Sequential Rationality and Ordinal Preferences*

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Abstract

Given a dynamic game with ordinal preferences, we deem a strategy sequentially rational if there exist a von Neumann-Morgenstern utility function that agrees with the assumed ordinal preferences and a conditional probability system with respect to which the strategy is a maximizer. We prove that this notion of sequential rationality is characterized by a notion of dominance, called Conditional B-Dominance, that extends Pure Strategy Dominance of Börgers (1993) to dynamic games represented in their extensive form. Additionally, we introduce an iterative procedure based on Conditional B-Dominance with a forward induction reasoning flavour, called Iterative Conditional B-Dominance, that we prove: (*i*) satisfies nonemptiness; (*ii*) algorithmically characterizes an 'ordinal' version of Strong Rationalizability *à la* Pearce (1984) and Battigalli (1997); (*iii*) selects the unique backward induction outcome in dynamic games with perfect information that satisfy the genericity condition called "No Relevant Ties". Finally, we show how our results on Iterative Conditional B-Dominance allow a 'forward induction reasoning' interpretation of the unique backward induction outcome obtained in binary agendas with sequential majority voting.

Keywords: Dynamic Games, Ordinal Preferences, Sequential Rationality, Forward Induction, Conditional B-Dominance, Iterative Conditional B-Dominance.
JEL Classification Number: C63, C72, C73.

1. Introduction

1.1 Motivation & Results

Imagine that you see the choices made by players in a dynamic strategic interaction: as an observer, when could you say that those players acted rationally? This rather natural question potentially arises every time we face an experiment based on a dynamic game: which behavior observable by the experimenter is compatible with the idea that her subjects are acting rationally?

Clearly, at the core of the problem lies the notion of rationality we employ in non-cooperative strategic interactions. According to the so called *Bayesian approach*, rationality in games has to be *explicitly*[1] defined as subjective expected utility maximization: given a game, a player is *rational* if she treats the strategic uncertainty present in the game like a decision maker who obeys the axioms of Savage (1954) approaches a (nonstrategic) decision problem under uncertainty. Hence, a rational player should choose from her set of strategies the strategy that maximizes her subjective expected utility, i.e., a strategy that maximizes her expected utility according to a subjective belief

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[1] An alternative way, proper of the *Refinement Program* initiated by Selten (1965) and of the *Stability Program* of Kohlberg & Mertens (1986), is to *not* define what rationality is, in order to capture it indirectly by means of the definition of a solution concept: in the specific case of the Stability Program, this solution concept should satisfy all those basic tenets that we intuitively associate to the idea of rational behavior (see Mertens (1989, Section 1.6)). Brandenburger (2007, Section 14, p.490) calls the Bayesian approach (related to the *Epistemic Program*) the *bottom-up* approach to rationality, as opposed to the *top-down* approach exemplified by the Refinement/Stability Programs.

she has formed regarding the strategies chosen by her opponents. Thus, two elements need to be present in order to meaningfully talk about the rationality of a player in a Bayesian context: a (cardinal) *utility function* and a *subjective belief*.

As a matter of fact, typically, to perform a game-theoretical analysis, we assume to have knowledge on our side—as analysts—of the players' von Neumann-Morgenstern utility functions,[2] which also encompass the players' risk attitudes. However, such assumption is unrealistic, since—quite often—we do not have the luxury to know the actual utility functions of the players and their risk attitudes in a game whose play we decide to observe. Also, an additional—important—point is in order: if we happen to know them in a decision-theoretical setting, then it is not necessarily the case that they can be adopted *as such* in a game-theoretical context as well.[3]

Given what is written above, whereas it seems that rather often we do not have all the elements (e.g., the utilities) needed to proceed with well-grounded inferences regarding the players' rationality, it would be important to still be in position to say something concerning this issue. Thus, the purpose of this paper is to provide a new—natural—notion of sequential rationality that bypasses the problems pointed out above and that is consistent with the usage of the expression in the game-theoretical literature.[4] For this purpose, we simply assume on our side knowledge of the players' *ordinal* preferences over outcomes and we extend the results in Börgers (1993) to dynamic games. Indeed, by taking as given and transparent[5] only players' ordinal preferences over outcomes, Börgers (1993) deems a strategy of a player *rational* if there exist a von Neumann-Morgenstern utility function that agrees with the assumed ordinal preferences and a probability measure over the strategies of the opponents according to which that strategy maximizes the player's expected utility. It is shown in the paper that a strategy is rational (according to this definition) if and only if it is not pure strategy dominated, i.e., we can find a subset of the strategies of the opponents relative to which the strategy is not weakly dominated by any pure strategy. Thus, in the present contribution, by assuming (again) only players' *ordinal* preferences over outcomes as transparent, in Definition 3.2 we deem a strategy of a player *sequentially rational* if there exist a von Neumann-Morgenstern utility function that agrees with the assumed ordinal preferences and a conditional probability system[6] over the strategies of the opponents according to which that strategy maximizes the expected utility of the player. Then we show in Proposition 1 that the above defined sequential rationality is characterized by a form of dominance, that we call Conditional B-Dominance (see Definition 2.1).

Two points are in order concerning Conditional B-Dominance. First of all, it is an *ordinal* concept: it does not involve any reference to cardinal utility as in von Neumann & Morgenstern (1953) and—as a byproduct—probability measures (representing either subjective beliefs or mixed strategies) play no role in this context, a crucial aspect that captures in what sense we broadly use the term "ordinal" in this paper. In second place, incidentally, in deriving from ordinal preferences both a (cardinal) utility function and a conditional probability system via our dominance notion, we perform an analog of the exercise of Savage (1954) in its derivation of a utility function and a belief starting from the purely ordinal concept of a preference relation.

Armed with our notion of Conditional B-Dominance as in Definition 2.1, it is natural to ask ourselves what are the behavioral predictions that we obtain by iterating this dominance notion, with the caveat that—to perform this analysis—we additionally have to assume transparency between the players of their ordinal preferences. However, it is crucial to notice that, as natural as this question is, in order to answer it, we have to take into account that there are different ways in which players can reason in a dynamic game. By polarizing these alternatives, we typically have two extreme scenarios: a player can reason according to *forward* induction (i.e., she is going to make deductions based on the opponents' rational behavior in the past)[7] or she can reason according to *backward* induction (i.e., she is going to be completely forward looking, dismissing

[2]See Footnote 28 in Section 3 for a clarification of the terminology employed.

[3]A similar point is made in Gilboa & Schmeidler (2003, Section 1.1, pp.185-186): *"If one were to measure a player's utility over such outcomes in a laboratory, one would have to generate outcomes that simulate all the interactive effects of a game. That is, one would have to measure utility in the context of the game itself"*.

[4]The standard notion of rationality employed in the literature is that of *sequential rationality* introduced by Kreps & Wilson (1982, Section 4, p.872), with the caveat that Kreps & Wilson (1982) assume players to have *correct beliefs* concerning the play, due to their working on an equilibrium-based solution concept.

[5]That is, *common knowledge* in the informal sense of the term.

[6]See Section 3 for the corresponding definition and Footnote 29 therein for references.

[7]Alternatively, forward induction can be informally captured via the *Best Rationalization Principle* of Battigalli (1996, Section 1.1, p.180).

altogether the opponents' behavior in the past, with deductions based on the opponents' rational behavior in the future) (see Kohlberg (1990, Section 2.4, p.8)). Both ways of reasoning have a counterpart in game-theoretical solution concepts and related algorithmic procedures:[8]

- forward induction is captured by Strong Rationalizability[9] of Pearce (1984, Definition 9, p.1042) and Battigalli (1997, Definition 2, p.46), whose algorithmic counterpart is the Iterated Conditional Dominance of Shimoji & Watson (1998, Section 3, pp.170-171);[10]

- backward induction is captured by Backward Rationalizability of Penta (2015, Appendix B, pp.302-303) or by the related algorithm known as Backward Dominance Procedure of Perea (2014, Algorithm 5.1, p.240).[11]

Thus, building on Conditional B-Dominance, we define an algorithmic procedure in Algorithm 1, called "Iterative Conditional B-Dominance" (henceforth, ICBD), that, by mimicking Iterated Conditional Dominance, captures forward induction reasoning in our purely 'ordinal' framework. Hence, whereas we show in Proposition 2 that ICBD satisfies nonemptiness, we also define in Algorithm 2 an 'ordinal' version of Strong Rationalizability, called "Ordinal Strong Rationalizability", that we prove in Proposition 3 is algorithmically characterized by ICBD. Additionally, given the conceptual link between ICBD and Iterated Conditional Dominance, we investigate the relation between the two. With respect to this point, we distinguish the following two cases:

- when we fix a utility function and we treat it alternatively as ordinal and cardinal, we have that the set of strategy profiles surviving a step of Iterated Conditional Dominance is necessarily a subset of the set of strategy profiles surviving a step of ICBD (as in Proposition 4), whereas the opposite inclusion cannot be established (see Example 5);

- when we do not fix a (cardinal) utility function and we contemplate as possible *all* the (cardinal) utility functions that are compatible with the assumed ordinal preferences, we have for every player that for every strategy that survives a step of ICBD there exists a (cardinal) utility function such that the aforementioned strategy survives a step of Iterated Conditional Dominance defined on that utility function and vice versa (see Lemma 5.1).

Given that this paper revolves around the idea of dropping the assumption of having access to the players' (cardinal) utility functions with related knowledge of their risk attitudes, it is most natural on our side to investigate if ICBD—as an ordinal solution concept—behaves like its cardinal counterparts in the context of dynamic games satisfying a genericity condition that makes knowledge of risk attitudes essentially irrelevant. As a consequence, we establish in Corollary 2 an ordinal analog of the so called Battigalli's theorem (see Battigalli (1997, Theorem 4, p.53)), namely, that ICBD selects the unique backward induction outcome in dynamic games with perfect information that satisfy a genericity condition known as the "No Relevant Ties" condition: a particularly appealing result, since—to the best of our knowledge—it is the first instance in which the connection between backward and forward induction reasoning is established in an ordinal setting for this class of games, where—indeed—risk attitudes should not play any role.

Since the proof of Corollary 2 relies on the more general Proposition 6 which concerns all dynamic games with perfect information that satisfy the so called "Transference of Decision-Maker Indifference" condition of Marx & Swinkels (1997, Equation 2, p.223) as in Definition 6.1,[12] it is already possible to see various potential applications for ICBD. Indeed, the class of games

[8] They are also additionally related to the fact that the epistemic notion of *Rationality and Common Belief in Rationality* can assume in dynamic games different forms. Regarding this point, see Section 8.4 and Section 8.6.

[9] A common name for this solution concept—only hinted in Pearce (1984), but adopted in Battigalli (1997) and subsequent work—is "Extensive-Form Rationalizability". Here we adopt a terminology introduced in Battigalli (1999), since it distinguishes it from other forms of Rationalizability that can be formalized for the analysis of dynamic games in their extensive form (i.e., Initial (or Weak) Rationalizability *à la* Ben-Porath (1997) and Backward Rationalizability *à la* Penta (2015)).

[10] See also Perea (2014, Algorithm 7.3, p.247) for a definition phrased in terms of the primitive objects belonging to the extensive form representation of a dynamic game, whereas the original definition is stated in a more general setting.

[11] See Section 8.6 for a discussion of the relations between Backward Rationalizability and the Backward Dominance Procedure.

[12] In particular (as it should be, given that our result is purely ordinal) on those results belonging to Marx & Swinkels (1997, Section IV), which deals *only* with dominance via *pure* strategies.

satisfying Definition 6.1 contains—for example—first price auctions and all games in which at some point in the interaction a player has the possibility to end the game choosing an outside option.[13] Nonetheless, in Section 7, we provide a careful study of an application of ICBD to voting theory, where the ordinal nature of our exercise is particularly appealing.[14] In particular, we introduce a framework to capture binary agendas (with sequential majority voting) and we show in Proposition 7 how our Proposition 6 happens to link *sophisticated voting à la* Farquharson (1969) to forward induction reasoning.[15]

Thus, going back to the scenario depicted in our incipit, what do the paragraphs above tell us? Take an experimenter that accepts our definition of sequential rationality and that is involved in running an experiment based on a dynamic game. By simply assuming knowledge of the subjects' ordinal preferences over the outcomes of the dynamic game implemented in the experiment, the experimenter can infer if a subject acted rationally. Also, by assuming transparency between the players of their ordinal preferences over the outcomes of the game, the experimenter can check if an outcome is compatible with forward induction reasoning.

1.2 Related Literature

This work is related to various papers belonging to different streams of literature. In its focus on providing a *Bayesian foundation* to the notion of sequential rationality, whereas it is—of course—related to Börgers (1993), it is also related to the Siniscalchi (2021). In its desire to provide a *decision-theoretical foundation* to utilities as employed in game theory, it is related to Gilboa & Schmeidler (2003) and Perea (2021).[16] Additionally, since we put a special emphasis on the fact that our exercise is grounded on the idea of dispensing a game-theoretical analysis from common knowledge assumptions of *players' risk attitudes*, it is related to Fishburn (1978) (see also Perea et al. (2006)). Concerning our introduction of an *iterative procedure* based on the dominance notion we introduce, i.e., Conditional B-Dominance, it is related to Shimoji & Watson (1998). Also, there is an immediate link to Battigalli (1997) with respect to the problem of establishing a relation between backward and forward induction reasoning via the *unique backward induction outcome* in dynamic games with perfect information satisfying the genericity condition known as "No Relevant Ties". Finally, with respect to the literature on *voting in binary agendas*, it is related to the seminal Farquharson (1969) (along with McKelvey & Niemi (1978), Moulin (1979), Gretlein (1983), and Sloth (1993)), where it has to be emphasized its relation to De Sinopoli (2004) for its focus on forward induction reasoning in voting games.

1.3 Synopsis

In Section 2, we introduce the primitive objects of our analysis, namely, dynamic games with ordinal preferences, we recall the definitions of various dominance notions for games in strategic form, and we formalize the dominance notion for dynamic games in their extensive form representation that we study in this paper, i.e., Conditional B-Dominance. In Section 3, we introduce our notion of sequential rationality and we show that a strategy that is sequentially rational according to our definition is not conditionally B-dominated and vice versa. In Section 4, we introduce ICBD, i.e., the algorithm that iteratively eliminates conditionally B-dominated strategies, we study the properties it satisfies, and we prove that it algorithmically characterizes a form of Strong Rationalizability defined for ordinal games called "Ordinal Strong Rationalizability", whereas in Section 5 we study its relation to Iterated Conditional Dominance. Section 6 is devoted to investigate the behavioral implications of ICBD in dynamic games with perfect information which satisfy a genericity condition called "No Relevant Ties", whereas in Section 7 we show how our work sheds light on previous results belonging to the study of sequential majority voting in binary agendas. Finally, in Section 8, we address various issues related to our analysis. All the proofs of the results established in the paper are relegated to Appendix A.

[13]See Marx & Swinkels (1997, Section III, pp.224-225) for additional classes of games satisfying this condition.

[14]See for example Moulin (1979, Footnote 3, p.1338) and all the related literature reviewed in Section 1.2.

[15]Concerning forward induction and voting, see the analysis of De Sinopoli (2004) in the context of the model of representative democracy of Besley & Coate (1997).

[16]We are grateful to an anonymous referee for having pointed out the relation between this work and these two contributions.

2. Dynamic Ordinal Games

2.1 Primitive Objects

The primitive object of our analysis is a finite dynamic game with ordinal preferences[17] and perfect recall in its extensive form representation

$$\Gamma := \langle I, (A_i)_{i\in I}, X, Z, (H_i, \mathbf{S}_i, S_i)_{i\in I}, \zeta, (\precsim_i)_{i\in I}\rangle, \tag{2.1}$$

which we simply refer to as *dynamic game*, where this definition possibly allows for simultaneous moves.[18] In Equation (2.1), I denotes the set of players and A_i, with $i \in I$, is the set of *actions* of player i. The set X is the set of *histories*, where a history x is either the empty sequence $\langle\varnothing\rangle$ (i.e., the initial history), or it is a sequence $(a^1, \dots, a^K)$, where $a^k := (a^k_i)_{i\in I}$ with $a^k_i \in A_i$ for every $i \in I$ and for every $1 \le k \le K$. We let $A(x) := \prod_{i\in I} A_i(x)$ denote the set of actions available to the players at history x: given that we let $|Y|$ denote the cardinality of an arbitrary set Y, if $|A_i(x)| \ge 2$, then player i is *active* at history x, otherwise she is inactive. The set $Z \subseteq X$ is the set of *terminal* histories, alternatively called *outcomes*, i.e., the set of histories such that $|A(z)| = 0$, for every $z \in Z$. The set of histories X is an *arborescence* when endowed with a binary relation capturing the notion of precedence (see Kreps & Wilson (1982, Section 2, pp.865-866)).[19]

We let H_i denote the set of *information sets* of player i, i.e., this is the partition of all those non-terminal histories where player i is active, where the elements of H_i satisfy the property that, if $x, x' \in h$, with $h \in H_i$, then $A_i(x) = A_i(x')$. We extend to information sets the notational convention introduced for histories and we let $A_i(h)$ denote the set of actions available to player i at information set $h \in H_i$. Given that $H := \bigcup_{i\in I} H_i$, we let $I_h \subseteq I$ denote the set of players that are active at information set $h \in H$. A dynamic game is said to be with *observable actions* if h is a singleton, for every $h \in H$. A dynamic game with observable actions is said to be with *perfect information* if I_h is a singleton, for every information set $h \in H$.

A *standard strategy* of player i is a function $\mathbf{s}_i : H_i \to \bigcup_{\overline{h}\in H_i} A_i(\overline{h})$ such that $\mathbf{s}_i(h) \in A_i(h)$, for every $h \in H_i$. We let $\mathbf{S}_i$ denote the set of standard strategies of player i and $\mathbf{S}_i(h)$ denote the set of standard strategies of player i that reach information set $h \in H$. Thus, given a standard strategy $\mathbf{s}_i \in \mathbf{S}_i$, we let $H(\mathbf{s}_i)$ denote the set of information sets that are not precluded by $\mathbf{s}_i$. Given two standard strategies $\mathbf{s}_i, \mathbf{s}'_i \in \mathbf{S}_i$, they are deemed behaviorally equivalent if $H(\mathbf{s}_i) = H(\mathbf{s}'_i)$ *and* $\mathbf{s}_i(h) = \mathbf{s}'_i(h)$ for every $h \in H(\mathbf{s}_i) = H(\mathbf{s}'_i)$. A *strategy*[20] s_i of player i is a maximal set of behaviorally equivalent standard strategies. We let S_i denote the set of all strategies of player i and—following standard notational conventions—we let $S_{-i} := \prod_{j\in I\setminus\{i\}} S_j$, with $S := \prod_{j\in I} S_j$. We extend the conventions introduced for standard strategies to strategies, i.e., we let $H(s_i)$ denote the set of information sets compatible with s_i, with $H_i(s_i)$ and $H_{-i}(s_i)$ defined accordingly. Given a nonempty subset $R \subseteq S$, R is deemed a *restriction for player* i if $R := R_i \times R_{-i}$, where we omit the reference to the player when it is clear from the context. A restriction $R \subseteq S$ for player i is called a *product restriction* if $R_{-i} = \prod_{j\in I\setminus\{i\}} R_j$, i.e., when R has a product structure.[21]

We let $S_i(h)$ denote the set of strategies of player i that allow information set $h \in H$. For every $h \in H_i$, we call $S_{-i}(h)$ a *conditioning event for player* i, with $\mathcal{H}_i := \{\, S_{-i}(h) \mid h \in H_i \,\}$, and $S(h)$ a *conditional problem for player* i. Two points need to be emphasized concerning $S(h)$. First, from assuming perfect recall, we have that $S(h) = S_i(h) \times S_{-i}(h)$, for every $h \in H$. In second place, $S(h)$ does not need to have a product structure: indeed, this is the case when $S_{-i}(h)$ does not have a product structure.[22] Observe that the notational conventions introduced in the paragraph

[17]See Bonanno (2018) for a textbook focusing on games with ordinal preferences.

[18]For a similar definition, see Battigalli & Friedenberg (2012, Section 2), inspired by Osborne & Rubinstein (1994, Definition 200.1, Chapter 11.1.2), with the caveat that the latter does not explicitly allow for simultaneous moves (see Osborne & Rubinstein (1994, Chapter 6.3.2) for the corresponding extension).

[19]It should be observed that—rather often—in the literature it can be alternatively found the term "tree" in place of "arborescence" (as in Osborne & Rubinstein (1994, Chapter 6.1.2, p.92)). See Fudenberg & Tirole (1991, Chapter 3.3.1, pp.77-79) for an explanation of the difference between these two notions.

[20]Alternatively called "plan of actions" as in the words of Rubinstein (1991, Section 2) or "reduced strategy". We use the—nonstandard—expression "standard strategy" to refer to an element of $\mathbf{S}_i$, for an arbitrary player i, where usually such an object is simply called "strategy", since we want to save that term for the actual primitive objects of interests for our analysis, which are the elements of S_i.

[21]Concerning our usage of the term "restriction" with respect to the related literature, see Section 8.3.

[22]See Mailath et al. (1993, Section 3, p.283) for an example based on the (3-player) Selten's Horse to see the potential lack of product structure of $S_{-i}(h)$.

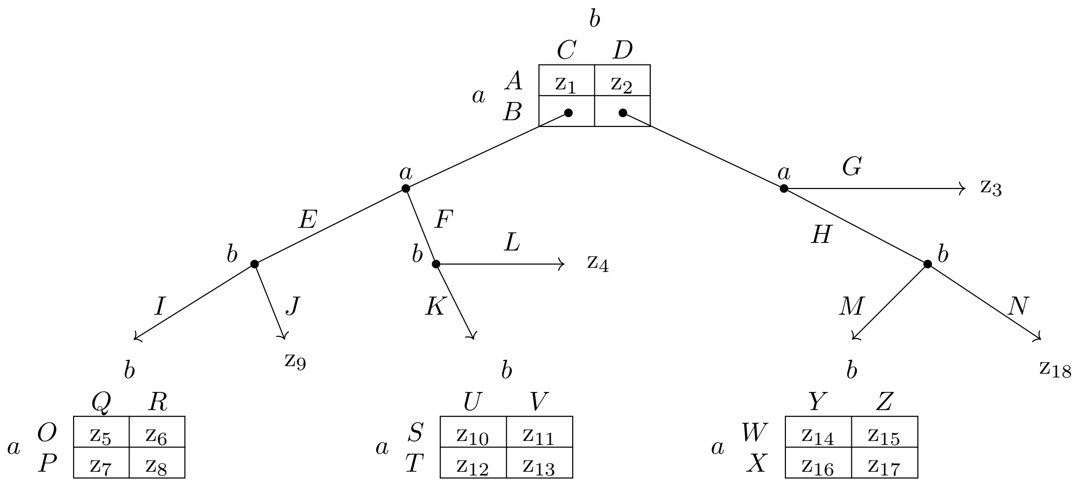


Figure 1: A dynamic game with simultaneous moves.

above extend naturally to $S_i(h)$, $S_{-i}(h)$, and $S(h)$. Also, given an arbitrary restriction $R \subseteq S$, we let $R_i(h) := R_i \cap S_i(h)$ and $R_{-i}(h) := R_{-i} \cap S_{-i}(h)$ (from what is written above, not necessarily a product restriction), with $R(h) := R_i(h) \times R_{-i}(h)$, for every $h \in H_i$, where all these sets can be empty.

Finally, we let $\zeta \in Z^S$ be an *outcome function* and, for every $i \in I$, we let $\precsim_i \subseteq Z \times Z$ be a preference relation over the outcomes of the game. As it is customary, we let $\succ_i$ (resp., $\sim_i$) denote the asymmetric (resp., symmetric) part of $\succsim_i$ and we let $\succsim := (\succsim_i)_{i \in I}$ denote a profile of preference relations of the players (where the symmetric and asymmetric parts are defined accordingly).

Example 1 (Dynamic Game with Simultaneous Moves). In order to build the intuition behind the objects introduced in this section, we focus on the dynamic game with simultaneous moves in Figure 1, whose rather involved structure proves to be useful inSection 2.3. This is a dynamic game with two players, namely, Ann (viz., a) and Bob (viz., b). Thus, for example, as instances of standard strategies, we have that $AEGOSW, BFHPTX \in \mathbf{S}_a$ and $CJKNQUY \in \mathbf{S}_b$. The corresponding strategies are simply $A, BFHTX \in S_a$ and $CJKU \in S_b$. Focusing on the players' conditioning events, for example we have $S_a(\langle BC \rangle) \in \mathcal{H}_a$ and $S_b(\langle DB, H \rangle) \in \mathcal{H}_b$. Now, given strategy $A \in S_a$, $H_a(A) = H_b(A) = \langle \varnothing \rangle$, while, given $BFHTX \in S_a$, we have both $\langle BC \rangle \in H_a(BFHTX)$ and $\langle BD \rangle \in H_a(BFHTX)$, since this strategy leaves open two possible paths: which one is going to be the actual one depends on the strategy taken by Bob at $S(\langle \varnothing \rangle)$. Regarding restrictions, for example, let $R := \{A, BFGS\} \times \{CILQ, DMY\}$. Then we have that $R(\langle BC, E \rangle) = \emptyset$, while $R(\langle BC, F \rangle) = \{BFGS\} \times \{CILG\}$. $\diamond$

2.2 Dominance Notions for the (Reduced) Strategic Form Representation

Given a dynamic game Γ, it is possible to bypass its sequential nature by means of what is called its *reduced strategic form*[23] (henceforth, strategic form), which is given by the tuple

$$\Gamma^r := \langle I, Z, (S_i), \zeta, (\succsim_i)_{i \in I} \rangle,$$

whose elements are defined as in Section 2.1.

Thus, fix a dynamic game Γ and focus on its strategic form. In particular, fix a player $i \in I$ and a restriction $R \subseteq S$. Then we say that a strategy $s_i \in R_i$ is *strictly dominated relative to* R_{-i} for player i by strategy $s_i^* \in R_i$ if $\zeta(s_i^*, s_{-i}) \succ_i \zeta(s_i, s_{-i})$ for every $s_{-i} \in R_{-i}$. A strategy $s_i \in R_i$ is *weakly dominated relative to* R_{-i} for player i by strategy $s_i^* \in R_i$ if $\zeta(s_i^*, s_{-i}) \succsim_i \zeta(s_i, s_{-i})$ for every $s_{-i} \in R_{-i}$ and there exists a $s_{-i}^* \in R_{-i}$ such that $\zeta(s_i^*, s_{-i}^*) \succ_i \zeta(s_i, s_{-i}^*)$. In the previous

[23] Traditionally, this is called the "*reduced* strategic form", since the relevant primitive object in the corresponding definition is what we call here "strategy" and is often called in the literature—as pointed out in Footnote 20—"reduced strategy".

cases we say that s_i^* strictly (resp., weakly) dominates s_i relative to R_{-i}. A strategy $s_i^* \in R_i$ that is not weakly dominated relative to R_{-i} is *admissible with respect to* $R_i \times R_{-i}$.[24]

Notation 1 (Set of Admissible Strategies with respect to a Restriction). We let $\mathsf{A}_i(R)$ denote the set of strategies of player i that are admissible with respect to $R \subseteq S$.

Börgers (1993, Definition 4, Section 3) introduced an additional dominance notion:[25] given a restriction $R \subseteq S$, strategy $s_i \in R_i$ is *pure strategy* or *Börgers dominated* (henceforth, B-dominated) *with respect to* R if $s_i \notin \mathsf{A}_i(R_i \times Q_{-i})$, for every nonempty subset $Q_{-i} \subseteq R_{-i}$. In general, strict dominance implies B-dominance, which in turn implies weak dominance, while the opposite directions do not hold.

Regarding all the dominance notions just introduced, it is understood that in the following the omission of the restriction with respect to which the dominance notion is defined stands for that restriction being equal to S.

Remark 2.1 (Singleton Sets). *Fix a player $i \in I$ and a restriction $R \subseteq S$. If a strategy $s_i \in R_i$ is weakly dominated relative to a singleton $Q_{-i} \subseteq R_{-i}$, then s_i is* strictly *dominated relative to Q_{-i}. Thus, a necessary condition for a strategy $s_i \in R_i$ to be B-dominated with respect to R is that for every singleton $Q_{-i} \subseteq R_{-i}$ there exists a strategy $s_i^* \in R_i$ that strictly dominates s_i relative to Q_{-i}.*

Notation 2 (Set of B-Dominated Strategies). Given a restriction $R \subseteq S$, we let $\mathsf{bd}_i(R)$ denote the set of strategies of player $i \in I$ that are B-dominated with respect to R.

Example 2 (Dynamic Game with Simultaneous Moves with an Outside Option). To see the dominance notions we have introduced at work, we take the dynamic game in Figure 2, with two players, again Ann (viz., a) and Bob (viz., b).

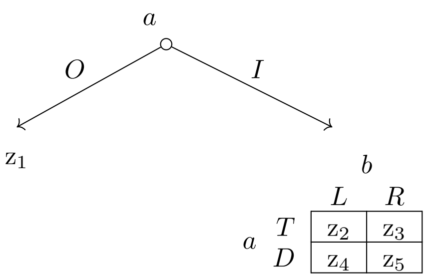


Figure 2: A dynamic game with simultaneous moves with an outside option.

We also provide its strategic form in Figure 3. We have that $S_a = \{O, IT, ID\}$ and $S_b = \{L, R\}$. Also, we posit that Ann has the following preferences over the terminal histories: $\mathrm{z}_2 \succ_a \mathrm{z}_1 \succ_a \mathrm{z}_5 \succ_a \mathrm{z}_3 \sim_a \mathrm{z}_4$.

$a \backslash b$	L	R
O	z_1	z_1
IT	z_2	z_3
ID	z_4	z_5

Figure 3: The game in Figure 2 in its strategic form representation.

Thus, focusing only on Ann, with the preferences over terminal histories given above, we have that strategy ID is strictly dominated by O: as a result, in light of what is written above concerning

[24] This terminology goes back to Luce & Raiffa (1957, Chapter 4.11, p.79).

[25] Observe that in the title it is called "Pure Strategy Dominance", while in the main body of the paper it is simply called "Dominance".

the relation between the various dominance notions just introduced, strategy ID is B-dominated. Alternatively, we can establish the fact that this strategy is B-dominated in a more explicit way by employing the actual definition of B-dominance.[26] Thus, given Bob's strategy L, we have that ID is strictly dominated by both O and ID; given Bob's strategy R, strategy ID is strictly dominated by O; finally, given S_b, we have that ID is strictly dominated by O. $\diamond$

2.3 Dominance Notions for the Extensive Form Representation

The dominance notions we introduced in Section 2.2 work at the level of strategies without taking into account the sequential nature of a dynamic game. That is, they do not take into consideration the extensive form representation of a dynamic game. The notion we provide next is—on the contrary—phrased specifically to address the presence of information sets.

Definition 2.1 (Conditional B-Dominance). *Fix a player $i \in I$ and a restriction $R \subseteq S$. Then strategy $s_i \in R_i$ is* conditionally B-dominated with respect to R *if there exists an information set $h \in H_i(s_i)$ such that $R(h)$ is nonempty and $s_i \in \mathsf{bd}_i(R(h))$.*

As for the dominance notions introduced in the previous section, the omission of the restriction with respect to which conditional B-dominance is defined stands for that restriction being equal to S.

Example 2 (Dynamic Game with Simultaneous Moves with an Outside Option, Continued). To see the definition of Conditional B-Dominance at work, we take the dynamic game in Figure 2 and—as before—we focus only on Ann's preferences over terminal histories. From Definition 2.1, we have to focus on the information sets where Ann is active, which—incidentally—in this specific case are all the information sets present in the game. Hence, we take Ann's conditional problems $S(\langle\varnothing\rangle)$ and $S(\langle I\rangle)$. Figure 4 provides the corresponding graphical representation.

$S(\langle\varnothing\rangle)$

a \ b	L	R
O	z_1	z_1
IT	z_2	z_3
ID	z_4	z_5

$S(\langle I\rangle)$

a \ b	L	R
IT	z_2	z_3
ID	z_4	z_5

Figure 4: A representation of Ann's conditioning events in the game in Figure 2.

Observe that only Ann is active at $\langle\varnothing\rangle$, hence the matrix that corresponds to $S(\langle\varnothing\rangle)$ is relevant *only* for Ann, while both players are active at $\langle I\rangle$ and the matrix that corresponds to $S(\langle I\rangle)$ is relevant for both. From the analysis we performed of this game (in its strategic form) in Section 2.2, we know that strategy ID is B-dominated. This can be rephrased by stating that strategy ID is B-dominated for Ann at $\langle\varnothing\rangle$. Hence, strategy ID is conditionally B-dominated with respect to S. $\diamond$

We now recall some definitions from Mailath et al. (1993) and Shimoji & Watson (1998), by opportunely modifying them to take into account the presence of restrictions.[27] Fix a restriction $R \subseteq S$, a player $i \in I$, and an information set $h \in H_i$ such that $R(h)$ is nonempty. Then we say that $s_i, s_i' \in R_i(h)$ *agree on* $R_{-i}(h)$ if

$$\zeta(s_i, s_{-i}) \sim \zeta(s_i', s_{-i})$$

[26] We are grateful to an anonymous referee for having pointed out the need to be more explicit regarding this issue.

[27] The original definitions are phrased in terms of *generic* information sets, where here we use the term "generic" with the meaning that the objects under scrutiny can be extensive form as well as *normal form* information sets (see Mailath et al. (1993, Definition 3, p.279)). Here, we state these definitions by referring directly to their extensive form representation (of course, by employing our notation).

for every $s_{-i} \in R_{-i}(h)$. Also, $s_i, s_i' \in R_i(h)$ are *$R(h)$-replacements* if s_i and s_i' agree on $R_{-i} \backslash R_{-i}(h)$, while they are *strong $R(h)$-replacements* if, in addition of being $R(h)$-replacements, for every $h' \in H_i$ and corresponding player i's conditional problem $R(h')$, if $R(h) \cap R(h') \neq \emptyset$, $s_i \in R_i(h')$ if and only if $s_i' \in R_i(h')$. Shimoji & Watson (1998, Section 2, p.168) prove for both standard strategies and strategies that the collection of conditional problems (i.e., with S as the reference restriction in our definitions) of a player satisfies the following property, that they call the "Strong Replacement Property": for every $i \in I$, for every conditional problem $S(h)$, and for every $s_i, s_i' \in S_i(h)$ there exists a strategy $s_i^* \in S_i(h)$ such that:

i) s_i and s_i^* are strong $S(h)$-replacements,

ii) s_i' and s_i^* agree on $S_{-i}(h)$.

In the following lemma, we restate the Strong Replacement Property by taking into account the presence of an arbitrary restriction $R \subseteq S$.

Lemma 2.1 (Strong Replacement Property Given a Restriction). *Given a restriction $R \subseteq S$, a player $i \in I$, an information set $h \in H_i$ such that $R(h)$ is nonempty, and strategies $s_i, s_i' \in R_i(h)$, there exists a strategy $s_i^* \in R_i(h)$ such that:*

i) s_i and s_i^* are strong $R(h)$-replacements,

ii) s_i' and s_i^* agree on $R_{-i}(h)$.

To develop an intuition for these definitions, we see them at work in the dynamic game in Figure 1.

Example 1 (Dynamic Game with Simultaneous Moves, Continued). Take the dynamic game in Figure 1. For example, we have that strategies $BFHSW, BFGS \in S_a$ agree on $S(\langle BC \rangle)$; also, strategies $BFHSW, BEHOW \in S_a$ are $S(\langle BC \rangle)$-Replacements. Concerning Strong $S(h)$-Replacements, $BEHOW, BEHPW \in S_a$ are Strong $S(\langle BC \rangle)$-Replacements: indeed, notice how they differ only at $S(\langle BC, E, I \rangle) \subsetneq S(\langle BC \rangle)$, but still satisfy the property that $s_a \in S_a(h')$ if and only if $s_a' \in S_a(h')$ for every $h' \in H_a$ such that $S(\langle BC \rangle) \cap S(h') \neq \emptyset$. Regarding the strong replacement property, it crucially depends on which strategies we take to have the role of s_i and s_i' in the definition above. For example, we now take strategies $BEGO, BFHSW \in S_a$. Thus, by taking the notation adopted in the definition above with $i := a$, if we let $s_a := BEGO$ and $s_a' := BFHSW$, then we can let $s_a^* := BFGS$; if we let $s_a := BFHSW$, with $s_a' := BEGO$, then we can let $s_a^* := BEHOW$ or $s_a^* := BEHOX$. ◇

The result that follows links the Strong Replacement Property given an arbitrary restriction $R \subseteq S$ as in Lemma 2.1 to weak dominance relative to R_{-i}.

Lemma 2.2 (Strong Replacement Property & Weak Dominance). *Given a player $i \in I$, a restriction $R \subseteq S$, and a strategy $s_i \in R_i$, if there exists an information set $h \in H_i(s_i)$ such that $R(h)$ is nonempty and there exists a strategy $s_i' \in R_i(h)$ such that s_i is weakly dominated relative to $R_{-i}(h)$ by s_i', then there exists a strategy $s_i^* \in R_i$ that agrees with s_i' on $R_{-i}(h)$ and that agrees with s_i on $R_{-i} \setminus R_{-i}(h)$ that weakly dominates s_i relative to R_{-i}.*

A rather immediate consequence of Lemma 2.2 is the following lemma, that establishes a link between Conditional B-Dominance and Weak Dominance and—in doing so—a connection between the extensive form and the strategic form representation of a dynamic game.

Lemma 2.3 (Conditional B-Dominance & Weak Dominance). *Given a dynamic game Γ, its strategic form representation Γ^r, and an arbitrary restriction $R \subseteq S$, if a strategy $s_i \in R_i$ is conditionally B-dominated given R, then it is weakly dominated relative to R_{-i} in Γ^r.*

3. Rationality

Given a dynamic game Γ, we let $u_i \in \Re^Z$ denote a *von Neumann-Morgenstern utility function*[28] of player i (henceforth, utility function) such that $u_i(z) \geq u_i(z')$ if and only if $z \succsim_i z'$, for every $z, z' \in Z$, where we write that such utility function u_i is *originated from the preference relation* $\succsim_i$. Also, for every $i \in I$, a *conditional probability system*[29] (henceforth, CPS) μ_i on $(S_{-i}, \mathcal{H}_i)$ is a mapping

$$\mu_i(\cdot|\cdot) : S_{-i} \times \mathcal{H}_i \to [0,1]$$

that satisfies the following axioms:

A1. for every $h \in H_i$, $\mu_i(S_{-i}(h)|S_{-i}(h)) = 1$;

A2. for every $h \in H_i$, $\mu_i(\cdot|S_{-i}(h))$ is a probability measure on $(S_{-i}, \mathcal{H}_i)$;

A3. (Chain Rule) for every $\widehat{S}_{-i} \subseteq S_{-i}$ and for every $S_{-i}(h), S_{-i}(h') \in \mathcal{H}_i$, if $\widehat{S}_{-i} \subseteq S_{-i}(h') \subseteq S_{-i}(h)$, then

$$\mu_i(\widehat{S}_{-i}|S_{-i}(h)) = \mu_i(\widehat{S}_{-i}|S_{-i}(h')) \cdot \mu_i(S_{-i}(h')|S_{-i}(h)).$$

Given that we let $\Delta(Y)$ denote the set of all probability measures over an arbitrary space Y, $[\Delta(S_{-i})]^{\mathcal{H}_i}$ denotes the set of all mappings from $\mathcal{H}_{-i}$ to $\Delta(S_{-i})$, while we let $\Delta^{\mathcal{H}_i}(S_{-i}) \subseteq [\Delta(S_{-i})]^{\mathcal{H}_i}$ denote the set of CPSs on $(S_{-i}, \mathcal{H}_i)$, i.e., the set of all those mappings from $\mathcal{H}_i$ to $\Delta(S_{-i})$ that satisfy the axioms above. Finally, given a restriction $R \subseteq S$, we let

$$\Delta^{\mathcal{H}_i}(R_{-i}) := \left\{ \mu_i \in \Delta^{\mathcal{H}_i}(S_{-i}) \;\middle|\; \forall h \in H_i \ \big(R(h) \neq \emptyset \Longrightarrow \mu_i(R_{-i}(h)|S_{-i}(h)) = 1\big) \right\}.$$

We can now introduce the following definition, which is the main building block behind the exercise performed in this section.

Definition 3.1 (Sequentially Rational Strategy Given a Restriction). *Given a restriction $R \subseteq S$, a strategy $s_i^* \in R_i$ is* sequentially rational given R *if there exist a utility function $u_i \in \Re^Z$ and a CPS $\mu_i \in \Delta^{\mathcal{H}_i}(R_{-i})$ such that for every $h \in H_i(s_i^*)$, if $R(h)$ is nonempty, then*

$$\sum_{s_{-i} \in R_{-i}} u_i(\zeta(s_i^*, s_{-i})) \cdot \mu_i(\{s_{-i}\}|S_{-i}(h)) \geq \sum_{s_{-i} \in R_{-i}} u_i(\zeta(s_i, s_{-i})) \cdot \mu_i(\{s_{-i}\}|S_{-i}(h)) \tag{3.1}$$

for every $s_i \in R_i(h)$.

Example 3 (4-legged Centipede). The game in Figure 5 is an example of a dynamic game with *perfect information* with two players, namely, Ann (viz., a) and Bob (viz., b).

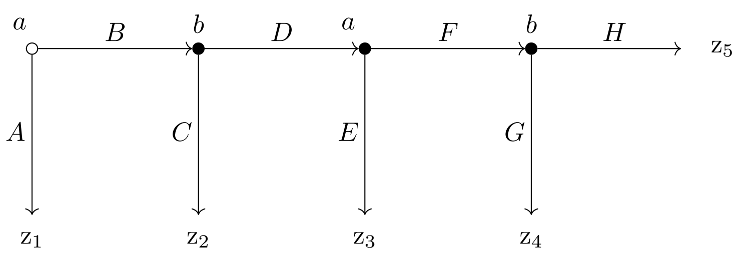


Figure 5: A 4-legged Centipede Game.

[28]Here, like Börgers (1993), we follow a rather common practice in the field that does not distinguish between the function u_i and the related function that captures expected utility, whereas—in the terminology of Mas-Colell et al. (1995, Chapter 6.B, p. 184)—the former would be deemed a *Bernoulli* utility function, with the latter being the actual von Neumann-Morgenstern utility function.

[29]This corresponds to one of the primitive elements of a *conditional probability space* of Rênyi (1955, Section 1.2–1.4). However, the name—that eventually stuck in the game-theoretical literature—actually comes from a related definition from Myerson (1986, Section 5). See Hammond (1994, Section 3.2) for a discussion of the relation between these two notions.

To see the intuition behind the definitions above, let Bob's preferences over terminal histories be captured by the following ordering: $z_4 \succ_b z_2 \succ_b z_3 \succ_b z_1 \sim_b z_5$. Also, let $s_b^* := DG$ and let $R := \{A\} \times S_b$ be a restriction for Bob. Clearly, for every $h \in H_b(s_b^*)$, $R_a(h)$ is empty. Thus, our definition asks for a utility function and an *arbitrary* CPS such that DG satisfies Equation (3.1). On the contrary, if Bob's restriction R is defined as $R := \{A, BE\} \times S_b$, since at $\langle B \rangle$ we have that $R_a(\langle B \rangle) = \{BE\}$, the definition asks for a utility function and a CPS μ_b such that $\mu_b(R_a(\langle B \rangle)|S_a(\langle B \rangle)) = 1$, which—in this specific case—would correspond to a CPS μ_b such that $\mu_b(BE|S_a(\langle B \rangle)) = 1$. Of course, in such an instance s_b^*, i.e., DG, is not sequentially rational given R, because it does not satisfy (subjected) expected utility maximization with respect to the CPS μ_b defined above. $\diamond$

Lemma 3.1. *Given a restriction $R \subseteq S$, a strategy $s_i^* \in R_i$ is sequentially rational given R if and only if it is not conditionally B-dominated with respect to R.*

We are now in position to introduce the notion of sequential rationality that lies at the core of this work, which further demonstrates that the building block of the present exercise is indeed Definition 3.1. Indeed, it obtains immediately from that definition by setting the restriction $R := S$.

Definition 3.2 (Sequentially Rational Strategy). *A strategy $s_i^* \in S_i$ is* sequentially rational *if there exist a utility function $u_i \in \Re^Z$ and a CPS $\mu_i \in \Delta^{\mathcal{H}_i}(S_{-i})$ such that*

$$\sum_{s_{-i} \in S_{-i}} u_i(\zeta(s_i^*, s_{-i})) \cdot \mu_i(\{s_{-i}\}|S_{-i}(h)) \geq \sum_{s_{-i} \in S_{-i}} u_i(\zeta(s_i, s_{-i})) \cdot \mu_i(\{s_{-i}\}|S_{-i}(h))$$

for every $h \in H_i(s_i^*)$ and for every $s_i \in S_i(h)$.

In the result that follows, we show that a strategy that is sequentially rational according to Definition 3.2 cannot be conditionally B-dominated. Thus, this result links our notion of sequential rationality to a dominance notion based solely on *pure* strategies that is employed in the extensive form representation of a dynamic game.

Proposition 1. *A strategy $s_i^* \in S_i$ is sequentially rational if and only if it is not conditionally B-dominated.*

4. Solution Concepts

In this section we investigate what are the game-theoretic implications of having players reasoning about each other's behavior in a context where only ordinal preferences are assumed to be transparent. In particular, we introduce a recursive procedure that works by iteratively eliminating strategies that are conditionally B-dominated.

To accomplish the task set forth above, we start by providing an abstract definition of the notion of reduction operator. Hence, a *reduction operator* is a mapping $\mathbb{O}$ such that

$$R \mapsto \mathbb{O}(R) \subseteq R$$

for every restriction $R \subseteq S$. Given a reduction operator $\mathbb{O}$ and two restrictions $R, R' \subseteq S$, R' is a *partial reduction* of R if $\mathbb{O}(R) \subseteq R' \subseteq R$, while R' is a *full reduction* of R if $R' = \mathbb{O}(R)$. An operator $\mathbb{O}$ that gives as output a full reduction of R, for every restriction $R \subseteq S$, is called a *full reduction* operator.

4.1 Iterative Conditional B-Dominance

Focusing on Conditional B-Dominance and recalling Notation 2, for every restriction $R \subseteq S$, we let $\mathbb{U}_i(R)$ denote the full reduction operator applied to R that captures those player i's strategies $s_i \in R_i$ that are not conditionally B-dominated with respect to R, i.e.,

$$\mathbb{U}_i(R) := \left\{ s_i \in R_i \;\middle|\; \forall h \in H_i(s_i) \; \big(R(h) \neq \emptyset \Longrightarrow s_i \notin \mathsf{bd}_i(R(h))\big) \right\}, \tag{4.1}$$

and we let $\mathbb{U}(R) := \prod_{j \in I} \mathbb{U}_j(R)$.

Remark 4.1. *By definition, for every restriction $R \subseteq S$, $\mathbb{U}(R) \subseteq R$.*

The operator $\mathbb{U}$ satisfies a property which is crucial for the proofs of the results that follow, namely, it does satisfy nonemptiness: it never gives the empty set as an output.

Lemma 4.1 (Operator Nonemptiness). *For every nonempty restriction $R \subseteq S$, $\mathbb{U}(R)$ is nonempty.*

We can now introduce the iterative elimination procedure under scrutiny in this paper, which can be seen as an analog of Iterated Conditional Dominance of Shimoji & Watson (1998) for our class of dynamic games (with ordinal preferences).

Algorithm 1 (Iterative Conditional B-Dominance (ICBD)). *Fix a dynamic game Γ and, for every restriction $R \subseteq S$, consider the following procedure, with $n \in \mathbb{N}$:*

- *(Step $n = 0$) Let $\mathbb{U}^0(R) := R$;*
- *(Step $n \geq 1$) assume that $\mathbb{U}^{n-1}(R)$ has been defined; then*
$$\mathbb{U}^n(R) := \mathbb{U}(\mathbb{U}^{n-1}(R)).$$

Then, for every $k \in \mathbb{N}$, we let $\mathbb{U}^k(S)$ denote the set of strategy profiles that survive the k-th iteration of the Iterative Conditional B-Dominance (henceforth, ICBD)[30] *Finally,*

$$\mathbb{U}^\infty(S) := \bigcap_{\ell \geq 0} \mathbb{U}^\ell(S)$$

is the set of strategy profiles that survive the ICBD algorithm.

To see Algorithm 1 at work, we take the dynamic games in Figure 2 and Figure 5 and we redraw them by representing the players' preferences over terminal histories via utility functions. The resulting dynamic games are two workhorses of the literature on dynamic games (in general) and on its part which focuses on the study of forward induction (in particular).

Example 2 (Dynamic Game with Simultaneous Moves with an Outside Option, Continued). We take the dynamic game in Figure 2 with two players and we set numbers to represent the ordinal preferences of both players to ease the analysis. The resulting dynamic game is the so called Battle of the Sexes with an Outside Option.

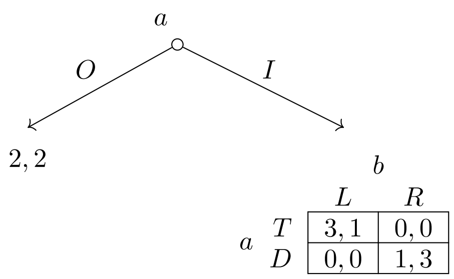


Figure 6: Battle of the Sexes with an Outside Option.

We proceed step by step to obtain $\mathbb{U}^\infty(S)$ for the game in Figure 6.

- ($n = 0$) By definition we have that $\mathbb{U}^0_a(S) := S_a$ and $\mathbb{U}^0_b(S) := S_b$.
- ($n = 1$) Strategy ID is strictly dominated—hence, B-dominated—with respect to Ann's conditional problem $S(\langle\varnothing\rangle)$. Thus, $\mathbb{U}^1_a(S) = \{O, IT\}$, while $\mathbb{U}^1_b(S) = \mathbb{U}^0_b(S)$.

[30] In a previous version of this paper, this algorithm was called "Iterative Forward B-Dominance", where "IFBD" was the corresponding acronym used throughout the paper.

- ($n = 2$) Strategy R is strictly dominated—hence, B-dominated—with respect to Bob's (only) conditional problem $S(\langle I \rangle)$ given $\mathbb{U}^1(S)$. Hence, $\mathbb{U}^2_a(S) = \mathbb{U}^1_a(S)$, while $\mathbb{U}^2_b(S) = \{L\}$.
- ($n = 3$) Strategy O is strictly dominated—hence, B-dominated—with respect to Ann's conditional problem $S(\langle \varnothing \rangle)$ given $\mathbb{U}^2(S)$. Hence, $\mathbb{U}^3_a(S) = \{IT\}$, while $\mathbb{U}^3_b(S) = \mathbb{U}^2_b(S)$.
- ($n \geq 4$) Nothing changes and the procedure ends.

Thus, we have that $\mathbb{U}^\infty_a(S) = \{IT\}$ and $\mathbb{U}^\infty_b(S) = \{L\}$ and—obviously—we have that $\mathbb{U}^\infty(S) = \{(IT, L)\}$. $\diamond$

Example 3 (4-legged Centipede, Continued). We now take the game in Figure 5 and—as in the previous example—we set numbers to represent the players' ordinal preferences. The result is the game in Figure 7, i.e., Reny's Centipede Game, introduced for the first time in Reny (1992).[31]

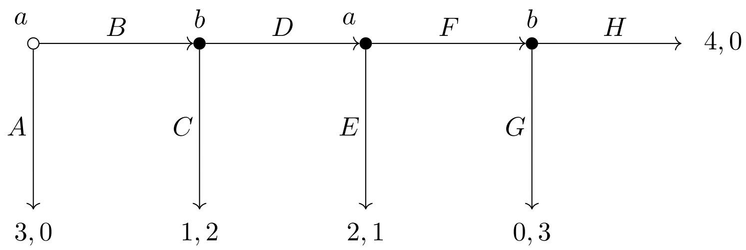


Figure 7: The 4-legged Centipede Game from Reny (1992).

As for the BoS with an outside option, we now proceed step by step to obtain $\mathbb{U}^\infty(S)$ for the game in Figure 7.

- ($n = 0$) By definition we have that $\mathbb{U}^0_a(S) := S_a$ and $\mathbb{U}^0_b(S) := S_b$.
- ($n = 1$) Strategy BE is strictly dominated—hence, B-dominated—with respect to Ann's conditional problem $S(\langle \varnothing \rangle)$. Also, strategy DH is strictly dominated—hence, B-dominated—with respect to Bob's conditional problem $S(\langle B \rangle)$. Thus, $\mathbb{U}^1_a(S) = \{A, BF\}$, while $\mathbb{U}^1_b(S) = \{C, DG\}$.
- ($n = 2$) Strategy BF is strictly dominated—hence, B-dominated—with respect to Ann's conditional problem $S(\langle \varnothing \rangle)$ given $\mathbb{U}^1(S)$. Hence, $\mathbb{U}^2_a(S) = \{A\}$, while $\mathbb{U}^2_b(S) = \mathbb{U}^1_b(S)$.
- ($n = 3$) Strategy C is strictly dominated—hence, B-dominated—with respect to Bob's conditional problem $S(\langle B \rangle)$ given $\mathbb{U}^2(S)$. Hence, $\mathbb{U}^3_a(S) = \mathbb{U}^2_a(S)$, while $\mathbb{U}^3_b(S) = \{DG\}$.
- ($n \geq 4$) Nothing changes and the procedure ends.

Thus, we have that $\mathbb{U}^\infty_a(S) = \{A\}$ and $\mathbb{U}^\infty_b(S) = \{DG\}$ and—obviously—we have that $\mathbb{U}^\infty(S) = \{(A, DG)\}$. $\diamond$

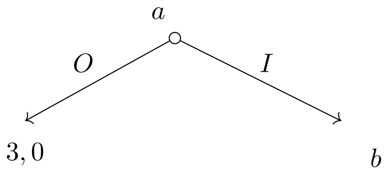
The next result establishes the nonemptiness of ICBD, a rather basic property that any solution concept should satisfy.

Proposition 2 (Nonemptiness). *For every dynamic game, for every $k \in \mathbb{N}$, $\mathbb{U}^k(S) \neq \emptyset$, i.e., $\mathbb{U}^\infty(S) \neq \emptyset$. In particular, for every dynamic game there exists a $K \in \mathbb{N}$ such that $\mathbb{U}^K(S) = \mathbb{U}^{K+1}(S) = \mathbb{U}^\infty(S) \neq \emptyset$.*

The example that follows shows that ICBD does not satisfy *order independence*, i.e., our algorithm is not robust to the order in which strategies are eliminated. In other words, different elimination orders can produce different outputs.

[31] We depict this game by following the representation in Battigalli & Siniscalchi (2002, Figure 2, p.271), while the original version of the game can be found in Reny (1992, Figure 3, p.637).

Example 4 (Order Dependence). To see that ICBD does not satisfy order independence, we take the dynamic game in Figure 8 from Perea (2014, Figure 4, p.247),[32] with two players, again Ann (viz., a) and Bob (viz., b).



a		L	C	R
	T	$2,2$	$2,1$	$0,0$
	D	$1,1$	$1,2$	$4,0$

Figure 8: A dynamic game from Perea (2014).

We also provide the related conditional problems in Figure 9, where $S(\langle\varnothing\rangle)$ is a conditional problem only for Ann, while $S(\langle I\rangle)$ is a conditional problem for both players.

$S(\langle\varnothing\rangle)$

a \ b		L	C	R
	O	$3,0$	$3,0$	$3,0$
	IT	$2,2$	$2,1$	$0,0$
	ID	$1,1$	$1,2$	$4,0$

$S(\langle I\rangle)$

a \ b		L	C	R
	IT	$2,2$	$2,1$	$0,0$
	ID	$1,1$	$1,2$	$4,0$

Figure 9: A depiction of the players' conditional problems present in the game in Figure 8.

We have that $\mathbb{U}^\infty(S) = \{(O, C)\}$. On the contrary, if we construct an ancillary operator $\mathbb{L}$ that on the first iteration deletes strategies only at the information set to which they belong and then for the remaining iterations works like $\mathbb{U}$, we would have that such operator would delete IT from $S(\langle\varnothing\rangle)$ *only*, without deleting it *also* from $S(\langle I\rangle)$ on its first iteration and—as a result—we would obtain $\mathbb{L}^\infty(S) = \{(O, L)\}$. ◇

4.2 Ordinal Strong Rationalizability

In Battigalli (1997, Footnote 9, p.50), it is mentioned that it is possible to define a version of Strong Rationalizability, call it *"Ordinal Strong Rationalizability"*, where only ordinal preferences over outcomes are transparent to the players. Given the tight link between Iterative Conditional Dominance and Strong Rationalizability (as described in Section 1.1) on one side and the one between Iterated Conditional Dominance and ICBD on the other side, it is natural to investigate the possibility of actually defining a form of Ordinal Strong Rationalizability.[33] This is exactly what we accomplish in this section.

We now introduce a new full reduction operator, which is going to be the focus of this section. Thus, for every restriction $R \subseteq S$, we let $\mathbb{S}_i(R)$ denote the full reduction operator applied to R that captures those player i's strategies $s_i \in R_i$ that are sequentially rational given R as in Definition 3.1, with $\mathbb{S}(R) := \prod_{j\in I} \mathbb{S}_j(R)$.

Algorithm 2 (Ordinal Strong Rationalizability). *Fix a dynamic game Γ and, for every restriction $R \subseteq S$, consider the following procedure, with $n \in \mathbb{N}$:*

- *(Step $n = 0$) Let $\mathbb{S}^0(R) := R$;*
- *(Step $n \geq 1$) assume that $\mathbb{S}^{n-1}(R)$ has been defined; then*

$$\mathbb{S}^n(R) := \mathbb{S}(\mathbb{S}^{n-1}(R)).$$

[32] The same dynamic game is used in Chen & Micali (2013, Example 4, p.138).
[33] We are grateful to an anonymous referee for having raised this issue.

Then, for every $k \in \mathbb{N}$, we let $\mathbb{S}^k(S)$ denote the set of strategy profiles that survive the k-th iteration of Ordinal Strong Rationalizability. Finally,

$$\mathbb{S}^\infty(S) := \bigcap_{\ell \geq 0} \mathbb{S}^\ell(S)$$

is the set of strategy profiles that survive Ordinal Strong Rationalizability.

Concerning the nonemptiness of Ordinal Strong Rationalizability, we are going to answer this question in the next section.

4.3 Relation between the Procedures

In light of the first paragraph of Section 4.2, armed with ICBD and Ordinal Strong Rationalizability, it seems natural to address the relation between these two solution concepts.

The following two results show that the relation between Iterated Conditional Dominance as an algorithmic counterpart of Strong Rationalizability applies *mutatis mutandis* to ICBD and Ordinal Strong Rationalizability.

Lemma 4.2. *Given a dynamic game Γ and a restriction $R \subseteq S$, $\mathbb{U}(R) = \mathbb{S}(R)$.*

Proposition 3. *For every dynamic game Γ,*

i) $\mathbb{U}^k(S) = \mathbb{S}^k(S)$, *for every $k \in \mathbb{N}$.*

ii) $\mathbb{U}^\infty(S) = \mathbb{S}^\infty(S)$.

Thus, we have the nonemptiness of Ordinal Strong Rationalizability as an immediate corollary of Proposition 3, stated next.

Corollary 1 (Nonemptiness of Ordinal Strong Rationalizability). *For every dynamic game, for every $k \in \mathbb{N}$, $\mathbb{S}^k(S) \neq \emptyset$, i.e., $\mathbb{S}^\infty(S) \neq \emptyset$. In particular, for every dynamic game there exists a $K \in \mathbb{N}$ such that $\mathbb{S}^K(S) = \mathbb{S}^{K+1}(S) = \mathbb{S}^\infty(S) \neq \emptyset$.*

5. Relation to Iterated Conditional Dominance

Something has to be observed concerning the last two sections. In moving from Section 3 to Section 4, we put actual numbers at the terminal histories of the games under scrutiny, as in Figure 6 and Figure 7. This is a natural step in our exercise, since we are producing utility functions (one for every player) that agree with the ordinal preferences over outcomes assumed to be transparent. However, and here there is a crucial point that has to be emphasized, those utility functions are a *particular* instance of the various that we could have produced.

The implications of this last point are immediately seen by comparing ICBD to its counterpart in the context of dynamic cardinal game, i.e., Iterated Conditional Dominance of Shimoji & Watson (1998). To define Iterated Conditional Dominance, we need additional notation. Thus, given a dynamic game Γ, we let Γ^u denote a *dynamic cardinal game derived from* Γ, where u_i is a utility function that agrees with player i's ordinal preferences, for every $i \in I$, that takes the place of $\succsim_i$ in Equation (2.1). Thus, Γ^u represents a dynamic game where us—as modelers—have 'more' information than in Equation (2.1) concerning the players' preferences, since we know also their *risk* preferences, that we are *fixing* exogenously. Now, we let $\overline{\Gamma}^u$ denote the—canonical—*mixed extension* of the dynamic cardinal game Γ^u as in Maschler et al. (2013, Chapter 5.1), where $\sigma_i \in \Delta(S_i)$ is an arbitrary *mixed strategy* of player i and

$$U_i(\sigma_i, s_{-i}) := \sum_{s_i \in S_i} u_i(\sigma_i, s_{-i}) \cdot \sigma_i(\{s_i\}),$$

for every $i \in I$. We can now introduce an additional dominance notion appropriate for cardinal games: given a $\overline{\Gamma}^u$, a player $i \in I$,and a restriction R_i, a strategy $s_i \in R_i$ is *strictly dominated* *relative to R_{-i} by a mixed strategy $\sigma_i^* \in \Delta(R_i)$* if $U_i(\sigma_i^*, s_{-i}) > U_i(s_i, s_{-i})$, for every $s_{-i} \in R_{-i}$. Given a restriction $R \subseteq S$, we let $\mathsf{md}_i(R)$ denote the set of strategies of player $i \in I$ that are strictly dominated relative to R_{-i} by a mixed strategy. Now, for every restriction $R \subseteq S$, we let $\mathbb{M}_i(R)$ denote the full reduction operator applied to R that captures those player i's strategies $s_i \in R_i$ that are not strictly dominated relative to R_{-i} by a mixed strategy, i.e.,

$$\mathbb{M}_i(R) := \{\, s_i \in R_i \mid \forall h \in H_i(s_i)\ (R(h) \neq \emptyset \Longrightarrow s_i \notin \mathsf{md}_i(R(h)))\,\}\,, \tag{5.1}$$

and we let $\mathbb{M}(R) := \prod_{j \in I} \mathbb{M}_j(R)$. As a result, Iterated Conditional Dominance is the algorithm defined as ICBD in Algorithm 1 with the operator $\mathbb{U}$ substituted by the newly introduced operator $\mathbb{M}$.

It is now natural to ask ourselves what is the relation between ICBD and Iterated Conditional Dominance. For this purpose, we distinguish two cases:

- *the particular case:* where a utility function is fixed and is treated at the same time as ordinal (to deal with ICBD) and cardinal (to deal with Iterated Conditional Dominance) to check the predictions that we obtain from the different solution concepts;
- *the general case:* where no utility function is fixed at the outset and we ask ourselves if for every strategy s_i of a player i that is not conditionally B-dominated given a restriction $R \subseteq S$ we can find a utility function u_i that agrees with the assumed ordinal preferences such that s_i is not conditionally dominated given R.

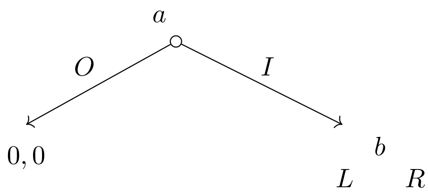
The two sections that follow address each case in turn, providing a complete picture of the relation between ICBD and Iterated Conditional Dominance.[34]

5.1 The Particular Case

As explained at the end of the previous section, here we compare the predictions obtained via ICBD and Iterated Conditional Dominance when a profile of utility functions $u := (u_i)_{i \in I}$ is fixed, with the caveat that in order to work with ICBD we treat u as a profile of *ordinal* utility functions, whereas to deal with Iterated Conditional Dominance we treat it as a profile of *cardinal* utility functions.

Thus, given an arbitrary restriction $R \subseteq S$, the first question we want to ask is if $\mathbb{U}(R) \subseteq \mathbb{M}(R)$. The answer is negative, as showed in the example that comes next.

Example 5 ($\mathbb{U}(R) \nsubseteq \mathbb{M}(R)$). To see that Iterated Conditional Dominance gives sharper predictions than ICBD, consider the dynamic game in Figure 10 with two players, again Ann (viz., a) and Bob (viz., b).



		L	R
a	T	$3,2$	$0,1$
	M	$0,1$	$3,0$
	D	$1,0$	$1,1$

Figure 10: Dynamic Game with an Outside Option with a fixed profile of utility functions.

First of all, we have that $O \notin \mathbb{U}_a^1(S)$ and $O \notin \mathbb{M}_a^1(S)$. Also, and rather crucially, we have that $D \in \mathbb{U}_a^1(S)$, but $D \notin \mathbb{M}_a^1(S)$, since D is strictly dominated relative to $S_b(\langle I \rangle)$ by any mixed strategy $\sigma_a(\alpha(T), (1-\alpha)(M))$, with $\alpha \in (\frac{1}{3}, \frac{2}{3})$. As a result, we have $\mathbb{U}^\infty(S) = (S_a \setminus \{O\}) \times S_b$, while $\mathbb{M}^\infty(S) = \{(T, L)\}$. $\diamond$

[34] We are grateful to an anonymous referee for having raised the issue of addressing the relation between these two solution concepts in its full generality.

Thus, what is happening in Example 5? When we analyze the game in Figure 10 with ICBD, as modelers we are *fixing* one possible $u := (u_i)_{i\in I}$ from the many we could have produced, but we are actually dismissing the 'cardinal' element: indeed, a perfectly natural step if we do not know what are the players' risk preferences. On the contrary, when we analyze the game with Iterated Conditional Dominance, we are—of course—explicitly taking the 'cardinal' element into account via mixed strategies. Hence, not surprisingly, assuming knowledge (and transparency between the players) of their risk attitudes leads to sharper predictions.

Regarding the opposite direction, on the contrary, we can establish that $\mathbb{M}(R) \subseteq \mathbb{U}(R)$ as stated in the following proposition.

Proposition 4 **($\mathbb{M}(R) \subseteq \mathbb{U}(R)$).** *Given a dynamic game Γ, a dynamic cardinal game $\overline{\Gamma}^u$ in its mixed extension derived from it, and a restriction $R \subseteq S$, $\mathbb{M}(R) \subseteq \mathbb{U}(R)$.*

As a matter of fact, Proposition 4 is hardly surprising, since it essentially captures the natural idea that taking into account *also* mixed strategies for (strict) dominance purposes is going to shrink the set of strategies that survive an elimination step.

Now, what conclusions can we draw from the analysis just performed in presence of a fixed profile u? What is written above tells us that it is really not possible to compare ICBD and Iterated Conditional Dominance, since the assumptions behind these solution concepts are completely different: ICBD is appropriate whenever we—as modelers—assume *only* that we know the players' ordinal preferences and they are transparent between them; on the contrary, Iterated Conditional Dominance is appropriate when we are confident concerning our knowledge of the players' risk preferences.

5.2 The General Case

In this section, we deal with what we call "the general case", where no utility function is fixed at the outset. In particular, the question is whether for every strategy s_i of a player i that is not conditionally B-dominated given a restriction $R \subseteq S$ there exists a utility function u_i that agrees with the assumed ordinal preferences such that s_i is not conditionally dominated (i.e., it does not survive an application $\mathbb{M}$ operator) given R.[35] Thus, for the analysis that follows, we let $\mathcal{U}_i$ denote the set of utility functions that are originated from the preference relation $\succsim_i$ as in Section 3.

As a matter of fact, given this setting, we can establish a relation between ICBD and Iterated Conditional Dominance, providing a positive answer to the question stated above.

Lemma 5.1. *For every dynamic game Γ, for every $R \subseteq S$, and $i \in I$,*

$$\mathbb{U}_i(R)[\succsim] = \bigcup_{u_i \in \mathcal{U}_i} \mathbb{M}_i(R)[u_i].$$

An immediate implication of Lemma 5.1 is the following proposition, that establishes a tight link between ICBD and Iterated Conditional Dominance.

Proposition 5. *For every dynamic game Γ and $i \in I$,*

$$\mathbb{U}_i^\infty(S)[\succsim] = \bigcup_{u_i \in \mathcal{U}_i} \mathbb{M}_i^\infty(S)[u_i].$$

6. The Backward Induction Outcome

Section 5.1 showed us that, when we fix a profile of utility functions u and we treat it alternatively as ordinal or cardinal depending on the solution concept we want to use, ICBD cannot really be compared with its 'cardinal' counterpart Iterated Conditional Dominance, since the two are based on different assumptions. However, there is an important class of games where—due to

[35] We are grateful to Andrés Perea for having raised this problem during the discussion following the presentation of this work at the GAMES2020 conference.

their nature—the behavioral predictions of these two solution concepts should really be the same, since risk preferences should not play any role in their analysis. Thus, the focus of this section is exactly on this class of games, i.e., generic[36] dynamic games with perfect information, where the notion of genericity we employ is the following, which goes back to Battigalli (1997, Section 4, p.48): a dynamic game with perfect information Γ satisfies the *"No Relevant Ties" condition* (henceforth, NRT condition) if, for every $z, z' \in Z$ and for every $i \in I$, if $z \neq z'$ and $i \in I_h$, where the information set $h \in H$ is the last common predecessor of z and z', then it is not the case that $z \sim_i z'$. In the rest of this section—and related proofs—a dynamic game that satisfies the NRT condition is called an NRT dynamic game (where it has to be observed that it is—*a fortiori*—of perfect information).

Example 3 (4-legged Centipede, Continued). The game in Figure 7 is an example of an NRT dynamic game. As we know from our previous analysis, we have that $\mathbb{U}^\infty(S) = \{(A, DG)\}$. Thus, we have that $\zeta(\mathbb{U}^\infty(S)) = \langle A \rangle$, that is, the ICBD procedure selects in this dynamic game the unique outcome $\langle A \rangle \in Z$. $\diamond$

In dynamic games with perfect information and in generic dynamic games with perfect information in particular, there is general agreement that the informal notion of backward induction reasoning applied to standard strategies is fully captured by Subgame Perfect Equilibrium of Selten (1965, Definition, p.308) (see Osborne & Rubinstein (1994, Section 6.2) for definitions and results). As it is well known, generic dynamic games with perfect information and NRT dynamic games[37] have a unique backward induction outcome, which we denote by z^{BI}, that can be obtained via Subgame Perfect Equilibrium. Interestingly, under a certain condition,[38] the unique backward induction outcome z^{BI} can be obtained by means of the algorithm known as Iterated Admissibility applied to the strategic form representation of an NRT dynamic game. Hence, before identifying this condition, first we recall some definitions apt for the study of games in their strategic form along with the definition of Iterated Admissibility.

Thus, we employ the following terminology (see Marx & Swinkels (1997, Definition 6, p.227)), which should be—not surprisingly—reminiscent of the one we used in Section 4 for operators. Given a game Γ^r in its strategic form representation, a set $\mathbf{R}$ is a *reduction of S by weak dominance* if $\mathbf{R}$ can be obtained from S as $\mathbf{R} := \bigcap_{\ell \geq 0} \mathbf{R}^\ell$,[39] by setting $\mathbf{R}^0 := S$, with $\mathbf{R}^{n+1}$ derived from $\mathbf{R}^n$ by letting $\mathbf{R}^{n+1} := \mathbf{R}^n \setminus \{s\}$, with $s := (s_j)_{j \in I}$ and there exists an $i \in I$ such that s_i is weakly dominated on $\mathbf{R}^n_i$ by an element of $\mathbf{R}^n_i \setminus \{s_i\}$. Also, $\mathbf{R}$ is a *full* reduction of S by weak dominance if $\mathbf{R}$ is a reduction of S by weak dominance and there exist no strategies in $\mathbf{R}$ that are weakly dominated. By proceeding recursively, we now introduce *Iterated Admissibility*, which is defined as the maximal[40] algorithm such that $\mathbf{A}^0_i := S_i$, $\mathbf{A}^{n+1}_i := \{\, s_i \in \mathbf{A}^n_i \mid s_i \in \mathsf{A}_i(\mathbf{A}^n) \,\}$, and $\mathbf{A}^\infty_i := \bigcap_{\ell \geq 0} \mathbf{A}^\ell_i$, where $\mathbf{A}^k := \prod_{j \in I} \mathbf{A}^k_j$ for every $k \geq 0$. As a result, $\mathbf{A}^\infty := \bigcap_{\ell \geq 0} \mathbf{A}^\ell$ is the set of strategy profiles that survive Iterated Admissibility, which can be proved via standard arguments to be nonempty.

Whereas it is obvious that $\mathbf{A}^\infty$ is a full reduction of S by weak dominance, it has to be observed that it is well-known that Iterated Admissibility is *not* order independent, i.e., it is not robust to the order in which strategies are eliminated.[41] As a result, given an arbitrary game in its strategic form Γ^r, it is possible to have many full reductions based on weak dominance, each giving a different set of strategy profiles as output. However, Marx & Swinkels (1997, Equation 2, p.223) identify a condition—which is the one we are after and we state next—that, if satisfied by a game, ensures that different full reductions based on weak dominance are equivalent outcome-wise.

Definition 6.1 (Transference of Decision-Maker Indifference). *The strategic form represen-*

[36] See Myerson (1991, Chapter 4.7, p.186) and Luo et al. (2021, Section 2, p.582, & Footnote 6).

[37] See, for example, Myerson (1991, Theorem 4.7, p.186).

[38] See also Section 8.7 for additional conceptual background.

[39] Marx & Swinkels (1997, Definition 6, p.227) employ a different notation: they write $\mathbf{R} := S \setminus \mathbf{R}^1, \dots, \mathbf{R}^\ell$, where $\mathbf{R}^k$ is the set of strategies *deleted* at the k-th iteration of the reduction process, for every $k \in \{1, \dots, \ell\}$. On the contrary, in our notation, $\mathbf{R}^k$ is the set of strategies that *survive* the reduction process at the k-th iteration.

[40] An algorithm is *maximal* with respect to a dominance notion if all strategies of all players that are dominated according to that notion are removed at every step.

[41] See the example in Osborne & Rubinstein (1994, Figure 63.1, p.63).

tation Γ^r of a dynamic game Γ satisfies "Transference of Decision-Maker Indifference" condition (henceforth, TDI) if, for every $i, j \in I$,

$$\zeta(s_i, s_{-i}) \sim_i \zeta(s'_i, s_{-i}) \Longrightarrow \zeta(s_i, s_{-i}) \sim_j \zeta(s'_i, s_{-i}), \tag{6.1}$$

for every $s_i, s'_i \in S_i$ and $s_{-i} \in S_{-i}$.

For the purpose of the present work, the TDI condition is particularly important in light of the remark that follows.

Remark 6.1 (NRT Dynamic Games & TDI). *The strategic form Γ^r of an NRT dynamic game Γ satisfies the TDI condition as in Definition 6.1 (see Battigalli & Friedenberg (2012, Section 8, p.77)).*

We now go back to Reny's Centipede Game in Figure 7 to see at work what is written above.

Example 3 (4-legged Centipede, Continued). Take the game in Figure 7 that we represent in its strategic form in Figure 11.

		b		
		C	DG	DH
	A	$3, 0$	$3, 0$	$3, 0$
a	BE	$1, 2$	$2, 1$	$2, 1$
	BF	$1, 2$	$0, 3$	$4, 0$

Figure 11: The strategic form representation of Reny's Centipede Game.

To see that this game satisfies Definition 6.1, we reformulate Equation (6.1) in terms of utility functions, i.e., we have

$$u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i}) \Longrightarrow u_j(s_i, s_{-i}) = u_j(s'_i, s_{-i}). \tag{6.2}$$

Now, it is easy to see that this game satisfies Equation (6.2). Concerning Iterated Admissibility, we have the following.

- $(n = 0)$ By definition we have that $\mathbf{A}^0_a := S_a$ and $\mathbf{A}^0_b := S_b$.
- $(n = 1)$ Strategy BE is strictly dominated by A. Thus, $\mathbf{A}^1_a = \{A, BF\}$, while $\mathbf{A}^1_b = \mathbf{A}^0_b(S)$.
- $(n = 2)$ Strategy C is weakly dominated by DG, that also strictly dominates DH. Hence, $\mathbf{A}^2_a = \mathbf{A}^1_a$, while $\mathbf{A}^2_b = \{DG\}$.
- $(n = 3)$ Strategy BF is strictly dominated by A. Hence, $\mathbf{A}^3_a = \{A\}$, while $\mathbf{A}^3_b = \mathbf{A}^2_b$.
- $(n \geq 4)$ Nothing changes and the procedure ends.

Thus, we have that $\mathbf{A}^\infty = \{(A, DG)\}$ and $\zeta(\mathbf{A}^\infty) = \{\langle A \rangle\} = \{z^{BI}\}$. ◇

What we showed above concerning the game in Figure 7, i.e., the fact that $\zeta(\mathbf{A}^\infty(S)) = \{z^{BI}\}$, does not obtain by accident. Indeed, not only TDI makes Iterated Admissibility order independent with respect to outcomes (as pointed out above), but it is also exactly the condition which ensures that Iterated Admissibility (again, no matter how performed—see Remark A.3) selects z^{BI} as the only outcome in the strategic form corresponding to an NRT dynamic game. This is captured by the result that we state next, that—indeed—links Iterated Admissibility and backward induction reasoning.[42]

Remark 6.2 (Marx & Swinkels (1997, Section VI, p.240)). *Given a dynamic game Γ whose strategic form representation Γ^r satisfies the TDI condition, $\zeta(\mathbf{A}^\infty) = \{z^{BI}\}$.*

[42]See Section 8.7 for additional references.

Remark 6.2 happens to be the missing ingredient for the result we are after—that we state next—in light of the tight connections between Conditional B-Dominance and Weak Dominance.

Proposition 6. *Given a dynamic game Γ whose strategic form representation Γ^r satisfies the TDI condition,* $\zeta(\mathbb{U}^\infty(S)) = \{z^{BI}\}$.

The—rather immediate—proof of Proposition 6 can be found in the appendix. Here, we just point out that we essentially already provided all the elements to derive it, since we mentioned above that Marx & Swinkels (1997) show that Iterated Admissibility is order independent with respect to outcomes in all games that satisfy TDI, i.e., no matter which algorithmic procedure based on weak dominance we employ, we always obtain the same outcome in this class of games. Thus, as soon as we realize that, on a dynamic game Γ whose strategic form representation Γ^r satisfies the TDI condition, ICBD induces the existence of an algorithmic procedure based on full reduction of weakly dominated strategies on Γ^r, the result is established.

We now collect a weaker result than Proposition 6, which is important in itself in the study of dynamic games with perfect information. As such, the following corollary is a translation in an ordinal setting of the celebrated Battigalli's theorem established in Battigalli (1997, Theorem 4, p.53).[43]

Corollary 2. *Given an NRT dynamic game* Γ, $\zeta(\mathbb{U}^\infty(S)) = \{z^{BI}\}$.

7. An Application: Binary Agendas with Sequential Majority Voting

As an example of an application of ICBD, in this section we show the importance of Proposition 6 in the context of binary agendas.[44] These are rather fundamental voting procedures that can be described informally as follows: a (finite) set of individuals vote on a (finite) set of alternatives, where the voting process is comprised itself of a sequence of voting procedures between two alternatives. Given their fundamental nature and their structure amenable to game-theoretical analysis, much work has been done on these voting procedures, starting from the seminal Farquharson (1969),[45] which has been actually one of the first contributions (along with Dummett & Farquharson (1961)) to frame voting as a non-cooperative game. A crucial aspect of this work has been the introduction of the notion of *sophisticated voting*, which corresponds to Iterated Admissibility applied to the strategic form representation of the voting process. Moving from this book, McKelvey & Niemi (1978), Moulin (1979), and Gretlein (1983)—among the others—have further investigated the relation between the analysis presented in Farquharson (1969) in terms of *sophisticated voting* and the backward induction outcome. Thus, given this background, in this section we study a particular form of binary agendas, where, at every step of the process, individuals have to vote *sequentially* between two alternatives.[46] As a result, this setting gives rise to a dynamic game with perfect information, whose class has been—of course—the focus of our analysis in Section 6. Related to our analysis, it is important to observe that our ordinal setting, based on preference relations, without any reliance on (cardinal) utilities and probability measures, is particularly well-suited for the analysis of voting procedures (e.g., see Moulin (1979, Footnote 3, p.1338)).

To link what is written above to the result we state in this section, we now provide a formal description of the following elements:

a) the structure of the voting process;

b) how individuals are going to vote at each step of the voting process;

c) according to which decision rule an alternative is chosen at every step of the voting process.

In what follows, we address each point to then provide a result in this context that is related to Proposition 6.

[43]See Section 8.7 for a discussion of this result and Footnote 50 therein for the related literature.

[44]See Myerson (1991, Chapter 4.10) for a definition of binary agendas similar to ours and a discussion of the model.

[45]See also Niemi (1983) for a reappraisal of this book.

[46]See Sloth (1993) for an analysis with a special focus of sequential voting.

Given the list above, first of all, we address point (a). To do so we describe the structure of the voting process by introducing the notion of binary agenda. A *binary agenda* Ψ is a tuple

$$\Psi := \langle K, I, (\trianglerighteq_i)_{i \in I}, (T, \sqsupseteq), \psi \rangle,$$

where K is finite set of (social) alternatives, I is a finite set of individuals having preferences over K represented for every $i \in I$ by a preference relation $\trianglerighteq_i \subseteq K \times K$, $(T, \sqsupseteq)$ is a finite arborescence with root n_0, and ψ is a correspondence $\psi : T \rightrightarrows K$ satisfying the following properties:

1) $\psi(n_0) = K$;

2) for every nonterminal $n \in T$, $|T(n)| = 2$, with $T(n)$ denoting the set of *immediate* successors of n;

3) for every nonterminal $n \in T$ and $n', n'' \in T(n)$, $\psi(n) = \psi(n') \cup \psi(n'')$.

The definition of binary agenda we just provided remains silent regarding point (b), which captures the form that the voting process takes at every step. In other words, the definition above does not describe how individuals are going to vote at every step, i.e., if simultaneously or sequentially. Thus, with respect to this point, we assume that at every step the voting process takes places *sequentially*, which is implemented by assuming that, every time players are called to cast their vote between two alternatives, a fix and exogenously imposed linear order $\gg$ on I is imposed, with $\bar{i}$ (resp., $\underline{i}$) denoting the first (resp., last) player to vote. As a result, a binary agenda Ψ induces a dynamic game with perfect information Γ_Ψ as in Equation (2.1), with $A_i := K$, $Z := K$, and $\succsim_i := \trianglerighteq_i$, where in the following we write that Ψ induces Γ_Ψ or that Γ_Ψ is induced by Ψ. Clearly, we have for every $i \in I$ that $A_i := K$ (since this represents the very act of casting a vote) and $Z := K$ (since a terminal history in Γ_Ψ represents an alternative $k \in K$ selected via the voting procedure). Regarding the translation of T in terms of the arborescence X of Equation (2.1) (with related—singleton—information sets belonging to H), the following points are in order:

- the empty history $\langle \varnothing \rangle$ corresponds to the root $n_0 \in T$;
- for every $n \in T$ there exists a subarborescence $X' \subseteq X$ with the history x corresponding to n being the initial history of this arborescence such that $I_x := \{\bar{i}\}$;
- for every $x \in X$, $|X(x)| = 2$, with $X(x)$ denoting the set of *immediate* successors of x.
- for every $x \in X$ with corresponding $h \in H$ such that $I_h := \{\underline{i}\}$, there exist $n', n'' \in T$ such that $X(x) := \{x', x''\}$ with x' corresponding to n' and x'' corresponding to n''.

We now have to specify according to which decision rule an alternative is chosen at every step of the voting process. Thus, to deal with point (c) above, we introduce a further assumption on the players' preferences over the alternatives in K. In particular, we assume that the following *"No Indifference" condition* holds: for every $i \in I$, $\trianglerighteq_i$ is a linear order represented by $\triangleright_i$. Now, armed with the "No Indifference" condition along with the additional assumption that $|I|$ is odd to avoid ties,[47] we introduce the decision rule that we are going to employ to choose between alternatives at every step of the voting process. Thus, given that the *majority preference relation* $\triangleright \subseteq K \times K$ is defined as $k \trianglerighteq k'$ if there exists a $J \subseteq I$ such that $|J| > |I \setminus J|$ and $k \triangleright_j k'$ for every $j \in J$, we define the *majority rule* as the decision rule that chooses—between two alternatives—the alternative that is (strictly) preferred according to the majority preference relation. In the following, we use the expression "majority voting" to refer to a voting procedure that employs the majority rule.

With the framework introduced above at our disposal, we can now start the process of linking the study of binary agendas with sequential majority voting to our work in Section 6. The following remark points out the relevancy of the TDI condition in this context as well.

Remark 7.1 (Hummel (2008, Section 3, p.269)). *A binary agenda Ψ with sequential majority voting satisfying the "No Indifference" condition induces a dynamic game with perfect information Γ_Ψ whose strategic form representation Γ^r_Ψ satisfies the TDI condition.*

[47] This assumption is actually unnecessary, since requiring $|I|$ to be odd can be avoided—for example—by giving to a particular voter the power to break ties. Nonetheless, we adopt it, since it is standard in the literature.

We can now state the main result of this section, that follows immediately from Remark 7.1 and Proposition 6. In linking the backward induction outcome obtained in binary agendas with sequential majority voting to forward induction reasoning (via the ICBD procedure), it sheds a new light on the notion of *sophisticated voting* of Farquharson (1969).

Proposition 7. *Given a dynamic game with perfect information Γ_Ψ induced by a binary agenda Ψ with sequential majority voting satisfying the "No Indifference" condition, $\zeta(\mathbb{U}^\infty(S)) = \{z^{BI}\}$.*

8. Discussion

8.1 Decision-Theoretical Foundation of Expected Utility in Games

Our exercise starts from assuming that every player has preferences over terminal histories of a dynamic game represented via a preference relation. Moving from this, we introduce in Definition 3.2 a notion of sequential rationality according to which a strategy of a player is sequentially rational if we can produce *both* a utility function which agrees with the payer's preference relation *and* a conditional subjective probability—in the form of a CPS—according to which the strategy is a subjective expected utility maximizer.

Two works, i.e., Gilboa & Schmeidler (2003) and Perea (2021), deal with similar issues and can be viewed as complementary to our approach in answering specifically the question of what is that utilities actually represent in a game-theoretical context. Thus, to answer this question, both papers work in the expected utility framework, with beliefs (as probability measures) assumed as a primitive notion. Moving from this very primitive notion, these works derive the main building block of their exercises, which are *conditional preference relations*, i.e., preference relations over acts and states of nature conditional on beliefs (as probability measures) on states of nature.

The main result of both papers is an axiomatization of expected utility maximization in a game-theoretical context (i.e., Gilboa & Schmeidler (2003, Theorem, p.189) and Perea (2021, Theorem 5.1, p.19)), thus providing an expected utility representation to conditional preference relations. However, while Gilboa & Schmeidler (2003) present a list of axioms on conditional preference relations that are sufficient—but not necessary—for an expected utility representation, the axioms introduced in Perea (2021) are necessary and sufficient for such representation, In general, Perea (2021, Sections 7(a) & 7(c)) discusses thoroughly the point of having beliefs (as probability measures) as a primitive notion in both frameworks, whereas Perea (2021, Section 7(b)) addresses its relation with related works (including Gilboa & Schmeidler (2003)).

8.2 Beliefs in Dynamic Games & Alternative Notions of Sequential Rationality

Whereas the two papers mentioned in Section 8.1 are complementary to the present work with respect to the issue of deriving utilities from preferences, Siniscalchi (2021) is complementary with respect to the problem of assessing a player's beliefs in the course of a dynamic game. Incidentally, this has immediate implications on the embraced notion of sequential rationality.

In particular, Siniscalchi (2021) moves from the following two observations: (1) according to the conceptual stance presented in Savage (1954), beliefs should be elicitable (i.e., it should be in principle possible to relate them to observable behavior); (2) dynamic games present a specific challenge in providing a notion of sequential rationality that takes into account that a player's beliefs cannot necessarily be observed at every information set (e.g., when such information set is off the predicted path of play). Thus, to address the two points described above, the author introduces in Siniscalchi (2021, Definition 3, p.8) a novel notion of sequential rationality that he calls *"structural rationality"* according to which a strategy is structurally rational given a belief if there is no other strategy of that player that gives her a higher *ex ante* expected payoff against all feasible perturbations—*à la* Selten (1975) and Bewley (2002)—of that player's belief.

The main result of the paper, namely, Siniscalchi (2021, Theorem 3, pp.18-19) essentially shows that under structural rationality offering side-bets at the beginning of the game allows to elicit beliefs in an incentive-compatible way at every information set and, as such, it provides a theoretical foundation to the usage of the strategy method as in Selten (1967).

8.3 Restrictions

In Section 2.1, we define a restriction of player i as a nonempty subset $R \subseteq S$ of the form $R_i \times R_{-i}$. In general, concerning the term "restriction", the usage in the literature varies.

In this paper we follow Shimoji & Watson (1998, Section 2, p.164) in dispensing restrictions from having a product structure, which is related to the fact that in that paper a restriction can eventually correspond to an instance of $S(h)$, as in our case.

Here, we use the expression "product restriction" for what Marx & Swinkels (1997, Section IV, p.226) call simply "restriction". Related to this point, the fact that the definition of Iterated Admissibility in Section 6 is based on product restrictions is an established convention, as mentioned also in Chen & Micali (2013, p.131).

8.4 Epistemic Foundation of ICBD

As pointed out in Börgers (1993, Section 5.1), the epistemic foundation of the iterative procedure based on B-dominance is essentially the same as the one of Correlated Rationalizability of Bernheim (1984) and Pearce (1984) made by Brandenburger & Dekel (1987) and Tan & da Costa Werlang (1988) (see also Böge & Eisele (1979)). The only difference lies in the definition of the event in the type structure that captures the rationality of a player: indeed, this has to be opportunely modified according to the notion of rationality introduced in Börgers (1993).

The same can be written concerning the epistemic foundation of ICBD, in the sense that the epistemic foundation of ICBD is captured as in Battigalli & Siniscalchi (2002, Sections 3-4) by the event *Rationality and Common Strong Belief in Rationality* in a belief-complete[48] type structure such as the Canonical Hierarchical Structure constructed in Battigalli & Siniscalchi (1999, Section 2), with the definition of sequential rationality changed accordingly.

8.5 Incomplete Information & Relation to Strong Δ-Rationalizability

As written in Section 8.4, a natural epistemic foundation for the iterative elimination of B-dominated strategies relies on the idea of modifying the notion of rationality and then moving on capturing *Rationality* (as newly defined) *and Common Belief in Rationality.*

In Dekel & Siniscalchi (2015, Section 6.5), it is mentioned that an alternative epistemic foundation of the procedure can be obtained by working in the 'standard' game-theoretical framework with von Neumann-Morgenstern utility functions as exogenously imposed, *without* modifying the notion of rationality (as subjective expected utility maximization), by assuming that players have *incomplete* information concerning the risk preferences of the opponents, but have complete information concerning their ordinal rankings. This exercise would rely on the notion of Δ-Rationalizability of Battigalli & Siniscalchi (2003, Definition 3.1).

An analog of what sketched above could be done for our ICBD procedure, by taking into account the differences coming from a dynamical setting. Hence, the appropriate game-theoretic notion we should rely on would be that of Strong Δ-Rationalizability, set forth in Battigalli (1999, Definition 3.2, p.22) and Battigalli (2003, Definition 3.2, p.17),[49] where the Δ in the name stands for a profile $\Delta := (\Delta_i)_{i \in I}$, where $\Delta_i \subseteq \Delta^{\mathcal{H}_i}(S_{-i})$ is called the *restriction for player i*.

It is important to point out that a straightforward adaptation of the exercise in Dekel & Siniscalchi (2015, Section 6.5), within the incomplete information setting described there, shows—additionally—what is the relation between ICBD and Strong Δ-Rationalizability and what kind of restrictions have to be imposed to obtain the strategies selected by ICBD from Strong Δ-Rationalizability.

8.6 Backward vs. Forward Induction Reasoning

From Section 8.4, we have that ICBD is intrinsically related to forward induction reasoning, something that comes from the fact that it mimics Iterated Conditional Dominance of Shimoji & Watson (1998), which algorithmically characterizes Strong Rationalizability. Nevertheless, as we observed

[48]See Siniscalchi (2008, Section 3), by bearing in mind that the author calls this notion "completeness".

[49]See Battigalli & Prestipino (2012, Section 3.3) and Battigalli & Friedenberg (2012, Footnote 6, pp.69-70) for a discussion of the various definitions present in the literature.

in Section 1, in dynamic games it is possible to reason also by following backward induction. Thus, why did we choose to focus on forward induction instead of backward induction reasoning?

First, it should be recalled from Section 1 that the Backward Dominance Procedure of Perea (2014) is the appropriate algorithmic procedure to capture backward induction. The notion that epistemically characterizes the Backward Dominance Procedure is that of *Rationality and Common Belief in the Opponents' Future Rationality*, which is not based on CPSs, but rather on mappings belonging to $[\Delta(S_{-i})]^{\mathcal{H}_i}$ that do not necessarily satisfy Axiom (A3). As pointed out in Perea (2012, Chapter 8.13, p.428), the conceptual reason behind this modelling decision is that the resulting notion is completely forward looking, since imposing the chain rule (that the author calls "Bayesian updating") would introduce a form of backward looking reasoning. Indeed, players should base their beliefs at a certain information set according to the beliefs held at a *previous* information set. Thus, our choice to focus on a forward induction reasoning procedure comes from the fact that our notion of rationality is crucially based on the existence of a CPS, that by definition has to satisfy the chain rule and—in doing so—is essentially backward looking.

Incidentally, what is written above is also the reason behind the fact that the Backward Dominance Procedure does *not* characterize Backward Rationalizability, much in the same spirit in which Rationality and Common Belief in the Opponents' Future Rationality does not epistemically characterize it. The point is that Backward Rationalizability employs CPSs in its definition, whereas Rationality and Common Belief in the Opponents' Future Rationality does not: however, Perea (2014) shows that a CPS-based notion of Rationality and Common Belief in the Opponents' Future Rationality (i.e., what he calls *"Rationality and Common Belief in the Opponents' Future Rationality & Common Belief in Bayesian Updating"*) characterizes Backward Rationalizability.

8.7 Genericity & The Backward Induction Outcome

In Section 6, we show that ICBD selects the unique backward induction outcome in dynamic games with perfect information that satisfy the genericity condition known as "No Relevant Ties". Our results can be related to an observation made in Battigalli (1997). As we already pointed out in Section 4.2, in Battigalli (1997, Footnote 9, p.50), it is mentioned the possibility of defining an 'ordinal' version of Strong Rationalizability, which is what we do in Algorithm 2. Additionally, it is written that—by means of Börgers (1993, Lemma, p.426)—it is possible to show that the corresponding ordinal version of Strong Rationalizability is outcome equivalent to Iterated Admissibility in generic dynamic games.

Also, since ICBD is the algorithmic counterpart of Ordinal Strong Rationalizability and captures forward induction reasoning in dynamic games with ordinal preferences, our Corollary 2 provides an 'ordinal' foundation to the so called Battigalli's theorem (see Battigalli (1997, Theorem 4, p.53)).[50] This fundamental result in the theory of dynamic games states that, given an NRT dynamic game (with cardinal preferences), the (unique) outcome z^{BI} can be induced by both Subgame Perfect Equilibrium *and* Strong Rationalizability. The importance of this result lies in the fact that Subgame Perfect Equilibrium in generic dynamic games is considered the prototypical application of *backward* induction reasoning, while Strong Rationalizability is considered a solution concept that captures *forward* induction reasoning. Thus, our 'ordinal' foundation of this theorem is particularly appealing in light of the fact that—as pointed out in Battigalli (1997, Section 4, p.52)—knowledge of cardinal preferences should not play any role in generic dynamic games with perfect information.

Closing on a bibliographical note, the Remark 6.2 in Section 6 has an ancestor in a similar result which relies on assuming that the dynamic game of perfect information is *generic*[51] (see Moulin (1986, Theorem 1, p.85)). Originally, this result can be found for the first time in Moulin (1979, Proposition 2), even if the idea of backward induction is not explicitly mentioned. However, Gretlein (1982) shows the presence of two problems in the proof of Moulin (1979, Proposition 2), amended in Gretlein (1983). Thus, as pointed out in Gretlein (1983, p.113), the first paper where the result is stated—by putting a special emphasis on the strict ordering over terminal histories of

[50] This result can be proved in various ways: Heifetz & Perea (2015, Theorem 2, Section 3), Perea (2018, Theorem 4.1, p.126), and Catonini (2020, Corollary 3, p.216) obtain it constructively. As mentioned in Heifetz & Perea (2015, Section 1), it is also possible to derive it by means of Theorem 1 in Marx & Swinkels (1997), as we do with Proposition 6 (see Appendix A.5). The original proof in Battigalli (1997, pp.59-60) borrowed arguments from Reny (1992), that—in turn—were based on properties of Fully Stable Sets of Kohlberg & Mertens (1986, Section 3.4).

[51] As in Footnote 36.

a given dynamic game with perfect information—and correctly proved is Rochet (1980),[52] where analogous bibliographical points can be found in Marx & Swinkels (1997, Section II, pp.221-222, & Section VI, p.240). It has to be observed that Moulin (1979), Rochet (1980), Gretlein (1983), and Moulin (1986) work with a condition actually stronger than the TDI condition, called the "One-to-One" Condition (see Moulin (1979, Section 1, p.1340)), which—as emphasized in Marx & Swinkels (1997, Section III, p.222)—is always satisfied by the strategic form of a generic dynamic game.

Appendix

A. Proofs

A.1 Proofs of Section 2

Proof of Lemma 2.1. Fix a restriction $R \subseteq S$, a player $i \in I$, an information set $\overline{h} \in H_i$ such that $R(\overline{h})$ is nonempty, and strategies $s_i, s_i' \in R_i(\overline{h})$. Now, for every $h \in H_i(s_i) \cup H_i(s_i')$, define strategy s_i^* as follows:

$$s_i^*(h) := \begin{cases} s_i(h), & \text{if } S(h) \subseteq S(\overline{h}), \\ s_i'(h), & \text{otherwise.} \end{cases}$$

Thus, we have by construction that $s_i^* \in R_i(\overline{h})$ is well-defined (i.e., it is defined for every $\widehat{h} \in H_i(s_i^*)$, beyond those information sets compatible with strategy profiles in R) and:

i) s_i and s_i^* are strong $R(\overline{h})$-replacements,

ii) while s_i' and s_i^* agree on $R_{-i}(\overline{h})$.

As a result, what is written above establishes the lemma. ■

Proof of Lemma 2.2. Let a player $i \in I$, a restriction $R \subseteq S$, and a strategy $s_i \in R_i$ be arbitrary and assume that there exists an information set $h \in H_i(s_i)$ such that $R_{-i}(h)$ is nonempty and there exists a strategy $s_i' \in R_i(h)$ such that s_i is weakly dominated relative to $R_{-i}(h)$ by s_i'. Thus, from Lemma 2.1, there exists a strategy $s_i^* \in R_i(h)$ that agrees with s_i' on $R_{-i}(h)$ and such that s_i and s_i^* are strong $R(h)$-replacements, which, in particular, implies that s_i and s_i^* agree on $R_{-i} \setminus R_{-i}(h)$. That s_i^* weakly dominates s_i relative to R_{-i} is an immediate consequence of the fact that—by construction—we have that:

- $\zeta(s_i^*, \overline{s}_{-i}) \sim_i \zeta(s_i, \overline{s}_{-i})$ for every $\overline{s}_{-i} \in R_{-i} \setminus R_{-i}(h)$, since s_i^* and s_i agree on $R_i \setminus R_i(h)$;
- $\zeta(s_i^*, s_{-i}) \succsim_i \zeta(s_i, s_{-i})$ for every $s_{-i} \in R_{-i}(h)$ and there exists a $s_{-i}^* \in R_{-i}(h)$ such that $\zeta(s_i^*, s_{-i}^*) \succ_i \zeta(s_i, s_{-i}^*)$, since s_i^* agrees with s_i' on $R_{-i}(h)$ and s_i' weakly dominates s_i relative to $R_{-i}(h)$.

Hence, what is written above establishes the result. ■

Proof of Lemma 2.3. Fix a dynamic game Γ with its corresponding strategic form Γ^r and let $R \subseteq S$ be an arbitrary restriction. Also, let $s_i \in R_i$ be arbitrary and assume that s_i is conditionally B-dominated given R. Thus, there exists and information set $h \in H_i(s_i)$ such that $R_{-i}(h)$ is nonempty and $s_i \in \mathsf{bd}_i(R(h))$. Thus, for every nonempty $Q_{-i} \subseteq R_{-i}(h)$, $s_i \notin \mathsf{A}_i(R_i(h) \times Q_{-i})$. Since the statement involves a universal quantifier, we have that it is—*a fortiori*—true for $R_{-i}(h)$, i.e., $s_i \notin \mathsf{A}_i(R_i(h) \times R_{-i}(h))$. Hence, there exists a strategy $\overline{s}_i \in R_i(h)$ that weakly dominates s_i relative to $R_{-i}(h)$. Now, we can invoke Lemma 2.2 in light of the fact that all its conditions are met. Thus, there exists a strategy $s_i^* \in R_i$ that agrees with $\overline{s}_i$ on $R_{-i}(h)$ and that agrees with s_i on $R_{-i} \setminus R_{-i}(h)$ that weakly dominates s_i relative to R_{-i}, which establishes the result. ■

[52] While writing these lines, we still had not have access to a copy of this paper: unfortunately, this work is presently unavailable, as confirmed by the author, who kindly tried to retrieve it without success (private communication). Hence, our decision of adding Moulin (1986) above (with full bibliographical details) as another reference for the result.

A.2 Proofs of Section 3

For the purpose of the proofs belonging to this section, we introduce a new—ancillary—notion. To introduce it, in the following, for every CPS $\mu_i \in \Delta^{\mathcal{H}_i}(S_{-i})$, we let $\operatorname{supp}\mu_i(\cdot|S_{-i}(h))$ denote its support given $S_{-i}(h)$. Thus, given a player $i \in I$ and a restriction R, a strategy $s_i^* \in R_i$ is *cautiously sequentially rational given* R if there exist a utility function $u_i \in \Re^Z$ and a CPS $\mu_i \in \Delta^{\mathcal{H}_i}(S_{-i})$, with $\operatorname{supp}\mu_i(\cdot|S_{-i}(h)) = R_{-i}(h)$,[53] for every $h \in H_i(s_i^*)$ for which $R_{-i}(h)$ is nonempty, such that

$$\sum_{s_{-i}\in R_{-i}} u_i(\zeta(s_i^*, s_{-i})) \cdot \mu_i(\{s_{-i}\}|S_{-i}(h)) \geq \sum_{s_{-i}\in R_{-i}} u_i(\zeta(s_i, s_{-i})) \cdot \mu_i(\{s_{-i}\}|S_{-i}(h)) \tag{A.1}$$

for every $s_i \in R_i(h)$.

Remark A.1. *Given a restriction $R \subseteq S$, if there exists a $Q_{-i} \subseteq R_{-i}$ such that strategy $s_i^* \in R_i$ is cautiously sequentially rational given $R_i \times Q_{-i}$, then s_i^* is sequentially rational given R.*

We can now proceed with the steps that lead to the proof of Proposition 1.

Lemma A.1. *Fix a player $i \in I$, a restriction $R \subseteq S$, a strategy $s_i^* \in R_i$, and assume that, for every information set $h \in H_i(s_i^*)$, if $R_{-i}(h)$ is nonempty, then s_i^* is not weakly dominated relative to $R_{-i}(h)$ by a strategy in $R_i(h)$. Then there exist a utility function $u_i \in \Re^Z$ and a CPS $\mu_i \in \Delta^{\mathcal{H}_i}(S_{-i})$ with $\operatorname{supp}\mu_i\big(\cdot\big|S_{-i}(\overline{h})\big) = R_{-i}(\overline{h})$, for every $\overline{h} \in H_i(s_i^*)$ for which $R_{-i}(\overline{h}) \neq \emptyset$, such that s_i^* maximizes the expected utility in R_i.*

Proof. We proceed by strong induction on the cardinality of R_{-i}. We let $\widetilde{\zeta}(s_i^*, \cdot) \in Z^{R_{-i}}$ denote the section of the restriction on $R \subseteq S$ of the outcome function $\zeta \in Z^S$ given $s_i^* \in R_i$. Also, we let $\underline{\alpha}$ denote the outcome for player i that is minimal with respect to the preference relation $\succsim_i$ restricted to the outcomes that can be obtained given $\widetilde{\zeta}(s_i^*, \cdot)$. Finally, we define

$$\underline{R}_{-i} := \left\{ s_{-i} \in R_{-i} \;\middle|\; \widetilde{\zeta}(s_i^*, s_{-i}) = \underline{\alpha} \right\}$$

and we let

$$\underline{S}_i := \{\, s_i \in S_i \mid \exists s_{-i} \in R_{-i} : \underline{\alpha} \succ_i \zeta(s_i, s_{-i}) \,\}\,.$$

- $|R_{-i}| = 1$. Let $\widetilde{s}_{-i}$ be the unique element of R_{-i} and assume that, for every information set $h \in H_i(s_i^*)$, if $R_{-i}(h)$ is nonempty, then s_i^* is not weakly dominated relative to $R_{-i}(h)$ by a strategy in $R_i(h)$. Trivially, since $\underline{R}_{-i} = \{\widetilde{s}_{-i}\}$, we have that $\widetilde{\zeta}(s_i^*, \widetilde{s}_{-i}) = \underline{\alpha}$. Now, define the utility function $\overline{u}_i$ as

$$\overline{u}_i(z) := \begin{cases} 1, & \text{if } z \succsim_i \underline{\alpha}, \\ 0, & \text{if } \underline{\alpha} \succ_i z, \end{cases}$$

  and, for every $h \in H_i(s_i^*)$ such that $R_{-i}(h) \neq \emptyset$, let $\mu_i(\cdot|R_{-i}(h)) := \delta_{\{\widetilde{s}_{-i}\}}$, where we let $\delta_{\{\widetilde{s}_{-i}\}}$ denote the Dirac measure over $\widetilde{s}_{-i}$. Clearly, μ_i trivially satisfies the required condition that $\operatorname{supp}\mu_i\big(\cdot\big|S_{-i}(\overline{h})\big) = R_{-i}(\overline{h})$, for every $\overline{h} \in H_i(s_i^*)$ for which $R_{-i}(\overline{h}) \neq \emptyset$. From the fact that s_i^* is by assumption not weakly dominated relative to $\{\widetilde{s}_{-i}\}$ and Remark 2.1, s_i^* is not strictly dominated relative to $\{\widetilde{s}_{-i}\}$. Hence, for every $s_i \in R_i$, $\zeta(s_i^*, \widetilde{s}_{-i}) \succsim_i \zeta(s_i, \widetilde{s}_{-i})$, i.e., $\underline{\alpha} \succsim_i \zeta(s_i, \widetilde{s}_{-i})$. Thus, opportunely rephrasing Equation (A.1), we have that

$$\begin{aligned}\sum_{s_{-i}\in R_{-i}} u_i(\zeta(s_i^*, s_{-i})) \cdot \mu_i(\{s_{-i}\}|R_{-i}(h)) &= \overline{u}_i(\zeta(s_i^*, \widetilde{s}_{-i})) \cdot \mu_i(\{\widetilde{s}_{-i}\}|R_{-i}(h)) = \\ &= \overline{u}_i(\zeta(s_i^*, \widetilde{s}_{-i})) \cdot \delta_{\{\widetilde{s}_{-i}\}} = \\ &= \overline{u}_i(\underline{\alpha}) \cdot \delta_{\{\widetilde{s}_{-i}\}} = 1 \geq \overline{u}_i(\zeta(s_i, \widetilde{s}_{-i})) \cdot \delta_{\{\widetilde{s}_{-i}\}},\end{aligned}$$

  for every $h \in H_i(s_i^*)$ and for every $s_i \in R_i(h)$, establishing the result.

[53] The usage of the term "cautious" in this definition is related to this full-support condition, as in Pearce (1984, Appendix B, p.1049) (see also Börgers (1993, Definition 3, p.426) and discussion thereafter).

- $|R_{-i}| = n$, with $n > 1$. Assume that the statement is true for every $k \in \mathbb{N}$ such that $1 < k \leq n$ and that there exists no information set $h \in H_i(s_i^*)$ such that s_i^* is weakly dominated relative to $R_{-i}(h)$ by a strategy in $R_i(h)$. We split the proof of the strong induction step in two cases.

  Case 1 ($R_{-i} \setminus \underline{R}_{-i} \neq \emptyset$). We have that $|R_{-i} \setminus \underline{R}_{-i}| < |R_{-i}|$ by assumption and that $s_i^* \in R_i \setminus \underline{S}_i$ by construction. By means of the strong induction hypothesis, this allows us to assume that the claim holds for $R_i \setminus \underline{S}_i \times R_{-i} \setminus \underline{R}_{-i}$. Thus, as a result of what is written above, we have in particular that there exist a utility function $\overline{u}_i \in \Re^Z$ and a CPS $\overline{\mu}_i \in \Delta^{\mathcal{H}_i}(S_{-i})$ with $\operatorname{supp} \overline{\mu}_i\big(\cdot \big| S_{-i}(\overline{h})\big) = (R_{-i} \setminus \underline{R}_{-i}) \cap S_{-i}(\overline{h})$, for every $\overline{h} \in H_i(s_i^*)$ for which $(R_{-i} \setminus \underline{R}_{-i}) \cap S_{-i}(\overline{h}) \neq \emptyset$ such that s_i^* is a maximizer of $\overline{u}_i$ in $R_{-i} \setminus \underline{R}_{-i}$ with respect to $\overline{\mu}_i$. We now define a utility function $u_i^\delta(z)$, with $\delta > 0$, as follows:

$$u_i^\delta(z) := \begin{cases} \overline{u}_i(z), & \text{if } z \succsim_i \underline{\alpha}, \\ \overline{u}_i(z) - \delta, & \text{if } \underline{\alpha} \succ_i z. \end{cases} \tag{A.2}$$

  Also, for every $\varepsilon \in (0,1)$, we define a CPS $\mu_i^\varepsilon \in \Delta^{\mathcal{H}_i}(S_{-i})$ as

$$\mu_i^\varepsilon\big(\{s_{-i}\}\big|S_{-i}(\overline{h})\big) := \begin{cases} (1-\varepsilon)\cdot \overline{\mu}_i\big(\{s_{-i}\}\big|S_{-i}(\overline{h})\big), & \text{if } s_{-i} \in R_{-i} \setminus \underline{R}_{-i}, \\ \frac{\varepsilon}{|\underline{R}_{-i}|}, & \text{if } s_{-i} \in \underline{R}_{-i}, \end{cases}$$

  for every $\overline{h} \in H_i(s_i^*)$ such that $R_{-i}(\overline{h}) \neq \emptyset$, and notice that $\operatorname{supp} \mu_i^\varepsilon\big(\cdot\big|S_{-i}(\overline{h})\big) = R_{-i}(\overline{h})$. To see that this is actually a CPS, let $\widehat{R}_{-i} \subseteq R_{-i}$ and observe that we have

$$\mu_i^\varepsilon\Big(\widehat{R}_{-i}\Big|S_{-i}(\overline{h})\Big) = (1-\varepsilon)\cdot\overline{\mu}_i\Big(\widehat{R}_{-i} \cap (R_{-i} \setminus \underline{R}_{-i})\Big|S_{-i}(\overline{h})\Big) + \varepsilon \tag{A.3}$$

  for every $\overline{h} \in H_i(s_i^*)$ such that $R_{-i}(\overline{h}) \neq \emptyset$. We now show that the definition above satisfies Axioms (A1)-(A3) in the definition of CPS. The fact that Axiom (A2) is satisfied is immediate from Equation (A.3). It is immediate also that Axiom (A1) is satisfied since, by letting $h \in H_i(s_i^*)$ be arbitrary, given $S_{-i}(h)$ as the conditioning event and $R_{-i}(h)$ nonempty, we have

$$\begin{aligned} \mu_i^\varepsilon(S_{-i}(h)|S_{-i}(h)) &= \mu_i^\varepsilon(R_{-i}(h)|S_{-i}(h)) = \\ &= (1-\varepsilon)\cdot\overline{\mu}_i\big(R_{-i}(h) \cap (R_{-i} \setminus \underline{R}_{-i})\big|S_{-i}(h)\big) + \varepsilon = 1, \end{aligned}$$

  since $\overline{\mu}_i\big(R_{-i}(h) \cap (R_{-i} \setminus \underline{R}_{-i})\big|S_{-i}(h)\big) = 1$. Focusing on Axiom (A3), let $\varepsilon \in (0,1)$ and the conditioning events $S_{-i}(h), S_{-i}(h') \in \mathcal{H}_i$ be arbitrary, assume that $\widehat{R}_{-i} \subseteq R_i$ as before and—additionally—that $\widehat{R}_{-i} \subseteq S_{-i}(h') \subseteq S_{-i}(h)$, and notice that we have that

$$\begin{aligned} \frac{\mu_i^\varepsilon\Big(\widehat{R}_{-i}\Big|S_{-i}(h)\Big) - \varepsilon}{\mu_i^\varepsilon(S_{-i}(h')|S_{-i}(h)) - \varepsilon} &= \frac{(1-\varepsilon)\cdot\overline{\mu}_i\Big(\widehat{R}_{-i} \cap (R_{-i} \setminus \underline{R}_{-i})\Big|S_{-i}(h)\Big)}{(1-\varepsilon)\cdot\overline{\mu}_i\big(S_{-i}(h') \cap (R_{-i} \setminus \underline{R}_{-i})\big|S_{-i}(h)\big)} = \\ &= (1-\varepsilon)\cdot\frac{\overline{\mu}_i\Big(\widehat{R}_{-i} \cap (R_{-i} \setminus \underline{R}_{-i})\Big|S_{-i}(h)\Big)}{\overline{\mu}_i\big(S_{-i}(h') \cap (R_{-i} \setminus \underline{R}_{-i})\big|S_{-i}(h)\big)} = \\ &= (1-\varepsilon)\cdot\overline{\mu}_i\Big(\widehat{R}_{-i} \cap (R_{-i} \setminus \underline{R}_{-i})\Big|S_{-i}(h')\Big) = \\ &= \mu_i^\varepsilon\Big(\widehat{R}_{-i}\Big|S_{-i}(h')\Big) - \varepsilon, \end{aligned}$$

  where the second to last equality comes from the fact that $\overline{\mu}_i$ is a CPS. Thus, notice that if we choose ε small enough, s_i^* continues to be an expected utility maximizer of $\overline{u}_i$ with respect to $R_i \setminus \underline{S}_i$ according to μ_i^ε, since utility is continuous with respect to CPSs at every $S_{-i}(h)$. We let $\overline{\varepsilon}$ denote such value, we fix it, and we let δ vary. Since increasing δ affects only the utility of the strategies in $\underline{S}_i$, which can be made in this way arbitrarily small, as a result, for a δ large enough, which we denote by $\overline{\delta}$, s_i^* is an expected utility maximizer in R_i of $u_i^{\overline{\delta}}$ with respect to the CPS $\mu_i^{\overline{\varepsilon}}$.

  Case 2 ($R_{-i} \setminus \underline{R}_{-i} = \emptyset$). Let $\mu_i \in \Delta^{\mathcal{H}_i}(S_{-i})$ be an arbitrary CPS with support on R_{-i}. Let $\overline{u}_i \in \Re^Z$ be an arbitrary utility function and let u_i^δ be defined as in Equation (A.2) from Case (1). Thus, with δ large enough, the utility function u_i^δ and the CPS μ_i satisfy the desired properties.

This completes the proof of the lemma. ■

Lemma A.2. *Fix a player $i \in I$ and a restriction $R \subseteq S$. Then a strategy $s_i^* \in R_i$ is cautiously sequentially rational given R if and only if, for every information set $h \in H_i(s_i^*)$, if $R_{-i}(h)$ is nonempty, then s_i^* is not weakly dominated relative to $R_{-i}(h)$ by a strategy in $R_i(h)$.*

Proof. [$\Rightarrow$] We assume that $s_i^* \in R_i$ is cautiously sequentially rational given R_{-i} and—proceeding by contradiction—that there exists an information set $\overline{h} \in H_i(s_i^*)$ such that s_i^* is weakly dominated relative to $R_{-i}(\overline{h})$, with $R_{-i}(\overline{h})$ nonempty. Hence, there exists a strategy $\overline{s}_i \in R_i(\overline{h})$ such that $\zeta(\overline{s}_i, s_{-i}) \succsim_i \zeta(s_i^*, s_{-i})$ for every $s_{-i} \in R_{-i}(\overline{h})$ and there exists a $s_{-i}^* \in R_{-i}(\overline{h})$ such that $\zeta(\overline{s}_i, s_{-i}^*) \succ_i \zeta(s_i^*, s_{-i}^*)$. From the assumption that s_i^* is cautiously sequentially rational given R, there exist a utility function $u_i \in \Re^Z$ and a CPS $\mu_i \in \Delta^{\mathcal{H}_i}(S_{-i})$, with $\operatorname{supp} \mu_i(\cdot | S_{-i}(h)) = R_{-i}(\overline{h})$ for every $h \in H_i(s_i^*)$ for which $R_{-i}(h) \neq \emptyset$, such that Equation (A.1) is satisfied by s_i^*. But, since $\overline{s}_i$ weakly dominates s_i^* at $\overline{h}$ and $\operatorname{supp} \mu_i\big(\cdot \big| S_{-i}(\overline{h})\big) = R_{-i}(\overline{h})$, we have that

$$\sum_{s_{-i} \in R_{-i}} u_i(\zeta(\overline{s}_i, s_{-i})) \cdot \mu_i\big(\{s_{-i}\} \big| S_{-i}(\overline{h})\big) > \sum_{s_{-i} \in R_{-i}} u_i(\zeta(s_i^*, s_{-i})) \cdot \mu_i\big(\{s_{-i}\} \big| S_{-i}(\overline{h})\}\big),$$

contradicting the fact that that Equation (A.1) is satisfied by s_i^* at every $h \in H_i(s_i^*)$.
[$\Leftarrow$] Assume that for every information set $h \in H_i(s_i^*)$, if $R_{-i}(h)$ is nonempty, then s_i^* is not weakly dominated relative to $R_{-i}(h)$ by a strategy in $R_i(h)$. Then the result follows immediately from Lemma A.1. ■

Proof of Lemma 3.1. [$\Rightarrow$] Let $R \subseteq S$ be an arbitrary restriction and assume that s_i^* is sequentially rational given R. Hence, there exist a utility function $u_i \in \Re^Z$ and a CPS $\mu_i \in \Delta^{\mathcal{H}_i}(R_{-i})$ such that for every $h \in H_i(s_i^*)$ if $R_{-i}(h)$ is nonempty, then Equation (3.1) is satisfied by s_i^* for every $s_i \in R_i(h)$. We proceed by contradiction and we assume that s_i^* is conditionally B-dominated with respect to R. Hence, there exists an information set $\overline{h} \in H_i(s_i^*)$ such that $R(\overline{h})$ is nonempty and $s_i^* \in \mathsf{bd}_i(R(\overline{h}))$. Thus, for every $Q_{-i} \subseteq R_{-i}(\overline{h})$, we have that s_i^* is weakly dominated relative to Q_{-i} by a strategy $s_i \in R_i(\overline{h})$. Let $Q_{-i}^{\mu_i} := \operatorname{supp} \mu_i\big(\cdot \big| S_{-i}(\overline{h})\big)$, with $Q_{-i}^{\mu_i} \subseteq R_{-i}(\overline{h})$ by definition, and let $\overline{s}_i \in R_i(\overline{h})$ denote the strategy that weakly dominates s_i^* relative to $Q_{-i}^{\mu_i}$. Thus, $\zeta(\overline{s}_i, s_{-i}) \succsim_i \zeta(s_i^*, s_{-i})$ for every $s_{-i} \in Q_{-i}^{\mu_i}$ and there exists a $s_{-i}^* \in Q_{-i}^{\mu_i}$ such that $\zeta(\overline{s}_i, s_{-i}^*) \succ_i \zeta(s_i^*, s_{-i}^*)$. But then we have that

$$\sum_{s_{-i} \in R_{-i}} u_i(\zeta(\overline{s}_i, s_{-i})) \cdot \mu_i\big(\{s_{-i}\} \big| S_{-i}(\overline{h})\big) > \sum_{s_{-i} \in R_{-i}} u_i(\zeta(s_i^*, s_{-i})) \cdot \mu_i\big(\{s_{-i}\} \big| S_{-i}(\overline{h})\big),$$

contradicting the fact that Equation (3.1) is satisfied by s_i^* at every $h \in H_i(s_i^*)$.
[$\Leftarrow$] Let $R \subseteq S$ be an arbitrary restriction and assume that s_i^* is not conditionally B-dominated with respect to R. Hence, for every $h \in H_i(s_i^*)$, if $R(h)$ is nonempty, then there exists a nonempty subset $Q_{-i}(h) \subseteq R_{-i}(h)$ such that $s_i^* \in \mathsf{A}_i(R_i(h) \times Q_{-i}(h))$. Let $Q_{-i} := \bigcup_{\overline{h} \in H_i(s_i^*)} Q_{-i}(\overline{h})$ and notice that, from Lemma A.2, s_i^* is cautiously sequentially rational given $R_i \times Q_{-i}$. Hence, from Remark A.1, s_i^* is—*a fortiori*—sequentially rational given R. ■

Proof of Proposition 1. The result follows immediately from Lemma 3.1 with $R := S$. ■

A.3 Proofs of Section 4

A.3.1 *Proofs of Section 4.1*

To prove Lemma 4.1 we recall some basic definitions from order theory, which we use to point out a property of weak dominance. Given an arbitrary set Y, a binary relation $\gtrdot \subseteq Y \times Y$ on Y is a *strict partial order* if: (Irreflexivity) for every $x \in Y$, it is not the case that $x \gtrdot x$; (Transitivity) for every $x, y, z \in Y$, if $x \gtrdot y$ and $y \gtrdot z$, then $x \gtrdot z$. An arbitrary (finite) set Y endowed with a strict partial order is called a *(finite) strict poset.*

Remark A.2. *Given a finite strict poset $(Y, \gtrdot)$, there exists a maximal element in Y.*

For every $i \in I$, weak dominance as in Section 2.2 is a binary relation on the strategy set S_i, which is a strict partial order.[54]

Proof of Lemma 4.1. Let $R \subseteq S$ be arbitrary and nonempty. Proceeding by contradiction, assume that $\mathbb{U}(R) = \emptyset$. Hence, there exists a player $i \in I$ such that $\mathbb{U}_i(R) = \emptyset$, i.e., for every $s_i \in R_i$ there exists an information set $h \in H_i(s_i)$ such that $R(h)$ is nonempty and $s_i \in \mathsf{bd}_i(R(h))$. Thus, for every $Q_{-i}(h) \subseteq R_{-i}(h)$ there exists a strategy $\overline{s}_i \in R_i(h)$ such that $\overline{s}_i$ weakly dominates s_i relative to $Q_{-i}(h)$. From Lemma 2.2, there exists a strategy $s_i^* \in R_i(h)$ that agrees with $\overline{s}_i$ on $R_{-i}(h)$ and that agrees with s_i on $R_{-i} \setminus R_{-i}(h)$ that weakly dominates s_i relative to R_{-i}. Since s_i was arbitrary, the same argument can be applied to every strategy in R_i, i.e., every strategy $\widetilde{s}_i \in R_i$ is weakly dominated relative to R_i by a strategy $\widetilde{s}_i^* \in R_i$. But this implies that there exists no maximal element in R_i with respect to weak dominance, which is a strict partial ordering. Hence, from the finiteness of the dynamic games under scrutiny and Remark A.2, we obtain the desired contradiction. ∎

Proof of Proposition 2. From Lemma 4.1, for every $R \subseteq S$, $\mathbb{U}(R)$ is nonempty. Also, from Remark 4.1, for every $R \subseteq S$, $\mathbb{U}(R) \subseteq R$ implies that $\mathbb{U}^k(R) \subseteq \mathbb{U}^{k-1}(R)$, for every $k \in \mathbb{N}$. Hence, for every $\ell \in \mathbb{N}$, $\mathbb{U}^\ell(S)$ is nonempty, i.e., $\bigcap_{\ell \geq 0} \mathbb{U}^\ell(S) \neq \emptyset$, and—since S is finite—there exists a $K \in \mathbb{N}$ such that $\mathbb{U}^K(S) = \mathbb{U}^{K+1}(S) = \mathbb{U}^\infty(S)$. ∎

A.3.2 *Proofs of Section 4.3*

Proof of Lemma 4.2. In the following, we fix a dynamic game Γ and we let $R \subseteq S$ be an arbitrary restriction.
[⇒] We prove the contrapositive. Thus, let $s \in R$ be arbitrary and assume that $s \notin \mathbb{S}(R)$. Thus, there exists a player $i \in I$ such that s_i is not sequentially rational given R. Hence, it follows from Lemma 3.1 that $s_i \notin \mathbb{U}_i(R)$, thus establishing the result.
[⇐] We prove the contrapositive. Thus, let $s \in R$ be arbitrary and assume that $s \notin \mathbb{U}(R)$. Thus, there exists a player $i \in I$ such that s_i is conditionally B-dominated with respect to R. Hence, from Lemma 3.1, s_i is not sequentially rational given R, i.e., $s_i \notin \mathbb{S}_i(R)$, that establishes the result. ∎

Proof of Proposition 3. Fix a dynamic game Γ.

i) This result follows immediately from Lemma 4.2.

ii) This result follows immediately from Lemma 4.2 along with the finiteness of the dynamic games under scrutiny. ∎

A.4 Proofs of Section 5

A.4.1 *Proof of Section 5.1*

Proof of Proposition 4. We proceed by proving the contrapositive. Thus, let $R \subseteq S$ and $s \notin \mathbb{U}(R)$ be arbitrary. Hence, there exists a player $i \in I$ and an information set $h \in H_i(s_i)$ with $R(h)$ nonempty such that $s_i \in \mathsf{bd}_i(R(h))$. By definition of B-dominance, this implies that for every $Q_{-i} \in R_{-i}(h)$ there exists a strategy $\widetilde{s}_i \in R_i(h)$ such that $\widetilde{s}_i$ weakly dominates s_i relative to Q_{-i}. In particular, from Remark 2.1, for every singleton subset $\{s_{-i}\} \subseteq R_{-i}(h)$ there exists a strategy $\overline{s}_i \in R_i(h)$ such that $\overline{s}_i$ strictly dominates s_i relative to $\{s_{-i}\}$. From the finiteness assumption and the fact that $R(h)$ is nonempty, there exists at least one s_{-i}^* such that $\{s_{-i}^*\} \subseteq R_{-i}(h)$. Let $s_i^* \in R_i(h)$ denote the corresponding strategy of player i such that s_i^* strictly dominates s_i relative to $\{s_{-i}^*\}$. Hence, recalling the notation of Dirac measure employed in the proof of Lemma A.1, strategy $s_i \in \mathsf{md}_i(R(h))$ via the (degenerate) mixed strategy $\delta_{\{s_i^*\}} \in \Delta(R_i(h))$. ∎

[54]See Gilboa et al. (1990, Section 3).

A.4.2 *Proofs of Section 5.2*

For the purpose of the proofs that follows, we introduce new notation. In particular, for every $i \in I$ and $R \subseteq S$, we let $\mathsf{md}_i^{u_i}(R)$ denote the set of strategies of player i that are strictly dominated relative to R_{-i} by mixed strategies given the utility function $u_i \in \mathcal{U}_i$.

Before proceeding with the proof, a clarification is in order. Whereas in the following we repeatedly refer to Börgers (1993, Proposition, p.427) in presence of a given restriction $R \subseteq S$, two points need to be emphasized regarding this usage.

1. Strictly speaking, no reference to restrictions $R \subseteq S$ is made in Börgers (1993, Proposition, p.427). However, It is immediate to establish such result from Lemma 3.1 opportunely modified to deal with the static case.
2. The "given" in the notion we use has to be read in light of the previous point and *not* in the way in which Börgers (1993) uses this term, which corresponds—has already mentioned in Appendix A.2 at Footnote 53—to our notion of *cautious* sequential rationality given a restriction.

Proof of Lemma 5.1. [$\Rightarrow$] We let $R \subseteq S$, we assume that $s_i \in \mathbb{U}_i(R)[\succsim]$, and we proceed by contradiction. Hence, we assume that for every $u_i \in \mathcal{U}_i$, $s_i \notin \mathbb{M}_i(R)[u_i]$. Thus, for every $u_i \in \mathcal{U}_i$ there exists an information set $h \in H_i(s_i)$ such that $R_{-i}(h)$ is nonempty and $s_i \in \mathsf{md}_i^{u_i}(R(h))$, where every such information set $\widehat{h} \in H_i(s_i)$ induces a 2-player (static) ordinal game with players i and $-i$ and strategy sets $R_i(\widehat{h}) \times R_{-i}(\widehat{h})$, (where we can disregard the payoff of $-i$) at which $s_i \notin \mathsf{bd}_i(R(\widehat{h})$. But the fact that $s_i \notin \mathsf{bd}_i(R(\widehat{h})$ implies in the corresponding 2-player static game by Börgers (1993, Proposition, p.427) and Pearce (1984, Lemma 3, p.1048) that there exists a utility function $\widehat{u}_i \in \mathcal{U}_i$ such that $s_i \notin \mathsf{md}_i^{\widehat{u}_i}(R(\widehat{h}))$, thus, reaching a contradiction and establishing the result.
[$\Leftarrow$] We proceed by proving the contrapositive. Thus, we let $R \subseteq S$ be arbitrary and we assume that $s_i \notin \mathbb{U}_i(R)[\succsim]$, that is, there exists a player $i \in I$ such that s_i is conditionally B-dominated with respect to R. Hence, there exists an information set $h \in H_i(s_i)$ such that $R_{-i}(h)$ is nonempty and $s_i \in \mathsf{bd}_i(R(h))$. Now, observe that the information set $h \in H_i(s_i)$ induces a 2-player (static) ordinal game with players i and $-i$ and strategy sets $R_i(h) \times R_{-i}(h)$, where we can disregard the payoff of $-i$. Thus, from Börgers (1993, Proposition, p.427) taking into account the presence of the restriction $R \subseteq S$, it follows that s_i is not rational given R for every $\mu_i \in \Delta(R_{-i}(h))$ and $u_i \in \Re^Z$. Hence, from Pearce (1984, Lemma 3, p.1048), it follows that for every utility function $u_i \in \mathcal{U}_i$ strategy s_i is strictly dominated relative to $R_{-i}(h)$ by a mixed strategy $\sigma_i^* \in \Delta(R_i(h))$, which establishes the result. ■

A.5 Proofs of Section 6

Fix a dynamic game Γ whose strategic form representation Γ^r satisfies the TDI condition as in Definition 6.1 and define the following algorithmic procedure on Γ^r: $\mathbf{U}_i^0 := S_i$, $\mathbf{U}_i^{n+1} := \mathbb{U}_i^{n+1}(S)$, and $\mathbf{U}_i^\infty := \bigcap_{\ell \geq 0} \mathbf{U}_i^\ell$, where $\mathbf{U}^k := \prod_{j \in I} \mathbf{U}_j^k$ for every $k \geq 0$. Finally, $\mathbf{U}^\infty := \bigcap_{\ell \geq 0} \mathbf{U}^\ell$ is the set of strategy profiles that survive the ICBD algorithm *in* Γ^r. Observe that, in the specific class of games under scrutiny, $\mathbf{U}^\infty$ is well-defined.

The—crucial for the purpose of our proof—fact that $\mathbf{U}^\infty$ gives rise to a full reduction by weak dominance on S is stated and proved next. To accomplish this goal, first of all, we let $\mathsf{cbd}_i(R)$ denote the set of player i's strategy that are conditionally B-dominated with respect to $R \subseteq S$, with $\mathsf{cbd}(R) := \prod_{j \in I} \mathsf{cbd}_j(R)$. This allows us to rewrite Lemma 2.3 in the following, more compact, form:

$$s_i \in \mathsf{cbd}_i(R) \Longrightarrow s_i \notin \mathsf{A}_i(R), \tag{A.4}$$

for every $R \subseteq S$, $i \in I$, and $s_i \in R_i$. Also, we can now rewrite $\mathbb{U}^n(R)$ as

$$\mathbb{U}^n(R) = \mathbb{U}^{n-1}(R) \setminus \mathsf{cbd}(\mathbb{U}^{n-1}(R)). \tag{A.5}$$

Lemma A.3. *Given a dynamic game Γ with corresponding strategic form representation Γ^r satisfying the TDI condition, $\mathbf{U}^\infty$ is a full reduction by weak dominance on S.*

Proof. Fix a dynamic game Γ with corresponding strategic form representation Γ^r satisfying the TDI condition as in Definition 6.1. We prove by induction on $n \in \mathbb{N}$ that $\mathbf{U}^\infty$ as defined above is a reduction by weak dominance.

- ($n = 0$) By definition, we have that $\mathbf{U}^0 := \mathbb{U}^0(S) = S$.
- ($n \geq 1$) Assume that the result has been established for $n \in \mathbb{N}$. To establish that $\mathbf{U}$ is a reduction by weak dominance, first of all, observe that we have by definition that $\mathbf{U}^{n+1} = \mathbb{U}^{n+1}(S)$. From Equation (A.5), this in turn is equivalent to $\mathbb{U}^{n+1}(S) = \mathbb{U}^n(S) \setminus \mathsf{cbd}(\mathbb{U}^n(S))$. The result follows immediately from Lemma 2.3 in the form of Equation (A.4).

Since the dynamic games under scrutiny are finite, what is written above establishes that $\mathbf{U}^\infty$ is a reduction by weak dominance. That $\mathbf{U}^\infty$ is additionally a *full* reduction by weak dominance is an immediate consequence of Proposition 2. Hence, this establishes the result. ■

Remark A.3 (Marx & Swinkels (1997, Corollary 1, p.230)). *Given a dynamic game with corresponding strategic form representation Γ^r satisfying the TDI condition, if $\mathbf{X}^\infty$ and $\mathbf{Y}^\infty$ are two full reductions by weak dominance on Γ^r, then $\zeta(\mathbf{X}^\infty) = \zeta(\mathbf{Y}^\infty)$.*

Proof of Proposition 6. Fix an NRT dynamic game with corresponding strategic form representation Γ^r. Since the dynamic game satisfies the NRT condition, from Remark 6.1, Γ^r satisfies the TDI condition. Hence, from Remark 6.2, $\mathbf{A}^\infty$ applied on Γ^r delivers the backward induction outcome, i.e., $\zeta(\mathbf{A}^\infty) = \{z^{BI}\}$. The result follows immediately from Lemma A.3 and Remark A.3 with $\mathbf{U}^\infty$ and $\mathbf{A}^\infty$. ■

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