Price Changes and Welfare Analysis: Measurement under Individual Heterogeneity

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Abstract

Accounting for general forms of unobserved heterogeneity is crucial when studying consumer demand, as neglect or misspecification might introduce substantial biases in the analysis. In this paper, we set-identify the distribution of exact consumer surplus from cross-sectional data when unobserved heterogeneity is essentially unrestricted. Knowledge of this distribution allows to study how the welfare gains (or losses) induced by price changes affect different sections of a population, something obfuscated by standard measures of welfare. Our approach exploits the information in the moments of demand, conditional on prices and income, thereby departing from the standard practice of obtaining identification through quantiles. In particular, we use the insight that cross-sectional data is informative about the average income effect at every given demand bundle. Our results can also be used to develop tests for stochastic rationalizability.

Keywords: nonparametric welfare analysis, individual heterogeneity, exact consumer surplus, deadweight loss, equivalent variation

JEL classification: C14, C31, D11, D12, D63, H22, I31

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1 Introduction

There are many settings in which it is essential to measure the impact of price changes on individual welfare. Price changes are ubiquitous in the economy and may occur through tax reforms, trade liberalization, or supply-side shocks.

In most cases, only (repeated) cross-sectional data on consumer demand is available to researchers. If individuals would have homogeneous preferences, standard tools could be used to calculate welfare directly from the data, given sufficient variation in prices and income. Hausman (1981) and Vartia (1983) provide analytical and numerical tools to calculate consumer surplus directly from the observed demand function.

In practice, however, individuals likely have heterogeneous preferences.[1] To accurately measure changes in welfare, it is crucial to account for this heterogeneity. Consider, for example, a scenario in which a carbon tax increases the price of gasoline. The welfare effects of this price change depend on her *idiosyncratic* willingness to substitute away from the use of gasoline or cars. By assuming the existence of a representative consumer, researchers may underestimate the actual cost of the tax by imposing a uniform rate of substitution across individuals. Moreover, it obfuscates the distributional consequences of the reform, which might be severely skewed.

Inferring the welfare impact of price changes under general forms of preference heterogeneity is challenging. If individuals' preferences are idiosyncratic, so is their demand. This significantly reduces the informational content of demand data since individuals are observed only once. More specifically, only a single point of every individual's demand function is recorded in cross-sectional data, and the heterogeneity renders it difficult to compare these points across individuals. This creates a *matching* problem: one may infer the marginal distribution of gasoline consumption before and after the price change, but one cannot match what people consumed earlier to what they consume now, which makes welfare analysis difficult.

In this paper, we study what can be learned about the welfare implications of price changes from cross-sectional data. We improve upon existing approaches by taking information on the distribution of income effects into account.[2] Income effects play a crucial role in analyzing welfare, as consumer surplus is calculated from compensated instead of uncompensated demands. Akin to the Slutsky equation, knowledge of income effects allows to infer one type of demand from the other.

More specifically, we exploit that cross-sectional data is informative about the average

[1] Allowing for heterogeneous preferences is thought to be important in empirical applications since traditional microeconometric models typically only explain a small part of the variation in consumer demand.

[2] Hausman and Newey (2016) also conduct welfare analysis under general forms of unobserved heterogeneity, but their approach does not rely on the distribution of income effects, which results in non-sharp bounds.

income effect at every given bundle.[3] These averages deliver first-order information on the compensated demands of the individuals located at these bundles, which can improve welfare estimates significantly. The latter is especially true when the analyst has no accurate a priori knowledge of the magnitude of the income effect, which leads to wide bounds with current methods.

By studying the moments of demand, conditional on prices and income, we characterize the informational content of demand data. Besides average welfare, our results also allow us to set-identify the distribution of welfare under unrestricted, unobserved heterogeneity.[4] This enables the distributional assessment of price changes. In particular, we derive approximations for all moments of exact consumer surplus in terms of the moments of demand.

As a methodological contribution, we derive a theoretical relationship between the moments of demand and the Slutsky equation. In doing so, we depart from the standard practice of obtaining identification and conducting welfare analysis through the quantiles of demand (e.g., see Dette, Hoderlein, and Neumeyer (2016)). Exploiting Slutsky symmetry, we show that the nth and $(n+1)$st moments of demand suffice to identify the first-order deviation of the nth moment of exact consumer surplus when we series expand it as a function of price changes.[5]

We improve upon Hausman and Newey (2016), who set-identify the average equivalent variation by imposing uniform bounds on individuals' income effect. While their method is robust in setups where preference heterogeneity is substantial, it can lead to wide bounds and makes any policy decision difficult. This is especially true when the analyst has no accurate a priori knowledge of the magnitude of the income effect.

Firstly, we derive a second-order approximation for the entire distribution of the equivalent variation, where the former only develop expressions for the average. This facilitates a more complete analysis of the distributional consequences of price changes. Unlike Hoderlein and Vanhems (2018), we do not assume demand to be monotonic in scalar-valued unobserved heterogeneity.

Second, if price changes are small, we show that our bounds on the average equivalent variation are strictly tighter. Our error terms are of third order importance, whereas those previously developed are second order. Our bounds have the most empirical bite when demand is convex in prices. Instead of relying on uniform bounds, which are unobserved in the data, we use conditional moments to inform us of some features of the distribution

[3] This result draws parallels with Hoderlein and Mammen (2007), who establish that local average structural derivatives are identified in nonseparable models.

[4] Notice that, from the viewpoint of the analyst, welfare changes are stochastic because individual preferences cannot be observed.

[5] Perhaps surprisingly, we find that even just the first and second moments of demand carry empirical content derived from utility maximization and can provide tight estimates of average welfare changes. This finding is in stark opposition to the first conditional moment of demand, which locally carries no empirical content. For a detailed review, see Rizvi (2006).

of income effects. Our results also suggest that the non-identification result generally holds but has limited empirical consequence. If two models are observationally equivalent, their welfare effects cannot be too far from each other in a way that we can quantify.

Third, the use of moments clarifies the informational content of panel data. More precisely, we characterize what can be learned from repeated observations, also elucidating the exact nature of the identification problem when analyzing cross-sectional data and what additional information is required to point-identify the distribution of exact surplus.

Fourth, our approach can also be employed in settings where researchers do not observe the entire demand distribution but only some coarse moments. This is especially useful in the many-good case and could, in principle, be used in settings where data is scarce and entire conditional quantile demands are hard to estimate.

The results developed in this paper are also useful to test stochastic rationalizability.[6] In the two good case, we can characterize rationalizable cross-sectional distributions of demand locally via a condition on several equations. Our method is computationally feasible once the moments are estimated and allows for an analytic characterization of rationality, in sharp contrast to the current results. We also use it to construct a computable semi-decidable test of rationality. With more than two goods, we can test the negativity of compensated demand but not symmetry.

To illustrate our results' empirical feasibility and usefulness, we present an application on gasoline demand. [TBA]

Related literature A long tradition in economics aims to estimate consumer surplus (or equivalent variation) from cross-sectional demand data. Most approaches solve a first-order differential equation in uncompensated demand. Hausman (1981) provides analytical solutions for several well-studied demand systems, and Vartia (1983) presents efficient algorithms. Combining these results with nonparametric demand estimation, Hausman and Newey (1995) obtain results for a flexible, representative demand function.

Our estimates also improve the approaches taken in Foster and Hahn (2000), Blundell, Browning, and Crawford (2003), and Schlee (2007), which treat unobserved heterogeneity as a regression error term and compute average equivalent from mean demands. These papers construct first-order approximations of welfare using mean demand, which our second-degree approximations improve upon. The most related to our approach is Schlee (2007), which gives assumptions under which equivalent variation of mean demand acts as a bound for average equivalent variation. Blundell, Browning, and Crawford (2003) show that if the idiosyncratic preference component in the demand equation is multiplicatively separable in preference type and the price-income pair, the EV of mean demand is a first-order approxi-

[6] See McFadden and Richter (1991); McFadden (2005) for a thorough treatment of stochastic rationalizability.

mation to the average EV. We differ from the above approaches by using approximations of compensating incomes; instead, we directly approximate compensated demands.

More recently, Hausman and Newey (2016) set-identify average consumer surplus when unobserved heterogeneity is unrestricted by imposing uniform bounds on individuals' income effect. They show that this average is not point-identified from cross-sectional data with a counterexample. Alternatively, Hoderlein and Vanhems (2018) derive the entire distribution of welfare effects under the assumption that demand is monotonic in scalar unobserved heterogeneity. In this case, the conditional quantile functions coincide with the true demand functions, such that point-identification is ensured. However, this is a somewhat restrictive and untestable assumption.

Alternatively, Kang and Vasserman (2022) provide bounds on welfare when only a few aggregate demand bundles are observed. They also assess the additional power that assumptions on the curvature of demand provide.

Finally, results in the context of discrete choice are obtained by Dagsvik and Karlström (2005), de Palma and Kilani (2011), and Bhattacharya (2015, 2018). The latter shows that the distribution of the equivalent variation can be written as a functional of uncompensated choice probabilities, even when unobserved heterogeneity is essentially unrestricted. This immediately implies that this distribution is nonparametrically point-identified from cross-sectional data. However, if choice is ordered, identification breaks down due to a lack of relative price variation in the data.[7]

From a broader perspective, our results are also related to the literature that derives observable restrictions on demand. In the multi-good case, Hoderlein and Stoye (2014, 2015); Dette, Hoderlein, and Neumeyer (2016), derive and test restrictions on marginal quantiles of demand. In a very related exercise, Hoderlein (2011) uses techniques very similar to ours to bound the proportion of individuals in a population who could satisfy rationality.

Focusing on aggregate demand, Hildenbrand (1983, 1994); Härdle, Hildenbrand, and Jerison (1991) impose restrictions on the variance of demand which guarantee that market demand obeys the so-called law of demand. In a similar setting, Grandmont (1987, 1992); Quah (1997) also find restrictions that guarantee the law of market demand and thus local stability of Walrasian equilibrium. We tackle the inverse problem; rather than putting restrictions on the fundamentals to discipline market demand, we observe the variance directly and see if we can test market demand directly for utility maximization.

We also view our results as a simple rebuke of the Sonnenchshein-Mantel-Debreu theorem. This theorem, stemming from Sonnenschein (1973) and then extended by Debreu (1974); Mantel (1974), suggested that rationality imposes no aggregate demand restrictions. We find that this is true only if only the first moment of demand is observable. If any

[7]Since continuous choice under a linear budget constraint can be seen as a limiting case of ordered discrete choice, this finding is consistent with the non-identification result in Hausman and Newey (2016).

other information on higher moments is available, rationality does have empirical content highlighting the fragility of the above claims.

Finally, from a methodological perspective, our results draw parallels with Hoderlein and Mammen (2007), who establish that local average structural derivatives can be recovered from quantiles in nonseparable models. Nonidentification of the higher-order terms of the distribution of surplus is akin to nonlinear transformations of the local average structural derivative not being identified.

Outline of the paper The remainder of this paper is organised as follows. Section 2 provides some motivating examples. In Section 3, our conceptual framework is laid out. Sections 4 and 5 presents the main results for small and large price changes, respectively. In Section 6, we derive the connection between conditional moments and individual rationality. Section 7 discusses estimation and implementation. In Section 8, we illustrate our results through an application on gasoline demand, using data from the U.S. National Household Transportation Survey. Finally, Section 9 concludes. Appendix A contains technical conditions omitted in the main text. In Appendix B, we extend our main results to economies with only finitely many people.

2 Individual demand and welfare

What can be learned about an individual's welfare from demand data? If only two price-quantity bundles are observed, a revealed preference argument can provide signs for the change in welfare.[8] However, if the individual's entire demand function is known, much more can be said.

Figure 1 illustrates the link between individual demand and welfare. For a single consumer, both uncompensated demand q and compensated demand h are shown.[9] Exact consumer surplus for a price change from p_0 to p_1 is equal to p_0abp_1, which is a surface between the vertical axis and compensated demand. Note that due to the presence of the income effect, this differs from inexact consumer surplus p_0cbp_1. In this case, calculating welfare based on uncompensated demand is biased downwards.

Hausman (1981) described the change in consumer surplus as a differential equation and showed that this can be solved analytically for commonly studied demand systems. However, when income effects differ across a population, this approach does not linearize. We can have 2 average demands which look identical, but the sum of the compensated demands of individuals can vary. This causes an identification problem when the analyst only has access

[8] For example, see the tabulations in Vartia and Weymark (1981).

[9] As the good is assumed to be normal, compensated demand is steeper than the associated uncompensated demand.

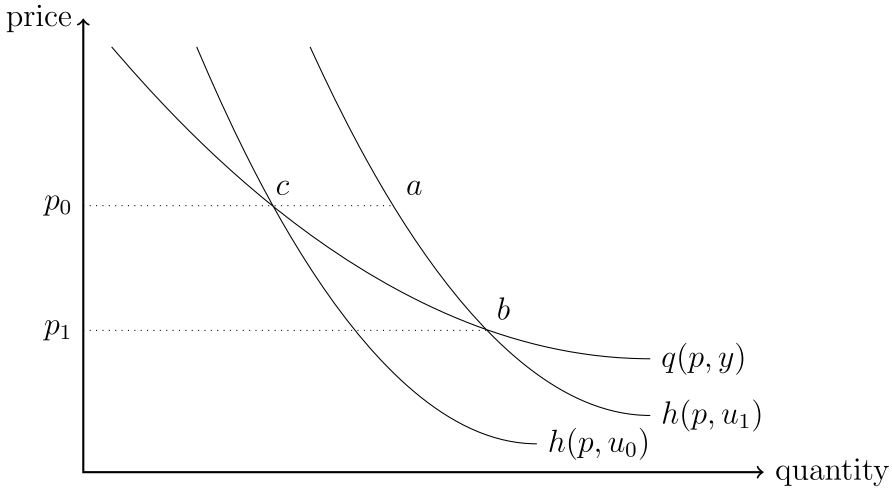


Figure 1: Consumer demand and welfare

to cross-sectional data. This problem was formalized in Hausman and Newey (2016), who show that the average exact consumer surplus is not identified from cross-sectional demand distributions. We circumvent this problem by considering an approximation of this quantity as a power series of price deviations.

Interestingly, we find that the coefficient of the second-order term is something that is linear in observable cross-sectional data, whereas the third-order term is not. This allows us to generate second-order estimates for welfare that are better than those previously known. Further we are able to zero in on the cause of the non-identification problem.

As the following example illustrates, it turns out that even the conditional variance contains considerable information about the average welfare when prices change.

Example 1. Consider a simple economy with a population of individuals with Cobb-Douglas preferences over two goods. We focus on the demand for good 1, fixing the price of good 2 to unity. The population has measure one and is further divided into three types of individuals:

- type ω_1 (with measure $\delta/2$) only consumes good 1, i.e. $q^{\omega_1}(p,y) = y/p$;
- type ω_2 (with measure $\delta/2$) only consumes good 2, i.e. $q^{\omega_2}(p,y) = 0$; and
- type ω_3 (with measure $1-\delta$) has equal weights on both goods, i.e. $q^{\omega_3}(p,y) = y/p$.

Notice that while average demand does not depend on δ,

$$M_1(p,y) = \frac{\delta}{2}\frac{y}{p} + \frac{\delta}{2}0 + (1-\delta)\frac{y}{2p} = \frac{y}{2p},$$

the variance of demand is increasing in this parameter,

$$M_2(p,y) = \frac{\delta}{2}\left(\frac{y^2}{p^2}\right) + \frac{\delta}{2}0 + (1-\delta)\left(\frac{y^2}{4p^2}\right) = (1+\delta)\left(\frac{y^2}{4p^2}\right).$$

Now consider an exogenous change in the price for good 1, and let $\Delta p = p_1 - p_0$. Straightforward calculations show that average exact consumer surplus is equal to

$$\mathbb{E}[\Delta CS] = \frac{\delta}{2}\frac{y\Delta p}{p_0} + (1-\delta)y\left[\left(1+\frac{\Delta p}{p_0}\right)^{\frac{1}{2}} - 1\right]. \tag{1}$$

A simple linear approximation to the change in average inexact consumer surplus yields

$$\mathbb{E}[\Delta CS] \approx M_1(p_0,y)\Delta p = \frac{y\Delta p}{2p_0}.$$

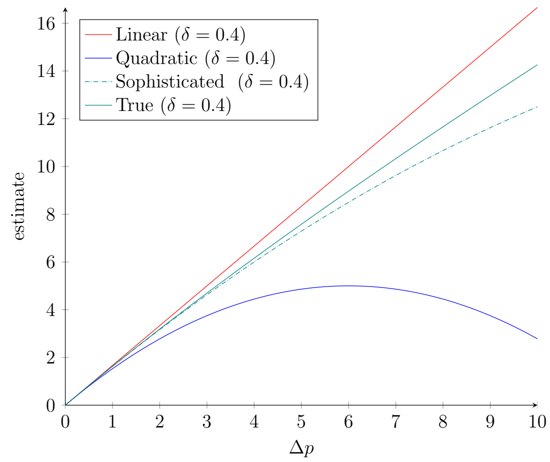
Alternatively, a quadratic approximation of inexact consumer surplus yields the more precise

$$\mathbb{E}[\Delta CS] \approx M_1(p_0,y)\Delta p + \frac{\Delta p^2}{2}D_p M_1(p_0,y) = \frac{y\Delta p}{2p_0} - \frac{y\Delta p^2}{4p_0^2}.$$

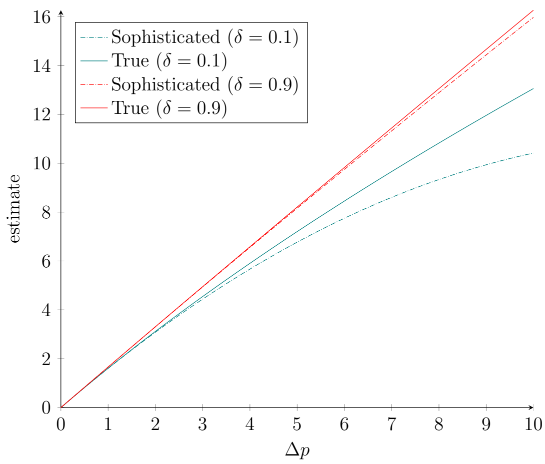
Our more sophisticated local approach, however, gives

$$\mathbb{E}[\Delta CS] \approx M_1(p,y)\Delta p + \frac{\Delta p^2}{2}\left(D_p M_1(p,y) + \frac{1}{2}D_y M_2(p,y)\right) = \frac{y\Delta p}{2p} + \frac{(\delta-1)y\Delta p^2}{4p^2},$$

which coincides with the first two terms of the power expansion of average exact consumer surplus in Equation (1). In the following section, we will show that this property is true in general.



(a) Relative performance



(b) Heterogeneity in income effects

Figure 2: Approximations to exact consumer surplus

Figure 2 provides a graphical illustration of these approximations. For a fixed value of δ,

Figure 2a shows the difference in performance between our estimate and those based on uncompensated demand. We find that our approximation outperforms the existing approaches for modest price changes. Figure 2b shows the impact of the degree of heterogeneity on our approximation.

3 Conceptual framework

Our conceptual framework allows for unrestricted, unobserved heterogeneity in individuals' preferences. For ease of exposition, we suppress all *observed* individual characteristics in the notation, as all results in this paper can be thought of as being conditional on these covariates.

3.1 Consumer demand and individual heterogeneity

We consider the standard model of utility maximization under a linear budget constraint. Let Ω denote the universe of preference types and let $F(\omega)$ denote the distribution of these preference types in the population. Every preference type ω can be thought of as a different individual who has idiosyncratic preferences over bundles of l goods $\mathbf{q} \in \mathcal{Q} = \mathbb{R}^l_+$. These preferences are assumed to be representable by a smooth, strongly quasi-concave utility function $u^\omega : \mathcal{Q} \to \mathbb{R}$. Note that this formulation of preferences is very general, as it allows the latter to differ arbitrarily across individuals. Prices $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^l_+$ and income $y \in \mathcal{Y} \subset \mathbb{R}_+$ are assumed to be positive; we call a pair $\mathbf{b} = (\mathbf{p}, y)$ a budget set.

Individual demand functions $\mathbf{q}^\omega : \mathcal{P} \times \mathcal{Y} \to \mathcal{Q}$ arise from individuals maximizing their utility subject to a linear budget constraint,

$$\mathbf{q}^\omega(\mathbf{b}) = \underset{\mathbf{p}.\mathbf{q} \leq y : \mathbf{q} \in \mathcal{Q}}{\arg\max}\ u^\omega(q).$$

These demand functions satisfy homogeneity of degree zero and Walras law,

$$\begin{aligned} \mathbf{q}^\omega(\alpha \mathbf{p}, \alpha y) &= \mathbf{q}^\omega(\mathbf{b}), \quad \forall \alpha \in \mathbb{R}_+, \\ \mathbf{p} \cdot \mathbf{q}^\omega(\mathbf{b}) &= y, \end{aligned}$$

for all budget sets. For every demand function $\mathbf{q}^\omega(\mathbf{b})$, there exists a compensated (or Hicksian) demand function $\mathbf{h}^\omega(\mathbf{p}, u) : \mathcal{P} \times \mathbb{R} \to \mathcal{Q}$ defined as

$$\mathbf{h}^\omega(\mathbf{p}, u) = \underset{\mathbf{q} \in \mathcal{Q}}{\arg\min} \{\mathbf{p} \cdot \mathbf{q} | u^\omega(q) \geq u\}.$$

As is well known, the Slutsky equation

$$D_p \mathbf{q}^\omega(\mathbf{b}) = D_p \mathbf{h}^\omega(\mathbf{p}, u) - D_y \mathbf{q}^\omega(\mathbf{b}) \mathbf{q}^\omega(\mathbf{b})^\intercal$$

provides a link between both demand functions.[10]

The indirect utility function $v^\omega : \mathcal{P} \times \mathcal{Y} \to \mathbb{R}$ is defined as

$$v^\omega(\mathbf{b}) = \max_{\mathbf{p}.q \leq y : q \in \mathcal{Q}} u^\omega(q),$$

i.e. the utility level obtained with budget $\mathbf{b}$, while the expenditure function $e^\omega(\mathbf{p}, u)$: $\mathcal{P} \times \mathbb{R} \to \mathcal{Y}$ is defined as

$$e^\omega(\mathbf{p}, u) = \min_{u \leq u^\omega(q) : q \in \mathcal{Q}} \mathbf{p}.q,$$

i.e. the minimum amount of income needed to achieve utility level u at prices $\mathbf{p}$.

We will assume that preference types are distributed independently of budget sets in the population. Intuitively, this can be thought of as prices and income being randomly assigned across preference types, giving rise to "treatment groups" with different budget sets. It is precisely this exogenous variation that will enable us to recover the impact of price changes on individual welfare from observational cross-sectional data.

Assumption 1 (budget set exogeneity)**.** The distribution of unobserved heterogeneity $F(\omega)$ is independent of prices and income:

$$F(\omega \mid \mathbf{b}) = F(\omega).$$

The exogeneity of budget sets is a strong but standard assumption in the literature on nonparametric identification of demand and welfare (e.g. see Hausman and Newey (2016); Blomquist, Newey, Kumar, and Liang (2021)). Indeed, to the best of our knowledge, there are no theoretical results that allow for general forms of endogeneity in the presence of unrestricted, unobserved heterogeneity. Some forms of endogeneity, however, can be mitigated by using a control function approach (see Section 7).

3.2 Welfare and deadweight loss

Our main object of interest is equivalent variation (or exact consumer surplus), which quantifies the impact of price changes on individual welfare.[11] It measures the minimum amount of additional income an individual needs to receive before the price change to be equally

[10] D_p and D_y denote the derivative operator with respect to prices and income, respectively.

[11] We focus on equivalent variation (instead of on compensating variation) because this measure allows comparing different reforms, as it is measured in baseline prices.

well-off as after the price change. Formally, for a price change from $\mathbf{p}_0$ to $\mathbf{p}_1$, it is defined as

$$\begin{aligned} EV^\omega(\mathbf{p}_0, \mathbf{p}_1, y) &= e^\omega(\mathbf{p}_1, v^\omega(\mathbf{p}_1, y)) - e^\omega(\mathbf{p}_0, v^\omega(\mathbf{p}_1, y)) \\ &= y - e^\omega(\mathbf{p}_0, v^\omega(\mathbf{p}_1, y)). \end{aligned}$$

Deadweight loss, which is a popular measure of economic efficiency, is defined as equivalent variation minus the tax receipts

$$DWL^\omega(\mathbf{p}_0, \mathbf{p}_1, y) = EV^\omega(\mathbf{p}_0, \mathbf{p}_1, y) - \Delta\mathbf{p} \cdot \mathbf{q}^\omega(\mathbf{p}_1, y),$$

where $\Delta\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0$.

3.3 Conditional moments of demand

Integrating out the unobserved preference heterogeneity, we can express the nth (non-central) conditional moment of demand as

$$\begin{aligned} \mathbf{M}_n(\mathbf{b}) &= \int \left(\bigotimes_{k=1}^{n} \mathbf{q}^\omega(\mathbf{b}) \right) dF(\omega \mid \mathbf{b}) \\ &= \int \left(\bigotimes_{k=1}^{n} \mathbf{q}^\omega(\mathbf{b}) \right) dF(\omega), \end{aligned} \tag{2}$$

in which $\bigotimes$ represents the Kronecker product and where the second equality follows directly from Assumption 1.[12] Since these conditional moments are essentially conditional expectation functions, they are nonparametrically identified from cross-sectional data. Note that in the two-good case, expression (2) simplifies to

$$M_n(b) = \int q^\omega(b)^n dF(\omega),$$

since then it suffices to consider scalar demand. It is clear from a preliminary inspection that the conditional moments inherit the following two conditions:[13]

$$\begin{aligned} \mathbf{M}_n(\alpha\mathbf{p}, \alpha y) &= \mathbf{M}_n(\mathbf{b}), \quad \forall \alpha \in \mathbb{R}_+, \\ \mathbf{p} \cdot \mathbf{M}_1(\mathbf{b}) &= y. \end{aligned}$$

We define a *moment sequence* as the (possibly infinite) sequence $\{M_i(\mathbf{b})\}_{i=1}^n$ of moments of demand of length n. We say that a sequence $\{a_i(\mathbf{b})\}_{i=1}^n$ is *rationalizable* if there exists a

[12] In the remainder of the paper, we will assume that all moments considered exist and are finite.

[13] Chiappori and Ekeland (1999b) show that these are the only restrictions on mean demand (i.e. $\mathbf{M}_1$).

universe of preference types Ω and a probability measure $dF(\omega)$ over these types such that

$$a_i(\mathbf{b}) = \int \left(\bigotimes_{k=1}^{i} \overline{\mathbf{q}}^{\omega}(\mathbf{b}) \right) dF(\omega), \quad \forall i,$$

and for all $\omega \in \Omega$, $\overline{\mathbf{q}}^{\omega}$ is a rational demand function.

4 Small price changes

This section demonstrates how the results developed above can be used to estimate the distribution of welfare changes in response to price changes. We show that the moments of this distribution can be approximated up to second order from the observed conditional moments of demand.

Before proceeding to the main results, the following lemma establishes a relation between income effects and the moments of demand.[14]

Lemma 1. Suppose Assumption 1 holds. For every $n \in \mathbb{N}_+$, it holds that

$$\mathbb{E}[(q^{\omega}(b_0))^n D_y q^{\omega}(b_0)] = (n+1)^{-1} D_y M_{n+1}(b_0).$$

Derivatives of observable moments with respect to income are related to the income effect terms that occur in many of our results. In particular, for $n = 1$ the lemma establishes a relation between the income effect term in the Slutsky equation and the conditional variance of demand.

Second-order approximation For clarity of exposition, we first consider the two-good case. Observe that by Shephards' lemma,

$$D_p e^{\omega}(p, u) = h^{\omega}(p, u),$$

we can write the equivalent variation in terms of Hicksian demand,

$$EV^{\omega}(p_0, p_1, y) = \int_0^1 h^{\omega}(\pi(t), v^{\omega}(b_0)) d\pi,$$

[14] A full exploration of the informational content of the moments. of demand is postponed to Section 6.

for some price path $\pi : [0,1] \to \mathcal{P}$. Without loss of generality, we will assume the path to be linear, i.e $\pi(t) = p_0 + (p_1 - p_0)t = p_0 + t\Delta p$, such that[15]

$$EV^\omega(p_0, p_1, y) = \Delta p \int_0^1 h^\omega(p_0 + t\Delta p, \overline{v}^\omega)dt,$$

where $\overline{v}^\omega = v^\omega(b_0)$.

A series representation of Hicksian demand around $\Delta p = 0$ gives

$$h^\omega(p_0 + t\Delta p, \overline{v}^\omega) = h^\omega(p_0, \overline{v}^\omega) + \sum_{n=1}^\infty \frac{(t\Delta p)^n}{n!} D_{p^n} h^\omega(p_0, \overline{v}^\omega),$$

such that

$$EV^\omega(p_0, p_1, y) = \Delta p q^\omega(b_0) + \sum_{n=1}^\infty \frac{(\Delta p)^{n+1}}{(n+1)!} D_{p^n} h^\omega(p_0, \overline{v}^\omega). \tag{3}$$

Retaining only on the first two terms, we get that

$$EV^\omega(p_0, p_1, y) \approx \Delta p q^\omega(b_0) + \frac{(\Delta p)^2}{2} \left[D_p q^\omega(b_0) + q^\omega(b_0) D_y q^\omega(b_0) \right].$$

Using Lemma 1, integrating out unobserved preference heterogeneity gives

$$\mathbb{E}[EV^\omega(p_0, p_1, y)] \approx \Delta p M_1(b_0) + \frac{(\Delta p)^2}{2} \left[D_p M_1(b_0) + \frac{1}{2} D_y M_2(b_0) \right].$$

This expression shows that the first moment of the equivalent variation is identified up to the second order from the conditional mean and variance functions.

Using an analogous derivation, we can also derive a second-order approximation for higher moments of welfare changes,

$$\mathbb{E}[EV^\omega(p_0, p_1, y)^n] \approx (\Delta p)^n \left(M_n(b_0) + \frac{\Delta p}{2} \left[D_p M_n(b_0) + \frac{n}{n+1} D_y M_{n+1}(b_0) \right] \right).$$

These findings are summarized in the following theorem.

Theorem 1. *Suppose Assumption 1 holds. Then the second-order approximation of the nth moment of the equivalent variation only depends on the nth and $(n+1)$th conditional moment of demand. It can be written as*

$$\mathbb{E}[EV^\omega(p_0, p_1, y)^n] \approx (\Delta p)^n \left(M_n(b_0) + \frac{\Delta p}{2} \left[D_p M_n(b_0) + \frac{n}{n+1} D_y M_{n+1}(b_0) \right] \right).$$

Remark 1. By a straightforward application of the mean value theorem, we can write

[15] The integral is path independent due to Slutksy symmetry.

average welfare as

$$\mathbb{E}[EV^\omega(p_0,p_1,y)^n] = \Delta p M_1(b_0) + \frac{(\Delta p)^2}{2}\left(D_pM_1(b_0) + \frac{1}{2}D_yM_2(b_0)\right) + \frac{(\Delta p)^3}{6}D_{pp}E[h^\omega(\overline{p},y)],$$

for some intermediate point $\overline{p} \in [p_0,p_1]$. If Hicksian demand is convex in prices, the second-order approximation yields a lower (upper) bound when $\Delta p > 0$ ($\Delta p < 0$).[16]

Higher-order approximations To obtain higher-order approximations for average welfare, it is clear from the series representation in Equation (3) that we need to be able to express $\{\mathbb{E}[D_{p^n}h^\omega(p,\overline{v}^\omega)]\}_{n=2}^\infty$ in terms of observable quantities. We now argue that this is impossible when only cross-sectional data is available.

Differentiating the identity $h^\omega(p,u) = q^\omega(p,e^\omega(p,u))$ twice with respect to price, we get

$$\begin{aligned} D_{p^2}h^\omega(p,\overline{v}^\omega) &= D_{p^2}q^\omega(b) + D_ph^\omega(p,\overline{v}^\omega)D_yq^\omega(b) + q^\omega(b)^2D_{y^2}q^\omega(b) + 2q^\omega(b)D_{p,y}q^\omega(b)\\ &= D_{p^2}q^\omega(b) + \left[D_pq^\omega(b) + q^\omega(b)D_yq^\omega(b)\right]D_yq^\omega(b) + q^\omega(b)^2D_{y^2}q^\omega(b) + 2q^\omega(b)D_{p,y}q^\omega(b), \end{aligned}$$

where the second equality follows from the Slutsky equation. Elementary calculations give that

$$\mathbb{E}[D_{p^2}h^\omega(p,\overline{v}^\omega)] = D_{p^2}M_1(b) + \frac{1}{2}D_{p,y}M_2(b) + \frac{1}{3}D_{y^2}M_3(b) - \mathbb{E}[q^\omega(b)(D_yq^\omega(b))^2], \qquad (4)$$

in which all terms, except for the last, are directly observable quantities. We extend this to higher-order approximations in the following lemma.

Lemma 2 (generalized Slutsky condition)**.** Let $C_n(b) = \mathbb{E}[q^\omega(b)(D_yq^\omega(b))^n \mid x]$. We have that

$$\mathbb{E}[D_{p^n}h^\omega(p_0,\overline{v}^\omega)] = B_n(b_0) - C_n(b_0)$$

where $B_n(b_0)$ is an known function of identifiable objects.

Proof. We have

$$D_{p^n}h^\omega(p,u) = D_{p^n}[q^\omega(p,e^\omega(p,u))] = \sum_{s,t:s+t=0}^{n}\left[D_{p^s,y^t}q^\omega(p,e^\omega(p,u))\right]B^n_{s,t}$$

[16] Convexity includes the common linear demand specification $q^\omega(b) = \omega_1 - \omega_2 p - \omega_3 y$ where $(\omega_1,\omega_2,\omega_3)$ varies across the population.

with (exponential) Bell polynomials

$$B^n_{s,t} = n! \sum_{\mathbf{s},\mathbf{t}\in\mathcal{K}^n_{s,t}} \prod_{j=1}^{\infty} \frac{1}{s_j!t_j!}\left(\frac{D_{p^j}p}{j!}\right)^{s_j}\left(\frac{D_{p^j}e^{\omega}(p,u)}{j!}\right)^{t_j}$$
$$= n! \sum_{\mathbf{s},\mathbf{t}\in\mathcal{K}^n_{s,t}} \prod_{j=1}^{\infty} \mathbb{I}[j=1 \text{ or } s_j=0]\frac{1}{s_j!t_j!}\left(\frac{D_{p^j}e^{\omega}(p,u)}{j!}\right)^{t_j}$$

where the sum is over

$$\mathcal{K}^n_{s,t} = \left\{\mathbf{s},\mathbf{t} : \sum_{j=1}^{\infty} j(s_j+t_j) = n \text{ and } \sum_{j=1}^{\infty}\begin{bmatrix} s_j \\ t_j\end{bmatrix} = \begin{bmatrix} s \\ t\end{bmatrix}\right\}$$

the set of tuples that exhaust the combinatorial division of n items into s and t groups. (Note that these tuples are in fact solutions to a Diophantine equation.)

We also have that

$$D_{p^n,y^m}[q^{\omega}(b)^r] = \sum_{s,t:s+t=0}^{n+m} \frac{r!}{(r-s-t)!} q^{\omega}(b)^{r-s-t} B^{n,m}_{s,t}$$

with bivariate Bell polynomials

$$B^{n,m}_{s,t} = n!m! \sum_{(\mathbf{s}_j,\mathbf{t}_j)\in\mathcal{K}^{n,m}_{s,t}} \prod_{j_1,j_2:j_1+j_2=1}^{\infty} \frac{1}{s_{j_1}!,t_{j_2}!}\left(\frac{D_{p^{j_1}y^{j_2}}q^{\omega}(b)}{j_1!j_2!}\right)^{s_{j_1}+t_{j_2}}$$

where the sum is now over two dimensional tuples

$$\mathcal{K}^{n,m}_{s,t} = \left\{\mathbf{s},\mathbf{t} : \sum_{j_1,j_2:j_1+j_2=1}^{\infty} \begin{bmatrix} j_1 \\ j_2\end{bmatrix}(s_{j_1}+t_{j_2}) = \begin{bmatrix} n \\ m\end{bmatrix} \text{ and } \sum_{j_1,j_2:j_1+j_2=1}^{\infty}\begin{bmatrix} s_{j_1} \\ t_{j_2}\end{bmatrix} = \begin{bmatrix} s \\ t\end{bmatrix}\right\}$$

.

Elementary combinatorial calculations lead to the claimed statement. □

Proposition 1 below shows that the terms $\{C_n(b)\}_{n=2}^{\infty}$ are not identified from cross-sectional data. Hence, the higher-order approximations of the average equivalent variation are also not identified. This fully characterizes the nature of the identification problem in recovering the average equivalent variation.

Proposition 1. Suppose Assumption 1 holds. Then $\{C_n(b)\}_{n=2}^{\infty}$ is not nonparametrically identified from cross-sectional data.

Proof. We show nonidentification of $\{C_n(b)\}_{n=2}^{\infty}$ by means of a counterexample. Suppose individual demand is linear in price and income

$$q^\omega(b) = \omega_1 - p + \omega_2 y,$$

and let $\omega_1 \sim U(0,1)$, and $\Pr[\omega_2 = 1/3] = \Pr[\omega_2 = 2/3] = 1/2$. Hausman and Newey (2016) show that for $y < 3$, an observationally equivalent specification is the quantile demand

$$\widetilde{q}^{\widetilde{\omega}}(b) = \begin{cases} -p + \mathbb{I}[y < 6\widetilde{\omega}](y/2 + \widetilde{\omega}) + \mathbb{I}[y \geq 6\widetilde{\omega}](y/3 + 2\widetilde{\omega}), & \widetilde{\omega} \leq 1/2, \\ -p + \mathbb{I}[y < 6(1-\widetilde{\omega})](y/2 + \widetilde{\omega}) + \mathbb{I}[y \geq 6(1-\widetilde{\omega})](2y/3 + 2\widetilde{\omega} - 1), & \widetilde{\omega} > 1/2, \end{cases}$$

where $\widetilde{\omega} \sim U(0,1)$.

For a budget set $(p, y) = (1, 2)$, elementary calculations show that

$$\begin{aligned} C_n(1,2) &= \mathbb{E}[(\omega_1 - p + \omega_2 y)\omega_2^n \mid p = 1, y = 2] \\ &= (\mathbb{E}[\omega_1] - 1)\mathbb{E}[\omega_2^n] + 2\mathbb{E}[\omega_3^n] \\ &= -1/4[(1/3)^n + (2/3)^n] + [(1/3)^{n+1} + (2/3)^{n+1}] \\ &= 1/12(1/3)^n + 5/12(2/3)^n \end{aligned} \tag{5}$$

holds for the original demand specification. However, after differentiating the quantile demand with respect to income, we obtain

$$\widetilde{q}^{\widetilde{\omega}}(b)(D_y \widetilde{q}^{\widetilde{\omega}}(b))^n \mid_{p=1,y=2} = \begin{cases} \mathbb{I}[1/3 < \widetilde{\omega}]\widetilde{\omega}(1/2)^n + \mathbb{I}[1/3 \geq \widetilde{\omega}](-1/3 + 2\widetilde{\omega})(1/3)^n, & \widetilde{\omega} \leq 1/2, \\ \mathbb{I}[2/3 > \widetilde{\omega}]\widetilde{\omega}(1/2)^n + \mathbb{I}[2/3 \leq \widetilde{\omega}](-2/3 + 2\widetilde{\omega})(2/3)^n, & \widetilde{\omega} > 1/2, \end{cases}$$

such that

$$\begin{aligned} \widetilde{C}_n(1,2) &= (1/3)^n \int_0^{1/3} (-1/3 + 2\widetilde{\omega}) + (1/2)^n \int_{1/3}^{1/2} \widetilde{\omega} + (1/2)^n \int_{1/2}^{2/3} \widetilde{\omega} + (2/3)^n \int_{2/3}^{1} (-2/3 + 2\widetilde{\omega}) \\ &= 1/6(1/2)^n + 1/3(2/3)^n. \end{aligned} \tag{6}$$

Expressions (5) and (6) are only equal for $n = 1$. Since two observationally equivalent models generate different results for $n \geq 2$, the latter are not nonparametrically identified. □

Remark 2. The unidentified terms can be given an intuitive interpretation. Using the law of iterated expectations, we can write

$$C_n(b) = \mathbb{E}\left[q^\omega(b)\mathbb{E}[(D_y q^\omega(b))^n \mid b, q^\omega(b)] \mid b\right].$$

This highlights that the nth term is equal to the (non-centered) covariance between demand

and the nth moment of the income effect at that bundle. Failure to identify the third-order approximation of average welfare is therefore due to cross-sectional data being uninformative about how the variance of the income effect changes across demand bundles. The same holds for the higher order approximations, mutatis mutandis.

Remark 3. Identification of these covariances requires knowledge on the joint distribution of demand and income effects, but cross-sectional data only delivers the associated marginal distributions. However, note that the unobserved terms could in principle be obtained from panel data with as little as two periods, given that sufficient intertemporal variation in income is present. We leave further consideration of the informational content of panel data to future research.

It is no coincidence that in the counterexample in the proof of Proposition 1, $C_1(b)$ is identical for the two observationally equivalent models. Direct application of Theorem 2.1 in Hoderlein and Mammen (2007) shows that the quantiles of demand identify the local average structural derivative. Specifically, it holds for every quantile $\overline{\tau}$ that

$$\mathbb{E}[D_y q^\omega(b) \mid x, q^\omega(b) = \widetilde{q}(\overline{\tau} \mid x)] = D_y \widetilde{q}(\overline{\tau} \mid x),$$

i.e the average derivative at bundle $\widetilde{q}(\overline{\tau} \mid x)$ is equal to the derivative with respect to income along the associated quantile demand. Again using the law of iterated expectations, we can write

$$C_1(b) = \mathbb{E}_\tau \left[\widetilde{q}(\tau \mid x) D_y \widetilde{q}(\tau \mid x) \mid x\right],$$

which is nonparametrically identified because it is the (non-centered) covariance of two estimable objects. Moreover, they also argue that nonlinear transformations of the local average structural derivative are not identified, which is similar to the nonidentification result we obtained in Proposition 1.

4.1 Many-goods case

We now analyze the distribution of the equivalent variation in the case where there are more than two goods. Analogously with the two-goods case, the following approximation holds,

$$EV^\omega(\mathbf{p}_0, \mathbf{p}_1, y) \approx \Delta\mathbf{p} \cdot \mathbf{q}^\omega(\mathbf{b}_0) + \frac{1}{2}\Delta\mathbf{p} \cdot [D_p \mathbf{q}^\omega(\mathbf{b}_0) + \mathbf{q}^\omega(\mathbf{b}_0) D_y \mathbf{q}^\omega(\mathbf{b}_0)] \cdot \Delta\mathbf{p} \tag{7}$$

but now we have vectors and matrices instead of scalars. Intergrating over types, we get

$$\mathbb{E}[EV^\omega(\mathbf{p}_0, \mathbf{p}_1, y)] = \Delta\mathbf{p} \cdot \mathbf{M}_1(\mathbf{b}_0) + \frac{1}{2}\Delta\mathbf{p} \cdot \left[D_p \mathbf{M}_1(\mathbf{b}_0) + (D_p \mathbf{M}_1(\mathbf{b}_0))^\intercal + D_y \mathbf{M}_2(\mathbf{b}_0)\right] \Delta\mathbf{p}.$$

For higher moments, this approximation is slightly more involved,

$$
\begin{aligned}
EV^{\omega}(\mathbf{p}_0,\mathbf{p}_1,y)^n &= \left[\Delta\mathbf{p}\cdot\mathbf{q}^{\omega}(\mathbf{b}_0)+\frac{1}{2}\Delta\mathbf{p}\cdot\left[D_p\mathbf{q}^{\omega}(\mathbf{b}_0)+\mathbf{q}^{\omega}(\mathbf{b}_0)D_y\mathbf{q}^{\omega}(\mathbf{b}_0)\right]\Delta\mathbf{p}\right]^n \\
&= \left[\Delta\mathbf{p}\cdot\mathbf{q}^{\omega}(\mathbf{b}_0)\right]^n \\
&\quad+\sum_{i=1}^{n}(\Delta\mathbf{p}\cdot\mathbf{q}^{\omega}(\mathbf{b}_0))^{n-i-1}\left\{\Delta\mathbf{p}\cdot\left[D_p\mathbf{q}^{\omega}(\mathbf{b}_0)+\mathbf{q}^{\omega}(\mathbf{b}_0)D_y\mathbf{q}^{\omega}(\mathbf{b}_0)\right]\cdot\Delta\mathbf{p}\right\}((\mathbf{q}^{\omega}(\mathbf{b}_0)\cdot\Delta\mathbf{p}))^i \\
&= \frac{1}{2}\Delta\mathbf{p}(**)\left\{\frac{1}{n}\left[\sum(\text{symmetrized moment derivative})\right]+\frac{n}{n+1}D_y\mathbf{M}_{n+1}\right\}
\end{aligned}
$$

where $(**)$ represents a generalized k tensor form.

Remark 4. Akin to Hausman (1981), if the price of only one good changes, it is clear from Equation (7) that this only requires knowledge on the income effects for that specific good. This might simplify the analysis considerably in the presence of many goods.

Remark 5. When the prices of all goods change, the second order approximation for average welfare requires the estimation of the entire variance-covariance matrix, which might be burdensome. It is possible, however, to bound the off-diagonal elements of this matrix from the marginal conditional variances. In particular one can impose the following three restrictions:

$$
\begin{aligned}
[\mathbf{M}^2(\mathbf{b})]_{ij} &= [\mathbf{M}^2(\mathbf{b})]_{ji} \\
\mathbf{p}\cdot D_y\mathbf{M}_2(\mathbf{b}) &= D_y\mathbf{M}_2(\mathbf{b})\cdot\mathbf{p}=0 \\
[\mathbf{M}^2(\mathbf{b})]_{ij} &\leq \sqrt{\mathbf{M}^2(\mathbf{b})]_{ii}\mathbf{M}^2(\mathbf{b})]_{jj}}
\end{aligned}
$$

Note that the Cauchy Schwarz inequality ensures that the off-diagonal elements have bounded support, even if we only observe the diagonal elements.

4.2 Link with sufficient statistic approach

TBA

5 Large price changes

In this section, we improve upon the second-order linear approximation, by exploiting the nonlinear structure of the problem. This is especially useful when the underlying demand functions are highly nonlinear in prices. Hausman and Newey (2016) assume the income effects to be linear and independent of prices, and constant across individuals. We show that this does not exploit all information that is available in cross-sectional data.

5.1 Nonlinear approach

Following Hausman (1981), consumer surplus can also be expressed in terms of the solution to a nonlinear differential equation. In particular, taking derivatives of both sides of

$$s^{\omega}(t) = y - e^{\omega}(\pi(t), v^{\omega}(b_1)), \quad t \in [0, 1],$$

with respect to t, together with Shephard's lemma, gives the first-order differential equation

$$D_t s^{\omega}(t) = -q^{\omega}(\pi(t), y - s^{\omega}(t)) D_t \pi(t), \quad t \in [0, 1], \tag{8}$$

with endpoint condition $s^{\omega}(1) = 0$. It then holds that $EV^{\omega}(p_0, p_1, y) = s^{\omega}(0)$.

However, there is an insurmountable problem with this approach when it comes to aggregating in a heterogeneous population. For ease of exposition, let us consider the simplest case, namely the one with constant income effects: i.e. $q^{\omega}(p(t), y - s^{\omega}(t)) = q^{\omega}(p(t), y) - \alpha s^{\omega}(t)$. We therefore have that

$$D_t s^{\omega}(t) = -[q^{\omega}(\pi(t), y) - \alpha s^{\omega}(t)] D_t \pi(t), \quad t \in [0, 1],$$

which is a first-order linear ODE, and can be solved analytically giving us

$$\begin{aligned} EV^{\omega}(p_1, p_2, y) &= \Delta p \int_0^1 q^{\omega}(p_0 + t\Delta p, y) \exp(-\alpha^{\omega} t \Delta p) dt \\ &= \int_{p_0}^{p_1} q(\pi(t), y) \exp(-\alpha(p_1 - p_0)) D_t \pi(t) dt. \end{aligned}$$

This formula directly illustrates the problem: when looking at aggregate data, the income derivatives are *additive*, and the EV is also additive, the but the dependence of EV on the income effects is non linear, leading to identification problems.

We now illustrate our approach by approximating the EV as follows

$$EV(p_0, p_1) = a_0(p_0 - p_1) + a_1(p_0 - p_1)^2 + a_3(p_0 - p_1)^3$$

for the specific case we consider above, we can find the constants we want, to do this, we first choose the linear price path of the form $\pi(t) = p_1 + t(\Delta p)$ where Δp is $p_1 - p_0$. The expression for the EV now reduces to

$$EV(p_0, p_1) = \Delta p \int_0^1 q(\pi(t), y) \exp(-\alpha \Delta p) dt$$

We can now expand the integrand as

$$
\begin{aligned}
EV(p_0, p_1) &= \Delta p \int_0^1 q(\pi(t), y) \left[1 - \alpha \Delta p + \frac{1}{2}\alpha^2 (\Delta p)^2 ...\right] dt \\
&= \Delta p \int_0^1 q(\pi(t), y) - (\Delta p)^2 \int_0^1 \alpha q(p(t), y) + o(\Delta p)^3
\end{aligned}
$$

further notice that $D_y q = \alpha$ implies

$$
\int_0^1 \alpha q(p(t), y) = \int_0^1 q(\pi(t), y) D_y q(\pi(t), y) = \int_0^1 D_y (q(\pi(t), y))^2
$$

which is interesting because differentiating with respect to income does linearize to a large population, so on the aggregate we can say that

$$
\mathbb{E}\left[\int_0^1 D_y (q(\pi(t), y))^2\right] = D_y \int_0^1 \mathbb{E}[(q(\pi(t), y]))^2]dt
$$

where $\mathbb{E}[q(\pi(t), y)^2]$ is just the second moment which is observable with aggregate data. This elads to the following formula

$$
\begin{aligned}
EV(p_0, p_1) &= \int_0^1 q(\pi(t), y) - \frac{(\Delta p)^2}{2} \int_0^1 D_y \mathbb{E}[(q(\pi(t), y]))^2]dt \\
&= CS(p_0, p_1) - (\Delta p)^2 \int_0^1 D_y M_2(p(t)) + o(\Delta p)^3
\end{aligned} \tag{9}
$$

We show that even if this setup is generalized to allow for unrestricted assumptions on income effects, the second order term is identified by the second moment of demand.

5.2 Carleman linearization

We now generalize these insights in a unified framework. Suppose that individual demand is analytic in income such that it can be expanded as follows around $t = 1$

$$
q^\omega(p(t), y - s(t)) = \sum_{k=0}^{\infty} a_k^\omega(t)(s(t))^k.
$$

Note that the function $a_k^\omega(t)$, which can vary very flexible with prices, can be recovered by taking the kth derivative with respect to income, evaluated in $t = 1$

$$
a_k^\omega(t) = D_y^k q^\omega(p(t), y - s(t))|_{t=1}.
$$

The expectations of these function are identified from the data

$$\mathbb{E}[a_k^\omega(t)] = D_y^k \mathbb{E}[q^\omega(p(t), y - s(t))]|_{t=1}.$$

Plugging this in the FOD equation in Expression (8), we can write that

$$D_t s(t) = -\sum_{k=0}^{\infty} a_k^\omega(t)(s(t))^k.$$

Note that it holds that

$$\begin{aligned} D_t(s(t))^k &= k(s(t))^{k-1} D_t s(t) \\ &= k(s(t))^{k-1} \sum_{l=0}^{n} a_k(t)(s(t))^k \\ &= k \sum_{l=0}^{n} a_l(t)(s(t))^{l+k-1}. \end{aligned}$$

For notational convenience, define $s_k(t) = s(t)^k$ for all k. This gives us the (potentially) infinitely dimensional system of linear FOD equations,

$$D_t \begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \\ s_3(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ a_0(t) & a_1(t) & a_2(t) & a_3(t) & \dots \\ 0 & 2a_0(t) & 2a_1(t) & 2a_2(t) & \dots \\ 0 & 0 & 3a_0(t) & 3a_1(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \\ s_3(t) \\ \vdots \end{bmatrix},$$

or more compactly

$$D_t \mathbf{s}(t) = \mathbf{A}(t)\mathbf{s}(t).$$

The approach represents the single nonlinear FOD equation by an infinite number of linear FOD equations. This representation is known as Carleman linearization.[17] Linearization techniques are instrumental in studying what can be recovered from cross-sectional data, as its results aggregate well across the population.

Remark 6. Note that the approximation of Hausman and Newey (2016) to the Hausman-Vartia differential equation can be seen as a special case of our approach. Let $q^\omega(p(t), y -$

[17]See Kowalski and Steeb (1991) for a detailed technical overview on Carleman linearization.

$s(t)) = a_0^\omega(t) + a_1 s(t)$ and consider the two-dimensional subsystem

$$D_t \begin{bmatrix} s_0(t) \\ s_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a_0^\omega(t) & a_1 \end{bmatrix} \begin{bmatrix} s_0(t) \\ s_1(t) \end{bmatrix}.$$

Since we have that that $s_0(t) = 1$, this system simplifies to the single first-order differential equation

$$D_t s_1(t) = a_0^\omega(t) + a_1 s_1(t),$$

which has the explicit solution

$$s_1(t) = -\exp(a_1 t) \int_t^1 \exp(-a_1 \tau) a_0^\omega(\tau) d\tau,$$

such that

$$s(0) = s_1(0) = -\int_0^1 \exp(-a_1 \tau) q^\omega(p(\tau), y) d\tau.$$

In fact, we can do away with the assumption that the income effect is constant across individuals and prices. Let $q^\omega(p(t), y - s(t)) = a_0^\omega(t) + a_1^\omega(t) s(t)$. Analogous arguments as in Remark 6 give

$$s(0) = s_1(0) = -\exp\left(-\int_0^1 a_1^\omega(\tau) d\tau\right) \int_0^1 \exp\left(\int_\tau^1 a_1^\omega(\sigma) d\sigma\right) q^\omega(p(\tau), y) d\tau.$$

Comparison with Hausman and Newey (2016) We now have a brief discussion comparing our results to the non identification results provided in Hausman and Newey (2016). They use known uniform bounds on income effects to bound average surplus using average demand. In particular, it is shown that for known constants $\underline{b}$ and $\bar{b}$ such that $\underline{b} \le D_y q^\omega(p, y) \Delta p \le \bar{b}$, it holds that $e_{\bar{b}} \le \mathbb{E}[EV^\omega] \le e_{\underline{b}}$ with

$$e_b = \Delta p \int_0^1 \mathbb{E}[q^\omega(p_0 + t\Delta p, y)] \exp(-b \Delta p t) dt.$$

This can be written as

$$\Delta p \int_0^1 \mathbb{E}[q^\omega(b_0) + D_p q(b_0) t \Delta p, y)](1 - b \Delta p t) dt$$

and collecting first and second order terms, we get

$$\Delta p \int_0^1 \mathbb{E}[q^\omega(b_0)] + \Delta p (D_p q(b_0) - b q^\omega(b_0)) t dt \approx \Delta p \mathbb{E}[q^\omega(b_0)] + \frac{(\Delta p)^2}{2} (D_p q(b_0) - b q^\omega(b_0))$$

such that aggregating over individuals gives us

$$\Delta p M_1 + \frac{(\Delta p)^2}{2}(D_p M_1(b_0) - bM_1(b_0))$$

which should be corrected to

$$\frac{(\Delta p)^2}{2}(D_p M_1(b_0) + \frac{1}{2} D_y M_2(b_0))$$

to account for the second-order effects. This gives us the "correction" term

$$\frac{(\Delta p)^2}{2}(bM_1(b_0) - \frac{1}{2} D_y M_2(b_0)).$$

This immediately leads to the following proposition, which improves the bounds of Hausman and Newey (2016) for the expected EV at one side (in most cases this will be the lower bound).

Proposition 2. Let the income effect be bounded, $\underline{b} \leq D_y q^\omega(p, y)\Delta p \leq \overline{b}$, for known constants $\underline{b}$ and $\overline{b}$. The average EV is bounded as follows:

$$e_{\overline{b}} + \max\left\{0, \frac{(\Delta p)^2}{2}(bM_1(b_0) - \frac{1}{2} D_y M_2(b_0))\right\}$$
$$\leq \mathbb{E}[EV^\omega] \leq$$
$$e_{\underline{b}} + \min\left\{0, \frac{(\Delta p)^2}{2}(bM_1(b_0) - \frac{1}{2} D_y M_2(b_0))\right\}$$

The following example illustrates the significant improvement that can be achieved when using the information from the higher conditional moments.

Example 2. Again consider the linear specification from Proposition 1. The first three conditional moments of this model are given by:

$$M_1(b) = 1/2 - p + 1/2y$$
$$M_2(b) = 1/3 + p^2 - p + 5/18y^2 + 1/2y - py$$
$$M_3(b) = 1/4 - p^3 + 3/2p^2 - p + 1/6y^3 + 5/12y^2 + 1/2y + 3/2p^2y - 5/6py^2 + 3/2py.$$

For a price change from $p_0 = 0.10$ to $p_1 = 0.11$ and at income $\overline{y} = 3/4$ we calculate the average consumer surplus.

Suppose the econometrician knows that the income effects are uniformly bounded as between minus one and one: $-0.1 = \underline{b} \leq D_y q^\omega(p, y)\Delta p \leq \overline{b} = 0.1$. Then the income effect bounds are as follows:

$$7.328\ 10^{-3} \leq \mathbb{E}[EV^{\omega}(0.1, 0.2, 0.75)] \leq 7.810\ 10^{-3}.$$

In contrast, our second-order approximation gives

$$\mathbb{E}[EV^{\omega}(0.1, 0.2, 0.75)] \approx 7.720\ 10^{-3}.$$

Higher moments Suppose we are looking for the second moment, under the same assumptions as Hausman and Newey (2016). We can do this by studying the solution of

$$D_t \begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_0(t) & a_1 & 0 \\ 0 & 2a_0(t) & 2a_1 \end{bmatrix} \begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \end{bmatrix}.$$

[TBA]

6 Conditional moments and rationality

In this section, we study rationality of a population of consumers and the conditional moments of demand.

In the case for where the analyst can observe not conditional moments but conditional quantile demand functions, this problem has been studied by Dette, Hoderlein, and Neumeyer (2016) and Hausman and Newey (2016). We expand the analysis by considering moments instead of quantiles. For clarity of exposition, we first consider the two-goods case, as the notation for the many-goods case is more involved.

6.1 Two-goods case

In settings with only two goods, it suffices to consider scalar demand, as one of both goods can be interpreted as the numeraire.

Claim (Hurwicz and Uzawa (1971))**.** Let $q^{\omega}(b)$ be a demand function, symmetry and negative semidefinitess of the slutsky matrix is necessary and sufficient for a demand function to locally be generated by utility maximization.

Further, because we are in the 2 good case, can we can treat demand as a scalar, symmetry always holds. Therefore the only observable restriction on demand q is that

$$D_p q^\omega(b) + q^\omega(b) D_y q^\omega(b) < 0 \tag{10}$$

for every preference type and budget set.

We now try to convert this expression and write it in terms of the conditional moments,in order to do this we use a technique we repeatedly exploit in this paper. Notice that multiplying both sides of the above equation by $q^\omega(b)^{n-1}$ for some $n \in \mathbb{N}_+$, we get that

$$\begin{aligned} 0 &\geq q^\omega(b)^{n-1} D_p q^\omega(b) + q^\omega(b)^n D_y q^\omega(b) \\ &= n^{-1} D_p q^\omega(b)^n + (n+1)^{-1} D_y q^\omega(b)^{n+1}. \end{aligned} \tag{11}$$

Interestingly, this expression can be written as a sum of the derivatives of conditional moments, allowing us to integrate over all preference types.

$$\begin{aligned} 0 &\geq \int \left[n^{-1} D_p q^\omega(b)^n + (n+1)^{-1} D_y q^\omega(b)^{n+1} \right] dF(\omega) \\ &= n^{-1} \int D_p q^\omega(b)^n dF(\omega) + (n+1)^{-1} \int D_y q^\omega(b)^{n+1} dF(\omega) \\ &= n^{-1} D_p M_n(b) + (n+1)^{-1} D_y M_{n+1}(b), \end{aligned} \tag{12}$$

where the inequality follows from the Slutsky equation being pointwise negative, the first equality is due to the linearity of the integral operator, and the second equality follows from changing the order of integration and differentiation and the definition of the conditional moments.[18]

This gives us a restriction for every two consecutive moments in our sequence. However, for a demand system to be rationalizable, the slutsky term must be negative *at each quantile*. (Dette, Hoderlein, and Neumeyer (2016)). Luckily, we can appeal to the density of polynomials in $\mathcal{C}^0(\mathbb{R})$ to prove point-wise negativity of the slustky term. However for this, we first need introduce some notation.

Definition 1 (Mononial translation)**.** For any monomial ξ^n, define its translation as

$$\begin{aligned} \Gamma(\xi^n)(b) &= \int_{\omega \in \Omega} \left[D_p q^\omega + q^\omega D_y q^\omega \right] \xi^n \\ &= (n+1)^{-1} D_p M_{n+1} + (n+2)^{-1} D_y M_{n+2} \end{aligned} \tag{13}$$

By the discussion we have had above, any monomial translation must be negative. This now allow us to state our first theorem characterizing rationalizable moment sequences.

[18]We assume the conditions for the dominated convergence theorem hold.

Definition 2 (Polynomial Translation)**.** For any $\pi(b) = \sum a_i x^i$ define its translation by breaking it up into monomials and adding back the results

$$\Gamma(\pi(b)) = \sum a_i \Gamma(x^i)$$

Theorem 2. *In the case of two goods, a demand distribution can be generated by a rational population if and only if any polynomial which is positive in the support of the distribution has a negative translation.*

Proof. The if part simply follows from any polynomial transformation being a sum of monomial transformations, thus requiring negativity. For the only if part, Hausman and Newey (2016) shows that negativity of the quantile demand function characterizes rationalizability. Now suppose our condition held, but some quantile contradicted negativity.

This would mean that there is some quantile $\overline{\tau}$, and some quantile demand $\widetilde{q}(\overline{\tau} \mid x)$ such that

$$D_p\widetilde{q}(\overline{\tau} \mid b) + \widetilde{q}(\overline{\tau} \mid b)D_y\widetilde{q}(\overline{\tau} \mid b) > 0$$

We can pick a sequence of polynomials $\{\pi_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \{\pi_n\} \to \delta_{b(\overline{\tau}|p,I)}.$$

Therefore, we have that

$$\lim_{n\to\infty} \{\Gamma(\pi_n)\} \to [D_p\widetilde{q}(\overline{\tau} \mid b) + \widetilde{q}(\overline{\tau} \mid b)D_y\widetilde{q}(\overline{\tau} \mid b)] > 0$$

which means at some finite n negativity must contradicted. This would contradict our assumption, hence proving the above theorem. □

A semi-decidable test Let $\mathbb{Q}[X]$ be the set of polynomials over the rational numbers. Further define

$$\mathbb{Q}_+[X] = \{p \in \mathbb{Q}[X] \quad | \quad x \geq 0 \implies p(x) \geq 0\}$$

Or simply the set of all polynomials which are positive in the positive x axis. We know from basic analysis that $\mathbb{Q}[X]$ and hence $\mathbb{Q}_+[X]$ must be countable. Therefore, we can pick an enumeration of $\mathbb{Q}_+[X]$ refer to it as $\{p_i\}_{i=1}^{\infty}$.

Now conduct the following procedure

1. At step n compute the the polynomial translation $\Gamma(p_n)$. If $\Gamma < 0$ move to step (n+1)

2. If $\Gamma > 0$ reject the distribution.

Theorem 3. *No rationalizable distribution is ever rejected and all non rationalizable distributions are eventually rejected.*

Proof. The first part follows trivially from the argument in the previous section and rationality implies negativity of all translations. For the second part, again by the above argument the system is not rationalizable, there is some polynomial p which has a positive translation, as $\{p_i\}_{i=1}^{\infty}$ is indeed an enumeration there must be some n where p_n has a positive translation, leading to rejection. □

The case with three moments Firstly observe that the restrictions are only on monomial translations of conditional moments. In the case where only the zeroeth and first monomial translation can be computed(or equivalently the first 3 moments can be observed) we only have to deal with linear polynomials which makes testing for rationality much simpler.

Theorem 4. *Let the support of demand at price p and income y be* $0 \leq q_{min} \leq q_{max} \leq \frac{I}{p}$ *in terms of the first 2 translations, only 4 polynomials need be checked for negativity, these are:-*

$$\begin{aligned} 1 \\ x \\ -q_{min} + x \\ q_{max} - x \end{aligned}$$

in our langauge this tranlates to

$$\begin{aligned} \Gamma(0) < 0 \\ \Gamma(1) < 0 \\ -q_{min}\Gamma(0) + \Gamma(1) \leq 0 \\ q_{max}\Gamma(0) - \Gamma(1) \leq 0 \end{aligned}$$

meaning that in additon to monomial negativity we only need to check

$$q_{max}\Gamma(0) \leq \Gamma(1) \leq q_{min}\Gamma(0)$$

6.2 Many-goods case

We now consider the case where we have multiple goods, in this case there are two distinct observable restrictions of rationality.

Claim. Let $\mathbf{q}^{\omega}(\mathbf{b})$ be a function which satisfies homogeneity and Walras' law it is (locally) rationalizable if and only if the matrix $\mathbf{S}(\mathbf{b}) = D_p\mathbf{q}^{\omega}(\mathbf{b}) + \mathbf{q}^{\omega}(\mathbf{b})(D_y\mathbf{q}^{\omega}(\mathbf{b}))^{\intercal}$ is symmetric and negative semidefinite.

Notice that symmetry means that

$$D_{p_j} q_i^\omega(\mathbf{b}) + q_i^\omega(\mathbf{b}) D_y q_i^\omega(\mathbf{b}) = D_{p_i} q_j^\omega(\mathbf{b}) + q_j^\omega(\mathbf{b}) D_y q_j^\omega(\mathbf{b}).$$

Now let us consider the variance of demand, $\mathbf{M}_2(\mathbf{b})$, in which the (i,j)th element is $\mathbf{E}[q_i^\omega(\mathbf{b})q_j^\omega(\mathbf{b}) \mid \mathbf{b}]$. However, $\mathbf{E}[q_i^\omega(\mathbf{b})q_j^\omega(\mathbf{b}) \mid \mathbf{b}] = \mathbf{E}[q_j^\omega(\mathbf{b})q_i^\omega(\mathbf{b}) \mid \mathbf{b}]$ or in other words, the moments of a random vector are symmetric. This symmetry, represents a loss of "degrees of freedom" available to the analyst.

Claim. Consider a sequence with the first two moments, $\{\mathbf{M}_1(\mathbf{b}), \mathbf{M}_2(\mathbf{b})\}$, in contrast with the two-goods case, the matrix $\mathbb{E}[\mathbf{S}(\mathbf{b}) \mid (\mathbf{b})]$ is not point identified.

Proof. The non-identification hinges on the second term $\mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal$ because indeed the first term is identified. But as we discussed above the analyst observes $\mathbf{M}_2(\mathbf{b})$ or equivalently $\mathbb{E}\left[\mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal + D_y\mathbf{q}^\omega(\mathbf{b})\mathbf{q}^\omega(\mathbf{b})^\intercal\right]$ if there are two demand models where

$$\forall\omega \in \Omega \quad \mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal + D_y\mathbf{q}^\omega(\mathbf{b})(\mathbf{q}^\omega(\mathbf{b}))^\intercal$$

is identical, they would generate the same two moments. □

This shows that if we remain agnostic about rationality, the Slutsky equation is not identified from the first two moments of demand. However, interestingly, if we assume that individuals satisfy Slutsky symmetry, this exactly identifies the Slustky terms, giving us just enough restrictions. This leads to the following theorem.

Theorem 5. *Consider a sequence with the first two moments,* $\{\mathbf{M}_1(\mathbf{b}), \mathbf{M}_2(\mathbf{b})\}$. *If individuals obey Slutsky symmetry, there always exists a demand system such that the matrix* $\mathbb{E}[\mathbf{S}(\mathbf{b}) \mid \mathbf{b}]$ *is point identified.*

Proof. The proof proceeds very simply. We know that

$$D_y\mathbf{M}_2(\mathbf{b}) = \mathbb{E}\left[\mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal + D_y\mathbf{q}^\omega(\mathbf{b})\mathbf{q}^\omega(\mathbf{b})^\intercal\right],$$

we also know from Slutsky symmetry that

$$D_p\mathbf{q}^\omega(\mathbf{b}) + \mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal = D_p\mathbf{q}^\omega(\mathbf{b}) + D_y\mathbf{q}^\omega(\mathbf{b})(\mathbf{q}^\omega(\mathbf{b}))^\intercal,$$

which implies that

$$\mathbb{E}[D_p\mathbf{q}^\omega(\mathbf{b}) - (D_p\mathbf{q}^\omega(\mathbf{b}))^\intercal] = \mathbb{E}[\mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal - D_y\mathbf{q}^\omega(\mathbf{b})(\mathbf{q}^\omega(\mathbf{b}))^\intercal].$$

Adding fetches us

$$\mathbb{E}[D_p\mathbf{q}^\omega(\mathbf{b}) - (D_p\mathbf{q}^\omega(\mathbf{b}))^\intercal] + D_y\mathbf{M}_2(\mathbf{b}) = 2\mathbb{E}[\mathbf{q}^\omega(\mathbf{b})(D_y\mathbf{q}^\omega(\mathbf{b}))^\intercal].$$

adding this to the fact that $\mathbb{E}\left[D_p\mathbf{q}^\omega(\mathbf{b})\right]$ is identified gives us the Slutsky matrix. □

Remark 7. Specifically, this means that symmetry is untestable from the first two moments, because there is always a symmetric demand model which rationalizes the first two moments. Also,

$$\mathbb{E}\left[\mathbf{S}(\mathbf{b}) \mid \mathbf{b}\right] = \frac{1}{2}\left[D_p\mathbf{M}_1(\mathbf{b}) + (D_p\mathbf{M}_1(\mathbf{b}))^\intercal + D_y\mathbf{M}_2(\mathbf{b})\right]$$

However, we can test for negative semi-definiteness of the population, this gives us the following lemma.

Lemma 3. Consider a sequence with the first two moments, $\{\mathbf{M}_1(\mathbf{b}), \mathbf{M}_2(\mathbf{b})\}$. If this is rationalizable

$$\mathbf{P}(\mathbf{b}) = D_p\mathbf{M}_1(\mathbf{b}) + \frac{1}{2}D_y\mathbf{M}_2(\mathbf{b})$$

must be NSD.

Proof. This follows because the average Slutsky matrix we identified above must by NSD and $\mathbf{P}(\mathbf{b}) + \mathbf{P}(\mathbf{b})^\intercal = \mathbf{S}(\mathbf{b})$. Which means for the slutsky matrix to be NSD this must be too. □

Higher moments Much like in the two-goods case, we have very similar restrictions for higher moments. The difference is that the monomial translation for any moment is now a tensor form.

Theorem 6. *For any $n > 0$ the following $n+1$ tensor form is negative semidefinite.*

$$n^{-1}D_p\mathbf{M}_n + (n+1)^{-1}D_y\mathbf{M}_{n+1}$$

Notice that the form is $n+1$ because differentiating a k form with respect to price increases the order of the form.

Remark 8 (magnitude independence). Notice that the decomposition does not depend on the exact magnitude of the variance but only its rate of change. This independence leads to 2 fundamental properties.

1. If there is any additively separable error in observations of demand data that we use to compute the variance, there is no effect on the restrictions. This observation provides a potential reason for using these restrictions even if micro-level data is available.
2. None of the results in this paper depends on "positivity" or the constraints that arise from the variance being close to zero and Chebyshev type tail inequalities.

Remark 9 (openness). Because our tests are simply tests of negative semidefinitess and not of symmetry, any small perturbation of a finite and rationalizable moment sequence is itself finite and rationalizable. This is in stark contrast to symmetry.

6.3 Statistical constraints

[TBA]

7 Estimation

[TBA]

8 Empirical application

[TBA]

9 Concluding remarks

We introduce a novel methodology to compute and study the welfare changes which are caused by changes in price. In order to do this, we introduce the moments of cross-sectional demand as useful estimands to conduct counterfactual exercises in applied welfare analysis.

Furthermore, we demonstrate that these moments can also be used to test rationality of aggregate data. In specific it can be used to confirm if individuals obey Slutsky symmetry.

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Appendix

"Individual Heterogeneity and the Distribution of Welfare"

A Regularity conditions

Demand $\mathbf{q}^\omega$ needs to be infinitely differentiable in $\mathbf{b}$ for all $\mathbf{b} \in \mathcal{P} \times \mathcal{Y}$. This is ensured by the following condition.

Assumption A.1. Individuals' preferences are continuous, strictly convex, and locally non-satiated. The associated utility functions u^ω are infinitely differentiable everywhere.

The following condition ensures that the dominated convergence theorem holds. This allows us to interchange limits and integrals.

Assumption A.2. There exists a function g such that for all $\mathbf{b} \in \mathcal{P} \times \mathcal{Y}$ and $n, m \in \mathbb{N}$ it holds that $\|\text{vec}(D_{p^n,y^m}\mathbf{q}^\omega(\mathbf{b}))\| \leq g(\omega)$ with $\int g(\omega)dF(\omega) < \infty$.

Finally, we require that all moments exist.

Assumption A.3. For all $n \in \mathbb{N}$, it holds that

$$\mathbb{E}\left[\left|\left(\bigotimes_{k=1}^{n}\mathbf{q}^\omega(\mathbf{b})\right)\right|\right] < \infty.$$

B Mean demand and the income effect

We first give a brief account of the disappearance of restrictions that apply to individual demand on aggregation. It is well known from Slutsky (1915) that a demand function arises from utility maximization if and only if it satisfies negative semi-definiteness and symmetry of the substitution matrix.

However, these restrictions do not survive when there is a population of individuals. To demonstrate this, we describe a theorem from Geanakoplos and Polemarchakis (1980) about testable restrictions "at a point".

Theorem 7. *Let the analyst observe an individual demand x(p,I) and the Jacobian of x, call it J(p,I) at one price income pair (p,I), this Jacobian and demand is rationalizable if and* *only if there exists a vector $v \in \mathbb{R}^l$ and a matrix $\mathcal{K} \in \mathbb{R}^{n^2}$ such that.*

$$J = \mathcal{K} - vx^{\intercal}$$

and

1. *$\mathcal{K}(p, I)$ is symmetric and negative semidefinite.*
2. *$\mathcal{K}(p, I)$ has rank (l-1) and $p\mathcal{K} = \mathcal{K}p = 0$*

Refer to the subspace of vectors orthogonal to $x(p, I)$ as $[x(p, I)]^{\perp}$, And the Jacobian of x as $\mathbf{D}_p x(p, I)$

The above theorem can be restated by as can be restated as follows

$$\forall V \in [x(p, I)]^{\perp}, \quad V^{\intercal}\mathbf{D}_p x(p, I)V \leq 0$$

and

$$\forall V \in [x(p, I), p]^{\perp}, \quad V^{\intercal}\mathbf{D}_p x(p, I)V < 0$$

The problem is that when we sum the demands of individuals, the individual demands need not be co-linear. Thus, the subspaces where the Jacobian is negative semi-definite are different. Given enough individuals, the intersection of these subspaces may be empty. Formally,

$$\bigcap_{i=n}^{n}[q^{\omega}(p, I)]^{\perp} = \phi$$

This problem causes the structure rationality places on individual demand to break down when aggregated.

In this vein, the following result was first demonstrated by Sonnenschein (1973), and then generalized by Diewert (1977) and Mantel (1975).

Theorem 8. *Let $\hat{X}(p, I)$ be any function that satisfies Walras' law and homogeneity. At a point $(\overline{p}, \overline{I})$ it "looks like" a mean demand function, meaning, there exist l individual, rationalizable demand functions $(x_1(p, I)...x_l(p, I))$, such that:-*

1. $\frac{1}{n}\sum x(\overline{p}, \overline{I}) = \hat{X}(\overline{p}, \overline{I})$
2. $\frac{1}{n}\sum \mathbf{D}_p q^{\omega}(p) = \mathbf{D}_p\hat{X}(\overline{p}, \overline{I})$

This theorem encapsulates the spirit of what we referred to in our intro as SMD theory. It shows that an analyst observing market demand "at a point" can never falsify the hypothesis of utility maximization. This problem occurs mainly because of the misbehaviour of "income effects," as all observable restrictions are placed on the substitution matrix. Andreu (1983) then showed that abstracting from non-negativity considerations allows one to extend the same result to finite price demand data.

We state one final result from Chiappori and Ekeland (1999a) which significantly generalizes this theorem to a small open set around a point; however, this requires that the observed mean demand to be analytic.

Theorem 9. *Consider some open set $\mathcal{U} \in \mathbf{R}^l_+ \times \mathbf{R}$ and an analytic mapping $\hat{X} : \mathcal{U} \to \mathbf{R}^l_+$ which satisfies Walras' law and homogeneity. For all $(\overline{p}, \overline{I}) in \mathcal{U}$ there exist n rationalizable individual demand function $(x_1, ... x_n)$ such that:-*

$$\sum q^{\omega}(\overline{p}, \overline{I}) = \hat{X}(\overline{p}, \overline{I})$$

for all p In some convex neighbourhood $\mathcal{V}$ of $\overline{p}$

This last theorem ends our short review of the main negative results in the specific market demand case we consider. We now state our main results and show how the (sample) variance rids us of the income effect problem.

B.1 Results for a finite population

Claim. Suppose there are finitely many people $\{1, ...n\}$ and demand functions $\{x_1, ...x_n\}$
Let $\mu(p, I) = \frac{1}{n} \sum_{i=1}^{n} x_n(p, I)$ and
$\sigma^2(p, I) = \frac{1}{n} \sum_{i=1}^{n} x_n(p, I) x_n^{\intercal}(p, I) - \mu(p, I)\mu(p, I)^{\intercal}$. The following relationship holds

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{D}_p h(p, u) = \frac{1}{2} \left[\left\{ \mathbf{D}_p \mu + \frac{\partial \mu}{\partial I} \mu^{\intercal} \right\} + \left\{ \mathbf{D}_p \mu + \frac{\partial \mu}{\partial I} \mu^{\intercal} \right\}^{\intercal} \right] + \frac{1}{2} \frac{\partial \sigma^2}{\partial I}$$

This claim shows the simplest version of our results because the left hand side matrix can be tested for negative semi definiteness and rank.