Competitive Nonlinear Pricing: A Random Search Model

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Abstract

This paper proposes a new framework for competitive nonlinear pricing: I incorporate the standard monopolistic screening problem in a dynamic random search model with explicit search cost. Firms' market power derives from search frictions and horizontal differentiation. I show that under private values firms' quality provision attains the first-best level, i.e., were one to maximize buyer's utility minus cost. This confirms and generalizes findings in the celebrated paper due to Rochet and Stole (2002) to a dynamic environment and a broader class of preferences and distributions. In contrast, first-best efficiency fails to obtain when there are common values. As a novel result I report that under adverse selection quality provision is less than the first-best whereas quality provision exceeds the first-best under advantageous selection for all but the buyers of least and greatest valuation. This both confirms and disproves the intuition derived from Rothschild and Stiglitz (1976) where there are two types only: There are downward distortions, but they are largest for intermediate types, not at the bottom. Also, unlike in Rotschild-Stiglitz, profit is not constant (i.e. zero) across types. Instead, under adverse (advantageous) selection firms make less (more) profit on high valuation buyers.

Keywords: nonlinear pricing, random search, competitive equilibrium, adverse selection

1 Introduction

The monopolistic model of nonlinear pricing (pioneered by Mussa and Rosen (1978)) has been hugely successful in explaining distortions to allocative efficiency when buyers are privately informed about their valuations: in order to raise prices on high quality goods bought by high valuation buyers, the monopolist must raise prices on goods of intermediate quality to inefficiently high levels. This begets the question: *does a competitive market structure alleviate the downward distortion in quality provision induced by the seller's profit-seeking motive?*

This question comes with conceptual challenges. Standard competitive equilibrium is oftentimes non-unique and sensitive to seemingly innocuous assumptions. For instance, is it reasonable to expect that all contracts offered are equally profitable? This point has been extensively scrutinized in models where there is adverse selection (due to Wilson (1977), Rothschild and Stiglitz (1976)), i.e., those buyers most eager to buy are also the most costly to serve. As to imperfect competition, Chade and Swinkels (2021) have argued that economists "lack [...] a standard workhorse for oligopolistic screening". Any such model must necessarily involve choices: what is firms' source of market power?

This paper, building on the theory of optimal screening with privately informed agents, proposes a new framework for modelling oligopoly markets. This framework incorporates the standard nonlinear pricing problem in a dynamic random search model with explicit search cost. Firms' market power derives from search frictions and horizontal differentiation.

Formally, there is a continuum of ex-ante heterogeneous buyers indexed by their valuation type $j \in [0, 1]$, each with unit demand, and many symmetric sellers that screen buyer valuations by offering a menu or tariff over price-quality pairs (q, p). Instead of instantaneous market clearing, trading is preceded by haphazard search. Buyers randomly bump into sellers in continuous time. Upon meeting a seller, each buyer draws an idiosyncratic brand preference shock ξ and exit the search pool upon making a purchase.[1] Purchase decisions follow a simple stopping rule: buy if indirect utility (consumption utility minus prices $U(q; j) - p$) plus the brand preference shock ξ exceeds one's continuation value of search $v(j)$. Notably, my analysis allows for both private and common values. There is adverse (advantageous) selection if the seller's cost $C(q; j)$ of providing quality q is increasing (decreasing) in the buyer's valuation j. To keep the analysis tractable I assume that the economy is in the steady state; exiting buyers are exactly replaced so that the distribution over buyer valuations does not change over time.

From a mathematical point of view this dynamic model is closely related to the static model proposed by Rochet and Stole (2002). In their model there is no search, buyers instantaneously draw brand preference shocks for all sellers present in the market. Perhaps surprisingly, the firms' optimization problem are largely identical (compare ((4)) with (11)

[1] In a random search context it is known that preference shocks are vital to avoid Diamond's paradox. This paradox asserts that absent preference shocks there is a unique equilibrium in which prices attain the monopoly level.

in their paper). This owes to two facts: First, in my model firms' pricing decisions are myopic; I assume that firms lack commitment over future prices to control future demand. Secondly, explicit search cost lead to symmetric stopping rules. A mimicking argument (see lemma 2) asserts that in equilibrium the threshold for which buyers make a purchase decision is identical across types.[2] The novelty of my approach is that I do not limit attention to the private value case, but also study adverse and advantageous selection.

Preliminary results are quite striking: under private values, i.e., absent adverse or advantageous selection there always exists an equilibrium where quality provision is at the first best, i.e., the level of quality attained were one to maximize utility minus cost (see proposition 2). Prices equal cost plus a common fee. This confirms the findings by Rochet and Stole for a broader class of preferences and distributions. There is also reason to believe that this equilibrium is unique. In contrast, in the presence of common values the set of types that consume their first-best quality is negligible in every equilibrium; profits are never constant in types (see proposition 3). Under the common parametrization of utility $U(q; j) = (\alpha + j)q - \frac{q^2}{4}$ and $C(q; j) = cjq - \frac{q^2}{4}$ where $|c| < 1$ and j is uniform in $[0, 1]$ I provide a full characterization of the unique equilibrium: there is first-best quality provision under private values, i.e., $c = 0$. Under common values there is no distortion at the top and no distortion at the bottom. For all other intermediate types quality provision is less than the first-best under adverse selection, $c > 0$; quality provision exceeds the first-best under advantageous selection, $c < 0$. This both confirms and disproves the common intuition under adverse selection gained from Rothschild-Stiglitz work where there are two types: There are downward distortions, but they are largest for intermediate types, not at the bottom. Also, unlike in Rotschild-Stiglitz, profit is not constant, i.e., zero, across types. Instead, under adverse selection high valuation buyers are less profitable, under advantageous selection they are more profitable.

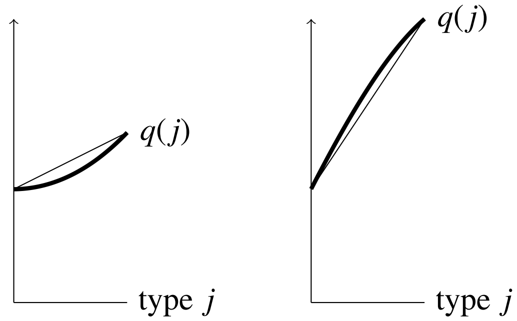

Figure 1: Adverse selection (left), advantageous selection (right) as characterized in section 4; The first-best is given by the affine functions.

[2]This does not render the random search framework superfluous. First, equivalence only arises because the economy is assumed to be stationary and there are explicit search cost, not discounting. Having proposed the framework, future work could well extend along those lines. Secondly, the equivalence affords a new interpretation of an important and well-known framework and thereby broadens its plausibility and appeal.

Related literature

There is a deep and insightful literature studying competitive nonlinear pricing. Two approaches stand out. The first approach builds on **?**-Wilson (1977) and uses arguments familiar from Bertrand competition: an equilibrium allocation must not allow for profitable one-shot deviations such as price undercutting, pooling or cross-subsidization. Later work provides game theoretic foundations under various trading rules (e.g., multiple or nonexclusive contracting Attar et al. (2014),Attar et al. (2019),Attar et al. (2021); withdrawal Netzer and Scheuer (2014)). Issues with the existence of pure strategy equilibria are well-documented; Azevedo and Gottlieb (2017) rely on a trembling hand perfection argument to bypass these.

The second approach perceives the seller's problem as an optimization problem subject to incentive constraints as in Mussa and Rosen (1978). The closest work is Rochet and Stole (2002). Another related paper is Garrett et al. (2019). They consider a sampling search model that leads to interim information asymmetries across buyers. In their model no symmetric equilibrium exists: ex-ante identical sellers offer distinct prices, some targeted at poorly informed buyers, others at competitive buyers. It is worth mentioning that their model can be recast as a ladder model with repeat purchases. Thereby it is complementary to my approach which considers one-time purchases. Finally, Chade and Swinkels (2021) study competitive nonlinear pricing with asymmetric sellers.

2 The model

Time is continuous. There is a continuum of consumer types $j \in [0, 1]$. The density of consumer type j is time-invariant and given by $\mu(j)$. Assume that $\mu(j)$ is continuously differentiable with uniformly bounded derivatives.

First-best benchmark

Buyer j derives utility $U(q; j)$ from consuming quality q. Providing quality q to buyer j costs $C(q; j)$. Assume both functions are differentiable as needed and $U(0; j) = C(0; j) = 0$. The first-best is attained for $q^{fb}(j) \equiv \arg\max_q U(q; j) - C(q; j)$. I assume that said maximum is uniquely characterized by first-order conditions:

Assumption 1. *$q \mapsto U(q; j)$ is strictly increasing and strictly concave; $q \mapsto C(q; j)$ is strictly decreasing and strictly convex.*

Further assume that $U(q; j)$ satisfies single-crossing:

Assumption 2 (Spence-Mirrlees). $\partial^2_{jq} U(q; j) \geq 0$ *for all q and j.*

Single-crossing provides an ordering on buyer types according to which higher buyer types have greater marginal utility from quality, and are thus more eager to trade. I will generally distinguish three cases:

Definition 1. *There are private values if $\partial_j C(q; j) = 0$ for all q and j, and common values otherwise. There is adverse selection if $\partial_j q C(q; j) > 0$ and advantageous selection if $\partial_j q C(q; j) < 0$ for all q and j.*

When there is advantageous selection, then $j \mapsto q^{fb}(j)$ is clearly increasing. Under adverse selection, this depends on the sign of $\partial^2_{jq}[U(q; j) - C(q; j)]$: if positive, then $j \mapsto q^{fb}(j)$ is increasing, if negative it is decreasing. In the former case refer to adverse selection as benign, in the latter case refer to adverse selection as severe.

Nonlinear pricing

Sellers compete in price-quality menus. Since the distribution of consumer types is time-invariant, so are sellers' menus. Let $(q(j), p(j))_{j\in[0,1]}$ a given seller's proposed menu, where trade $(q(j), p(j))$ is targeted at consumer type j. This provides ex-ante utility

$$w(j) \equiv U(q(j); j) - p(j)$$

to consumer type j. A menu is incentive compatible if

$$w(j) - w(i) \geq U(q(i); j) - U(q(i); i).$$

As is standard, due to single-crossing, incentive compatibility admits a characterization as a differential equation:

Lemma 1. *A menu is incentive compatible if and only if*

(i) $q(j)$ *is non-decreasing in* j*;*

(ii) *and* $w(j)$ *is absolutely continuous with derivative* $\dot{w}(j) = \partial_j U(q; j)_{|q=q(j)}$.

Note that it follows from (i) that the first-best is implementable by decentralized markets or by a centralized planner if and only if adverse selection is nowhere severe.

The seller's problem

Competition is organized through random search whereby sellers can only serve those buyers they meet.

A given buyer meets other sellers at Poisson rate λ. As λ does not depend on buyers' types, a given seller's meetings with agent types j are proportional to the mass of agents in the search pool, μ. The identity of sellers is of little importance. Sellers are myopic, either because they are infinitesimally small and cannot affect the rate at which they meet consumers, or because they lack commitment power over future prices.

Incorporating ideas from search theory (Wolinsky (1986), Anderson and Renault (1999)), consumers have idiosyncratic brand preferences, as expressed by a shock $\xi \sim H$ drawn at the time of a meeting.

Assumption 3. *The cdf* H *admits a continuous density* h *with unbounded support* $\mathbb{R}_+$ *and finite expectation.*

For further simplicity, assume that sellers are ex-ante identical so that H is identical across sellers. Finally, let $v(j)$ denote the continuation value of search of consumer type j, determined endogenously in equilibrium. Thus, a consumer j buys quality $q(j)$ at price $p(j)$ if and only if

$$w(j) + \xi \geq v(j).$$

$1 - H(v(j) - w(j))$ is the meeting-contingent probability of consumer type j making a purchasing decision.

The seller's problem is to choose an incentive compatible menu which maximizes profits:

$$\sup_{(q(j),p(j))} \int_0^1 [p(j) - C(q(j); j)]\,\mu(j)[1 - H(v(j) - w(j))]dj \quad \text{s.t. incentive compatibility.} \tag{1}$$

Here $C(q(j); j)$ the cost of providing quality $q(j)$ to consumer type j. To abbreviate notation, denote $\pi(j) = p(j) - C(q(j); j)$ the per-type profit the seller makes. Equivalently,

$$\pi(j) = U(q(j); j) - w(j) - C(q(j); j).$$

The buyer's option value of search

Consumer search until a purchase decision has been made, i.e., until they meet a seller and draw a brand preference shock which is sufficiently high. While searching, they meet a seller at Poisson rate λ. Contemporaneously they incur a flow cost s. The continuation value of search is then given by $v(j)$:

$$v(j) = \int_0^\infty [w(j) + \xi(j) - s\,t]dF(t; j), \tag{2}$$

and $F(t; j)$ is the probability that buyer j meets a seller in time interval $[0, t)$ and draws an idiosyncratic brand preference shock ξ high enough such that he buys from this seller; $\xi(j)$ is the expected draw of the idiosyncratic brand preference shock conditional on buying:

$$\xi(j) = \frac{\int_{\xi \geq v(j)-w(j)} \xi\, h(\xi)d\xi}{1 - H(v(j) - w(j))} \quad \text{and} \quad F(t; j) = 1 - \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \exp(-\lambda t)[H(v(j) - w(j))]^k.$$

Since search is costly and buyers must make a purchase decision to exit the search pool, the value of search can well be negative. To avoid such indeterminacy, I normalize $v(0) = 0$.

As it turns out, the value of search admits a surprising property in the steady state. The following show that $v(j) - w(j)$ is (i) identical across types[3] and (ii) uniquely pinned down by λ and s in any symmetric equilibrium.

[3]Compare with Atakan (2006), who finds a related property in random search models with explicit search cost in the steady state. The result does not obtain under discounting:

Remark 1. *In a stationary model with discounting rather than explicit search cost $v(j) - w(j) \leq v(i) - w(i)$ for $j > i$.*

Proof. As under explicit search cost, the stopping rule is to make a purchasing decision whenever $\xi \geq v(j) - w(j)$. Then consider

$$H(t; j) = \int_0^t e^{-\rho\tau} dF(\tau; j)$$

to be the discounted probability of making a purchase decision in time interval $[0, t)$. Denote $H(j)$ the limit as t goes to infinity. Unlike before, the limit $H(j)$ is strictly smaller than one. The value of search under discounting is given by

$$v(j) = \int_0^\infty [w(j) + \xi(j)]dH(t; j).$$

An identical mimicking argument as under explicit search cost implies that

$$(w(j) - w(i))H(j) \geq v(j) - v(i) \geq (w(j) - w(i))H(i).$$

Since $U(q; j)$ is increasing in j (this follows from single-crossing and the fact that zero-consumption utilities are zero across types), it follows that $w(j) \geq w(i)$, thus $H(j) \geq H(i)$, whenever $j > i$. Or, this means that greater types j purchase faster than types i, requiring them to have a lower threshold, i.e., $v(j) - w(j) \leq v(i) - w(i)$ as was to be shown. $\square$

Lemma 2 (mimicking argument)**.** *$v(j) - w(j) = v(i) - w(i)$ for all i, j.*

The proof is simple enough to be stated in full here.

Proof. Consider a simple mimicking argument: let buyer type j trade $(q(j), p(j))$ for all those brand preference realizations ξ for which buyer type i would buy $(q(i), p(i))$. The ensuing distribution over purchases is thus given by $F(t; i)$. Such choice, being in buyer type j's original choice set, gives a lower bound on buyer type j's value of search.

Taking differences gives the following result:

$$v(j) - v(i) \geq \int_0^\infty (w(j) - w(i)) dF(t; i)$$

for all i, j. Note that $t \mapsto F(t; i)$ integrates to one. Then reversing the roles of j and i implies that $w(j) - w(i) \geq v(j) - v(i) \geq w(j) - w(i)$, whence the result. $\square$

Lemma 3. *The threshold $v(j) - w(j)$ is uniquely defined by the ratio $\frac{s}{\lambda}$ across all equilibria.*

Proof. Step 1: Fix an arbitrary firms' menu and ensuing indirect utility $w(j)$. I show that

$$\frac{s}{\lambda} = (w(j) + \xi(j) - v(j))(1 - H(v(j) - w(j))) \tag{3}$$

so that the value of search for all buyer types is identical for any two (s_1, λ_1) and (s_2, λ_2) so that $\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2}$. Standard dynamic programming arguments give

$$v(j) = \int_0^\delta [w(j) + \xi(j) - s\tau] dF(\tau; j) + [1 - F(\delta; j)](v(j) - s\delta).$$

Dividing by δ and letting $\delta \to 0$ establishes

$$0 = (w(j) + \xi(j))\lambda(1 - H(v(j) - w(j))) - v(j)\lambda(1 - H(v(j) - w(j))) - s.$$

Then re-arranging gives the result.

Step 2: Careful inspection of (3) reveals that (i) a solution $w(j) - v(j)$ to (3) exists due to the intermediate value theorem and (ii) and is unique due to monotonicity of the right hand side in $w(j) - v(j)$ for arbitrary given $\frac{s}{\lambda}$. $\square$

Equilibrium [Sketch]

Definition of equilibrium

I have considered a model in which the presence of buyers μ is time-invariant. This means that those buyers that enter exactly replace those that exit.[4]

I will henceforth treat the distribution μ, required to be continuous and bounded in type, as an exogenous parameter. This modeling choice owes to the greater analytical tractability encountered in the steady state.

Like a Walrasian equilibrium, a symmetric stationary equilibrium specifies prices for all quantities traded. Here, this takes the form of a menu.

Definition 2 (Equilibrium)**.** *A stationary symmetric equilibrium is a menu* $(q(j), p(j))$*, solution to the sellers' problem* (1) *provided the buyers' values of search* $v(j)$ *are given by* (2).

Equilibrium existence [Sketch]

An equilibrium can be recast as a fixed point: a value of search $v(j)$ maps into a profit-maximizing menu $(q(j), p(j))$, solution to (1), from which one can compute the buyer's value of search via (2).

To establish existence of an equilibrium one must first define a one-to-one mapping between the value of search $v(j)$ and the profit-maximizing menu $(q(j), p(j))$. This relies on additional assumptions

Assumption 4. *H is log-concave.*

Assumption 5. *Under adverse selection* $\partial_j U(q; j)$ *is weakly convex; under advantageous selection* $\partial_U(q; j)$ *is wealy concave.*

Proposition 1. *The seller's problem* (1) *admits a unique solution.*

The proof of this result requires the further conjecture that any profit-maximizing quality satisfies $q(j) \leq q^{fb}(j)$ under adverse selection and $q(j) \leq q^{fb}(j)$ under advantageous selection. I show this for the parametrization of preferences and cost discussed in the introduction. The general proof of this conjecture remains to be established.

Proof. Suppose there is adverse selection. The result under advantageous selection follows from symmetric arguments. Define

$$\mathcal{L}(\pi) = \Big\{(w, q)\Big|\text{absolutely continuous, } q(0) = q^{fb}(0),\ \dot{w}(j) = \partial_j U(q(j); j) \text{ and } J((w, q), z) = \pi\Big\}.$$

[4] The hazard rate at which a given buyer type j exits the search pool is the meeting rate times the probability of drawing a sufficiently large brand preference. Whence the outflow of buyer types j is

$$\mu(j)\lambda(1 - H(v(j) - w(j))).$$

Stationarity amounts to assuming that the outflow of all buyer types j is magically replaced by new entrants.

Then for arbitrary $(w_n, q_n)_{n=1}^N : (w_n, q_n) \in \mathcal{L}(\pi)$ and arbitrary weights $(\alpha_n)_{n=1}^N : \alpha_n \geq 0, \sum_{n=1}^N \alpha_n = 1$, define q^*:

$$q^*(0) = q^{fb}(0) \quad \text{and} \quad \partial_j U(q^*(j); j) = \sum_{n=1}^N \dot{w}_n(j)\alpha_n = \sum_{n=1}^N \partial_j U(q_n(j); j)\alpha_n \leq \partial_j U(\sum_{n=1}^N q_n(j)\alpha_n; j).$$

where the latter inequality follows because $\partial_j U(q; j)$ is convex. Then $q^{fb}(j) \geq q^*(j) \geq \sum_{n=1}^N q_n(j)\alpha_n$ because $q_n(j) \leq q^{fb}(j)$ and $\partial_j U(q; j)$ is increasing in q (single-crossing). Then, due to the concavity-convexity of $U(q; j) - C(q; j)$, it follows that $U(q^*(j); j) - C(q^*(j); j) \geq U(\sum_{n=1}^N q_n(j)\alpha_n; j) - C(\sum_{n=1}^N q_n(j)\alpha_n; j)$. Then

$$J((\sum_{n=1}^N w_n\alpha_n, q^*), z) \geq J((\sum_{n=1}^N w_n\alpha_n, \sum_{n=1}^N q_n\alpha_n), z) > \min\{J((w_n, q_n), z) : n \in \{1, ..., N\}\} = \pi,$$

because J is strictly quasi-concave (to see this note that $(q(j), w(j)) \mapsto U(q(j); j) - w(j) - C(q(j); j)$ is concave and $x \mapsto H(x)$ is log-concave, so the integrand of J is the product of two log-concave functions, whence J is strictly quasi-concave). □

Further application of the Schauder (1930) fixed point theorem to establish existence requires more work and will be completed in the near future.

3 Equilibrium nonlinear pricing

I consider here the seller's problem:

$$\begin{aligned}
\text{maximize} \quad & J((w,q),z) = \int_0^1 [U(q(j); j) - w(j) - C(q(j); j)]\,\mu(j)[1 - H(v(j) - w(j))]dj \\
\text{subject to} \quad & (w,q) \text{ is absolutely continuous and } z \text{ is measurable} \\
& \dot{w}(j) = \partial_j U(q(j); j) \quad \text{and} \quad \dot{q}(j) = z(j) \quad \text{a.e.} \qquad (4) \\
& z(j) \geq 0 \quad \text{a.e.} \\
& w(0),\ w(1) \text{ and } q(0),\ q(1) \text{ are free.}
\end{aligned}$$

Observe that the optimal $j \mapsto w(j)$ must be continuous as is readily implied by incentive compatibility.

I obtain a solution drawing on the maximum principle (refer to Clarke, theorems 22.2, 22.17, 22.26 on pages 438, 454, 465, sufficiency corollary 24.2)). Choose as control the derivative of quality in type, denoted $\dot{q}(j)$. Incentive compatibility then requires that this control be non-negative. This approach has been pioneered by Guesnerie and Laffont (1984) in their celebrated work on bunching.

Assumption 6 (The classical regularity hypotheses)**.** *I require the following:*

- *$U(q;j)$ admits partial derivates $\partial_j U(q;j)$ and $\partial^2_{jq}U(q;j)$, and all three functions are continuous in $(q;j)$;*
- *$C(q;j)$ admits partial derivative $\partial_j C(q;j)$, and both are continuous in $(q;j)$;*

Theorem 1 (The maximum principle)**.** *Let the process (w,q) be a local maximizer for problem* (4)*. Then there exist absolutely continuous functions $\lambda_w, \lambda_q : [0,1] \to \mathbb{R}$ and a scalar $\lambda_0 \in \{0,1\}$ (as given in the Hamiltonian) such that*

$$
\begin{aligned}
&(\lambda_0, \lambda_w, \lambda_q) \neq 0 \quad \forall j \in [0,1] \\
&\qquad \lambda_w(0) = \lambda_w(1) = \lambda_q(0) = \lambda_q(1) = 0 \\
&\qquad \dot{\lambda}_w(j) = -\lambda_0[\mu(j)(1 - H(v(j) - w(j))) - \pi(j)\mu(j)h(v(j) - w(j))] \\
&\qquad \dot{\lambda}_q(j) = \lambda_0[\partial_q U(q(j);j) - \partial_q C(q(j);j)]\mu(j)(1 - H(v(j) - w(j))) - \lambda_w(j)\partial^2_{jq}U(q(j);j) \\
&\qquad z(j) \begin{cases} = 0 & \text{if } \lambda_q(j) > 0 \\ \in \mathbb{R}_+ & \text{if } \lambda_q(j) = 0 \\ = +\infty & \text{if } \lambda_q(j) < 0 \end{cases}
\end{aligned}
$$

for almost all $j \in [0,1]$.

Refer to the first condition as non-degeneracy, the second as transversality, the third and fourth as the adjoint equations, and the last as the maximum condition.

Proof. Refer to theorem 22.17 in Clarke. The Hamiltonian function $\mathcal{H}((w,q), z, (\lambda_w, \lambda_q), \lambda_0)$ associated to the problem (4) is defined by:

$$
-\lambda_0[U(q(j);j) - w(j) - C(q(j);j)]\mu(j)(1 - H(v(j) - w(j))) + \lambda_w(j)U_j(q(j);j) + \lambda_q(j)z(j).
$$

Adjoint equations follow from $\dot{\lambda}_w(j) = -D_w\mathcal{H}(\cdot)$ and $\dot{\lambda}_q(j) = -D_q\mathcal{H}(\cdot)$ for almost all j, and the characterization of $z(j)$ from the maximum condition. □

As is usually the case with free endpoint constraints, non-degeneracy implies that $\lambda_0 = 1$.

The following proposition provides a characterization of equilbrium pricing under private values. This characterization is identical to results due to Rochet and Stole (2002) in a static framework. (They exclusively focus on the private value case.) I view this as re-assuring, for it confirms that the random search view of competitive nonlinear pricing does not fundamentally alter the economic forces present in the much better explored static framework.

Proposition 2. *There exists an equilibrium with private values, i.e., $C(q;j) \equiv C(q)$, where*

- *each buyer type j consumes the first-best allocation $q^{fb}(j)$;*
- *per-type profits $\pi(j)$ are constant;*
- *prices equal cost plus a common fee, i.e., $p(j) = p_0 + C(q(j))$.*

Proof. To see that the necessary conditions are satisfied, set $\dot{\lambda}_w(j) = \dot{\lambda}_q(j) = 0$ for all j. The first-adjoint equation implies that $\pi(j) = \frac{1-H(v(j)-w(j))}{h(v(j)-w(j))}$ which is constant due to the mimicking argument. The initial conditions $\lambda_w(0) = \lambda_w(1)$ further imply that $\lambda_w(j) = 0$ for all $j \in [0,1]$. The second adjoint equation then establishes that $q(j) = q^{fb}(j)$. Lastly, to see that the maximum condition is satisfied it suffices to note that $q^{fb}(j)$ is non-decreasing. This follows from two observations. First, $q \mapsto S(q;j) \equiv U(q;j) - C(q;j)$ is single-peaked (due to concavity of $U(q;j)$ and convexity of $C(q;j)$). Secondly, $S(q;j)$ has increasing differences (due to $U(q;j)$ satisfying the Spence-Mirrlees condition and $C(q;j)$ satisfying private values). Therefore, if there existed $j' > j$ such that $q^{fb}(j') < q^{fb}(j)$, then

$$0 > S(q^{fb}(j);j') - S(q^{fb}(j');j') \geq S(q^{fb}(j);j) - S(q^{fb}(j');j) > 0,$$

thus posing the desired contradiction.

That this solution is the unique solution that satisfies the necessary conditions remains to be seen. □

That the optimistic lessons drawn from the private value case do not carry over to common values, a common pre-occupation in insurance economies or corporate lending for instance, is affirmed by the following proposition:

Proposition 3. *In any equilibrium with common values, i.e., $\partial_j C(q;j) \neq 0$,*

- *the set of qualities q that coincide with the first-best, i.e., $q = q(j) = q^{fb}(j)$ for some j, is negligible;*
- *per-type profits $\pi(j)$ are different from average profits for almost all j;*
- *Equilibrium quantities $q(j)$ are such that $q(0) \leq q^{fb}(0)$ and $q(1) \geq q^{fb}(1)$.*

Proof. I prove each point in turn.

First point: Suppose there exist $l < h$ so that $q(j) = q^{fb}(j)$ (and therefore $\partial_q U(q(j);j) = \partial_q C(q(j);j)$) for (almost) all $j \in (l,h)$.

For the set of qualities for which the first-best and the optimum coincide to not be negligible, $q^{fb}(j)$ must be non-constant in a subset of (l,h). If so, the maximum condition requires $\lambda_q(j) = 0$ for almost all $j \in (l,h)$. Differentiating means that $\dot{\lambda}_q(j) = 0$, and so $\lambda_w(j) = 0$ for all $j \in (l,h)$ due to the second adjoint equation. Since $\lambda_w(j)$ is constant, the first adjoint equation implies that $\pi(j) = (1 - H(v(j) - w(j)))/h(v(j) - w(j))$. Now recall that the mimicking argument (see lemma 2) establishes that $(1 - H(v(j) - w(j)))/h(v(j) - w(j))$ is constant in j, so that $\pi'(j) = 0$. Further recall that $\pi(j) = U(q(j);j) - w(j) - C(q(j);j)$. First-best efficiency then stipulates that $0 = \pi'(j) = -\partial_j C(q;j)_{|q=q^{fb}(j)}$. But $\partial_j C(q;j)_{|q=q^{fb}(j)} \neq 0$ under common values, thus posing the desired contradiction.

Second point: Suppose per-type profits $\pi(j)$ were equal to average profits $\int_0^1 \pi(i)\mu(i)di$ for all $j \in [0,1]$. Denote $\kappa(j) \equiv (1 - H(v(j) - w(j))) - \int_0^1 \pi(i)\mu(i)di h(v(j) - w(j))$. The first-adjoint equation re-writes as $\dot{\lambda}_w(j) = \mu(j)\kappa(j)$. Integrating from 0 to 1 and noting that $\lambda_w(0) = \lambda_w(1) = 0$ this implies that $\int_0^1 \mu(i)\kappa(i) = 0$. The mimicking argument establishes that $\kappa(j)$ is constant, from which it follows that $\kappa(j) = 0$ throughout. Therefore $\lambda_w(j) = 0$ for all $j \in [0, 1]$, thus implying that $\partial_q U(q(j); j) = \partial_q C(q(j); j)$ throughout. Therefore $0 = \pi'(j) = -\partial_j C(q; j)_{|q=q^{fb}(j)}$, which is impossible under common values.

Third point: I distinguish between two cases. First, observe that absent bunching at the top or bottom, quality provision at the top or bottom respectively is at the first-best level. Indeed, absent bunching in some interval $[0, a]$ at the bottom, the maximum condition implies that $\lambda_q(j) = 0$ for all $j \in [0, a]$. In particular, $\dot{\lambda}_q(0) = 0$. The transversality condition further requires that $\lambda_w(0) = 0$. Then the second adjoint equation gives the result. A symmetric reasoning applies at the top.

Now suppose there is bunching in some maximal interval $[0, a]$ at the bottom. Recall the transversality condition $\lambda_q(0) = 0$, and the maximum condition $\lambda_q(j) \geq 0$ for all $j \in [0, a]$. It follows that $\dot{\lambda}_q(0) \geq 0$. Noting that the transversality condition further requires that $\lambda_w(0) = 0$, the second adjoint equation implies that

$$0 \leq \partial_q U(q(0); 0) - \partial_q C(q(0); 0).$$

Since $q \mapsto U(q; j)$ is concave, and $q \mapsto C(q; j)$ is convex, it follows that $q(0) \leq q^{fb}(0)$. Once more a symmetric reasoning applies at the top. □

Sufficient conditions for separation

As under monopolistic price-discrimination, it is easiest to characterize nonlinear pricing in the absence of pooling. I call a separating segment an interval of types that all consume distinct qualities.

In a separating segment, the maximum principle applies: $\lambda_q(j) = 0$ throughout. Then by the fundamental theorem of calculus (noting that $\lambda_w(0) = 0$) and plugging the first adjoint equation into the second,

$$\begin{aligned}0 = &-[\partial_q U(q(j); j) - \partial_q C(q(j); j)]\mu(j)(1 - H(v(j) - w(j))) \\ &- \int_0^j [(1 - H(v(\ell) - w(\ell))) - \pi(\ell)h(v(\ell) - w(\ell))]\mu(\ell)d\ell\, \partial^2_{jq} U(q(j); j).\end{aligned}$$

Dividing by $\partial^2_{jq} U(q(j); j)$ and $h(v(j) - w(j)$, noting that $v(j) - w(j)$ is constant across types due to the mimicking argument, and differentiating with respect to j yields

$$\left\{\frac{\partial^2_{jq}(U(q(j);j)-C(q(j);j))\partial^2_{jq}U(q(j);j)-\partial_q(U(q(j);j)-C(q(j);j))\partial^2_j\partial_q U(q(j);j)}{(\partial^2_{jq}U(q(j);j))^2}\right.$$
$$-\frac{\partial_q(U(q(j);j)-C(q(j);j))\partial_j\partial^2_q U(q(j);j)\dot q(j)}{(\partial^2_{jq}U(q(j);j))^2}$$
$$\left.+\frac{\partial^2_q(U(q(j);j)-C(q(j);j))\dot q(j)\partial^2_{jq}U(q(j);j)}{(\partial^2_{jq}U(q(j);j))^2}+\frac{\partial_q(U(q(j);j)-C(q(j);j))}{\partial^2_{jq}U(q(j);j)}\frac{\dot\mu(j)}{\mu(j)}\right\}$$
$$\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))}=\pi(j)-\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))}. \tag{5}$$

If one can rule out pooling, the differential equation (5) fully characterizes the competitive $q(j)$. In fact, this is a Dirichlet problem, since the transversality conditions jointly with the maximum condition imply that $q(0)=q^{fb}(0)$ and $q(1)=q^{fb}(1)$. As to the remaining parameters, $v(j)-w(j)$ was uniquely determined for given s,λ by the HJB equation. $w(0)$ is free and must be chosen as to ensure that $\lambda_w(1)=1$.

But how to rule out pooling? It suffices that $\dot\lambda_q(j)$ is upcrossing in any interval where $q(j)$ is constant.

Remark 2. *If $\dot\lambda_q(j)$ is upcrossing, i.e., $\dot\lambda_q(j_1)\geq 0$ implies $\dot\lambda_q(j_2)>0$ for all $j_2>j_1$ in any interval (a,b) where $q(j)$ is constant, then there cannot be pooling in equilibrium; $q(j)$ is characterized by the solution to the Dirichlet problem* (5) *jointly with $q(0)=q^{fb}(0)$ and $q(1)=q^{fb}(1)$.*

Proof. If there is a maximal pooling interval (a,b), i.e., (a,b) is the greatest interval where $q(j)$ is constant throughout, then $\lambda_q(a)=\lambda_q(b)=0$: if $a=0$ or $b=1$ this follows from the transversality condition, otherwise from the maximum condition; moreover, it must be that $\lambda_q(j)>0$ for all $j\in(a,b)$ due to the maximum condition. This is impossible when $\dot\lambda_q(j)$ is upcrossing. □

Sufficient conditions under which $\dot\lambda_q(j)$ is upcrossing remain to established.

Perfect competition and the Diamond paradox

I end this section on general properties of equilibrium with a note of caution. One would ideally like to draw on the oligopolistic analysis to characterize perfect competition. What happens when the meeting rate grows large, or, equivalently, search cost fall to zero?

One may think of this limit as perfect competition because search cost determine the extent to which sellers act as a monopolist over the buyers they meet contemporaneously. Quite intuitively, a seller that charges more than what is expected in equilibrium will retain only the few buyers whose brand preference are large. But if search cost are low, only those with large brand preferences will buy in the first place, whereas all the other buyers keep searching until they, too, draw a large ξ, i.e., find a seller they particularly like. In effect, any seller that raises prices every slightly above his competitors risks selling to no one. This is formalized by the following:

Lemma 4. *As $\frac{s}{\lambda}$ converges to zero, so does* $1 - H(v(j) - w(j))$.

Proof. Set $\frac{s}{\lambda} = \frac{1}{n^2}$ and denote w_n, v_n and $\xi_n(j)$ associated equilibrium indirect utiliy, value of search and expected brand preference shock conditional on buying respectively. The preceding HJB equation (3) implies that

$$\begin{aligned}\frac{1}{n^2} &= (w_n(j) + \xi_n(j) - v_n(j))(1 - H(v_n(j) - w_n(j))) \\ &= \int_{v_n(j)-w_n(j)}^{v_n(j)-w_n(j)+\frac{1}{n}} (\xi - (v_n(j) - w_n(j)))h(\xi)d\xi + \int_{v_n(j)-w_n(j)+\frac{1}{n}}^{\bar{\xi}} (\xi - (v_n(j) - w_n(j)))h(\xi)d\xi \\ &\geq \frac{1}{n}(1 - H(v_n(j) - w_n(j) + \frac{1}{n})).\end{aligned}$$

Letting n go to infinity gives the result. □

I do not really know what happens in this limit. But I feel that I risk encountering the Diamond paradox again: the quantity $\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))}$ must not converge to zero. If it did, buyers' brand preference shocks are predictable and nothing prevents sellers from charging monopoly prices.

4 Equilibrium characterization: an example

The preceding analysis was most insightful in revealing what can not happen in equilibrium. I finally parametrize the model to be able to characterize an equilibrium under common values.

Assumption 7. $U(q; j) = (\alpha + j)q - \frac{q^2}{4}$ *and* $C(q; j) = cjq + \frac{q^2}{4}$ *where* $\alpha > 0$.

There is adverse selection if $c > 0$. Adverse selection is severe if $c \geq 1$. There is advantageous selection if $c < 0$. This parametrization is closely related to the one studied in Rochet and Stole (2002) where $c = 0$. The first-best is as follows:

$$q^{fb}(j) = \alpha + (1 - c)j \tag{6}$$

Under this assumption the separation-characterizing equation (5) simplifies to

$$(2 - c - \dot{q}(j))\frac{1 - H(v(j) - w(j))}{h(v(j) - w(j))} = \pi(j).$$

Differentiating yields (noting the mimicking argument holds in equilibrium) $-\ddot{q}(j)\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))} = \dot{\pi}(j) = \frac{d}{dj}[U(q(j); j) - w(j) - C(q(j); j)]$:

$$-\ddot{q}(j)\frac{1 - H(v(j) - w(j))}{h(v(j) - w(j))} = (q^{fb}(j) - q(j))\dot{q}(j) - c\,q(j), \tag{7}$$

where, remember, $\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))}$ is constant across types and uniquely determined by $\frac{\lambda}{s}$ (see lemmata 2,3).

Next I leverage the maximum condition to arrive at a powerful conclusion.

Remark 3. *No equilibrium involves pooling.*

Proof. Consider a maximal pooling interval $[a, b]$. Notice that

$$\dot{\lambda}_q(j) = \alpha + (2-c)j - q(a)(1 - H(v(a) - w(a))) - \int_a^j U(q(a); a) - w(a) - c\ell\, q(a) - \frac{(q(a))^2}{2} d\ell.$$

Or $\dot{\lambda}_q(j)$ is of the form $A + Bj + Cj^2$ where $C > 0$. It follows that $\dot{\lambda}_q(j)$ is upcrossing, i.e., $\dot{\lambda}_q(j_1) \geq 0$ implies $\dot{\lambda}_q(j_2) > 0$ for all $j_2 > j_1$. It follows that it can impossibly be that $\lambda_q(a) = \lambda_q(b) = 0$ and $\lambda_q(j) \geq 0$ for all $j \in (a, b)$, as is required in a maximal bunching interval due to the maximum condition. □

In conclusion, $q(j)$ is given by the solution to the differential equation (7) with the Dirichlet boundary condition $q(0) = q^{fb}(0)$ and $q(1) = q^{fb}(1)$.

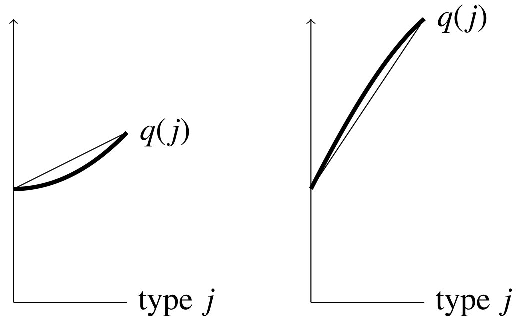

Figure 2: Adverse selection (left) where $\alpha = 1$, $c = 0.5$, $\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))} = 0.7$; advantageous selection (right) where $\alpha = 1$, $c = -0.5$, $\frac{1-H(v(j)-w(j))}{h(v(j)-w(j))} = 0.7$. The first-best is given by the affine functions.

5 Conclusion

In this work I studied competitive nonlinear pricing. The main results are summarized by figure 2: under adverse selection there is downward quality distortion; under advantageous selection there is upward quality distortion.

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