Endogenous Liquidity and Volatility*

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Abstract

Is asset liquidity a source of price volatility? We answer this question within a continuous-time, New Monetarist economy under extrinsic uncertainty and endogenous asset liquidity. We consider single or multiple assets, risk-free or risky assets, assets that have a positive intrinsic value, no intrinsic value, or even a negative intrinsic value. If assets are both perfectly liquid and intrinsically valuable, then their price is invariant to extrinsic uncertainty. Sunspot equilibria exist only if assets have a non-positive intrinsic value (e.g., fiat monies) or if their liquidity (acceptability, pledgeability) is imperfect, e.g., due to informational frictions.

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1 Introduction

Does the liquidity of an asset – the ease with which it can be traded to finance expenditures – make its price excessively volatile relative to fundamentals?[1] The recent literature in monetary theory answers positively based on the following logic. The price of an asset is composed of the value of its dividend stream and the value of its liquidity services. To the extent that liquidity is based on self-referential beliefs, the liquidity value of an asset can fluctuate as beliefs change over time. This view has been formalized in the context of discrete-time New Monetarist models by Lagos and Wright (2003) and Rocheteau and Wright (2013), among several others reviewed in Section 1.1, by showing the existence of equilibria where asset prices vary with changes in *extrinsic* states unrelated to economic fundamentals.

In this paper, we revisit the liquidity-volatility relationship in a continuous-time setting. We construct a New Monetarist economy under extrinsic uncertainty where centralized and decentralized (over-the-counter) markets open concurrently and continuously over time. Our first result contradicts the common wisdom described above. It shows that the liquidity of a Lucas tree yielding a constant, positive, dividend flow does not make its price volatile. Formally, extrinsic uncertainty does not matter.

This negative result casts a shadow on the ability of liquidity considerations to contribute to our understanding of asset price volatility. The rest of the paper explores the robustness of the result to alternative assumptions regarding assets' characteristics. We consider single or multiple assets, risk-free or risky assets, assets that have a positive intrinsic value, no intrinsic value, or even a negative intrinsic value. We also explore alternative microfoundations of the acceptability and pledgeability of an asset.

A first insight is that the intrinsic value of an asset is critical for the relationship between its liquidity and its volatility. While extrinsic uncertainty does not matter when the assets' intrinsic value is positive, it does matter when assets have zero or negative intrinsic value. In other words, liquidity might not explain the volatility of stocks or housing prices but it could explain the volatity of crypto-currencies. Formally, if an asset has no fundamental value, e.g., a fiat money, and if its supply is not shrinking, then price volatility can occur in equilibrium but only if there is a state where the asset becomes valueless, either instantly or asymptotically. In contrast to discrete-time models (e.g., Lagos and Wright, 2003), however, there is no proper stationary sunspot equilibrium where the value of money is always positive. Our result has a direct implication for the high volatility of crypto-currency prices. It suggests that a crypto-currency is volatile when it is anticipated that there is a state where it will ultimately fail.

If the growth rate of the money supply is negative, there exist other sunspot equilibria that generate recurrent bubbles. Along these equilibria that satisfy the transversality condition for optimality, the market capitalization of the currency grows at a constant rate until it bursts and returns to a positive, constant value. Hence, our model can explain repeated boom-bust cycles of currency prices in deflationary environments.

[1] The notion of excess volatility of asset prices has been introduced by Shiller (1981) and captures the idea that the observed volatility of asset prices is larger than the one that would be predicted by a frictionless asset pricing model.

A last case to consider is when the asset has a negative intrinsic value, e.g., a fiat money with a storage cost or a real asset with a negative net present value.[2] We show there can exist multiple steady states with different asset prices as well as sunspot equilibria exhibiting asset price fluctuations.[3] Interestingly, this set of equilibria is equivalent to the one obtained when money creation finances a constant flow of expenditure. We generalize the model to the case where the asset takes the form of a risky Lucas tree with a dividend that alternates between a positive and negative value. Provided that the fundamental value of the Lucas tree in the low-dividend state is negative, there can exist multiple equilibria with different asset price volatilities.

Equilibria $\longrightarrow$ / Asset intrinsic value $\downarrow$	Steady states with positive asset price	Stationary, positive recurrent sunspot equilibria	Nonstationary sunspot equilibria
Positive (Lucas trees)	Unique	$\times$	$\times$
Zero (Fiat monies)	Unique	$\times$	$\surd$
Negative	Multiple	$\surd$	$\surd$

Table 1: Assets' intrinsic value and the relevance of extrinsic uncertainty when assets are perfectly liquid. "$\surd$" means equilibria exist under some conditions.

The main result so far, summarized in Table 1, is that liquidity in environments where assets are traded continuously can only explain the volality of assets with nonpositive fundamental values, which limits considerably the relevance of the theory. This conclusion, however, has been reached under the assumption that assets are universally accepted and fully pledgeable. Our second insight is that liquidity can be a source of volatility when it is *truly endogenous*, i.e., it can be partial and depends on primitives, policy, as well as equilibrium outcomes. In order to illustrate this point, the second part of the paper endogenizes the degree of asset liquidity by assuming informational frictions and moral hazard along the lines of Li et al. (2012).[4] We derive an endogenous pledgeability constraint that features a pecuniary externality according to which pledgeability increases with the expected rate of return of the asset. Mathematically, the asset price obeys an ODE that takes the form of a correspondence. This property allows our model to generate deterministic, periodic equilibria even when the steady-state equilibrium is unique. More generally, there exist a variety of deterministic and stochastic equilibria along which the asset price and pledgeability fluctuate. If the asset has a positive intrinsic value and the moral hazard problem is sufficiently severe, then multiple steady states can exist across which asset prices and pledgeability are negatively correlated. The results are summarized in the following table.

[2]While this case might seem a mere theoretical curiosity, firms incurring operating losses comprise the majority of equity issuers in the U.S. (Denis and McKeon, 2018).

[3]These results are consistent with the literature in discrete time (e.g., Lagos and Wright, 2003; Gu, Menzio, Wright, and Zhu, 2021) but we emphasize the need of a disposal cost in order to sustain a negative price for the asset.

[4]We study endogenous acceptability as in Lester et al. (2012) in Appendix D.

Equilibria $\longrightarrow$ / Asset intrinsic value $\downarrow$	Steady states	Deterministic, periodic equilibria	Stationary sunspot equilibria
Zero (Fiat monies)	Unique	$\surd$	$\surd$
Positive (Lucas trees)	Unique or multiple	$\surd$	$\surd$

Table 2: Endogenous asset liquidity and the relevance of extrinsic uncertainty.
"$\surd$" means equilibria exist under some conditions.

We conclude by studying economies with multiple liquid assets. If all assets have a positive intrinsic value, the irrelevance of extrinsic uncertainty still holds. If two assets with fundamental values of opposite sign coexist, then sunspot equilibria only exist when the real interest rate is negative. In that case, the volatility of the price of the bad asset (the one with negative intrinsic value) makes the price of the good asset (the one with positive intrinsic value) also volatile.

1.1 Literature

The literature on extrinsic uncertainty and sunspot equilibria is surveyed by Shell (2008) and, for macroeconomic applications, by Benhabib and Farmer (1999) and Farmer (2020). The notion of sunspot equilibrium is due to Cass and Shell (1983).[5] While the earlier work on extrinsic uncertainty took place in the context of discrete-time, competitive overlapping generations (OLG) economies, we describe a continuous-time New Monetarist economy with infinitely-lived agents who face liquidity constraints when they interact within randomly-formed pairwise meetings.[6] We conjecture that the results regarding the relevance of extrinsic uncertainty in continuous-time environments are fairly general and are not overly dependent on some specificities of the environment.

In the recent generation of search-theoretic models of liquidity, Lagos and Wright (2003) study deterministic and stochastic dynamic monetary equilibria.[7] Our main innovations consist in: (i) formalizing asset markets opened continuously through time; (ii) providing a characterization of stationary and nonstationary sunspot equilibria; (iii) endogenizing asset liquidity with explicit informational frictions.[8] The importance of continuous time for the set of deterministic equilibria is emphasized in Choi and Rocheteau (2021b). In order to focus on the liquidity-volatility relationship, our paper adds extrinsic uncertainty so as to determine the existence of sunspot equilibria and it endogenizes asset liquidity with informational frictions. Moreover,

[5] Contributions that are relevant for our work include Azariadis (1981) who showed the existence of stationary sunspot equilibria, Azariadis and Guesnerie (1986) and Guesnerie (1986) who established links between the existence of deterministic cycles and stationary sunspot equilibria. Our extension in Appendix D with costly participation is related to Balasko, Cass, and Shell (1995) who showed the existence of sunspot equilibria when market participation is restricted.

[6] Closer to the class of models we are working with, Woodford (1986) proved the existence of sunspot equilibria in an economy where infinitely-lived agents face liquidity constraints.

[7] Wright (1994) was the first to show the existence of sunspot equilibria in the model with indivisible money and goods of Kiyotaki and Wright (1989, 1993). Corbae, Temzelides, and Wright (2002) considered sunspot equilibria in a version with endogenous matching. Shi (1995) established the existence of sunspot equilibria in a model with indivisible money and divisible goods. Ennis (2001, 2004) studied sunspot equilibria in the Shi (1995) and Trejos and Wright (1995) models in the presence of barter trades and under the extensive-form bargaining game of Coles and Wright (1998).

[8] The continuous-time New Monetarist model has been developped by Craig and Rocheteau (2008) and Rocheteau and Rodriguez-Lopez (2014). These papers only consider steady-state equilibria. Choi and Rocheteau (2021a) investigate deterministic non-stationary equilibria when money is privately produced.

it is not limited to the study of fiat money and considers assets with positive and negative intrinsic values.

There are several applications of the Lagos-Wright model with extrinsic uncertainty that are directly related to our work.[9] Relative to Rocheteau, Rupert, and Wright (2007) and Rocheteau, Rupert, Shell and Wright (2008) who consider sunspot equilibria in a version of the Lagos and Wright (2005) model with indivisible labor, we do not need to introduce nonconvexities to generate sunspot equilibria.[10] In Appendix D, we consider the entry of sellers, as in Rocheteau and Wright (2005, 2013), and show that the multiplicity of steady states is robust to continuous time. However, the sunspot equilibria obtained from the backward-bending phase line are not robust. The case of assets with negative dividends is studied in Lagos and Wright (2003) and Gu, Menzio, Wright, and Zhu (2021). We generalize the model to have stochastic dividends and the coexistence of assets with positive and negative intrinsic values. Related to that latter extension, Altermatt, Iwasaki, and Wright (2021) study sunspot equilibria with multiple assets (money, Lucas trees, and capital). While sunspot equilibria disappear in continuous time, we show how to recover them by providing microfoundations for asset liquidity.

While most of the literature above assumes that assets are either perfectly liquid or liquidity (acceptability or pledgeability) is exogenous, we endogenize asset liquidity based on Li, Rocheteau and Weill (2012). The approach in Appendix D is based on Lester, Postlewaite, and Wright (2012). The role of endogenous liquidity in generating multiple steady states under both approaches is illustrated in Rocheteau, Wright, and Xiao (2018) in economies with money and bonds. Alternative approaches to asset liquidity in New Monetarist models include Rocheteau (2011), Bajaj (2018), and Madison (2019), based on adverse selection, and Jacquet (2021) based on endogenous market segmentation.

2 Environment

Time is continuous and indexed by $t \in \mathbb{R}_+$. The economy is populated by a measure two of infinitely-lived agents divided evenly between *buyers* and *sellers*. There are two perishable goods: the first good, taken as the numéraire, is traded in a competitive market opened continuously through time; the second good is exclusively produced and consumed in pairwise meetings. The labels *buyer* and *seller* refer to agents' roles in pairwise meetings.

Buyers' preferences are represented by the following lifetime expected discounted utility:

$$\mathcal{U}^b = \mathbb{E}\left\{\int_0^{\infty} e^{-\rho t} dC(t) + \sum_{n=1}^{+\infty} e^{-\rho T_n} u\left[y(T_n)\right]\right\}, \tag{1}$$

where $C(t)$ is a measure of the cumulative net consumption of the numéraire good and $y(t)$ is consumption in

[9] Other extensions of the Lagos-Wright model with extrinsic uncertainty that are not directly related to what we do include Lagos (2013) on the coexistence of money and interest-bearing bonds, Gu, Mattesini, Monnet, and Wright (2013) and Bethune, Hu, and Rocheteau (2018) on endogenous credit cycles.

[10] Sunspots play a similar role as lotteries in convexifying the commodity space, as in Garratt (1995). Dong (2011) adopts the same environment under the notion of competitive search equilibrium. Other applications of nonconvexities (increasing returns to scale) to generate sunspot equilibria include Benhabib and Farmer (1994) and Howitt and McAfee (1992).

pairwise meetings.[11] Negative consumption of the numéraire good is interpreted as production. The integral on the right side of (1) accounts for the utility of consuming, or producing if $dC(t) < 0$, the numéraire good. The discrete sum represents the discounted utility of consumption in pairwise meetings. The utility from consuming $y \in \mathbb{R}_+$ units of goods in a pairwise meeting is $u(y)$, where u is continuously differentiable, strictly increasing, strictly concave, $u(0) = 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$. (We will weaken these conditions in some of our examples.) We denote y^* the solution to $u'(y^*) = 1$. Pairwise meetings occur according to a Poisson process, $\{T_n\}$, with arrival rate $\alpha > 0$.

Sellers' preferences are represented by the following lifetime expected utility:

$$\mathcal{U}^s = \mathbb{E}\left\{\int_0^\infty e^{-\rho t} dC(t) - \sum_{n=1}^{+\infty} e^{-\rho T_n} y(T_n)\right\}. \tag{2}$$

The integral on the right side of (2) is the discounted linear utility from the consumption and production of the numéraire good. The discrete sum corresponds to the disutility of producing y in pairwise meetings.

In pairwise meetings, the buyer does not have access to the technology to produce the numéraire. Moreover, in the absence of public monitoring and enforcement technology, she is not trusted to repay her debt in the future. These frictions create a need for liquid assets.

Assets take the form of long-lived Lucas trees that are storable and durable. Each Lucas tree generates a dividend flow equal to d. The supply of Lucas trees at time t is denoted M_t. If $d > 0$, then the asset is intrinsically valuable. If $d < 0$, then the asset has negative intrinsic value and can be interpreted as a fiat money with a storage cost. In these two cases, we assume its supply is fixed, $M_t = M$ for all t. (This assumption is necessary for the existence of a steady state.) If $d = 0$, then the asset has no intrinsic value, i.e., it corresponds to a fiat money. The constant money growth rate is $\pi \equiv \dot{M}_t/M_t$ and new money is injected into the economy through lump-sum transfers (or taxes if $\pi < 0$) to buyers. These transfers, expressed in terms of the numeraire, are denoted τ_t. We will also generalize the environment to include multiple assets.

We introduce extrinsic uncertainty as follows. There is a finite set, S, of sunspot states that do not affect the fundamentals of the economy, e.g., preferences or technology. Transitions across states obey a continuous-time Markov chain. We denote $\lambda_{ss'} \in \mathbb{R}_+$ the Poisson arrival rate of state s' condition on the current state being s. In general, allocations and prices could depend on the entire history of sunspot shocks, s^t. We will restrict our attention to equilibria where agents only pay attention to the current realization of the sunpost state, s_t, and calendar time, t. Hence, the price of the asset in terms of the numéraire in state s at time t is denoted $\phi_{s,t}$. This formulation will encompass the standard notions of equilibria under extrinsic uncertainty used in the literature (e.g., deterministic equilibria, stationary sunspot equilibria).

[11] This specification is flexible enough to accommodate consumption (or production) in flows and consumption or production of discrete quantities. If consumption (or production) of the numéraire happens in flows, then $C(t)$ admits a density, $dC(t) = c(t)dt$. If the buyer consumes or produces a discrete quantity of the numéraire good at some instant t, then $C(t^+) - C(t^-) \neq 0$.

3 When sunspots don't matter: Assets with positive intrinsic value

We start our investigation by considering an economy where the liquid asset takes the form of a risk-free Lucas tree with a constant dividend flow, $d > 0$. Hence, the fundamental price in the absence of liquidity considerations is $d/\rho > 0$ and is constant through time. It follows that any fluctuation of the asset price in equilibrium can be interpreted as some excess volatility.

Consider a buyer with real asset holdings equal to m (expressed in the numéraire). From the linearity of the preferences with respect to the numéraire good, the value function of a buyer at time t and in state s is linear in m, i.e., $V^b_{s,t}(m) = m + V^b_{s,t}$ where $V^b_{s,t}$ is a function of time and the state (see Appendix B). In the following, we characterize the intercept of buyers' value functions, $V^b_{s,t}$.

We first define the expected rate of return of the asset in state s at time t as

$$r_{s,t} = \frac{d + \dot{\phi}_{s,t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\phi_{s',t} - \phi_{s,t}\right)}{\phi_{s,t}}. \tag{3}$$

It is equal to the dividend plus the expected capital gain divided by the value of the asset. The capital gains on the numerator have two components. The first component is the change in the value of the asset over time conditional on the sunspot state being constant while the second component is the change in the value of the asset due to a change in the sunspot state.

Given (3), the value function of the buyer solves the following HJB equation:

$$\rho V^b_{s,t} = \max_{m \geq 0} \left\{ -(\rho - r_{s,t})m + \alpha v(m) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(V^b_{s',t} - V^b_{s,t}\right) + \tau_{s,t} + \dot{V}^b_{s,t} \right\}, \tag{4}$$

where

$$v(m) = \max_{y \geq 0} \left\{ u(y) - p(y) \text{ s.t. } p(y) \leq m \right\}, \tag{5}$$

is the buyer's surplus from a bilateral trade. At every point in time the buyer chooses her (real) asset holdings to maximize the right side of (4). The first term is the flow cost of holding liquid assets. It is the difference between the rate of time preference, ρ, and the expected rate of return of the asset, r_s, multiplied by the real asset holdings, m. The second term is the expected surplus from a bilateral trade which, from (5), is equal to the difference between the utility of consumption, $u(y)$, and the payment, $p(y)$, subject to the feasibility condition that the payment does not exceed the buyer's liquid wealth, m. The third term represents the changes in the value functions as the sunspot transitions between states. The change from state s to state s' occurs at Poisson arrival rate $\lambda_{ss'}$. In that event, the buyer's continuation value changes from V^b_s to $V^b_{s'}$. The fourth term represents a lump-sum transfer by the government (which can be set to zero for now). Finally, the last term is the change of the value function over time conditional on the state.

The payment function, $p(y)$, can take different forms corresponding to different bargaining solutions.[12] For instance, if buyers make take-it-or-leave-it offers, $p(y) = y$. If terms of trade are determined according

[12] Here we do not provide strategic foundations for $p(y)$. We just require that the payment is individually rational and pairwise

to the Kalai proportional solution, then $p(y) = \theta y + (1-\theta)u(y)$ for some $\theta \in (0,1)$. If assets are negotiated gradually according to the Nash solution (Rocheteau et al. 2021),

$$p(y) = \int_0^y \frac{u'(x)}{\theta u'(x) + 1 - \theta} dx \quad \text{for all } y \leq y^*,$$

where y^* is the solution to $u'(y^*) = 1$. For all the solutions above, $p(y)$ is continuously differentiable with $p'(y) > 0$ for all $y \in (0, y^*)$, $p'(y) = 0$ for all $y > y^*$. The individual surpluses, $u(y) - p(y)$ and $p(y) - y$ are non-decreasing in y.

The first-order condition for the choice of asset holdings, assuming interiority, is

$$\rho - r_{s,t} = \rho - \frac{d + \dot{\phi}_{s,t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\phi_{s',t} - \phi_{s,t}\right)}{\phi_{s,t}} = \alpha \left[\frac{u'(y_{s,t})}{p'(y_{s,t})} - 1 \right]. \tag{6}$$

The left side of (6) is the cost of holding assets. In the middle term, r_s has been replaced by its expression given by (3). The right side of (6) is the expected liquidity value of a unit of real asset in pairwise meetings.

The value function of a producer (labelled "f" for firm) solves

$$\rho V^f_{s,t} = \alpha \left[p(y_{s,t}) - y_{s,t} \right] + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(V^f_{s',t} - V^f_{s,t} \right) + \dot{V}^f_{s,t}. \tag{7}$$

In this formulation, we used the result that sellers have no incentive to accumulate assets since y_s only depends on the buyer's real balances and holding real balances is costly.[13] The interpretation of (7) is analogous to (4).

By market clearing, $m_{s,t} = \phi_{s,t} M$. We can then rewrite (6) as a system of ODEs in $m_{s,t}$, i.e.:

$$\rho - \frac{\dot{m}_{s,t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(m_{s',t} - m_{s,t}\right) + dM}{m_{s,t}} = \alpha L(m_{s,t}), \tag{8}$$

for all $s \in S$, where $L \equiv u'(y)/p'(y) - 1$ with $p(y) = \min\{p(y^*), m\}$. An equilibrium is a list of time paths, $(m_{s,t})_{s \in S}$, that solves the system of ODEs, (8), and the transversality condition

$$\lim_{t \to +\infty} \mathbb{E}_{s_0} \left[e^{-\rho t} m_{s_t,t} \right] = 0, \tag{9}$$

where the expectation is with respect to s^t conditional on s_0. According to (9), the expected present value of real asset holdings must approach zero as time goes to infinity.[14]

Pareto-efficient. The strategic foundations for the Kalai and gradual Nash solutions in economies with liquidity constraints are provided by Hu and Rocheteau (2020) and Rocheteau et al. (2021). The game is composed of a sequence of take-it-or-leave-it-offers without discounting between rounds, and hence it can be interpreted as taking place at one point in time. Finally, Coles and Wright (1998) characterize the non-stationary equilibria of a monetary economy with indivisible money where agents negotiate the output in pairwise meetings according to a Rubinstein alternating-offer game with an explicit time dimension. They show that it is not equivalent to applying the Nash solution in a static sense.

[13] A formal proof in the context of a discrete-time model with fiat money can be found in Rocheteau and Wright (2005). In the version of the model with Lucas trees, sellers could be indifferent between holding assets or not if the asset supply is abundant. Provided surpluses are nondecreasing with liquid assets, as assumed above, we can still assume without loss of generality that all assets will held by buyers.

[14] We show in Appendix B that the transversality condition is sufficient to solve the buyer's problem recursively as shown in the Bellman equation (4). Choi and Rocheteau (2021b) show it is also necessary for optimality. Wilson (1979) and Lagos (2010b) derive a related transversality condition for discrete-time monetary economies. Lagos et al. (2011) derive a similar condition for a continous-time model of an over-the-counter market.

The following proposition studies the existence of deterministic and sunspot equilibria. Deterministic equilibria are such that allocations and prices do not depend on the realization of the sunspot state. Formally, $m_{s,t} = m_t$ for all $s \in S$. Among those equilibria, we can distinguish stationary from nonstationary equilibria. In contrast, in a proper sunspot equilibrium, $m_{s,t}$ varies with the realization of the sunspot state. In the following, we assume that s_t obeys a positive recurrent Markov chain.[15]

Proposition 1 ***(Assets with positive intrinsic value and the absence of excess volatility.)*** *Suppose $d > 0$ and the continuous-time Markov chain for the sunspot state is positive recurrent.*

1. *Deterministic equilibria. The only deterministic equilibrium is the steady state where $m_t = m^*$ for all t where m^* is the unique solution to*
$$\rho - \frac{dM}{m^*} = \alpha L(m^*). \tag{10}$$
The asset pays a positive liquidity premium, $L(m^*) > 0$, if $dM/\rho < p(y^*)$.

2. *Sunspot equilibria. There is no proper stationary sunspot equilibrium.*

Even though the Lucas tree can be valued for its liquidity services and can pay a liquidity premium, this liquidity role does not generates asset price volatility. This result contrasts with models in discrete time, e.g., Lagos and Wright (2003), Rocheteau and Wright (2013), and Altermatt, Iwasaki, and Wright (2022), where the presence of liquid Lucas trees generate deterministic cycles and sunspot equilibria. In order to explain it, we now go through the proof of the proposition.

Consider first the existence of nonstationary deterministic equilibria. A steady-state equilibrium is a deterministic equilibrium where real asset holdings are constant through time, i.e., $m_{s,t} = m^*$ for all $s \in S$ and all $t \in \mathbb{R}_+$. From (8), m^* is the solution to (10). The left side is increasing in m^* from $-\infty$ to ρ while the right side is decreasing from $\alpha L(0)$ to 0. Hence, there is a unique $m^* \geq dM/\rho$ solution to (10).[16] Next, a nonstationary deterministic equilibrium is a m_t solution to
$$\frac{\dot{m}_t}{m_t} = \rho - \alpha L(m_t) - \frac{dM}{m_t}. \tag{11}$$
The right side is increasing in m and it is equal to 0 when $m = m^*$. It is represented by a red curve in Figure 1. The dashed orange curve corresponds to the right side without the liquidity premium, $\rho - dM/m$. It intersects the horizontal axis at the fundamental price of the asset. If $m_0 < m^*$, then m decreases over time and violates the constraint that the asset must be priced above its fundamental value, $m \geq dM/\rho$. If $m_0 > m^*$ then m grows asymptotically at rate ρ and violates the transversality condition (9). Hence, there is a unique deterministic equilibrium and it corresponds to the steady state.

[15] A Markov chain for which all states communicate, i.e., from each state all other states can be reached, is called *irreducible*. A state is *recurrent* if starting from that state the event that consists in returning to it in finite time occurs with probability one. A state is *positive recurrent* if the expected time to return to it is finite. An irreducible Markov chain with a finite state space, as we are considering here, is always recurrent, i.e., all states are recurrent.

[16] This equilibrium can be interpreted as the continuous-time version of the steady-state equilibrium of the Lagos-Wright model (2005) with Lucas trees as in Geromichalos et al. (2007).

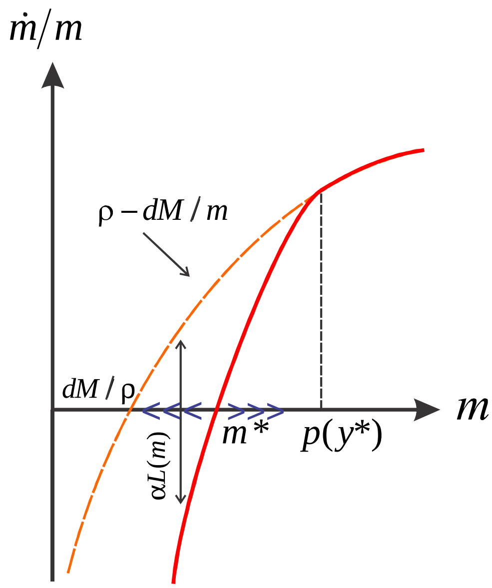

Figure 1: Deterministic equilibria of the model with liquid Lucas trees

Consider next stationary sunspot equilibrium defined as a list of positive real numbers, (m_s), that solves

$$\rho = \overbrace{\frac{\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} (m_{s'} - m_s) + dM}{m_s}}^{\text{pecuniary return}} + \overbrace{\alpha L(m_s)}^{\text{liquidity premium}} . \tag{12}$$

In order to establish that there is no proper sunspot equilibrium where $m_s \neq m_{s'}$ in two distinct states, we denote $\bar{s}$ the state where m_s is maximum and $\underline{s}$ the state where m_s is minimum. Then,

$$r_{\bar{s}} + \alpha L(m_{\bar{s}}) = r_{\underline{s}} + \alpha L(m_{\underline{s}}) = \rho. \tag{13}$$

where

$$r_s = \frac{\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} (m_{s'} - m_s) + dM}{m_s}.$$

In all states, the sum of the expected rate of return of the asset and its liquidity premium must be equal to the agent's rate of time preference, which is independent of s. In a proper sunspot equilibrium, $m_{\bar{s}} > m_{\underline{s}}$. The expected rate of return is lower in state $\bar{s}$ than in state $\underline{s}$ since the asset is more expensive and agents expect capital losses, i.e., $r_{\bar{s}} < r_{\underline{s}}$. Moreover, the liquidity premium, which is decreasing in liquid wealth, is the lowest in state $\bar{s}$, i.e., $L(m_{\bar{s}}) \leq L(m_{\underline{s}})$. Hence, it follows immediately that (13) can only hold if $m_{\bar{s}} = m_{\underline{s}}$.

Comparison to the discrete-time model The impossibility of sunspot equilibria in Proposition 1 is in contrast to the results in the discrete-time models of Lagos and Wright (2003) and Rocheteau and Wright (2013). In order to see where the difference is coming from, consider a discrete-time Lagos-Wright model with Lucas trees and extrinsic uncertainty. The sunspot, $s \in S$, is observed by all agents at the beginning of the centralized market (CM) before buyers make their choice of asset holdings. The length of a period of time is Δ, where Δ is small, and the discount factor is $\beta = (1+\rho\Delta)^{-1}$. Denote

$$\bar{\phi}_s = d\Delta + \phi_s + \sum_{s' \in S/\{s\}} \lambda_{ss'}\Delta(\phi_{s'} - \phi_s), \tag{14}$$

the expected cum dividend price of the Lucas tree in the next CM conditional on the current state s. It is the relevant price of the asset in pairwise meetings. Note that the dividend and transition probabilities are proportional to the length of a period of time. From (14), the expected cum dividend price of the asset is equal to the sum of the dividend, the price of the asset conditional on the state not changing, and the capital gains if the state changes. The first-order condition of the buyer's problem, together with the market-clearing condition, gives

$$\phi_s = \beta\bar{\phi}_s \left\{ 1 + \alpha\Delta \left\{ \frac{u'\left[y(\bar{\phi}_s M)\right]}{p'\left[y(\bar{\phi}_s M)\right]} - 1 \right\} \right\} \qquad \forall s \in S. \tag{15}$$

The value of the asset is equal to its expected discounted value in the next CM multiplied by a liquidity premium factor. A stationary sunspot equilibrium is a list, $(\phi_s; s \in S)$, that satisfies (15) for all s. The possibility of sunspot equilibria arises from the fact that an increase in $\bar{\phi}_s$ has two opposite effects on the right side of (15). First, an increase in $\bar{\phi}_s$ corresponds to an increase in the discounted expected future price of the asset, which tends to increase the current price. Second, an increase in $\bar{\phi}_s$ corresponds to an increase in the buyer's liquid wealth, $\bar{\phi}_s M$, which increases y and reduces the liquidity premium.

The equilibrium condition (15) converges to the equilibrium condition of the continuous-time model as Δ goes to 0. To see it, rewrite (15) as

$$\rho - \frac{d + \sum_{s' \in S/\{s\}} \lambda_{ss'}(\phi_{s'} - \phi_s)}{\phi_s} = \frac{\bar{\phi}_s}{\phi_s}\alpha \left\{ \frac{u'\left[y(\bar{\phi}_s M)\right]}{p'\left[y(\bar{\phi}_s M)\right]} - 1 \right\}. \tag{16}$$

From (14), as Δ goes to 0, $\bar{\phi}_s \to \phi_s$, i.e., $\bar{\phi}_s/\phi_s \to 1$. This suggests that the non-existence of sunspot equilibria in continuous is also true in discrete time provided that Δ is sufficiently small.

In order to illustrate this point, we plot (15) for different values of Δ. We consider an example with two sunspot states, $S = \{\ell, h\}$, and the following transition rates $\lambda_{\ell h} = 1$, and $\lambda_{h\ell} = 1.5$. Moreover, $u(y) = [(y+b)^{1-a} - b^{1-a}]/(1-a)$, $p(y) = \theta y + (1-\theta)u(y)$, with $a = 1.5$, $b = 0.005$, $\rho = 0.1$, $\alpha = 1$, $\theta = 0.5$, $d = 0.01$, and $M = 1$. In the top left panel of Figure 2, we set $\Delta = 1$. The red and the blue curves represent the relationship between m_ℓ and m_h obtained by combining (14) and (15). The two curves are backward-bending, and intersect twice outside of the 45° line. These intersections correspond to proper sunspot equilibria. In the top right panel, we reduce Δ to 0.7 and show that the two proper sunspot equilibria still exist. As we reduce Δ to 0.3 and 0.1 in the bottom panels, the proper sunspot equilibria disappear.

4 When sunspots do matter: Assets with nonpositive fundamental value

We now study the relationship between liquidity and volatility for assets with a non-positive fundamental value, $d \leq 0$. We start with fiat money, $d = 0$, and study the case $d < 0$ separately. The money supply is

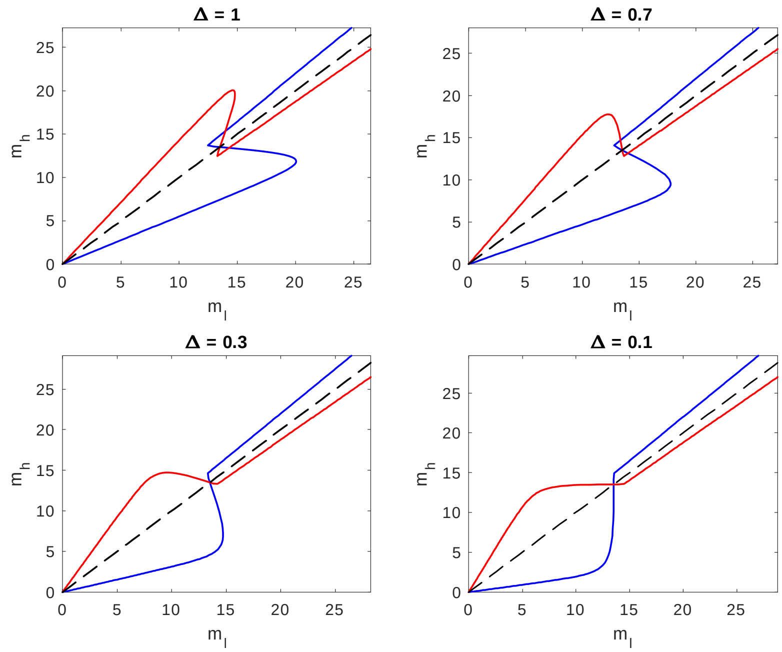

Figure 2: Nonexistence of proper sunspot equilibria when Δ is sufficiently small

growing at rate π. From market clearing, $m_s = M\phi_s$. Hence, $\dot{m}_s/m_s = \pi + \dot{\phi}_s/\phi_s$ and we can rewrite the first-order condition (6) as:

$$\rho + \pi - \frac{\dot{m}_s}{m_s} - \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\frac{m_{s'} - m_s}{m_s} \right) = \alpha L(m_s), \quad \forall s \in S. \tag{17}$$

An equilibrium is a list of time-paths, $(m_{s,t})$, solution to (17) that satisfy the transversality condition (9).

4.1 Deterministic equilibria

Our presentation of deterministic equilibria is analogous to the one in Choi and Rocheteau (2021b). We distinguish stationary from nonstationary equilibria. From (17) a steady-state equilibrium is a m^* solution to

$$\rho + \pi = \alpha L(m^*), \tag{18}$$

Using that L is decreasing, provided that $i < \alpha L(0)$, there exists a unique steady-state monetary equilibrium. The determination of m^* is illustrated graphically in the left panel of Figure 3.

Let's now consider nonstationary equilibria. A deterministic equilibrium is a time-path, m_t, solution to (17), i.e.,

$$\frac{\dot{m}}{m} = \rho + \pi - \alpha L(m). \tag{19}$$

The right side is increasing in m and is equal to 0 when $m = m^*$. Hence, as shown in the right panel of Figure 3, there is a continuum of equilibria, indexed by $m_0 \in (0, m^*)$, such that m_t decreases over time

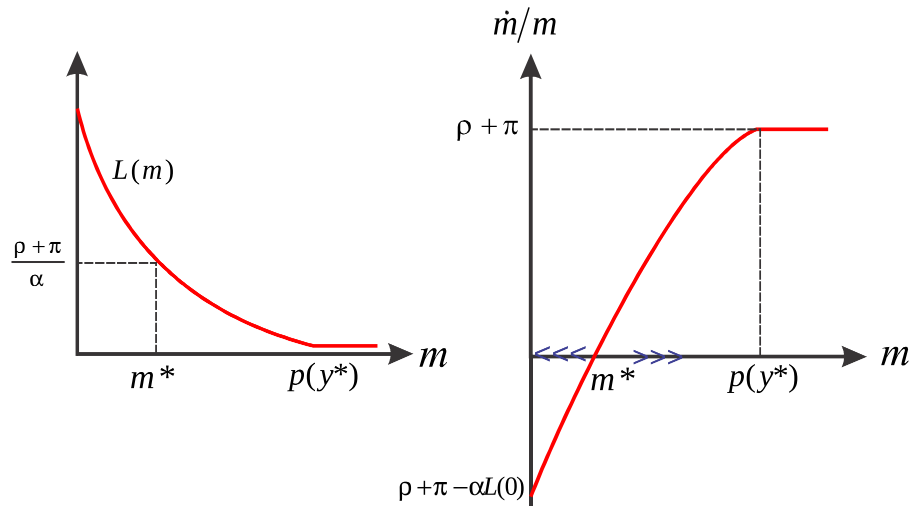

Figure 3: Deterministic equilibria. Left panel: steady state. Right panel: nonstationary equilibria.

toward 0. If $\pi < 0$, then there are a continuum of equilibria where $m_0 > m^*$ and m_t keeps growing over time. The transversality condition, (9), holds because $e^{-\rho t} m_t \leq e^{-\rho t} m_0 e^{(\rho+\pi)t} = m_0 e^{\pi t}$, which tends to zero when $\pi < 0$.[17] In contrast to a monetary economy in discrete time, there are no equilibria where the value of money evolves in a nonmonotone fashion, e.g., there are no periodic cycles or chaotic dynamics.

4.2 Stationary sunspot equilibria

We turn to stationary sunspot equilibria where the aggregate real balances in a given sunspot state s, $\phi_{s,t} M_t$, are constant. Hence, $\dot{\phi}_s / \phi_s = -\pi$. From (3), the expected rate of return of money is

$$r_s = r_{s,t} = \frac{\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\phi_{s',t} - \phi_{s,t}\right)}{\phi_{s,t}} - \pi. \tag{20}$$

The rate of return of money is equal to the capital gains or losses triggered by the changes in the sunspot state net of the growth rate of the money supply. We assume that the continuous-time Markov chain describing the evolution of sunspot states is irreducible, positive recurrent. In particular, there is no absorbing state, i.e., $\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} > 0$ for all $s \in S$. Hence, a stationary equilibrium can only be monetary if $m_s > 0$ for all $s \in S$. From (17) a stationary sunspot equilibrium is a list of positive real numbers, $(m_s)_{s \in S}$, solution to

$$\rho + \pi = \alpha L(m_s) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\frac{m_{s'} - m_s}{m_s}\right). \tag{21}$$

Proposition 2 ***(Nonexistence of stationary sunspot equilibria.)*** *Suppose the Markov chain describing the evolution of the sunspot states is irreducible, positive recurrent. There are no proper stationary sunspot monetary equilibria. The unique stationary monetary equilibrium is the steady state, $m_s = m^*$ for all $s \in S$.*

[17] These equilibria are the analog to the equilibria in the discrete-time model of Lagos (2010b). He shows that these equilibria implement the first best provided that $m_0 \geq p(y^*)$.

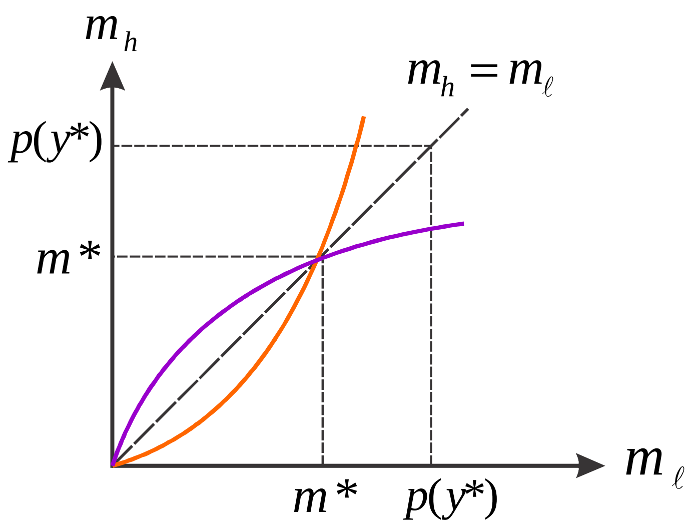

Figure 4: Nonexistence of stationary proper sunspot equilibria

In contrast to monetary economies in discrete time that can feature stationary sunspot equilibria (e.g., Lagos and Wright, 2003), such equilibria do not exist in continuous time. The logic is intuitive. Consider two-state sunspot equilibria, $S = \{\ell, h\}$ with the convention $m_h \geq m_\ell$. A proper sunspot equilibrium is a pair, (m_ℓ, m_h), solution to

$$\rho + \pi - \alpha L(m_\ell) = \lambda_{\ell h}\left(\frac{m_h - m_\ell}{m_\ell}\right) \tag{22}$$

$$\rho + \pi - \alpha L(m_h) = \lambda_{h\ell}\left(\frac{m_\ell - m_h}{m_h}\right). \tag{23}$$

In a proper sunspot equilibrium, the right sides of (22) and (23) are of opposite sign. It means that real balances fluctuate around their value at the steady-state equilibrium (which corresponds to the solution to either (22) or (23) when the right side is 0). Suppose $m_\ell < m^* < m_h$ and consider state h. Agents anticipate that the value of money will shrink when the new sunspot state is realized. The anticipation of a capital loss raises the cost of holding money above its value at a steady-state equilibrium, which is inconsistent with $m_h > m^*$. The same logic holds to rule out $m_\ell < m^*$.

Graphically, in Figure 4, the orange curve representing (22) is below the 45$^{\text{o}}$ line for all $m_\ell < m^*$ and above it for all $m_\ell > m^*$. Indeed, if $m_\ell < m^*$ then the left side of (22) is negative which implies $m_h < m_\ell$. The opposite is true for the purple curve representing (23). As a result, the only intersections between the two curves are located on the 45$^{\text{o}}$ line. In the proof of Proposition 2, we generalize this logic to show that there are no sunspot equilibrium irrespective of the number of states.

4.3 Sunspot equilibria with absorbing states

We now construct sunspot equilibria when (s_t) has one or multiple absorbing states. It means that there exists a $s \in S$ such that $\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} = 0$. Once state s has been reached, there is no transition to other states.

In the following, we construct equilibria where an absorbing state, s_0, triggers coordination to the non-monetary equilibrium. Until state s_0 is reached, real balances are constant over time conditional on the state. To illustrate such equilibria, consider three states, $S = \{0, \ell, h\}$. The transitions between states h and ℓ are $\lambda_{\ell h}$ and $\lambda_{h\ell}$. State h does not transition directly into state 0, $\lambda_{h0} = 0$, while state ℓ transitions into state 0 at rate $\lambda_{\ell 0} > 0$. These transitions are represented in the top left graph in Figure 5. A sunspot equilibrium is a pair, (m_ℓ, m_h), solution to

$$\rho + \pi - \alpha L(m_\ell) = \lambda_{\ell h}\left(\frac{m_h - m_\ell}{m_\ell}\right) - \lambda_{\ell 0} \tag{24}$$

$$\rho + \pi - \alpha L(m_h) = \lambda_{h\ell}\left(\frac{m_\ell - m_h}{m_h}\right). \tag{25}$$

Proposition 3 ***(Sunspot equilibrium with absorbing state.)*** *Suppose $S = \{0, \ell, h\}$ and $\lambda_{0\ell} = \lambda_{0h} = \lambda_{h0} = 0$. For all $0 < m_\ell < m_h < m^*$ and $\lambda_{\ell h} > 0$, there exists $(\lambda_{h\ell}, \lambda_{\ell 0}) \in \mathbb{R}_{2+}$ such that $(0, m_\ell, m_h)$ is a sunspot equilibrium.*

Proposition 3 shows that there exists sunspot monetary equilibria where the value of money alternates in a stochastic fashion between high and low values as illustrated in the left panel of Figure 5. This equilibrium requires that money loses all its value in the absorbing state. Indeed, the value of money can only fluctuate if agents believe that there is a positive probability that it becomes valueless in the future. We think this result can help interpret the volatility of the price of crypto-currencies: this volatility reveals that agents' beliefs do not rule out that cryptocurrencies will burst. If agents did not believe in such possibility, then the value of real balances would be constant (provided fundamentals do not change).

Suppose now that there are two absorbing states, $s = 0$ and $s = *$. In state 0, the economy settles down in the nonmonetary steady state while in state $*$ the economy reaches the steady-state monetary equilibrium with $m = m^*$. In addition, we impose $\lambda_{\ell *} = 0$ and $\lambda_{h0} = 0$. These transitions are represented in the top right graph in Figure 5. A sunspot equilibrium is a pair, (m_ℓ, m_h), solution to

$$\rho + \pi - \alpha L(m_\ell) = \lambda_{\ell h}\left(\frac{m_h - m_\ell}{m_\ell}\right) - \lambda_{\ell 0} \tag{26}$$

$$\rho + \pi - \alpha L(m_h) = \lambda_{h\ell}\left(\frac{m_\ell - m_h}{m_h}\right) + \lambda_{h*}\left(\frac{m^* - m_h}{m_h}\right). \tag{27}$$

Proposition 4 ***(Sunspot equilibrium with two absorbing states.)*** *Suppose $S = \{0, \ell, h, *\}$ and $\lambda_{0s} = 0$ for all $s \neq 0$, $\lambda_{*s} = 0$ for all $s \neq *$, $\lambda_{\ell *} = 0$ and $\lambda_{h0} = 0$. For all $0 < m_\ell < m_h < m^*$, $\lambda_{h*} > 0$ and $\lambda_{\ell h} > 0$, there exists $(\lambda_{h\ell}, \lambda_{\ell 0}) \in \mathbb{R}_{2+}$ such that $(0, m_\ell, m_h, m^*)$ is a sunspot equilibrium.*

This equilibrium describes a situation where agents are uncertain whether the currency will establish itself permanently or will collapse. These two possibilities are represented by the two absorbing states. As long as the economy has not reached any of these two states, the value of money fluctuates in a stochastic fashion.

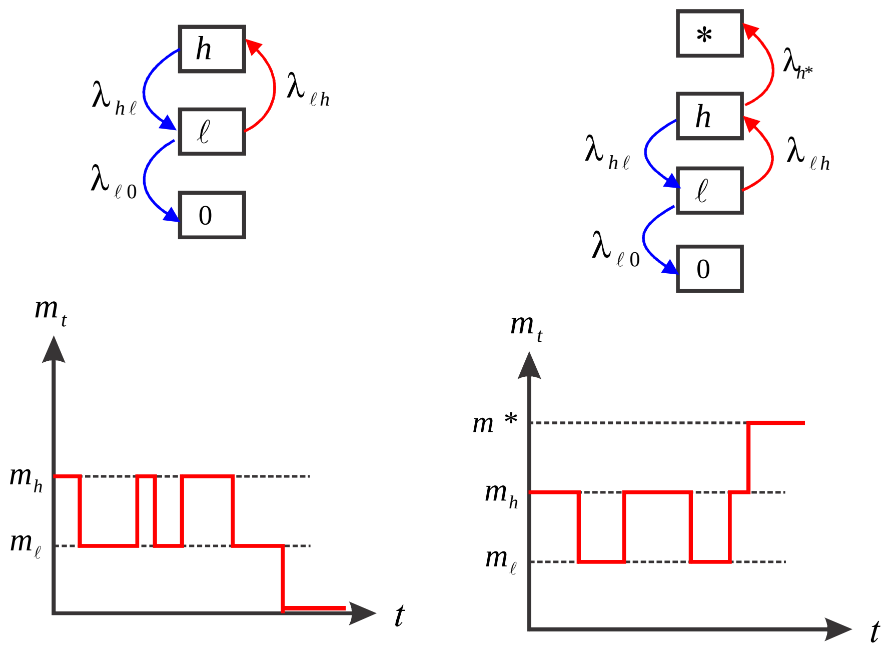

Figure 5: Sunspot equilibria with absorbing states

4.4 Nonstationary sunspot equilibria

We now allow for the non-stationarity of real balances given a sunspot state in order to capture new phenomena such as temporary and recurrent belief-driven (hyper-)inflation or bubbly asset price trajectories.

Transitory belief-driven inflation We describe a sunspot equilibrium where in the initial state real balances decrease over time due to a temporary inflationary episode fueled by self-fulfilling beliefs. The episode ends with either real balances reaching a high steady-state value or real balances depreciating fully. Suppose the state space is $S = \{0, i, *\}$ with $\lambda_{0s'} = 0$ for all $s' \in \{i, *\}$ and $\lambda_{*s'} = 0$ for all $s' \in \{0, i\}$. State 0 corresponds to the nonmonetary steady state while state $*$ corresponds to the monetary steady state. Both states are absorbing. The initial state, $s_0 = i$, is a transient state. From (17), $m_{i,t}$ obeys the following ODE:

$$\frac{\dot{m}_i}{m_i} = \rho + \pi + \lambda_{i0} + \lambda_{i*} - \lambda_{i*}\left(\frac{m^*}{m_i}\right) - \alpha L(m_i). \tag{28}$$

The right side of (28) increases from $-\infty$ to $\rho + \pi + \lambda_{i0} + \lambda_{i*}$ as m_i covers $\mathbb{R}_+$. Hence, there is a unique stationary solution to (28), $m_i^* \in (0, m^*)$. There are a continuum of equilibria, indexed by $m_{i,0} \in (0, m_i^*)$, where $m_{i,t}$ decreases over time. Hence, real balances decrease over time until the sunspot state transitions to either 0, in which case money loses all its value, or $*$ in which case the value of money jumps to its steady state. These equilibria are represented in the top left panel of Figure 6.

Recurrent inflation In Proposition 2, we saw that in order to obtain a stationary sunspot equilibrium, there must exist an absorbing state (or absorbing subset of S) where money becomes valueless. We now

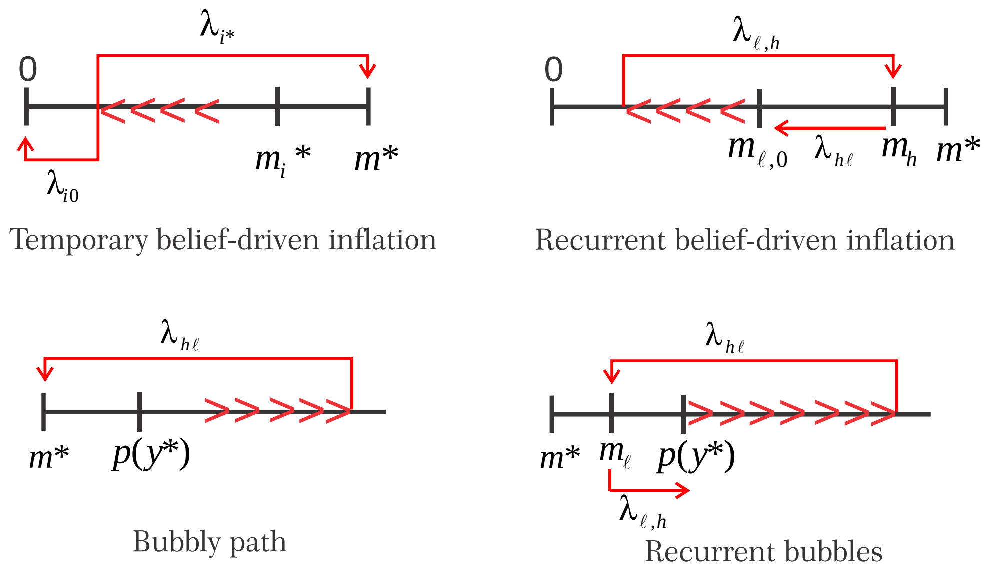

Figure 6: Examples of nonstationary sunspot equilibria

construct a non-stationary sunspot equilibrium where the value of real balances fluctuates over time but remains positive in all states and all dates. We think of such an equilibrium as hyper-inflationary episodes. In-between these episodes, aggregate real balances are stable. Suppose the state space is $S = \{\ell, h\}$ where we interpret ℓ as the inflationary state and h as a steady state with constant real balances. In state h,

$$\rho + \pi - \lambda_{h\ell}\left(\frac{m_{\ell,0} - m_h}{m_h}\right) = \alpha L(m_h). \tag{29}$$

When state ℓ occurs, real balances fall instantly to a lower level $m_{\ell,0} < m^*$. The unique solution to (29) is $m_h \in (m_{\ell,0}, m^*)$. In state ℓ,

$$\rho + \pi - \frac{\dot{m}_\ell}{m_\ell} - \lambda_{\ell h}\left(\frac{m_h - m_\ell}{m_\ell}\right) = \alpha L(m_\ell). \tag{30}$$

Given m_h, there is a unique stationary solution, $m_\ell^* \in (m_h, m^*)$. It follows that $m_{\ell,0} < m_\ell^*$ so that $m_{\ell,t}$ decreases over time. We illustrate the dynamics in the top right panel of Figure 6.

Bubbly path We now describe equilibria featuring bubble-like trajectories where the price of money keeps increasing until it bursts and ends up at a positive steady-state value. Suppose $S = \{\ell, h\}$ with $\lambda_{h\ell} > 0$ and $\lambda_{\ell h} = 0$. So state ℓ is absorbing. The initial state is $s_0 = h$. We construct an equilibrium where $m_{h,t} > p(y^*)$ and $m_\ell = m^*$. From (17),

$$\frac{\dot{m}_h}{m_h} = \rho + \pi + \lambda_{h\ell}\left(\frac{m_h - m^*}{m_h}\right), \tag{31}$$

where we used $L(m_h) = 0$. Given in state h real balances grow unbounded, we need to check the transversality condition (9) holds. Note first $0 < \dot{m}_h/m_h \leq \rho + \pi + \lambda_{h\ell}$. Hence,

$$\mathbb{E}\left[e^{-\rho t} m_t\right] = e^{-(\rho+\lambda_{h\ell})t} m_{h,t} + e^{-\rho t}(1 - e^{-\lambda_{h\ell} t}) m^* \leq e^{\pi t} m_{h,0} + e^{-\rho t}(1 - e^{-\lambda_{h\ell} t}) m^*.$$

Real balances at time t are equal to $m_{h,t}$ if the transition to state ℓ did not occur, with probability $e^{-\lambda_{h\ell}t}$, and they equal m^* otherwise. So, provided $\pi < 0$, the transversality condition (9) holds. In state h, liquidity needs are satiated as real balances grow at a rate that compensates for time preference (ρ), money growth ($\pi < 0$), and the capital loss when the bubble bursts ($\lambda_{h\ell}\left(m_h - m^*\right)/m_h$). In that case, the nominal interest rate on an illiquid nominal bond is zero. When state ℓ occurs, the economy jumps to the steady state. The trajectories are represented graphically in the bottom left panel of Figure 6.

Recurrent bubbles Suppose $S = \{\ell, h\}$ with $\lambda_{h\ell} > 0$ and $\lambda_{\ell h} > 0$. The equilibrium is stationary in state ℓ and bubbly in state h. More precisely, in state ℓ, m_ℓ solves

$$\rho + \pi - \lambda_{\ell h}\left[\frac{m_h^0 - m_\ell}{m_\ell}\right] = \alpha L(m_\ell), \tag{32}$$

where $m_h^0 \geq p(y^*)$. So when state h occurs, real balances jump to m_h^0 where liquidity needs are satiated. (In Figure 6, we assume $m_h^0 = p(y^*)$.) In state h real balances keep growing according to

$$\frac{\dot{m}_h}{m_h} = \rho + \pi + \lambda_{h\ell}\left(\frac{m_h - m_\ell}{m_h}\right). \tag{33}$$

As before, provided $\pi < 0$, the transversality condition (9) holds. So the economy alternates between a state where liquidity constraints bind and aggregate real balances are constant over time and a state where liquidity constraints are slack and real balances grow over time. If we price an illiquid bond, the nominal interest rate is positive in state ℓ and equal to 0 in state h. As the economy transitions from h to ℓ, aggregate real balances and output fall and the nominal interest rate jumps instantly to a positive level. The trajectories are represented in the bottom right panel of Figure 6.

4.5 Assets with negative intrinsic value

Suppose now $d < 0$, the asset imposes a cost on the owner, e.g., a commodity with a storage cost, a vacant property, or a firm that incurs losses.[18] In Appendix C, we provide an alternative but equivalent interpretation where the asset is the stock of a firm that pays no dividend. The losses incurred by the firm, $-M_0 d$, are financed by issuing additional stocks.[19]

Irrespective of the interpretation, the buyer's value function solves the same HJB equation as before, (4). The key difference is that the asset is only acceptable as means of payment in pairwise meetings if its price is positive. Hence, $p(y_s) = \min\{p(y^*), \mathbb{I}_{\{\phi_s>0\}}m_s\}$ where $\mathbb{I}_{\{\phi_s>0\}}$ is an indicator function equal to one when the asset price is positive and zero otherwise. In addition, while the asset can be traded at no cost, we assume there is a utility cost, $\zeta > 0$, to dispose of it. An equilibrium is a list of time-paths, $(m_{s,t})_{s\in S}$, solution to

[18] The idea that monies can have a storage cost dates back to Kiyotaki and Wright (1989) and Ayagari and Wallace (1991). In the context of discrete-time models with divisible money, the idea as been studied in Lagos and Wright (2003), Nosal and Rocheteau (2011, Chapter 5), and Gu et al. (2021) where such assets are called toxic.

[19] This second interpretation is also formalized by Farmer and Woodford (1997) in an OLG model where a flow of real government expenditure, $g > 0$, is financed with money creation. In that case, there are two monetary steady states, as shown in the bottom panel of their Figure 1.

(8) where $L(m_s) \equiv \mathbb{I}_{\{m_s>0\}}\left[u'(y_s)/p'(y_s) - 1\right]$. We assume that primitives are such that $\lim_{m\to 0} L(m)m = 0$ and $L(m)m$ is unimodal.

A steady state is a m^* solution to

$$\rho m^* - dM = \alpha L(m^*)m^*. \tag{34}$$

As shown in the left panel of Figure 7, the left side is linear in m^* with a positive intercept while the right side is hump-shaped and is equal to 0 at both $m^* = 0$ and $m^* \geq p(y^*)$. Hence, provided d is not too negative, there are two steady-state monetary equilibria. In Figure 7, the two steady states are denoted m_ℓ^* and m_h^*. There is also a nonmonetary equilibrium in which the price of the asset is $\phi = d/\rho < 0$ if it is costly to dispose of the asset. Agents hold onto the asset if $\zeta > -d/\rho$ and dispose of it otherwise.

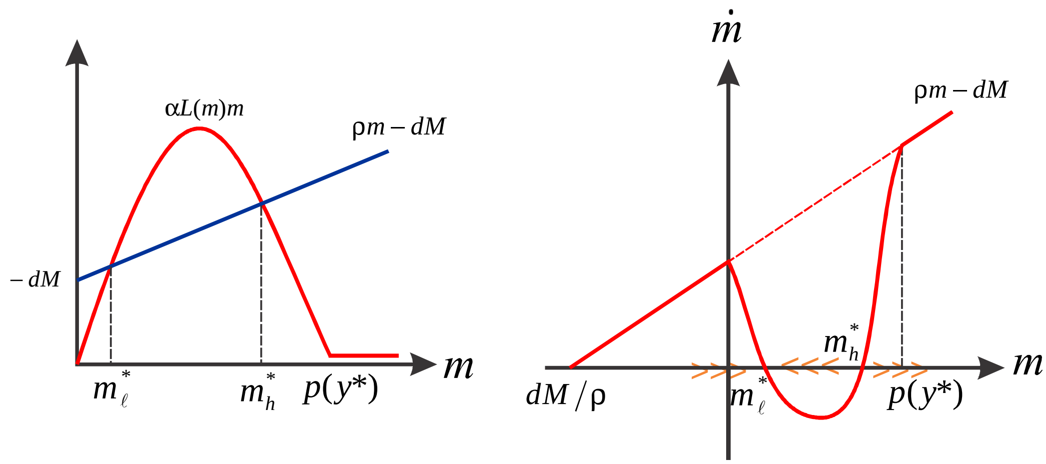

Figure 7: Deterministic equilibria with toxic assets ($d < 0$)

A deterministic, nonstationary equilibrium is a m_t solution to

$$\dot{m} = \rho m - \alpha L(m)m - dM. \tag{35}$$

The right side has a positive intercept and is located below the upward-sloping line $\rho m - dM$ for all $m \in (0, p(y^*))$. As shown in the right panel of Figure 7, provided that d is not too negative, it intersects twice with the horizontal axis at m_ℓ^* and m_h^*. There are a continuum of equilibria indexed by $m_0 \in (dM/\rho, m_h^*)$ that converge to m_ℓ^*. (Trajectories with $m_0 < 0$ can also be sustained if there is a cost to dispose of the asset.) The unique equilibrium leading to m_h^* is the steady-state one, $m_0 = m_h^*$.

Consider two-state stationary sunspot equilibria with $0 < m_\ell < m_h$. The pair, (m_ℓ, m_h), solves

$$\rho m_\ell - dM - \alpha L(m_\ell)m_\ell = \lambda_{\ell h}(m_h - m_\ell) \tag{36}$$

$$\rho m_h - dM - \alpha L(m_h)m_h = \lambda_{h\ell}(m_\ell - m_h). \tag{37}$$

Suppose d is not too negative so that two steady states, $0 < m_\ell^* < m_h^*$, exist. For all (m_ℓ, m_h) such that $0 < m_\ell < m_\ell^* < m_h < m_h^*$ one can find $(\lambda_{\ell h}, \lambda_{h\ell})$ such that (36)-(37) hold. Graphically in Figure 8, the curves representing (36) and (37) are U-shaped with a positive intercept when $d < 0$. Hence, they can intersect outside of the 45^o line in the positive quadrant. These intersections correspond to proper sunspot equilibria. If the asset is costly to dispose of, there are also sunspot equilibria where $m_\ell < 0$, in which case the asset is not traded and $L(m_\ell) = 0$. Such equilibria are the continuous-time analogues of the recurrent market freezes in Gu, Menzio, Wright, and Zhu (2021).

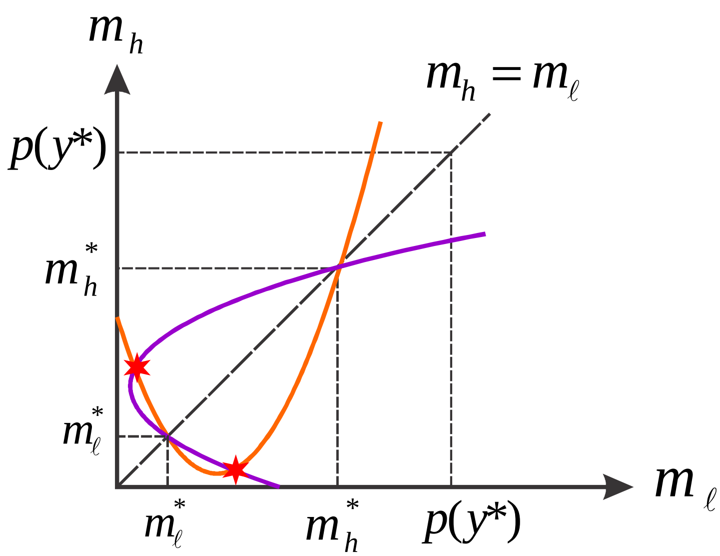

Figure 8: Sunspot equilibria with toxic assets

Proposition 5 ***(Sunspot equilibria when assets have a negative intrinsic value.)*** *Suppose $S = \{\ell, h\}$. The per unit cost to dispose of the asset is $\zeta > 0$. There exists a $\underline{d}< 0$ such that for all $d \in (\underline{d}, 0)$, there exists two steady states, $0 < m_\ell^* < m_h^*$.*

1. *For all (m_ℓ, m_h) such that $0 < m_\ell < m_\ell^* < m_h < m_h^*$, there exists $(\lambda_{\ell h}, \lambda_{h\ell}) \in \mathbb{R}^2_{++}$ such that (m_ℓ, m_h) is a sunspot equilibrium.*

2. *For all (m_ℓ, m_h) such that $\max\{dM/\rho, -\zeta M\} < m_\ell < 0 < m_\ell^* < m_h < m_h^*$, there exists $(\lambda_{\ell h}, \lambda_{h\ell}) \in \mathbb{R}_{2+}$ such that (m_ℓ, m_h) is a sunspot equilibrium. In the low state, no trade takes place, $y_\ell = 0$.*

5 Intrinsic uncertainty, liquidity, and volatility

So far, we assumed the dividend of the asset, d, was constant so that any volatility could be interpreted as excess volatility. We now consider risky Lucas trees where the fundamental value is state dependent, as in Lagos (2010a). In contrast to the sunspot states described earlier, states in $S = \{\ell, h\}$ correspond to changes in fundamentals: in state ℓ the dividend is d_ℓ while in state h the dividend is $d_h > d_\ell$. Uncertainty about the state is no longer extraneous, but states can still be used by agents to coordinate their beliefs in the presence of multiple equilibria.

The fundamental values of the asset in different states, v_s, in an economy without liquidity needs (e.g., $\alpha = 0$), are given by the solutions to the following asset pricing equations:

$$\rho v_s = d_s + \lambda_{ss'} \left(v_{s'} - v_s \right), \quad \text{for } s, s' \in S. \tag{38}$$

If assets pay no liquidity premium, the discount rate on the left side is the rate of time preference. Solving for v_h and v_ℓ in closed form we obtain:

$$v_h = \frac{\left(\rho + \lambda_{\ell h}\right) d_h + \lambda_{h\ell} d_\ell}{\rho \left(\rho + \lambda_{h\ell} + \lambda_{\ell h}\right)}, \quad v_\ell = \frac{\left(\rho + \lambda_{h\ell}\right) d_\ell + \lambda_{\ell h} d_h}{\rho \left(\rho + \lambda_{h\ell} + \lambda_{\ell h}\right)}.$$

In the following, we price the asset in an economy with liquidity needs and we compare its equilibrium volatility to its fundamental volatility. We distinguish two cases depending on whether d_ℓ is positive or negative.

5.1 The case $d_\ell > 0$

By the same logic as in Section 3, there is a unique equilibrium represented by a list, $(m_s)_{s \in S}$, solution to

$$\rho - \frac{\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(m_{s'} - m_s\right) + d_s M}{m_s} = \alpha L(m_s), \text{ for all } s \in S.$$

The asset price is now fundamentally volatile and the liquidity premium is state dependent. We illustrate with a numerical example how the liquidity role of the asset affects its volatility. The utility function is $u(y) = y^{(1-a)}/(1-a)$ with $a = 1/3$ and $\rho = 4\%$, and terms of trade are determined according to proportional bargaining, $p(y) = \theta y + (1-\theta) u(y)$, where $\theta = 0.2$. The frequency of trade is $\alpha = 4$. The total supply of the asset, M, is normalized to 1. The dividends are set to $d_h = 0.04$ and $d_\ell = 0.02$. Finally, we set $\lambda_{h\ell} = 0.2$ and $\lambda_{\ell h} = 0.5$. The fundamental values of the asset (in the absence of liquidity considerations) in the two states are $\mathcal{F} = (0.84, 0.86)$. The unique equilibrium is $\mathcal{E} = (m_\ell, m_h) = (1.33, 1.35)$. The asset pays a positive liquidity premium in each state. We measure the volatility of asset prices by the standard deviation of m when m is distributed between m_ℓ and m_h according to its limiting distribution, $(2/7, 5/7)$. The standard deviation of m at $\mathcal{E}$ and $\mathcal{F}$ are

$$\sigma_{\mathcal{E}} = 0.0083 < \sigma_{\mathcal{F}} = 0.0122.$$

For this example, liquidity reduces the volatility of the asset price.

5.2 The case $d_\ell < 0$

We now assume fundamentals are such that $v_\ell < 0 < v_h$, i.e.,

$$\rho d_\ell + \lambda_{h\ell} d_\ell + \lambda_{\ell h} d_h < 0 < \rho d_h + \lambda_{\ell h} d_h + \lambda_{h\ell} d_\ell.$$

The fundamental value is negative in the low state and positive in the high state. This condition is satisfied, e.g., if $d_\ell + d_h = 0$ and $\lambda_{h\ell} = \lambda_{\ell h}$. We assume that the cost to dispose of the asset is larger than $-v_\ell$.

Consider first equilibria where $\phi_\ell < 0$. In that case the asset cannot serve as medium of exchange in the low state. Market capitalization evolves according to

$$\rho m_\ell - d_\ell M = \lambda_{\ell h}(m_h - m_\ell) \tag{39}$$

$$\rho m_h - d_h M - \alpha L(m_h) m_h = \lambda_{h\ell}(m_\ell - m_h). \tag{40}$$

The difference between (39) and (40) is that the asset pays no liquidity premium in the low state because its price is negative. If $v_h M \geq p(y^*)$ then $L(m_h) = 0$ and the asset is priced at its fundamental value in all states, $(\phi_\ell, \phi_h) = (v_\ell, v_h)$. If $v_h M < p(y^*)$ then the asset exhibits a liquidity premium in the high state, $L(m_h) > 0$, $\phi_\ell > v_\ell$ and $\phi_h > v_h$. The condition $v_\ell < 0$ requires that M is not too low.

Consider next equilibria where $\phi_\ell > 0$. Market capitalization evolves according to

$$\rho m_\ell - d_\ell M - \alpha L(m_\ell) m_\ell = \lambda_{\ell h}(m_h - m_\ell) \tag{41}$$

$$\rho m_h - d_h M - \alpha L(m_h) m_h = \lambda_{h\ell}(m_\ell - m_h). \tag{42}$$

If $m_h \geq p(y^*)$ then $L(m_h) = 0$ but $L(m_\ell) > 0$. Equation (41) is represented in Figure 9 by a red curve while equation (42) is represented by a green curve. The departure of each curve from the dashed lines corresponds to the size of the liquidity premium in each state. In the low state, the asset pays a liquidity premium when $m_\ell > 0$ since otherwise it is not accepted in payment for good y. In our example, there are three equilibria, two where $m_\ell > 0$ and one where $m_\ell < 0$.

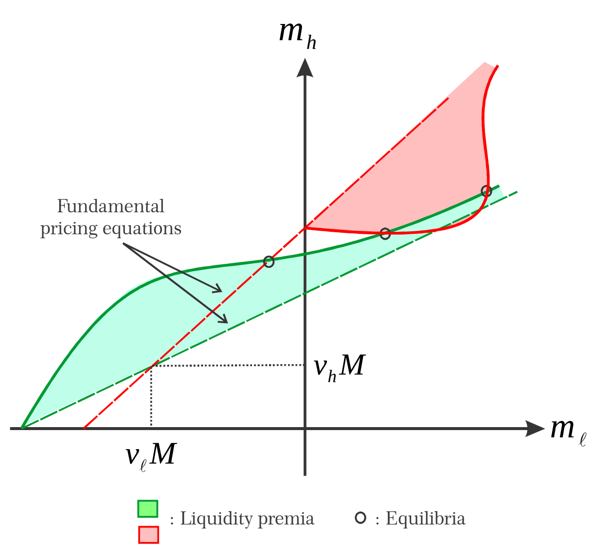

Figure 9: Equilibria with aggregate shocks

We provide a numerical example to illustrate the relation between liquidity and volatility across multiple equilibria. The utility function is $u(y) = y^{1-a}/(1-a)$ with $a = 1/3$ and $\rho = 4\%$. We assume proportional bargaining, $p(y) = \theta y + (1-\theta)u(y)$. We set $\theta = 0.2$ and $\alpha = 4$. The asset supply is $M = 1$ and dividends are $d_h = 0.25$ and $d_\ell = -0.65$ with $\lambda_{h\ell} = 0.2$ and $\lambda_{\ell h} = 0.5$. Given this dividend process, the fundamental market capitalization is

$$\mathcal{F} \equiv (v_\ell M, v_h M) = (-1.05, 0.17).$$

There are three steady-state equilibria, denoted $\mathcal{E}_j = (m_\ell^j, m_h^j)$ for $j \in \{1, 2, 3\}$:

$$\mathcal{E}_1 = (0.33, 1.37), \quad \mathcal{E}_2 = (0.06, 1.28), \quad \mathcal{E}_3 = (-0.06, 1.23).$$

In equilibria $\mathcal{E}_1$ and $\mathcal{E}_2$, the asset has a positive price in all states. In contrast, in equilibrium $\mathcal{E}_3$, the asset price is negative in the low state. Hence, it is only accepted in the high state. Again, we measure the volatility of asset prices by the standard deviation of m when m is distributed between m_ℓ^i and m_h^i according to the limiting distribution, $(2/7, 5/7)$. We find

$$\sigma_{\mathcal{E}_3} = 0.585 > \sigma_{\mathcal{F}} = 0.549 > \sigma_{\mathcal{E}_2} = 0.548 > \sigma_{\mathcal{E}_1} = 0.467.$$

So liquidity raises volatility relative to its fundamental value in equilibrium $\mathcal{E}_3$. In contrast, volatility is lower than the fundamental value in equilibria $\mathcal{E}_1$ and $\mathcal{E}_2$.

6 Endogenous liquidity

In the sections above, assets are perfectly liquid, i.e., they are universally accepted and fully pledgeable (unless the asset price is negative and the asset is not freely disposable as in Section 4.5). This assumption, however, is not innocuous because it prevents asset liquidity from varying with equilibrium outcomes, including prices and rates of return. Hence, we now go deeper into the microfoundations of asset liquidity and explain why some assets are only imperfectly liquid in the presence of informational asymmetries. By formalizing the partial liquidity of assets, we are activating a feed-back loop between the liquidity of the asset and its price.

We distinguish two types of microfoundations for asset liquidity. In Appendix D, we assume sellers incur a cost to participate in the market or, equivalently, to be able to accept the asset in payment, as in Lester et al. (2012) and Rocheteau and Wright (2013). We establish the existence of multiple steady states and sunspot equilibria.[20] The reason for this multiplicity is that acceptability increases with the asset price while the asset price increases with the fraction of sellers accepting it, which creates strategic complementarities between sellers' acceptability choices. In the following, we provide a novel explanation for asset price volatility that does not require the multiplicity of steady states.

We explain asset pledgeability based on informational frictions and moral hazard (hidden actions) as in Li et al. (2012). At the time a match is formed, the buyer can produce fake assets at a fraction, $\kappa \in (0, 1)$, of the cost of acquiring genuine ones. A fake asset is defined as an asset disguised as genuine that generates no dividend. For simplicity, we assume fake assets are destroyed or confiscated instantly after the match is

[20] Gu, Menzio, Wright, and Zhu (2021) use a related model in discrete time to explain the existence of recurrent market freezes. Choi and Rocheteau (2021a) formalize acceptability in the context of a continuous-time model of privately produced monies (e.g., cryptocurrencies) and show the existence of multiple nonstationary equilibria.

dissolved.[21] For example, a fake asset can be a counterfeited note or a fraudulent mortgage loan or asset-backed security.[22] Only buyers have the technology to produce fake assets and, importantly, their actions are not observable to sellers.

The timing of the counterfeiting game, which is illustrated in Figure 10, is as follows. Before pairwise meetings take place, buyers choose their holdings of genuine assets, m. Over a small interval of time of length dt, the buyer enters a match with probability αdt. Once a match is formed, the buyer decides whether or not to produce fake assets. At this point, the buyer's asset holdings are observable, but the seller does not know whether assets are genuine or fake. Then, the buyer makes a take-it-or-leave-it offer, (y, p). If the offer is accepted, then the units of the asset used for the payment p are drawn randomly from the buyer's asset holdings. Once an offer has been accepted or rejected, the buyer and the seller split apart and any fraudulent asset that has been produced is confiscated.

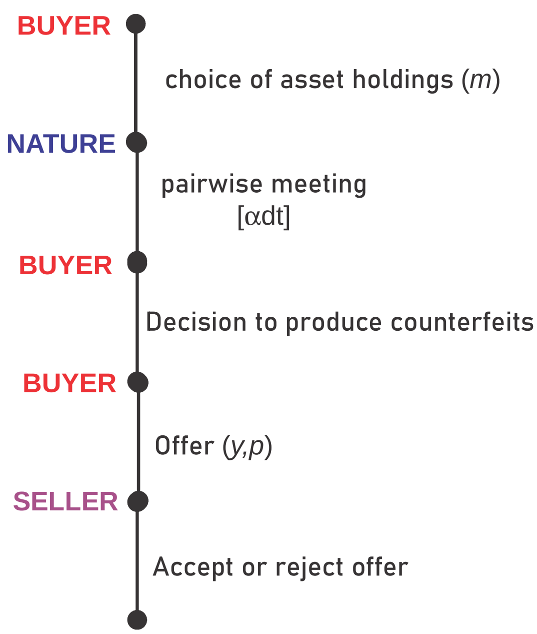

Figure 10: Tree of the counterfeiting game

Since the buyer has all the bargaining power, the payment covers the seller's cost, $p(y) = y$. Moreover, Li et al. (2012) show that fraud is prevented from occurring in equilibrium by an endogenous liquidity constraint.[23] According to this no-fraud constraint, the buyer's offer, y, and his choice of asset holdings, m, must satisfy:

$$-(\rho - r_s)m + \alpha [u(y) - y] \geq \alpha [-\kappa m + u(y)] . \tag{43}$$

The left side is the net surplus of the buyer who invests in genuine assets. It is composed of the flow cost of

[21] The assumption that counterfeits are confiscated is borrowed from Nosal and Wallace (2007). Its purpose is to prevent counterfeits from circulating in equilibrium. See Li and Rocheteau (2011) for a version of the model where this assumption is relaxed.

[22] In their Appendix B1, Li et al. (2012) consider a variant of their model where agents can issue asset backed securities.

[23] Kang (2017) studies a version of the model with costly verification where fraud can occur in equilibrium.

holding assets and the expected surplus of a match. The right side is the expected surplus of a buyer who mimics the offer of an honest buyer but produces fake assets instead at cost κm. So (43) guarantees it is not profitable for the buyer to offer fake assets. The no-fraud constraint can be reexpressed as $y \leq \chi m$ where χ is an endogenous pledgeability coefficient equal to

$$\chi_s = \left(\kappa - \frac{\rho - r_s}{\alpha} \right)^+, \tag{44}$$

where $[x]^+ = \max\{x, 0\}$. By overcollaterizing the payment (which we interpret as $m > p$), the buyer is able to signal the quality of her assets. Indeed, it is more costly to overcollateralize the payment with fraudulent assets than genuine ones provided that $\alpha\kappa > \rho - r_s$. A key feature of the pledgeability coefficient in (44) is that it depends positively on the rate of return of the asset. As r_s increases, the asset becomes less costly to hold and, from (43), buyers have less incentives to counterfeit it. As a result, for given y, the no-fraud constraint can be satisfied for a lower m, i.e., pledgeability increases. In the special case where $r_s = \rho$, the pledgeability coefficient reduces to $\chi_s = \kappa$. From (44), $\alpha\kappa$ is an upper bound for the cost of holding the asset, $\rho - r_s$, above which the asset is no longer acceptable. Equivalently, there is a lower bound for the rate of return of the asset, $\underline{r} = \rho - \kappa\alpha$.

The HJB equation for the value function of the buyer is given by (4) where the surplus function, $\upsilon(m)$, incorporates the pledgeability constraint and can now be rewritten as:

$$\upsilon(m) = \max_{y \geq 0} \{ u(y) - y \text{ s.t. } y \leq \chi m \}. \tag{45}$$

If $\chi m < y^*$, then $\upsilon'(m) = \chi [u'(\chi m) - 1]$. If $\chi m > y^*$, then $\upsilon'(m) = 0$. Hence, the FOC for the choice of asset holdings is

$$\rho - \frac{d + \dot{\phi}_{s,t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} (\phi_{s'} - \phi_s)}{\phi_{s,t}} = \alpha \chi_s [u'(\chi_s m_s) - 1]. \tag{46}$$

The liquidity premium on the right side depends on the pledgeability coefficient, χ_s, that is a function of time and the sunspot state. On the one hand, pledgeability, χ_s, increases the liquidity premium by allowing a larger fraction of the asset to serve as means of payment, i.e., for given y_s, $\alpha\chi_s [u'(y_s) - 1]$ increases. On the other hand, it decreases the liquidity premium by raising consumption toward its efficient level, i.e., $u'(y_s)$ decreases. As a result of these two opposite effects, the right side of (46) can be nonmonotone in χ_s.

6.1 The case of fiat money

We start with the case where the asset pays no dividend, $d = 0$, and M is constant. At a steady state, $r = 0$ and $\chi = (\kappa - \rho/\alpha)^+$. A steady state is a m^* solution to

$$u' \left[\left(\kappa - \frac{\rho}{\alpha} \right) m^* \right] = \frac{\alpha\kappa}{\alpha\kappa - \rho}. \tag{47}$$

Provided $\alpha\kappa > \rho$, there is a unique steady-state monetary equilibrium. In nonstationary deterministic equilibria, the rate of return of money is $r = \dot{m}/m$ and, from (44), the pledgeability coefficient is

$$\chi_t = \left(\kappa - \frac{\rho - \dot{m}_t/m_t}{\alpha} \right)^+. \tag{48}$$

An equilibrium is a m_t solution to

$$\left(\kappa - \frac{\rho - \dot{m}_t/m_t}{\alpha}\right) u'\left[\left(\kappa - \frac{\rho - \dot{m}_t/m_t}{\alpha}\right) m_t\right] = \kappa \text{ if } \frac{\dot{m}_t}{m_t} < \rho \tag{49}$$

$$\kappa m_t \geq y^* \text{ if } \frac{\dot{m}_t}{m_t} = \rho. \tag{50}$$

Equation (49) corresponds to the case where the liquidity constraint binds. We can then express m_t as a function of $r_t = \dot{m}_t/m_t$, i.e.,

$$m_t = \Gamma(r_t) \equiv \frac{\alpha u'^{-1}\left(\frac{\alpha\kappa}{\alpha\kappa - \rho + r_t}\right)}{\alpha\kappa - \rho + r_t}. \tag{51}$$

Equation (50) corresponds to the case where the liquidity constraint is slack. The rate of return of money is equal to ρ, which corresponds to the Friedman rule, and pleadgeable real balances on the left side of the inequality are greater than y^*.

Proposition 6 ***(Deterministic equilibria under the threat of fraud.)*** *Suppose $d = \pi = 0$ and $u(y) = \left[(y+b)^{1-a} - b^{1-a}\right]/(1-a)$. Moreover, $\alpha\kappa > \rho$.*

1. *If $a < 1$ and $b = 0$, there is a unique stationary monetary equilibrium, $m_t = m^*$ for all t, and a continuum of equilibria indexed by $m_0 \in (0, m^*)$ where $\dot{m}_t < 0$ and $\lim_{t\to 0} m_t = 0$.*

2. *If $a > 1$ and $b \in (0,1)$ with*

$$\rho + \alpha\kappa\left[\left(\frac{ab}{a-1}\right)^a - 1\right] < 0, \tag{52}$$

then there exists a continuum of periodic equilibria.

If the coefficient of risk aversion, a, is not too large, the dynamics of m_t are qualitatively similar to the ones of the economy without informational frictions. When a is larger than one, however, the equilibrium set changes qualitatively. The phase line is backward-bending as illustrated in Figure 11. For intermediate values of m_t there are multiple solutions for $\dot{m}_t/m_t$ of opposite sign. So, for the same m_t, one can construct trajectories where m_t increases and trajectories where it decreases. It is then possible to construct periodic equilibria.

We illustrate two periodic equilibria in the panels of Figure 11. In the left panel, in the high part of the phase line, real balances increase over time as the rate of return of money and pledgeability decrease. In the low part of the phase line, real balances, the rate of return of money, and pledgeability all decrease over time. As the economy switches from the high part to the low part of the phase line, the rate of return of money jumps downward, and vice versa as the economy switches from the low part to the high part of the cycle. Such discontinuities in r_t happen a countable number of times. In contrast, m_t and ϕ_t are continuous along the periodic equilibrium.

In the right panel, the high part of the cycle is such that $r = \rho$ and $\chi = \kappa$. So agents are not liquidity constrained, the rate of return of money and pledgeability are constant. The value of money grows at rate

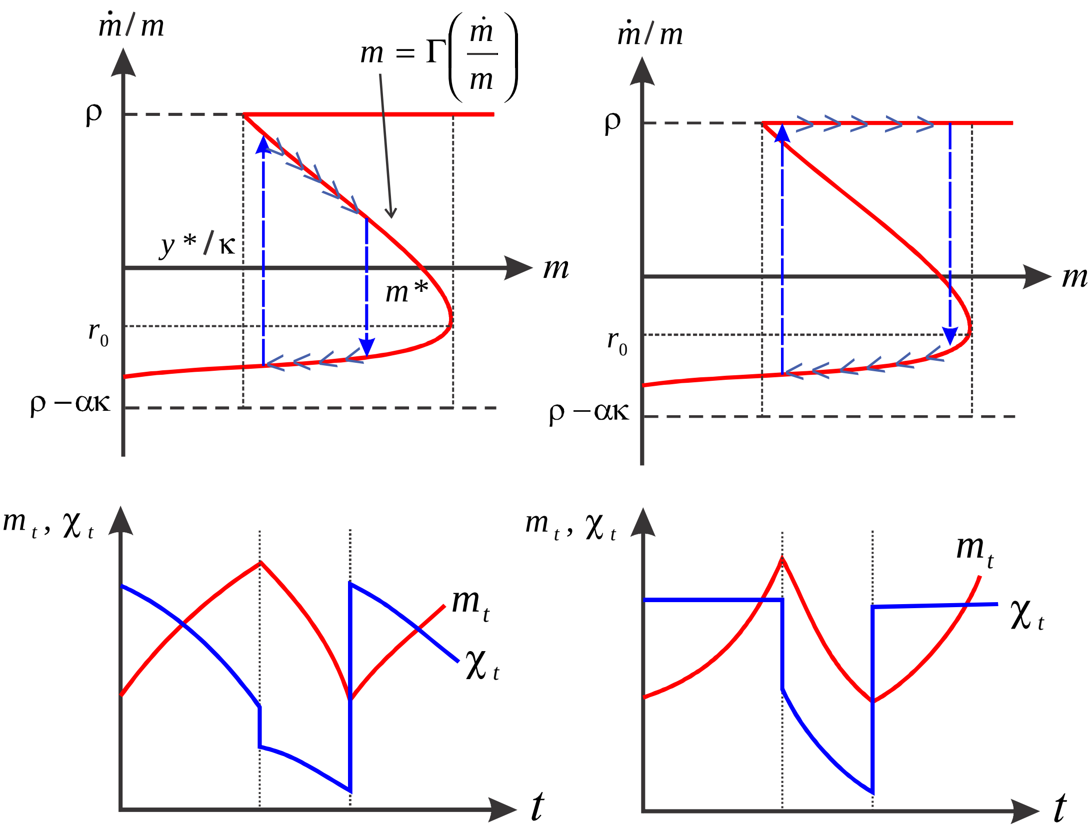

Figure 11: Phase diagram when the conditions of Proposition 6, part 2, are satisfied. Two examples of periodic equilibria.

ρ, resembling a bubble, until the economy switches to the low part of the cycle at which point the "bubble" bursts. Figure 12 provides a numerical example of such a periodic equilibrium.[24]

The occurrence of periodic solution to a one-dimensional ODE that admits a single steady state is generally hard to obtain. (We are not aware of other examples.) In particular, if the ODE takes the form of $\dot{m}$ being a function of m, then deterministic cycles cannot occur in continuous time. In our model, $\dot{m}$ is a correspondence of m, which, under some conditions, allows m to be associated with two distinct values of opposite sign for $\dot{m}$.[25] It is this mathematical property of the ODE that allows for the existence of equilibrium cycles.

From an economic standpoint, the existence of such equilibria relies critically on the endogenous asset pledgeability and its dependence on the asset rate of return. Suppose agents anticipate the rate of return of money is high. In this case buyers want a large y_t since holding liquidity is inexpensive. But since from

[24] It is based on the following parametrization: $u(y) = [(y+b)^{1-a} - b^{1-a}]/(1-a)$ with $a = 2$ and $b = 0.1$; $\rho = 2\%$; $\alpha = 4$; and $\kappa = 0.5$. In the high part of the cycle, pledgeability is maximum at 50%, liquidity needs are satiated, and real balances grow at rate ρ. As the economy transitions in the low part of the cycle, pledgeability drops and recovers gradually afterwards.

[25] This mathematical representation of the dynamic system is somewhat related to the multiplicity of equilibria of the discrete-time neoclassical growth model under adverse selection of Azariadis and Smith (1998) where the indeterminacy is due to the existence of multiple laws of motion for the capital stock.

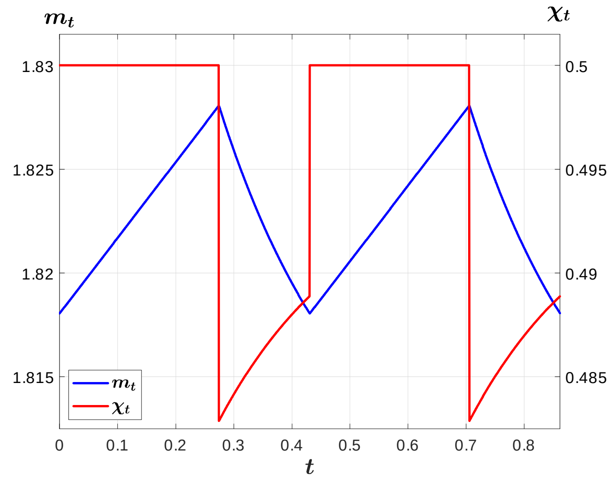

Figure 12: Asset price dynamics when pledgeability is endogenous: example of a periodic equilibrium

(48) pledgeability is high, a large y does not require a very large m. At the opposite, if agents anticipate the rate of return of money is low, then they choose a low consumption that they finance with sufficiently high m. Since $m = y/\chi$ and both y and χ increase with r, it is then possible to have the same m for two distinct pairs, (χ_h, r_h) and (χ_ℓ, r_ℓ).

We now turn to stationary sunspot equilibria. A two-state sunspot equilibrium is a pair, (m_ℓ, m_h), solution to

$$\rho - \frac{\lambda_{\ell h}(m_h - m_\ell)}{m_\ell} = \alpha\chi_\ell\left[u'(\chi_\ell m_\ell) - 1\right] \quad (53)$$

$$\rho - \frac{\lambda_{h\ell}(m_\ell - m_h)}{m_h} = \alpha\chi_h\left[u'(\chi_h m_h) - 1\right] \quad (54)$$

where

$$\chi_s = \kappa - \frac{\rho - \lambda_{ss'}(m_{s'} - m_s)/m_s}{\alpha} \in [0, 1] \quad \text{for } s \in \{\ell, h\}.$$

Proposition 7 ***(Sunspot equilibria under the threat of fraud.)*** *Suppose $d = \pi = 0$ and $u(y) = \left[(y+b)^{1-a} - b^{1-a}\right]/(1-a)$ and (52) holds. There exists $0 < m_0 < m_1$ such that for all $(m_\ell, m_h) \in \left\{(x, y) \in (m_0, m_1)^2 : x < y\right\}$ there is $(\lambda_{\ell h}, \lambda_{h\ell}) \in \mathbb{R}_{2+}$ such that (m_ℓ, m_h) is a sunspot equilibrium.*

We describe the construction of sunspot equilibria from the example in the top left panel of Figure 11. Take m_ℓ as the lowest real balances when the economy transitions from the lower part of the cycle to the high part. Take m_h as the highest real balances when the economy transitions from the high part of the cycle to the low part. We show in the proof of Proposition 7 that one can find positive transition rates, $(\lambda_{\ell h}, \lambda_{h\ell})$, such that (m_ℓ, m_h) solves (53)-(54), i.e.,it is a sunspot equilibrium.[26]

Sunspot equilibria constructed in Proposition 7 have the following properties. Real money balances alternate between a high (m_h) and a low (m_ℓ) value. In the high state, the expected rate of return of money

[26] For instance, for the same parametrization as the one used in Figure 12, we construct a sunspot equilibrium where $\lambda_{\ell h} = \lambda_{h\ell} = 4$, $(m_\ell, \chi_\ell) = (1.81, 0.5)$, and $(m_h, \chi_h) = (1.82, 0.49)$.

is negative (i.e., agents anticipate inflation) and pledgeability is low. Agents accumulate large real balances to compensate for the low pledgeability. At the opposite, in the low state, the expected rate of return of money is positive and pledgeability is high allowing buyers to hold lower real balances.

6.2 The case of assets with positive intrinsic value

We now show that the existence of periodic and sunspot equilibria is robust to the case where assets are intrinsically valuable, $d > 0$. Deterministic equilibria are given by $m_t = \Gamma(r_t)$ where $r_t = (\dot{m}_t + dM)/m_t$. We start by providing conditions under which there are multiple steady states. From (44) and (46), a steady state solves

$$\rho - \frac{dM}{m^*} = \alpha\chi\left[u'(\chi m^*) - 1\right]^+ \quad \text{with} \quad \chi = \left(\kappa - \frac{\rho - dM/m^*}{\alpha}\right)^+. \tag{55}$$

A key difference relative to the case of fiat money is that when $dM > 0$, pledgeability depends on the asset price even at the steady state. Hence, the asset price and liquidity can interact and generate multiple steady states.

Proposition 8 ***(Steady states with endogenous pledgeability and intrinsically valuable assets.)*** *Suppose $d > 0$ and $u(y) = \left[(y + b)^{1-a} - b^{1-a}\right]/(1 - a)$ with $a > 1$ and $b \in (0, 1)$.*

1. *If $\rho - \alpha\kappa < 0$, then the steady state is unique.*

2. *Suppose*

$$\kappa < \left(\frac{CRRA - 1}{CRRA}\right)\frac{\rho}{\alpha}, \tag{56}$$

where $CRRA \equiv -u''(y^*)y^*/u'(y^*)$. If dM is neither too low nor too high, then there are multiple steady-state equilibria.

Graphically, the steady-state equilibria are obtained at the intersection of Γ and the locus $r = dM/m$, which is monotone decreasing. When Γ is backward bending, it is possible to obtain multiple steady-state equilibria when κ is low and dM is in some intermediate range.[27] We illustrate this possibility in the right panel of Figure 13 where steady states are at the intersection of the blue and red curves. In our example, there are three distinct steady states. For the functional form considered in the proposition, if $b \approx 0$, then $y^* \approx 1$ and (56) becomes $\kappa < (a - 1)\,\rho/(\alpha a)$. As risk aversion increases, multiple steady states are more likely.

By the same logic as in Proposition 6, one can construct periodic equilibria and equilibria where ϕ_t fluctuates over time irrespective of whether the steady state is unique or not. Examples of periodic equilibria are depicted in Figure 13 with purple arrows. In the left panel, the steady state is unique while on the right panel there are three steady states. In the first part of the cycle, along the orange curve, ϕ_t increases while

[27] Suppose $u(y) = [(y + b)^{1-a} - b^{1-a}]/(1 - a)$ with $a = 2.5$ and $b = 0.15$; $\rho = 5\%$; and $\alpha = 1$. We constructed examples such that when $\kappa = 0.025$, there are three steady states provided that dM is in some intermediate range. As κ increases 0.045, then the steady state is unique.

r_t and χ_t decrease. In the second part of the cycle, ϕ_t, r_t, and χ_t decrease. As the economy transitions to from the second part to the first part, r_t and χ_t jump upward.

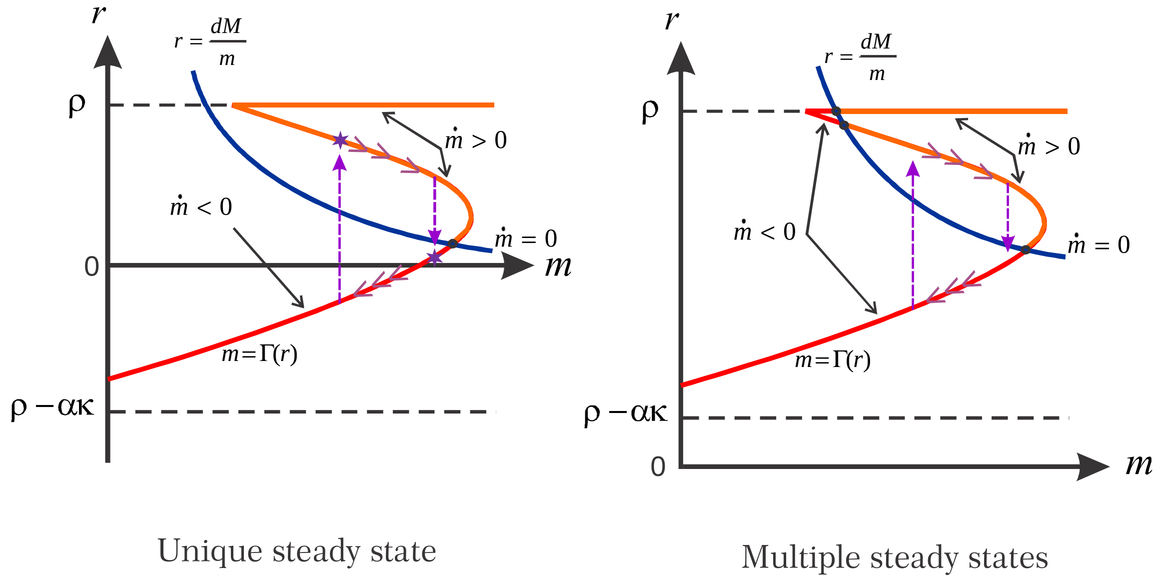

Figure 13: Liquid and the threat of fraud: the case of Lucas trees ($d > 0$)

In Figure 14, we provide a numerical example of a periodic equilibrium.[28] This example is different from the one in Figure 13 in that in the high part of the cycle liquidity needs are fully satiated and pledgeability is maximum at $\chi = 0.5$. In the low part of the cycle, the real interest rate and pledgeability drop while the asset price declines gradually.

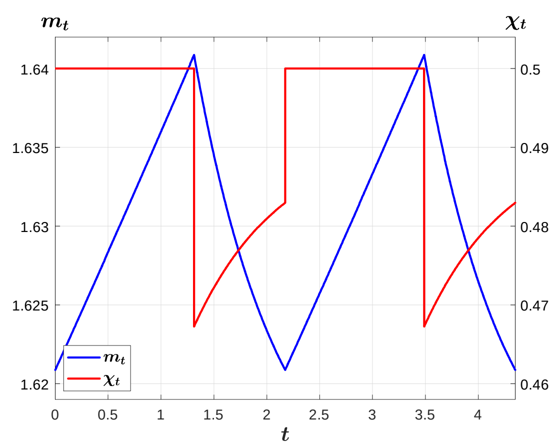

Figure 14: Periodic equilibrium under endogenous pledgeability when assets have positive intrinsic value

[28] It is based on the parametrization: $u(y) = [(y + b)^{1-a} - b^{1-a}]/(1 - a)$ with $a = 2$ and $b = 0.2$; $\rho = 4\%$; $\alpha = 1; \kappa = 0.5$ and $dM = 0.05$.

Suppose the sunspots state space is $S = \{\ell, h\}$. Stationary sunspot equilibria are solutions to $m_s = \Gamma(r_s)$ where

$$r_s = \frac{\lambda_{ss'}\left(m_{s'} - m_s\right) + dM}{m_s}, \;\; s \in \{\ell, h\}.$$

We construct such equilibria following the same logic as in the proof of Proposition 7. We take the two extreme points, $(\underline{m}, \bar{r})$ and $(\bar{m}, \underline{r})$, of the cycle marked by purple stars in Figure 13 and corresponding to the lowest real balances paired with the highest rate of return and vice-versa. We assign $(\underline{m}, \bar{r})$ to state $s = h$ and $(\bar{m}, \underline{r})$ to state $s = \ell$. We can then compute $(\lambda_{h\ell}, \lambda_{\ell h})$ that solve $r_h = \bar{r}$ and $r_\ell = \underline{r}$, i.e.,

$$\lambda_{h\ell} = \frac{\bar{r}\underline{m} - dM}{\bar{m} - \underline{m}}, \;\; \lambda_{\ell h} = \frac{\underline{r}\bar{m} - dM}{\underline{m} - \bar{m}}.$$

Using that $(\underline{m}, \bar{r})$ is located above the locus $r = dM/m$, it follows that $\bar{r}\underline{m} > dM$, and hence $\lambda_{h\ell} > 0$. Using that $(\bar{m}, \underline{r})$ is located below the locus $r = dM/m$, it follows that $\underline{r}\bar{m} < dM$, and hence $\lambda_{\ell h} > 0$.[29] The existence of proper sunspot equilibria when assets have a positive intrinsic value is in contrast with the result from Proposition 1 according to which the price of perfectly liquid assets is uniquely determined. It highlights the role of endogenous liquidity to generate self-fulfilling volatility.

7 Multiple-asset economies

So far we have restricted our attention to economies with a single asset. We now check whether our results carry over to economies with multiple assets. Is extrinsic uncertainty relevant in the presence of multiple liquid assets? Can an asset with a negative intrinsic value coexist with assets with positive intrinsic values and what are the implications for volatility?[30]

We consider an economy with a continuum of Lucas trees, represented by a measurable space, $(\mathcal{J}, \Sigma_{\mathcal{J}})$, where $\mathcal{J} \equiv [0, 1]$. Assets are indexed by $j \in \mathcal{J}$ and are characterized by their supplies expressed as a density measure, $\mu(j)$, their dividends, $d(j)$, and their death rates, $\delta(j)$. Dividends can be positive or negative. Our formulation encompasses the case with a discrete number of assets. For instance, a dual asset economy is one where all assets on intervals $[0, \hat{\varepsilon})$ and $[\hat{\varepsilon}, 1]$ have the same characteristics in terms of dividend and duration, the total supply of the first asset is $M^1 = \int_0^{\hat{\varepsilon}} \mu(\varepsilon) d\varepsilon$ and the supply of the second asset is $M^2 = \int_{\hat{\varepsilon}}^1 \mu(\varepsilon) d\varepsilon$.

The value of asset j at time t in state s is $\phi_{s,t}(j)$. Hence, the expected rate of return of asset j in state s at time t is

$$r_{s,t}(j) = \frac{d(j) + \dot{\phi}_{s,t}(j) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}\left[\phi_{s'}(j) - \phi_s(j)\right] - \delta(j)\phi_s(j)}{\phi_{s,t}(j)}. \tag{57}$$

The novelty is the last term that corresponds to the death of the asset, $\delta(j)$.

The buyer's portfolio is now represented by a density measure $m(j)$ over $(\mathcal{J}, \Sigma_{\mathcal{J}})$. The buyer's value

[29] Using the same parametrization as the numerical example above, if $(\lambda_{lh}, \lambda_{hl}) = (1.91, 0.74)$, there exists a sunspot equilibrium such that $(m_\ell, r_\ell, \chi_\ell) = (1.64, 0.01, 0.47)$, and $(m_h, r_h, \chi_h) = (1.62, 0.04, 0.5)$.

[30] In Appendix E, we also investigate multiple currencies and multiple Lucas trees with endogenous pledgeability constraints.

function is $V^b_{s,t}(m) = \int_0^1 m(j)dj + V^b_{s,t}$ where V^b_s obeys:

$$\rho V^b_s = \max_m \left\{ -\int_0^1 \left[\rho - r_s(j)\right] m(j)dj + \alpha v\left(m\right) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}\left(V^b_{s'} - V^b_s\right) + \tau_s + \dot{V}^b_s \right\}, \tag{58}$$

with

$$v\left(m\right) = \max_{y \geq 0} \left\{ u(y) - p(y) \text{ s.t. } p(y) \leq \int_0^1 \chi(j)m(j)dj \right\}. \tag{59}$$

The first term on the right side of (58) is the flow cost of holding a portfolio of assets, m. The coefficients $\chi(j)$ in (59) indicate whether assets j is acceptable in payment, which is determined endogenously.

The first-order condition, assuming interiority, is

$$\rho - \frac{d(j) + \dot{\phi}_{s,t}(j) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}\left[\phi_{s'}(j) - \phi_s(j)\right] - \delta(j)\phi_s(j)}{\phi_{s,t}(j)} = \alpha\chi(j)\left[\frac{u'(y_s)}{p'(y_s)} - 1\right]. \tag{60}$$

From market clearing, $m_s(j) = \mu(j)\phi_s(j)$. Hence, $\dot{m}_s(j)/m_s(j) = \dot{\phi}_s(j)/\phi_s(j)$ and we can rewrite (60) as:

$$\rho - \frac{\dot{m}_s(j) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}\left[m_{s'}(j) - m_s(j)\right] - \delta(j)m_s(j) + d(j)\mu(j)}{m_s(j)} = \alpha\chi(j)L(m), \tag{61}$$

for all $s \in S$, and all $j \in \mathcal{J}$, where $L \equiv u'(y)/p'(y) - 1$ with $p(y) = \min\left\{p(y^*), \int_0^1 \chi(j)m(j)dj\right\}$. An equilibrium is a list of time paths, $(m_{s,t}(j))_{j \in [0,1], s \in S}$, that solves the system of ODEs, (61).

7.1 Long-lived Lucas trees

We consider first the case where all assets are productive, $d(j) > 0$, and long lived, $\delta(j) = 0$. Because asset prices are bounded below by their fundamental value, $d(j)/\rho > 0$, all assets are accepted in bilateral meetings, $\chi(j) = 1$. It implies that assets are perfect substitutes as means of payments and their rates of return are equalized.

A steady state is a density measure, $m^*(j)$, that solves

$$\rho - \frac{d(j)\mu(j)}{m^*(j)} = \alpha L\left(m^*\right). \tag{62}$$

Multiplying both sides by $m^*(j)$ and integrating over j, it is easy to check that m^* is uniquely determined. Hence, $m^*(j)$ is unique for all j, and there exists a unique steady-state equilibrium. With a slight abuse of notation, we also use $m \equiv \int_0^1 m(j)dj$ to denote market capitalization. The following proposition shows that there is a unique equilibrium and it is deterministic and stationary.

Proposition 9 ***(Equilibria with a continuum of liquid Lucas trees.)***

1. *There exists a unique deterministic equilibrium and it is such that*

$$m_t(j) = \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)} \quad \text{for all } t \text{ and all } j, \tag{63}$$

where $m^* \equiv \int_0^1 m^*(j)dj$ solves

$$\rho = \alpha L(m^*) + \frac{\int_0^1 d(j)\mu(j)dj}{m^*}. \tag{64}$$

2. *There is no proper stationary sunspot equilibrium where $m_s(j)$ differs across states for a positive measure of assets, i.e.,*

$$m_s(j) = \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)} \text{ for all } s \in S \text{ and all } j. \tag{65}$$

The existence of multiple liquid assets does not expand the set of equilibria relative to the one-asset case. There is a unique deterministic equilibrium and it corresponds to the steady state.

7.2 Coexistence of assets with positive and negative fundamental values

We now consider the case of two assets with opposite intrinsic values: asset 1 generates a positive dividend, $d^1 > 0$, while asset 2 generates a negative dividend, $d^2 < 0$. This extension will show that the volatility of the price of the asset with negative intrinsic value can be passed onto the asset with positive fundamental value when the real interest rate is negative.

If asset 1 is long lived, $\delta^1 = 0$, then the real interest rate is positive and asset 2 cannot have a positive price. Hence, we assume that both assets are short lived, $\delta^1 > 0$ and $\delta^2 > 0$. We start with the equilibrium where asset 2 has a negative price and hence is not accepted as a means of payment. The market capitalization of asset 1, m^1, is the solution to

$$\rho - \frac{d^1 M^1 - \delta^1 m^1}{m^1} = \alpha L(m^1).$$

If $r^1 = d^1 M^1 / m^1 - \delta^1 > 0$, i.e., $d^1 M^1$ is sufficiently large, then it is the only equilibrium. If $r^1 < 0$, then another equilibrium might exist where asset 2 becomes a medium of exchange and acquire a positive price. In the following we consider such equilibria.

Proposition 10 ***(Equilibria where assets with positive and negative fundamental values coexist.)*** *Assume $\delta^1 > \delta^2$. If $d^1 M^1 < (\delta^1 - \delta^2) L^{-1}\left[\left(\rho + \delta^2\right)/\alpha\right]$ and d^2 is not too negative, then:*

1. *There exist two steady states where both assets have a positive price and the real interest rate is negative. Across steady-state equilibria, m^1 and m^2 are negatively correlated.*

2. *There are a continuum of deterministic equilibria where (m^1_t, m^2_t) converges to the steady state with high m^1 and low m^2. Along deterministic, nonstationary equilibria, m^1 and m^2 are negatively correlated.*

3. *Suppose $S = \{\ell, h\}$ and $\lambda_{ss'}$ are close to zero. There exists stationary sunspot equilibria where (m^1_s, m^2_s) is in the neighborhood of a steady state for each state $s \in S$.*

If the asset with positive intrinsic value is in low supply relative to the liquidity needs of the economy, then multiple steady-state equilibria with negative real interest rates exist. In the equilibrium where the asset that lacks fundamental value has a low price, then the asset with positive intrinsic value has a high price, and vice versa. This result shows that, in contrast to Proposition 9, the price of assets with positive intrinsic values can be indeterminate when liquidity is scarce and real interest rates are negative if there exists assets with negative intrinsic value that can supplement the existing liquidity.

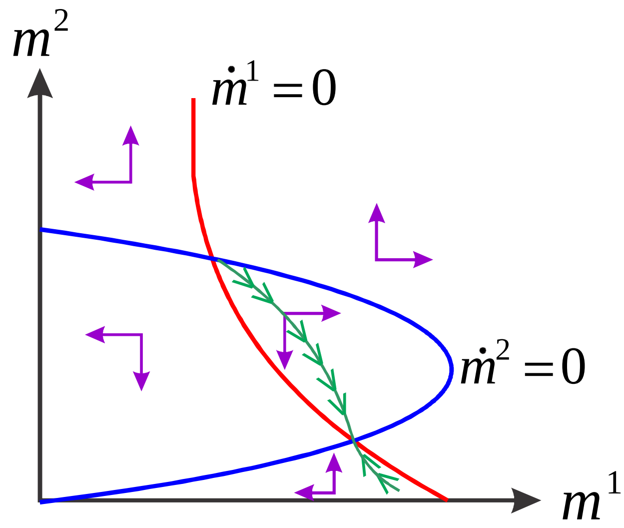

Figure 15: Phase diagram of the economy with productive and toxic assets

The second part of Proposition 10 shows that there are a continuum of deterministic, nonstationary equilibria that converge to the steady state where the asset with negative fundamental value has a low price. Along these equilibria, if one asset experiences a capital gain then the other asset experiences a capital loss, which reflects the substituability of the two assets as media of exchange.

Finally, the last part of Proposition 10 establishes the existence of sunspot equilibria constructed by continuity from the two steady states. As discussed above, this result highlights the possibility that the price of an asset with positive intrinsic value can be volatile if aggregate liquidity is scarce and there exists other liquid assets with negative intrinsic value.[31]

8 Conclusion

In this paper, we asked whether asset liquidity is a source of volatility. We approached this question by constructing a continuous-time, New Monetarist model under extrinsic uncertainty. We obtained the following answers. First, if assets have a positive intrinsic value and are perfectly liquid (i.e., universally accepted and fully pledgeable), then asset prices are uniquely determined and extrinsic uncertainty does not matter. Second, sunspot equilibria exist when assets lack fundamental value. If assets have zero intrinsic value (e.g., fiat monies), then sunspot equilibria emerge only if there is a state where the asset becomes valueless. If the asset has a negative intrinsic value, then there can exist both multiple steady states and stationary sunspot equilibria. Third, extrinsic uncertainty matters when asset liquidity is endogenous, i.e., liquidity which is impaired by informational frictions can be partial and depend on fundamentals and policy. Endogenous asset acceptability and pledgeability generate pecuniary externalities that can lead to periodic

[31] While we focused on sunspot equilibria where $m^2 > 0$ in all states, one could also construct sunspot equilibria where in the low state asset 2 has a negative price, $m^2_\ell < 0$. In that case, asset is not accepted as means of payment.

and sunspot equilibria in continuous time even when the steady state is unique. Fourth, if assets with positive intrinsic values coexist with assets with negative intrinsic values, then the volatility created by the latter can affect the former, i.e., volatility is contagious across liquid assets. However, this volatility only arises when real interest rates are negative.

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Appendix A: Proofs of propositions

Proof of Proposition 2. Suppose a proper sunspot equilibrium, $(m_s)_{s\in S}$, exists. Since the Markov chain is irreducible, positive recurrent, the expected time for any state s' to be reached from s is finite. Hence, the equilibrium is monetary if and only if $m_s > 0$ for all s. (An equilibrium is monetary if $m_s > 0$ in some state s. Given that the time it takes to reach state s from any state s_0 is finite in expectation, $m_s > 0$ implies $m_{s_0} > 0$ for all $s_0 \neq s$.) We denote $\bar{s}$ the state with the highest real balances and $\underline{s}$ the state with the lowest real balances, i.e., $m_{\bar{s}} > m_{\underline{s}}$. From (21), in the highest state, $s = \bar{s}$,

$$\rho + \pi - \alpha L(m_{\bar{s}}) = \sum_{s' \in S\setminus\{\bar{s}\}} \lambda_{\bar{s}s'} \left(\frac{m_{s'} - m_{\bar{s}}}{m_{\bar{s}}} \right).$$

Recall that $\rho + \pi - \alpha L(m^*) = 0$ and $L'(m^*) < 0$. In a proper sunspot equilibrium the right side is negative, which implies $m_{\bar{s}} < m^*$. From (21), in the lowest state,

$$\rho + \pi - \alpha L(m_{\underline{s}}) = \sum_{s' \in S\setminus\{\underline{s}\}} \lambda_{\underline{s}s'} \left(\frac{m_{s'} - m_{\underline{s}}}{m_{\underline{s}}} \right).$$

The right side is positive and hence $m_{\underline{s}} > m^*$. Hence, $m_{\bar{s}} < m_{\underline{s}}$, which contradicts $m_{\bar{s}} > m_{\underline{s}}$. So, there are no proper stationary sunspot equilibria. ■

Proof of Proposition 3. From (25) the transition rate from h to ℓ solves

$$\lambda_{h\ell} = \frac{m_h \left[\rho + \pi - \alpha L(m_h)\right]}{m_\ell - m_h}.$$

By assumption, $m_h < m^*$ which implies $\rho + \pi - \alpha L(m_h) < 0$. The denominator is negative since $m_\ell < m_h$, hence $\lambda_{h\ell} > 0$. From (24), the transition rate from ℓ to 0 is equal to

$$\lambda_{\ell 0} = \left[\alpha L(m_\ell) - \rho - \pi\right] + \lambda_{\ell h} \left(\frac{m_h - m_\ell}{m_\ell} \right).$$

By the assumption $m_\ell < m^*$, $\alpha L(m_\ell) - \rho - \pi > 0$, which implies $\lambda_{\ell 0} > 0$ for all $\lambda_{\ell h} > 0$. ■

Proof of Proposition 4. From (26),

$$\lambda_{\ell 0} = \lambda_{\ell h} \left(\frac{m_h - m_\ell}{m_\ell} \right) + \alpha L(m_\ell) - \rho - \pi.$$

By the assumption $m_h > m_\ell$, the first term of the right side is positive for all $\lambda_{\ell h} > 0$. By the assumption $m_\ell < m^*$, $\alpha L(m_\ell) > \alpha L(m^*) = \rho + \pi$. Hence, $\lambda_{\ell 0} > 0$. From (27),

$$\lambda_{h\ell} = \frac{m_h}{m_h - m_\ell} \left\{ \left[\alpha L(m_h) - \rho - \pi\right] + \lambda_{h*} \left(\frac{m^* - m_h}{m_h} \right) \right\}.$$

By the assumption $m_h < m^*$, $\alpha L(m_h) > \alpha L(m^*) = \rho + \pi$. It follows that $\lambda_{h\ell} > 0$. ■

Proof of Proposition 5. Part 1. If $m_\ell < m_\ell^*$ then the left side of (36) is positive. Hence,

$$\lambda_{\ell h} = \frac{\rho m_\ell - dM - \alpha L(m_\ell) m_\ell}{m_h - m_\ell} > 0.$$

If $m_h \in (m_\ell^*, m_h^*)$, then the left side of (37) is negative. Hence,

$$\lambda_{h\ell} = \frac{\rho m_h - dM - \alpha L(m_h) m_h}{m_\ell - m_h} > 0.$$

Part 2. If $m_\ell > dM/\rho$, then, from (36),

$$\lambda_{\ell h} = \frac{\rho m_\ell - dM}{m_h - m_\ell} > 0.$$

If $m_h \in (m_\ell^*, m_h^*)$, then the right side of (35), $\rho m - \alpha L(m) m - dM$, is negative, as illustrated in the right panel of Figure 7. It follows that the left side of (37) is negative and hence

$$\lambda_{h\ell} = \frac{\rho m_h - dM - \alpha L(m_h) m_h}{m_\ell - m_h} > 0.$$

Finally, buyers in the low state have no incentive to dispose of the asset since $m_\ell > -\zeta M$, i.e., $\phi_\ell > -\zeta$. ■

Proof of Proposition 6. The ODE (49)-(50) can be reexpressed as $m_t = \Gamma\left(\dot{m}_t/m_t\right)$ for all $\dot{m}_t/m_t < \rho$ where

$$\Gamma(r) \equiv \left(\kappa - \frac{\rho - r}{\alpha}\right)^{-1} \left[\frac{\left(\kappa - \frac{\rho - r}{\alpha}\right)^{\frac{1}{a}}}{\kappa^{\frac{1}{a}}} - b\right],$$

and $m_t \in [(1-b)/\kappa, +\infty)$ if $\dot{m}_t/m_t = \rho$.

Case #1: $a < 1$ and $b = 0$. In that case,

$$\Gamma(r) = \frac{\left(\kappa - \frac{\rho - r}{\alpha}\right)^{\frac{1-a}{a}}}{\kappa^{\frac{1}{a}}}$$

Since $1 - a > 0$, $\Gamma'(r) > 0$. As $r \to (\rho - \alpha\kappa)^+$ then $\kappa - (\rho - r)/\alpha \to 0^+$. Hence, $\lim_{r \to (\rho - \alpha\kappa)^+} \Gamma(r) = 0$. Moreover, $\Gamma(\rho) = \kappa^{-1}$. For all $r \in (\rho - \alpha\kappa, \rho)$, the function Γ can be inverted so that m_t solves

$$\frac{\dot{m}_t}{m_t} = \Gamma^{-1}(m_t).$$

For all $m_t > m^*$, $\dot{m}_t/m_t > 0$. For all $m_t < m^*$, $\dot{m}_t/m_t < 0$. Hence, there is a unique stationary monetary equilibrium corresponding to the steady state and a continuum of equilibria indexed by $m_0 \in (0, m^*)$ where the value of money decreases over time and converges asymptotically to 0.

Case #2: $a > 1$ and $b \in (0,1)$. In that case,

$$\Gamma(r) = \frac{\left(\kappa - \frac{\rho - r}{\alpha}\right)^{\frac{1-a}{a}}}{\kappa^{\frac{1}{a}}} - \frac{b}{\kappa - \frac{\rho - r}{\alpha}}.$$

As $r \to (\rho - \alpha\kappa)^+$ then $\kappa - (\rho - r)/\alpha \to 0^+$and $\lim_{r \to (\rho - \alpha\kappa)^+} \Gamma(r) = -\infty$. Moreover, $\Gamma(\rho) = (1-b)/\kappa$. The function Γ reaches a maximum, $\Gamma'(r) = 0$, at $r = r_0$ where

$$r_0 = \rho + \alpha\kappa \left[\left(\frac{ab}{a-1} \right)^a - 1 \right].$$

Provided $ab/(a-1) < 1$, $r_0 < \rho$ and $\Gamma(r)$ is hump-shaped over the interval $(\rho - \alpha\kappa, \rho)$. The function Γ is represented by a red curve in Figure 11.

We now provide sufficient conditions for nonmonotone solutions to (49)-(50). Under the condition (52), $r_0 < 0$. As shown in Figure 11, for all $m \in ((1-b)/\kappa, \Gamma(r_0))$, there exists at least two solutions, r_1 and r_2, to

$$\begin{aligned} m &= \Gamma(r) \text{ if } r < \rho \\ m &\geq \frac{1-b}{\kappa} \text{ if } r = \rho \end{aligned}$$

with $r_1 < 0 < r_2 \leq \rho$. Hence, for all $m_t \in ((1-b)/\kappa, \Gamma(r_0))$, the sign of $\dot{m}_t/m_t$ is indeterminate and there are a continuum of solutions where m_t is nonmonotone as illustrated in Figure 11. ■

Proof of Proposition 7. This proof builds onto the proof of Proposition 6. The equilibrium conditions, (53)-(54), can be rewritten as $m_s = \Gamma(r_s)$ where

$$r_s = \frac{\lambda_{ss'}(m_{s'} - m_s)}{m_s}, \tag{66}$$

if $r_s < \rho$, and $m_s \geq (1-b)/\kappa$ if $r_s = \rho$. Denote $m_0 = (1-b)/\kappa$ and $m_1 = \Gamma(r_0)$ where $r_0 < 0$ is defined in the proof of Proposition 6. As shown in the proof of Proposition 6, for all $m \in (m_0, m_1)$, there exists two values of r of opposite sign, $\underline{r}(m) < 0 < \bar{r}(m)$, such that m is part of an equilibrium, i.e., $m = \Gamma(r)$. We use this result to show that any $(m_\ell, m_h) \in \left\{ (x,y) \in (m_0, m_1)^2 : x < y \right\}$ can be part of a sunspot equilibrium constructed as follows. We assign the positive value, $r_\ell = \bar{r}(m_\ell) > 0$, to m_ℓ, i.e., $m_\ell = \Gamma(r_\ell)$, and the negative value, $r_h = \underline{r}(m_h) < 0$, to m_h, i.e., $m_h = \Gamma(r_h)$. It follows from (66) that the arrival rates of the sunspot states consistent with r_ℓ and r_h are given by:

$$\begin{aligned} \lambda_{\ell h} &= \frac{r_\ell m_\ell}{m_h - m_\ell} > 0 \\ \lambda_{h\ell} &= \frac{r_h m_h}{m_\ell - m_h} > 0. \end{aligned}$$

Hence, (m_ℓ, m_h) is a sunspot equilibrium associated with the sunspot states, $S = \{\ell, h\}$, and the positive transition rates, $(\lambda_{\ell h}, \lambda_{h\ell})$. ■

Proof of Proposition 8. Part 1. From (55), assuming $\chi > 0$,

$$\rho - \frac{dM}{m^*} = \left(\alpha\kappa - \rho + \frac{dM}{m^*} \right) \left\{ u' \left[\left(\kappa - \frac{\rho}{\alpha} \right) m^* + \frac{dM}{\alpha} \right] - 1 \right\}^+.$$

Suppose $\alpha\kappa - \rho > 0$. The right side is nonincreasing in m^* while the left side is increasing in m^*. Moreover, the left side varies from 0 to ρ as m^* varies from dM/ρ to $+\infty$. The right side varies from some nonnegative number to 0. Hence, there is a unique m^* solution to the equation above, i.e., the steady state is unique.

Part 2. From (51), if the liquidity constraint, $y \leq \chi m$, binds, we can rewrite $m = \Gamma(r)$ as:

$$(\alpha\kappa - \rho + r)\, u'\left[\frac{(\alpha\kappa - \rho + r)\, m}{\alpha}\right] = \alpha\kappa.$$

We differentiate this expression with respect to m and r in order to obtain the slope, $\partial r/\partial m$, and we evaluate it at $r = \rho$ and $\kappa m = y^*$:

$$\left.\frac{\partial r}{\partial m}\right|_{m=\Gamma(r)} = \frac{-u''(y^*)\,\alpha\kappa^2}{1 + u''(y^*)\, y^*}.$$

Along the locus $r = dM/m$ evaluated at $r = \rho$ and $\kappa m = y^*$:

$$\left.\frac{\partial r}{\partial m}\right|_{r=dM/m} = \frac{-\rho\kappa}{y^*}.$$

Provided that $a > 1/(1-b)$, the curve representing $\Gamma(r)$ is backward bending. As illustrated in the right panel of Figure 13, if

$$\left.\frac{\partial r}{\partial m}\right|_{r=dM/m} < \left.\frac{\partial r}{\partial m}\right|_{m=\Gamma(r)},$$

there exist multiple of steady states for dM in some intermediate range. It can be reexpressed as:

$$\frac{-\rho\kappa}{y^*} < \frac{-u''(y^*)\,\alpha\kappa^2}{1 + u''(y^*)\, y^*}.$$

This condition can be rewritten as (56). Condition (56) implies that $CRRA > 1$, or equivalently, $a > 1/(1-b)$. ■

Proof of Proposition 9. Part 1. A deterministic equilibrium is a time-path, $m_t(j)$, solution to (61), i.e.,

$$\frac{\dot{m}(j)}{m(j)} = \rho - \alpha L(m) - \frac{d(j)\mu(j)}{m(j)}, \tag{67}$$

where $m = \int_0^1 m(j)dj$. Summing (67) across assets, market capitalization, m, solves the following ODE:

$$\dot{m} = \rho m - \alpha m L(m) - \int_0^1 d(j)\mu(j)dj. \tag{68}$$

Equation (68) is identical to the one in the one-asset case where dM is replaced with $\int_0^1 d(j)\mu(j)dj$. Hence, there is a unique solution and it corresponds to the steady-state solution, m^*. Solving for (67),

$$m_t(j) = \left[m_0(j) - \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)}\right] e^{[\rho - \alpha L(m^*)]t} + \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)}.$$

Summing across $j \in [0,1]$,

$$m_t = \left[\int_0^1 m_0(j)dj - \frac{\int_0^1 d(j)\mu(j)dj}{\rho - \alpha L(m^*)}\right] e^{[\rho - \alpha L(m^*)]t} + \frac{\int_0^1 d(j)\mu(j)dj}{\rho - \alpha L(m^*)}$$

Since $m_t = m^*$ for all t,

$$\int_0^1 \left[m_0(j) - \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)} \right] dj = 0.$$

Suppose

$$m_0(j) \neq \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)} \text{ for a positive measure of } j.$$

Then, there is a positive measure of assets such that

$$m_0(j) < \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)}.$$

Since $\rho - \alpha L(m^*) > 0$, $m_t(j) < d(j)\mu(j)/\rho$ in finite time. It is a contradiction since the price of an asset cannot fall below its fundamental value. Hence, (63) holds.

Part 2. Stationary sunspot equilibria are lists, $m_s(j)$, that solve

$$\rho - \frac{\sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left[m_{s'}(j) - m_s(j) \right] + d(j)\mu(j)}{m_s(j)} = \alpha L(m_s), \quad \forall s \in S, \ \forall j \in [0,1], \tag{69}$$

where $m_s \equiv \int_0^1 m_s(j)dj$. Sum (69) across j's to obtain:

$$\rho m_s - \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(m_{s'} - m_s \right) + \int_0^1 d(j)\mu(j)dj = \alpha m_s L(m_s), \quad \forall s \in S.$$

It is the same equation as in the one asset case where dM is replaced with $\int_0^1 d(j)\mu(j)dj$. Hence, $m_s = m^*$ for all s. Suppose there is an asset j and a state s_0 such that $m_{s_0}(j) \geq m_s(j)$ for all $s \neq s_0$. From (69),

$$\rho - \frac{d(j)\mu(j)}{m_{s_0}(j)} \leq \alpha L(m^*), \text{ i.e., } m_{s_0}(j) \leq \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)}.$$

Similarly, suppose there is a state s_1 such that $m_{s_1}(j) \leq m_s(j)$ for all $s \neq s_1$. Then,

$$m_{s_1}(j) \geq \frac{d(j)\mu(j)}{\rho - \alpha L(m^*)}.$$

This proves $m_{s_0}(j) = m_{s_1}(j)$ and (65). ∎

Proof of Proposition 10. Part 1. A steady state, (m^1, m^2), solves

$$\rho + \delta^1 - \frac{d^1 M^1}{m^1} = \alpha L(m) \tag{70}$$

$$\rho + \delta^2 - \frac{d^2 M^2}{m^2} = \alpha L(m), \tag{71}$$

where $m = m^1 + m^2$. For such an equilibrium to exist, rates of return must be equalized, which requires that $\delta^1 - \delta^2 > 0$. Equation (70) gives a negative relationship between m^1 and m^2 with $m^1 > 0$ when $m^2 = 0$ and $m^1 = d^1 M^1 / \left(\rho + \delta^1 \right)$ when $m^2 = p(y^*)$. This relationship is represented by the red curve labelled $\dot{m}^1 = 0$ in Figure 15. Equation (71) can be reexpressed as:

$$\left(\rho + \delta^2 \right) m^2 - d^2 M^2 = \alpha m^2 L(m^1 + m^2). \tag{72}$$

The left side is linear in m^2 with a positive intercept while the right side is hump-shaped and is equal to 0 when $m^2 = 0$ and $m^2 = p(y^*) - m^1$. Provided d^2 is not too negative and m^1 is not too large, then there are two values of m^2 that solve (72). As m^1 increases, the right side decreases, so that the lowest solution for m^2 increases while the largest solution decreases. The two solutions for m^2 given m^1 are represented by the hump-shaped blue curve labelled $\dot{m}^2 = 0$ in Figure 15. A steady-state equilibrium corresponds to an intersection of the blue and red curves in Figure 15. If a solution exists with $m^2 > 0$, then there are two solutions as illustrated in Figure 15. In each equilibrium, the real interest rate is $r = d^2M^2/m^2 - \delta^2 < 0$. In order to establish existence of a solution with $m^2 > 0$ when $d^1M^1 < (\delta^1 - \delta^2)L^{-1}\left[\left(\rho + \delta^2\right)/\alpha\right]$ and d^2 not too negative, note that if $d^2 = 0$, then, $L(m^1 + m^2) = \left(\rho + \delta^2\right)/\alpha$ and $m^1 = d^1M^1/(\delta^1 - \delta^2)$, which implies $m^2 = L^{-1}\left[\left(\rho + \delta^2\right)/\alpha\right] - d^1M^1/(\delta^1 - \delta^2) > 0$, since $d^1M^1 < (\delta^1 - \delta^2)L^{-1}\left[\left(\rho + \delta^2\right)/\alpha\right]$. Let $\varepsilon = L^{-1}\left[\left(\rho + \delta^2\right)/\alpha\right] - d^1M^1/(\delta^1 - \delta^2)$. Then, by continuity, there exists a neighborhood around 0, $N(0)$, such that for all $d^2 \in N(0)$, $m^2 \in (\varepsilon/2, 3\varepsilon/2)$, so that $m^2 > 0$.

Part 2. A deterministic equilibrium is a pair, (m_t^1, m_t^2), solution to

$$\begin{aligned} \frac{\dot{m}^1}{m^1} &= \rho + \delta^1 - \alpha L\left(m\right) - \frac{d^1M^1}{m^1} \\ \frac{\dot{m}^2}{m^2} &= \rho + \delta^2 - \alpha L\left(m\right) - \frac{d^2M^2}{m^2}. \end{aligned}$$

The loci for $\dot{m}^1/m^1 = 0$ and $\dot{m}^2/m^2 = 0$ are represented by the red and blue lines in Figure 15. We also represent the arrows of motion for m_1 and m_2 within each region delimited by these two curves. While the steady-state equilibrium with a low m_1 and high m_2 is a diverging node, the steady state with high m_1 and low m_2 corresponds to a saddle point with a unique saddle path, represented by the green curve in Figure 15, leading to it. As a result, there are a continuum of deterministic equilibria, with initial conditions located on the saddle path, that converge to the saddle steady state. Along these equilibria, the prices of the two assets vary in opposite directions.

Part 3. A two-state, stationary sunspot equilibrium is a $(m_s^1, m_s^2)_{s\in\{\ell,h\}}$ solution to

$$\begin{aligned} \frac{\lambda_{ss'}(m_{s'}^1 - m_s^1)}{m_s^1} &= \rho + \delta^1 - \alpha L\left(m_s\right) - \frac{d^1M^1}{m_s^1} \\ \frac{\lambda_{ss'}(m_{s'}^2 - m_s^2)}{m_s^2} &= \rho + \delta^2 - \alpha L\left(m_s\right) - \frac{d^2M^2}{m_s^2}, \end{aligned}$$

for $s \in \{\ell, h\}$. When $\lambda_{\ell h} = \lambda_{h\ell} = 0$, a solution is such that (m_h^1, m_h^2) is equal to the steady state with high m^1 and (m_ℓ^1, m_ℓ^2) is equal to the steady state with low m^1. By continuity, if $\lambda_{\ell h} \approx 0$ and $\lambda_{h\ell} \approx 0$, then there exists a proper sunspot equilibrium, $(m_s^1, m_s^2)_{s\in\{\ell,h\}}$, where (m_h^1, m_h^2) is in the neighborhood of the steady state with high m^1 and (m_ℓ^1, m_ℓ^2) is in the neighborhood of the steady state with low m^1. ■

Appendix B: Bellman equations in continuous time

In this appendix we provide a detailed description of the buyer's problem. We start by writing the problem recursively and derive the optimal path for asset holdings. We then derive the Hamilton-Jacobi-Bellman equation used in the main text.

Recursive formulation

We start by writing down the recursive Bellman equation that determines $V^b_{s,t}$. Without loss of generality, we only allow the buyer to consume or produce a discrete amount of the numéraire, $\Delta C_0 \in \mathbb{R}$, at the initial date. Afterwards, and until the next event occurs, the buyer consumes or produces the numéraire as a flow, c_t. The next event is either a meeting with a seller or a change in the sunspot state. The buyer's value function solves:

$$V^b_{s,0}(m_0) = \max_{m_t, c_t, \Delta C_0} \left\{ \Delta C_0 + \mathbb{E} \int_0^T e^{-\rho t} c_t dt + e^{-\rho T} W^b_{s,T}(m_T) \right\} \tag{73}$$

$$\text{s.t.} \quad \dot{m}_t = \varrho_{s,t} m_t - c_t + \tau_{s,t} \quad \text{for all } t > 0 \tag{74}$$

$$\Delta C_0 = m_0 - m_{0^+} \tag{75}$$

$$m_0 \text{ given},$$

where T is the next random time at which the buyer is matched with a seller or the sunspot state changes. The random variable, T, is exponentially distributed with mean $1/\mu_s$ where $\mu_s = \alpha + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}$. According to (73) the buyer chooses her real asset holdings, m_t, and consumption, c_t and ΔC_0, so as to maximise her lifetime expected discounted utility. The first two terms in the objective function correspond to the discounted utility of consumption until T. The last term is the discounted expected value of the consumer when the first event occurs at time T. The budget identity, (74), states that the change in real asset holdings is equal to the return on assets plus transfers net of consumption. The return, $\varrho_{s,t}$, is conditional on the sunspot state remaining unchanged and is equal to $\varrho_{s,t} = (\dot{\phi}_{s,t} + d)/\phi_{s,t}$. In order to guarantee that value functions are bounded, we assume that

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \left| \tau_{s,t} \right| dt \right] < +\infty. \tag{76}$$

According to (76), the discounted sum of the transfers is bounded.

From the Principle of Optimality, the buyers' value functions are the solution to the Bellman equations (73)-(75) if $\lim_{t\to\infty} \mathbb{E}\left[e^{-\rho t} V^b_{s,t}(m_t)\right] = 0$ for all asset plans, (m_t), where a plan is a function of the history of sunspot realizations. In order to guarantee that this condition holds, we restrict the feasible plans to those that satisfy the transversality condition

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\rho t} m_t\right] = 0. \tag{77}$$

In Choi and Rocheteau (2021), we show that this condition is necessary for optimality.

The value function $W^b_{s,t}$ represents the expected utility of the buyer when an event occurs at time t. It solves

$$\mu_s W^b_{s,t}(m) = \alpha \max_{p(y)\leq m} \left\{u(y) + V^b_{s,t}\left[m - p(y)\right]\right\} + \sum_{s'\in S/\{s\}} \lambda_{ss'} V^b_{s',t}\left(\frac{\phi_{t,s'}}{\phi_{t,s}} m\right). \tag{78}$$

Conditional on an event occuring, there are two possibilities: either the buyer meets a seller with probability α/μ_s or there is a change in the sunspot state with probability $\sum_{s'\in S/\{s\}} \lambda_{ss'}/\mu_s$. If the buyer meets a seller, she chooses consumption y and reduces her asset holdings by $p(y)$. Her continuation value is then $V^b_{s,t}\left[m - p(y)\right]$. If the sunspot state changes from s to s', the asset price jumps from $\phi_{t,s}$ to $\phi_{t,s'}$ and asset holdings grow by a factor $\phi_{t,s'}/\phi_{t,s}$.

Using that T is exponentially distributed we can rewrite (73) as follows:

$$V^b_{0,s}(m_0) = m_0 + \max_{m_t, c_t}\left\{-m_{0^+} + \int_0^{+\infty} e^{-(\rho+\mu_s)t}\left[c_t + \mu_s W^b_{t,s}(m_t)\right] dt\right\}. \tag{79}$$

Note that the effective discount rate in (79) is $\rho + \mu_s$ and the instantaneous utility is $c_t + \mu W^b_{t,s}(m_t)$. From (74), $c_t = \varrho_{s,t} m_t - \dot{m}_t + \tau_{t,s}$ which allows us to write

$$\int_0^{+\infty} e^{-(\rho+\mu_s)t} c_t dt = \int_0^{+\infty} e^{-(\rho+\mu_s)t}\left(\varrho_{s,t} m_t + \tau_{t,s}\right) dt - \int_0^{+\infty} e^{-(\rho+\mu_s)t}\dot{m}_t dt.$$

By integration by parts, we can rewrite the last term as

$$\int_0^{+\infty} e^{-(\rho+\mu_s)t}\dot{m}_t dt = -m_0^+ + \int_0^{+\infty} (\rho+\mu_s) e^{-(\rho+\mu_s)t} m_t dt,$$

where we used the transversality condition which implies $\lim_{t\to\infty} e^{-(\mu_s+\rho)t} m_t = 0$. Hence,

$$\int_0^{+\infty} e^{-(\rho+\mu_s)t} c_t dt = m_0^+ + \int_0^{+\infty} e^{-(\rho+\mu_s)t}\left[\left(\varrho_{s,t} - \rho - \mu_s\right) m_t + \tau_{t,s}\right] dt.$$

We substitute this expression into (79) to obtain:

$$V^b_{s,0}(m_0) = m_0 + \max_{m_t}\left\{\int_0^{+\infty} e^{-(\rho+\mu_s)t}\left\{-\left(\rho - \varrho_{s,t}\right) m_t + \tau_{s,t} + \mu_s\left[W^b_{s,t}(m_t) - m_t\right]\right\} dt\right\}. \tag{80}$$

Using the linearity of $V^b_{s,t}$ and the definition of $\upsilon(m)$, we can reexpress the last term in the maximation as

$$\mu_s\left[W^b_{s,t}(m_t) - m_t\right] = \alpha\upsilon(m) + \alpha V^b_{s,t} + \sum_{s'\in S\setminus\{s\}} \lambda_{ss'}\left[V^b_{s',t} + \left(\frac{\phi_{s',t} - \phi_{s,t}}{\phi_{s,t}}\right) m\right].$$

Substituting this expression into (80), and using the definition of the expected rate of return of the asset,

$$r_{s,t} = \varrho_{s,t} + \sum_{s'\in S\setminus\{s\}} \lambda_{ss'}\left(\frac{\phi_{s',t} - \phi_{s,t}}{\phi_{s,t}}\right),$$

we obtain:

$$V^b_{s,0}(m_0) = m_0 + \max_{m_t}\left\{\int_0^{+\infty} e^{-(\rho+\mu_s)t}\left[-\left(\rho - r_{s,t}\right) m_t + \tau_{s,t} + \alpha\upsilon(m_t) + \alpha V^b_{s,t} + \sum_{s'\in S\setminus\{s\}} \lambda_{ss'} V^b_{s',t}\right] dt\right\}. \tag{81}$$

It follows that the optimal solution maximizes the terms between squared brackets at all points in time, i.e.,

$$m_t \in \arg\max \left\{ - \left(\rho - r_{s,t} \right) m_t + \alpha v(m_t) \right\}. \tag{82}$$

The optimal asset holdings maximize the expected surplus of the buyer in a meeting with a seller net of the cost of holding assets as measured by the difference between the rate of time preference and the expected rate of return of the asset.

Uniqueness of value functions

We now establish the uniqueness of $V^b_{s,t}$. We define the mapping T from the set of bounded function, $\mathcal{B}(S \times \mathbb{R}_+)$, into itself as:

$$Tf(s,t) = \int_0^{+\infty} e^{-(\rho+\mu_s)(t+x)} \left[U(r_{s,t+x}) + \tau_{s,t+x} + \alpha f(s,t+x) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} f(s', t+x) \right] dx,$$

where

$$U(r) \equiv \max_{m \geq 0} \left\{ - \left(\rho - r \right) m + \alpha v(m) \right\}.$$

The function Tf is bounded because $U(r) \leq u(y^*) - p(y^*)$, f is bounded, and from (76) the expected discounted sum of the transfers is also bounded. The mapping T satisfies the monotonicity and discounting Blackwell's sufficient conditions for a contraction. Since $\mathcal{B}(S \times \mathbb{R}_+)$ is a complete metric space, by Banach fixed point theorem, T admits a unique fixed points that corresponds to $V^b_{s,t}$.

Derivation of the Hamilton-Jacobi-Bellman equation

We now derive the Hamilton-Jacobi-Bellman equations used in the text to characterize $V^b_{s,t}$. From (81) the buyer's value function at time t when $m_0 = 0$ is equal to

$$V^b_{s,t} = \int_0^{+\infty} e^{-(\rho+\mu_s)x} \left[- \left(\rho - r_{s,x+t} \right) m^*_{x+t} + \tau_{s,x+t} + \alpha v(m^*_{x+t}) + \alpha V^b_{s,x+t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} V^b_{s',x+t} \right] dx, \tag{83}$$

where m^*_t indicates the optimal choice of real asset holdings given by (82). Differentiating with respect to t:

$$\dot{V}^b_{s,t} = \int_0^{\infty} e^{-(\rho+\mu_s)x} \left[\frac{\partial \left\{ - \left(\rho - r_{s,x+t} \right) m^*_{x+t} + \tau_{s,x+t} + \alpha v(m^*_{x+t}) + \alpha V^b_{s,x+t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} V^b_{s',x+t} \right\}}{\partial t} \right] dx.$$

Using integration by parts:

$$\begin{aligned} \dot{V}^b_{s,t} \quad = \quad & \left(\rho - r_{s,t} \right) m^*_t - \tau_{s,t} - \alpha v(m^*_t) - \alpha V^b_{s,t} - \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} V^b_{s',t} \\ & + (\rho + \mu_s) \int_0^{\infty} e^{-(\rho+\mu_s)x} \left[- \left(\rho - r_{s,x+t} \right) m^*_{x+t} + \tau_{s,x+t} + \alpha v(m^*_{x+t}) + \alpha V^b_{s,x+t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} V^b_{s',x+t} \right] dx, \end{aligned} \tag{84}$$

where we have used that

$$\lim_{x \to +\infty} e^{-(\rho+\mu_s)x} \left[U(r_{s,x+t}) + \tau_{s,x+t} + \alpha V^b_{s,x+t} + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} V^b_{s',x+t} \right] = 0.$$

From the expression for $V_{t,s}^b$ in (83), we can rearrange (84) to obtain:

$$\dot{V}_{t,s}^b = (\rho - r_{s,t})\, m_t^* - \tau_{s,t} - \alpha v(m_t^*) - \alpha V_{s,t}^b - \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} V_{s',t}^b + (\rho + \mu_s) V_{s,t}^b.$$

By using the definition of $\mu_s = \alpha + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}$ and rearranging the terms we obtain:

$$\rho V_{s,t}^b = \tau_{s,t} - (\rho - r_{s,t})\, m_t^* + \alpha v(m_t^*) + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(V_{s',t}^b - V_{s,t}^b \right) + \dot{V}_{s,t}^b.$$

Appendix C: An alternative interpretation of assets with a negative dividend

We showed in the main text that assets that yield a negative dividend are fundamentally different from fiat monies that generate no dividend. In particular, the equilibrium set differs qualitatively: assets with negative dividends generate multiple steady states and stationary sunspot equilibria while fiat monies do not (in continuous time). Why aren't assets with a negative dividend equivalent to a fiat money with a growing money supply that generates a negative rate of return? Another difficulty with assets that generate a negative dividend is in terms of their real-world interpretation. In practice, because of limited liability, owners of publicly-traded stocks are not subject to negative dividends to cover the losses of the companies they invested in. Instead, losses are covered by issuing equity or corporate debt. Hence, it is not clear what the prototypical example of an negative-dividend asset is, other than a fiat money with a storage cost.

In this appendix we address these different questions. Consider the version of the model where the asset has an initial supply, M_0, and generates total losses equal to $dM_0 < 0$.[32] Because the asset generates losses and because the owners of the asset benefit from limited liability, the asset yields no dividend. The losses of the asset are financed by raising the asset supply according to

$$\dot{M}_t \phi_{t,s} = -dM_0.$$

The left side is the revenue from equity issuance while the right side is the total loss generated by the asset. It implies that the rate of growth of the asset supply is endogenous and equal to

$$\pi_{t,s} \equiv \frac{\dot{M}_t}{M_t} = \frac{-dM_0}{m_{t,s}},$$

where $m_{t,s} = M_t \phi_{t,s}$ and the asset supply, M_t, is a function of the history of shocks up to t, s^t. The rate of growth of the asset supply is a decreasing function of its market capitalization, $m_{t,s}$.

The dynamics for the market capitalization of the asset is now identical to the one of fiat money, i.e.,

$$\rho + \pi_{t,s} - \frac{\dot{m}_{t,s}}{m_{t,s}} - \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\frac{m_{t,s'} - m_{t,s}}{m_{t,s}} \right) = \alpha L(m_{t,s}), \quad \forall s \in S.$$

Substituting $\pi_{t,s}$ by its expression, the ODE for $m_{t,s}$ can be reexpressed as:

$$\rho - \frac{dM_0}{m_{t,s}} - \frac{\dot{m}_{t,s}}{m_{t,s}} - \sum_{s' \in S \setminus \{s\}} \lambda_{ss'} \left(\frac{m_{t,s'} - m_{t,s}}{m_{t,s}} \right) = \alpha L(m_{t,s}), \quad \forall s \in S.$$

This equation is identical to the one where the asset has a constant supply, M_0, but pays a negative dividend, d.

At the steady state, the pair (π, m) is determined by

$$\begin{aligned} \pi m &= -dM_0 \\ L(m) &= \frac{\rho + \pi}{\alpha}. \end{aligned}$$

[32] Denis and McKeon (2018) document that U.S. firms incurring operating losses comprise over 30% of all the Compustat firms and such losses last for a median of four years. They also document that firms incurring operating losses comprise the majority of equity issuers.

We see that the loss of the asset is financed by the seigniorage revenue arising from the growth of its supply. Because the seigniorage revenue, πm, is an hump-shaped Laffer curve of π, provided that $-dM_0$ is not too large, there are two solutions, (π, m), of this system of equations. This result shows that an asset with a negative dividend is equivalent to a fiat money with a growing money supply when the money growth rate is endogenous and determined so as a finance a constant expenditure flow.

Appendix D: Acceptability and market participation

We describe the decision of sellers to accept an asset in payment as in Lester et al. (2012). In order to accept an asset, sellers must incur a flow cost, $k > 0$. An alternative interpretation is in term of market participation (e.g., Rocheteau and Wright, 2005, 2009): in order to participate in the market, the seller incurs the cost k. Assume that $k < \alpha\left[p(y^*) - y^*\right]$. The individual decision to incur that cost is denoted $\varkappa \in [0,1]$. Aggregate acceptability is $\chi = \int_0^1 \varkappa(j)dj$.

The value functions of the seller solve:

$$\rho V_s^f = \max_{\varkappa \in [0,1]} \left\{ -k\varkappa + \alpha\varkappa\left[p(y_s) - y_s\right] + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}\left(V_{s'}^f - V_s^f\right) + \dot{V}_s^f \right\}. \tag{85}$$

The seller's acceptability decision is represented in the right side of (85) by the choice of $\varkappa \in [0,1]$. Let $\underline{m}$ denote the value of real asset holdings such that

$$\underline{m} - p^{-1}(\underline{m}) = \frac{k}{\alpha}.$$

Since $k/\alpha < p(y^*) - y^*$, $\underline{m} \in (0, p(y^*))$. It represents the value of the payment, $m = p(y)$, such that the seller's expected surplus, $\alpha\left[p(y) - y\right]$, is exactly equal to k. Provided that the seller's surplus is increasing in m, if $m > \underline{m}$, then the seller has strict incentives to invest in the technology to accept the asset or, equivalently, to participate in the market. At the opposite, if $m < \underline{m}$, the seller is better-off by not accepting the asset, or not participating. The seller's decision to accept the asset is then

$$\varkappa \begin{cases} = 1 & \\ \in [0,1] & \text{if} \\ = 0 & \end{cases} \begin{matrix} m > \underline{m} \\ m = \underline{m} \\ m < \underline{m} \end{matrix}. \tag{86}$$

According to (86), the seller invests in the technology to accept the asset if buyers' real asset holdings are sufficiently large. The aggregate acceptability correspondence, $\chi(m) = \int_0^1 \varkappa(j)dj$, is equal to 1 if $m > \underline{m}$, 0 if $m < \underline{m}$, and $\chi(\underline{m}) = [0,1]$. The value function of the buyer solves (4) when α is replaced with $\alpha\chi$. The ODE for m is

$$\rho - \frac{\dot{m}_s + \sum_{s' \in S \setminus \{s\}} \lambda_{ss'}\left(m_{s'} - m_s\right) + dM}{m_s} = \alpha\chi(m_s)L(m_s). \tag{87}$$

An equilibrium is a list of time paths, $(m_{s,t})$, solution to (87).

A steady state is a m^* solution to

$$\rho - \frac{dM}{m^*} = \alpha\chi(m^*)L(m^*).$$

Let m_1^* denote the solution to this equation when $\chi = 1$. If $\underline{m} \in (dM/\rho, m_1^*)$, then there are three steady states: dM/ρ, $\underline{m}$, and m_1^*. We illustrate the determination of the steady states in the left panel of Figure 16. The red curve represents the liquidity value of the asset while the blue curve represents its holding cost. If $m < \underline{m}$, the asset is not accepted and hence its liquidity value is zero – the red curve coincides with the horizontal axis. If $m = \underline{m}$, acceptability can be anything in $[0,1]$, hence the red curve is vertical. If $m > \underline{m}$, the asset is universally accepted, in which case the liquidity premium is positive and decreases with m until m reaches $p(y^*)$. The blue curve is upward sloping because as m increases, the return of the asset decreases and its holding cost increases. The three intersections between the red and blue curves represent the three steady-state equilibria.

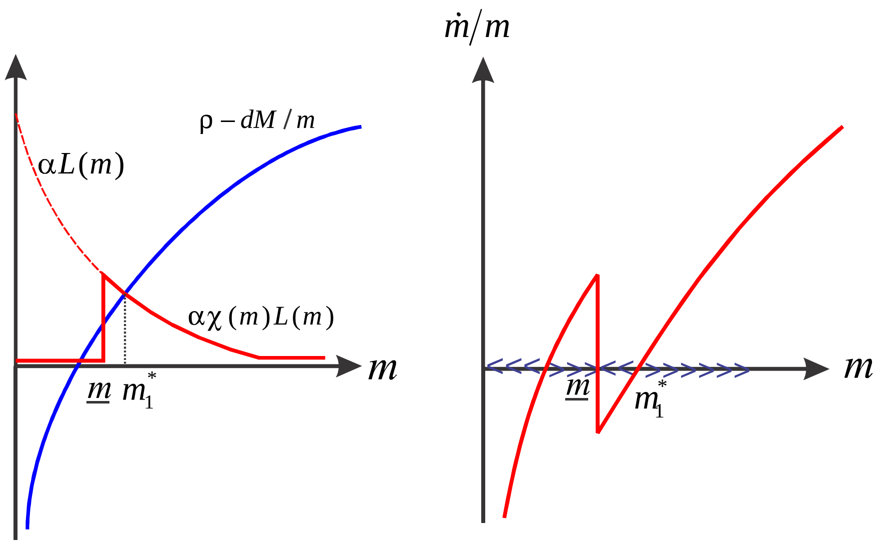

Figure 16: Deterministic equilibria with endogenous acceptability

If we don't impose stationarity, a deterministic equilibrium is a m_t solution to

$$\frac{\dot{m}}{m} = \rho - \alpha\chi(m)L(m) - \frac{dM}{m}. \tag{88}$$

The phase diagram corresponding to this ODE is represented in the right panel of Figure 16. The red curve corresponds to the right side of (88). For all $m < \underline{m}$, $\chi(m) = 0$ and the red curve is the spread between the rate of time preference and the ratio of the dividend over the asset price. It is equal to zero at the fundamental value, $m = dM/\rho$. For all $m > \underline{m}$, $\chi(m) = 1$ and the red curve is the difference between the spread, $\rho - dM/m$, and the liquidity premium, $\alpha L(m)$. It is equal to zero at the steady-state value, m_1^*. From the phase diagram, if $\underline{m} \in (dM/\rho, m_1^*)$, then there are a continuum of equilibria indexed by $m_0 \in (dM/\rho, m_1^*)$ that converge to $\underline{m}$.

We now move to the construction of two-state, stationary, sunpost equilibria, (m_ℓ, m_h), where the acceptability of the asset varies between 0 and 1. It solves:

$$\rho - \frac{\lambda_{h\ell}(m_\ell - m_h) + dM}{m_h} = \alpha L(m_h) \tag{89}$$

$$\rho - \frac{\lambda_{\ell h}(m_h - m_\ell) + dM}{m_\ell} = 0. \tag{90}$$

In the high state, the asset is accepted with probability one and hence it pays a liquidity premium on the right side of (89). In the low state, the asset is not accepted and hence it pays no liquidity premium.

Proposition 11 ***(Sunspot equilibria with endogenous acceptability.)*** *Suppose* $\underline{m} \in (dM/\rho, m_1^*)$. *For all* $m_h \in (\underline{m}, m_1^*)$ *and* $m_\ell \in (dM/\rho, \underline{m})$ *there exists* $(\lambda_{h\ell}, \lambda_{\ell h}) \in \mathbb{R}_{2+}$ *such that* (m_ℓ, m_h) *is a sunspot equilibrium.*

Proof. From (89)-(90),

$$\begin{aligned}\lambda_{h\ell} &= \frac{\rho m_h - \alpha L(m_h) m_h - dM}{m_\ell - m_h} \\ \lambda_{\ell h} &= \frac{\rho m_\ell - dM}{m_h - m_\ell}.\end{aligned}$$

The quantity, m_1^*, is defined as steady-state market capitalization when the asset is universally accepted. Hence, m_1^* solves

$$\rho = \alpha L(m_1^*) + \frac{dM}{m_1^*}.$$

Using that the right side is decreasing in m_1^* and $m_h < m_1^*$, it follows that numerator of $\lambda_{h\ell}$ is negative. Hence, $\lambda_{h\ell} > 0$. From $m_\ell > dM/\rho$, the numerator of $\lambda_{\ell h}$ is positive and hence $\lambda_{\ell h} > 0$. ■

Appendix E: More on multiple asset economies

In this appendix we consider two extensions of the model with multiple liquid assets. First, we consider an economy with multiple currencies. Second, we describe an economy with two Lucas trees that are subject to their specific, endogenous pledgeability constraints.

Extension #1: Multiple-currency economies

Consider the case of fiat monies, $d(j) = 0$. Deterministic equilibria are analogous to the ones of the economy with Lucas trees except that the composition of m between different currencies is indeterminate, i.e., the exchange rates between the currencies are not pinned down by the model. Aggregate real balances, $m \equiv \int_0^1 m(j)dj$, solve the following ODE:

$$\frac{\dot{m}}{m} = \rho - \alpha L(m).$$

By the same logic as in the one-currency economy, there are a continuum of equilibria where all currencies lose their value asymptotically.

Consider an equilibrium where aggregate real balances are at their steady-state value, $m = m^*$. Suppose currency j's price responds to extrinsic uncertainty. In state h, currency j has a positive value while in the absorbing state ℓ its value is zero. Then,

$$\rho - \frac{\dot{m}_h(j) - \lambda_{h\ell} m_h(j)}{m_h(j)} = \alpha L(m^*). \tag{91}$$

The liquidity premium on the right side of (91) is constant because the supply of currency j is of measure 0. Hence,

$$\frac{\dot{m}_h(j)}{m_h(j)} = \lambda_{h\ell}.$$

The value of currency j appreciates at rate $\lambda_{h\ell}$ before it collapses at the same rate, $\lambda_{h\ell}$. Because currency j does not affect aggregate liquidity, real allocations are invariant to the sunspots.

We now turn to the case of a dual currency economy composed of currencies, 1 and 2, with respective supplies, M^1 and M^2. The price of each currency is now relevant for aggregate liquidity. The fact that aggregate liquidity is composed of two distinct monies allows for new equilibria. We illustrate this point by constructing a nonstationary, two-state, $S = \{\ell, h\}$, sunspot equilibrium where only currency 1 becomes valueless in state ℓ. The initial state is $s_0 = h$ and ℓ is an absorbing state, i.e., $\lambda_{h\ell} > 0$ and $\lambda_{\ell h} = 0$. When state ℓ occurs, $m^1 = 0$ and $m^2 = m^*$. In the initial state h, (m_h^1, m_h^2), solves:

$$\rho - \frac{\dot{m}_h^1 - \lambda_{h\ell} m_h^1}{m_h^1} = \alpha L(m_h) \tag{92}$$

$$\rho - \frac{\dot{m}_h^2 + \lambda_{h\ell}\left(m^* - m_h^2\right)}{m_h^2} = \alpha L(m_h). \tag{93}$$

According to the left side of (92), the rate of return of currency 1 is composed of the change in its value over time, $\dot{m}_h^1$, and the capital loss when state ℓ occurs, $\lambda_{h\ell} m_h^1$. According to (93), the rate of return of currency

2 is composed of its appreciation over time, $\dot{m}_h^2$, and its capital gain when state ℓ occurs and currency 2 becomes the only valued currency, $\lambda_{h\ell}\left(m^* - m_h^2\right)$.

Proposition 12 ***(Nonstationary sunspot equilibria with dual currencies.)*** *Suppose $S = \{\ell, h\}$ where h is the initial state, $\lambda_{h\ell} > 0$ and $\lambda_{\ell h} = 0$. There exist a continuum of sunspot equilibria where initial aggregate real balances are $m_{h,0} < m^*$, the path of $m_{h,t}^1$ is nonmonotone along some equilibria, $\lim_{t\to\infty} m_{h,t}^1 = \lim_{t\to\infty} m_{h,t}^2 = 0$. In the absorbing state ℓ, $m_\ell^2 = m^*$ and $m_\ell^1 = 0$.*

Proof. Summing (92) and (93), the ODE for total real balances in the high state is:

$$\rho m_h - \dot{m}_h - \lambda_{h\ell}(m^* - m_h) = \alpha m_h L(m_h). \tag{94}$$

The phase diagram of the system of ODEs composed of (93) and (94) that describes total real balances and real holdings of currency 2 in the high state, $(m_{h,t}, m_{h,t}^2)$, is represented in the left panel of Figure 17. From (94) the locus of the points such that $\dot{m}_h = 0$ is $m_h = m^*$ where $\rho = \alpha L(m^*)$. From (93) the locus of the points such that $\dot{m}_h^2 = 0$ is

$$\rho + \lambda_{h\ell} - \frac{\lambda_{h\ell} m^*}{m_h^2} = \alpha L(m_h).$$

It gives a negative relationship between m_h^2 and m_h. The phase diagram of the system of ODEs composed of (92) and (94) that describes $(m_{h,t}, m_{h,t}^1)$ is represented in the right panel of Figure 17. From (92) the locus of the points such that $\dot{m}_h^1 = 0$ is given by $\rho + \lambda_{h\ell} = \alpha L(m_h)$. It is represented by an horizontal line below m^*. As shown in the left panel of Figure 17, if $m_{h,t} < m^*$, then $m_{h,t}$ and $m_{h,t}^2$ are decreasing over time and converge to 0. From the phase diagram in the right panel of Figure 17, $m_{h,t}^1$ is nonmontone if $m_{h,0}^1$ is small and $m_{h,0}^2$ is close to m^*: it increases first and then it decreases.

The bottom panel of Figure 17 represent the dynamic system in terms of $(m_{h,t}^1, m_{h,t}^2)$, i.e.,

$$\begin{aligned} \rho - \frac{\dot{m}_h^1 - \lambda_{h\ell} m_h^1}{m_h^1} &= \alpha L(m_h^1 + m_h^2) \\ \rho - \frac{\dot{m}_h^2 + \lambda_{h\ell}\left(m^* - m_h^2\right)}{m_h^2} &= \alpha L(m_h^1 + m_h^2). \end{aligned}$$

The m_h^1-isocline, $\dot{m}_h^1 = 0$, is:

$$m_h^1 + m_h^2 = L^{-1}\left(\frac{\rho + \lambda_{h\ell}}{\alpha}\right).$$

Hence, the m_h^1-isocline is downward-sloping with slope -1. Above the m_h^1-isocline, $\dot{m}_h^1 > 0$ while below the m_h^1-isocline, $\dot{m}_h^1 < 0$. The m_h^2-isocline, $\dot{m}_h^2 = 0$, is:

$$\rho + \lambda_{h\ell} - \frac{\lambda_{h\ell} m^*}{m_h^2} = \alpha L(m_h^1 + m_h^2).$$

Note that for all $m_h^2 > 0$, the m_h^2-isocline is located above the m_h^1-isocline. Moreover, the m_h^2-isocline is downward-sloping. If $m_h^2 = \lambda_{h\ell} m^*/(\rho + \lambda_{h\ell})$, then $m_h^1 \geq p(y^*) - m_h^2$ and the m_h^2-isocline is vertical in the space (m_h^2, m_h^1). Above the m_h^2-isocline, $\dot{m}_h^2 > 0$ while below the m_h^2-isocline, $\dot{m}_h^2 < 0$. One can check from this phase diagram that $m_{h,t}^1$ is nonmontone if $m_{h,0}^1$ is small and $m_{h,0}^2$ is close to m^*. ■

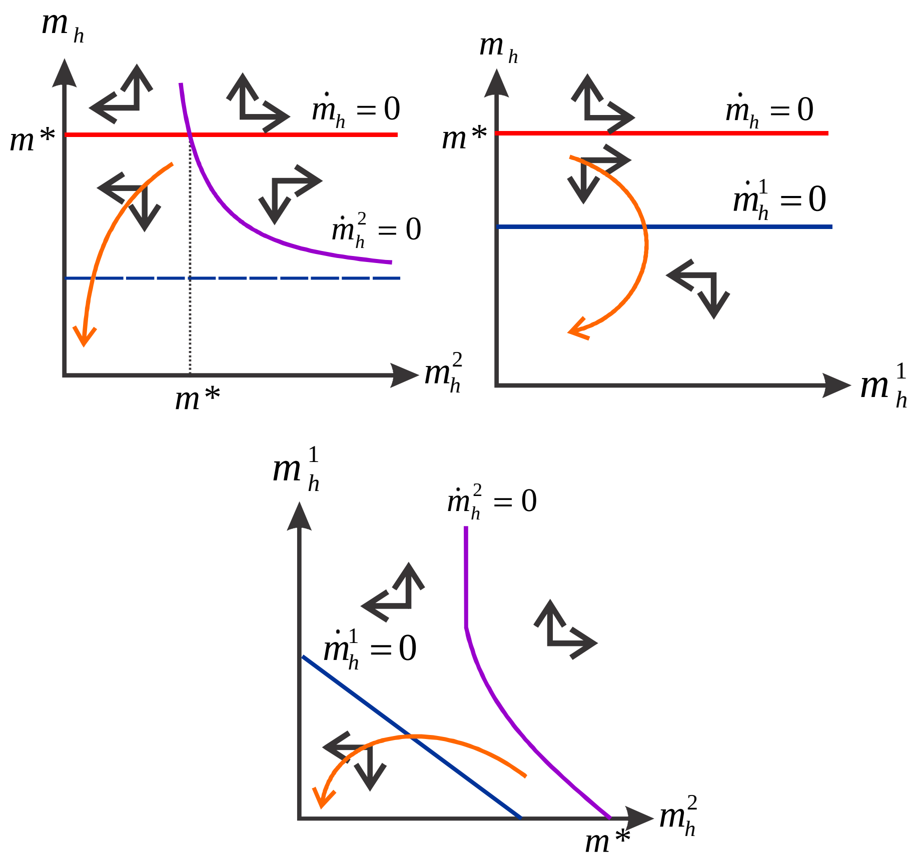

Figure 17: Dual currency sunspot equilibrium. Phase diagram of the high state. In the low state currency 1 becomes valueless.

In the equilibria described in Proposition 12, only a fraction of the liquid assets lose their value in an absorbing sunspot state. Because liquid assets are substitutes as means of payment, changes in the price of one liquid asset affect the prices of all liquid assets. The phase diagrams in Figure 17 represent the system (92)-(93) in different spaces for the endogenous variables, $(m^1_{h,t}, m^2_{h,t}, m_{h,t})$. The top left panel shows that $m^2_{h,t}$ and $m_{h,t}$ are decreasing over time while the top right and bottom panels show that $m^1_{h,t}$ is non-monotone. A key insight of Proposition 12 is the possibility of non-monotone price trajectories that did not exist in the one asset case.

Endogenous differences in asset liquidity

We now consider the case of two long-lived assets where $d^j > 0$ and $\delta^j = 0$ for $j \in \{1, 2\}$. As in Section 6, that each asset j can be counterfeited at a cost equal to a fraction κ^j of the value of a genuine asset. A buyer's offer is now a list, (y, p^1, p^2, m^1, m^2), where output y is paid for with p^1 units of asset 1 (expressed in terms of numéraire) and p^2 units of asset 2 drawn from a portfolio composed of m^1 units of asset 1 and

m^2 units of asset 2 (expressed in the numéraire.) The no-fraud constraint can be generalized as

$$-(\rho - r_s^1)m^1 - (\rho - r_s^2)m^2 + \alpha\left[u(y) - p^1 - p^2\right] \geq -(\rho - r_s^{j'})m^{j\prime} + \alpha\left[u(y) - \kappa^j m^j - p^{j'}\right],$$

for all $j \in \{1,2\}$ and $j' \neq j$. The left side of the inequality is the net expected surplus of the buyer if he acquires genuine assets. The right side is a deviation where only asset j is fraudulent. The constraint can be reexpressed as $p^j \leq \chi_s^j m^j$ where

$$\chi_s^j = \left[\kappa^j - \frac{\rho - r_s^j}{\alpha}\right]^+. \tag{95}$$

As in the one asset case, pledgeability increases with the rate of return of the asset.

Consider first steady-state equilibria. An equilibrium is a pair, (m^1, m^2), solution to

$$\rho - r^j = \alpha\chi^j L(\chi^1 m^1 + \chi^2 m^2) \tag{96}$$

$$\text{with } r^j = \frac{d^j M^j}{m^j} \text{ and } \chi^j = \left[\kappa^j - \frac{\rho - r^j}{\alpha}\right]^+.$$

If $\kappa^j > \alpha/\rho$ for $j = 1, 2$, i.e., both assets are sufficiently costly to counterfeit, then there exists a unique steady state. Otherwise, there might exist multiple steady states. We illustrate this possibility with a numerical example. The utility function is $u(y) = [(y+b)^{1-a} - b^{1-a}]/(1-a)$, with $a = 2$ and $b = 0.1$. We set $\rho = 0.04$ and $\alpha = 1$. The two assets are characterized by $d^1M^1 = d^2M^2 = 0.8$, $\kappa^1 = 0.025$ and $\kappa^2 = 0.02$. In Figure 18, we plot the asset pricing condition, (96), for both assets. There are three steady states across which m^1 and m^2 are positively correlated and m^j and χ^j are negatively correlated.[33] In the highest steady state, asset pledgeability is close to 0 but agents compensate for the low pledgeability by accumulating large amount of the asset. So, paradoxically, asset prices are very high even though the assets are almost illiquid.

One can use the multiplicity of steady states to construct stationary sunspot equilibria. The rate of return of the asset is now

$$r_s^j = \frac{d^j M^j + \sum_{s' \in S\setminus\{s\}} \lambda_{ss'}\left(m_{s'}^j - m_s^j\right)}{m_s^j}.$$

As an example, suppose sunspot states are drawn from $\{\ell, h\}$. If $(\lambda_{\ell h}, \lambda_{h\ell}) = (0.1, 0.2)$, then there exists a sunspot equilibrium with the outcome in each state, $O_s \equiv (m_s^1, m_s^2, \chi_s^1, \chi_s^2)$, being

$$O_\ell = (48.71, 37.86, 0.00047, 0.00037), \quad O_h = (48.25, 37.58, 0.0035, 0.0028).$$

In this example, asset prices jump up when assets become almost illiquid. Sunspot equilibria exist also in cases where the steady state is unique. Suppose $(\kappa^1, d^1M^1) = (0.6, 0.0262)$ and $(\kappa^2, d^2M^2) = (0.5, 0.0262)$. There is a unique steady state where $(m^1, m^2, \chi^1, \chi^2) = (0.84, 0.80, 0.59, 0.49)$. If $(\lambda_{\ell h}, \lambda_{h\ell}) = (0.413, 1.586)$, then there exists a sunspot equilibrium where

$$O_\ell = (0.85, 0.81, 0.6, 0.5), \quad O_h = (0.87, 0.82, 0.56, 0.46).$$

[33] If we define a steady state by a list $\mathcal{E} = (m^1, m^2, \chi^1, \chi^2)$, then the three steady states are $\mathcal{E}_1 = (20, 20, 0.025, 0.02)$, $\mathcal{E}_2 = (36.46, 31.31, 0.0069, 0.0056)$, and $\mathcal{E}_3 = (51.61, 39.21, 5 \times 10^{-4}, 4 \times 10^{-4})$.

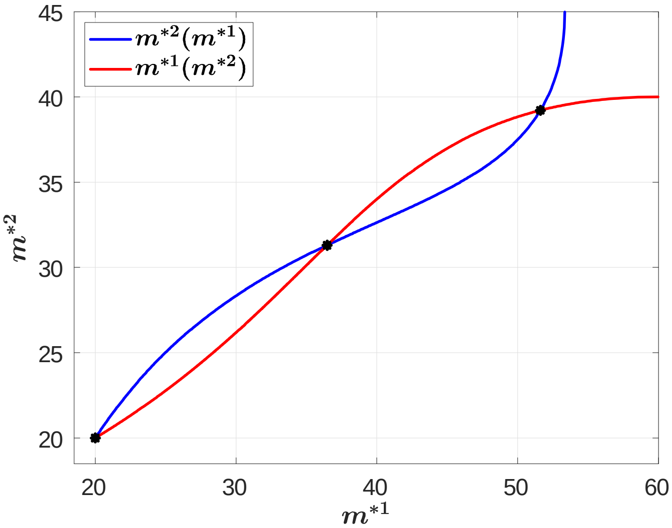

Figure 18: Dual asset economy under the threat of fraud: Steady-state equilibria.

In the ℓ-state, both asset prices and pledgeability are larger for both assets than their steady-state values. In the h-state, asset prices are even higher than the ones in the ℓ-state, but the pledgeability ratios fall below their steady-state values for both assets.