Seller-paid Ratings

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Abstract

A rater is paid by a seller, observes a signal about the seller's product and issues a public cheap-talk rating for potential buyers. I characterize the equilibrium partition of the rater's information into ratings in two regimes: when payments from the seller to the rater are publicly disclosed, and when payments are not disclosed and remain private. Public payments are compatible with precise ratings and can reveal rater's information perfectly. Private payments tend to be inflated for high ratings, which endogenously leads to coarse ratings in equilibrium. I characterize optimal contract offered by a competitive when payments are public, it results in a coarse rating for signals below a threshold which is free for the seller, and precise ratings for signals above the threshold with payments increasing in ratings.

Keywords: cheap talk, rating agency, mechanism design, transparency, limited commitment.

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Introduction

In assessing financial products potential buyers often rely on ratings provided by credit rating agencies. The three largest credit rating agencies Moody's, Standard & Poor's and Fitch, operate under the seller-pays business model, they get around 90 percent of their revenue from fees paid by the issuers according to Partnoy (2006). The potential buyers of the rated financial products typically access the ratings for free. The financial crisis of 2007-2008 raised concerns over the conflict of interest that could arise in the seller-pays business model. Credit rating agencies were blamed for catering to issuers and inflating the ratings for mortgage backed CDOs (collateralized debt obligations).[1] At first blush, concerns about rating inflation seem to contradict rationality of investors purchasing highly rated financial products: it is well known that rating agencies are paid by issuers and rational investors should correctly infer the informational content of their rating.

This paper focuses on the transparency aspect of the seller-pays business model of a rater, and shows that non-transparent payments can cause a loss of information and inefficiencies, even assuming rational buyers and no external forces, such a rating based regulation, that would make ratings valuable.[2] Specifically, the paper characterizes how the rater's information about financial products maps into ratings (feasible partitions) in transparent and non-transparent contracting environments between the seller and the rater. It finds that precise ratings are feasible only when payments from the seller to the rater are transparent, i.e. publicly disclosed. These results show that even with rational buyers seller-paid ratings may be inefficient when payments are not-transparent, yet the inefficiency is not due to "rating inflation" or regulatory arbitrage, but due to the fact that equilibrium ratings are coarse and part of the information is lost. The analysis argues in favor of a regulation mandating transparent payments.

The 2010 report on credit rating agencies by the U. S. Government Accountability Office recommended that "An effective compensation model should be transparent to market participants to help them understand it and to increase market acceptance." Yet, current regulation under the Dodd-Franck act does not require credit rating agencies to disclose

[1] Griffin and Tang (2012) employ a model used by one of the credit rating agencies and document that just before 2007 the AAA tranches of CDOs were larger than what the rating agency model would deliver. Furthermore, according to the 2009 report by International Monetary Fund of all asset-backed security collateralized debt obligations issued in 2005-2007 in the U.S. and rated AAA by Standard & Poor's, only 10% maintained AAA rating in 2009. Pagano and Volpin (2010) documents similar evidence.

[2] Several recent papers argued that the conflict of interest in credit rating agencies may have been exacerbated by regulation based on ratings Opp et al. (2013) or by seller's forum shopping and "naive" nature of some buyers Skreta and Veldkamp (2009), Bolton et al. (2012).

their fees. This is striking, given that the role of credit ratings in marketing financial products is almost as important as the role of underwriters. Underwriters of securities are required to disclose their commissions under the 1933 Securities Act. Buyers of securities can check underwriters' commissions in the prospectus published on the SEC website (www.sec.gov). This paper suggests that similar requirements may be imposed on credit rating agencies.

Borrowing ideas from the two-sided market literature started by Rochet and Tirole (2003) I consider the rater as an information platform which needs to attract both sides of the market: product sellers that pay for ratings, and product buyers that use ratings in their decisions. In the baseline two-period model the rater receives a payment from the seller in the first period when he issues a rating about the product and gets a reputational payoff the next period when the product payoff realizes. The rater's reputational payoff in the second period is proportional to his *user base* (the number of buyers that use his ratings) in the second period.[3] The rater's user base in the second period in turn depends on the payoff experienced by the buyers that bough the product in the first period.[4]

First, the paper shows that in the absence of payments from the seller, the rater who maximizes his reputational payoff is cautious with ratings. Formally, in the terminology of cheap talk literature the rater is "downward biased", but unlike in the classic Crawford and Sobel (1982) and many following papers, here the bias is not assumed but arises endogenously. Intuitively, the rater's user base and reputational payoff at the second period are affected by the return experienced by the first period buyers. The return depends on the product's actual payoff, its price and the quantity purchased by buyers. A rating published by the rater does not effect the product's actual quality, but it affects its perceived quality and, as a result, it affects the buyers' payoff through the quantity purchased and through the product's price. The quantity is optimally chosen by the buyers, so the marginal effect of quantity on their payoff is of second order by Envelope theorem. Yet, the price is set by the seller, and the price effect on the buyers' net payoff is of the first order and is negative. Indeed, for a given actual payoff of the product an increase in the product's

[3]In Section A.2 I study the infinite horizon model and show that the reputational payoff in the two-period model can be viewed as a shortcut for the net present value of future payments from sellers to the rater.

[4]Assuming the rater's user base to depend on past returns experienced by rating users is natural, albeit hard to gauge empirically because ratings are public and the exact number of investors using them is uncertain. In the context of mutual equity funds Sirri and Tufano (1998) documents that investors flock to the funds with the highest recent returns, suggesting that the investors using ratings of different rating agencies may exhibit a similar behavior.

rating increases its price and reduces the net return accruing to the buyers. As a result, the rater has an incentive to marginally downplay his information about the product's payoff in an attempt to lower the equilibrium price. Of course rational buyers understand this bias and use the Bayes' rule to interpret ratings, in equilibrium ratings must be correct on average and cannot be systematically lower than the expected payoff. Yet, this bias can lead to a coarse rating in equilibrium.

Second, the paper finds that precise ratings are feasible when payments from the seller to the rater are allowed and required to be transparent, that is publicly disclosed. Intuitively, the rater's "downward bias" can be compensated with a payment schedule increasing in ratings, which would induce the rater to perfectly reveal his information. This can be done, for instance, with a payment which is proportional to the seller's revenue. Then I consider a competitive rater who tries to attract the seller and designs ratings and the payment schedule that deliver the highest expected profit to the seller (seller-optimal ratings). Seller-optimal ratings take a very natural form with an imprecise rejection rating being issued when the product's expected payoff is below a threshold and perfectly precise ratings being issued above the threshold. The optimal payment above the threshold is approximately proportional to the seller's revenue and resembles the actual fee structure of credit ratings agencies.[5]

Third, only coarse partially informative ratings are feasible when the payments are private to the rater and the seller, that is are not transparent. Essentially, private payments result in an endogenous "bias" in the terminology of Crawford and Sobel (1982) which causes partition of the rater's information in coarse ratings in equilibrium. The bias arises because the seller can manipulate the rater's reports by secretly increasing payments for certain ratings without the buyers observing this and their beliefs reacting. Rational buyers anticipate these potential manipulations and form believes that pool distinct realizations of rater's information into coarse ratings. To better understand why precise ratings are not feasible, suppose ratings were precise and perfectly revealed the rater's information. In this case the seller would have an incentive to elicit the highest rating by secretly inflating the payment for this rating. This would bias the rater's reports away from the truth, which couldn't happen in equilibrium. To prevent such manipulations the ratings must be imprecise. Intuitively, when the highest rating is issued for a wide range of rater's signals it is costly to inflate the payment for this rating because for each realization of the rater's

[5]Credit rating agencies' fees for corporate bonds correspond to 3-4 basis points of the issuer's proceeds.

signal within the rating's range the seller has to make the inflated payment. At the same time, the expected gain is limited because the buyers' perception of the rating is equal to the average signal within the corresponding rating's range. The highest rating being issued for a wide range of signals resembles the phenomenon of "rating inflation," even though rational buyers interpret it's informational content correctly. Interestingly, a competitive rater who maximizes the expected profit of the seller would offer the uninformative rating. Implementing partially informative ratings with private payments requires significant payments from the seller, and results in a lower profit than she can get with the uninformative rating.

Related literature

This paper contributes to several strands of literature. On the methodological side it extends the literature on cheap talk communication by introducing contracts in a communication game with multiple receivers. First, it shows that strategic interaction among multiple receivers can endogenously give rise to the properties of payoffs in the spirit of Crawford and Sobel (1982). Moreover, to the best of my knowledge this paper is the first to analyze optimal compensation contracts in a cheap-talk model a la Crawford and Sobel (1982) with multiple receivers and to consider private contracts.[6]

Krishna and Morgan (2008) introduces contracts between the sender and receiver in the single-receiver Crawford and Sobel (1982) set-up with exogenous bias and shows that the optimal contract can compensate the bias and facilitate communication. Differently from Krishna and Morgan (2008), in this paper the rater communicates with multiple receivers, the seller and the buyers, that interact strategically and affect payoffs of each other. This allows me to study private (i.e. unobserved to the buyers) contracts between the seller and the rater. This comparison is not possible in the single-receiver set-up of Krishna and Morgan (2008). Moreover, in my paper the rater's bias is not assumed but is a result of the strategic interaction between the buyers and the seller.

The paper contributes to the growing literature about credit rating agencies. The cheap-talk rating agency model studied in Goel and Thakor (2013) is related to the one studied here. In their paper the rating agency's objective function includes a fixed fraction of the seller's profit, but no contracts are allowed. Here instead, the seller signs a contract with

[6]Cheap-talk communication with multiple audiences without contracts has been analyzed in Farrell and Gibbons (1989) and Goltsman and Pavlov (2011).

the rater, and the degree to which the rater cares about the seller's profit depends on the contract signed. Moreover, here public and private contracts are considered, and it is shown that optimal contracts in both regimes are different from the fixed fraction of the seller's profit. The resulting information partition in ratings is also different from Goel and Thakor (2013) where ratings are imprecise, here instead the optimal public contract induces precise ratings, while private contracts lead to imprecise ratings.

This paper also relates to Inderst and Ottaviani (2012) in which the information intermediary internalizes buyers' payoff and recommends one of the two competing sellers A and B. The paper shows that private payments can tilt the recommendations towards the cost efficient seller and improve social welfare compared to public payments. Their application is very different from an issuer seeking a credit rating. Also they do not study multiple ratings and their informativeness, which is the focus of this paper. This paper stresses the adverse effect of private payments on the number and on the informativeness of ratings. Endogenous multiple ratings and private payments also distinguish this paper from Bizzotto and Vigier (2017), which considers public payments to the rating agency and shows that requiring the fees to be payed upfront may lower the rating agency's incentives to acquire information and social welfare.

Several rating agency papers study cheap-talk ratings without considering rating contingent contracts between the seller and the rater. Frenkel (2015) shows that repeated interaction between some sellers and the rating agency may allow these sellers to receive inflated ratings. Bouvard and Levy (2017) argues that reputational concerns have a nontrivial effect on the precision of the rater's ratings, and can lead to imprecise ratings and low welfare. At the same time Bar-Isaac and Deb (2014) shows that reputational concerns for raters work well when all audiences observe the same outcomes, while inefficiencies may arise when audiences separately observe different outcomes. These studies focus on the "double reputation" that credit rating agencies can build with different audiences observing different information about the performance of the product, but do not study compensation contracts of the rating agencies. This paper, instead, assumes that the buyers and the seller have the same information about ratings and the product performance, but the buyers may or may not see the compensation contract signed between the rater and the seller.

Literature on forum shopping is also distantly related. Lerner and Tirole (2006) considers pass/fail ratings, without contracts, it finds that weak applicants go to tougher raters and make more concessions. This paper allows arbitrary ratings and contracts between the seller and the rater, and it finds that pass/fail ratings are not optimal. Skreta and Veldkamp (2009) and Bolton et al. (2012) show that ratings can be biased when sellers can sample several raters for independent noisy ratings and make only the best ratings public. Bolton et al. (2012) shows that the competition between raters can exacerbate the bias, while Skreta and Veldkamp (2009) shows that the bias increases with the product's complexity. Both papers assume some investors to be naive, and do not allow sellers and raters to contract, while my paper considers rational investors and focuses on the transparency of the contracts between the seller and the rater.

This paper also relates a strand of literature on certification started by Lizzeri (1999), which assumes that raters can commit to a disclosure rule and ignores the issue of credibility. These papers are silent about how information production, credibility and welfare are affected when the rater's reporting incentives are affected by a contract. See Kartasheva and Yilmaz (2013) and Farhi et al. (2013) for recent certification models with commitment.

The paper is organized as follows. In Section 1 I introduce the basic two-period model.[7] Section 2 considers public payments between the seller and the rater, while Section 3 studies private payments. Section 4 is devoted to welfare analysis. Section 5 studies extensions of the main model when the rater's signal is costly and imprecise. Section 6 concludes.

1. Baseline model

Consider a baseline model with two periods $\tau = 0, 1$. At period $\tau = 0$ the seller offers a new product for sale. The product's payoff $y \geq 0$ is drawn from some cdf F, y is unknown at $\tau = 0$ and realizes only the next period $\tau = 1$. The discount factor in the economy is $\delta \in (0, 1)$, so that the product's discounted payoff at the moment of sale is δy. Naturally, in case of a zero-coupon bond, y stands for the bond's realized payoff at maturity, and δ is the discount factor between the bond's date of issue and it's maturity date.

In the beginning of period $\tau = 0$ the seller can hire a rater, who learns a signal $\theta \sim U[0, 1]$ informative about the product: $E[y|\theta] = \theta$. For example, in the case of a zero-coupon bond the final payoff is binary $y \in \{0, 1\}$ (the bond either defaults or not), and one can think of θ as the rater's estimate of the probability of no default. In the baseline model I assume that the rater observes the signal at no cost. In the Section 5 I discuss what happens when

[7] The general infinite-horizon model can be found in the Online Appendix A.2.

the signal is costly, and when the precision of the signal may vary.

Assumption 1. *The signal about the product's future payoff θ is private to the rater.*

After learning the signal θ the rater issues a public rating m from an arbitrary set M. For tractability I assume the rater's rating is the only information about the product's future payoff available to the buyers and the seller.

I consider the rater as a two-sided platform that needs to attract both the sellers of new products and the buyers that will use ratings for their purchasing decisions. In order to attract the seller at $\tau = 0$ the rater publicly announces the set of ratings M and proposes the payment schedule $t : M \to R^+$. The payment schedule may be public or private to the seller and the rater. In the seller-paid business model buyers access public ratings for free and the rater can't attract buyers with price instruments. To capture the rater's need to attract buyers I follow the two-sided platform models and introduce the notion of the rater's *user base* n_τ, $\tau = 0, 1$ which corresponds to the mass of buyers that rely on the rater's rating in their purchasing decisions. At period $\tau = 0$ mass n_0 of identical buyers relies on the rater, while at $\tau = 1$ the mass of buyers relying on the rater (n_1) depends on the buyers' payoff from the rating's performance. Specifically, each buyer $i \in n_0$ using the rating at $\tau = 0$ and buying q_i units of the product at price p gets the payoff

$$S_i = (\delta y - p)q_i - \rho q_i^2/2.$$

Parameter ρ and the quadratic term capture decreasing marginal value of each additional unit of the product. In the case of financial products this is due to buyers' risk-aversion and the idiosyncratic risk associated with the product. For other products this captures the decreasing marginal utility.

When the rated product benefits (harms) the buyers the mass of buyers relying on the rater grows (decreases) in accordance with:

Assumption 2. $n_1 = n_0 + \varphi \int_{i \in n_0} S_i di$, $\varphi > 0$.

In the case of financial products S_i corresponds to the investors' return and, naturally, investors are more willing to rely on the rater and spread good word about him when they get a positive return on the rated product than when they get a negative return.[8] In order

[8]Note, that positive returns also increase the investors' wealth, so they may scale up their investments accordingly after positive returns, while the opposite may happen after negative returns. In fact, if the returns are sufficiently negative some investors can leave the market, as happened during the 2007-2008 financial crisis, presumably reducing the number of investors that are relying on the rater.

to prevent the user base turning negative after low returns, I assume $\varphi < \frac{8\rho}{3\delta^2}$.

The seller of the product is monopolistic and has zero marginal cost of production for simplicity. When she hires the rater she agrees to pay $t(m)$ to the rater depending on the rating $m \in M$. After the rating is issued the seller sets a price $p \geq 0$ and each buyer $i \in n_0$ chooses what quantity $q_i \geq 0$ to purchase. The resulting seller's profit is:

$$p \int_{i \in n_0} q_i di - t(m), \; \forall m \in M.$$

The rater cares about the payments he receives from the seller $t(m)$ at $\tau = 0$ and about the value of his reputation at $\tau = 1$, which I assume to be linear in his user base n_1.

Assumption 3. $U = t(m) + \delta u n_1, \; \forall m \in M.$

Parameter $u > 0$ captures rater's reputation. In Online Appendix A.2 I endogenize u in an infinite time horizon model, in which the rater maximizes the present value of payments he receives from sellers arriving at dates $\tau = 0, ..., \infty$. In a stationary equilibrium of the infinite horizon model the seller's payment to the rater is proportional to the user base at any date $\tau = 0, ..., \infty$. As a result, the rater's discounted expected stream of payments from $\tau + 1$ onward is also proportional to user base $n_{\tau+1}$. This is equivalent to the reputation term u in the baseline two-period model studied here (see formula (30) for details).

Outside options. The rater gets non-negative payments $t(m) \geq 0$ for ratings and is always willing to rate the seller.[9] If the seller refuses to be rated, she can always sell her product to a mass of buyers $\underline{n} \leq n_0$ that are willing to buy unrated products of uncertain quality. It is easy to check that her expected profit in this case is $\underline{\Pi} = \underline{n}\delta^2/16\rho$.

Public versus private payments. The set of ratings M is publicly known. The payment schedule $t : M \to R^+$ can be either publicly known, or privately known to the seller and the rater. In the second case the buyers do not observe the payment schedule and their beliefs about ratings must be consistent with the actual payment schedule offered in equilibrium. Intuitively, any outcome achievable when t is private should also be achievable when t is public, therefore, I start the analysis with the case of public payments.

[9] Negative payments may provoke bogus applications and are unrealistic.

2. Public payments

The rater publicly announces a set of ratings M and a payment schedule t in the beginning of period $\tau = 0$. The pair M, t induces a game $\Gamma(M, t)$ between the rater, the buyers and the seller described below in detail.

The buyers and the seller do not observe the rater's private signal $\theta \in [0, 1]$ and form beliefs about it $M \to \Delta([0, 1])$, that specify for each rating m a probability distribution over possible signals $\theta \in [0, 1]$ that the rater may receive about the seller's product, with a density function $\mu(\theta|m)$, $\int_0^1 \mu(\theta|m)d\theta = 1$.

If the seller's expected profit with the rater is higher or equal than her outside option she hires the rater and commits to pay according to t: the rater is hired if

$$\Pi = \int_0^1 E_\mu[pq - t(m)|m]\sigma(m|\theta)d\theta \geq \underline{\Pi},$$

here σ is the rater's reporting strategy described below. In principle, all beliefs and strategies are equilibrium specific and should be indexed accordingly. Here I omit these indexes to shorten notation.

If the rater is not hired the game ends. The seller gets his outside option $\underline{\Pi}$, the buyers in the rater's user base get zero, so that $n_1 = n_0$, and the rater's payoff is $U = \delta u n_0$. If the rater is hired, he observes signal θ and publishes a rating $m \in M$: his strategy $[0, 1] \to \Delta(M)$ assigns to each signal θ a probability distribution over possible ratings M with a density function $\sigma(m|\theta)$, $\int_M \sigma(m|\theta)dm = 1$.

Having observed m the seller pays $t(m)$ to the rater, then she sets the price p for her product, her strategy is $\hat{p} : M \to R^+$. Having observed p and m, each buyer $i \in n_0$ decides on quantity q_i, hence individual demand functions of buyers from the rater's user base are given by $q_i : R^+ \times M \to R^+$, $i \in n_0$. Individual demand functions generate the market demand $q = \int_{i \in n_0} q_i di$ and the seller gets profit $qp - t(m)$. At $\tau = 1$ the product's quality y and the buyers' payoffs realize. The rater's user base n_1 adjusts according to Assumption 2 and he receives the reputation benefit un_1. I look for a Perfect Bayesian Equilibrium in which the rater is hired on the equilibrium path.

2.1. Grading equilibrium

In the cheap talk environment Crawford and Sobel (1982) proved that one can focus on *partition* equalibria without loss of generality. I extend this idea and prove that any equilibrium in my model is outcome equivalent to a *grading equilibrium.* The notion of *grading* is, essentially, a generalization of the partition concept, which permits intervals of perfect revelation alongside coarse intervals.

Definition 1. A set of ratings G is a *grading* if each rating $r \in G$ corresponds to a convex set of signals (grade) $g(r)$ and $r = E[\theta|\theta \in g(r)]$. Grades satisfy $g(r') \cap g(r) = \emptyset$ for any $r' \neq r$ and $\underset{r\in G}{\cup} g(r) = [0, 1]$.[10]

For example, a completely uninformative grading has a single rating $r = E[\theta] = 1/2$ and a single grade $g(1/2) = [0, 1]$. A perfectly informative grading, in contrast, has a continuum of perfectly precise ratings and grades $g(r) = \{r\}$, $r = \theta \in [0, 1]$.

Grading equilibrium is an equilibrium where the set of ratings used by the rater is a *grading.*

Lemma 1. *For any equilibrium under any message space M, there exists an outcome-equivalent equilibrium in pure strategies under some grading G, in which the rater announces rating r whenever θ is in grade $g(r)$.*

All omitted proofs are in the Appendix. Intuition is the following. In principle, the rater's set of ratings M can be arbitrary. However, in equilibrium, the uninformed parties (the buyers, and the seller) need to infer from a rating only the range of rater's signals (grade) for which this rating is issued. The single-crossing condition holds for the rater, so that his ratings monotonically increase with his signals θ and define a natural grading of an interval $[0, 1]$.

2.2. Feasible grading under public payments

For each rating $r \in G$ the grading G effectively fixes the beliefs of the buyers and the seller about the rater's information $\theta \in [0, 1]$: the buyers and the seller believe the seller's signal to be in the corresponding grade $\theta \in g(r)$.

[10] Labeling of ratings can be arbitrary as far as distinct grades correspond to distinct ratings. Here, with no loss of generality, each rating corresponds to the expected value of the signal in the corresponding grade.

A grading G is *feasible* if for some payment schedule t the resulting game $\Gamma(G,t)$ has an equilibrium, in which rater's reports are consistent with the grading, and the equilibrium behavior of other agents. Let's describe the equilibrium behavior of all agents under a *feasible* grading, starting with the market reaction to a rating.

Market reaction to a rating. Given a rating $r \in G$, the buyers believe the rater's signal θ to be within the grade $g(r)$. Therefore, for a given price p each buyer $i \in n_0$ buys quantity q_i which maximizes his expected payoff:

$$\hat{q}_i(r,p) = \arg\max_{q_i \geq 0} E[(\delta y - p)q_i - \rho q_i^2/2 | \theta \in g(r)],\ \forall r \in G,\ p \geq 0. \tag{1}$$

Note, that in a grading equilibrium $E[y|\theta \in g(r)] = r$ and $\hat{q}_i(r,p) = (\delta r - p)/\rho$. Aggregation of individual demands of all buyers in the rater's user base n_0 determines the total market demand. The seller takes into account the buyers' reaction and sets the profit maximizing price:

$$\hat{p}(r) \in \arg\max_{p \geq 0} \int_{i \in n_0} p\hat{q}_i(r,p)di,\ \forall r \in G. \tag{2}$$

For any rating r the equilibrium price is $\hat{p}(r) = \delta r/2$, and each buyer $i \in n_0$ buys quantity

$$\hat{q}(r) = \hat{q}_i(r, \hat{p}(r)) = \delta r/2\rho.$$

Rater's reporting incentives. At $\tau = 0$ the rater observes signal θ and decides on the rating. He anticipates the market reaction to his rating, and how the performance of the rating will affect his user base and his payoff next period $\tau = 1$. If he issues a rating r when his signal is θ his expected payoff is

$$E[U|\theta, r] = t(r) + \delta u E[n_1|\theta, r].$$

His user base in the next period grows if the buyers' experience positive payoff and shrinks if their payoff is negative in accordance with Assumption 2

$$E[n_1|\theta, r] = n_0 + \varphi \int_{i \in n_0} E[S_i|\theta, r]di. \tag{3}$$

The market reaction to a rating $r \in G$ is fully characterized by the price $\hat{p}(r)$ and by the quantity bought by each buyer $\hat{q}(r)$. The expectation of the product's payoff conditional on the rater's signal is $E[y|\theta] = \theta$, hence the rater's expectation of a buyer's payoff is

$$E[S_i|\theta, r] = (\delta\theta - \hat{p}(r))\hat{q}(r) - \rho\hat{q}(r)^2/2.$$

Substituting for $E[S_i|\theta, r]$ in the rater's objective function one obtains

$$E[U|\theta, r] = t(r) + \delta u n_0[1 + \varphi(\delta\theta - \hat{p}(r) - \rho\hat{q}(r)/2)\hat{q}(r)],$$

and the rater's reporting constraint becomes:

$$\hat{r}(\theta) \in \arg\max_{r \in G} \{t(r) + \delta u n_0[1 + \varphi(\delta\theta - \hat{p}(r) - \rho\hat{q}(r)/2)\hat{q}(r)]\}, \ \forall\theta \in [0, 1]. \tag{4}$$

Bayesian updating. The rater's reporting strategy $\hat{r} : \theta \to G$ must be consistent with the beliefs of the buyers and the seller. In a pure strategy *grading equilibrium* the buyers and the seller expect the rater to report according to the grading G, which partitions interval $[0, 1]$ into non-overlapping grades $g(r)$, $r \in G$ (Definition 1). Bayesian updating requires:

$$\forall\theta \in g(r), \ r \in G, \ \hat{r}(\theta) = r. \tag{5}$$

Seller's participation constraint. The seller applies for a rating only if she expects to get a profit greater than her outside option:

$$\int_0^1 [n_0\hat{p}(\hat{r}(\theta))\hat{q}(\hat{r}(\theta)) - t(\hat{r}(\theta))] \, d\theta \geq \underline{\Pi}. \tag{6}$$

Limited liability of the rater. Finally, the payments must be non-negative:

$$t(r) \geq 0, \ \forall r \in G. \tag{7}$$

Formally, a grading G is *feasible* if for some t conditions (1), (2), (4), (5), (6) and (7) are satisfied. Naturally, these conditions characterize an equilibrium of the game induced by G, t. It is easy to see that the set of feasible gradings is not empty. For instance, the uninformative grading with a single rating $r = 1/2$ and grade $g(1/2) = [0, 1]$ is feasible, as all equilibrium conditions trivially hold.

2.3. Rater's intrinsic bias

Here I show that the non-trivial reaction of market participants to ratings, makes the rater intrinsically downward "biased". To illustrate this bias, I consider the case when the rater charges a fixed payment from the seller $t = t_1 \geq 0 = const$. One may conjecture that under a fixed payment the rater reports his information perfectly $\hat{r}(\theta) = \theta \in [0, 1]$. This conjecture is false. Under a fixed payment only the uninformative grading with single rating $r = 1/2$ and grade $g(1/2) = [0, 1]$ is feasible. Uninformative communication and equilibria are common in the cheap talk environment, for instance, in Crawford and Sobel (1982) the uninformative babbling equilibrium prevails when the sender's bias is extreme. In essence, the reason behind uninformative grading under fixed payments in my model is the rater's endogenous *bias*, which is similar in it's effect to an exogenous bias assumed in Crawford and Sobel (1982). Let me now explain why the rater tends to be "biased" under a fixed payment.

Consider a perfect grading $G = [0, 1]$, in which the rater could send a perfectly precise rating $r \in G = [0, 1]$ for any realization of his signal $\theta \in [0, 1]$.

Definition 2. The rater is *downward* biased for a given $\theta \in [0, 1]$ if under the perfect grading $G = [0, 1]$ he reports $\hat{r}(\theta) < \theta$.

In equilibrium ratings must be consistent with the Bayesian updating (5): $\hat{r}(\theta) = r$ if and only if $\theta = r$ for any $\theta \in [0, 1]$. Let's show, that the perfect grading is not feasible under a fixed payment because the rater is *downward* biased, and his reports are not consistent with Bayesian updating.

Lemma 2. *Under a fixed payment the rater is downward biased for any* $\theta \in (0, 1]$.

Proof. Intuitively a public rating r affects both the equilibrium quantity $\hat{q}(r)$ and price $\hat{p}(r)$, that in turn affect the future payoff of the buyers, the rater's user base at $\tau = 1$ (Assumption 2) and, ultimately, the rater's expected payoff given by (3).

Therefore, the rater takes into account the effect of his ratings on future payoff of the buyers. The quantity $\hat{q}(r)$ purchased by each buyer $i \in n_0$ is chosen optimally by this buyer By Envelope theorem the marginal effect of changes in quantity on the buyers' payoff is zero. Consequently, only the price effect on the buyers' payoff is relevant. For a given θ, an increase in the rating r raises the price $\hat{p}(r) = \delta r/2$, and lowers the buyers' net expected payoff. This negative price effect makes the rater downward biased (cautions about ratings): for any $\theta \in (0, 1]$ the rater prefers to report $\hat{r} < \theta$. Q.E.D.

Due to this intrinsic bias, the rater would understate his signal if the grading were perfect. In an equilibrium with Bayesian buyers this is not possible. Hence, under a fixed payment the perfect grading can't happen in equilibrium.[11] In fact, one can establish a stronger result:

Proposition 1. (Monotonicity of payments.) *In a feasible grading with at least two ratings the payments strictly increase with ratings: for any $r > r'$ one has $t(r) > t(r')$.*

In light of the rater's intrinsic downward bias discussed above, Proposition 1 has a natural interpretation: an informative grading is feasible only if the rater's intrinsic downward bias is compensated with an increasing payment schedule. When the rater's signal about the product is on the borderline between two ratings, other things being equal, the rater is reluctant to issue the high rating. Indeed, the quality of the product is fixed, and the high rating results in buyers paying a high price for the product, experiencing a low net payoff, and lowering the rater's reputational gain at $\tau = 1$. However, if at $\tau = 0$ the rater receives a higher payment for the high rating than for the low one, his reporting incentives are better balanced, and informative ratings are feasible. One can show that a perfect grading is feasible with an appropriate choice of the payment schedule. The perfect grading is the most informative, however, as is illustrated in the next section, a competitive rater may not implement the most informative grading.

2.4. Competitive grading under public payments

Any feasible grading can take place in equilibrium, it can be uninformative or contain perfect ratings. In order to narrow down possible outcomes I consider a competitive rater. This is natural in the rating agency industry where a seller of a financial product can freely choose among several raters and each rater wants to get the rating fees from the seller. Without modeling the competition explicitly, I assume that at $\tau = 0$ the rater offers a grading and payments G, t that result in the highest expected profit for the seller. Let's us first establish a helpful technical result.

Lemma 3. *The equilibrium under a feasible grading G is fully characterized by G and the payment for the lowest rating t_1.*

[11] Publicity of ratings is key for this result, Farrell and Gibbons (1989) showed that in cheap-talk models with multiple audiences private communication may be easier than public communication. If in my model the rater privately reports r to buyers, the seller cannot react to the reported r, and informative private ratings may be feasible. Certain raters do advise their clients privately; investment banks or consultants are among the examples. The analysis of private raters is out of the scope of this paper.

The detailed proof is in the Appendix.[12] The seller's expected profit in the equilibrium can be expressed as:

$$\Pi(G, t_1) = \int_0^1 [n_0 \hat{p}(\hat{r}(\theta))\hat{q}(\hat{r}(\theta)) - t(\hat{r}(\theta))]d\theta,$$

here equilibrium reports $\hat{r}(\theta)$, prices $\hat{p}(r)$, quantities $\hat{q}(r)$ and the transfer schedule $t(r)$ are consistent with the grading G. The grading offered by a competitive rater maximizes the seller's expected profit:

$$\max_{\{G,t_1\}} \Pi(G, t_1),\ s.t.\ (1), (2), (4), (5), (6), (7). \tag{8}$$

Solution to this problem characterizes ratings offered by a competitive rater.

Proposition 2. *If $\delta u\varphi < 2$ the competitive rater with public payments offers a coarse rating for low signals $\theta \leq \theta_1 = \frac{6\delta u\varphi}{2+5\delta u\varphi}$, and precise ratings ($r = \theta$) for high signals ($\theta > \theta_1$). The coarse rating is free, while precise ratings command positive payments to the rater*

$$t(r) = n_0\delta^3 u\varphi(4r^2 - 3\theta_1^2)/32\rho,\ for\ r = \theta > \theta_1.$$

If $\delta u\varphi \geq 2$ the rater offers the uninformative grading for free.

When $\delta u\varphi < 2$ the grading has a natural interpretation: a seller with a low quality product $\theta \leq \theta_1$ is rejected by the rater and pays nothing, while a sellers with the product's quality above the minimal threshold θ_1 gets a precise rating and pays for it. The intuition is as follows, the seller's gross expected profit is the highest when the grading is perfect: $r = \theta$ for any $\theta \in [0, 1]$, however such a grading would require significant payments to the rater from the seller, that reduce seller's expected profit net of the payments. As a result, the competitive grading is not perfect. Because of the rater's limited liability the payment from the seller to the rater cannot be negative. According to Proposition 1 payments must increase with ratings, and if the seller is paying for low ratings she must pay even more for high ratings.

A way to economize on expected payments is to pay nothing for the lowest rating and increase the likelihood $\Pr(\theta \leq \theta_1) = \theta_1$ that the seller's product drops in the corresponding

[12] *Feasible* grading G effectively determines equilibrium reporting strategy of the rater (5), the pricing strategy of the seller (2) and quantity bought by buyers (1). The only aspect of the equilibrium that remains to be defined is the payment schedule $t : G \to R^+$. This schedule is fully determined by t_1 and the rater's reporting constraint (4), as is formally shown in the Appendix.

grade $[0, \theta_1]$. In the optimal grading, the choice of θ_1 trades off the loss in expected profit when θ_1 goes up and interval $(\theta_1, 1]$ with precise ratings shrinks, against the savings in the expected payment to the rater when interval $[0, \theta_1]$ with a "free" coarse rating expands.

The competitive grading is partially informative when $\delta u \varphi < 2$, that is when the reaction of the rater's user base to the rating performance (φ) is moderate, and when the sensitivity of the rater's future payoff to the user base (u) is not too high. These parameters are key, because they determine how fast the payments grow with precise ratings $t(r) = n_0 \delta^3 u \varphi (4r^2 - 3\theta_1^2)/32\rho$, and how expensive it is to implement them. When $\delta u \varphi \geq 2$ it is prohibitively expensive, and the uninformative grading is optimal for the seller. Interestingly enough, in the latter case the rater does not get paid, as the optimal uninformative grading requires no payments from the seller. In Online Appendix A.2 I consider an infinite horizon version of the model in which the rater is paid by a new seller each period $\tau = 0, ..., \infty$. In this model u is determined endogenously and an analog of condition $\delta u \varphi < 2$ holds.

Somewhat contrary to the conventional wisdom, a contingent payment, supposedly feeding the rater's conflict of interest, causes no apparent harm to the buyers. When the payments are publicly known the ratings can be precise for a wide range of the rater's signals ($\theta > \theta_1$). The subsequent analysis shows that this result hinges on the fact that the payments are public. Section 3 shows that with private payments precise ratings are not feasible, and a phenomenon similar to the "rating inflation" can take place.

3. Private payments

Certain raters do not disclose the compensation they receive from sellers, for instance main credit rating agencies do not reveal the issuers' payments for ratings. Essentially, the payments are private and are not observed by the buyers. In an attempt to attract the seller, the rater may secretly inflate payments for high ratings and issue high ratings even for low values of his signal thus benefiting the seller (loosely speaking the rater can be willingly bribed into issuing high ratings).

First, I characterize feasible gradings under private payments, assuming that the grading G is exogenously given and publicly known. Note that, any grading feasible under private payments is also feasible under public payments, hence Lemma 1 applies, and one consider grading equilibrium without loss of generality. Grading G fixes the beliefs of the buyers for each rating $r \in G$. Given these beliefs the rater competes for the seller and privately proposes payment schedule t, which maximizes the expected profit of the seller given G.[13]

3.1. Feasible grading under private payments

A feasible grading G must be consistent with equilibrium behavior of all agents. The grading G pins down the buyers' equilibrium beliefs: for each rating $r \in G$ they believe the rater's signal to be in the corresponding grade $\theta \in g(r)$. For a given G the rater privately proposes payments t maximizing the seller's expected profit. The payments must be non-negative, that is (7) must hold. The seller decides whether to accept G, t or get her outside option, if (6) holds, she accepts and agrees to pay according to t. The rater learns actual θ, reports $r \in G$ and gets paid $t(r)$, his reports are given by (4) . The seller sets a price p according to (2) and each buyer $i \in n_0$ buys quantity q_i given by (1). Formally, the rater takes G as given and privately proposes t, which solves the following problem:

$$\max_{\{t\}} \int_0^1 E[pq - t|G, t]d\theta, \ s.t.(1), (2), (4), (6), (7). \tag{9}$$

The buyers are Bayesian, in equilibrium their beliefs must be consistent with the rater's reporting strategy. Grading G is feasible if the payments solving (9) and the corresponding rater's reporting strategy satisfy the Bayesian updating constraint (5).

As noted before, any grading feasible under private payments satisfies (1), (2), (4), (5), (6), (7) and is feasible under public payments, while the opposite is not true. Unlike with public payments here the buyers do not observe the payment schedule t, which determines the rater's reporting incentives. Loosely speaking, the grading G which determines the buyers' beliefs, must be robust to hidden payment manipulations by the rater.

For instance, if the buyers believe that a certain rating corresponds to high values of θ, then the rater may set a high payment for this rating and start issuing this rating for values of θ that are lower than what the buyers believe it corresponds to. Such payment manipulation should not be profitable in equilibrium, otherwise Bayesian updating constraint would be violated. To shorten the notation, let me introduce a parameter $\lambda = \delta\varphi u$, which measures the strength of the reputational concern for the rater. From now on I consider $\lambda = \delta u\varphi \leq 2$ for brevity. Note, that according to Proposition 2 when $\delta u\varphi > 2$ the uninformative grading is the best for the seller under public payments, and, therefore,

[13] In Section 3.2 I make G endogenous and characterize G offered by a competitive rater.

it is also the best for the seller under private payments.

Proposition 3. *A feasible grading G under private payments has no precise ratings and contains at most a countable number N of coarse ratings.*

1) If $\lambda \in (0, \frac{2}{11}]$ then $N = 1$.

2) If $\lambda \in (\frac{2}{11}, \frac{2}{3}]$ then $N \in \{1, 2\}$. If $N = 2$ the border between ratings is $\theta_1 = \frac{11\lambda-2}{10\lambda+4}$.

3) If $\lambda \in (\frac{2}{3}, 2]$ then $N \in \{1, ..., \infty\}$. If $N = 2$ the border is $\theta_1 = \frac{11\lambda-2}{10\lambda+4}$, if $N \geq 3$ the borders are:

$$\theta_i = \sum_{j=1}^{i} \frac{(8 - 4\lambda + 8\lambda D^N)D^{N+1-j} + (8\lambda + (8 - 4\lambda)D^N)D^j}{(3\lambda - 2)(1 - D^{2N})(1 - D)}, \ i = 1, ..., N - 1,$$

where

$$D = \frac{5\lambda + 2 - 4\sqrt{\lambda(\lambda + 2)}}{3\lambda - 2}.$$

To illustrate the idea behind the proof note that the seller's revenue $p(r)q(r)$ increases with ratings, thus for each border point θ_i, $i = 1, ..., N - 1$ per se the seller prefers rating r_i over rating r_{i-1}. She may increase her chance of receiving rating r_i instead of r_{i-1} if she agrees to inflate rater's compensation $t(r_i)$ for rating r_i. Loosely speaking, in equilibrium the rater proposes to inflate some payments and the seller agrees to do so, until she is indifferent in expectation between ratings r_i and r_{i-1}.

In a feasible grading the border points are such that the buyers' beliefs are consistent with the rater's reports induced by the private payments. The incentive to inflate payments for high ratings declines when the ratings become less precise. This is so because any additional payment promised for a rating has to be paid whenever the rating is issued. The less precise the rating is, the broader is the range of signals $(\theta_{i-1}, \theta_i]$ when this rating is issued, and the higher is the expected additional expense of the seller associated with this rating. If the rating is sufficiently coarse it is not profitable for the seller to accept an inflated payment schedule. Therefore, in any feasible grading ratings are coarse, so that the seller's expected gains from payment manipulation away from the equilibrium payments are zero.

One can interpret the incentive to inflate private payments and the resulting imprecise ratings as a manifestation of the so called "rating inflation". In my framework all agents are rational and nobody is deceived, yet "rating inflation" can be harmful as the informational value of ratings is limited. For instance, the informational value of the highest rating is reduced not because buyers fail to interpret it, but because a high payment from the seller induces the rater to issue the highest rating even when his signal is not the highest.

Role of reputation under private payments

The parameter $\lambda = \delta\varphi u$, which captures the degree to which the rater cares about his reputation, is crucial when the payments are private. It reflects how easy it is to influence rater's ratings with payments and determines the informativeness of ratings in equilibrium. When the rater is easy to influence $\lambda \leq \frac{2}{11}$ only the uninformative grading is feasible. If the rater were to use more than one rating, then a small private payment would induce the rater to report a high rating even when the quality is low. Such a manipulation would be profitable, therefore, the informative grading is not feasible in equilibrium. When $\lambda \in (\frac{2}{11}, \frac{2}{3}]$ the rater cares sufficiently about his reputation so that a grading with two ratings is feasible. The high rating is issued for a wide range of qualities and involves more pooling than the low rating: $\theta_1 = \frac{11\lambda-2}{10\lambda+4} \leq 1/2$. When $\lambda \in (\frac{2}{3}, 2]$, the rater cares a lot about his reputation and infinitely many ratings are feasible. Yet, even in this case all ratings are not precise and involve some pooling, unlike under public payments that make precise ratings feasible.

3.2. Competitive grading under private payments

Thus far the grading G was exogenous. Here I endogenize it. As with public payments, I consider a competitive rater who designs a grading G which is feasible under a private payments and delivers the highest profit to the seller:

$$\max_{\{G\}} E[pq - t|G], \ s.t.(5), (9). \tag{10}$$

The next Proposition characterizes the competitive grading under private payments.

Proposition 4. *The competitive grading with private payments is uninformative and free.*

Formal proof is in the appendix. The result is striking because the seller is information loving per se. Intuitively, the seller's gross profit conditional on a rating $r \in G$ is a convex function of r, hence in expectation the seller's gross profit is high when grading G has many precise ratings. However, in any feasible grading under private payments precise ratings command a payment schedule which is rapidly increasing with ratings. Loosely speaking, the resulting seller's profit net of the payment becomes a concave function of ratings : under a private payment the seller becomes information averse. It turns out, if the seller has an opportunity to privately pay the rater, then from an ex ante perspective she prefers not to have an informative rater around. If such a rater was present then the seller would not resist the temptation to offer high payments for high ratings. Consequently, in expectation the seller would end up paying the rater more than the extra profit she would gain from informative ratings.

This result critically depends on the assumption that the rater is competitive and offers the best grading from the point of view of the seller. If the rater has some market power, then potentially some other feasible grading may be chosen by the rater in equilibrium.

4. Welfare analysis

The social welfare is defined as a sum of payoffs of all parties. Any payment t is a transfer from the seller to the rater and does not affect the total welfare. In the Online Appendix A.2 I show that the rater's reputational payoff ($\delta u n_1$) can be micro-founded as a present value of the future payments from the seller in an infinite horizon model. Hence, in the two-period model the payment t and the rater's reputational payoff do not affect the total welfare (they just represent current and future transfers between agents).

When the product payoff is y and each buyer $i \in n_0$ purchases quantity q_i the resulting total welfare can be expressed as the sum of the seller's gross profit and the buyers' payoffs

$$\int\limits_{i \in n_0} (p q_i + (\delta y - p) q_i - \rho q_i^2/2) di = \int\limits_{i \in n_0} (\delta y q_i - \rho q_i^2/2) di.$$

In an equilibrium the grading G determines the rater's equilibrium report $r(\theta)$ for any $\theta \in [0,1]$. For a given rating $r \in G$ each buyer purchases the same equilibrium quantity $q(\theta) = q(r(\theta)) = \delta r(\theta)/2\rho$. Since $E[y|\theta] = \theta$, the ex ante expected total welfare can be computed as

$$W(G) = n_0 \int\limits_0^1 (\delta \theta q(\theta) - \rho q(\theta)^2/2) d\theta.$$

Remark 1. A social planner with the objective functions W is information loving.

Intuitively information is beneficial for the buyers because high-payoff products have higher value and should be purchased in larger quantities than low-payoff products. Whenever ratings allow to better distinguish products with different payoffs, the ex ante expected welfare increases.

Remark 2. If the social planner with the utilitarian objective function W were to impose a grading, she would induce perfect ratings for all $\theta \in [0, 1]$.

In reality, a regulator acting as a social planer can hardly impose the set of rating to the rater to use, and force the rater to adopt a particular reporting strategy. Nevertheless, the regulator may be able to influence contracts between the parties. For instance, the regulator can mandate full disclosure of payments from the seller to the rater. I proceed assuming that the social planner can impose two contacting regimes: private or public payments. I also assume that under any regime the grading is decided by a competitive rater. Proposition 4 implies that under private payments the competitive rater offers the uninformative grading. According to Proposition 2, under public payments the competitive grading is informative if and only if $\delta u \varphi < 2$, thus one gets:

Corollary 1. *If $\delta u \varphi < 2$ then public contingent payments strictly dominate private contingent payments from the welfare perspective. If $\delta u \varphi \geq 2$ both regimes lead to the uninformative grading and deliver the same level of welfare.*

To get the intuition consider $\delta u \varphi < 2$. Private payments allow payment manipulations not observed by the buyers and, loosely speaking, create a temptation for bribes. Due to these potential payment manipulations, informative ratings become very expensive for the seller under, so that the competitive rater induces the uninformative grading. Public payments make payment manipulations costly, because the buyers observe them and update their beliefs accordingly. If buyers suddenly see that the rater is promised an inflated payment for a rating, they understand that the rater may issue this rating even if his signal is below the minimal threshold for this rating. They value of the rating drops, as it becomes associated with a lower expected payoff in the eyes of the buyers. This in turn mitigates the seller's benefits from inflating a payment for the rating. Without payment manipulations perfect ratings are feasible, and are not very expensive for the seller. The competitive rater induces an informative grading, in which perfect ratings for high values of his signal. The resulting social welfare is higher than under private payments.

5. Rater's information

In this section I discuss alternative assumptions about the rater's information. First, I consider the case when the rater has to spend a cost in order to acquire his information, and then I discuss what happens if the precision of his information varies.

5.1. Costly information

Throughout the paper I have assumed that the rater gets the signal about the product payoff at no cost. Given that the rater always receives non-negative payments main results will not change even if the signal was moderately costly to the rater. For instance, optimal grading under public payments delivers to the rater expected revenue

$$t^E = \int_{\theta_1}^{1} \delta^3 u\varphi(4\theta^2 - 3\theta_1^2)/32\rho d\theta > 0,$$

$\theta_1 = \frac{6\delta u\varphi}{2+5\delta u\varphi}$, and the rater is happy to spend up to t^E for acquiring the signal. Clearly, if the signal costs more than t^E this has to be taken into account and the equilibrium payment schedule is likely to be affected. Intuitively, when the rater shirks and acquires no signal, his information coincides with the ex ante prior $E[\theta] = 1/2$, and he would report the corresponding rating $r(1/2)$. One way to motivate the rater to acquire the signal is to lower the payment for this rating. For instance, if the threshold of the free coarse rating θ_1 is close to 1/2, it may be optimal to raise the threshold to $\theta_1' > 1/2$. In this case the uninformed rater would optimally issue the coarse rating corresponding to the grade $[0, \theta_1)$ and would not get paid, and this would motive him to get informed. However, the general analysis of the rater's information acquisition incentives is rather complex and is out of the scope of this paper.

5.2. Precision of the rater's information

Throughout the analysis I assumed the rater's signal (θ) to be informative about the product's payoff y. The only assumption about the distribution F of y was that it was compatible with $E[y|\theta] = \theta$, and $\theta \sim U[0, 1]$. This general specification does not parametrize the precision of the rater's signal explicitly. One can parametrize this precision by specifying a particular destribution F for y. For instance, one can assume that $y \sim U[0, 1]$ and the rater's signal $\theta \sim U[0, 1]$ correctly predicts y with probability $\beta \in [0, 1]$ and with the complementary probability it is noise. In this case the expected payoff of the product conditional on the signal increases linearly with the signal $y_\theta = E[y|\theta] = 1/2 + \beta(\theta - 1/2)$. The parameter β captures the informativeness of the signal, for instance when $\beta = 0$ the signal is uninformative, while $\beta = 1$ corresponds to the perfectly informative signal.

It is easy to see that the parameter β does not affect the main analysis qualitatively. Indeed, the rater's information relevant for buyers is given by y_θ, and one can consider the rater's communication about $\theta' = y_\theta \sim U[\frac{1-\beta}{2}, \frac{1+\beta}{2}]$. With such relabeling, the expected payoff of the product conditional on signal θ' is given by the same formula as in the main case studied in the paper $E[y|\theta] = \theta$ and most of the analysis will remain the same. However, unlike the original signal θ, the new uniform signal θ' is distributed on an interval $[\frac{1-\beta}{2}, \frac{1+\beta}{2}]$, and the exact rating borders in Propositions 2 and 3 should depend on β. This extension naturally combines with the rater's incentives to acquire information, and can result in a separate project about the rater's incentives to acquire information, the precision of the rater's information, and the revelation of the information through ratings. This analysis is out of the scope of this paper, which focuses on the information revelation through ratings and can be seen as a first step towards a more general analysis of seller-paid raters.

6. Conclusion

This paper analyzes the seller-pays business model used by many raters in the real world, with the most prominent example being the three largest credit rating agencies. The analysis highlights that per se the fact of sellers paying for ratings does not lead to the "rating inflation" or causes inefficiencies when the payments are public, and rational buyers that use ratings can see the fees paid to the rater. Somewhat contrary to the conventional wisdom, the paper shows that perfectly precise ratings that make the best use of the rater's information can be achieved in an equilibrium with public payments. Even when the rater is competitive and offers ratings and the payment schedule that deliver the highest expected profit to the seller, perfectly precise ratings are implemented for high quality products (Proposition 2).

The paper shows that private payments between the seller and the rater are problematic and lead to coarse ratings. In this case the buyers do not observe actual payments between the seller and the rater, which creates a scope for hidden payments manipulations. This potential payment manipulations make the rater endogenously upward biased and lead to coarse ratings in equilibrium. Intuitively, the seller gets high profit from high ratings and is ready to pay a high fees for these ratings. Since the rater gets high fees for high ratings, she is motivated to issue high ratings and does so even when the product's quality is not the highest. As a result high ratings are coarse and pool products of a wide range of qualities in a single rating. This phenomenon is similar to the "rating inflation", even though the buyers are rational and correctly infer the informational content of the ratings. Yet, coarse ratings imply that part of the rater's information is lost and the ex ante expected welfare is low.

The main policy implication of this analysis is about the transparency of the rater's fees. Public payments between the rater and the seller result in more informative ratings and higher welfare than private payments, therefore a regulation mandating disclosure of the rater's compensation might be beneficial. Current regulation of credit rating agencies requires no such disclosure, yet such regulation is not anheard of. For instance, underwriters of securities disclose their fees under the 1933 Securities Act, and the fees are published in the prospectus on the SEC website (www.sec.gov). This analysis suggests that a similar regulation may benefit the credit rating industry.

A. Appendix

A.1. Auxiliary results and omitted proofs

Lemma A.1. *In an equilibrium, for a given product payoff (y) and rater's message (m), the outcome (payoffs of all parties) is pinned down by the buyers' expectation of the product payoff ($\hat{y} = E[y|m]$).*

Proof of Lemma A.1.

First, note that $E[y|\theta] = \theta$. Suppose the rater has sent message m so that

$$\hat{y} = E[y|m] = \int_0^1 \theta\mu(\theta|m)d\theta.$$

For a given p, each buyer $i \in n_0$ maximizes his expected payoff

$$E[S_i|m] = (\delta\hat{y} - p)q_i - \rho q_i^2/2,$$

and his demand function is

$$\hat{q}_i(\hat{y}, p) = (\delta\hat{y} - p)/\rho.$$

Total market demand for the product is

$$\hat{q}(\hat{y}, p) = \int_{i \in n_0} \hat{q}_i(\hat{y}, p) = \frac{n_0}{\rho}(\delta\hat{y} - p),$$

the seller chooses the price

$$\hat{p}(\hat{y}) = \arg\max_{p \geq 0} \frac{n_0}{\rho}(\delta\hat{y} - p)p = \delta\hat{y}/2,$$

and his profit is

$$\frac{n_0}{4\rho}\delta^2\hat{y}^2 - t(m).$$

Each buyer $i \in n_0$ purchases $\hat{q}_i(\hat{y}) = q_i(\hat{y}, \hat{p}(\hat{y})) = \delta\hat{y}/2\rho$. Substituting for $\hat{p}$ and $\hat{q}_i$ the equilibrium payoff of each buyer $i \in n_0$ can be expressed as

$$S_i(y, \hat{y}) = (\delta y - 3\delta\hat{y}/4)\delta\hat{y}/2\rho.$$

It remains to show that the rater's payoff is also a function of y, m and $\hat{y}$. Using Assumption 2 one can express

$$n_1(y, \hat{y}) = n_0[1 + \varphi(\delta y - 3\delta\hat{y}/4)\delta\hat{y}/2\rho] \tag{11}$$

Combining the above expression with Assumption 3 one obtains

$$U = t(m) + \delta u n_1(y, \hat{y}).$$

It follows that for a given m and y the equilibrium outcome is pinned down by the buyers' expectation of the product payoff $\hat{y} = E[y|m]$. Q.E.D.

Lemma A.2. *In any equilibrium the buyers' expectation of the product payoff ($\hat{y}$) weakly increases with the rater's signal $\theta \in [0, 1]$.*

Proof of Lemma A.2.

In an equilibrium for each $\theta = [0, 1]$ the rater sends some message $m \in M$ with positive probability ($\sigma(m, \theta) > 0$), which induces the buyers' expectation of the product's payoff $\hat{y} = E[y|m] = \int_0^1 \theta\mu(\theta|m)d\theta$.

Consider $\theta_2 > \theta_1$ and take any $m_2 : \sigma(m_2|\theta_2) > 0$ and any $m_1 : \sigma(m_1|\theta_1) > 0$. Denote $\hat{y}_1 = E[y|m_1]$ and $\hat{y}_2 = E[y|m_2]$. Note that $E[y|\theta] = \theta$. For θ_1 the rater's chooses to send message m_1, that is $E[U|\theta_1, m_1] \geq E[U|\theta_1, m_2]$:

$$t(m_1) + \delta u n_1(\theta_1, \hat{y}_1) \geq t(m_2) + \delta u n_1(\theta_1, \hat{y}_2).$$

Analogously, for θ_2 message m_2 is chosen:

$$t(m_2) + \delta u n_1(\theta_2, \hat{y}_2) \geq t(m_1) + \delta u n_1(\theta_2, \hat{y}_1).$$

These two inequalities imply

$$n_1(\theta_2, \hat{y}_2) - n_1(\theta_2, \hat{y}_1) \geq n_1(\theta_1, \hat{y}_2) - n_1(\theta_1, \hat{y}_1).$$

Substituting for $n_1(\theta, \hat{y})$ from (11) one gets

$$n_0\varphi\delta^2\theta_2(\hat{y}_2 - \hat{y}_1)/2\rho \geq n_0\varphi\delta^2\theta_1(\hat{y}_2 - \hat{y}_1)/2\rho,$$

which implies $\hat{y}_2 \geq \hat{y}_1$. It follows that $\hat{y}$ is weakly increasing with θ. Q.E.D.

Proof of Lemma 1.

If the rater is not hired in equilibrium the result is trivial. Suppose the rater is hired. First, according to Lemma A.1 for a given message m and y the equilibrium outcome is pinned down by $\hat{y} = E[y|m]$. Second, Lemma A.2 implies that $\hat{y}$ weakly increases with $\theta \in [0, 1]$.

The weakly increasing function $\hat{y}(\theta)$ naturally defines a grading of the signal space $[0, 1]$. Intuitively for any value of this function x a grade $g(x)$ is the set of signals θ such that $\hat{y}(\theta) = x$, and the corresponding rating can be labeled as $r(x) = E[\theta|\theta \in g(x)]$ without loss of generality. So constructed grades are convex sets and may only overlap at points where $y(\theta)$ jumps. Because $\hat{y}$ is monotone it has countably many jumps, and *overlap* points are countably many.

To avoid grade overlap, any *overlap* point θ can be excluded from the grade to the right of it. This way, for any message $m \in M$ resulting in $\hat{y} = E[y|m]$ in equilibrium, one can define the corresponding grade $g(\hat{y})$ and rating $r(\hat{y})$.

By Lemma A.1, for any $m \in M$ and y the outcome is pinned down by $\hat{y}$. Therefore, for any $\theta \in [0, 1]$ instead of a message $m \in M$ inducing $\hat{y}$ the rater can use rating $r(\hat{y}) = \hat{y}$, and the outcome would be equivalent at almost every point (except maybe for countably many points).

The grades $g(x)$, $x \in [0, 1]$ are not overlapping by construction, and the constructed equilibrium is in pure strategies. This completes the proof. Q.E.D.

Proof of Lemma 3.

Under a feasible grading, the market equilibrium (prices and quantities) is fully characterized by a rating $r \in G$. It remains to show that for a given t_1 the payment schedule $t : G \to R^+$ can be fully recovered from the rater's reporting incentive constraint (4). In equilibrium the rater's expected payoff conditional on his signal (indirect utility) is a continuous function of his signal

$$U(\theta) = E[U|\theta] = \max_{r \in G} \{t(r) + \delta u n_0[1 + \varphi(\delta\theta - \hat{p}(r) - \rho\hat{q}(r)/2)\hat{q}(r)]\}, \ \forall \theta \in [0, 1]. \quad (12)$$

Using envelope theorem one gets

$$U(\theta) = U(0) + \int_0^\theta \delta^2 u n_0 \varphi \hat{q}(\hat{r}(x))dx, \ \theta \in [0, 1],$$

$$t(\hat{r}(\theta)) = U(0)+\int_0^{\theta} \delta^2 u n_0 \varphi \hat{q}(\hat{r}(x))dx - \delta u n_0[1+\varphi(\delta\theta-\hat{p}(r)-\rho\hat{q}(r)/2)\hat{q}(r)], \ \forall\theta \in [0,1]. \quad (13)$$

This formula determines the equilibrium payment schedule t up to a constant $U(0)$, which can be expressed as

$$U(0) = t_1 + \delta u n_0[1 - \varphi(\hat{p}(r_G) + \rho\hat{q}(r_G)/2)\hat{q}(r_G)]. \quad (14)$$

Thus, the equilibrium under feasible grading G is fully characterized by G and t_1. Q.E.D.

Proof of Proposition 1.

Take two consecutive ratings in a feasible grading (considering intervals is without loss of generality)

$$r_{i-1} = E[y|\theta \in [\theta_{i-2}, \theta_{i-1})] = \frac{\theta_{i-2}+\theta_{i-1}}{2}, \ r_i = E[y|\theta \in [\theta_{i-1}, \theta_i)] = \frac{\theta_{i-1}+\theta_i}{2}.$$

At the border point θ_{i-1} the rater is indifferent between the ratings, and (4) implies

$$\frac{t(r_i) - t(r_{i-1})}{\varphi\delta u n_0} = \delta\theta_{i-1}[\hat{q}(r_{i-1}) - \hat{q}(r_i)] + \hat{q}(r_i)[\hat{p}(r_i) + \frac{\rho}{2}\hat{q}(r_i)] - \hat{q}(r_{i-1})[\hat{p}(r_{i-1}) + \frac{\rho}{2}\hat{q}(r_{i-1})].$$

Substituting for $\hat{p}(r) = \delta r/2$ and $\hat{q}(r) = \delta r/2\rho$ one obtains

$$t(r_i) - t(r_{i-1}) = \frac{\varphi\delta^3 u n_0}{2\rho}(r_i - r_{i-1}) \left[\frac{3}{4}(r_i + r_{i-1}) - \theta_{i-1}\right]. \quad (15)$$

Given that

$$\frac{3}{4}(r_i + r_{i-1}) - \theta_{i-1} = \frac{3}{4}(\frac{\theta_{i-2}+\theta_{i-1}}{2} + \frac{\theta_{i-1}+\theta_i}{2}) - \theta_{i-1} = \frac{3}{8}(\theta_{i-2}+\theta_i) - \frac{1}{4}\theta_{i-1} > 0,$$

one concludes that $t(r_i) > t(r_{i-1})$ for any two consecutive ratings. This in turn implies that $t(r') > t(r)$ for any two ratings $r' > r$ that belong to the grading G. Q.E.D.

Proof of Proposition 2.

The proof consists of three parts. First, I reformulate the seller's optimization problem and show that his expected profit is affected by two distinct parts of a grading G: the lowest grade $[0, \theta_1]$, and the remaining part $G \setminus r_G$, which partitions interval $(\theta_1, 1]$ into grades. Second, I show that the part $G \setminus r_G$ consists of perfect ratings $r = \theta \in (\theta_1, 1]$. This allows to express the seller's expected profit as a function on a single parameter $\theta_1 \in [0, 1]$

and characterize the optimal grading G.

1. Reformulating the seller's problem.

Take a feasible grading G and consider the rater's indirect utility $U(\theta)$ given by (12). Equilibrium payments $t(\hat{r}(\theta))$ for each $\theta \in [0,1]$ are given by (15). One can express the seller's expected profit as

$$\Pi(G,t_1) = \int_0^1 [n_0\hat{p}(\hat{r}(\theta))\hat{q}(\hat{r}(\theta)) - t(\hat{r}(\theta))]d\theta = \int_0^1 n_0\hat{p}(\hat{r}(\theta))\hat{q}(\hat{r}(\theta))d\theta - U(0)$$
$$- \int_0^1\int_0^\theta \delta^2 u n_0 \varphi \hat{q}(\hat{r}(x))dx d\theta + \int_0^1 \delta u n_0[1+\varphi(\delta\theta - \hat{p}(\hat{r}(\theta)) - \frac{\rho}{2}\hat{q}(\hat{r}(\theta)))\hat{q}(\hat{r}(\theta))]d\theta. \tag{16}$$

Rewrite

$$\int_0^1\int_0^\theta \delta^2 u n_0 \varphi \hat{q}(\hat{r}(x))dx d\theta = \left(\theta\int_0^\theta \delta^2 u n_0 \varphi \hat{q}(\hat{r}(x))dx\right)\Bigg|_0^1 - \int_0^1 \delta^2 u n_0 \varphi \hat{q}(\hat{r}(\theta))\theta d\theta$$
$$= \int_0^1 \delta^2 u n_0 \varphi \hat{q}(\hat{r}(\theta))(1-\theta)d\theta.$$

Let

$$\tilde{\pi}(\theta) = n_0\{\hat{p}(\hat{r}(\theta))\hat{q}(\hat{r}(\theta)) + \delta u[1+\varphi(\delta\theta - \hat{p}(\hat{r}(\theta)) - \frac{\rho}{2}\hat{q}(\hat{r}(\theta)))\hat{q}(\hat{r}(\theta))] - \delta^2 u\varphi\hat{q}(\hat{r}(\theta))(1-\theta)\},$$

and rewrite the seller's expected profit as

$$\Pi(G,t_1) = \int_0^1 \tilde{\pi}(\theta)d\theta - U(0).$$

Finally, substitute for $U(0)$ from (14) in $\Pi(G,t_1)$ and reformulate the seller's problem as:

$$\max_{\{G,\, t_1 \geq 0\}} \int_0^1 \tilde{\pi}(\theta)d\theta - t_1 - \delta u n_0[1-\varphi(\hat{p}(r_G) + \frac{\rho}{2}\hat{q}(r_G))\hat{q}(r_G)], \ s.t. \ (1),(2),(5),(6). \tag{17}$$

This problem is equivalent to (8). The rater's reporting constraint (4) holds because constructed $U(\theta)$ is continuous and weakly increasing. Limited liability constraint (7) is equivalent to $t_1 \geq 0$ according to Proposition 1. Therefore, solution to (17) delivers the optimal grading.

Before proceeding with the rest of the poof let me make two observations. First, there is no gain from increasing t_1 and the solution has $t_1 = 0$. Second, the uninformative grading is feasible, therefore, the seller can always guarantee the expected profit $\Pi = n_0\delta^2/14\rho \geq \underline{\Pi}$. Hence, the seller's participation constraint (6) is not binding and can be ignored.

2. Two parts of the grading G.

For any rating $r \in G$ and corresponding grade $g(r)$ the Bayesian updating condition (5) implies

$$E[\tilde{\pi}|r] = E[\tilde{\pi}(\theta), \theta \in g(r))] = n_0[\hat{p}(r)\hat{q}(r) + \delta u[1 + \varphi(\delta r - \hat{p}(r) - \frac{\rho}{2}\hat{q}(r))\hat{q}(r)] - \delta^2 u\varphi\hat{q}(r)(1-r)],$$

because $E[\theta|\theta \in g(r)] = r$ for any $r \in G$. Using (1) and (2) one obtains $\hat{p}(r) = \delta r/2$ and $\hat{q}(r) = \delta r/2\rho$, which allows to express

$$E[\tilde{\pi}|r] = n_0\delta u + n_0\frac{\delta^2}{8\rho}[(2 + 5\delta u\varphi)r^2 - 4\delta u\varphi r],$$

$$(\hat{p}(r_G) + \frac{\rho}{2}\hat{q}(r_G))\hat{q}(r_G) = \frac{3\delta^2}{8\rho}r_G^2.$$

Let $\lambda(r)$ be the Lebesgue measure of the grade $g(r)$ for any $r \in G$, then one can write

$$\int_0^1 \tilde{\pi}(\theta)d\theta = \int_G E[\tilde{\pi}|r]\lambda(dr).$$

With no loss of generality, a grading G of the interval $[0, 1]$ consists of the lowest grade $[0, \theta_1]$ and of the remaining part $G \setminus r_G$, which partitions interval $(\theta_1, 1]$ into grades. The lowest rating $r_G = \theta_1/2$ (in case $\theta_1 = 0$ the grade is a singleton), and one can write

$$\int_G E[\tilde{\pi}|r]\lambda(dr) = \int_{G\setminus r_G} E[\tilde{\pi}|r]\lambda(dr) + E[\tilde{\pi}|r_G]\theta_1.$$

Substituting for $E[\tilde{\pi}|r]$ one can express the seller's expected profit as

$$\begin{aligned}\Pi(G,0) = &\int_{G\setminus r_G} n_0\frac{\delta^2}{8\rho}[(2+5\delta u\varphi)r^2 - 4\delta u\varphi r]\lambda(dr)\\ &+n_0\frac{\delta^2}{8\rho}[(2+5\delta u\varphi)r_G^2 - 4\delta u\varphi r_G]\theta_1 + \delta u n_0\varphi\frac{3\delta^2}{8\rho}r_G^2.\end{aligned} \tag{18}$$

In order to solve for optimal grading G, I first prove that for any $\theta_1 < 1$ the part of the grading $G \setminus r_G$ must be perfect, and then I optimize over $\theta_1 \in [0,1]$ to characterize the optimal grading G.

Consider the part of the sellers expected profit for $\theta \in (\theta_1, 1]$ affected by $G \setminus r_G$ and given by the first line in (18)

$$\Pi(G\setminus r_G) = \int_{G\setminus r_G} n_0\frac{\delta^2}{8\rho}[(2+5\delta u\varphi)r^2 - 4\delta u\varphi r]\lambda(dr).$$

This part of the profit is the highest when grades are perfect, that is $r = \theta$ for any $\theta \in (\theta_1, 1]$, and

$$\Pi((\theta_1,1]) = \int_{\theta_1}^{1} n_0\frac{\delta^2}{8\rho}[(2+5\delta u\varphi)\theta^2 - 4\delta u\varphi\theta]d\theta = n_0\frac{\delta^2}{8\rho}[(2+5\delta u\varphi)(1-\theta_1^3)/3 - 4\delta u\varphi(1-\theta_1^2)/2].$$

Let's show that $\Pi((\theta_1,1]) > \Pi(G\setminus r_G)$ if at least one grade $g(r)$, $r \in G\setminus r_G$ is not perfect (not a singleton). By Definition 1 each grade $g(r)$, $r \in G \setminus r_G$ is a convex set. Suppose grade $g(r)$ is not perfect, that is it contains at least two points $\theta_a(r) < \theta_b(r)$, then by convexity it also contains all points $\theta \in [\theta_a(r), \theta_b(r)]$ and has a positive mass $\lambda(r) > 0$. There maybe at most countably many not perfect grades (ratings) in $G \setminus r_G$. Denote the set of these ratings by R, and for each rating $r \in R$ define it's lower bound $\underline{\theta}(r) = \inf g(r)$ and it's upper bound $\overline{\theta}(r) = \sup g(r)$. This allows to express:

$$\Pi((\theta_1,1]) - \Pi(G\setminus r_G) = \sum_{r\in R}\int_{\underline{\theta}(r)}^{\overline{\theta}(r)} n_0\frac{\delta^2}{8\rho}[(2+5\delta u\varphi)(\theta^2 - r^2) - 4\delta u\varphi(\theta - r)]d\theta.$$

Note that $r = E[\theta|\theta \in g(r)] = \frac{\underline{\theta}(r)+\overline{\theta}(r)}{2}$, and

$$\Pi((\theta_1, 1]) - \Pi(G \setminus r_G) = n_0 \frac{\delta^2}{8\rho}(2 + 5\delta u\varphi) \sum_{r \in R} \int_{\underline{\theta}(r)}^{\overline{\theta}(r)} (\theta^2 - r^2) d\theta > 0,$$

because

$$\int_{\underline{\theta}(r)}^{\overline{\theta}(r)} (\theta^2 - r^2) d\theta = \frac{1}{12}(\overline{\theta}(r) - \underline{\theta}(r))^3 > 0, \ \forall r \in R.$$

Therefore, the optimal grading has perfect ratings $\theta \in (\theta_1, 1]$. Using perfect ratings $\theta \in (\theta_1, 1]$ and substituting for $r_G = \theta_1/2$ into $\Pi(G, 0)$, the seller's expected profit can be written as a function of θ_1:

$$\Pi(\theta_1) = n_0 \frac{\delta^2}{8\rho}[(2 + 5\delta u\varphi)(\frac{1}{3} - \frac{1}{12}\theta_1^3) - 2\delta u\varphi + \delta u\varphi \frac{3}{4}\theta_1^2].$$

3. Solving for the optimal grading.

Maximizing $\Pi(\theta_1)$ over $\theta_1 \in [0, 1]$ one finds optimal value θ_1 which fully characterizes the optimal grading. Compute

$$\Pi'(\theta_1) = n_0 \frac{\delta^2}{32\rho}[6\delta u\varphi - (2 + 5\delta u\varphi)\theta_1]\theta_1.$$

$\Pi'(\theta_1) \geq 0$ for $\theta_1 \in [0, \hat{\theta}]$, $\hat{\theta} = \frac{6\delta u\varphi}{2+5\delta u\varphi}$.

If $\delta u\varphi < 2$ then $\hat{\theta} < 1$, and the expected profit is maximized for $\theta_1 = \hat{\theta}$, otherwise it is maximized for $\theta_1 = 1$.

If $\delta u\varphi \geq 2$ the optimal grading is uninformative, it contains single rating $r = 1/2$.

If $\delta u\varphi < 2$ the optimal grading has the lowest rating $r_G = \theta_1/2$, $\theta_1 = \frac{6\delta u\varphi}{2+5\delta u\varphi}$, corresponding to the grade $[0, \theta_1]$, and to the payment $t_1 = 0$. The optimal grading also contains a continuum of perfect ratings $r = \theta \in (\theta_1, 1]$. The payment schedule can be expressed from (15) and (14). For any $r \in (\theta_1, 1]$ one gets

$$t(r) = n_0 \frac{\delta^3 u\varphi}{32\rho}(4r^2 - 3\theta_1^2).$$

Q.E.D.

Proof of Proposition 3.

The proof proceeds in several steps. First, I show that perfect ratings are not possible, and one can consider coarse ratings without loss of generality. Then I show that borders points of ratings are given by solutions of difference equations. Finally, I solve the difference equations and characterize admissible gradings.

1. Perfect ratings are not possible.

Note, that the payment schedule $t(.)$ pins down the rater's reporting strategy $\hat{r}:[0,1]\to G$ via (4). Therefore, using the rater's indirect payoff $U(\theta)$ given by (12), (14), and the seller's virtual profit

$$\tilde{\pi}(\theta,\hat{r}) = n_0\{\hat{p}(\hat{r}(\theta))\hat{q}(\hat{r}(\theta))+\delta u[1+\varphi(\delta\theta-\hat{p}(\hat{r}(\theta))-\frac{\rho}{2}\hat{q}(\hat{r}(\theta)))\hat{q}(\hat{r}(\theta))]-\delta^2u\varphi\hat{q}(\hat{r}(\theta))(1-\theta)\},$$

one can rewrite the seller's expected profit in problem (9) as

$$\Pi(\hat{r},t_1) = \int_0^1 \tilde{\pi}(\theta,\hat{r})d\theta - t_1 - \delta u n_0[1-\varphi(\hat{p}(r_G)+\frac{\rho}{2}\hat{q}(r_G))\hat{q}(r_G)].$$

For details see the proof of Proposition 2. Here r_G is the lowest rating in G, and t_1 is the corresponding payment. Problem (9) is equivalent to

$$\max_{\{\hat{r},\, t_1\geq 0\}} \Pi(\hat{r},t_1),\ s.t.\ (1),(2),(4),(6),(7). \tag{19}$$

Let's show that perfect ratings are not feasible. Suppose there was an interval with perfect ratings $r\in(a,b)$. Then for $r=\theta\in(a,b)$ one must have

$$\tilde{\pi}(\theta,r) = n_0\{\hat{p}(r)\hat{q}(r)+\delta u[1+\varphi(\delta\theta-\hat{p}(r)-\frac{\rho}{2}\hat{q}(r))\hat{q}(r)]-\delta^2u\varphi\hat{q}(r)(1-\theta)\},$$

and the maximization of Π would require $\frac{\partial\tilde{\pi}(\theta,r)}{\partial r}=0$ for $r=\theta\in(a,b)$. Substitute $\hat{p}(r)=\delta r/2$, $\hat{q}(r)=\delta r/2\rho$ and observe that the necessary condition

$$\frac{\partial\tilde{\pi}(\theta,r)}{\partial r} = \frac{n_0\delta^2}{4\rho}[(2-3\delta u\varphi)r+4\delta u\varphi\theta-2\delta u\varphi]=0$$

can't hold for all $r=\theta\in(a,b)$. In fact it only holds at one point $r=\theta=\frac{2\delta u\varphi}{2+\delta u\varphi}$. Thus, there can't be an interval with perfect ratings in equilibrium. A perfect rating corresponding to a single point is possible. There can be at most countably many such ratings (for details see the proof of Lemma 1), they have zero mass and can be merged with neighboring ratings without loss of generality.

2. Border points are given by solutions to difference equations.

From now on I consider ratings $r_i \in G$, $i = 1, ..., N-1$ corresponding to grades $[\theta_{i-1}, \theta_i)$, and rating r_N corresponding to grade $[\theta_{N-1}, 1]$. The solution to problem (19) has $t_1 = 0$ and, given that

$$\frac{\partial^2 \tilde{\pi}(\theta, r)}{\partial\theta\partial r} = \frac{n_0 \delta^3 u\varphi}{\rho} > 0,$$

the rater's reports are given by an increasing function

$$r^*(\theta) = \arg\max_{\{r \in G\}}[\tilde{\pi}(\theta, r)],\ \forall\theta \in [0, 1].$$

The rater's reporting strategy r^* defines the effective border points θ^*_{i-1}, $i = 2, ..., N$, so that rating r_i is issued when $\theta \in [\theta^*_{i-1}, \theta^*_i)$. At these border points the seller's virtual profit is the same for the two bordering ratings

$$\tilde{\pi}(\theta^*_{i-1}, r_{i-1}) = \tilde{\pi}(\theta^*_{i-1}, r_i),\ i = 2, ..., N. \tag{20}$$

In a feasible grading G the rater's reports must be consistent with Bayesian updating constraint (5), which is equivalent to

$$\theta^*_{i-1} = \theta_{i-1},\ i = 2, ..., N.$$

In other words, the effective border points must coincide with the border points in G that determine the buyers' beliefs. Substituting for $\tilde{\pi}(\theta, r)$ in $\tilde{\pi}(\theta_{i-1}, r_{i-1}) = \tilde{\pi}(\theta_{i-1}, r_i)$ one gets

$$(r_i - r_{i-1})[(2 - 3\delta u\varphi)(r_i + r_{i-1}) + 8\delta u\varphi\theta_{i-1} - 4\delta u\varphi] = 0,\ i = 2, ..., N. \tag{21}$$

Finally, using the fact that $r_i = (\theta_i + \theta_{i-1})/2$ one obtains

$$(3\delta u\varphi - 2)\theta_i - 2(5\delta u\varphi + 2)\theta_{i-1} + (3\delta u\varphi - 2)\theta_{i-2} = -8\delta u\varphi,\ i = 2, ..., N.$$

With notation $\lambda = \delta u\varphi$ the above condition becomes

$$(3\lambda - 2)\theta_i - 2(5\lambda + 2)\theta_{i-1} + (3\lambda - 2)\theta_{i-2} = -8\lambda,\ i = 2, ..., N. \tag{22}$$

3. Solving difference equations and characterizing feasible gradings.

Clearly, the uninformative grading with one rating ($N = 1$) and $\theta_0 = 0$, $\theta_1 = 1$ is feasible for any $\lambda \in (0, 2]$.

Since $\theta_0 = 0$, $\theta_N = 1$ a solution to (22) with $N = 2$ exists if and only if $\lambda \in (\frac{2}{11}, 6)$ and is characterized by $\theta_1 = \frac{11\lambda-2}{10\lambda+4}$.

Consider, $N \geq 3$ and take the first difference of (22)

$$(3\lambda - 2)d\theta_i - 2(5\lambda + 2)d\theta_{i-1} + (3\lambda - 2)d\theta_{i-2} = 0.$$

One must have $d\theta_i \geq 0$ for $i = 1, .., N$, which is not possible if $\lambda \leq \frac{2}{3}$, therefore $N \in \{1, 2\}$ for $\lambda \leq \frac{2}{3}$.

Consider $\lambda \in (\frac{2}{3}, 2]$, denote $x = \frac{5\lambda+2}{3\lambda-2}$, then $1 + x = \frac{8\lambda}{3\lambda-2}$, rewrite the first difference as

$$d\theta_i - 2xd\theta_{i-1} + d\theta_{i-2} = 0,\ i = 3, ..., N.$$

Solving the characteristic polynomial $D^2 - 2xD + 1 = 0$ one obtains

$$D = x - \sqrt{x^2 - 1} < 1,\ D' = 1/D.$$

Solution to the difference equation is

$$d\theta_i = AD^i + A'D^{-i},\ i = 1, ..., N.$$

Conditions $\theta_0 = 0$, $\theta_N = 1$ correspondingly require

$$(d\theta_2 + d\theta_1) - 2xd\theta_1 = -(1 + x),$$

$$1 - 2x(1 - d\theta_N) + (1 - d\theta_N - d\theta_{N-1}) = -(1 + x).$$

Substituting for $d\theta_1$, $d\theta_2$, $d\theta_{N-1}$ and $d\theta_N$ one obtains

$$A' = D(A - \frac{1 + x}{1 - D}),\ A' = D(AD^{2N} + \frac{x - 3}{1 - D}D^N).$$

Which implies

$$A = \frac{(1 + x) + (x - 3)D^N}{(1 - D)(1 - D^{2N})},\ A' = D\frac{(x - 3)D^N + (1 + x)D^{2N}}{(1 - D)(1 - D^{2N})}.$$

After substitutions one gets

$$d\theta_i = \frac{(8-4\lambda+8\lambda D^N)D^{N+1-i}+(8\lambda+(8-4\lambda)D^N)D^i}{(3\lambda-2)(1-D)(1-D^{2N})},\ i=1,...,N,$$

$$\theta_i = \sum_{j=1}^{i} d\theta_j,\ i=1,...,N.$$

Partition with $N>3$ ratings is admissible only if $d\theta_i \geq 0$, $i=1,...,N$ that is if

$$(8-4\lambda)D^N + 8\lambda\frac{D^{2N+1}+D^{2i}}{D+D^{2i}} \geq 0,\ i=1,...,N,$$

which is equivalent to

$$8-4\lambda+8\lambda\frac{D^N(1+D)}{D+D^{2N}} \geq 0.$$

For $\lambda \in (\frac{2}{3},2]$ the above condition holds and any $N=3,...,\infty$ is admissible.

As was shown before $N=2$ is admissible if and only if $\lambda \in (\frac{2}{11},6)$, that is for $\lambda \leq \frac{2}{11}$ the uninformative grading prevails. Q.E.D.

Proof of Proposition 4.

In order to show that the uninformative grading maximizes the seller's expected profit under private payments, I first obtain a useful expression for the seller's profit. Then I find an upper bound for the seller's expected profit by proving that the expected profit would only increase if all ratings except the lowest one are pooled together. Finally, I show that this upper bound on the seller's expected profit never exceeds the seller's expected profit with the uninformative grading.

1. Deriving a useful expression for the seller's profit (23).

Proposition 3 sates that grading G has at most countable number of coarse ratings r_i, $i=1,...,N$. For any $\theta \in (\theta_{i-1},\theta_i) \subset g(r_i)$, $i=1,...,N$ the seller's virtual profit

$$\tilde{\pi}(\theta) = \tilde{\pi}(\theta,r_i) = n_0\{\hat{p}(r_i)\hat{q}(r_i) + \delta u[1+\varphi(\delta\theta - \hat{p}(r_i) - \frac{\rho}{2}\hat{q}(r_i))\hat{q}(r_i)] - \delta^2 u\varphi\hat{q}(r_i)(1-\theta)\}$$

has a derivative

$$\frac{\partial\tilde{\pi}(\theta,r_i)}{\partial\theta} = 2n_0\delta^2 u\varphi\hat{q}(r_i).$$

Hence $\tilde{\pi}(\theta)$ is piece-wise differentiable with respect to θ on $[0,1]$. The seller's expected profit can be expressed as

$$\Pi(G) = \int_0^1 \tilde{\pi}(\theta)d\theta - t_1 - \delta u n_0[1 - \varphi(\hat{p}(r_1) + \frac{\rho}{2}\hat{q}(r_1))\hat{q}(r_1)].$$

For details see the proof of Proposition 2. Clearly, payment for the lowest rating must be zero ($t_1 = 0$).

According to (20) the virtual profit $\tilde{\pi}(\theta)$ is continuous at border points θ_i, $i = 1, ..., N-1$, and one can express

$$\int_0^1 \tilde{\pi}(\theta)d\theta = (\theta - 1)\tilde{\pi}(\theta)\Big|_0^1 - \int_0^1 (\theta - 1)\frac{\partial\tilde{\pi}(\theta)}{\partial\theta}d\theta$$

$$= n_0\{\hat{p}(r_1)\hat{q}(r_1) + \delta u[1 - \varphi(\hat{p}(r_1) + \frac{\rho}{2}\hat{q}(r_1))\hat{q}(r_1)] - \delta^2 u\varphi\hat{q}(r_1)\} + 2\tau\int_0^1 (1-\theta)\hat{q}(\hat{r}(\theta))d\theta,$$

here $\tau = n_0\delta^2 u\varphi$. Substituting in the seller's profit one obtains

$$\Pi(G) = n_0\hat{p}(r_1)\hat{q}(r_1) - \tau\hat{q}(r_1) + 2\tau\int_0^1 \hat{q}(\hat{r}(\theta))(1-\theta)d\theta.$$

For any $\theta \in [\theta_{i-1}, \theta_i)$, $i = 1, ..., N$ one has

$$\hat{r}(\theta) = r_i = E[\theta|\theta \in [\theta_{i-1}, \theta_i)] = \frac{\theta_{i-1} + \theta_i}{2},$$

and $\hat{q}(r_i) = \delta r_i/2\rho$. Therefore,

$$\int_0^1 \hat{q}(\hat{r}(\theta))(1-\theta)d\theta = \sum_{i=1}^N \frac{\delta r_i}{2\rho}\int_{\theta_{i-1}}^{\theta_i}(1-\theta)d\theta = \sum_{i=1}^N \int_{\theta_{i-1}}^{\theta_i}\frac{\delta r_i}{2\rho}(1-r_i)d\theta =$$

$$= \frac{\delta}{2\rho}\sum_{i=1}^N \int_{\theta_{i-1}}^{\theta_i}(\theta - r_i^2)d\theta = \frac{\delta}{4\rho} - \frac{\delta}{2\rho}\sum_{i=1}^N \int_{\theta_{i-1}}^{\theta_i} r_i^2 d\theta.$$

Together with $\hat{p}(r_1) = r_1/2 = \delta\theta_1/4$ and notations $\tau = n_0\delta^2 u\varphi$, $\lambda = \delta u\varphi$ this allows to obtain a useful expression for the seller's expected profit

$$\Pi(G) = \frac{n_0\delta^2}{16\rho}\left\{\theta_1^2 - 4\lambda\theta_1 + \lambda[8 - 4\sum_{i=1}^{N}(\theta_i + \theta_{i-1})^2(\theta_i - \theta_{i-1})]\right\} \tag{23}$$

2. Upper bound on the expected profit with informative gradings $\bar{\Pi}(\theta_1)$.

The seller's expected profit would increase if ratings above r_1 are pooled together. First, let's show that $\Pi(G)$ would increase if any any two consecutive ratings $r_i = (\theta_i + \theta_{i-1})/2$, $r_{i-1} = (\theta_{i-1} + \theta_{i-2})/2$, $i \geq 3$ are pooled together in a single rating $\tilde{r} = (\theta_i + \theta_{i-2})/2$. To see this take the sum

$$\psi(\theta_{i-1}) = -[(\theta_i + \theta_{i-1})^2(\theta_i - \theta_{i-1}) + (\theta_{i-1} + \theta_{i-2})^2(\theta_{i-1} - \theta_{i-2})],\ \theta_{i-1} \in [\theta_{i-2}, \theta_i],$$

and compute

$$\psi'' = 2(\theta_i - \theta_{i-2}) > 0.$$

Hence, $\psi(\theta_{i-1})$ is the highest when $\theta_{i-1} = \theta_i$ or $\theta_{i-1} = \theta_{i-2}$, that is when the two ratings r_i and r_{i-1} are replaced with a single pooling rating $\tilde{r}$. Since this argument can be applied for any rating $r_i \geq 2$, one obtains that for any $N \geq 2$ the following is true

$$\Pi(G) \leq \overline{\Pi}(\theta_1) = \frac{n_0\delta^2}{16\rho}\left\{\theta_1^2 - 4\lambda\theta_1 + \lambda[8 - 4[\theta_1^3 + (1+\theta_1)^2(1-\theta_1)]\right\}.$$

3. The uninformative grading delivers a higher expected profit than the upper bound.

The uninformative grading has a single rating $r_1 = 1/2$, the corresponding profit can be obtain by substituting $\theta_1 = 1$ in the expression $\Pi(\theta_1)$, which delivers $\Pi_0 = \frac{n_0\delta^2}{16\rho}$. In order to show that the seller's expected profit is the highest with the uninformative grading it suffices to prove that $\overline{\Pi}(\theta_1) < \Pi_0$, for any $N \geq 2$.

According to Proposition 3 one needs to consider two cases. Case 1: $\lambda \in (\frac{2}{11}, 2]$, $N = 2$ and $\theta_1 = \frac{11\lambda - 2}{10\lambda + 4}$. Case 2: $\lambda \in (\frac{2}{3}, 2]$, $N \geq 3$, $D = \frac{5\lambda + 2 - 4\sqrt{\lambda(\lambda+2)}}{3\lambda - 2}$ and

$$\theta_1(N) = \frac{(8 - 4\lambda)(1 + D)D^N + 8\lambda(D + D^{2N})}{(3\lambda - 2)(1 - D^{2N})(1 - D)}. \tag{24}$$

Case 1: $N = 2$, $\lambda \in (\frac{2}{11}, 2]$, $\theta_1 = \frac{11\lambda - 2}{10\lambda + 4}$. Denote

$$S(\theta_1) = \theta_1^2 - 4\lambda\theta_1 + \lambda(8 - 4(\theta_1^3 + (1+\theta_1)^2(1-\theta_1))) =$$

$$= \theta_1^2 - 4\lambda\theta_1 + \lambda(8 - 4(\theta_1^3 + 1 + 2\theta_1 + \theta_1^2 - \theta_1 - 2\theta_1^2 - \theta_1^3)) =$$

$$= \theta_1^2 - 4\lambda\theta_1 + \lambda(8 - 4(1 + \theta_1 - \theta_1^2)) =$$

$$= \theta_1^2 - 4\lambda\theta_1 + 4\lambda - 4\lambda\theta_1 + 4\lambda\theta_1^2 = \theta_1^2 + 4\lambda(\theta_1 - 1)^2 =$$

$$= \frac{(11\lambda - 2)^2}{(10\lambda + 4)^2} + 4\lambda\frac{(\lambda - 6)^2}{(10\lambda + 4)^2} =$$

$$= \frac{121\lambda^2 - 44\lambda + 4 + 4\lambda^3 - 48\lambda^2 + 144\lambda}{100\lambda^2 + 80\lambda + 16} =$$

$$= \frac{4\lambda^3 + 73\lambda^2 + 100\lambda + 4}{100\lambda^2 + 80\lambda + 16}.$$

In order to prove $\overline{\Pi}(\theta_1) < \Pi_0$ one needs to show to $S(\theta_1) < 1$, which is equivalent to $4\lambda^3 - 27\lambda^2 + 20\lambda - 12 < 0$. Denote the latter function by

$$f(\lambda) = 4\lambda^3 - 27\lambda^2 + 20\lambda - 12 = \lambda^2(4\lambda - 27) + 4(5\lambda - 3).$$

So, if $\frac{1}{4} < \lambda \leq \frac{3}{5}$, then both summands are negative and $f(\lambda) < 0$. Next I consider the case $\frac{3}{5} \leq \lambda \leq 2$. Take derivatives

$$f'(\lambda) = 12\lambda^2 - 54\lambda + 20, \; f''(\lambda) = 24\lambda - 54.$$

For $\lambda \leq 2$, one has $f''(\lambda) < 0$. So, f' is decreasing for $\frac{3}{5} \leq \lambda \leq 2$. In particular,

$$f'(\lambda) \leq f'(\frac{3}{5}) = \frac{108}{25} - \frac{162}{5} + 20 = -\frac{202}{25} < 0$$

and f is also decreasing. In particular

$$f(\lambda) \leq f(\frac{3}{5}) = \frac{108}{125} - \frac{243}{25} = -\frac{1107}{125} < 0$$

It means that for $\lambda \in (\frac{2}{11}, 2]$ one has $S(\theta_1) < 1$ and $\overline{\Pi}(\theta_1) < \Pi_0$.

Case 2: $\lambda \in (\frac{2}{3}, 2]$, $N \geq 3$ and $\theta_1 = \theta_1(N)$.

Consider function $S(\theta_1) = \theta_1^2 + 4\lambda(\theta_1 - 1)^2$. Note that it is convex and reaches the highest value either at $\theta_1 = 1$ or at the lowest possible $\theta_1(N)$, $N = 3, ..., \infty$. First, I show that $\theta_1(N)$ is decreasing with N and take the limit $\theta_1(\infty) < \theta_1(N)$, for any $N \geq 3$. Then, I show that $S(\theta_1(\infty)) < S(1)$, that is $N = 1$ and the uninformative grading ($\theta_1 = 1$) delivers the highest expected profit to the seller.

To prove that $\theta_1(N)$ is decreasing in N, observe that

$$D = \frac{(2\sqrt{\lambda} - \sqrt{\lambda+2})^2}{(2\sqrt{\lambda} - \sqrt{\lambda+2})(2\sqrt{\lambda} + \sqrt{\lambda+2})} = \frac{2\sqrt{\lambda} - \sqrt{\lambda+2}}{2\sqrt{\lambda} + \sqrt{\lambda+2}} < 1.$$

The denominator in (24) $(3\lambda - 2)(1 - D^{2N})(1 - D)$ is increasing with N as $\lambda > 2/3$. The first derivative of the numerator in (24) is

$$N\left[(8 - 4\lambda)(1 + D) + 16\lambda)\right]\ln(D).$$

Given that $\ln(D) < 0$, and $8(1 + D) + 4\lambda(3 - D) > 0$ the numerator decreases with N. This proves that $\theta_1(N)$ decreases with N.

Compute the limit

$$\theta_1(\infty) = \lim_{N\to\infty} \theta_1(N) = \frac{8\lambda D}{(3\lambda - 2)(1 - D)}.$$

If one can prove $S(\theta_1(\infty)) < S(1)$ this would also imply $S(\theta_1(N)) < S(1)$ for any $N \geq 3, ..., \infty$ because $S(\theta_1)$ is convex. $S(\theta_1(\infty)) < S(1)$ is equivalent to

$$\Phi(\lambda) = 16\lambda D^2 + \left[(11D - 3)\lambda + 2(1 - D)\right]^2 - \frac{(3\lambda - 2)^2}{4\lambda}(1 - D)^2 < 0.$$

Introduce new variable $t = D$, then

$$t = 1 - \frac{1}{\sqrt{\frac{\lambda}{\lambda+2} + \frac{1}{2}}}, \text{ and } \lambda = \frac{2(1+t)^2}{4(1-t)^2 - (1+t)^2}.$$

When λ varies from 2/3 to 2, the variable t varies from 0 to $\frac{\sqrt{2}-1}{\sqrt{2}+1}$. The function Φ becomes

$$\Psi(t) = 16t^2 \frac{2(1+t)^2}{4(1-t)^2 - (1+t)^2} + \left((11t - 3)\frac{2(1+t)^2}{4(1-t)^2 - (1+t)^2} + 2(1 - t)\right)^2 -$$

$$-\frac{(8(1+t)^2 - 8(1-t)^2)^2}{8(1+t)^2(4(1-t)^2 - (1+t)^2)}(1 - t)^2.$$

We will prove that $\Psi(t) < 0$ for $0 < t \leq \frac{\sqrt{2}-1}{\sqrt{2}+1}$. Rewrite the function $\Psi(t)$ in the form

$$\Psi(t) = \frac{32t^2(1+t)^2}{4(1-t)^2 - (1+t)^2} + \frac{((22t - 6)(1+t)^2 + 8(1-t)^3 - 2(1-t)(1+t)^2)^2}{(4(1-t)^2 - (1+t)^2)^2} -$$

$$-\frac{128t^2(1-t)^2}{(1+t)^2(4(1-t)^2-(1+t)^2)} =$$

$$= \frac{32t^2(1+t)^2}{4(1-t)^2-(1+t)^2} + \frac{256t^2(t^2+4t-1)^2}{(4(1-t)^2-(1+t)^2)^2} -$$

$$-\frac{128t^2(1-t)^2}{(1+t)^2(4(1-t)^2-(1+t)^2)} = \frac{32t^2}{4(1-t)^2-(1+t)^2}\Psi_1(t),$$

where

$$\Psi_1(t) = (1+t)^2 + \frac{8(t^2+4t-1)^2}{4(1-t)^2-(1+t)^2} - \frac{4(1-t)^2}{(1+t)^2} =$$

$$= \frac{(1+t)^4-4(1-t)^2}{(1+t)^2} + \frac{8(t^2+4t-1)^2}{4(1-t)^2-(1+t)^2} =$$

$$= \frac{(t^2+4t-1)(t^2+3)}{(1+t)^2} + \frac{8(t^2+4t-1)^2}{4(1-t)^2-(1+t)^2} = (t^2+4t-1)\Psi_2(t)$$

where

$$\Psi_2(t) = \frac{t^2+3}{(1+t)^2} + \frac{8(t^2+4t-1)}{4(1-t)^2-(1+t)^2}.$$

So,

$$\Psi(t) = \frac{32t^2(t^2+4t-1)}{4(1-t)^2-(1+t)^2}\Psi_2(t)$$

The multiplier near $\Psi_2(t)$ is negative on $(0, \frac{\sqrt{2}-1}{\sqrt{2}+1}]$, so it is enough to check that

$$\Psi_2(t) > 0, 0 < t \leq \frac{\sqrt{2}-1}{\sqrt{2}+1}.$$

Writing under the common denominator,

$$\Psi_2(t) = \frac{1-14t+76t^2+38t^3+11t^4}{(1+t)^2(4(1-t)^2-(1+t)^2)}$$

It is enough to check that the numerator is > 0, i.e. there are no real roots of the quartic equation

$$11t^4+38t^3+76t^2-14t+1=0.$$

Denote coefficients of this equation by $a = 11$, $b = 38$, $c = 76$, $d = -14$, $e = 1$. Then discriminants are calculated as follows

$$\Delta = 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de +$$

$$+18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2 = 2027749376 > 0$$

$$P = 8ac - 3b^2 = 2356 > 0$$

In such case there are no real roots. So, $\Psi_2(t) > 0$ on the needed interval. This proves that $S(\theta_1(\infty)) < S(1)$ and $S(\theta_1(N)) < S(1)$ for any $\lambda \in (\frac{2}{3}, 2]$ and $N \geq 3$. This in turn implies $\overline{\Pi}(\theta_1) < \Pi_0$.

This completes the proof. Q.E.D.

A.2. Infinite time horizon model

Here I develop the infinite-horizon model that provides foundations for Assumption 3 and the reputational term u in the rater's objective function in the baseline two-period model. For simplicity, I consider public payments and a competitive rater.

Each period $\tau = 0, .., \infty$ a new seller arrives to the market with a product. The payoff of product $y_\tau \geq 0$ on R^+ offered by seller $\tau = 0, .., \infty$ is unknown at τ and is distributed according to distribution function F, it realizes at $\tau + 1$. We assume that the rater receives a signal $\theta_\tau = E[y_\tau|\theta_\tau]$ at moment τ about the payoff y_τ. For technical reasons I assume the signal to be sufficiently precise, so that the realization of y_τ can't be much lower than the signal θ_τ: $F(y_\tau|\theta_\tau)$ is such that $y_\tau|\theta_\tau \geq \frac{3}{4}\theta_\tau - \frac{2\rho}{\delta^2 \varphi \theta}$.

At each τ a long-lived rater can issue a rating informative about the product payoff, which is followed by buyers in his user base n_τ. For simplicity we assume that each seller and each cohort or buyers $\tau = 0, .., \infty$ live for one period only, and their payoffs are the same as in the two-period model (alternatively we can assume that they interact with the rater only once). As before, $n_0 > 0$ is the size of the rater's user base at $\tau = 0$. For simplicity, here the seller's outside option is equal to her expected profit with uninformative grading and zero payments $\underline{\Pi}(n_\tau) = n_\tau \delta^2 / 16\rho$.

At any period $\tau \geq 1$ the payoffs of the previous cohort of buyers determines the rater's current user base (dynamic version of Assumption 2):

$$n_{\tau+1} = n_\tau + \varphi \int_{i \in n_\tau} S_i di, \ \forall \tau = 0, ..., \infty. \tag{25}$$

Markov Strategies. The timing of the game within each period τ is similar to the two-period model. Given that sellers and buyers are short-lived, and the payoffs of products y_τ, $\tau \in 0, ..., \infty$ are i.i.d., the payoff relevant information in the beginning of each period is fully captured by the rater's user base n_τ. Therefore, I focus on equilibria in Markov strategies that depend only on user base n_τ and drop subscript τ.

For each level of n and any possible combination of G, t the buyers and the seller form beliefs about the rater's signal for each rating $r \in G$. Formally, for each $r \in G$ the beliefs specify a probability distribution over possible signals $G \to \Delta([0, 1])$ with the corresponding density $\mu^n_{Gt}(\theta|r)$.

The rater's Markov strategy for each level of the user base n specifies the grading G_n and the corresponding payments $t_n(r)$, $r \in G_n$ offered to the seller. Moreover, when the

G_n, t_n is accepted the rater's strategy also specifies his reporting function, which maps each possible realization of his signal $\theta \in [0,1]$ into report $\hat{r}^n_{Gt}(\theta) \in G_n$.

The seller's Markov strategy specifies whether to accept G, t, and in case of acceptance it defines the price function which determines the product price $\hat{p}^n_{Gt}(r)$ for each rating $r \in G$ potentially issued by the rater.[14] The Markov strategy of each buyer $i \in n$ is a function indexed by n that for any G, t maps each rating $r \in G$ and price $p \in R^+$ into the quantity this buyer is willing to purchase $\hat{q}^n_{Gt}(r,p)$.

Equilibrium. Markov strategies G_n, t_n, $\hat{p}^n_{Gt}$, $\hat{r}^n_{Gt}$, $\hat{q}^n_{Gt}$, and the dynamics of the user base determined by (25) constitute a Markov equilibrium of the dynamic game described above if the following conditions hold

i) the quantity purchased by each buyer $i \in n$ maximizes his expected payoff

$$\hat{q}^n_{Gt}(r,p) = \arg\max_{q_i \geq 0} \int_0^1 S_i(y_\tau, p, q_i)\mu^n_{Gt}(\theta|r)d\theta,\ \forall r \in G,\ p \geq 0, \tag{26}$$

ii) the seller price function $\hat{p}^n_{Gt}$ maximizes his profit

$$\hat{p}^n_{Gt}(r) \in \arg\max_{p \geq 0} \{np\hat{q}^n_{Gt}(r,p)\},\ \forall r \in G, \tag{27}$$

iii) the rater's reporting strategy at each $\tau = 0, ..., \infty$ maximizes his expected payoff

$$\hat{r}^n_{Gt}(\theta) \in \arg\max_{r \in G} \{t_\tau(r) + E\left(\sum_{s=1}^{\infty} \delta^s t_{\tau+s}|\theta, \hat{p}^n_{Gt}, \hat{q}^n_{Gt}, \hat{r}^n_{Gt}, G_n, t_n, n\right)\},\ \forall \theta \in [0,1], \tag{28}$$

iv) the beliefs $\mu^n_{Gt}(\theta|r)$ are consistent with the reports $\hat{r}^n_{Gt}$ on the equilibrium path,

v) payments are non-negative $t(r) \geq 0$, $r \in G$,

vi) the user base is not negative $n_\tau \geq 0$, $\tau = 0, ..., \infty$,

vii) the grading and the payment schedule maximize the seller's expected profit given n, denoting $p^n_{Gt}(\theta) = \hat{p}^n_{Gt}(\hat{r}^n_{Gt}(\theta))$, $q^n_{Gt}(\theta) = \hat{q}^n_{Gt}(\hat{r}^n_{Gt}(\theta), p^n_{Gt}(\theta))$ one must have

$$\{G_n, t_n\} \in \arg\max_{\{G,t\}} \{\int_0^1 [np^n_{Gt}(\theta)q^n_{Gt}(\theta) - t(\hat{r}^n_{Gt}(\theta))]\, d\theta\},\ \forall n, \tag{29}$$

Since rater's messages are cheap-talk multiple equilibria are possible. For instance, the

[14]As in the two-period model, the seller can't commit to a pricing rule in the beginning of the period and sets the ex-post profit maximizing price once rating r is published.

non-informative or "babbling" equilibrium always exits. In this equilibrium the payment to the rater is zero and the grading contains a single uninformative rating, which the rater issues independently of his information.

Stationary grading equilibrium. The goal of this section is to provide foundations for the rater's objective function in the two-period model (Assumption 3). To this end, I characterize a class of Markov equilibria where the sum of the rater's future expected discounted payments is equivalent to the parameter u in Assumption 3. I consider *stationary grading equilibria* in which the rater's grading remains stable in time $G_n = G$, and the payments to the rater are proportional to his user base $t_n(r) = nx(r)$, $x(r) > 0$.[15]

In a stationary grading equilibrium at any date τ the rater's objective function in (28) can be written as

$$U = n_\tau x(r) + \delta E_\tau[n_{\tau+1}x(\hat{r}_G(\theta_{\tau+1}))] + \sum_{s=2}^{\infty} \delta^s E_\tau[n_{\tau+s}x(\hat{r}_G(\theta_{\tau+s}))].$$

Once the rating r is issued, the market price and the quantity bought by each buyer $i \in n_\tau$ are $\hat{p}(r) = \delta r/2$, $\hat{q}(r) = \delta r/2\rho$. The rater knows $E[y_\tau|\theta_\tau] = \theta_\tau$ when he chooses the rating, and using (25) he expects the user base next period to be

$$E[n_{\tau+1}(\theta_\tau, r)|\theta_\tau] = n_\tau w(\theta_\tau, r), \; w(\theta, r) = 1 + \varphi[(\delta\theta - \hat{p}(r))\hat{q}(r) - \rho\hat{q}(r)^2/2].$$

Since θ_τ, $\tau = 0, ..., \infty$ are i.i.d. in a stationary grading equilibrium for any $s = 1, ..., \infty$ the expected per user payment at $\tau + s$ can be computed as

$$x^E = E_\tau\left(x(\hat{r}_G(\theta_{\tau+s}))\right) = \int_0^1 x(\hat{r}_G(\theta))d\theta.$$

Denote

$$\zeta = E_\tau\left(w(\theta_{\tau+s}, r_{\tau+s})\right) = \int_0^1 w(\theta, \hat{r}_G(\theta))d\theta$$

and rewrite the rater's expected payoff as:

$$U = n_\tau x(r) + \delta E[n_{\tau+1}(\theta_\tau, r)|\theta_\tau]\frac{x^E}{1 - \delta\zeta}. \tag{30}$$

An equivalent of this objective function in the two-period model is $U = n_0 x(r) + \delta n_1 u$, in

[15] In the Online Appendix it is proven that in a stationary grading equilibrium the payment schedule must be of the form $t_n(r) = \underline{t} + nx(r)$.

which the reputation parameter $u = x^E/(1-\delta\zeta)$ captures the expected future payments to the rater. This observation immediately implies that the rater's reporting incentives in the two models are similar. In other words, the two-period model where the rater cares about the buyer's surplus directly (Assumption 3 with $u > 0$) can be seen as a shortcut for a more complex infinite horizon model in which the rater only cares about the payments he receives form sellers. As is shown below, under certain conditions the infinite horizon model has stationary grading equilibria that correspond to the seller's optimal grading characterized in the two-period model.

Remark 3. Any grading G feasible in a stationary grading equilibrium of the infinite time horizon model is feasible in an equivalent two-period model with parameter $u = x^E/(1-\delta\zeta)$. In other words, any possible stationary grading equilibrium of the infinite horizon model has a corresponding equivalent equilibrium of the two-period model with some exogenous reputation parameter u.

Consider now the grading offered by a competitive rater in the two-period model and let's show that under certain conditions this grading can be a part of the stationary grading equilibrium in the infinite time horizon model. This is obviously true if the competitive grading is uninformative. When $u < \frac{2}{\delta\varphi}$ the competitive grading G^* is informative, according to Proposition 2. The grading has a single imprecise rating with no payment for low $\theta \leq \underline{\theta} = \frac{6\delta u\varphi}{2+5\delta u\varphi}$, and perfect ratings ($r = \theta$) for $\theta > \underline{\theta}$ with positive payments given by

$$t_n^*(r) = n\delta^3 u\varphi(4r^2 - 3\underline{\theta}^2)/32\rho.$$

Under certain conditions the infinite horizon model has a stationary grading equilibrium with exactly the same grading and payment schedule as the competitive grading in the two-period model with some reputation parameter $u \in (0, \frac{2}{\delta\varphi})$.[16]

Proposition 5. *If $\delta \geq \frac{3}{4}$ and $32 > \frac{\delta^3\varphi}{(1-\delta)\rho} > 12$ there exists a stationary grading equilibrium with grading G^* which has a single imprecise rating with no payment for $\theta \leq \underline{\theta} = \frac{6\delta u\varphi}{2+5\delta u\varphi}$, and perfect ratings ($r = \theta$) for $\theta > \underline{\theta}$ with positive payments*

$$t_n^*(r) = n\delta^3 u\varphi(4r^2 - 3\underline{\theta}^2)/32\rho.$$

[16]The case with $u = 0$ is trivial. In this case the payments are zero and the rater get's no compensation. Since he is indifferent between ratings his reports can be consistent with any grading G in equilibrium.

The threshold $\underline{\theta}$ solves

$$8 + 4\underline{\theta}^3 - 9\underline{\theta}^2 = \frac{(1-\delta)96\rho}{\varphi\delta^3}. \tag{31}$$

Proof of Proposition 5.

We need to prove that there exists a reputation parameter $u \in (0, \frac{2}{\delta\varphi})$ and the corresponding competitive grading in the two-period model with public payments that can be implemented in a stationary grading equilibrium of the infinite-horizon model. Suppose such u exists and consider grading G^* and payment schedule $t_n^*(r)$ optimal in the two-period model.

As in the two-period model, in the infinite horizon model once the rating r is issued equilibrium conditions i) and ii) determine the market price and the quantity bought by each buyer $i \in n_\tau$ are $\hat{p}(r) = \delta r/2$, $\hat{q}(r) = \delta r/2\rho$. The rater's reports $\hat{r}(\theta)$ are determined by grading G^*.

This allows to compute

$$x^E = \int_{\underline{\theta}}^{1} \delta^3 u\varphi(4\theta^2 - 3\underline{\theta}^2)/32\rho d\theta = u\varphi\frac{\delta^3}{96\rho}(4 + 5\underline{\theta}^3 - 9\underline{\theta}^2)$$

and

$$\zeta = \int_0^1 w(\theta, \hat{r}_G(\theta))d\theta = 1 + \varphi\int_0^{\underline{\theta}}[(\delta\theta - \frac{\delta\underline{\theta}}{4})\frac{\delta\underline{\theta}}{4\rho} - \frac{\delta^2\theta^2}{32\rho}]d\theta + \varphi\int_{\underline{\theta}}^{1}\frac{\delta^2\theta^2}{8\rho}]d\theta$$

$$= 1 + \varphi\frac{\delta^2}{32\rho}\underline{\theta}^3 + \varphi\frac{\delta^2}{24\rho}(1-\underline{\theta}^3) = 1 + \varphi\frac{\delta^2}{96\rho}(4-\underline{\theta}^3).$$

The rater's objective function is (30) and his reporting constraint (28) is equivalent to the reporting constraint in the two-period model (4) with $u' = x^E/(1-\delta\zeta)$. By construction G^* and $t_n^*(r)$ maximize the seller's profit in the two-period model when the rater has the reputation parameter u. In a stationary grading equilibrium the rater's reports must be consistent with G^* and payments $t_n^*(r) = nx(r)$. Hence one must have $u' = x^E/(1-\delta\zeta) = u > 0$, which requires (31)

$$8 + 4\underline{\theta}^3 - 9\underline{\theta}^2 = \frac{(1-\delta)96\rho}{\varphi\delta^3}.$$

The above condition is necessary for a stationary grading equilibrium with G^*, $t_n^*(r)$ to exist, it is required for equilibrium condition iii) to hold. Conditions iv) and v) hold because G^* maximizes the seller's profit in the two-period model. Two more conditions must be satisfied vi) and vii).

Condition vi) requires that the rater's user base is never negative:

$$n_{\tau+1} = w(y_\tau, r_\tau)n_\tau \geq 0,\ \tau = 0, ..., \infty.$$

This is so only if $w(y_\tau, \hat{r}(\theta_\tau)) \geq 0$ for any $\theta_\tau \in [0,1]$ and $y_\tau \geq 0$. Under grading G^* for $\theta_\tau > \underline{\theta}$ one has

$$w(y_\tau, \hat{r}(\theta_\tau)) = 1 + \varphi\frac{\delta^2\theta_\tau}{2\rho}(y_\tau - \frac{3}{4}\theta_\tau) \geq 0,$$

because we have assumed that $F(y_\tau|\theta_\tau)$ is such that $y_\tau|\theta_\tau \geq \frac{3}{4}\theta_\tau - \frac{2\rho}{\delta^2\varphi\theta}$.

For $\theta_\tau \leq \underline{\theta}$ one gets

$$w(y_\tau, \hat{r}(\theta_\tau)) = 1 + \varphi[(\delta y_\tau - \frac{\delta\underline{\theta}}{4})\frac{\delta\underline{\theta}}{4\rho} - \frac{\delta^2\underline{\theta}^2}{32\rho}],$$

with the lowest value of $1 - \varphi\frac{3\delta^2}{32\rho}\underline{\theta}^2$ at $y_\tau = 0$. Thus, in equilibrium one must have

$$\underline{\theta}^2 \leq \frac{32\rho}{3\varphi\delta^2}. \tag{32}$$

Finally, vii) requires the seller's expected profit to be at least as high as under the uninformative grading. One can fix the out of equilibrium beliefs in such a way that whenever the seller proposes a grading different from G^*, the buyers believe rating to be uninformative, which in turn would guarantee that the seller proposes G^* on equilibrium path. Note that for $\underline{\theta} \in (0,1]$ the seller expected profit under G^* is at least as high as with the uninformative grading, because G^* is optimally chosen in the two-period model where the uninformative grading is feasible. Therefore vii) is satisfied.

One can conclude that when (31) and (32) hold for some $\underline{\theta} \in (0,1]$ there exists a Markov equilibrium with stationary grading G^* and payments $t_n^*(r)$ because equilibrium conditions i)-vii) are satisfied.

Now we show that when $\delta \geq \frac{3}{4}$ and $32 > \frac{\delta^3\varphi}{(1-\delta)\rho} > 12$ one can find $u \in (0, \frac{2}{\delta\varphi})$ and $\underline{\theta} \in (0,1)$ that satisfy (31) and (32). Note that the left hand side of (31) is decreasing with $\underline{\theta}$, it is equal to 8 when $\underline{\theta} = 0$ and it is equal to 3 when $\underline{\theta} = 1$. It follows that whenever $32 > \frac{\delta^3\varphi}{(1-\delta)\rho} > 12$ solution to (31) delivers $\underline{\theta} \in (0,1)$ and the corresponding

$$u = \frac{2}{\delta\varphi}\frac{\underline{\theta}}{6 - 5\underline{\theta}} \in (0, \frac{2}{\delta\varphi}).$$

Finally, $32 > \frac{\delta^3\varphi}{(1-\delta)\rho}$ implies

$$\frac{32\rho}{3\varphi\delta^2} > \frac{\delta}{3(1-\delta)},$$

which together with $\delta \geq \frac{3}{4}$ guarantee $\frac{32\rho}{3\varphi\delta^2} > 1 > \underline{\theta}^2$, and (32) is satisfied. Q.E.D.

In the infinite horizon game the equivalent of reputation parameter u is endogenously determined: u must solve a fixed point problem. Proposition 5 provides the sufficient condition for this fixed point problem to have a solution with $\underline{\theta} \in (0,1)$. To gain the intuition behind the sufficient condition let us first establish

Corollary 2. *Equilibrium grading described in Proposition 5 becomes less informative ($\underline{\theta}$ increases) when the rater cares more about future payments (δ increases), when rater's user base becomes more sensitive to past payoffs (φ/ρ increases).*

The proof immediately follows from the fact that the left-hand-side of (31) is decreasing with $\underline{\theta} \in (0,1)$, while the right-hand side decreases with δ and φ/ρ. Intuitively, when δ (or φ/ρ) increases the rater cares a lot about the payoff of current buyers, because they affect his future user's base and his future revenue. In terms of the two-period model the rater is pro-buyers and is tough with the seller (high u), he requires significant compensation for issuing high precise ratings. As a result the competitive grading (described in Proposition 2) economizes on expensive precise ratings $r = \theta > \underline{\theta}$ by increasing the threshold $\underline{\theta}$ for the imprecise rating.

The above results are somewhat counterintuitive. Indeed, one might expect that a rater who cares a lot about future payoffs (high δ) or whose users are very sensitive to their payoffs (high φ/ρ) is likely to issue very informative ratings. However, the analysis suggests that a rater with low δ and low φ/ρ may use more informative grading in equilibrium, than a rater with high δ and high φ/ρ. This is not so surprising once one takes into account that the rater with high δ and high φ/ρ is very protective of the buyers and requires significant compensation for high precise ratings. This endogenously forces the grading to be less informative in order to minimize the seller's expected expenses from a rating and make certification attractive from the seller's point of view. All in all, the rater with low δ and low φ/ρ requires a low payment in expectation and induces a relatively informative grading in equilibrium.

Finally, note that Corollary 2 helps to explain the conditions in Proposition 5 for a stationary grading equilibrium with G^* to exist. Indeed, by definition $\underline{\theta}$ solving (31) must be inside the interval $(0,1)$. Given that $\underline{\theta}$ increases with δ and φ/ρ, in order to guarantee

$\underline{\theta} \in (0,1)$ one must have $32 > \frac{\delta^3 \varphi}{(1-\delta)\rho} > 12$.

B. Online Appendix

Lemma B.1. *In a stationary grading equilibrium the payment schedule must be of the form* $t_n(r) = \underline{t} + nt(r)$.

Proof of Lemma B.1. General idea behind the proof is simple. The rater's objective in (28) is to maximize the expected present value of payments. If the payments depend on n in a non-linear way, the rater's maximization problem may deliver different reporting strategies for different n. This can't happen in a stationary grading equilibrium where the rater's equilibrium reporting strategy must be independent of n. Hence, the payment schedule must be linear in n.

Consider the rater's reporting constraint (28) at some $\tau = 0, ..., \infty$. As in the two-period model once the rating r is issued, the market price and the quantity bought by each buyer $i \in n_\tau$ are $\hat{p}(r) = \delta r/2$, $\hat{q}(r) = \delta r/2\rho$. The rater knows $E[y_\tau|\theta_\tau] = \theta_\tau$ when he chooses the rating, he perfectly anticipates the dynamics of the user base next period

$$E[n_{\tau+1}] = n_\tau w(\theta_\tau, r),\ w(\theta, r) = 1 + \varphi[(\delta\theta - \hat{p}(r))\hat{q}(r) - \rho\hat{q}(r)^2/2].$$

The rater's reporting strategy must be consistent with the grading: at any period $\tau = 0, ..., \infty$, $\hat{r}(\theta) = r$ if and only if the signal is in the corresponding grade $\theta \in g(r)$. As a result, the dynamics of the rater's user base under a stationary grading in subsequent periods can also be easily characterized $E[n_{\tau+1}] = n_\tau z(\theta)$, with $z(\theta) = w(\theta, \hat{r}(\theta))$. With a minor change of notation $t_n = t(n)$ and $t_n(r) = t(n, r)$ the rater's objective function in (28) can be written as

$$\begin{aligned} U(n_t, \theta_t, r) &= t(n_t, r) + \delta E_t \left(t(n_\tau w(\theta_\tau, r), \hat{r}(\theta_{\tau+1}))\right) \\ &+ \sum_{s=2}^{\infty} \delta^s E_\tau \left(t(n_\tau w(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}), \hat{r}(\theta_{\tau+s})) \right). \end{aligned}$$

The equilibrium reporting strategy

$$\hat{r}(\theta) = \arg\max_{r \in G} U(n_t, \theta_t, r)$$

must be consistent with a stationary grading G, that is the solution must not depend on n_τ. Clearly the trivial uninformative grading and a fixed payment satisfy this condition. Here I characterize the general from of payment schedule which is compatible with a non-trivial grading. Suppose the equilibrium grading G is non-trivial and contains an interval of perfect ratings $r = \theta \in (\underline{\theta}, \overline{\theta})$. Now it will be shown that in this case, $t(n, r)$ is either linear in n or is a constant.

For any θ within this interval the FOC must hold:

$$\frac{\partial U(n_t, \theta_t, r)}{\partial r} = t_2(n_t, r) + \delta E_t \left(t_1(n_\tau w(\theta_\tau, r), \hat{r}(\theta_{\tau+1}))n_\tau w_2(\theta_\tau, r)\right)$$
$$+\sum_{s=2}^{\infty} \delta^s E_\tau \left(t_1(n_\tau w(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}), \hat{r}(\theta_{\tau+s}))n_\tau w_2(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}) \right) = 0.$$

Moreover, the FOC must hold for any $n > 0$, hence additional necessary condition

$$\frac{\partial^2 U(n_t, \theta_t, r)}{\partial r \partial n_t} = t_{12}(n_t, r) + \delta E_t \left(t_1(n_\tau w(\theta_\tau, r), \hat{r}(\theta_{\tau+1}))w_2(\theta_\tau, r)\right)$$
$$+\delta E_t \left(t_{11}(n_\tau w(\theta_\tau, r), \hat{r}(\theta_{\tau+1}))n_\tau w_2(\theta_\tau, r) w(\theta_\tau, r)\right)$$
$$+\sum_{s=2}^{\infty} \delta^s E_\tau \left(t_{11}(n_\tau w(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}), \hat{r}(\theta_{\tau+s}))n_\tau w_2(\theta_\tau, r) w(\theta_\tau, r) \prod_{j=1}^{s-1} z^2(\theta_{\tau+j}) \right)$$
$$+\sum_{s=2}^{\infty} \delta^s E_\tau \left(t_1(n_\tau w(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}), \hat{r}(\theta_{\tau+s}))w_2(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}) \right) = 0, \ \forall n > 0$$

Together the above conditions require:

$$\frac{\partial^2 U(n_\tau, \theta_\tau, r)}{\partial r \partial n_\tau} = t_{12}(n_\tau, r) - \frac{t_2(n_\tau, r)}{n_\tau} + \delta E_t \left(t_{11}(n_\tau w(\theta_\tau, r), \hat{r}(\theta_{\tau+1}))n_\tau w_2(\theta_\tau, r) w(\theta_\tau, r)\right)$$
$$+\sum_{s=2}^{\infty} \delta^s E_\tau \left(t_{11}(n_\tau w(\theta_\tau, r) \prod_{j=1}^{s-1} z(\theta_{\tau+j}), \hat{r}(\theta_{\tau+s}))n_\tau w_2(\theta_\tau, r) w(\theta_\tau, r) \prod_{j=1}^{s-1} z^2(\theta_{\tau+j}) \right) = 0, \ \forall n > 0$$

The condition can't hold for all $\theta \in (\underline{\theta}, \overline{\theta})$ and any $n > 0$ unless $t_{11} = 0$ because $t_{12}(n_\tau, r) - \frac{t_2(n_\tau, r)}{n_\tau}$ is independent of θ_τ, while the remaining part depends on both θ_τ and n_τ.

It follows that $t_{11}(n, r) = 0$ and the equilibrium payment schedule must be of the form $t(n, r) = \underline{t} + nt(r)$. Note, that $\underline{t}$ can't depend on r because for the FOC to hold the term $t_2(n_\tau, r)$ must be linear in n_τ. Also note, that the interval of perfect ratings in the grading is not required and a similar argument can be made for coarse ratings. Finally, if the grading is trivial and uninformative the payment schedule is just a constant, that is it is of the form $t_n(r) = \underline{t} + nt(r)$. QED.

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