Welfare Measurements with Heterogenous Agents*

Marek Weretka† Marcin Dec‡

October 5, 2021

Abstract

The canonical infinite horizon framework with heterogeneous consumers, used in macro and financial literature, lacks a preference-based index that consistently quantifies the welfare impacts of economic policies. In particular, the classic money-metric indices, equivalent as well as compensating variations, are not additive on the set of policies, and predictions may depend on the assumed *status quo* or order in which alternatives are implemented. This paper offers a positive result. We show that, for arbitrary heterogenous von Neumann-Morgenstern preferences with a common discount factor, the equivalent (compensating) variation is nearly additive and aggregates as long as consumers are patient. As a result, the index gives consistent quantitative welfare predictions for a wide variety of short-lived policies studied in the macro and finance literature.

Key words: Money-metric welfare, additivity, aggregation

JEL classification numbers: D43, D53, G11, G12, L13

*We thank Dilip Abreu, Andrés Carvajal, Lukasz Drozd, John Geanakoplos, Krzysztof Makarski and Joanna Tyrowicz for their insightful comments. We have also benefited from feedback from the 2019 SAET conference, Ischia. The support of National Center for Science (grant UMO-2017/27/B/HS4/00868) is gratefully acknowledged.

†University of Wisconsin-Madison, Department of Economics, 1180 Observatory Drive, Madison, WI 53706, U.S.A. and Grape/Fame, Warsaw, Poland. E-mail: `weretka@wisc.edu`; web: `www.ssc.wisc.edu/ mweretka/`

‡Grape/Fame, Warsaw, Poland. E-mail: `m.dec@grape.org.pl`

1 Introduction

Imagine a policymaker designing an optimal stimulus package in response to an adverse economic shock. The available alternatives often involve several policy variants that differ in size, duration, targeted consumers, etc. How should a decision-maker quantify the welfare impact of different scenarios? In the macroeconomic literature, the canonical framework to study economic policies is a stochastic infinite-horizon model with heterogenous endowment shocks Aiyagari (1994), Heathcote et al. (2009). This framework offers innumerable important insights regarding an abstract economy in which consumers are subject to heterogeneous income shocks. Unfortunately, the usefulness of this approach for real-world policymaking is hampered by the fact that analyses are performed from the standpoint of mitochondrial Eve. Policies are evaluated in terms of period-zero expected utility of a typical consumer, whose multiple "selves" become heterogeneous only in subsequent periods after they receive idiosyncratic shocks. When decision-makers implement policies, consumers already differ in terms of wealth and preferences. As a result, at this stage, the *ex-ante* utility of the "original consumer" is irrelevant for the policy decision. A welfare index that is appropriate for evaluating policy interventions has to reflect the *current state of an economy* by taking into account all the existing consumers' heterogeneity.

Welfare economists consider normative analysis with heterogenous consumers a daunting task as predictions require interpersonal comparisons of preferences of different individuals. In this paper, we follow the money-metric tradition that quantifies social welfare as average equivalent variation (or alternatively willingness-to-accept), Hicks (1939), McKenzie (1983), McFadden (2004). In this literature, the welfare effect of a policy is given by a transfer of numeraire (money) in *status quo*, which makes a consumer indifferent to the implementation of the counterfactual policy itself. Social welfare is then an average of the effects in the population of all individuals.

One of the main challenges in applying money-metric criterion to the standard infinite horizon framework is that the index generates ambiguous predictions. Normative recommendations involving several variants of a policy may depend on the assumed choice of the *status quo* policy or order in which alternatives are implemented. Formally, the index is not additive on a set of policies, Roberts (1980), Slesnick (1991).[1] In this paper, we offer a positive result: in a small open economy with heterogeneous consumers, under fairly general conditions *money-metric welfare is nearly additive for transient policies when consumers are patient*. As a result, the index unambiguously quantifies welfare impacts of variants for poli-

[1] A welfare index is additive if for any three policies p, p', and p'', of policy p'' relative to p is equal to the sum of the effects of policy p' relative to p and policy p'' relative to p'. Money-metric index satisfies this property in the settings with quasilinear preferences, but it does not hold in the infinite horizon model with strictly concave utilities.

cies that are sufficiently short-lived. We derive a simple formula to calculate such welfare effects and test our approximation in the context of Polish economy.

We consider a canonical infinite-horizon, small, open economy with complete markets. We allow for (discounted) heterogenous von Neumann-Morgenstern preferences and Markov endowments. Policies are modeled as perturbations to stationary prices and endowments and, among others, can include technological and income shocks, sales and service taxes, lump-sum transfers, subsidies, public spending, and social safety net programs. Our theorem shows that each policy can be associated with a value of social surplus. The impact of a policy relative to any alternative is then approximated well by the difference between the corresponding surplus values. The accuracy of the approximation improves with a discount factor, and it becomes exact in the limit with a discount factor equal to one.

For the intuition behind our approximation, consider two policies that affect fundamentals in the initial period. In this period, the changes in savings induced by policies are distributed over infinitely many subsequent periods. With a discount factor close to one, finite differences in savings can have only a negligible impact on consumption in other periods. As a result, the marginal utility of money is nearly identical in the two scenarios. The infinite horizon framework is indistinguishable in terms of equivalent variation from the reduced-form, quasilinear one. In the latter model, however, equivalent variation is additive. This mechanism has been effectively used in Bewley (1976) to establish the permanent income hypothesis in the general equilibrium setting. Vives (1987) relied on a similar argument to characterize income effects in individual markets. These classic results are not sufficient to establish approximate additivity of equivalent variation due to the problem of the fallacy of composition.[2]

Our results can be utilized in policymaking as follows. A discount factor is defined for a prespecified unit of time (e.g., a year). By redefining a period as a shorter time unit (e.g., a quarter or a month), one can make the value of the observed discount factor arbitrary close to one. In this light, one can use our approximation method to make predictions with arbitrary accuracy by restricting attention to policies with sufficiently short duration. In Section 4, we apply this logic to the context of the Polish economy. We extract preference and income distributions from the available micro-econometric data. We derive normative predictions for different variants of a stimulus package and contrast them with the approximated additive values. Our simulations suggest that, for the policies that affect the economy within the first four (twelve) quarters, the approximation error is no greater than 1.5% (5%) of the total welfare effect.

As a byproduct, our paper contributes to the literature on the aggregation of money-metric welfare. One of the most celebrated results in welfare economics shows that average

[2]For the discussion, see, e.g., Mas-Colell et al. (1995), p. 89.

equivalent variation depends on the distribution of endowments only though aggregate endowment if and only if consumers' preferences are in Gorman polar form, Gorman (1953).[3,4] Extensive empirical studies strongly reject such preferences, Blundell et al. (2007), and Lewbel and Pendakur (2009). Our theorem provides an alternative justification for the aggregation of money metric-welfare. In our framework, the surplus and hence limit equivalent variation is not affected by income distribution. Thus, the aggregation of equivalent variation holds for arbitrary preferences when policies are short-term relative to the empirical discount factor.

The rest of this paper is organized as follows. Section 2 explains the main idea within a simple example. Section 3 states the approximation theorem, and Section 4 tests this approximation in the context of the Polish economy. Section 5 concludes the paper.

2 Motivating Example

We first explain the key ideas in an example of an open economy with two consumers. Each consumer $i = 1, 2$ maximizes preferences over infinite (deterministic) consumption streams, represented by utility:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{1-\theta^i} - 1}{1 - \theta^i}.$$

In each period the consumers faces budget constraints

$$\text{s.t. } c_t^i + q_{t+1} w_{t+1}^i = w_t^i + A_t^i l_t^i.$$

Prices of bonds q_{t+1} in each t are exogenously determined in international markets. A consumer inelastically supplies one unit of labor $l_t^i = 1$. A_t^i gives labor productivity, and w_t^i is wealth at the beginning of period t. The baseline economy is stationary: Labor productivity is $A_t^i = 2$ for $i = 1, 2$, the price of a one-period bond is $q_{t+1} = \beta$ for all $t \geq 0$, and aggregate output is $Y_t \equiv \sum_{i=1,2} A_t^i l_t^i = 4$. The considered economic policies perturb exogenous fundamentals in period zero. The welfare impact of policies is measured as an average equivalent variation (a money-metric welfare, where "money" is a numearaire, here given by consumption in period two)) We examine two important properties of the welfare index: additivity and normative aggregation.

[3]Preferences are in Gorman polar form if they admit indirect utility representation that gives rise to parallel Engel curves (lines), e.g., for homogenous CRRA preferences.

[4]The Gorman condition on preferences that results in aggregation of money-metric welfare is considered as one of the central results in welfare economics. For example, McFadden describes it as "the most enduring legacy of research" and compares it to Einstein's formula $E = mc^2$, see McFadden (2004).

2.1 Additivity

We first test for the additivity of equivalent variation. For this, we consider two policies. Under the factual policy p, the fundamentals are not affected by any shocks, meaning the economy is stationary. For counterfactual policy, p', period-zero productivity increases to $A_0^{i\prime} = 3$ while international price adjusts to $q_1' = 2$. For the experiment, $p \to p'$, equivalent variation is given by a transfer of consumption in period one, which makes factual policy as attractive as the counterfactual one. The index is denoted by $EV_{p,p'}$.

With the additive equivalent variation, a round trip, $p \to p' \to p$, yields zero welfare change (i.e., $EV_{p,p'} + EV_{p',p} = 0$). Motivated by this observation, we quantify non-additivity as a difference between the two values in percentage terms:

$$Ad\% \equiv \frac{|EV_{p,p'} + EV_{p',p}|}{|EV_{p,p'}|}.$$

We call this measure an additivity gap.

Note that for policy experiment $p \to p'$, an alternative index, compensating variation, can be written as $CV_{p,p'} \equiv -EV_{p',p}$. Therefore, additivity gap also measures the percentage difference between the two classic welfare indices. In particular, zero additivity gap implies equality of compensating and equivalent variation. It follows that equivalent variation is additive if and only if this property holds for compensating variation as well.

Table 1 reports the welfare effects and the additivity gap for identical homothetic preferences, $\theta^1 = \theta^2 = 2$. Note that the magnitudes of equivalent variation in the considered policy experiments is not equal to each other, $|EV_{p,p'}| \neq |EV_{p',p}|$ and the additivity gap is non-zero. Accordingly, the welfare index is not additive regardless of the value of the discount factor.

Table 1: Welfare effects for homogenous preferences

β	0.5	0.7	0.9	0.95	0.98	0.99	Limit
$EV_{p,p'}$	1.7778	1.0826	0.7663	0.7152	0.6882	0.6798	0.6715
$EV_{p',p}$	-1.0000	-0.8335	-0.7167	-0.6932	-0.6800	-0.6753	-0.6715
$Ad\%$	0.4375	0.2300	0.0648	0.0308	0.0120	0.0059	0.0000

Note: The table reports values for for risk aversion coefficients $\theta^1 = \theta^2 = 2$. The first two rows report equivalent variations for counterfactual policy p'. The third row reports the additivity gap.

We next look at the limit values of the welfare effects as the discount factor approaches one. Although utilities achieved under the considered policies become unbounded, the equivalent variations have well-defined limits, as reported in the last column of Table 1. Importantly, such limits satisfy $|\lim_{\beta\to 1} EV_{p,p'}| = |\lim_{\beta\to 1} EV_{p',p}|$, and the aggregation gap converges to zero.

This observation extends to arbitrary policies that affect fundamentals in period zero. Indeed, the limits of welfare effects can be derived using the following simple formula. For policy p, let surplus function be given by:

$$S_p = \frac{Y_0}{2q_1} + \frac{1}{2}\sum_{i=1,2} 4^{\frac{1}{\theta^i}} \frac{\theta^i}{1-\theta^i}(q_1)^{\frac{1-\theta^i}{\theta^i}} - \frac{2}{1-\theta^i} = \frac{Y_0}{2q_1} + 4(1 - \frac{1}{\sqrt{q_1}}), \tag{1}$$

where Y_0 and q_1 are aggregate income and the price of the bond that pays in period one, under assumed policy. For any policy experiment, $p \to p'$, the limit equivalent variation is given by the difference $S_{p'} - S_p$. As a result, the equivalent variation in economy with the two patient consumers is nearly additive.

2.2 Aggregation

We next examine the problem of aggregation of equivalent variation. This property holds in a framework whenever social welfare effects are measurable with respect to aggregate income; in other words, welfare is invariant to wealth redistribution. We test this hypothesis by comparing the impact of the policy from the previous section, $p \to p'$, with an alternative experiment, $p \to p''$. Under policy p'', the productivity of consumer one is $A_0^{1''} = 4$, whereas for the other consumer productivity is $A_0^{2''} = 2$. Note that the two counterfactual policies give rise to the same aggregate output: $Y_0 = 6$. The policies differ, however, in how they allocate income among consumers. Consequently, a potential welfare differential between the policy experiments indicates lack of aggregation.

Table 2 reports values for homogenous preferences $\theta^i = 2$ for $i = 1, 2$. The two scenarios generate the same normative predictions. In particular, the aggregation gap, which quantifies the departure from the aggregation benchmark,

$$Ag\% \equiv \frac{|EV_{p,p''} - EV_{p,p'}|}{|EV_{p,p'}|}$$

is zero for all $\beta < 1$.

Unfortunately, the aggregation does not hold for more general heterogenous preferences. As Table 3 reveals, for risk aversion $\theta^1 = 0.5$ and $\theta^2 = 5$, the welfare effects in the two experiments diverge, and the aggregation gap is nonzero. Indeed, it is well-known that equivalent variation aggregates if and only if preferences are in Gorman polar form. Within our parametric family of preferences Gorman condition is equivalent to identical CRRA coefficient among consumers.

Importantly, regardless of the assumed values of risk aversion, the aggregation gap vanishes as the discount factor becomes one. Indeed, the surplus formula (1) and, hence, limits

Table 2: Welfare effects for homogenous preferences

β	0.5	0.7	0.9	0.95	0.98	0.99	Limit
$EV_{p,p'}$	1.7778	1.0826	0.7663	0.7152	0.6882	0.6798	0.6715
$EV_{p,p''}$	1.7778	1.0826	0.7663	0.7152	0.6882	0.6798	0.6715
$Ag\%$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Note: The table reports values for risk aversion coefficients $\theta^1 = \theta^2 = 2$. The first two rows report equivalent variations for counterfactual policies p' and p'', respectively. The third row reports the aggregation gap.

of average equivalent variation are measurable with respect to aggregate income, suggesting approximate aggregation of equivalent variation with sufficiently high $\beta < 1$.

Table 3: Welfare effects for heterogenous preferences

β	0.5	0.7	0.9	0.95	0.98	0.99	Limit
$EV_{p,p'}$	3.4549	1.9924	1.2619	1.1384	1.0728	1.0521	1.0320
$EV_{p,p''}$	3.6635	2.0659	1.2765	1.1451	1.0752	1.0533	1.0320
$Ag\%$	0.0604	0.0369	0.0119	0.0058	0.0023	0.0011	0.0000

Note: The table reports values for risk aversion coefficients $\theta^1 = 0.5$ and $\theta^2 = 5$. The first two rows report equivalent variations for counterfactual policies p' and p'', respectively. The third row reports the aggregation gap.

To summarize, our example shows that equivalent variation is not additive, and it aggregates only in the instance of non-generic (Gorman) preferences. The model also indicates that the classic welfare index acquires these two fundamental properties, even with heterogenous risk aversion as consumers become patient. As a result, the equivalent variation can be closely approximated with surplus when the discount factor is sufficiently high. Moreover, the surplus function depends exclusively on market-level data. In the next section, we formalize this idea for arbitrary von Neumann-Morgenstern preferences and fundamental shocks. We derive the surplus formula for the limit welfare index that gives consistent predictions in complex economic settings.

3 General Small Open Economy

3.1 Framework

We now introduce the general framework of a small open economy. Consider an infinite-horizon economy with $i = 1, 2, ..., I$ consumers. Each consumer i has preferences over random, strictly positive consumption flows $c^i = \{c_t^i\}_{t=0}^{\infty}$, represented by the expected utility function

$$U^i\left(c^i\right) = E\sum_{t=0}^{\infty}\beta^t u^i(c_t^i). \tag{2}$$

Instantaneous utility function satisfies standard assumptions; the function $u^i : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly concave, and satisfies Inada conditions. Preferences can be heterogeneous, with a discount factor common for all consumers.[5]

Each consumer is endowed with one unit of time per period that can be used to produce output. Individual output in period t is given by $y_t^i = A_t^i f^i(l_t^i)$, where labor choice is $l_t^i \in [0, 1]$. Production function $f^i(\cdot)$ is non-negative and strictly increasing. The process of labor productivity is denoted by $A^i = \{A_t^i\}_t$.

Consumers hedge productivity shocks by trading assets in international markets that are dynamically complete. In the event after history $h_t = \{s_0, s_1, ..., s_t\}$, a consumer faces budget constraint:

$$c_t^i + E(q_{t+1}w_{t+1}^i|h_t) \le w_t^i + A_t^i f^i(l^i).$$

Random variable w_{t+1}^i is wealth in different states of the subsequent period, and q_{t+1} are prices of the corresponding state-contingent claims.

Consumers enter period zero with no wealth. Fundamentals $\{A_t^i\}_t$ and $\{q_t\}_t$ are measurable with respect to an underlying Markov chain $s = \{s_t\}_t$ with finite state space $\mathcal{S} = \{0, 1, 2, ..., S\}$. For simplicity we assume that the transition matrix has real eigenvalues, and that the process has a unique stationary distribution with full support. We denote the stationary distribution over state space $\mathcal{S}$ by $\bar{s}$.

For the neutral policy (in the absence of shocks), the economy is stationary. In particular, there exist functions $q : \mathcal{S} \to \mathbb{R}_{++}$ and $A^i : \mathcal{S} \to \mathbb{R}_{++}$ for each i such that, after history $h_t = \{s_0, s_1, ..., s_t\}$, the price of contingent consumption in s_{t+1} is given by $q_{s_{t+1}}|h_t = \beta q(s_{t+1})/q(s_t)$, and the realization of the productivity of the consumer is $A^i(s_t)$, where s_t is the final state of history h_t. This specification permits arbitrary correlations of consumers' productivities as well as correlations with international prices. The assumed prices, standard in the literature, ensure that optimal consumption flows are martingales.

[5] Common discount factor is a technical assumption that allows us to avoid the complications related to different speeds of convergence for various consumers.

Economic policies are broadly defined as perturbations of consumers' productivities and international prices. Formally, a policy is represented by a tuple of random processes $p = (\Delta^q, \{\Delta^{A^i}\}_i)$, adapted to the natural filtration of underlying process s. Note that admissible policy shocks are history dependent: Under policy p, after history $h_t = \{s_0, s_1, ..., s_t\}$ the price of contingent consumption for state s_{t+1} is equal to

$$q_{s_{t+1}}|h_t = \beta \frac{q(s_{t+1}) + \Delta^q_{h_{t+1}}}{q(s_t) + \Delta^q_{h_t}} > 0 \tag{3}$$

where $h_{t+1} = (h_t, s_{t+1})$. The price of consumption in period zero is $q_0 = 1/(q(s_0) + \Delta^q_{h_0})$. Similarly, productivity of consumer i in event h_t is perturbed to $A_{h_t} = A^i(s_t) + \Delta^{A^i}_{h_t} > 0$.

Formulation (3) does not impose any restrictions on considered policies; any positive measurable price process can be written as a perturbation of some stationary Markov chain. Using the example from Section 2, price processes for policy p' is generated by function $q(s_t) = 1$ and perturbations $\Delta^q_{h_0} = -0.5$ in the initial period and zero in all subsequent periods. The next assumption restricts admissible policies to those for which perturbations of fundamental vanish over time.

Assumption 1. *(Vanishing perturbations) Consider policy p. There exist constants $C > 0$ and $\Delta \in (0, 1)$ such that $|\Delta^q_{h_t}| \leq C \times (\Delta)^t$ and $|\Delta^{A^i}_{h_t}| \leq C \times (\Delta)^t$ for all period t, histories h_t, and i.*

The collection of all policies that satisfy Assumption 1 is denoted by $\mathcal{P}$.[6]

Following Hicks (1939) and the subsequent literature, we define *money-metric welfare* as follows. Fix a consumption flow x that defines a numeraire. For policy experiment $p \to p'$, an equivalent variation $EV^i_{p,p'}$ is a transfer of numeraire x, for which a consumer is indifferent between policy p and the counterfactual one, p', assuming current (i.e., p prices). The aggregate index is then calculated as the sum of individual effects for all consumers,[7]

$$EV_{p,p'} \equiv \frac{1}{I} \sum_i EV^i_{p,p'}.$$

We provide the formal definition of the welfare index in Appendix A.2.

[6] Price process consistent with Assumption 1 can be observed in a representative agent international economy with preferences represented by a function of the form (2) and the endowment following a Markov chain, potentially perturbed by shocks satisfying the assumption analogous to Assumption 1. Note that, in this micro-foundation of prices, the assumed discount factor in international markets coincides with the one from the considered small open economy.

[7] This is a slight generalization of the definition of money-metric welfare, as we allow for arbitrary numeraire x. The standard definition is obtained by choosing x, for which market value at current pricing kernel is equal to one.

The money-metric index has been extensively used in the microeconometric and industrial organization literature to measure the welfare impacts of different policies.[8] Its attractiveness stems from simplicity and interpretability in terms of consumers' behavior. In particular, welfare is not affected by monotonic transformations of utility functions; predictions do not require the comparability of consumers' utilities. In addition, welfare effects are expressed in real terms (consumption flow x); hence, their values are invariant to price normalizations. As a result, social welfare can be (potentially) inferred from consumers' behavior (or its derivative, such as prices) in markets without *ad hoc* assumptions regarding individual utilities or price levels.[9] Finally, the index is Paretian; whenever a counterfactual policy improves the welfare of all consumers, the equivalent variation is positive.

We conclude this section by showing that the welfare index is uniquely defined for an arbitrary pair of policies.

Proposition 1. *For any pair $p, p' \in \mathcal{P}$ and discount factor $\beta < 1$, equivalent variation $EV_{p,p'}$ exists and is unique.*

Proof of Proposition 1: The proof is in Appendix A.2.

Consequently, the problem in which we examine the behavior of equivalent variation is well-posed.

3.2 Approximation Theorem

We first define a surplus function consisting of two components; The first component captures welfare gains/losses resulting from trade in international markets. The second component reflects changes in consumers' nominal income. For the first component, we define $\bar{q} \equiv q(\bar{s})$ and $\bar{A}^i \equiv A^i(\bar{s})$ as stationary price and productivity, respectively. Consider equality

$$E\left[\bar{q} u^{i\prime -1}(\bar{q}\lambda^i)\right] = E\left[\bar{q}\bar{A}^i f^i(1)\right], \tag{4}$$

[8]The Hicksian notion of money-metric welfare was further developed by Samuelson (1948) and Hurwicz and Uzawa (1971). The literature that adopts the criterion to make normative predictions is too large to cite all relevant papers. For applications in the microeconometric literature see for example Deaton and Muellbauer (1980), for discrete choice applications, see Diamond et al. (1974), Small and Rosen (1981), McFadden (2004), and in the context of industrial organization problems see Berry et al. (1995), and the literature that follows.

[9]Money-metric index (or more generally welfare criteria derived from ordinal consumer preferences) has been sometimes criticized in the social choice literature using the following argument. Ordinal preferences are not sufficiently rich to contain enough information to make conclusive judgments about social welfare, as preferences ignore the intensity of consumers' overall "pleasure," Sen (1979). For the defense of the money-metric criterion, see, e.g., McKenzie et al. (1983).

where $u^{i\prime -1}$ is an inverse of the marginal utility. Equation (4) has a unique solution, $\bar{\lambda}^i$, that is strictly positive.[10]

Fix history $h_t = \{s_0, s_1, ..., s_t\}$. Let $\bar{c}^i(\Delta^q_{h_t}) = u^{i\prime -1}([q(s_t) + \Delta^q_{h_t}]\bar{\lambda}^i)$ be an optimal consumption in event h_t, assuming price perturbation $\Delta^q_{h_t}$ and marginal utility of money $\bar{\lambda}^i$. The trade component is given by:

$$s_{trade}(\Delta^q_{h_t}) \equiv \frac{1}{I}\sum_i \left(\frac{u^i(\bar{c}^i(\Delta^q_{h_t}))}{\bar{\lambda}^i} - (q(s_t) + \Delta^q_{h_t})\bar{c}^i(\Delta^q_{h_t})\right). \tag{5}$$

For an individual consumer, the surplus is geometrically represented by the area under the ($\bar{\lambda}$-normalized) marginal utility and the price of consumption. The aggregate surplus is the sum of such areas for all consumers. The second component is equal to nominal income:

$$s_{income}(\Delta^q_{h_t}, \Delta^Y_{h_t}) \equiv (q(s_t) + \Delta^q_{h_t}) \times \frac{Y_{h_t} + \Delta^Y_{h_t}}{I}. \tag{6}$$

For policy p, the approximate social surplus is given by a sum of the two components for all date-events, normalized by the corresponding value for the neutral policy:

$$S(\Delta^q, \Delta^Y) \equiv \sum_{t=0}^{\infty} E\left[s_{trade}(\Delta^q_t) - s_{trade}(0) + s_{income}(\Delta^q_t, \Delta^Y_t) - s_{income}(0,0)\right]. \tag{7}$$

Assumption 1 implies that the surplus is finite for any $p \in \mathcal{P}$. The formula is additive across consumers and histories. It is also measurable with respect to market-level data: aggregate output and prices of contingent claims.

Surplus equation (7) involves infinite sums; without appropriate normalization by neutral policy, its value would be unbounded. Note that for policies that affect fundamentals in a finite time, in periods after the effects take place, surplus values are zero and, hence, can be dropped. As a result, for finite policies, the neutral-policy normalization can be ignored as well. Indeed, in the example from Section 2, policies perturb fundamentals in period zero, and formula (1) involves period zero surplus.

We are ready to state our theorem. Consider experiment $p \to p'$, where policies $p, p' \in \mathcal{P}$. Pick welfare numeraire $x = \{x_t\}_t$ that takes zero value in all events, in which perturbations of prices for both policies are non-zero (i.e., $x_{h_t}\Delta^q_{h_t} = x_{h_t}\Delta^{q'}_{h_t} = 0$). We also require that the limit present value (as $\beta \to 1$) is positive and finite (i.e., $\bar{v}^x \equiv \sum_{t=0}^{\infty} E[q_t x_t] \in \mathbb{R}_{++}$). If the considered policies perturb prices after period zero, then the natural welfare numeraire is a unit of consumption in period zero. Next we state the main result of the paper.

[10]The left-hand side of equation (4) is a continuous function strictly decreasing in λ^i with range $\mathbb{R}_+$. On the right-hand side, the real number is strictly positive. It follows that a solution exists and is unique. Scalar $\bar{\lambda}^i$ gives the marginal utility of money—a Lagrangian multiplier in the optimization problem — for which optimal consumption satisfies budget constraint in a steady-state, period-by-period.

Theorem 1. *Aggregate equivalent variation has an additive limit, measurable with respect to aggregate income.*

$$\lim_{\beta \to 1} EV_{p,p'} = \frac{S(\Delta^{q'}, \Delta^{Y'}) - S(\Delta^{q}, \Delta^{Y})}{\bar{v}^x}.$$

Proof of Theorem 1: The proof is in the Appendix.

For all policies in $\mathcal{P}$, the limit of equivalent variation admits a surplus representation. As such, the index is additive. Moreover, its magnitude is measurable with respect to aggregate income. One can easily find the value of equivalent variations using the surplus approximation.

We conclude this section by commenting on welfare numeraire. The choice of flow x affects predictions only up to a normalization constant. The set of admissable x is restricted in two ways. First, to eliminate the differential effects of shocks on the value of the numeraire, it is zero in periods for which policies perturb prices. Furthermore, the present value of the numeraire in the limit has to be bounded; otherwise, the numeraire would be infinitely more preferred relative to the welfare impact, and welfare effects would vanish. Any numeraire that takes non-zero values only in a finite number of periods satisfies this restriction. The assumption, however, rules out stationary consumption flows.

4 Approximation in Practice

Our main result identifies a trade-off: One can achieve the desired prediction accuracy by considering policies with a sufficiently short timespan relative to empirically observed patience. In this section, we test this trade-off in the context of the stylized model of the Polish economy.

We consider the Polish economy for three reasons. First, it is large enough to be regarded as relevant. Moreover, Poland is well integrated within the EU markets, although its impact on the European markets and, therefore, prices is relatively limited. This observation motivates the small open economy assumption. Finally, we test the approximation within the context of a once-and-for-all intervention — namely a stimulus package of the EU during the coronavirus response. As the external EU transfers fully fund the policy, we can ignore the financing source, thereby simplifying the simulations.

The simple numerical framework is as follows. Consumers are heterogeneous in two dimensions: productivity and risk aversion (CRRA preferences). We borrow the distributions of the respective parameters A^i and θ^i from the available micro-econometric data for Poland.[11] Risk aversion and productivity are independently distributed. Finally, a quarterly

[11] We borrow the productivity distribution from **?**. For risk aversion, as a basis we use micro-data on Poland's risk-taking preferences from Falk et al. (2018). Note that the survey does not offer directly the

discount factor is $\beta = 0.9924$, which corresponds to the annual value of 0.97, that is typically assumed for the Polish economy.

We introduce uncertainty to our model via global and idiosyncratic shocks. The labor markets can be in either a normal state or a slowdown. An individual consumer is employed or unemployed. As a result, in each period, a consumer can be in one of four states: an employed consumer during normal time ($s = 1$), an unemployed consumer during a normal time ($s = 2$), an employed consumer during an economic downturn ($s = 3$), or an unemployed consumer during economic downturn ($s = 4$). Furthermore, the price of consumption is $q(s) = 1$ for all states, which is consistent with empirical values observed in actual markets.

We construct an empirical transition matrix for the four states from the Polish panel data on the activity in the labor market.[12] For the aggregate states, we first classify the Polish labor market in different periods as being either in a *normal state* (N) or a *recession*, (R) to approximate the empirical frequencies with which the labor market transitions between the aggregate states. We then derive the transition matrices for the employment status from the panel data by averaging individual probabilities, conditional on an aggregate state. A slowdown in labor markets (R) elevates the probability with which a consumer becomes and remains unemployed.

For the four-state Markov process, the matrix is obtained by element-wise multiplication of an augmented transition matrix for the aggregate states. The block matrix consists of the conditional probabilities of retaining and losing a job by an individual. Thus constructed matrices yield an unconditional employment rate of 0.9161, which matches the average BAEL rates in Poland.

In the experiment, we compared various recovery paths from the slowdown triggered by the COVID-19 pandemic. To this end, we assumed that the labor market was initially in a state of recession (R). Within each homogenous group of consumers, period zero's employment rate was equal to 0.94, the empirical value in January 2021 and the initial beliefs were determined accordingly.

The factual policy is an economic recovery path without any intervention. In this scenario, an employed worker's productivity during regular times is reported in the first row of Table 3. When a slowdown hits the economy, productivity is uniformly reduced by a fraction, reflecting an estimated drop in real wages during economic downturns. The unemployed worker receives a continual unemployment benefit, regardless of the aggregate state of the economy. We assumed that his productivity is then 0.15. A counterfactual stimulus package proposed by the EU aimed to preserve jobs, albeit at a lower productivity level. We incorporate such a policy in our environment in a stylized way by introducing an additional

values of θ^i, but only relative deviations of risk aversion. We center the values of risk aversion around 2 - a standard value of theta in macroeconomic studies of Polish economy with representative agent.

[12]BAEL - Population Economic Activity Research in Poland.

state for a consumer, namely *supported employed* - referring to the ones who would become unemployed in the absence of the stimulus package.

When a job protection policy is in place, consumers are either employed or move to the supported employed state. We considered a variant of the policy in which the productivity of these agents in this new state equalled 50% of their productivity in *normal* employment states. Once the policy expires, the supported employed lose their benefit and become unemployed in that period. For the counterfactual policy, we increased the benefit for unemployed agents to a new constant value of 0.25. As the stimulus was financed from the European budget, the policy's costs were ignored in the analyses. We considered the policy variants that last $T = 1, 2, 4, 8, 12$ quarters.

Table 4 reports equivalent variation for policies of different lengths. The table illustrates the trade-off we posited in the introduction. For policies with a shorter time horizon, the surplus approximation is more accurate. For policies that last less than one year, the error is no greater than 1.5 percentage points, while this error increases to 5 percentage for policies that last four years.

Table 4: Aggregated EV convergence

	number of quarters				
	1	2	4	8	12
EV	0.0060	0.0163	0.0458	0.1250	0.2162
L	0.0060	0.0164	0.0465	0.1291	0.2272
[EV/L -1] $\times$100	0.00	-0.48	-1.41	-3.17	-4.82

Note: The first two rows of the table report the actual equivalent variation and the predicted limit value. The third row gives the difference in percentage terms.

5 Discussion

This paper has demonstrated in the standard framework that average equivalent variation is well-behaved when consumers are patient. In particular, the index is approximately additive, and it aggregates. We next discuss some of the assumptions under which we established our result.

In our analysis, we considered an exchange economy with exogenous endowments. Our result straightforwardly carries over to more complex production economies, as long as the resulting endogenous income flows satisfy the assumptions regarding endowments detailed in this paper.

Another strong assumption that we made in this paper is complete financial markets. This assumption is technically very convenient; it allows us to recast recursive optimization problems of each consumer as a static choice of a lifetime consumption flow subject given a pricing kernel. Therefore we could utilize static optimization methods to characterize equilibrium outcomes. Unfortunately, this analytical tool is not feasible in more realistic, incomplete markets settings, in which optimal choices depend in a complex way on a span of the available asset structure. Nevertheless, the additivity of the limit equivalent variation relies on a simple and robust mechanism that operates under fairly general conditions. In a separate note, we use numerical methods to demonstrate additivity in the polar economy with an extreme form of market incompleteness, in which consumers trade riskless assets of different maturity in period zero. We expect our additivity results to hold for a large class of incomplete asset structures.[13] We should emphasize that, in incomplete market settings, aggregation results break down.

References

Aiyagari, S. R. (1994): "Uninsured idiosyncratic risk and aggregate saving," *The Quarterly Journal of Economics*, 109, 659–684.

Berry, S., J. Levinsohn, and A. Pakes (1995): "Automobile prices in market equilibrium," *Econometrica*, 841–890.

Bewley, T. (1976): "The permanent income hypothesis: A theoretical formulation." Tech. rep., HARVARD UNIV CAMBRIDGE MASS.

Blundell, R., X. Chen, and D. Kristensen (2007): "Semi-nonparametric IV estimation of shape-invariant Engel curves," *Econometrica*, 75, 1613–1669.

Deaton, A. and J. Muellbauer (1980): *Economics and consumer behavior*, Cambridge university press.

Diamond, P. A., D. L. McFadden, et al. (1974): "Some uses of the expenditure function in public finance," *Journal of Public Economics*, 3, 3–21.

Falk, A., A. Becker, T. Dohmen, B. Enke, D. Huffman, and U. Sunde (2018): "Global evidence on economic preferences," *The Quarterly Journal of Economics*, 133, 1645–1692.

Gorman, W. M. (1953): "Community preference fields," *Econometrica*, 63–80.

[13] We believe that our result does not go through when consumers face borrowing constraints.

Heathcote, J., K. Storesletten, and G. L. Violante (2009): "Quantitative macroeconomics with heterogeneous households," *Annu. Rev. Econ.*, 1, 319–354.

Hicks, J. R. (1939): *Value and Capital*.

Hurwicz, L. and H. Uzawa (1971): "On the integrability of demand functions, in "Preferences, Utility and Demand"(Chipman, Hurwicz, Richter and Sonnenschein, Eds.)," .

Lewbel, A. and K. Pendakur (2009): "Tricks with Hicks: The EASI demand system," *American Economic Review*, 99, 827–63.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*, New York: Oxford University Press.

McFadden, D. (2004): "Welfare Economics at the Extensive Margin," *working paper.*

McKenzie, G. W. (1983): *Measuring economic welfare: new methods*, Cambridge University Press.

McKenzie, G. W. et al. (1983): *Measuring economic welfare: new methods*, Cambridge University Press.

Roberts, K. (1980): "Price-independent welfare prescriptions," *Journal of Public Economics*, 13, 277–297.

Samuelson, P. A. (1948): "Foundations of economic analysis," *Science and Society*, 13.

Sen, A. (1979): "The welfare basis of real income comparisons: A survey," *Journal of economic Literature*, 17, 1–45.

Slesnick, D. T. (1991): "Aggregate deadweight loss and money metric social welfare," *International Economic Review*, 123–146.

Small, K. A. and H. S. Rosen (1981): "Applied welfare economics with discrete choice models," *Econometrica*, 105–130.

Vives, X. (1987): "Small Income Effects: A Marshallian Theory of Consumer Surplus and Downward Sloping Demand," *The Review of Economic Studies*, 54, 87–103.

Weretka, M. (2018): "Ordial Minimum Theorem," *working paper.*

A Appendices

Proof of Theorem 1: The proof of the theorem proceeds as follows. In Section A.1 we reformulate the consumer's problem as a static problem. We define equivalent variation and demonstrate its existence in Section A.2. In the next two sections, we restrict attention to policies that perturb fundamentals in finite time. In Section A.3, we define a reduced form of a problem and demonstrate its equivalence in terms of observables to the infinite horizon problem. In Section A.4, in the reduced form model we demonstrate convergence of equivalent variation form model to the additive limit. Section A.5 shows absolute continuity of the limit surplus function in the assumed policy horizon. Finally, Section A.5 extends the result to all policies in $\mathcal{P}$.

A.1 A static problem

In this section we recast the recursive problem of a consumer from Section 3.1 as a static choice of a consumption flow $c_i = \{c_t^i\}_t$, from the set of consumptions flows adapted with respect to a natural filtration of s, i.e.,

$$X^i \equiv \{c^i | c_t^i > 0 \text{ for all } t \text{ and } U^i(c^i) \in \mathbb{R}\},$$

given endowments and prices of state contingent claims.

First observe that with no disutility of labor and non-satiated preferences, a consumer is going to supply the maximal labor, $l_t^i = 1$. Consequently, for each policy income is given by endowment flow $e^i = \{e_t^i\}_t$, for each history given by $e_{h_t}^i \equiv (A^i(s_t) + \Delta_{h_t}^{A^i}) f^i(1) > 0$.

Consider an event followed by history $h_t = \{s_0, s_1, ..., s_t\}$. In the recursive problem, using a rollover strategy that relies on contingent claims, a consumer can transfer one unit of consumption to this event, paying in terms of consumption in s_0

$$\begin{aligned} Price(h_t) &= \pi_{s_1} q_{s_1}|_{s_0} \times \pi_{s_2} q_{s_2}|_{s_1} \times ... \times \pi_{s_t} q_{s_t}|_{s_{t-1}} \\ &= \pi_{s_1}|_{s_0} \beta \frac{q(s_1) + \Delta_{h_1}^q}{q(s_0) + \Delta_{h_0}^q} \times \pi_{s_2}|_{s_1} \beta \frac{q(s_2) + \Delta_{h_2}^q}{q(s_1) + \Delta_{h_1}^q} \times ... \times \pi_{s_t}|_{s_{t-1}} \beta \frac{q(s_t) + \Delta_{h_t}^q}{q(s_{t-1}) + \Delta_{h_{t-1}}^q} \\ &= \pi_{h_t} \beta^t \frac{q(s_t) + \Delta_{h_t}^q}{q(s_0) + \Delta_{h_0}^q}. \end{aligned}$$

where π_{h_t} is the unconditional probability of event identified by h_t. For all histories one can normalize prices $Price(h_t)$ by factor $1/(q(s_0) + \Delta_{h_0}^q)$. The recursive problem with dynamic trading strategy is equivalent to a static choice of a lifetime consumption plan in period zero. The process of state contingent prices $\zeta = \{\zeta_t\}_t$ for history h_t is given by $\zeta_{h_t} \equiv q(s_t) + \Delta_{h_t}^q > 0$. Note that for for history h_0 the corresponding price is $\zeta_{h_0} = q(s_0) + \Delta_{h_0}^q$

It follows that under policy p, and with additional transfer of α units of welfare numeraire x, consumer is effectively choosing from the budget constraint is given by

$$b_p^i(c^i, \beta) \equiv E \sum_{t=0}^{\infty} \beta^t \zeta_t (c_t^i - e_t^i - \alpha x_t) \leq 0.$$

Budget set is a collection of measurable, strictly positive processes that satisfy budget constraint, i.e., $B_p^i \equiv \{c^i | b_p^i(c^i, \beta) \leq 0\}$. The recursive consumer's problem is then equivalent to a static problem

$$\max_{c^i \in B_p^i \cap X^i} E \sum_{t=0}^{\infty} \beta^t u^i(c_t^i). \tag{8}$$

In the rest of the appendix we use the static formulation of the problem. We also adopt the following notation: for policy, p, and discount factor β, present value of flow c^i is defined as

$$PV^{\beta,p}(c^i) \equiv E \sum_{t=0}^{\infty} \beta^t \zeta_t c_t^i.$$

For the welfare numeraire and the individual endowment we uses the following compact notation: $v^x \equiv PV^{\beta,p}(x)$ and $v^{e^i} \equiv PV^{\beta,p}(e^i)$, respectively. Observe that under Assumption 1 present value of endowment is an increasing sequence of sums, bounded from above and consequently $v^{e^i} \in \mathbb{R}_{++}$. Similarly, it is straightforward to show that for any policy $p \in \mathcal{P}$ by Assumption 1 there exist scalars $0 < \underline{\zeta} < \overline{\zeta}$ and $0 < \underline{e} < \overline{e}$ such that for all periods t, all histories h_t and all consumers i, one has $\underline{\zeta} < \zeta_{h_t} < \overline{\zeta}$ and $\underline{e} < e_{h_t}^i < \overline{e}$.

A.2 Equivalent variation

We first state a definition of equivalent variation in terms of preferences. Then we reformulate the definition using a utility representation. Consider an abstract problem of a consumer with set of alternatives $X^i \subset \mathbb{R}^N$, where N can be finite of infinite. Let B_p^i be a budget set associated with factual policy p and let $\Psi_{p'}^i$ be the upper countur set of an optimal alternative that is attained under the counterfactual policy p'. Equivalent variation is a minimal transfer of welfare numeraire $x \in X^i$, shifting B_p^i that allows to attain a bundle in $\Psi_{p'}^i$. Formally, the equivalent variation is a solution to the following problem:

$$EV_{p,p'}^i \equiv \min_{z \in X^i, \alpha \in \mathbb{R}} \alpha, \tag{9}$$

subject to $z \in \Psi_{p'}^i$ and $z \in B_p^i + \alpha x$. We say that equivalent variation is attained at $\bar{z} \in X^i$ if tuple $(\bar{z}, EV_{p,p'}^i)$ is a solution to program (9). Note that equivalent variation is defined in real terms (upper countur set and budget set) and hence it is not affect by normalization of utility or prices.

The consumer preferences considered in this paper admit a strictly monotone, continuous utility representation and the budget sets are determined by linear inequality constraints. In this instance equivalent variation can be simplified as follows. Define value function

$$V^i(p,\alpha) = \max_{c^i \in X^i} E \sum_{t=0}^{\infty} \beta^t u^i(c^i_t) \text{ s.t. } E \sum_{t=0}^{\infty} \beta^t \zeta_t c^i_t \leq E \sum_{t=0}^{\infty} \beta^t \zeta_t (e^i_t + \alpha x_t) = v^{e^i} + \alpha v^x. \quad (10)$$

For policies p, p' equivalent variation is given by as a solution to the following equation

$$V^i(p, EV^i_{p,p'}) = V^i(p', 0). \quad (11)$$

Next, we prove Proposition 1 by showing that equivalent variation is well defined for any pair of policies p, p' that satisfy our assumptions.

Proof of Proposition 1:

Step 1. In this step we characterize properties of function $V^i(p,\cdot)$. Derivative $u^{i\prime} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a continuous and strictly decreasing bijection, therefore its inverse $u^{i\prime -1}$ is well-defined, is continuous and strictly decreasing. The solution to program (10), if it exists, satisfies the first order conditions in the standard Lagrangian problem. For an event identified by history h_t the first order condition with respect to consumption, $\beta^t \pi_{h_t} u^{i\prime}\left(c^i_{h_t}\right) = \pi_{h_t} \beta^t \zeta_{h_t} \lambda^i$, can be equivalently reformulated as $c^i_{h_t} = u^{i\prime -1}(\lambda^i \zeta_{h_t})$. Replacing the latter conditions in the budget constraint implicitly defines scalar λ^i,

$$E \sum_{t=0}^{\infty} \beta^t \zeta_t u^{i\prime -1}(\lambda^i \zeta_t) = v^{e^i} + \alpha v^x. \quad (12)$$

The limit sum on the left hand side is well defined for any $\lambda^i > 0$ and $\beta \in (0,1)$, since the sequence is increasing in t and it is bounded from above by $E\sum_{t=0}^{\infty} \beta^t \bar{\zeta} u^{i\prime -1}(\lambda^i \underline{\zeta}) = \frac{\bar{\zeta} u^{i\prime -1}(\lambda^i \underline{\zeta})}{1-\beta} < \infty$. Moreover the limit sum is a strictly decreasing continuous bijection in λ^i, mapping $\mathbb{R}_{++} \to \mathbb{R}_{++}$. Consequently equation (12) has a unique solution if and only if the constant on the right hand side is strictly positive, or, in terms of parameter, $\alpha > -v^{e^i}/v^x$. Given strictly convex separable preferences, solution $\lambda^i > 0$, along with stochastic consumption flow $c^i = \{c^i_t\}_{t=0}^{\infty}$ defined as $c^i_t = u^{i\prime -1}(\lambda^i \zeta_t)$ satisfy necessary and sufficient conditions for optimality. For policy p, solution is uniformly bounded from above by $\bar{u}^{i\prime -1}(\lambda^i \underline{\zeta}) < \infty$ and from below by $u^{i\prime -1}(\lambda^i \bar{\zeta}) > 0$. As a result, for any $\alpha > -v^{e^i}/v^x$ limit $V^i(p,\alpha) = E\sum_{t=0}^{\infty} \beta^t u^i(u^{i\prime -1}(\lambda^i \zeta_t))$ exists. Moreover, since $V^i(p,\cdot)$ is a sum of continuous bijections, and itself it is a continuous bijection mapping $(-v^{e^i}/v^x, \infty) \to (\inf_{c^i} u^i(c^i)/(1-\beta), \infty)$. Importantly, the target set is independent of a particular policy.

Step 2. By the previous step $V^i(p', 0) \in (\inf_{c^i} u^i(c^i)/(1-\beta), \infty)$. Since $V^i(p,\cdot)$ is a bijection, its inverse exists and is a bijection as well. It follows that equation (11) has the unique solution, given by $EV^i_{p,p'} = V^{i,-1}(p, V^i(p',0)) \in (-v^{e^i}/v^x, \infty)$. □

A.3 Temporary policies.

Fix $\tau < \infty$. In this and the next section we characterize equivalent variation for a set of temporary policies $\mathcal{P}^\tau \subset \mathcal{P}$, whose effects vanish after finite time τ, i.e., for which $\Delta^q_{h_t} = 0$ and $\Delta^{A^i}_{h_t} = 0$ for all $t > \tau$, h_t and i. We extend our characterization to all polices in $\mathcal{P}$ in Section A.5.

We first introduce a reduced from of the static problem from Section A.1. For any $w^i \in \mathbb{R}$, consider the following problem

$$v^i(w^i) \equiv \max_{\{c^i_t\}^\infty_{t=\tau+1}} E \sum_{t=\tau+1}^{\infty} \beta^t u^i(c^i_t), \text{ s.t. } E \sum_{t=\tau+1}^{\infty} \beta^t \zeta_t c^i_t \leq E \sum_{t=\tau+1}^{\infty} \beta^t \zeta_t e^i_t + w^i \tag{13}$$

Since by assumption prices and endowments after τ are the same for all considered policies, function $v^i(\cdot)$ is independent of a particular policy. The set of consumption flows that satisfy budget constraint is empty, whenever borrowing constraint fails, i.e., $w^i \leq \underline{w}^i \equiv -E \sum_{t=\tau+1}^{\infty} \beta^t \zeta_t e^i_t$. The next lemma shows the converse: the domain is non-empty, and the solution is uniquely defined whenever the borrowing constraint is satisfied.

Lemma 1. *Program (13) has a unique solution if and only if $w^i > \underline{w}^i$.*

Proof of Lemma 1:

We essentially follow the steps of the proof of Proposition 1. By Inada assumption constraints $c^i_t > 0$ are not binding for $t > \tau$ and the solution to the program is given by the first order conditions in the Lagrangian problem. For each t and history h_t optimal consumptions satisfies $c^i_{h_t} = u^{i\prime-1}(\lambda^i \zeta_{h_t})$ where shadow price λ^i can be derived from the budget constraint (multiplied by constant $1-\beta$):

$$\eta\left(\beta, \lambda^i\right) \equiv (1-\beta)\, E \sum_{t=\tau+1}^{\infty} \beta^t \zeta_t u^{i\prime-1}(\lambda^i \zeta_t) = (1-\beta)\, w^i + (1-\beta)\, E \sum_{t=\tau+1}^{\infty} \beta^t \zeta_t e^i_t. \tag{14}$$

For any fixed $\lambda^i > 0$, the left hand side is a limit of an increasing sequence bounded from above, and, hence it is well defined and finite. Function $\eta(\beta, \cdot)$, is a strictly decreasing bijection, mapping $\mathbb{R}_{++} \to \mathbb{R}_{++}$. The right-hand side of the equality gives a real number. Equation (14) has a solution if and only if the constant is strictly positive, or $w^i > \underline{w}^i$. Given strictly convex separable preferences solution $\lambda^i > 0$, along with random consumption flow $c^i = \{c^i_t\}^\infty_{t=\tau+1}$ such that $c^i_t = u^{i\prime-1}(\lambda^i \zeta_t)$ satisfy necessary and sufficient conditions of optimality. □

For the set of policies truncated to the first τ periods the reduced-form problem consists of three elements: consumption space, preferences and budget set correspondence defined as follows. Consider consumption flows $c^i = (w^i, \{c^i_t\}^\tau_{t=0})$ where $\{c^i_t\}^\tau_{t=0}$ is a stochastic process that satisfies the respective measurability conditions with respect to $\{s\}^\tau_{t=0}$. In the reduced-form model consumption space is $\tilde{X}^i \equiv \{c^i | w^i > \underline{w}^i \text{ and } c^i_t > 0 \text{ for } t = 0, 1, ..., \tau\}$. Reduced-form preferences, over consumption flows in the reduced from are represented by utility function

$$\tilde{U}^i(c^i) \equiv v^i(w^i) + E\sum_{t=0}^{\tau} \beta^t u^i(c^i_t). \tag{15}$$

Finally, for policy p and a monetary transfer αv^x, in the reduced form problem budget set $\tilde{B}^i_p$ is derived from the constraint

$$\tilde{b}^i_p\left(c^i, \beta\right) \equiv w^i + \alpha v^x + E\sum_{t=0}^{\tau} \beta^t \zeta_t (c^i_t - e^i_t) \leq 0. \tag{16}$$

By $\tilde{EV}^i_{p,p'}$ we denote an equivalent variation in the reduced-form, expressed in monetary units w^i, or $\tilde{x} = (1, 0, ..., 0)$. We next demonstrate the sufficiency of the reduced-form problem for equivalent variation in the infinite horizon model.

Lemma 2. *Fix $\beta \in (0, 1)$ and welfare numeraire x for which $v^x < \infty$. Equivalent variation in the infinite horizon problem is well defined if and only if equivalent variation is well defined in the reduced form. Moreover, the indices are related accordingly:*

$$EV^i_{p,p'} = \frac{\tilde{EV}^i_{p,p'}}{v^x}.$$

Proof of Lemma 2:

We demonstrate the lemma in three steps. We show the equivalence of the two representations of the problem in terms of budget sets (Step 1), optimal choices (Step 2), and welfare index (Step 3).

For a stochastic process $c^i = \{c^i_t\}^{\infty}_{t=0} \in X^i$ in the infinite horizon problem (henceforth referred to as IH), define a corresponding reduction $c^{i-} \equiv (w^i_{c^{i-}}, \{c^{i-}_t\}^{\tau}_{t=0}) \in \tilde{X}^i$ as follows: $w^i_{c^{i-}} \equiv E\sum^{\infty}_{t=\tau+1} \beta^t \zeta_t (c^i_t - e^i_t)$ is the value of consumption in periods after τ and $c^{i-}_t = c^i_t$ for $t = 0, 1, ..., \tau$. For a process $c^i = (w^i, \{c^i_t\}^{\tau}_{t=0}) \in \tilde{X}^i$ in the reduced form (RF) define its extension $c^{i+} \equiv \{c^{i+}_t\}^{\infty}_{t=0} \in X^i$ as $c^{i+}_t \equiv c^i_t$ for $t = 0, 1, \ldots, \tau$ while $\{c^{i+}_t\}^{\infty}_{t=\tau+1}$ is a solution to Program (13) given w^i.

Step 1. We first demonstrate the equivalence of the two representations in terms of budget sets, shifted by vector αx.

Claim 1. *Suppose consumption flow in IH satisfies $c^i \in X^i \cap (B^i_p + \alpha x)$. Reduction c^{i-} is well-defined in RF and satisfies $c^{i-} \in \tilde{X}^i \cap (\tilde{B}^i_p + \alpha v^x \tilde{x})$. Conversely, for $c^i \in \tilde{X}^i \cap (\tilde{B}^i_p + \alpha v^x \tilde{x})$ in RF, its extension, c^{i+}, is well-defined and satisfies $c^{i+} \in X^i \cap (B^i_p + \alpha x)$.*

Proof of Claim 1:

Fix $c^i \in X^i \cap (B^i_p + \alpha x)$ in IH. Since $c^i \in X^i$, for all $t = 0, \dots, \tau$ one has $c^i_t > 0$ and $w^i_{c^{i-}} \equiv E\sum_{t=\tau+1}^{\infty} \beta^t \zeta_t (c^i_t - e^i_t) > -E\sum_{t=\tau+1}^{\infty} \beta^t \zeta_t e^i_t = \underline{w}^i$. Moreover, $c^i \in B^i_p + \alpha x$ and hence $E\sum_{t=0}^{\infty} \beta^t \zeta_t (c^i_t - e^i_t - \alpha x_t) \leq 0$. This implies

$$w^i_{c^{i-}} \equiv E\sum_{t=\tau+1}^{\infty} \beta^t \zeta_t (c^i_t - e^i_t) < E\sum_{t=0}^{\tau} \beta^t \zeta_t e^i_t + \alpha v^x < \infty.$$

Consequently $\underline{w}^i < w^i_{c^{i-}} < \infty$, and reduction $c^{i-} \in \tilde{X}^i$ is well-defined. Moreover,

$$w^i_{c^{i-}} - \alpha v^x + E\sum_{t=0}^{\tau} \beta^t \zeta_t (c^{i-}_t - e^i_t) = E\sum_{t=0}^{\infty} \beta^t \zeta_t (c^i_t - \alpha x_t - e^i_t) \leq 0$$

where the last inequality holds since $c^i \in B^i_p + \alpha x$. It follows that $c^{i-} \in \tilde{B}^i_p + \alpha v^x \tilde{x}$.

Next fix $c^i \in \tilde{X}^i \cap (\tilde{B}^i_p + \alpha v^x \tilde{x})$ in RF. Since $c^i \in \tilde{X}^i$, one has $c^i_t > 0$ for $t = 0, \dots, \tau$. Moreover, $w^i > \underline{w}^i$, and by Lemma 1 solution to Program (13) exists $\{c^{i+}_t\}_{t=\tau+1}^{\infty}$ that are strictly positive. It follows that extension $c^{i+} \in X^i$ is well-defined. Moreover,

$$\begin{aligned} E\sum_{t=0}^{\infty} \beta^t \zeta_t (c^{i+}_t - e^i_t - \alpha x_t) &= E\sum_{t=\tau+1}^{\infty} \beta^t \zeta_t (c^{i+}_t - e^i_t) - \alpha E\sum_{t=0}^{\infty} \beta^t \zeta_t x_t + E\sum_{t=0}^{\tau} \beta^t \zeta_t (c^i_t - e^i_t) \\ &\leq w^i - \alpha v^x + E\sum_{t=0}^{\tau} \beta^t \zeta_t (c^i_t - e^i_t) \leq 0, \end{aligned}$$

where the last inequality holds by the fact that $c^i \in \tilde{B}^i_p + \alpha v^x \tilde{x}$. Therefore, the extension of the consumption profile satisfies $c^{i+} \in B^i_p + \alpha x$. □

Step 2. We next demonstrate the equivalence of the two formulations in terms of optimal choices.

Claim 2. *Suppose c^i is optimal in IH on set $B^i_p \cap X^i$. Then reduction c^{i-} is well-defined and optimal on $\tilde{B}^i_p \cap \tilde{X}^i$ in RF. Conversely, if in RF c^i is optimal on $\tilde{B}^i_p \cap \tilde{X}^i$ then c^{i+} is well-defined and optimal on $B^i_p \cap X^i$ in IH.*

Proof of Claim 2:

Let c^i be optimal on set $B^i_p \cap X^i$ in IH. Since $c^i \in B^i_p \cap X^i$, by Step 1 reduction is well-defined and $c^{i-} \in \tilde{B}^i_p \cap \tilde{X}^i$. Suppose c^{i-} is not optimal on this set. This implies that there exists $y^i \in \tilde{B}^i_p \cap \tilde{X}^i$ strictly preferred to c^{i-}. By Step 1 extension y^{i+} is well-defined and satisfies $y^{i+} \in B^i_p \cap X^i$. Finally,

$$\begin{aligned} U^i(y^{i+}) &= \sum_{t=0}^{\infty} \beta^t u^i(y^{i+}_t) = v^i(w^i_{y^i}) + E\sum_{t=0}^{\tau} \beta^t u^i(y^i_t) \\ &> v^i(w^i_{c^{i-}}) + E\sum_{t=0}^{\tau} \beta^t u^i(c^{i-}_t) \geq E\sum_{t=0}^{\infty} \beta^t u^i(c^i_t) = U^i(c^i), \end{aligned}$$

where $w^i_{y^i}$ and $w^i_{c^{i-}}$ are the first components of vectors y^i and c^{i-}, respectively, and the strict inequality holds by the fact that y^i is strictly preferred to c^{i-} in RF. This contradicts optimality of c^i on $B^i_p \cap X^i$.

Let c^i be optimal on set $\tilde{B}^i_p \cap \tilde{X}^i$ in RF. Since $c^i \in \tilde{B}^i_p \cap \tilde{X}^i$, by Step 1 extension is well-defined and $c^{i+} \in B^i_p \cap X^i$. Suppose c^{i+} is not optimal on this set. It follows that there exists $y^i \in B^i_p \cap X^i$ strictly preferred to c^{i+}. By Step 1 reduction y^{i-} is well-defined and satisfies $y^{i-} \in \tilde{B}^i_p \cap \tilde{X}^i$. Finally,

$$\begin{aligned}\tilde{U}^i(y^{i-}) &= v^i(w^i_{y^{i-}}) + E\sum_{t=0}^{\tau}\beta^t u^i(y^i_t) \geq \sum_{t=0}^{\infty}\beta^t u^i(y^i_t) \\ &> E\sum_{t=0}^{\infty}\beta^t u^i(c^{i+}_t) = v^i(w^i_{c^i}) + E\sum_{t=0}^{\tau}\beta^t u^i(c^i_{\tau+1}c^i_t) = \tilde{U}^i\left(c^i\right)\end{aligned}$$

where the strict inequality holds by the fact that y^i is strictly preferred to c^{i+} in IH. This contradicts the optimality of c^i on $\tilde{B}^i_p \cap \tilde{X}^i$ in RF. □

Step 3. Finally, we demonstrate the equivalence of the frameworks in terms of equivalent variation.

Claim 3. *Suppose equivalent variation $EV^i_{p,p'}$ in IH is attained on z^i. Then in RF equivalent variation is attained on z^{i-} and satisfies $\tilde{EV}^i_{p,p'} = v^x \times EV^i_{p,p'}$. Conversely, if in RF equivalent variation $\tilde{EV}^i_{p,p'}$ is attained on z^i, then in IH equivalent variation is attained on z^{i+} and satisfies $EV^i_{p,p'}(\beta) = \tilde{EV}^i_{p,p'}/v^x$.*

Proof of Claim 3:

Suppose in IH equivalent variation $\alpha \equiv EV^i_{p,p'}$ is attained on z^i. By definition of equivalent variation $z^i \in B^i_p + \alpha x$ and $z^i \in \Psi^i_{p'} \subset X^i$. By Step 1 reduction $z^{i-} \in \tilde{X}^i$ is well-defined and it satisfies $z^{i-} \in \tilde{B}^i_p + \alpha v^x \tilde{x}$. Let $o^i \in \Psi_{p'}$ be an optimal choice in IH under policy p'. By definition of upper countur set, $U^i(z^i) \geq U^i(o^i)$. By Step 2 reduction o^{i-} is well-defined and optimal under p' in RF. Then

$$\begin{aligned}\tilde{U}^i(z^{i-}) &= v^i(w^i_{z^{i-}}) + E\sum_{t=0}^{\tau}\beta^t u^i(z^{i-}_t) \geq E\sum_{t=0}^{\infty}\beta^t u^i(z^i_t) \\ &\geq E\sum_{t=0}^{\infty}\beta^t u^i(o^i_t) = v^i(w^i_{o^{i-}}) + E\sum_{t=0}^{\tau}\beta^t u^i(o^{i-}_t) = \tilde{U}^i(o^{i-}).\end{aligned}$$

Consequently $z^{i-} \in \tilde{\Psi}^i_{p'}$ in RF. It follows that $(z^{i-}, \alpha v^x)$ satisfy constraints of Program (9) within RF. Suppose that $(z^{i-}, \alpha v^x)$ does not solve this program. It follows that there exists $z^{i\prime} \in \tilde{\Psi}^i_{p'} \subset \tilde{X}^i$ in RF satisfying $z^{i\prime} \in \tilde{B}^i_p + \alpha' v^x \tilde{x}$ for some $\alpha' < \alpha$. By Step 1, extension to

IH of $z^{i\prime+} \in X^i$ is well-defined and satisfies $z^{i\prime+} \in B_p^i + \alpha' x$. Next, let $o^i \in \tilde{\Psi}_{p'}$ be an optimal choice in RF under policy p'. Then

$$\begin{aligned} U^i(z^{i\prime+}) &= \sum_{t=0}^{\infty} \beta^t u^i(z_t^{i\prime+}) = v^i(w^i_{z^{i\prime}}) + E\sum_{t=0}^{\tau} \beta^t u^i(z_t^{i\prime}) \\ &\geq v^i(w^i_{o^i}) + E\sum_{t=0}^{\tau} \beta^t u^i(o_t^i) = \sum_{t=0}^{\infty} \beta^t u^i(o_t^{i+}) = U^i(o^{i+}), \end{aligned}$$

and hence $z^{i\prime+} \in \Psi^i_{p'}$ in the IH problem. Thus $(z^{i\prime+}, \alpha')$ satisfies constraints of Program (9) in IH and gives a smaller value, contradicting that (z^i, α) is a solution to a minimization problem. It follows that $\tilde{EV}^i_{p,p'} = v^x \times EV^i_{p,p'}$.

Suppose equivalent variation $\tilde{EV}^i_{p,p'}$ is attained on z^i in RF and let $\alpha \equiv \tilde{EV}^i_{p,p'}/v^x$. By the definition of equivalent variation, $z^i \in \tilde{B}^i_p + \alpha v^x \tilde{x}$ and $z^i \in \tilde{\Psi}^i_{p'} \subset \tilde{X}^i$. By Lemma 1 extension $z^{i+} \in X^i$ is well-defined and satisfies $z^{i+} \in B^i_{p'} + \alpha x$. Let o^i be an optimal choice in RF under policy p'. By Lemma 2, extension o^{i+} is well-defined and optimal in IH under p' as well. Then

$$\begin{aligned} U^i(z^{i+}) &= E\sum_{t=0}^{\infty} \beta^t u^i(z_t^{i+}) = v^i(w^i_{z^i}) + E\sum_{t=0}^{\tau} \beta^t u^i(z_t^i) \\ &\geq v^i(w^i_{o^i}) + E\sum_{t=0}^{\tau} \beta^t u^i(o_t^i) \geq E\sum_{t=0}^{\infty} \beta^t u^i(o_t^{i+}) = U^i(o^{i+}) \end{aligned}$$

which implies that $z^{i+} \in \Psi^i_{p'}$ in IH. It follows that (z^{i+}, α) satisfy constraints of Program (9) in IH. Suppose that (z^{i+}, α) is not a solution to the problem in IH. It follows that there exists $z^{i\prime} \in \Psi^i_{p'} \subset X^i$ satisfying $z^{i\prime} \in B^i_{p'} + \alpha' x$ for some $\alpha' < \alpha$. By Lemma 1, reduction of $z^{i\prime}$ to RF, $z^{i\prime-} \in \tilde{X}^i$ is well-defined and satisfies $z^{i\prime-} \in \tilde{B}^i_{p'} + \alpha' v^x \tilde{x}$. Let o^i be an optimal choice in IH under policy p'. By Lemma 2, reduction o^{i-} is well-defined and optimal in RF under p' as well.

Moreover,

$$\begin{aligned} \tilde{U}^i\left(z^{i\prime-}\right) &= v^i(w^i_{z^{i\prime-}}) + E\sum_{t=0}^{\tau} \beta^t u^i(z_t^{i\prime}) = E\sum_{t=0}^{\infty} \beta^t u^i(z_t^{i\prime}) \\ &\geq E\sum_{t=0}^{\infty} \beta^t u^i(o_t^{i+}) = v^i(w^i_{o^i}) + E\sum_{t=0}^{\tau} \beta^t u^i(o_t^i) = \tilde{U}^i(o^{i-}), \end{aligned}$$

hence $z^{i\prime-} \in \tilde{\Psi}^i_{p'}$ in RF. Thus $(z^{i\prime-}, \alpha' v^x)$ satisfied constraints of Program (9) in RF and attains a smaller value than $\tilde{EV}^i_{p,p'} = \alpha v^x$, contradicting that $z^i, \tilde{EV}^i_{p,p'}$ is a solution in the reduced form problem. It follows that $EV^i_{p,p'} = \tilde{EV}^i_{p,p'}/v^x$. □

The three claims jointly imply the result in Lemma 2. □

Corollary 1. *Fix $\beta \in (0,1)$. For any policies $p, p' \in \mathcal{P}^\tau$ equivalent variation in the reduced form, $\tilde{EV}^i_{p,p'}$ problem is well-defined.*

Proof of Corollary 1: In the infinite horizon problem pick x that pays one unit in $\tau+1$ and zero otherwise. Note that $v^x = E\beta^{\tau+1}\zeta_{\tau+1} \in \mathbb{R}_{++}$. By Proposition 1 equivalent variation is well defined in IH. Then by Lemma 2 equivalent variation in the reduced form exists and is equal to $\tilde{EV}^i_{p,p'} = EV^i_{p,p'} \times v^x$.

A.4 Ordinal convergence

In this section we argue that the reduced form preferences (locally, on a compact box) continuously transform into the quasilinear limits as consumers become patient. As a result the indifference curves become aligned with the quasilinear ones. In the reduced form model consider arbitrary compact box that gives a collection of measurable flows $c^i = (w^i, \{c^i_t\}_{t=0}^{\tau})$ defied as

$$\tilde{X}^{i,b} = \{c^i_t | \underline{w}_b \leq w^i < \overline{w}_b \text{ and } \underset{\sim}{c}_b \leq c^i_t \leq \underset{\sim}{c}_b \text{ for all } t = 0, ..., \tau\},$$

where finite bounds satisfy $\underline{w}_b < \overline{w}_b$ and $0 < \underset{\sim}{c}_b < \underset{\sim}{c}_b$.

For $\underline{\beta} \in (0,1)$ define weakly-better-than-c^i correspondence, mapping $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \rightrightarrows X^{i,b}$ as follows:

$$\Psi^i(c^i, \beta) \equiv \{c^{i\prime} \in X^{i,b} | c^{i\prime} \succsim^i_\beta c^i\},$$

where preferences $\succsim^i_\beta$, for all $\beta < 1$ are represented by utility function (15). For $\beta = 1$ preferences $\succsim^i_1$ are given by the quasilinear utility

$$\tilde{U}^{i,Q}(c^i) = \bar{\lambda}^i w^i + E\sum_{t=0}^{\tau} u^i(c^i_t), \tag{17}$$

where $\bar{\lambda}^i > 0$ solves equality (4).

Observe that for some values of a discount factors, one might have, $\underline{w}_b < \underline{w}^i$ and the reduced-form preferences, and hence, the weakly-better-than-c^i correspondence might not be well-defined on the entire domain. The next lemma shows that for sufficiently patient consumers, the correspondence is well-defined. Moreover, the result demonstrates that the preferences continuously transforms into the quasilinear ones.

Lemma 3. *There exists threshold $\underline{\beta} \in (0,1)$, such that weakly-better-than-c^i correspondence $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \to X^{i,b}$ is well-defined and continuous on its domain (including at $\beta = 1$).*

Proof of Lemma 3:

For considered policies fundamentals are not altered after period τ and so prices and endowments follow the underlying Markov process. In particular, for any $h_t = \{s_0, s_1, ..., s_t\}$ contingent prices and endowments are determined by a realization of a state in period t, i.e.,

$\zeta_{h_t} \equiv q(s_t)$ and $e^i_{h_t} = A^i(s_t)f^i(1)$. As a result, for all histories with the same last state are equivalent in terms of contingent prices and endowments. Let $\pi(t,s)$ be the unconditional probability of all histories h_t for which the realization of the Markov process in period t is $s_t = s$ (alternatively unconditional probability of state $s_t = s$ in period t). For the considered Markov chain, a transition matrix is diagonalizable with S independent eigenvectors and real eigenvalues. The largest eigenvalue is equal to one, while other, possibly repeated, eigenvalues $m = 2, 3, ..., S$ satisfy $|r_m| < 1$. It follows that the unconditional probability of s at t can be written as $\pi(t,s) = \bar{\pi}_s + \sum_{m=2}^{S} \gamma_m (r_m)^t v_{m,s}$ where $\bar{\pi}_s$ denotes the stationary probability of state s, derived from the eigenvector with the largest eigenvalue, $v_{m,s}$ is the s element of an eigenvector corresponding to r_m and γ_m is a constant that expresses the initial distribution in terms of eigenvector basis.

Step 1. We first show that the borrowing constrain is not binding on the box, for sufficiently high discount factor. Consider the bound in the the borrowing constraint

$$\begin{aligned}
\underline{w}^i &\equiv -E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t e^i_t = -\sum_{t\geq\tau+1}\beta^t\sum_{s=1}^{S}\pi(t,s)q(s)A^i(s)f^i(1) \\
&= = -\sum_{t\geq\tau+1}\beta^t\sum_{s=1}^{S}[\bar{\pi}_s + \sum_{m=2}^{S}\gamma_m (r_m)^t v_{m,s}]q(s)A^i(s)f^i(1) \\
&= -\sum_{t\geq\tau+1}\beta^t E(\bar{q}\bar{A}^i f^i(1)) - \sum_{m=2}^{S}\sum_{s=1}^{S}\gamma_m v_{m,s}q(s)A^i(s)f^i(1)\sum_{t\geq\tau+1}(r_m\beta)^t \\
&= -\frac{\beta^{\tau+1}}{(1-\beta)}E(\bar{q}\bar{A}^i f^i(1)) - \sum_{m=2}^{S} r_m^{\tau+1}\gamma_m\beta^{\tau+1}\frac{1}{1-r_m\beta}\sum_{s=1}^{S}v_{m,s}q(s)A^i(s)f^i(1).
\end{aligned}$$

By assumption $A^i(s)f^i(1) > 0$ and $q(s) > 0$, for any s therefore $E(\bar{q}\bar{A}^i f^i(1)) > 0$ and the first term in the equation converges to $-\infty$ as $\beta \to 1$. Since other eigenvalues are strictly smaller than one, one has $1/(1-r_m\beta) \to 1/(1-r_m) < \infty$ and therefore the second term converges to a finite limit. It follows that $\lim_{\beta\to 1}\underline{w}^i = -\infty$ and there exists $\beta^w < 1$ such that for all $\beta \in [\beta^w, 1)$ the borrowing constraint is satisfied for all $c^i \in X^{i,b}$ and correspondence $\Psi^i : X^{i,b} \times [\beta^w, 1] \rightrightarrows X^{i,b}$ is well defined. In the next two steps we demonstrate that the correspondence is continuous.

Step 2. In this step we give an auxiliary result in which we characterize the slope of the value function $v^i(\cdot)$. In terms of eigenvalues of the transition matrix, function $\eta(\beta, \lambda^i)$ from

(14) can be written as

$$\begin{aligned}\eta\left(\beta,\lambda^{i}\right) &\equiv (1-\beta)\,E\sum_{t=\tau+1}^{\infty}\beta^{t}\zeta_{t}u^{i\prime-1}(\lambda^{i}\zeta_{t}) = (1-\beta)\sum_{t=\tau+1}^{\infty}\beta^{t}\sum_{s=1}^{S}\pi(t,s)q(s)u^{i\prime-1}(\lambda^{i}q(s)) \\ &= (1-\beta)\sum_{t=\tau+1}^{\infty}\beta^{t}\sum_{s=1}^{S}[\bar{\pi}_{s}+\sum_{m=2}^{S}\gamma_{m}\left(r_{m}\right)^{t}v_{m,s}]q(s)u^{i\prime-1}(\lambda^{i}q(s)) \\ &= (1-\beta)\sum_{t=\tau+1}^{\infty}\beta^{t}E[\bar{q}u^{i\prime-1}(\bar{q}\lambda^{i})] + (1-\beta)\sum_{m=2}^{S}\sum_{s=1}^{S}\gamma_{m}v_{m,s}q(s)u^{i\prime-1}(\lambda^{i}q(s))\sum_{t=\tau+1}\left(r_{m}\beta\right)^{t} \\ &= \beta^{\tau+1}E[\bar{q}u^{i\prime-1}(\bar{q}\lambda^{i})] + \sum_{m=2}^{S}\omega_{m}\sum_{s=1}^{S}v_{m,s}q(s)u^{i\prime-1}(\lambda^{i}q(s)),\end{aligned}$$

where corresponding weights ω_m are given by

$$\omega_{m}\equiv\gamma_{m}(r_{m}\beta)^{\tau+1}\frac{1-\beta}{1-r_{m}\beta}.$$

Since $|r_m|<1$, for $m=2,...,S$ the weights are finite in a neighborhood of $\beta=1$. Therefore, the weights, as well as function $\eta(\beta,\lambda)$ itself, are well-defined and differentiable with respect to β in the neighborhood of $\beta=1$. Similarly, for arbitrary value $w^i\in\mathbb{R}$ the constant on the right hand side of equality (14) can be written as

$$(1-\beta)\,w^{i}+\beta^{\tau+1}E(\bar{q}\bar{A}^{i}f^{i}(1))+\sum_{m=2}^{S}\omega_{m}\sum_{s=1}^{S}v_{m,s}q(s)A^{i}(s)f^{i}(1).$$

For $\beta=1$ condition (14) reduces to $E(\bar{q}u^{i\prime-1}(\bar{q}\lambda^{i}))=E(\bar{q}\bar{A}^{i}f^{i}(1))$ and is independent of initial wealth. By the arguments analogous to the ones in Lemma 1, this equation has unique solution denoted by $\bar{\lambda}^i$. Moreover, since $u^{i\prime-1}(\cdot)$ is strictly decreasing, the derivative $\partial\eta(1,\bar{\lambda}^{i})/\partial\lambda^{i}=E(\bar{q}^{2}u^{i\prime-1}(\bar{q}\bar{\lambda}^{i}))<0$ is non-zero. By the implicit function theorem there exists threshold $\beta^{w^i}<1$, a neighborhood of $\bar{\lambda}^i$, denoted by V and a continuous bijection $\lambda^{w^i}:[\beta^{w^i},1]\to V$ such that $\lambda^{w^i}(\beta)$ is a unique solution to equation (14) for each $\beta\in[\beta^{w^i},1]$. Note that by continuity of this function $\lim_{\beta\to1}\lambda^{w^i}(\beta)=\lambda^{w^i}(1)=\bar{\lambda}^i$ for arbitrary value w^i, i.e., the family of bijections $\lambda^{w^i}(\cdot)$ for various w^i has the same limit.

Step 3. Let $\beta^0,\beta^{\underline{w}_b},\beta^{\overline{w}_b}$ be the thresholds from Step 2, derived for particular values of wealth equal to $0,\underline{w}_b$ and $\overline{w}_b$ respectively (recall that $\underline{w}_b$ and $\overline{w}_b$ are the bounds on wealth that define box $X^{i,b}$). Let functions $\lambda^0(\cdot)$, $\lambda^{\underline{w}_b}(\cdot)$ and $\lambda^{\overline{w}_b}(\cdot)$ be the corresponding bijections. Finally define $\underline{\beta}\equiv\max\{\beta^0,\beta^{\underline{w}_b},\beta^{\overline{w}_b},\beta^{w}\}\in(0,1)$ where the last element is defined in Step 1.

We next define a monotonic transformation of the reduced-form utility function, that maps $\tilde{\tilde{U}}^i:X^{i,b}\times[\underline{\beta},1]\to\mathbb{R}$ as follows. For $\beta\in[\underline{\beta},1)$ let function $\tilde{\tilde{U}}^i(c^i,\beta)\equiv\tilde{U}^i\left(c^i\right)-v^i(0)$ where the latter utility function is defined in (15). For $\beta=1$ function is given by $\tilde{\tilde{U}}^i\left(c^i,1\right)\equiv$

$\tilde{U}^{i,Q}$. Note that preferences represented by function $\tilde{\tilde{U}}^i$ coincide with the ones represented by $\tilde{U}^i$ and hence the function defines correspondence Ψ^i.

We now show that the representation $\tilde{\tilde{U}}^i(\cdot,\cdot)$ is jointly continuous. Clearly, $\tilde{\tilde{U}}^i(c^i,\beta)$ is jointly continuous for all $c^i, \beta \in X^{i,b} \times [\underline{\beta}, 1)$ by the standard maximum theorem. Therefore it suffices to verify joint continuity for the elements in the box for which $\beta = 1$. Consider an arbitrary sequence $\{c^{i,h}, \beta^h\}_{h=1}^{\infty} \subset X^{i,b} \times [\underline{\beta}, 1]$ such that $c^{i,h}, \beta^h \to \bar{c}, 1 \in X^{i,b} \times [\underline{\beta}, 1]$. By the envelope theorem, the derivative of the value function is given by the Lagrangian multiplier $\partial v^i(0)/\partial w^i = \lambda^0(\beta)$. Difference $v^i(w^i) - v^i(0)$ is strictly concave and it attains zero at $w^i = 0$. Hence, for any element of the sequence $h = 1, 2, ...$ utility function is bounded from above by

$$\tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \leq \max[\lambda^0(\beta^h) w^{i,h}; \bar{\lambda}^i w^{i,h}] + E \sum_{t=0}^{\tau} \left(\beta^h\right)^t u^i(c_t^{i,h}). \tag{18}$$

For all $\beta \in [\underline{\beta}, 1]$ function $\lambda^0(\beta)$, is well-defined and continuous and hence $\lim_{h\to\infty} \lambda^0(\beta^h) = \lambda^0(\lim_{h\to\infty} \beta^h) = \lambda^0(1) = \bar{\lambda}^i$. It follows that both elements of the max function have the same limit and $\lim_{h\to\infty} \tilde{\tilde{U}}^i \left(c^{i,h}, \beta^h\right) \leq \bar{\lambda}^i \bar{w} + E \sum_{t=0}^{\tau} u^i(\bar{c}_t^i) = \tilde{\tilde{U}}^i(\bar{c}^i, 1)$.

By strict concavity of $v^i(\cdot)$ for all values of wealth $w^i \in [\underline{w}_b, 0]$ value function satisfies $v^i(w^i) - v^i(0) \geq \lambda^{\underline{w}_b}(\beta^h) w^i$, while for all $w^i \in [0, w_b]$ one has $v^i(w^i) - v^i(0) \geq \lambda^{\overline{w}_b}(\beta^h) w^i$ and hence

$$\tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \geq \min[\lambda^{\underline{w}_b}(\beta^h) w^{i,h}; \lambda^{\overline{w}_b}(\beta^h) w^{i,h}; \bar{\lambda}^i w^{i,h}] + E \sum_{t=0}^{\tau} \left(\beta^h\right)^t u^i(c_t^{i,h}) \tag{19}$$

Each of the three elements of the min function has the same limit. Taking the limit gives $\lim_{h\to\infty} \tilde{\tilde{U}}^i(c^{i,h}, \beta^h) \geq \bar{\lambda}^i \bar{w}^i + E \sum_{t=0}^{\tau} u^i(\bar{c}_t^i) = \tilde{\tilde{U}}^i(\bar{c}^i, 1)$. Limits of inequalities (18) and (19) imply that $\lim_{h\to\infty} \tilde{\tilde{U}}^i \left(c^{i,h}, \beta^h\right) = \tilde{\tilde{U}}^i(\bar{c}^i, 1)$ and utility representation $\tilde{\tilde{U}}^i$ is jointly continuous on $X^{i,b} \times [\underline{\beta}, 1]$. Since for all $\beta \in [\underline{\beta}, 1]$ preferences $\succsim_\beta^i$ are strictly monotone and they admit jointly continuous utility representation, by Lemma 1 in Weretka (2018) weakly-better-than-c^i correspondence $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \to X^{i,b}$ is continuous. □

Define surplus function for consumer i as:

$$S^{i,\tau}(p) \equiv \sum_{t=0}^{\tau} E\left[u^i(u^{i\prime -1}(\zeta_t \bar{\lambda}^i))/\bar{\lambda}^i - \zeta_t u^{i\prime -1}(\zeta_t \bar{\lambda}^i) + \zeta_t e_t^i)\right] \tag{20}$$

Lemma 4. *Fix $\tau < \infty$. Consider arbitrary $p, p' \in \mathcal{P}^\tau$. In the reduced problem with quasi-linear preferences (17) equivalent variation is well-defined and given by*

$$\tilde{EV}_{p,p'}^{i,Q} = S^{i,\tau}(p') - S^{i,\tau}(p) \in \mathbb{R}.$$

Proof of Lemma 4:

In Step 1 we show that for the quasilinear preferences represented by $\tilde{U}^{i,Q}(c^i)$ optimal choice and equivalent variation on the unrestricted domain $\tilde{X}^{i,Q} \equiv \{c^i | w^i \in \mathbb{R} \text{ and } c^i_t > 0 \text{ for all } t \leq \tau\}$ are well-defined. For policy p', optimal choice $c^{i\prime}$ is uniquely defined by the necessary and sufficient conditions: consumption in the event after history h_t is given by $c^{i\prime}_{h_t} = u^{i\prime-1}(\bar{\lambda}^i \zeta'_{h_t})$ and consumption of wealth is determined from budget constraint $w^i_{c^{i\prime}} = -E\sum_{t=0}^{\tau} \zeta'_t(c^{i\prime}_t - e^{i\prime}_t)$.

Next consider policies p and p'. Program (9) specializes to $\min_{z^i,\alpha} \alpha$ subject to two constraints $\tilde{U}^{i,Q}(z^i) \geq \tilde{U}^{i,Q}(c^{i\prime})$ and $w^i_{z^i} - \alpha + E\sum_{t=0}^{\tau} \zeta_t(z^i_t - e^i_t) \leq 0$. With strictly monotone preferences, both constraints must hold with equality. Solving the second equation for α and plugging it into the objective function reduces the problem to

$$\tilde{EV}^{i,Q}_{p,p'} = \min_{z^i} w^i_{z^i} + \sum_{t=0}^{\tau} E\zeta_t \left(z^i_t - e^i_t\right) \text{ s.t. } \tilde{U}^{i,Q}(z^i) = \tilde{U}^{i,Q}(c^{i\prime}). \tag{21}$$

Solution to program (21), denoted by z^{i*} is given by first order conditions: $z^{i*}_{h_t} = u^{i\prime-1}(\bar{\lambda}^i \zeta_{h_t})$ and $w^i_{z^{i*}} = w^i_{c^{i\prime}} + \frac{1}{\bar{\lambda}^i} E\sum_{t=0}^{\tau}(u^i(c^{i\prime}_t) - u^i(z^{i*}_t))$. Under Inada assumptions these conditions define unique $z^{i*} \in X^{i,Q}$. Plugging $z^{i*}, c^{i\prime}$ in objective function (21) gives

$$\begin{aligned}
\tilde{EV}^{i,Q}_{p,p'} &= w^i_{c^{i\prime}} + E\sum_{t=0}^{\tau} \frac{u^i(c^{i\prime}_t) - u^i(z^{i*}_t))}{\bar{\lambda}^i} + \sum_{t=0}^{\tau} E\zeta_t \left(z^{i*}_t - e^i_t\right) \\
&= E\sum_{t=0}^{\tau} \frac{u^i(c^{i\prime}_t) - u^i(z^{i*}_t))}{\bar{\lambda}^i} - E\sum_{t=0}^{\tau}[\zeta_t c^{i\prime}_t - \zeta_t z^{i*}_t] + E\sum_{t=0}^{\tau}[\zeta'_t e^{i\prime}_t - \zeta_t e^i_t], \\
&= S^{i,\tau}(p') - S^{i,\tau}(p).
\end{aligned}$$

□

Lemma 5. *Fix $\tau < \infty$. Consider arbitrary $p, p' \in \mathcal{P}^\tau$. Equivalent variation in the reduced form economy converges to the quasiliner limit*

$$\lim_{\beta \to 1} \tilde{EV}^i_{p,p'} = \tilde{EV}^{i,Q}_{p,p'}.$$

Proof of Lemma 5: We restrict attention to discount factors from $[\underline{\beta}, 1]$ as defined in Lemma 3. We verify sufficient conditions for the convergence of the equivalent variation for individual agent (Assumptions 2-3 in Weretka (2018)) within the reduced-form problem. First note that since $\tau < \infty$, and $S < \infty$, a collection of all histories, h_t such that $t \leq \tau$ is finite. Consequently, box $\tilde{X}^{i,b}$ can be reinterpreted as a subset of $\mathbb{R}^N$. The first order condition is uniformly bounded partial derivatives of the budget constraint (16). For any history h_t, one has $\partial \tilde{b}^i_p / \partial c^i_{h_t} = \pi_{h_t} \beta^t \zeta_{h_t}$ and $\partial \tilde{b}^i_p / \partial w^i = 1$. Therefore, $\bar{b} \geq \partial \tilde{b}^i_p / \partial c^i_{h_t} \geq \underline{b}$ and $\bar{b} \geq \partial \tilde{b}^i_p / \partial w^i \geq \underline{b}$, where bounds $\bar{b} \equiv \max(1, \max_{h_t : t \leq \tau} \zeta_{h_t} \pi_{h_t}) < 0$ and $\underline{b} \equiv \min(1, (\underline{\beta}^i)^\tau \min_{h_t : t \leq \tau} \zeta_{h_t} \pi_{h_t}) > 0$ are well-defined since $\tau < \infty$ and $S < \infty$, and $\pi_{h_t} > 0$ for all date-events h_t. Thus, Assumption 2 is satisfied. In the reduced-form representation for each $\beta \in [\underline{\beta}, 1]$, preferences are strictly convex on the respective domains $\tilde{X}^i$. In Step 1, we demonstrated that optimal choice and equivalent variation for the quasilinear model ($\beta = 1$) are well-defined. Fix arbitrary convex box $X^{i,b}$ such that the optimal choice and equivalent variation point with quasilinear preferences are in the interior. For policy p and $\beta \in [\underline{\beta}, 1]$ function $\tilde{b}^i_p(\cdot, \beta)$ is linear in c^i, and hence it is quasi-convex. Finally, by Lemma 5 correspondence $\Psi^i : X^{i,b} \times [\underline{\beta}, 1] \to X^{i,b}$ is continuous. By Proposition 1 in Weretka (2018), equivalent variation in the reduced form model satisfies $\lim_{\beta \to 1} \tilde{EV}^i_{p,p'} = EV^{i,Q}_{p,p'}$. □

We now specialize the results to truncations of policies. For a pair of policies $p, p' \in \mathcal{P}$ let $EV^{i,\tau}_{p,p'}$ be the equivalent variation for truncations of the policies to the first $\tau < \infty$ periods, i.e., for periods $t > \tau$ perturbations for both policies are replaced by zero, i.e., the endowments and prices follow the baseline Markov process. Consider x for which $\lim_{\beta \to 1} v^x \equiv \bar{v}^x \in \mathbb{R}_{++}$.

Corollary 2. *Fix $\tau < \infty$ and $p, p' \in \mathcal{P}$. Equivalent variation for truncations of p, p' policies (point-wise, given τ) converges to a finite limit, i.e.,*

$$\lim_{\beta \to 1} EV^{i,\tau}_{p,p'} = EV^{i,Q,\tau}_{p,p'} / \bar{v}^x \in \mathbb{R}$$

where $EV^{i,Q,\tau}_{p,p'}$ is the equivalent variation in the quasilinear problem, for policies p, p' truncated to the first τ periods.

Proof of Corollary 2 : The result follows from Lemma 2 and 5, and the facts that truncations of policies to the first τ periods are in $\mathcal{P}^\tau$.

A.5 Convergence for general policies

Our next lemma shows that the the welfare index derived for the truncated policies approximates well equivalent variation $EV^i_{p,p'}$ when τ is sufficiently high.

Lemma 6. *Fix arbitrary $\underline{\beta} \in (0, 1)$. Equivalent variation derived for truncated policies converges, i.e.,*

$$\lim_{\tau \to \infty} EV^{i,\tau}_{p,p'} = EV^i_{p,p'},$$

uniformly for $\beta \in (\underline{\beta}, 1)$.

Proof of Lemma 6: Fix arbitrary $\varepsilon > 0$. Pick τ for which

$$\frac{\Delta^\tau}{1 - \Delta} C \left[1 + \frac{\bar{\zeta}}{\underline{\zeta}} \frac{u^{i\prime}(\underline{c})}{u^{i\prime}(\bar{c})} \frac{1}{\underline{\beta}} \right] \frac{\bar{c} + \bar{e} + \bar{\zeta} + C\Delta}{\bar{v}^x} \leq \varepsilon \tag{22}$$

Since $\Delta < 1$, the corresponding τ exists and it does not depend on β. Consider arbitrary $\beta \in (\underline{\beta}, 1)$.

Step 1. By c^i denote a solution to problem (10) for policy p with transfer $EV^i_{p,p'}$, while $c^{i,\tau}$ is a solution for truncation of this policy p^τ with transfer $EV^{i,\tau}_{p,p'}$. Suppose that c^i is weakly preferred to $c^{i,\tau}$ (For the reverse preferences the argument is symmetric.) Under policy p^τ net cost of consumption flow c^i is given by

$$
\begin{aligned}
E\sum_{t=0}^{\infty}\beta^t\zeta^\tau_t(c^i_t - e^{i,\tau}_t) &= E\sum_{t=0}^{\infty}\beta^t\zeta_t(c^i_t - e^i_t) \\
&+ E\sum_{t=\tau+1}^{\infty}\beta^t\zeta^\tau_t(c^i_t - e^{i,\tau}_t) - E\sum_{t=\tau+1}^{\infty}\beta^t\zeta_t(c^i_t - e^i_t) \\
&\leq EV^i_{p,p'}v^x + E\sum_{t=\tau+1}^{\infty}\beta^t\left[|\Delta^\zeta_t|c^i_t + |\Delta^\zeta_t|e^i_t + |\Delta^e_t|\zeta_t + |\Delta^e_t\Delta^\zeta_t\right] \\
&\leq EV^i_{p,p'}\bar{v}^x + \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta),
\end{aligned} \tag{23}
$$

For all $\tau' \geq \tau$ flow c^i is affordable given the transfer and by assumption it is preferred to solution to $V^i(p^\tau, EV^{i,\tau}_{p,p'})$. Consequently, by (22) one has

$$EV^{i,\tau}_{p,p'} \leq EV^i_{p,p'} + \varepsilon.$$

Step 2. We next prove the other inequality. Let $c^{i\prime}$ be a solution to (10) for policy p' with no transfer. Using the arguments form Step 1 one can show that under policy $p^{\tau\prime}$ the net cost of the flow cannot exceed

$$E\sum_{t=0}^{\infty}\beta^t\zeta^{\tau\prime}_t(c^{i\prime}_t - e^{i\prime\tau}_t) \leq \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)$$

Consequently $V^i(p'^\tau, \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)/\bar{v}^x) \geq V^i(p', 0)$. The difference in utility

$$
\begin{aligned}
V^i(p, EV^i_{p,p'}) - \bar{V}(p^\tau, EV^{i,\tau}_{p,p'}) &= V^i(p', 0) - V^i(p'^\tau, 0) \\
&\leq V^i(p'^\tau, \delta(\tau)) - V^i(p'^\tau, 0) \\
&\leq \frac{u^{i\prime}(\underline{c})}{\underline{\zeta}}\frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)
\end{aligned} \tag{24}
$$

is bounded, where the first equality follows from (11) the inequality from the previous observation and the last inequality from the fact that the marginal utility of a dollar is bounded from above by $u^{i\prime}(\underline{c})/\underline{\zeta}$. On the other hand, for policy p for any $\gamma > 0$ within the considered range the difference in utility is

$$V^i(p^\tau, EV^{i,\tau}_{p,p'} + \gamma) - V^i(p^\tau, EV^{i,\tau}_{p,p'}) \geq \gamma\frac{\beta u^{i\prime}(\bar{c})}{\bar{\zeta}}\bar{v}^x. \tag{25}$$

Equating the two constants on the right hand sides of (24) and (25) gives

$$\gamma(\tau) = \frac{\bar{\zeta}}{\underline{\zeta}} \frac{u^{i'}(\underline{c})}{\underline{\beta} u^{i'}(\bar{c})} \frac{C\Delta^{\tau}}{1-\Delta} (\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)/\bar{v}^{x} \tag{26}$$

for which $V^i(p^\tau, EV^{i,\tau}_{p,p'} + \gamma(\tau)) \geq V^i(p, EV^i_{p,p'})$. Applying the argument from Step 1 one can show that the solution to problem (10) under policy p^τ and transfer $(EV^{i,\tau}_{p,p'} + \gamma(\tau))\bar{v}^x$ is affordable under policy p with transfer $(EV^{i,\tau}_{p,p'} + \gamma(\tau))\bar{v}^x + \frac{C\Delta^\tau}{1-\Delta}(\bar{c} + \bar{e} + \bar{\zeta} + C\Delta)$ and it gives higher utility than $\bar{V}(p, EV^i_{p,p'})$. Consequently, by (22) one has

$$EV^i_{p,p'} \leq EV^{i,\tau}_{p,p'} + \varepsilon.$$

The inequalities derived in Steps 1 and 2 imply $|EV^i_{p,p'} - EV^{i,\tau}_{p,p'}| \leq \varepsilon$ for all $\beta \in (\underline{\beta}, 1)$. □

A.6 Concluding argument

The following equality concludes the proof:

$$\lim_{\beta\to 1} EV_{p,p'} \overset{(1)}{=} \frac{1}{I}\sum_i \lim_{\beta\to 1} EV^i_{p,p'} \overset{(2)}{=} \frac{1}{I}\sum_i \lim_{\beta\to 1}\lim_{\tau\to\infty} EV^{i,\tau}_{p,p'} \tag{27}$$

$$\overset{(3)}{=} \frac{1}{I}\sum_i \lim_{\tau\to\infty}\lim_{\beta\to 1} EV^{i,\tau}_{p,p'} \overset{(4)}{=} \frac{1}{I}\sum_i \lim_{\tau\to\infty} \frac{\tilde{EV}^{i,Q,\tau}_{p,p'}}{\bar{v}^x} \tag{28}$$

$$\overset{(5)}{=} \frac{\lim_{\tau\to\infty}\frac{1}{I}\sum_i [S^{i,\tau}(p') - S^{i,\tau}(p)]}{\bar{v}^x} \overset{(6)}{=} \frac{S(\Delta q', \Delta Y') - S(\Delta q, \Delta Y)}{\bar{v}^x} \tag{29}$$

In (27) equality (1) follows from the definition of the aggregate equivalent variation and the sum law for limits, and equality (2) from Lemma 6. In (3) the interchange of limits is justified by Moore-Osgood theorem along with Lemma 6 and Corollary 2. Equation (4) is implied by Corollary 2 while (4) by follows from Lemma Lemmas 5 and the fact that policies p, p' truncated to the first τ periods are in $\mathcal{P}^\tau$. Replacing, prices and endowments from the static model with the recursive counterparts gives gives (6). □