2014], in which $\beta_1, \ldots, \beta_k$ are unrestricted. In contrast, a k-hyperbolic utility requires $\beta_1, \ldots, \beta_k$ to be an increasing sequence, which then translates the existence of a present bias.

The advantage of considering this class of present-biased utilities is at least twofold. First, any present-biased TSU can be approximated by some k-hyperbolic utility. Therefore, replacing TSUs with this class of utilities does not lead to a loss of generality. Second, this parametrized utility is data friendly. One can apply, for instance, MPL (multiple price list) to elicit $\beta_1, \ldots, \beta_k$ and, therefore, fully recover the form of social utility.[17]

k**-Stationarity.** A lifetime utility function U is k-*delayed stationary* if, for all $x \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{c}, \hat{\mathbf{c}} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}_k \mathbf{c}) \geq U(\mathbf{z}_k \hat{\mathbf{c}}) \text{ if and only if } U(x, \mathbf{z}_k \mathbf{c}) \geq U(x, \mathbf{z}_k \hat{\mathbf{c}}).$$

This property, k-delayed stationarity, generalizes classical stationarity, which states that if two consumption streams are identical up to period t, then the ranking between these two streams is preserved after adding the same consumption in the current period and delaying both streams one period further. It is clear that a k-hyperbolic utility satisfies k-stationarity, but it is not true that any utility satisfying k-stationarity has a k-hyperbolic form. Delayed stationarity does not impose any restriction on the rate of impatience before period k. Next, it is natural that when k grows, the stationarity-like property becomes stronger. In other words, if $k > \ell$, then k-stationarity implies ℓ-stationarity. Now, we can formally state our result.

[17] The empirical elicitation question is beyond the scope of this article. We refer to Montiel Olea and Strzalecki [2014] for MPL and Cohen, Ericson, Laibson, and White [2020] for more general methods.

Theorem 5. *The k-stationary and k-*IAI-PC *are satisfied if and only if there exist nonnegative numbers α_i and γ_i such that social preferences are represented by a k-hyperbolic social lifetime utility U as in (5), with*

$$u = \sum_i \alpha_i u_i \tag{6}$$

$$\delta = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \tag{7}$$

$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \cdot \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}} \qquad \textit{for all } 1 \le \ell \le k. \tag{8}$$

Theorem 5 proves that a society respecting both k-stationarity and $(k+2)$-IPC has a social lifetime utility with a form that is k-hyperbolic. Furthermore, social instantaneous utility is a weighted average of individual utility. Additionally, the social discount factor at horizon ℓ *before* horizon k is a proportion of the weighted average of individual discounting function values at horizon ℓ to that at horizon $(\ell - 1)$. The social discount factor at horizon ℓ *after* horizon k is constant and defined as δ, which is a proportion of the weighted average of individual discounting function values at horizon $(k+1)$ to that at horizon k. Therefore, the social discount factor at horizon $\ell \le k$ can be decomposed into δ and a period $(\ell - 1)$ bias denoted by β_ℓ.

In fact, Theorem 5 includes Theorem 4 as a special case of k-IAI-PC for $k = 3$. When k goes to infinity, k-stationarity has no bite on stationarity, and k-IAI-PC becomes IPC. Therefore, when $k \to \infty$, Theorem 5 is a special case of Theorem 1, in which each individual lifetime utility is an EDU.

Since a social k-hyperbolic lifetime utility displays decreasing impatience, i.e., present bias, it is natural to explore how the degree of decreasing impatience changes when k increases. Let us first provide a notion of comparative present bias.

Definition 5. A utility U is *more present biased* than utility V if, for any t, s in $\mathbb{N}$ and $x, y, x', y' \in \mathcal{L}$, $U(x, \bar{z}_*) = U(y_t, \bar{z}_{*-t})$, $V(x', \bar{z}_*) = V(y'_t, \bar{z}_{*-t})$, and $V(x'_s, \bar{z}_{*-s}) \le V(y'_{t+s}, \bar{z}_{*-\{t+s\}})$ implies $U(x_s, \bar{z}_*) \le U(y_{t+s}, \bar{z}_{*-\{t+s\}})$.

The intuition behind this definition is the following:[18] suppose that one utility U equivalently evaluates two streams, one with consumption x at the current time and the other with further consumption y at t. In contrast, another lifetime utility V ranks similarly, i.e., equivalently evaluates

[18] Prelec [2004] and Quah and Strulovici [2013] suggest different notions of comparative decreasing impatience but based on continuous times.

two other streams, the current stream x' and a further stream y' at t. Suppose that all consumption is postponed by the time interval s. Whenever utility V prefers further consumption y' at period $(t+s)$ to closer consumption x' at period s, it is always the case that utility U also prefers further consumption y at $t+s$ than closer consumption x at s. Since U has earlier preference reversal than V, U is said to be more present biased than V.

Proposition 4. *Let there be nonnegative numbers α_i and γ_j such that $\sum_{i\in\mathcal{I}}\alpha_i = \sum_{j\in\mathcal{I}}\gamma_j = 1$. If $k \leq \hat{k}$, then a society characterized by $(\hat{u}, \hat{\delta}, \{\hat{\beta}_\ell\}_{\ell=1}^{\hat{k}})$ defined as in (6,7,8) is more present biased than a society characterized by $(u, \delta, \{\beta_\ell\}_{\ell=1}^{k})$ defined as in (6,7,8).*

This result indicates that when k increases, the social lifetime utility becomes more present biased. Next, we observe that k-stationarity is necessary to characterize a k-hyperbolic social utility. If we replace delayed stationarity with the standard stationarity, then society has to be a dictatorship.

Corollary 1. *Let $k > 2$. The k-*IAI-PC *and stationarity are satisfied if and only if society is dictatorial.*

As an illustration, consider a society with 2 individuals who have identical instantaneous utilities. For simplicity, consider that QAI-PC holds, where $k=3$. We know that $u(x)+\delta u(y)+\delta^2 u(z)$ is a convex combination of corresponding individual utilities. Consequently, there exists $\lambda \in [0,1]$ such that

$$\lambda\delta_1 + (1-\lambda)\delta_2 = d_2 \quad \text{and} \quad \lambda\delta_1^2 + (1-\lambda)\delta_2^2 = d_3.$$

Stationarity requires that $d_2^2 = d_3$. The only solution must be either $\lambda = 1$ or $\lambda = 0$, which are exactly the two polar cases of dictatorship.

5.4 Perfect or Imperfect Altruism?

With perfect altruism, a society would admit a constant discounting in economics of evaluating future gains and losses. As we discussed, the appeal of constant discounting is not that it simplifies the analysis, but that it demonstrates normatively sounding properties, such as time consistency and stationarity. Therefore, it is not surprising that constant discounting schemes are widely accepted for cost-benefit analysis by many countries. However, the shortcoming of constant discounting is negligible, in particular, in the face of environmental policy making. As Karp [2005], Gerlagh and Liski [2017] and many others pointed out, the optimal policy is sensitive to the choice of discount factor, which may make a society unwilling to pay even moderate costs now to prevent disaster in the future.

Consider a society has to choose between two consumption streams

$$\mathbf{z} = (\underbrace{1, 0, \ldots, 0, -100}_{\text{11 periods}}, 0, \ldots) \quad \text{and} \quad \mathbf{z}' = (1.1, -0.4, 0, \ldots).$$

Stream $\mathbf{z}$ illustrates that consumption is 1 in the first period, -100 in period 11, and zero for the rest of time. In stream $\mathbf{z}'$, the consumption in the first two periods are 1.1 and -0.4 and the consumption in the rest period is also zero. Suppose that society is a constant discounter with $\delta = 0.5$[19] and has an identity instantaneous utility function. Then, it is immediate to see that a society prefers $\mathbf{z}$ to $\mathbf{z}'$:

$$1 - 100 \times 0.5^{10} = 0.9032 > 0.9 = 1.1 - 0.4 \times 0.5.$$

In contrast, if a society is, for example, quasi-hyperbolic discounter with present bias parameter $\beta = 0.8$, then society would prefer $\mathbf{z}'$ to $\mathbf{z}$:

$$1 + 0.8 \times 0.5^{10} \times (-100) = 0.9219 < 0.94 = 1.1 + 0.8 \times 0.5 \times (-0.4)$$

In this numerical example, $\mathbf{z}$ can be regarded as a policy that generates short-run benefit and long-run damage. Alternatively, we can regard $\mathbf{z}'$ as a policy that generates a moderate short-run cost to avoid a long-run damage. We believe that it is intuitive for a society to take responsibility to endure short-run loss to avoid long-run disaster. However, if a society choose to be time consistent, then it values the distant future damage as a marginal loss and, therefore, chooses a policy not to avoid damage. In contrast, quasi-hyperbolic discounting society provides a plausible trade-off between benefit and costs in the distant future. In this example, quasi-hyperbolic discounting ameliorates the defects of constant discounting and can be interpreted as normative recommendations for society.

Perfect altruism is the ubiquitous ethic principle for social policy making. However, we argue that it is not the unique rational principle to adopt. In particular, in some environmental policy making, imperfect altruism is more compelling and effective to trade-off between costs and benefits in the distant future.

[19] In fact, this example is not sensitive to the choice of δ. For instance, if δ increases, we can delay the damage -100 further to maintain our argument.

6 Related Literature

In two different settings, Zuber [2011] and Jackson and Yariv [2014] show that a constant discounting society that respects the PC cannot aggregate individual lifetime preferences in a nondictatorial manner if individual discount factors and instantaneous utilities are heterogeneous. For our part, even if we consider a setting quite similar to Jackson and Yariv's, we show in Theorem 1 that a nondictatorial aggregation is possible if the social lifetime utility satisfies the IPC, which is weaker than the PC. However, if all individuals are EDUs, a linearly separate aggregation rule results in present-biased social lifetime utility. Since constant discounting is a simple and tractable assumption for policy making, we argue in Theorem 3 that if a TSU society respects a slightly modified IPC, for instance, restricted to pairs of streams only differing in two periods, then society is time consistent and its associated discount factor is a weighted average of individual factors.[20] It has been observed by Phelps and Pollak [1968], Barro [1974], Kimball [1987], Saez-Marti and Weibull [2005], and more recently by Galperti and Strulovici [2017] that altruism with respect to the immediate generation would lead to time consistency. Although our issue and setting are substantially different from theirs, the fundamental insight we obtain is that such time consistency can be derived from altruism between two arbitrary generations, namely, not necessarily between two consecutive generations.

Chambers and Echenique [2018] literally disregard the heterogeneity of individual instantaneous utilities and suggest three aggregation rules for discount factors. One of them proposes aggregation by means of a weighted average method, which can then be viewed as an alternative approach to a special case of our Theorem 3. However, due to the significant difference between the two settings, the visions conveyed by the respective social principles are fundamentally different. By contrast, **?** argue that preferences of successive generations should be counted. Hence, they suggest an intergenerational PC and characterize a constant social discount factor that is greater than any individual factor.[21] Chichilnisky, Hammond, and Stern [2020] consider an extinction threat for future generations and accordingly propose 'extinction' social discounting. There are many other approaches to studying social time consistency. Millner and Heal [2018] demonstrates that a society can be time consistent if the assumption that social consumption is time invariant is dropped. The same exercise, but in a continuous-time setting, is conducted by Drouhin [2020]. More recently, Hayashi and Lombardi [2021] in parallel develops a consensus Pareto condition in

[20] This result can be regarded as an axiomatization of the exponential social discounter. The first axiomatization of this kind of discounting behavior is due to Koopmans [1960]. It was subsequently extended by Fishburn and Rubinstein [1982] and Bleichrodt, Rohde, and Wakker [2008] in a different way.

[21] Drugeon and Wigniolle [2020] studies a similar collective decision problem by assuming hyperbolic discounting individuals.

the context of EDU individuals and society, which characterize a utilitarian social instantaneous utility and a dictatorial social constant discount factor.

Although constant social discounting is an irresistible form, it is nevertheless rarely observed in policy making. In reality, either the institutional rotation of political power (see, Harstad [2020]) or the cost for commitment (see, Laibson [2015]) would be responsible for triggering present-biased policies. To understand the underlying behavioral mechanism, we show in Theorem 4 that if the social lifetime utility satisfies the IPC for any pair of consumption streams that are different in the first 3 periods, our QAI-PC, then, along with quasi-stationarity, this social lifetime utility admits a quasi-hyperbolic discounting form as in Phelps and Pollak [1968], Laibson [2015] and many others.[22] In fact, Gollier and Zeckhauser [2005] and Jackson and Yariv [2014] show that a social lifetime utility satisfying the PC is present biased if individuals have heterogeneous discount factors. Therefore, our Theorem 5 generalizes this observation in showing that various versions of the IPC may characterize comparatively different levels of present bias. This result relates to Millner [2020], where it is argued that a society may feel insecure under various normative arguments. Theorem 5 can then be interpreted as an axiomatic judgment about multiple social principles. In contrast to the present bias approach, recent papers by Gonzalez, Lazkano, and Smulders [2018] and Ray [2018] show that society may exhibit future bias if there is a conflict of interest among future generations.

To the best of our knowledge, we are the first to identify the social principles necessary to obtain a separate aggregation of discount factors and instantaneous utilities. As Jackson and Yariv [2015] note, TSU is quite analogous to subjective expected utility, namely, we can interpret time as states and a discount function as a probability distribution over these states. In this regard, our result, by relaxing the PC to avoid Jackson-Yariv's and Zuber's impossibilities and obtaining separate aggregation, is conceptually related to Gilboa, Samet, and Schmeidler [2004] and Billot and Qu [2020], who show that a relaxed PC leads to separately aggregate heterogeneous beliefs and tastes. However, the issues are drastically different. In particular, the rule of aggregation for heterogeneous discount factors is different from those used in a belief aggregation context since the nature and the measure of the two respective notions, a belief and an impatience rate, are not alike.

Weitzman [2001] illustrate the undeniable and substantial reality of individual heterogeneity, that is, the natural disagreement between individuals' feelings or experts' opinions over lifetime preferences, which should encapsulate the tradeoff between current benefits and future benefits

[22] This result can be regarded as an axiomatization of the quasi-hyperbolic social discounter. The axiomatization of this kind of behavior can be found in Hayashi [2003], Attema, Bleichrodt, Rohde, and Wakker [2010], Noor [2009] and Montiel Olea and Strzalecki [2014].

from a social perspective. However, Weitzman [2001] and Freeman and Groom [2014] applied gamma discounting method to form the social discount factor, which is a different method from ours. Indeed, it is well known from Harsanyi [1955] and, more recently, Zuber [2011] that a society respecting the PC cannot make decisions in the same way as individuals when they are heterogeneous in lifetime preferences, whether it comes from the heterogeneity of discount factors, or instantaneous utilities or both. Our paper demonstrates, however, that a society satisfying another kind of Paretian unanimity, namely, the so-called impartial Pareto condition, behaves in the same way as individuals when individuals' preferences are time separable. In addition, the social discount function and instantaneous utility are shown to correspond to a weighted average of individual discount functions and individual utilities, respectively.

7 CONCLUSION

The principle of Pareto condition is not always compelling, in particular when individuals are heterogenous in both discount function and instantaneous utilities. We highlight the limitations of the principle in the context of heterogeneity and provide conditions under which a society should follow a separate aggregation rule and admit a time consistent or inconsistent representation form.

While we find that the plausibility of time consistency depends on the problems in hands, the empirical literature demonstrates that neither the collective time consistency nor inconsistency prevail in both field and lab studies. Adams, Cherchye, De Rock, and Verriest [2014] discover that household consumption behavior can be described by time consistent preference. However, Jackson and Yariv [2014] demonstrate the collectively violation of time consistency. Therefore, in practice, we can test collective decision making by justifying which one of our principles is applied.

APPENDIX

A PRELIMINARY RESULTS

We first prove a general aggregation result. One may find different versions of this proof, but we choose to present this result for two reasons. On the one hand, it is expressed in our setting, and on the other hand, it will be used repeatedly in the following proofs. To avoid tiresome duplication, this result is then singled out at the beginning.

If $k \in \mathbb{N}$, for $ij \in \mathcal{I} \times \mathcal{I}$, let $U^k_{ij} : \mathcal{L}^k \to \mathbb{R}$ and $U^k : \mathcal{L}^k \to \mathbb{R}$ be two real-valued functions defined on $\mathcal{L}^k$ with a convex range. Consider now a unanimity postulate.

k-Unanimity: Fix $k \in \mathbb{N}$; for any $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^k$, if $U^k_{ij}(\mathbf{z}) \geq U^k_{ij}(\mathbf{z}')$, for all $i, j \in \mathcal{I} \times \mathcal{I}$, then $U^k(\mathbf{z}) \geq U^k(\mathbf{z}')$. Furthermore, if there exists a fictitious ij such that $U^k_{ij}(\mathbf{z}) > U^k_{ij}(\mathbf{z}')$, then $U^k(\mathbf{z}) > U^k(\mathbf{z}')$.

Proposition A1. *Under MAC,*[23] *k-unanimity holds if and only if there exist positive numbers λ_{ij} and a real number μ such that, for $\mathbf{z} \in \mathcal{L}^k$:*

$$U^k(\mathbf{z}) = \sum_{ij \in \mathcal{I} \times \mathcal{I}} \lambda_{ij} U^k_{ij}(\mathbf{z}) + \mu.$$

Proof. Let

$$Y = \{\mathbf{y} \in \mathbb{R}^{n^2+1} | y_0 \leq -1 \text{ and } y_{ij} \geq 0, \text{ for } ij \in \mathcal{I} \times \mathcal{I}\},$$

and

$$A = \Big\{\big(U^k(\mathbf{z}) - U^k(\hat{\mathbf{z}}), U^k_{11}(\mathbf{z}) - U^k_{11}(\hat{\mathbf{z}}), \ldots, U^k_{nn}(\mathbf{z}) - U^k_{nn}(\hat{\mathbf{z}})\big) \in \mathbb{R}^{n^2+1} \Big| \mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^k\Big\}.$$

Since $\mathcal{L} = \Delta(X)$, the set $\{(U^k(\mathbf{z}), U^k_{11}(\mathbf{z}), \ldots, U^k_{nn}(\mathbf{z})) \big| \mathbf{z} \in \mathcal{L}\}$ is convex. Therefore, A is convex and symmetric with respect to vector $\mathbf{0}$. According to k-unanimity, we have $Y \cap A = \emptyset$. Now, define the vector space spanned by A:

$$\text{span}(A) = \Big\{\sum_{\ell=1}^{m} r_\ell \mathbf{a}_\ell \Big| m \in \mathbb{N}, r_\ell \in \mathbb{R} \text{ and } \mathbf{a}_\ell \in A\Big\}.$$

It is immediately clear that $Y \cap \text{span}(A) = \emptyset$. Since Y and $\text{span}(A)$ are polyhedral, nonempty and mutually disjoint, the strictly separating theorem (see, e.g., Rockafellar, Corollary 19.3.3) means that there exist $\boldsymbol{\pi} = (\pi, \pi_{11}, \ldots, \pi_{nn})$ such that, for all $\mathbf{a} \in \text{span}(A)$ and $\mathbf{y} \in Y$, $\boldsymbol{\pi} \cdot \mathbf{y} > \boldsymbol{\pi} \cdot \mathbf{a}$. Note that for all $\mathbf{a} \in \text{span}(A)$, we have $\boldsymbol{\pi} \cdot \mathbf{a} = 0$. (Suppose the opposite. Then, there must exist $\hat{\mathbf{a}} \in \text{span}(A)$ such that $\boldsymbol{\pi} \cdot \hat{\mathbf{a}} > 0$, and this is wlog since set A is symmetric. Therefore, there exists a large enough $r \in \mathbb{R}$ such that $\boldsymbol{\pi} \cdot r\hat{\mathbf{a}} > \boldsymbol{\pi} \cdot \mathbf{y}$, which is a contradiction.)

Select $\mathbf{y} = (-1, 0, \ldots, 0) \in Y$. The above inequality, i.e., $\boldsymbol{\pi} \cdot \mathbf{y} > 0$, implies that $-\pi > 0$, that is, $\pi < 0$. Thus, for all $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^k$,

$$U^k(\mathbf{z}) - U^k(\hat{\mathbf{z}}) = \sum_{ij \in \mathcal{I} \times \mathcal{I}} \frac{\pi_{ij}}{-\pi} \big[U^k_{ij}(\mathbf{z}) - U^k_{ij}(\hat{\mathbf{z}})\big].$$

[23] Recall the MAC: There exists $x^*, x_* \in \mathcal{L}$ such that, for all $k \in \mathbb{N}$ and all $ij \in \mathcal{I} \times \mathcal{I}$, $U_{ij}(x^*) > U_{ij}(x_*)$.

Fix $\hat{\mathbf{z}}$. For $ij \in \mathcal{I} \times \mathcal{I}$, define λ_{ij} as $\frac{\pi_{ij}}{-\pi}$. Define also μ as $U^k(\hat{\mathbf{z}}) - \sum_{ij \in \mathcal{I} \times \mathcal{I}} \lambda_{ij} U^k_{ij}(\hat{\mathbf{z}})$. Therefore, for all $\mathbf{z} \in \mathcal{L}^k$, we have $U^k(\mathbf{z}) = \sum_{ij} \lambda_{ij} U^k_{ij}(\mathbf{z})$.

To verify that each λ_{ij} is positive, let $\mathbf{y}$ be such that $y = -1, y_{ij} = r > 0$ and $y_{i'j'} = 0$ for $ij \neq i'j'$. The existence of $\mathbf{y}$ is guaranteed by MAC. Therefore, $\boldsymbol{\pi} \cdot \mathbf{y} > 0$ implies $-\pi + r\pi_{ij} > 0$. Hence, $\pi_{ij} > 0$, for ij, which then entails that each $\lambda_{ij} > 0$. □

B PROOF OF SECTION 3: PROPOSITION 1

The necessity part whereby a dictatorial society satisfies the PC is straightforward. We only prove the sufficiency part. Note that if we assume that $k = \infty$ and $U_{ij}(\mathbf{z}) = U_i(\mathbf{z}) = \sum_t d_{it} u_i(z_t)$, for all $j \in \mathcal{I}$, then the PC is equivalent to k-unanimity, for $k = \infty$. Since the PC is satisfied, by Proposition A1, for $k = \infty$, there exist nonnegative $\{\lambda_i\}_{i \in \mathcal{I}}$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$:

$$\sum_t d_t u(z_t) = \sum_{i \in \mathcal{I}} \lambda_i \sum_t d_{it} u_i(z_t) + \mu.$$

By normalization, $u_i(x_*) = 0$ and $u_i(x^*) = 1$ for all i. First, take $\mathbf{z} = (x_*, \ldots, x_*)$. Then, we have $\mu = 0$. Second, take $\mathbf{z} = (x^*, x_*, x_*, \ldots)$. Then, it implies $\sum_{i \in \mathcal{I}} \lambda_i = 1$. Therefore, for all $t \in \mathbb{N}$, $d_t = \sum_{i \in \mathcal{I}} \lambda_i d_{it}$. Also, Harsanyi aggregation theorem on $\mathcal{L} = \Delta(X)$ requires that for all $z \in \mathcal{L}$, $u(z) = \sum_{i \in \mathcal{I}} \lambda_i u_i(z)$. Hence, it requires that $\sum \lambda_i d_{it} u_i(z) = \sum \lambda_i d_{it} \cdot \sum \lambda_i u_i(z)$, i.e. :

$$\sum_{i \in \mathcal{I}} \lambda_i \big(d_{it} - \sum_{i \in \mathcal{I}} \lambda_i d_{it}\big) u_i(z) = 0.$$

By regularity, we know that $\{u_i\}_{i \in \mathcal{I}}$ are independent. Thus, for every $i \in \mathcal{I}$, $d_{it} - \sum \lambda_i d_{it} = 0$. Hence, $\lambda_i = 0$ or 1.

C PROOFS OF SECTION 4

C.1 Proof of Theorem 1

Necessity part. Suppose that for $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, all $ij \in \mathcal{I} \times \mathcal{I}$, $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$. Since each α_i and γ_j are nonnegative, then we have $\alpha_i \gamma_j U_{ij}(\mathbf{z}) \geq \alpha_i \gamma_j U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$. Now, $U = \sum_{ij} \alpha_i \gamma_j U_{ij}$ implies $U(\mathbf{z}) \geq U(\mathbf{z}')$, which proves the IPC.

Sufficiency part. Suppose that the IPC is satisfied. The IPC is equivalent to k-unanimity for $k = \infty$, where $U_{ij}(\mathbf{z}) = \sum_t d_{jt} u_i(z_t)$, for all $\mathbf{z} \in \mathcal{L}^\infty$. Then, according to Proposition A1, there exist nonnegative $\{\lambda_{ij}\}_{ij\in\mathcal{I}\times\mathcal{I}}$ such that, for $\mathbf{z}\in\mathcal{L}^\infty$, $U(\mathbf{z})=\sum_{ij}\lambda_{ij}U_{ij}(\mathbf{z})$, i.e.:

$$\sum_{t=1}^{\infty} d_t u(z_t) = \sum_{ij}\lambda_{ij}\sum_{t=1}^{\infty} d_{jt}u_i(z_t).$$

Recall that, by normalization, $u_i(x_*)=0$ and $u_i(x^*)=1$ for all i. Accordingly, defining $\mathbf{z}$ as $z_1=x^*$ and $z_t=x_*$, for $t\neq 1$, implies $\sum_{ij}\lambda_{ij}=1$. Let $\alpha_i=\sum_j\lambda_{ij}$ and $\gamma_j=\sum_i\lambda_{ij}$. Then, for $z\in\mathcal{L}$, $u(z)=\alpha_i u_i(z)$. For $t\in\mathbb{N}$, consider the stream $\mathbf{z}$ such that $z_t=x^*$ and $z_s=x_*$, for $s\neq t$. Therefore, $d_t=\sum_j\gamma_j d_{jt}$. Hence, U separately aggregates instantaneous individual utilities and individual discount functions.

C.2 Proof of Theorem 2

Since the necessity part is straightforward, we only show the sufficiency part. Suppose that CV-PC is satisfied. First, consider CV-PC restricted to pairs of constant common-valued consumption streams. Then, it is equivalent to: for all $x,y\in\mathcal{L}$, if $u_i(x)\geq u_i(y)$ for all $i\in\mathcal{I}$, then $u(x)\geq u(y)$. Clearly, this is PC in the context of lotteries. Therefore, Harsanyi's Aggregation Theorem implies that there exist nonnegative numbers α_i with $\sum_{i\in\mathcal{I}}\alpha_i=1$ such that $u(x)=\sum_i\alpha_i u_i(x)$ for $x\in\mathcal{L}$.

Second, consider CV-PC restricted to pairs of non-constant common-valued consumption streams. By MAC, $u_i(x^*)=1$ and $u_i(x_*)=0$ for all $i\in\mathcal{I}$. Since each individual has expected utility representation over $\mathcal{L}$, for all $z\in\Delta(\{x^*,x_*\})$, $u_i(z)=u_j(z)$ for all $i,j\in\mathcal{I}$. The set of common-valued streams is defined by

$$\mathcal{L}^\infty_{\text{cv}}=\{\mathbf{z}\in\mathcal{L}^\infty : z_t\in\Delta(\{x^*,x_*\}) \text{ for all } t\in\mathbb{N}\}.$$

It is immediate to see that $\mathcal{L}^\infty_{\text{cv}}$ is convex and, therefore, the set $\{(U(\mathcal{L}^\infty_{\text{cv}}),U_1(\mathcal{L}^\infty_{\text{cv}}),\ldots,U_n(\mathcal{L}^\infty_{\text{cv}})\}$ is also convex. Applying the separation theorem, there exist nonnegative γ_i with $\sum_i\gamma_i=1$ such that $U(\mathbf{z})=\sum_i\gamma_i U_i(\mathbf{z})$ for all $\mathbf{z}\in\mathcal{L}^\infty_{\text{cv}}$. Let $\mathbf{z}=(x_*,x^*,x_*,x_*,\ldots)$. We replace it into the expression and get $d_2=\sum_i\gamma_i d_{i2}$. Similarly, we can replace $(x_*,x_*,x^*,x_*,\ldots)$ into the expression and get $d_3=\sum_i\gamma_i d_{i3}$. Repeat the process, we have $d=\sum_i\gamma_i d_i$. Hence, social discount function is a convex combination of individual discount functions.

C.3 *Proof of Lemma 1*

Necessity part. Suppose that $U(x_t, \bar{z}_{*-t}) = U(y_s, \bar{z}_{*-s})$, where $t > s$. This implies, for $k > 0$:

$$\begin{aligned}\frac{u(y)}{u(x)} = \frac{d_t}{d_s} &= \frac{d_t}{d_{t-1}} \times \frac{d_{t-1}}{d_{t-2}} \times \cdots \times \frac{d_{s+1}}{d_s} \\ &= \delta(t) \times \delta(t-1) \times \cdots \times \delta(s+1) \\ &\leq \delta(t+k) \times \delta(t+k-1) \times \cdots \times \delta(s+k+1) \\ &= \frac{d_{t+k}}{d_{s+k}}.\end{aligned}$$

Therefore, it is clear that $U(x_{t+k}, \bar{z}_{*-(t+k)}) \geq U(y_{s+k}, \bar{z}_{*-(s+k)})$.
Sufficiency part. Since u is continuous and has a range containing $[0,1]$, for $t > 1$, there exist $x, y \in \mathcal{L}$ such that:

$$\delta(t) = \frac{d_t}{d_{t-1}} = \frac{u(x)}{u(y)}.$$

Then, present bias implies, for $k \in \mathbb{N}$, $d_{t+k}u(x) \geq d_{t+k-1}u(y)$. Therefore, we have:

$$\frac{d_t}{d_{t-1}} \geq \frac{d_t}{d_{t-1}}.$$

Since this expression is valid for all t, the discount factor $\delta(t)$ is increasing in t.

A similar process can be easily applied to prove the case of constant impatience.

C.4 *Proof of Proposition 2*

By Lemma 1, it suffices to show that $\delta(t)$ is increasing. Let $\delta_i(t)$ be the discount factor of individual i at horizon t. Then,

$$\begin{aligned}\delta(t+1) = \frac{d_{t+1}}{d_t} &= \frac{\sum \gamma_i d_{i(t+1)}}{\sum \gamma_i d_{it}} \\ &= \frac{\sum \gamma_i \delta_i(t+1) d_{it}}{\sum \gamma_i d_{it}}.\end{aligned}$$

Similarly, we have:

$$\delta(t) = \frac{\sum \gamma_i d_{it}}{\sum \frac{\gamma_i}{\delta_i(t)} d_{it}}.$$

Since $\delta_i(t)$ is increasing, for all i, it follows therefrom:

$$\frac{\delta_j(t+1)}{\delta_i(t)} + \frac{\delta_i(t+1)}{\delta_j(t)} \geq 2\sqrt{\frac{\delta_j(t+1)}{\delta_i(t)} \times \frac{\delta_i(t+1)}{\delta_j(t)}} \geq 2.$$

Now, since coefficients γ_i are nonnegative, it is also true that:

$$\Big(\frac{\delta_j(t+1)}{\delta_i(t)} + \frac{\delta_i(t+1)}{\delta_j(t)}\Big)\gamma_i\gamma_j \geq 2\gamma_i\gamma_j \quad \text{for all } i,j \in \mathcal{I}.$$

Therefore:

$$\Big(\sum \gamma_i\delta_i(t+1)d_{it}\Big) \times \Big(\sum \frac{\gamma_i}{\delta_i(t)}d_{it}\Big) \geq \Big(\sum \gamma_i d_{it}\Big)^2,$$

which, in turn, implies that $\delta(t+1) \geq \delta(t)$.

D Proofs of Section 5

D.1 Proof of Theorem 3

Necessity part is straightforward, therefore, we only prove the sufficiency part. Suppose that PAI-PC and stationarity hold.

(i) We have to show that the social lifetime utility U is an EDU. For this, it is sufficient to demonstrate that U satisfies all postulates of Koopmans [1960]. First, Postulate 1 is implied by the continuity of U. Next, Postulates 3 and 3′ are implied by the time separability of U. Moreover, stationarity corresponds to Postulate 4. Since $u_i(x^*) > u_i(x_*)$, for all $i \in \mathcal{I}$, PAI-PC implies that $u(x^*) > u(x_*)$. Therefore, by time additivity, for $\mathbf{z} \in \mathcal{L}^\infty$, $U(x^*, \mathbf{z}) > U(x_*, \mathbf{z})$, which implies Postulate 2. Finally, since $u(x^*) \geq u(z) \geq u(x_*)$, for all $z \in \mathcal{L}$ and all $\mathbf{z} \in \mathcal{L}^\infty$, we have $U(x^*, \ldots, x^*) \geq U(\mathbf{z}) \geq U(x_*, \ldots, x_*)$, that is, Postulate 5. Hence, by Koopmans' Theorem, the social lifetime utility U is an EDU.

(ii) For $k = 2$, PAI-PC is equivalent to 2-unanimity, where $U^2_{ij}(x,y) = u_i(x) + \delta_j u_i(y)$ and $U^2(x,y) = u(x) + \delta u(y)$, for all $(x,y) \in \mathcal{L}^2$. Therefore, Proposition A1 implies that there exist nonnegative λ_{ij} and a real number μ such that $U^2 = \lambda_{ij}U^2_{ij} + \mu$, that is, for any $(x,y) \in \mathcal{L}^2$:

$$u(x) + \delta u(y) = \sum_{ij} \lambda_{ij}u_i(x) + \sum_{ij} \lambda_{ij}\delta_j u_i(y) + \mu.$$

Let $x = y = x_*$. Then, $u(x_*) = u_i(x_*) = 0$ implies $\mu = 0$. Now, take $y = x_*$, and let $\alpha_i = \sum_j \lambda_{ij}$. The above equation becomes $u(x) = \sum_i \alpha_i u_i(x)$,, which proves that the social instantaneous utility u is a convex combination of individual instantaneous utilities. Suppose that $x = x_*$ and $y = x^*$. Then, $u(x^*) = u_i(x^*) = 1$, for all i. Let $\gamma_j = \sum_i \lambda_{ij}$. The above equation then becomes $\delta = \sum_j \gamma_j \delta_j$,, which proves that the social discount factor δ is a convex combination of individual discount factors.

D.2 Proof of Theorem 5

If $k \in \mathbb{N}$, we define the function $U^k : \mathcal{L}^\infty \to \mathbb{R}$ for $\mathbf{z} \in \mathcal{L}^\infty$ as follows:

$$U^k(\mathbf{z}) = U(x_*, \ldots, x_*, \mathbf{z}) = \sum_{t=1}^{\infty} d_{t+k} u(z_t).$$

We want first to show that this so-defined function U^k satisfies all five postulates of Koopmans [1960] and, therefore, is an EDU.

(i) Postulate 1 follows from the definition of U^k and continuity of u.

(ii) Since $u_i(x^*) > u_i(x_*)$, for all $i \in \mathcal{I}$, k-unanimity implies that $u(x^*) > u(x_*)$. Therefore, due to the time additivity of U^k and $d_{k+1} > 0$, for $\mathbf{z} \in \mathcal{L}^\infty$, $U^k(x^*, \mathbf{z}) > U^k(x_*, \mathbf{z})$, which implies Postulate 2.

(iii) Postulate 3 follows immediately from the time additivity of U^k. That is, for all $x, y \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$:

$$U^k(x, \mathbf{z}) \geq U^k(y, \mathbf{z}) \Leftrightarrow u(x) \geq u(y) \Leftrightarrow U^k(x, \mathbf{z}') \geq U^k(y, \mathbf{z}').$$

Similarly, we have:

$$U^k(x, \mathbf{z}) \geq U^k(x, \mathbf{z}') \Leftrightarrow \sum_{t=2}^{\infty} u(z_t) \geq \sum_{t=2}^{\infty} u(z_t') \Leftrightarrow U^k(y, \mathbf{z}) \geq U^k(y, \mathbf{z}').$$

(iv) Let $x \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, such that:

$$\begin{aligned}
U^k(\mathbf{z}) \geq U^k(\mathbf{z}') &\Leftrightarrow U(x_*, \ldots, x_*, \mathbf{z}) \geq U(x_*, \ldots, x_*, \mathbf{z}') \\
&\Leftrightarrow U(x_*, \ldots, x, \mathbf{z}) \geq U(x_*, \ldots, x, \mathbf{z}') \\
&\Leftrightarrow U(x_*, \ldots, x, \mathbf{z}) \geq U(x_*, \ldots, x, \mathbf{z}') \\
&\Leftrightarrow U^k(x, \mathbf{z}) \geq U^k(x, \mathbf{z}').
\end{aligned}$$

The first and last equivalence relations are given by definition. The second equivalence stems from the time additivity of U. The third equivalence is induced by the property of k-stationarity and proves that Postulate 4 holds.

(v) Note that $u(x^*) > u(x_*)$. Therefore, for two streams $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$ such that $z_t = x^*$ and $z'_t = x_*$, for all $t \in \mathbb{N}$, because d_t is positive, for any $\hat{\mathbf{z}} \in \mathcal{L}^\infty$, we have:

$$U^k(\mathbf{z}) \geq U^k(\hat{\mathbf{z}}) \geq U^k(\mathbf{z}'),$$

which demonstrates Postulate 5.

(vi) Finally, we have to demonstrate Postulate $3'$. Let $x, y, x', y' \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$. Hence:

$$\begin{aligned} U^k(x, y, \mathbf{z}) \geq U^k(x', y', \mathbf{z}) &\Leftrightarrow d_{k+1}u(x) + d_{k+2}u(y) \geq d_{k+1}u(x') + d_{k+2}u(y') \\ &\Leftrightarrow U^k(x, y, \mathbf{z}') \geq U^k(x', y', \mathbf{z}'). \end{aligned}$$

Similarly:

$$\begin{aligned} U^k(x, y, \mathbf{z}) \geq U^k(x', y, \mathbf{z}') &\Leftrightarrow d_{k+1}u(x) + \sum_{t=2}^{\infty} d_{t+2}u(z_t) \geq d_{k+1}u(x') + \sum_{t=2}^{\infty} d_{t+2}u(z'_t) \\ &\Leftrightarrow U^k(x, y', \mathbf{z}) \geq U^k(x', y', \mathbf{z}'). \end{aligned}$$

Therefore, U^k defined on $\mathcal{L}^\infty$ satisfies Postulates 1-5 and $3'$. Then, according to Koopmans' Theorem, there exist $a \in (0, 1)$ and a continuous function u on $\mathcal{L}$, such that:

$$U^k(\mathbf{z}) = \sum_{t=1}^{\infty} a^{t-1}u(z_t).$$

Since the representation is unique, there exists $b > 0$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$:

$$U(\mathbf{z}) = \sum_{t=1}^{k} d_t u(z_t) + b \sum_{t=k+1}^{\infty} a^{t-k-1}u(z_t).$$

For $\mathbf{z} \in \mathcal{L}^\infty$, we write:

$$U^{k+2}(\mathbf{z}) = \sum_{t=1}^{k} d_t u(z_t) + bu(z_{k+1}) + bau(z_{k+2}).$$

Therefore, $(k + 2)$-unanimity can be equivalently written as follows: for any $z_1, \ldots, z_{k+2}$ and

$z'_1, \ldots, z'_{k+2}$ in $\mathcal{L}$,

$$U^{k+2}_{ij}(\mathbf{z}) \geq U^{k+2}_{ij}(\mathbf{z}'), \text{ for all } i, j \in \mathcal{I} \Longrightarrow U^{k+2}(\mathbf{z}) \geq U^{k+2}(\mathbf{z}').$$

Hence, there exist nonnegative λ_{ij} such that:

$$U^{k+2}(\mathbf{z}) = \sum_{ij} \lambda_{ij} U^{k+2}_{ij}(\mathbf{z}). \tag{9}$$

Let $\mathbf{z}$ be such that $z_1 = x^*$ and $z_t = x_*$ for $t \neq 1$. Then, (9) implies $\sum_{ij} \lambda_{ij} = 1$. For $i \in \mathcal{I}$, denote $\alpha_i = \sum_j \lambda_{ij}$. Therefore, for all $z \in \mathcal{L}$, $u(z) = \sum_i \alpha_i u_i(z)$, which proves that u is a convex combination of u_i.

For $j \in \mathcal{I}$, denote $\gamma_j = \sum_i \lambda_{ij}$. Clearly, $\sum_j \gamma_j = 1$. Now, let $\mathbf{z}, \mathbf{z}'$ be such that $z_{k+1} = x^*$ and $z_t = x_*$, for $z \neq k+1$, $z'_{k+2} = x^*$ and $z_t = x_*$, for $z \neq k+2$. Substituting $\mathbf{z}$ and $\mathbf{z}'$ into (9) implies $b = \sum_j \gamma_j \delta_j^k$ and $ba = \sum_j \gamma_j \delta_j^{k+1}$. Define $\delta = a$ as follows:

$$\delta := a = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k}. \tag{10}$$

Let $\overline{\delta} = \max_j \delta_j$ and $\underline{\delta} = \min_j \delta_j$. Therefore, since γ_j is nonnegative, for any j:

$$\frac{\sum_j \gamma_j \delta_j^k \underline{\delta}}{\sum_j \gamma_j \delta_j^k} \leq \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \leq \frac{\sum_j \gamma_j \delta_j^k \overline{\delta}}{\sum_j \gamma_j \delta_j^k},$$

hence, $\underline{\delta} \leq \delta \leq \overline{\delta}$.
Let $\hat{b}$ be such that $\hat{b} \cdot a^k = b$. Therefore:

$$\hat{b} = \frac{(\sum_j \gamma_j \delta_j^k)^{k+1}}{(\sum_j \gamma_j \delta_j^{k+1})^k}.$$

Let $\mathbf{z}$ be such that $z_k = x^*$ and $z_t = x_*$ for $t \neq k$. Substituting $\mathbf{z}$ into (9) implies $d_k = \sum_j \gamma_j \delta_j^{k-1}$.

Similarly, for all $\ell = 2, \ldots, k-1$, $d_\ell = \sum_j \gamma_j \delta_j^{\ell-1}$. Now, define $\beta_k, \ldots, \beta_1$ recursively:

$$\beta_k = \hat{b} \times \frac{\delta^{k-1}}{d_k}$$
$$\beta_{k-1} = \frac{\hat{b}}{\beta_k} \times \frac{\delta^{k-2}}{d_{k-1}}$$
$$\vdots$$
$$\beta_\ell = \frac{\hat{b}}{\beta_k \beta_{k-1} \cdots \beta_{\ell+1}} \times \frac{\delta^{\ell-1}}{d_\ell}$$
$$\vdots$$
$$\beta_1 = \frac{\hat{b}}{\beta_k \beta_{k-1} \cdots \beta_2}.$$

Hence, for every $\ell = 2, \ldots, k$, it yields $d_\ell = \beta_1 \beta_2 \cdots \beta_{\ell-1} \delta^{\ell-1}$. Thus, for $\mathbf{z} \in \mathcal{L}^\infty$, U should take the following form:

$$U(\mathbf{z}) = u(z_1) + \beta_1 \delta u(z_2) + \beta_1 \beta_2 \delta^2 u(z_3) + \cdots + \prod_{j=1}^{k} \beta_j \sum_{t=j+1}^{\infty} \delta^{t-1} u(z_t),$$

in which δ is given by (10). Furthermore, substituting $\hat{b}$ and each d_ℓ into the above expression implies:

$$\beta_k = \frac{(\sum_j \gamma_j \delta_j^k)^2}{(\sum_j \gamma_j \delta_j^{k-1})(\sum_j \gamma_j \delta_j^{k+1})}$$
$$\beta_{k-1} = \frac{\sum_j \gamma_j \delta_j^{k-1}}{\sum_j \lambda_j \delta_j^{k-2}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}$$
$$\vdots$$
$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}$$
$$\vdots$$
$$\beta_1 = \frac{\sum_j \gamma_j \delta_j}{\sum_j \gamma_j} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}.$$

Now, we need to show that $0 < \beta_\ell < 1$, for each $\ell = 1, \ldots, k$. Let $1 < \ell < k$. Denote by A the term $\left[\sum_j \gamma_j \delta_j^\ell)^2 - (\sum_j \gamma_j \delta_j^{\ell-1})(\sum_j \gamma_j \delta_j^{\ell+1}\right]$. Then, we have:

$$\begin{aligned} A &= \sum_j (\gamma_j \delta_j^\ell)^2 + 2\sum_{i<j} \gamma_i \gamma_j \delta_i^\ell \delta_j^\ell - \big(\sum_j (\gamma_j \delta_j)^2 + \sum_{i<j} \gamma_i \gamma_j \delta_i^{\ell-1} \delta_j^{\ell+1} + \sum_{i<j} \gamma_i \gamma_j \delta_i^{\ell+1} \delta_j^{\ell-1}\big) \\ &= \sum_{i<j} \gamma_i \gamma_j (\delta_i \delta_j)^2 \big(2\delta_i \delta_j - \delta_i^2 - \delta_j^2\big) = -\sum_{i<j} \gamma_i \gamma_j (\delta_i \delta_j)^2 \big(\delta_i - \delta_j^2\big)^2 < 0. \end{aligned}$$

Since $A < 0$, it implies:

$$\frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \leq \frac{\sum_j \gamma_j \delta_j^{\ell+1}}{\sum_j \gamma_j \delta_j^\ell}.$$

By induction, we obtain:

$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}} \leq \frac{\sum_j \gamma_j \delta_j^{k+1}}{(\sum_j \gamma_j \delta_j)^k} \times \frac{\sum_j \gamma_j \delta_j^k}{(\sum_j \gamma_j \delta_j)^{k+1}} = 1.$$

D.3 Proof of Theorem 4

Note that Theorem 4 is a special case of Theorem 5 in which $k = 3$. Therefore, its proof follows directly from the proof of Theorem 5.

D.4 Proof of Proposition 4

Let $k < k'$. Let α_i and γ_j be nonnegative numbers such that $\sum_{i \in \mathcal{I}} \alpha_i = \sum_{j \in \mathcal{I}} \gamma_j = 1$. Therefore, according to (6,7,8):

$$\begin{aligned} u = \sum_i \alpha_i u_i \quad &= \quad \hat{u} \\ \delta = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \quad &< \quad \hat{\delta} = \frac{\sum_j \gamma_j \delta_j^{\hat{k}+1}}{\sum_j \gamma_j \delta_j^{\hat{k}}} \\ \beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}} \quad &> \quad \hat{\beta}_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^{\hat{k}}}{\sum_j \gamma_j \delta_j^{\hat{k}+1}}. \end{aligned}$$

Let $t, s \in \mathbb{N}$ with $t > 1$. Consider consumptions $x, y, x', y' \in \mathcal{L}$, such that $U(x, \bar{z}_*) = U(y_t, \bar{z}_{*-t})$, $\hat{U}(x', \bar{z}_*) = \hat{U}(y'_t, \bar{z}_{*-t})$, and $U(x_s, \bar{z}_{*-s}) \leq U(y_{t+s}, \bar{z}_{*-(t+s)})$. Equivalently, we have:

$$\begin{aligned} u(x) &= d_t u(y) \\ \hat{u}(x') &= \hat{d}_t \hat{u}(y') \\ d_s u(x) &\leq d_{t+s} u(y). \end{aligned}$$

Therefore,

$$d_t d_s \leq d_{t+s}. \tag{11}$$

We need to show that $\hat{d}_t \hat{d}_s \leq \hat{d}_{t+s}$. Hence, consider the three following cases:

Case 1: $k \geq t + s$.

Then, (11) implies:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_{t+s}} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_{t+s}} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}}.$$

Since $\delta < \hat{\delta}$, it is straightforward that:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} < \hat{\delta},$$

which implies: $\hat{d}_t \hat{d}_s \leq \hat{d}_{t+s}$.

Case 2: $k < t + s \leq \hat{k}$.

Assume that $t \leq k$ and $s \leq k$. (For the case of $t \geq k$ or $s \geq k$, the proof is quite similar.) Then, (11) implies:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} \times \left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s}.$$

Note that, for $1 \leq \ell \leq t + s - k$:

$$\left(\frac{\hat{\delta}}{\delta}\right) \times \hat{\beta}_{k+\ell} \geq 1,$$

which implies:

$$\left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s} \geq 1.$$

Therefore:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} < \hat{\delta}.$$

<u>Case 3:</u> $t + s > \hat{k}$.

We only prove the case corresponding to $t \leq k$ and $s \leq k$ since the rest are similar. Again, we have:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} < \delta.$$

In this case, we know that:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{\hat{k}}} \times \left(\frac{\hat{\delta}}{\delta}\right)^{t+s-\hat{k}} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{\hat{k}}.$$

By the same argument as the one used in Case 2, we have:

$$\left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s} \geq 1.$$

Therefore:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{\hat{k}}} < \hat{\delta}.$$

D.5 Proof of Corollary 1

The necessity part is immediate. We only show *sufficiency* part. Suppose the social lifetime utility U satisfies k-IAI-PC for $k > 3$. Since individual lifetime utilities are EDUs, k-IAI-PC is equivalent to k-unanimity, and $U_{ij}^k = u_i + \delta_j u_i + \cdots + \delta_j^{k-1} u_i$. By Proposition A1 and previous arguments, there exist nonnegative λ_{ij} with $\sum_{ij} \lambda_{ij} = 1$ such that, for all $\mathbf{z} \in \mathcal{L}^k$,

$$u(z_1) + \delta u(z_2) + \cdots + \delta^{k-1} u(z_k) = \sum_{ij} \lambda_{ij} \Big(u_i(z_1) + \delta_j u_i(z_2) + \cdots + \delta_j^{k-1} u_i(z_k) \Big).$$

Now, let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, we obtain:

$$u = \sum_i \alpha_i u_i \quad \text{and} \quad \delta^k = \sum_j \gamma_j \delta_j^k \quad \text{for all } k.$$

Therefore, we must have $\gamma_j = 0$ or 1. That is, the rule is dictatorial for the conception of the social discount factor.

REFERENCES

Abi Adams, Laurens Cherchye, Bram De Rock, and Ewout Verriest. Consume Now or Later? Time Inconsistency, Collective Choice, and Revealed Preference. *American Economic Review*, 104(12):4147–83, December 2014.

Arthur E. Attema, Han Bleichrodt, Kirsten I. M. Rohde, and Peter P. Wakker. Time-Tradeoff Sequences for Analyzing Discounting and Time Inconsistency. *Management Science*, 56(11):2015–2030, 2010.

Robert J. Barro. Are Government Bonds Net Wealth? *Journal of Political Economy*, 82(6):1095–1117, November 1974.

B. Douglas Bernheim. Intergenerational Altruism, Dynastic Equilibria and Social Welfare. *The Review of Economic Studies*, 56(1):119–128, 01 1989.

Antoine Billot and Xiangyu Qu. Utilitarian aggregation with heterogeneous beliefs. *American Economic Journal: Microeconomics*, 2020. forthcoming.

Han Bleichrodt, Kirsten I.M. Rohde, and Peter P. Wakker. Koopmans' Constant Discounting for Intertemporal Choice: A Simplification and a Generalization. *Journal of Mathematical Psychology*, 52(6):341–347, December 2008.

J.M. Buchanan and G. Tullock. *The Calculus of Consent: Logical Foundations of Constitutional Democracy*. Ann Arbor paperbacks. University of Michigan Press, 1962.

Andrew Caplin and John Leahy. The Social Discount Rate. *Journal of Political Economy*, 112(6):1257–1268, 2004.

Christopher P. Chambers and Federico Echenique. On Multiple Discount Rates. *Econometrica*, 86(4):1325–1346, 2018.

Christopher P. Chambers, Federico Echenique, and Alan D. Miller. Decreasing impatience. working paper, 2021.

Graciela Chichilnisky, Peter J. Hammond, and Nicholas Stern. Fundamental utilitarianism and intergenerational equity with extinction discounting. *Social Choice and Welfare*, 54(2):397–427, 2020.

Jonathan Cohen, Keith Marzilli Ericson, David Laibson, and John Myles White. Measuring Time Preferences. *Journal of Economic Literature*, 58(2):299–347, June 2020.

Nicolas Drouhin. Non-stationary additive utility and time consistency. *Journal of Mathematical Economics*, 86:1 – 14, 2020.

Jean-Pierre Drugeon and Bertrand Wigniolle. On Markovian collective choice with heterogeneous quasi-hyperbolic discounting. *Economic Theory*, 2020.

Moritz A. Drupp, Mark C. Freeman, Ben Groom, and Frikk Nesje. Discounting Disentangled. *American Economic Journal: Economic Policy*, 10(4):109–34, November 2018.

Emmanuel Farhi and Iván Werning. Inequality and Social Discounting. *Journal of Political Economy*, 115(3):365–402, 2007.

M. S. Feldstein. The Social Time Preference Discount Rate in Cost Benefit Analysis. *The Economic Journal*, 74(294): 360–379, 1964.

Tangren Feng and Shaowei Ke. Social Discounting and Intergenerational Pareto. *Econometrica*, 86(5):1537–1567, 2018.

Peter C. Fishburn and Ariel Rubinstein. Time Preference. *International Economic Review*, 23(3):677–694, 1982.

Shane Frederick, George Loewenstein, and Ted O'Donoghue. Time Discounting and Time Preference: A Critical Review. *Journal of Economic Literature*, 40(2):351–401, June 2002.

Mark C. Freeman and Ben Groom. Positively Gamma Discounting: Combining the Opinions of Experts on the Social Discount Rate. *The Economic Journal*, 125(585):1015–1024, 07 2014.

Simone Galperti and Bruno Strulovici. A Theory of Intergenerational Altruism. *Econometrica*, 85(4):1175–1218, 2017.

Reyer Gerlagh and Matti Liski. Consistent climate policies. *Journal of the European Economic Association*, 16(1): 1–44, 03 2017.

Itzhak Gilboa, Dov Samet, and David Schmeidler. Utilitarian Aggregation of Beliefs and Tastes. *Journal of Political Economy*, 112(4):932–938, August 2004.

Christian Gollier and Richard Zeckhauser. Aggregation of Heterogeneous Time Preferences. *Journal of Political Economy*, 113(4):878–896, August 2005.

Francisco M. Gonzalez, Itziar Lazkano, and Sjak A. Smulders. Intergenerational altruism with future bias. *Journal of Economic Theory*, 178:436–454, nov 2018.

Yoram Halevy. Time Consistency: Stationarity and Time Invariance: Time Consistency. *Econometrica*, 83(1):335–352, January 2015.

John C Harsanyi. Cardinal utility in welfare economics and in the theory of risk-taking. *Journal of Political Economy*, 61(5):434–435, 1953.

John C. Harsanyi. Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. *Journal of Political Economy*, 63(4):309–321, 1955.

Bård Harstad. Technology and Time Inconsistency. *Journal of Political Economy*, 128(7):2653–2689, July 2020.

Takashi Hayashi. Quasi-stationary cardinal utility and present bias. *Journal of Economic Theory*, 112(2):343 – 352, 2003.

Takashi Hayashi and Michele Lombardi. Social discount rate: spaces for agreement. *Economic Theory Bulletin*, 08 2021.

Matthew O. Jackson and Leeat Yariv. Present Bias and Collective Dynamic Choice in the Lab. *American Economic Review*, 104(12):4184–4204, December 2014.

Matthew O. Jackson and Leeat Yariv. Collective Dynamic Choice: The Necessity of Time Inconsistency. *American Economic Journal: Microeconomics*, 7(4):150–178, November 2015.

Larry Karp. Global warming and hyperbolic discounting. *Journal of Public Economics*, 89(2):261–282, 2005.

Miles S. Kimball. Making sense of two-sided altruism. *Journal of Monetary Economics*, 20(2):301 – 326, 1987.

Tjalling C. Koopmans. Stationary Ordinal Utility and Impatience. *Econometrica*, 28(2):287, April 1960.

David Laibson. Golden Eggs and Hyperbolic Discounting. *The Quarterly Journal of Economics*, 112(2):443–477, 1997.

David Laibson. Why don't present-biased agents make commitments? *American Economic Review Papers and Proceedings*, 105(5):267–272, 2015.

Stephen A. Marglin. The Social Rate of Discount and The Optimal Rate of Investment. *The Quarterly Journal of Economics*, 77(1):95–111, 1963.

Antony Millner. Nondogmatic Social Discounting. *American Economic Review*, 110(3):760–775, March 2020.

Antony Millner and Geoffrey Heal. Discounting by Committee. *Journal of Public Economics*, 167:91–104, November 2018.

Philippe Mongin. Consistent Bayesian Aggregation. *Journal of Economic Theory*, 66(2):313 – 351, 1995.

J.L. Montiel Olea and Tomasz Strzalecki. Axiomatization and Measurement of Quasi-hyperbolic Discounting. *Quarterly Journal of Economics*, 129:1449–1499, 2014.

Frikk Nesje. Cross-dynastic intergenerational altruism. working paper, 2021.

Jawwad Noor. Hyperbolic discounting and the standard model: Eliciting discount functions. *Journal of Economic Theory*, 144(5):2077 – 2083, 2009.

Jawwad Noor and Norio Takeoka. Optimal discounting. *Econometrica*, 2021. forthcoming.

William D. Nordhaus. A Review of the Stern Review on the Economics of Climate Change. *Journal of Economic Literature*, 45(3):686–702, September 2007.

E. S. Phelps and R. A. Pollak. On Second-Best National Saving and Game-Equilibrium Growth. *The Review of Economic Studies*, 35(2):185–199, 1968.

Drazen Prelec. Decreasing Impatience: A Criterion for Non-stationary Time Preference and Hyperbolic Discounting. *Scandinavian Journal of Economics*, 106(3):511–532, October 2004.

John K.-H. Quah and Bruno Strulovici. Discounting, Values, and Decisions. *Journal of Political Economy*, 121(5): 896–939, 2013.

F. P. Ramsey. A Mathematical Theory of Saving. *The Economic Journal*, 38(152):543–559, 1928.

Debraj Ray. Hedonistic altruism and welfare economics. working paper, 2018.

Maria Saez-Marti and Jörgen W. Weibull. Discounting and altruism to future decision-makers. *Journal of Economic Theory*, 122(2):254 – 266, 2005.

Paul A. Samuelson. A Note on Measurement of Utility. *The Review of Economic Studies*, 4(2):155–161, 1937.

Richard Thaler. Some empirical evidence on dynamic inconsistency. *Economics Letters*, 8(3):201 – 207, 1981.

Martin L. Weitzman. Gamma Discounting. *American Economic Review*, 91(1):260–271, March 2001.

Stéphane Zuber. The aggregation of preferences: can we ignore the past? *Theory and Decision*, 70(3):367–384, March 2011.