Product Reallocation and Market Concentration

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Abstract

This paper connects the transfer of product ownership across firms to product market concentration and efficiency. We employ a dataset of the universe of products across firms to document new facts central to the ownership of products: Consumer markets are dominated by large firms and old brands; There is a lot of exchange of products across firms, brands are transacted when they are older and have more sales, yet brands are becoming younger at transaction over time; Brand transactions are followed by increases in the market share and price of the transferred products. These facts point to how brands are distinct from technologies in that they build consumer goodwill. While singular technological advancements depreciate in value over time, the value of a brand can grow with a rising customer base over time. We embed this intuition in a model that contains multi-product firms that hold gradually maturing brands. Firms grow by either creating new products, holding maturing products, or purchasing products from competitors. When firms expand their product scope, there is both an increase in their market share and the markup charged for their products. The life-cycle of brands is a critical component of the efficiency implications of product exchange. In markets with fast product churn, consolidation is less costly due to the entry of new products. In markets where old brands dominate, the costs of consolidation are higher because of the lag of product maturity.

Key Words: Concentration, Markups, Products, Firm Dynamics, Reallocation

JEL Code: D22, D43, L11, L13, L22

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1 Introduction

Large firms are multi-product in nature. The product scope of firms explains a large share of sales variation across firms (Hottman et al., 2016) and creation of new products is important for growth and welfare (Romer, 1990; Grossman and Helpman, 1991a). While the literature has documented firm growth through the product creation and destruction margin (Argente et al., 2020), less is known regarding the reallocation of existing products across firms. With the rising interest in product market concentration and its implications on welfare, a careful analysis of this reallocation process is needed. This project documents the empirical patterns of brand reallocation in the US, and quantify its welfare impacts through the lens of an endogenous growth model with product creation and firm dynamics.

To do so, we make three distinct contributions. First, we integrate two datasets that are important to understand branding activity: USPTO Trademark data (to study product creation and transfer) and Nielsen retail scanner data (to study prices and quantities of products). We use these datasets to document persistent facts relating products and firms over their life-cycles. Second, we introduce a new endogenous growth model with variable markups and product reallocation across firms, differentiating the role of individual products and the holding firms. Some strong firms hold weak products, and vice versa. In general, this connects to the fact that product reallocation across firms plays a crucial role in how the market for brands develops. Third, we quantify the model with the micro-data on products to understand the role of reallocation across firms and optimal anti-trust and innovation policies.

Our findings inform an important and growing debate in macroeconomics on the role of concentration, markups, and innovation. The paper stresses that understanding the life-cycle of products has important implications for optimal anti-trust activity. Integrating the brand life-cycle with brand transactions connects the *stock* of products, which make up an overwhelming share of sales, to the overall efficiency of product markets and their growth. Due to the product life-cycle, product innovation becomes only the tip of the iceberg.

We motivate our analysis with three facts about firms and products from USPTO Trademark data and Nielsen Scanner data. Each dataset provides unique ingredients into our analysis of how brands move across firms and the role of large firms in each market.

Fact 1 *Trademark and consumer product markets are dominated by large firms; these firms accumulate products through transactions of brand ownership.*

Fact 2 *Brands are dynamic; few products survive, but those that mature are more likely to be transacted and persistently make up a large share of sales in the market. Yet, brands are entering the exchange market at younger ages over time.*

Fact 3 *Product transactions are more likely to go from small to large firms; upon transaction, sales and prices both increase.*

Few firms dominate product markets (see Fact 1). These firms build their portfolios through both product development and product acquisition (also see Fact 1). They tend to purchase products that exhibit strong sales and the transaction rate increases in age (see Fact 2). Product sales growth and product purchases contribute to the reallocation of products across firms and product dynamism (also see Fact 2). When firms consolidate, markups increase, and this is particularly true for large firms (Fact 3).

We interpret these facts in a model of endogenous growth with product creation and evolution. In the model, firms grow by engaging in costly development of products that are imperfect substitutes to existing products, aggregated by two-layer constant elasticity of substitution (CES) utility function within and across product groups. Each product group has a dominant multi-product firm and many fringe firms with free entry. We highlight the difference between branding and process innovation through explicitly modeling the life-cycle of brands: (1) the initial quality of products are different, as some brands are better to start; (2) brand quality grows per unit of time, as brands accumulate its customer base; (3) brands can retire from the market with exogenous rate.

We allow firms to trade brands with each other. The group leaders choose their search efforts by paying a cost, which directly affects the selling rate of fringe brands. When brands are reallocated, the customer base they accumulated so far is entirely transferred to its new owner.

In our model, the fate of brands is distinguished from the fate of multi-product firms. A firm with different productivity can engage in faster or slower speed of innovation or acquisition. Firms engage in either price competition or quantity competition. The fringes charge a constant markup according to the within-group substitution elasticity, because they are small; The multi-product leader charges a variable markup that is increasing in its within-group market share. As a result, expanding product scope creates value to the firm through two margins: (1) they increase the total sales of the firm holding markups unchanged; (2) it enables firms to charge a higher markup on all products due to the reduced competition from their own.

We then understand the welfare implications through a planner's problem that aims to maximize the welfare of representative household. The existence of variable markups across firms in the decentralized equilibrium creates two wedges when compared to the planner. First, firms that are more productive or have large product scope charge higher markups and thus produce too little compared to the efficient allocation. We refer to this as the static distortion; Second, firms do not produce to an efficient scale after they create new products and thus under-innovate. We refer to this as the dynamic distortion. The reallocation of products interacts with these distortions and presents a tension on the trade-off between optimal allocation and market concentration.

With this model in hand, we ask: *How does the reallocation of products across firms affect efficiency of the economy?* We answer this question both analytically and quantitatively. In a special case where there is no firm-level fluctuation of productivity, we show the steady state of the economy is characterized by two relationships between quality and competition. First, a faster transaction rate from fringe firms to leaders leads to higher concentration of the market, we refer to this as *consolidation effect*. Second, a faster transaction rate leads to higher option value of selling for fringes, and thus encourages entry of fringe firms, which we refer to as *entry effect*. In net, the welfare impact is ambiguous. One important primitive that governs the comparison between the consolidation effect and the entry effect is the life-cycle of brands. When it is slower for new brands to accumulate customer base, the consolidation effect becomes stronger. In the limit, if the new brands are infinitely unappealing compared to existing ones, the entry effect is muted.

Guided by this analytical model, we find the key parameters that are relevant. The prediction depends on (1) the innovation elasticity, search elasticity, and entry elasticity, (2) substitution elasticity among varieties in the market and, (3) crucially, the life-cycle of brands, which is changing over time. Our empirical context allows us to identify these parameters well. First, we match the innovation and search elasticity to the correlation of buying rate and creation rate in data, and the entry elasticity to the average market share; Second, we take the substitution elasticity from the demand curve estimations in the literature; Lastly, we match the life-cycle of brands to the average market share of new brands and the sales and transaction evolution of products.

Through the lens of the quantified model, we ask the welfare incidence of product reallocation. We find this relationship to be non-monotonic. Discouraging transactions of products could lead to welfare gains locally, but can eventually decrease efficiency.

The paper is structured as follows. The rest of this section reviews the literature. Section 2 introduces the USPTO Trademark Dataset, Nielsen Scanner Data, and Refinitiv Merger & Acquisition data. Section 3 documents some key empirical facts that frame our investigation. Section 4 introduces a novel model of product creation and transfer with variable firm productivity and variable markups. Section 6 builds a bridge from the empirical facts to the model to investigate the role of shutting down specific channels of product exchange. We use this to study policy counterfactuals. Section 7 concludes.

Related Literature

This paper builds on and contributes to several literatures: the study of endogenous growth and firm dynamics, the study of multi-product firms and their implications, the study of the life-cycle of products, and the study of firm market power.

The introduction of new products is a central element of economic growth. This is a bedrock component of much of modern endogenous growth theory Romer (1990); Grossman and Helpman (1991a), as well as consumer welfare (Jaravel, 2018). Product creation has also been noted as a key empirical component of both economic growth and gains from trade Bils and Klenow (2001); Broda and Weinstein (2006). Further, the ability of individuals to exchange products allows for products to expand into new markets and may spur upstream innovation Eaton and Kortum (1996). We contribute to this literature by documenting facts of the reallocation of products across multi-product firms and its implications on welfare.

The quantitative model in this paper is based on the endogenous product creation model developed by Grossman and Helpman (1991b) and oligopolistic competition model by Atkeson and Burstein (2010). There is long theoretical literature linking markups to endogenous growth. Most of these models assume limit pricing in order to gain tractability, for example Peters (2020). A recent paper by (Liu et al., 2019) differs by considering a model with duopolistic competition. Having the welfare question and concentration in mind, our paper is related to (Liu et al., 2019) with endogenous markup dispersion. The markup dispersion in our framework

Intellectual property transfer plays an important role in the distribution of technologies and products across firms. Our paper is thus related to Akcigit et al. (2016), who study the effect of patent transfers on productivity growth, where the gains from trade in patent transfers come from matching firms to technologies. Shi and Hopenhayn (2017) study how the appropriability of innovation, e.g. the ability to license or sell intellectual property induces upstream incentives. This is related to Abrams et al. (2019), who illustrates the how middlemen in intellectual property transfers can have competing negative and positive effects. We focus this paper on the demand side by documenting facts of trademark transfers; this allows us to turn to variable markups depending on consumer demand and firm market power, adding intellectual property transfer to a framework developed in and Liu et al. (2019). This extension enables us to discuss the potential welfare loss of transfers due to the shifting dispersion of markups.

This paper connects the literature on economic growth and intellectual property to the study of multi-product firms. Hottman et al. (2016) study multi-product firms and find the scope of products explains a large share of sales variations across firms, while Berger et al. (2019) apply this reasoning to study the role of market power in the labor market. In this paper, the sources of market power come from oligopolistic competition across firms, which follows Atkeson and Burstein (2008).

Our paper relates to recent literature integrating the life-cycle of products into frameworks of economic growth. Argente et al. (2018, 2020) explore how product creation and destruction are pervasive in product markets. Argente et al. (2021) and Einav et al. (2021) document that the expansion of product sales is largely due to expansion of the customer base. Persistent brand preferences (Bronnenberg et al., 2012) and consumer growth naturally leads to a product life-cycle. Products are born, build a consumer base, and then live on those persistent preferences. As a result, a firm must incorporate their set of products into how they optimize (Dhingra, 2013). Gourio and Rudanko (2014) note how as a result, consumer goodwill is a relevant state variable for firms and products. Our current paper points out that when mature brands are transacted, this life-cycle element becomes essential for understanding concentration.

This paper applies insights from search theory to study the market for trademark exchange. Some previous work has stressed the importance of reallocation and labor market frictions in driving economic growth. For instance, Lentz and Mortensen (2008) apply a random search framework to uncover the importance of entry, exit, and reallocation in how labor markets interact with firm dynamics. This current paper considers the frictions in the market for intellectual property, applying competitive search theory as developed in Menzio and Shi (2011).

A discussion of the reallocation of products naturally connects to a rich empirical literature on firm dynamics. Many researchers have noted a declining reallocation in the economy. For example, the reallocation rate of jobs has been decreasing, and the entry and exit rate of firms has been decreasing (Decker et al., 2014, Davis and Haltiwanger, 2014, and Decker et al., 2020). Our reallocation measure follows the work of Davis and Haltiwanger (1992) and Davis et al. (1996). We show that even during recent periods of diminishing dynamism we also experience a rise in the reallocation of brands.

A lot of product reallocation is due to exchanges from small firms to large firms. This connects to work on rising concentration and markups which has been studied extensively, both empirically (Barkai, 2020; De Loecker et al., 2020; Traina, 2018) and theoretically (Edmond et al., 2018; Peters, 2020; Akcigit and Ates, 2021). Some papers have deployed detailed methods to focus on the transfer of products and firms. Cunningham et al. (2021) focus on killer acquisitions, where incumbents purchase small firms in order to keep concentration high. However, this does not match the observation that large firms pay high premiums and often deploy the products from the firms they buy David (2020). This current paper integrates these two viewpoints by informing new empirical facts and a theoretical perspective that links these facts to current hypotheses on product market concentration. Further, we connect this framework to time trends on brand evolution in the data and apply it to anti-trust policies. There are two recent papers discussing the role of anti-trust policies on growth, from the perspective of technology innovation (Cavenaile et al., 2021, Fons-Rosen et al., 2021). Our theoretical framework follows these in integrating the dynamic effects of transactions, but differs in both focus and mechanism. Our attention on product markets and the life-cycle of brands puts the development of products at the center. This induces an essential ingredient to our model which predicts the timing and nature of transactions. On the data side, we are able to connect these mechanisms to prices and sales at the product-level over time and around the time of transactions. This allows us to study at a more micro-level the effect of acquisitions.

Lastly, we extend a new literature on the role of trademarks in marketing and strategy to a macroeconomic context. Graham et al. (2013) provide a general overview of the dataset and provide insights about the uses of trademarks. Schautschick and Greenhalgh (2016), who document the importance of trademarks to firms, review other literature that confirms the growing recognition of the importance of trademarks. Dinlersoz et al. (2018) document the newly available USPTO bulk dataset on trademarks and document facts about trademarks over a firm's life cycle. Heath and Mace (2019) focus the role of trademarks in the strategic interaction of firms. Castaldi (2019) discusses the potential of this rich dataset in providing empirical analogs of a host of subjects in management research. Kost et al. (2019) introduce trademarks in the context of macroeconomics, focusing on markups through the lens of trademarks. In this current paper, we integrate trademarks to more common datasets in order to understand in more granular detail the distribution of products across firms.

2 Data

This paper uses two main datasets in order to track the creation, distribution, and prices and quantities of products. This section discusses the datasets and their unique contributions to our analysis. The most novel dataset in our analysis is from the US Patent and Trademark Office (USPTO) Trademark data. This carries with it details on brand creation, brand transfer, and cancellation.

To focus on the response of prices and quantities, we connect this firm-product level data to specific information on product prices and quantities sold by store in RMS Nielsen Scanner Data. The following two sections discuss the datasets in turn.

2.1 USPTO Trademark Data

USPTO Trademark data provides a unique and comprehensive insight into brand-building. Trademarks are a central and dynamic arena of the economy: firms register for trademarks whenever they want their brand protected.

In this paper, we direct attention to how trademark creation and exchange contribute to the growth of firms. When firms create new products, this is highly correlated with applications to protect the brand related to the product. Firms want to ensure that their consumer goodwill cannot be infringed by other firms. Further, when firms buy the rights to sell products from other firms, the trademark is reassigned across firms.

To register for a trademark, a firm must undergo the following process. First, an individual who applies must pay a fee that ranges from $225-$400. Within three months of filing, an examining attorney checks for compliance, and if the application is approved, it "publishes for opposition." After this, there is a 30-day period during which third parties affected by the trademark registration can step forward to file an "Opposition Proceeding" to stop the registration. This process is again evaluated by an examiner. If it clears this process, the trademark is registered.

With a registered trademark in hand, the owner now has exclusive rights to use the mark within the sphere of activity designated in the process. The main principle underlying trademark law is to minimize consumer confusion. If consumer confusion is possible, the trademark owner has a case against infringers. However, one can still petition to cancel a trademark and end the exclusive rights of the owner. The petition to cancel often comes from competing firms that think the intellectual property is too broad. Trademarks are also canceled if firms are not actively using them. Cancellations are a significant share of overall trademark activity. In addition to registration and cancellation, firms exchange a large share of trademarks, which delivers the rights to brand and sell the product.

The USPTO Trademark data consists of more than 5.3 million unique trademark registrations since 1870. Table 1 provides summary statistics for the dataset. Overall, one million unique firms in our dataset have produced at least one trademark in the past. Lots of firms are active, but the median firm has only two trademarks.

One striking feature of the data noted in Table 1 is the number of cancellations and transactions.

Table 1: Summary statistics on Trademarks from USPTO

	Overall
# unique firms	1.35M
# unique registrations	5.36M
# unique transactions by bundle	915076
# unique transactions by ID	4.46M
# unique cancels	2.12M
99th percentile firm size	83
75th percentile firm size	5
Median firm size	1
Mean firm size	5

Note: Firm size is defined as the number of trademarks within a firm

This activity indicates that the market for trademarks is highly contested and dynamic. Cancels either require that other firms are concerned about the territory – many cancellations suggest a competitive market for accruing goodwill, or that a firm is not using its trademark. The contested aspect of the trademark market has been noted in prior literature as an important component of firm dynamics (Fosfuri and Giarratana, 2009). Kost et al. (2019) discuss the institutional aspects of trademarks in greater detail, and we later discuss these facts in Section 3.

2.2 Nielsen Scanner Data

The most comprehensive store-product level data comes from Kilts-Nielsen Retail Measurement Services Data from the University of Chicago Booth School of Business. The data is large and comprehensive in the consumer product space from years 2006-2018. We observe more than 100 billion observations at the product $\times$ store $\times$ time level. Product is defined by a UPC identifier, 12 digits that are uniquely assigned to each specific good. The store is defined at the local level with over 40,000 total; time is defined weekly. Total sales are approximately \$300 billion per year, covering around half of consumption in the consumer goods industry, which itself covers approximately 8% of total consumption in GDP.

The barcodes from UPC provide a unique identifier for each product. Changes in any attribute of a good corresponds to a new barcode. Barcodes are widespread and thus cover a large amount of the Consumer Packaged Goods Industry. However, the unique identifying feature of the barcodes may not be as relevant for our analysis. For instance,

A key departure from the literature in our case is identifying *brands* rather than *products*. There are three reasons for this. First, consumer goodwill tends to be brand rather than product-specific. Coke 12oz relies on the same core-branding that Coke 20oz relies on. Thus, when it comes to how the consumer interacts with the product, we think brand is a more core indicator. Second, when firms transact products, e.g. the right to sell a specific brand, they systematically transfer the full rights on the consumer goodwill, making the specific product differentiation within the brand less relevant. Third, our data enables identification at the brand-level in both Nielsen data and USPTO trademark data. Nielsen provides brand identifiers in addition to product identifiers. We collapse this information into brand sales by product group by year, with less geographical focus.

The volume represents over half of all transactions in grocery stores and drug stores, and slightly less than half in convenience and mass merchandise stores (Argente et al., 2020). We apply a dataset from GS1 US to link parent firms to products through UPCs. While this links to most parent companies, the trademark dataset helps complement this to ensure the correct company is allocated to the correct brand.

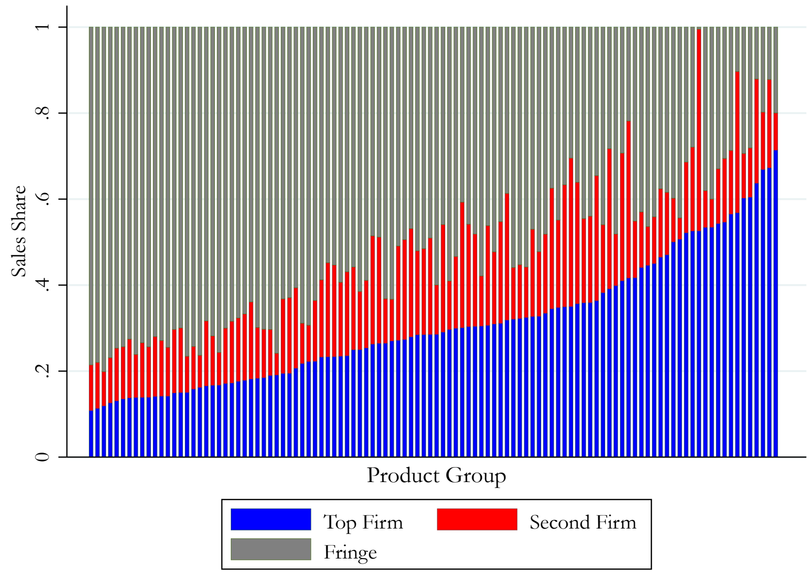
3 Empirical Facts

This section discusses the main empirical facts that frame our investigation. We focus primarily on how firms grow and decline through trademarks. We address the five facts discussed in the introduction. This includes the overall dynamism of the market, the role of reallocation in determining the firm size distribution, trends in reallocation and the market for trademarks, and the marginal effect of a trademark transaction.

Large Firms and Product Holdings

The trademark market is active, as many firms enter (register or buy their first trademark), exit (cancel or trade their last trademark). Once firms enter, they create and transact brands throughout their life-cycle. However, the distribution is skewed – few firms hold many brands. We first address this fact in detail.

Figure 1: Sales Share of Leader, by Product Group



Note: This figure shows the sales share by product group (ordered by % share of leader) in 2010.
Source: RMS Kilts-Nielsen Data Center & GS1 firm-product merge

While trademark markets are contested, they are still quite skewed and dominated by large firms. Once can see this more clearly in the Nielsen Scanner data.

Figure 1 maps out sales share of the product leader, the second firm, and the rest of firms in the market. This split by product group contains 116 unique product group categories (such as e.g. “ICE CREAM" or “BEER”). The average top firm share is 34% of the total market, though in many markets the top firm holds a significantly larger share. Thus, understanding how large and fringe firms interact is essential to understanding concentration.

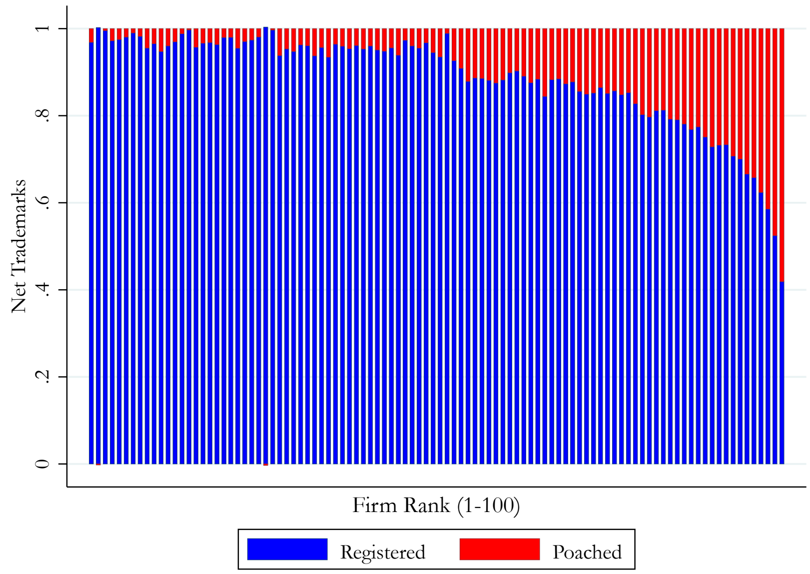
Further, Figure 2 shows through the USPTO Trademark data that large firms become large through transaction of products. For the largest firms, net buying contributes to about half of their portfolio. For the smallest firms, less than 10% of their portfolio comes from buying, as most small firms create their own brands.

These two graphs taken together lead to the first fact in our empirical discussion.

Fact 1 *Trademark and consumer product markets are dominated by large firms, who consolidate products through market transactions.*

We now turn to the dynamics of the product life-cycle to understand the drivers of the firm size distribution and pricing power at the firm.

Figure 2: Contribution of Net Poaching and Registration to Firm Size



Note: This figure shows the share of net poaching contribution to the stock of trademarks by firm size percentile.
Source: USPTO Trademark Data

Product Market Dynamism: Age, Sales, and Transactions

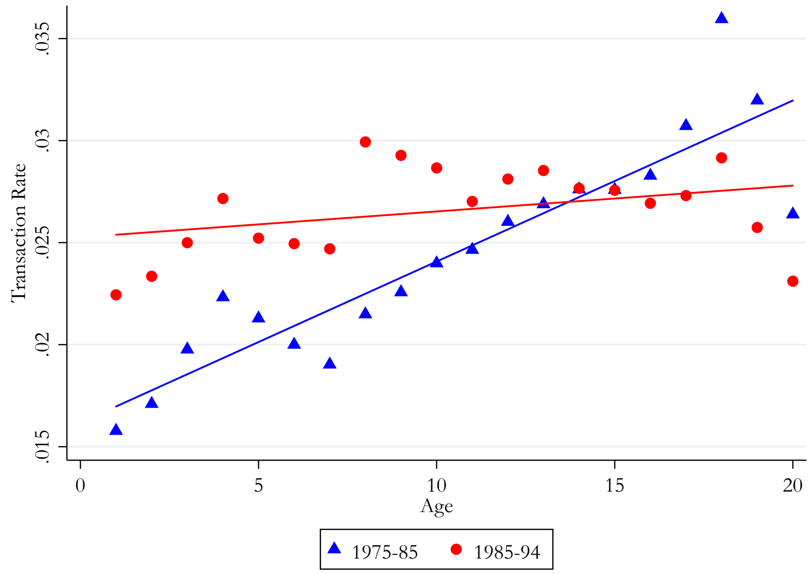
Products change over their life-cycle. Figure 3 documents the probability of a live trademark being exchanged in each year of its life from age 1-20, split by the period they were born in. Figure 3 illustrates how as products survive, they are more likely to be sold over time. This is becoming less true over time, however, as products born in 1985-1994 have a flatter age-transaction profile that trademarks born from 1975-1985.

One can also see the importance of old products when it comes to sales in Nielsen Scanner Data. Figure 4 the sales share of products from 2006-2017 and splits them by the brand cohort. For brands created before 2006, they maintain large sales share into the future. By 2018, these brands are less than 40% of the total brands but make up over 70% of the sales, suggesting that old brands are relevant not only for transactions, but also for taking up a large sales share of the market.

Lastly, Figure 5 illustrates the shifting dynamism of the market for trademark transactions. As time goes on, younger trademarks tend to be purchased, indicating that the life-cycle of brands are potentially changing. As we turn to the analytical framework, this shifting age profile may be relevant for product market transactions.

These three graphs lead us overall to Fact 2, which focuses on the nature of product markets.

Figure 3: Transaction Rate by Age, Period



Fact 2 *Products are dynamic; few products survive, but those that mature are more likely to be transacted and persistently make up a large share of sales in the market. Yet, brands are entering the exchange market at younger ages over time.*

Event Study: Product Transactions and Market Response

The life-cycle of products plays an essential role in the propensity of products to be transacted and the market power of firms. Events move products mostly from small to large firms on almost 70% of the transactions. An essential question related to efficiency immediately arises: how do prices and sales respond to brand transactions?

To evaluate this question, we build an event study regression that evaluates the response of prices and sales to transaction events, focusing on a balanced panel.

$$\log y_{it} = \sum_{\tau=-5}^{\tau=5} \beta_\tau \text{Transaction}_{i,t+\tau} + \Gamma' \mathbf{X}_{i,t} + \xi_i + \phi_t + \epsilon_{i,t}. \quad (1)$$

In Equation (1), we focus on two outcome variables of interest y_{it}. We evaluate regressions with both sales and the price of the product, before and after transaction. $\text{transaction}_{i,t+\tau}$ is an indicator for the quarters that are τ quarters away from a transaction event. We control for other firm-level variables indicated by $\mathbf{X}_{i,t}$, and include a brand, group, and time fixed effect.

We use log sales and look at the first event of trademarking. Figure 6 plots two separate

Figure 4: Old Product Persistence

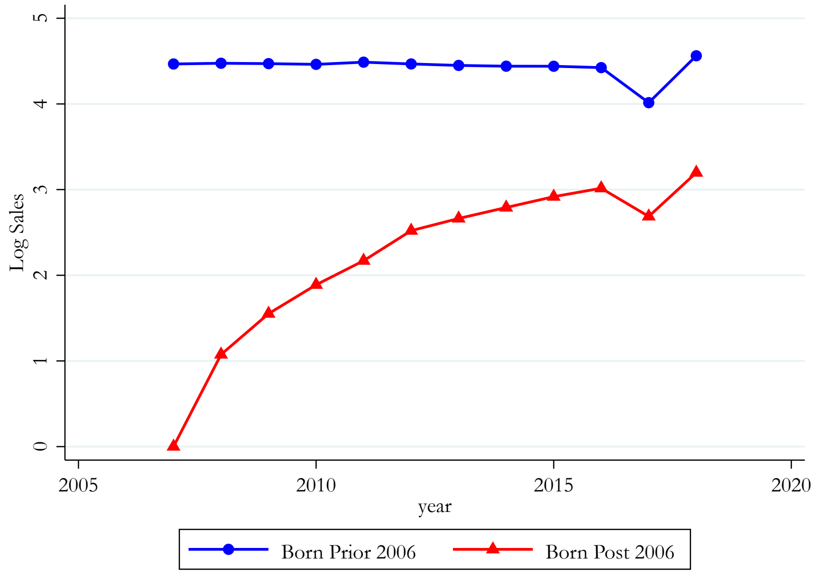


Figure 5: Age of Transaction Over Time

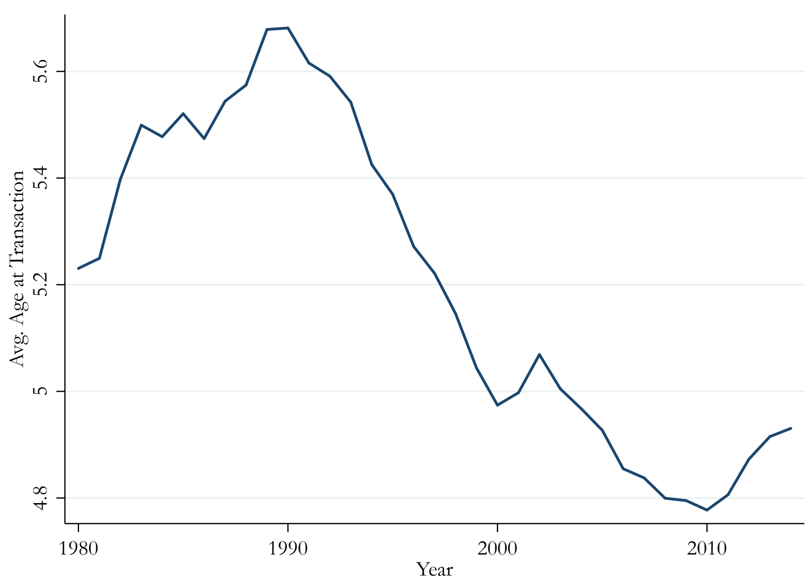
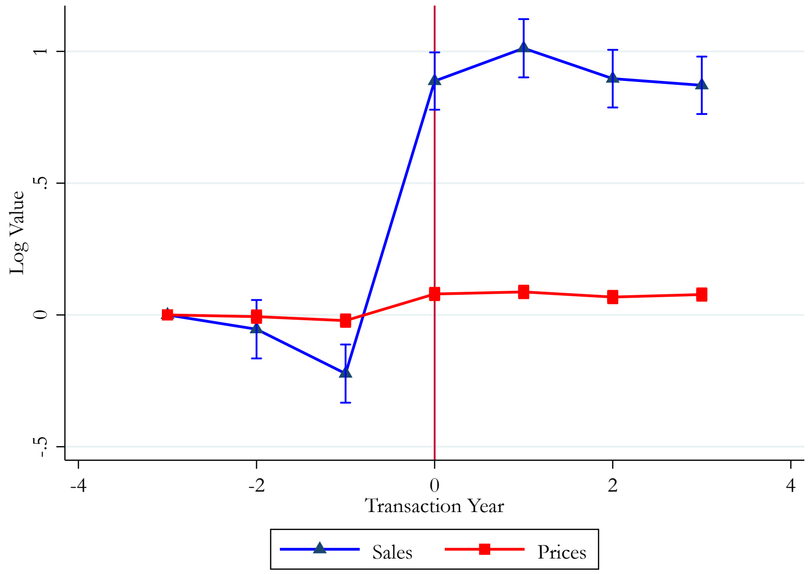


Figure 6: Event in Nielsen



Note: This figure shows the regression coefficients on a brand transaction across firms in RMS Nielsen.

regressions on one graph with different outcome variables of interest: prices and sales. We plot each coefficient with the clustered standard error.

After the event, both prices and sales move strongly, with sales moving significantly more so. With the increase in prices, the results in Figure 6 provide evidence that after adding additional brands, firms may increase their market power over time. Combining this with the rising rate of transfer from small to large firms can help connect the importance of brand dynamism with the aggregate distribution of markups across firms. Further, the change in markups will be a key outcome of our model, which we turn to next. Thus, this leads to the fact that summarizes this section:

Fact 3 *Product transactions are more likely to go from small to large firms; upon transaction, sales and prices both increase.*

With our three empirical facts in hand, we turn to a model that incorporates these essential ingredients.

4 Model

We introduce a firm dynamics model with endogenous (1) product creation, (2) variable markups, and (3) transfer of brand ownership. Through both analytical and quantitative analysis, we show how reallocation of products affect welfare and its policy implications.

4.1 Environment

Time is continuous. There is a representative household that endogenously supplies labor to the economy at each instant t, denoted as $\mathbf{L}_t$. The household receives income from their labor and dividends from the corporate sector, according to a representative portfolio of firms. The income of the household is spent on differentiated varieties from measure 1 of product groups, indexed by $k \in [0,1]$. At each instant of time t, there are measure n_{kt} varieties available for consumption. Each variety is indexed by (i,k) with $i \in [0, n_{kt}]$. Given the consumption choice $\{c_{ikt}\}_{[0,\infty]\times[0,n_{kt}]\times\in[0,1]}$, the household's real consumption is given by the following nested CES function:

$$\mathbf{C}_t = \left(\int_0^1 C_{kt}^{\frac{\theta-1}{\theta}} dk \right)^{\frac{\theta}{\theta-1}}, \ \ \theta > 1, \tag{2}$$

$$C_{kt} = \left(\int_0^{n_{kt}} [\psi_{ikt} c_{ikt}]^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, \ \ \sigma > \theta. \tag{3}$$

In equation (2), the real consumption from different product groups are aggregated according to a CES function, with θ being the substitution elasticity. In equation (3), the real consumption is aggregated across varieties with substitution elasticity σ. Products are different in their appeals ψ_{ikt}, which we use to capture the time-varying customer goodwills of products. We will discuss the evolution of the appeals in later sections on brand lifecycle.

The household can freely choose to save or borrow, in order to maximize its discounted lifetime utility. The household has separable period utility in its real consumption $\mathbf{C}_t$ and labor $\mathbf{L}_t$ per instant, with discount rate ρ. Throughout the paper, we normalize the price index for aggregate real consumption to be 1. Thus the utility maximization problem of the representative household can be summarized as following:

$$\max_{c_t(i,k),\dot{a}_t} \int_0^\infty e^{-\rho t} \left(\frac{\mathbf{C}_t^{1-\gamma}}{1-\gamma} - \frac{\mathbf{L}_t^{1+1/\varphi}}{1+1/\varphi} \right) dt, \tag{4}$$

s.t.

$$\int_0^1 \int_0^{n_t(k)} p_{ikt} c_{ikt} di dk + \dot{a}_t \leq r_t a_t + \mathbf{w}_t \mathbf{L}_t + \Pi_t,$$

$$\mathbf{C}_t \text{ is given by equation (2) and (3).}$$

Firms.- In each product group, there is one multi-product firm and many fringe firms. We refer to the multi-product firm also as leader. The leader and fringe firms are different in the following aspects: (1) Capacity in operating varieties; The leaders are able to own and operate positive measure of products, while the fringe firms are only able to operate a infinitesimal product; (2) Entry; The leaders are not subject to entry and exit, while there is free entry of fringe firms. The entering firms pay a lump-sum by hiring κ units of labor. (3) Pricing; The leaders are big relative to their product groups, and they internalize their impacts on the group-level price index. The fringe firms are small relative to the market, and they behave as monopolistic competitive firms.

In addition to product scope, there are other types of intangible assets that are not modeled in our paper, for example the general productivity of firms or their relationship with retail distributors. We summarize these factors as firm-specific and non-transferable labor productivity $exp(y_{kt}/(\sigma-1))$ for the leader in product group k and normalize the fringe firms' productivity to be 1. We assume the productivity of the leader follows geometric Brownian motion with ς being its volatility. For each product group k, we denote the collection of products operated by group leader as $\mathcal{Z}_{kt}$ and the collection operated by fringes as $\mathcal{X}_{kt}$. We define a quality index of a group leader as $z_{kt} = exp(y_{kt}) \int_{i \in \mathcal{Z}_{kt}} \psi_{ikt}^{\sigma-1} di$ and the quality index for all fringe firms as $x_{kt} = \int_{i \in \mathcal{X}_{kt}} \psi_{ikt}^{\sigma-1} di$. These two quality indices are sufficient for us to characterize the equilibrium.

Products.- Products go through its lifecycle. For a product that is born in instant τ, it starts with an initial draw from a fixed distribution $G_0(\psi)$ with mean ψ_0. Over time, the products accumulate customer base. We capture this process by a constant growth rate of α. Meanwhile, existing products retire from the market with Poisson rate δ. This is a tractable way to model the features of brand lifecycle we observe in data. For a product that has been surviving in the market for $t-\tau$ years with initial draw $\psi(0)$, it will have appeal $\psi(t-\tau) = \psi(0)e^{\alpha(t-\tau)}$. With the assumption $\delta > \alpha$, the age profile of average sales for each cohort is hump-shaped as documented in data.

Innovation and Acquisition.- The product group leaders can endogenously change its product scope through (1)innovation and (2)acquisition.

First, for a leader with current quality z, it chooses the flow of new products $\tilde{\eta}_t$ by paying cost $D\left(\frac{\tilde{\eta}}{z}\right)z$, where D is increasing and concave with $D(0) = 0$. We normalize the internal innovation cost by the current quality index. This assumption ensures that the model, in the case where $\sigma \to \theta$ is comparable to the models with endogenous firm growth in the literature. Due to this normalization, we could also directly assume firms choose the innovation rate $\eta = \frac{\tilde{\eta}}{z_l}$ and pay cost $D(\eta)z$.

Second, the leader could acquire new products from fringe firms. The leader of a group and fringe firms meet through a frictional search process, which captures the time or efforts taken in order to locate the brands to buy. The total measure of meetings at each instant t is λx, where x is the measure of fringe firms and λ is the search intensity of leaders. By choosing the intensity λ, the group leader pays a flow cost $R(\lambda)x$. When two sides meet, they bargain over gains from trades, with the sellers' bargaining power being γ.

4.2 Equilibrium

Household Decision.- We first characterize the optimal consumption and saving decisions of the representative household. Given the problem in (4), the optimal choice household is relatively standard. The real consumption of each instant should follow the Euler equation:

$$\gamma \frac{\dot{\mathbf{C}}_t}{\mathbf{C}_t} = r_t - \rho. \tag{5}$$

The labor supply for each instant equalizes the marginal utility from consumption and marginal disutility of labor:

$$\frac{\varphi'(\mathbf{L}_t)}{\mathbf{C}_t^{-\gamma}} = \mathbf{w}_t \tag{6}$$

The within-period optimization leads to a standard demand curve for variety (i, k):

$$c_{ikt} = p_{ikt}^{-\sigma} P_{kt}^{\sigma-\theta} \mathbf{C}_t \quad . \tag{7}$$

The price indices are given as:

$$P_{kt} = \left[\int_0^{n_{kt}} \psi_{ikt}^{\sigma-1} p_{ikt}^{1-\sigma} di \right]^{\frac{1}{1-\sigma}}, \tag{8}$$

$$\mathbf{P}_t = \left[\int_0^1 P_{kt}^{1-\sigma} dk \right]^{\frac{1}{1-\sigma}} = 1. \tag{9}$$

Pricing Equilibrium.- Firms take the demand curve as given from the representative household, and choose the prices for their products in order to maximize the profit. Our assumption regarding fringe firms implies that they will price according to the constant markup $\frac{\sigma}{\sigma-1}$ multiplied to the marginal cost of production $\mathbf{w}_t$. The market leader solves a joint pricing problem for all of its products:

$$\max_{p_t(i,k)} \int_{i \in \mathcal{Z}_{kt}} p_{ikt} \left[c_{ikt} - \frac{1}{e^{y_{kt}/(\sigma-1)}} \right] di, \tag{10}$$

s.t.

$$\textit{equation } (7)$$

As the leaders internalize their impacts on the group-level price index, their optimal markup is no longer constant. The firm-level markup is an increasing function of its total market share within product group: $\mu_{kt} = \frac{\epsilon_{kt}}{\epsilon_{kt}-1}$, where μ_{kt} is the markup of the leader within product group k and ϵ_{kt} is the perceived substitution elasticity of the leader. Under different assumption on competition, this elasticity differs, but in both cases is function of leader's within-group market share. Under price competition, the leader has elasticity $\epsilon_{kt} = \theta s_{kt} + \sigma(1 - s_{kt})$ and under quantity competition, the leader's elasticity is $\epsilon_{kt} = \left[\frac{1}{\theta} s_{kt} + \frac{1}{\sigma}(1 - s_{kt})\right]^{-1}$. With the two quality index as defined, we define the gap between the two quality indices as $\phi_{kt} = z_{kt}/x_{kt}$. We out the results from within-group competition in the following lemma:

Lemma 1 *Given the quality gap ϕ, the within-group market share of leader and markup (s, μ) jointly solve:*

$$s = \frac{\mu^{1-\sigma}}{\mu^{1-\sigma} + \phi^{-1}[\frac{\sigma}{\sigma-1}]^{1-\sigma}},$$

$$\mu = \frac{\epsilon(s)}{\epsilon(s) - 1}.$$

Denote the solution as $s(\phi)$ and $\mu(\phi)$.

As the firms (both leaders and fringes) are all atomless compared to the competition across groups, the total sales share of group k with respect to the aggregate expenditure $\mathbf{C}_t$ is entirely determined by the group-level price index:

$$P(z, x) = \left(z\mu\left(\frac{z}{x}\right)^{1-\sigma} + x\left[\frac{\sigma}{\sigma-1}\right]^{1-\sigma} \right)^{1/(1-\sigma)}. \tag{11}$$

The profit for leaders and followers with state (z^l, z^f) are as following:

(Leader Profit)

$$\Pi(z,x) = z\mu\left(\frac{z}{x}\right)^{1-\sigma}\left[1-\mu\left(\frac{z}{x}\right)^{-1}\right] \times P(z,x)^{\sigma-\theta} \times \mathbf{C}_t \tag{12}$$

(Fringe Profit)

$$\pi(z,x) = \frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \times P(z,x)^{\sigma-\theta} \times \mathbf{C}_t \tag{13}$$

Dynamics Decision.- Our characterization so far implies that the two relevant states of any product group are leader quality z and fringe quality x. If the leader choose innovation rate η and search intensity λ. These two states evolve according to:

$$\dot{z} = \Gamma^Z(\eta,\lambda,z,x) = \underbrace{\psi_0\eta z}_{\text{Creation}} + \underbrace{(\alpha-\delta)z}_{\text{Brand Growth}} + \underbrace{\lambda x}_{\text{Reallocation}} \tag{14}$$

$$\dot{x} = \Gamma^X(\lambda,z,x) = \underbrace{\psi_0 v z}_{\text{Entry}} + (\alpha-\delta)x - \lambda x \tag{15}$$

We write out the Bellman equation for group leaders and followers. Denote the real value of a leader (normalized by the aggregate expenditure) as $V(z,x)$ and the value of a fringe firm with quality ψ as $\tilde{v}(\psi,z,x)$. The following linearity property simplifies our characterization:

Lemma 2 $\tilde{v}(\psi,z,x) = \psi v(z,x)$.

In the following equations, we omit time as an argument of value functions, and denote all time-dependence to the time derivatives. The value for a group leader with state (z,x) is:

$$\begin{aligned}\left[\rho+(\gamma-1)\frac{\dot{\mathbf{C}}}{\mathbf{C}}\right]V(z,x) = &\underbrace{\Pi(z,x)}_{\text{Flow Profit}} + \underbrace{\frac{\varsigma^2}{2}z^2V_{zz} + \dot{V}(z,x)}_{\text{Productivity Shock and Aggregate Shift}} \\ &+ \max_{\lambda}\ \underbrace{\lambda(1-\gamma)x\Big[V_z(z,x) - V_x(z,x) - v(z,x)\Big] - R(\lambda)x\frac{\mathbf{w}}{\mathbf{C}}}_{\text{Product Acquisition}} \\ &+ \max_{\eta}\ \underbrace{\eta z V_z(z,x) - D(\eta)z\frac{\mathbf{w}}{\mathbf{C}}}_{\text{Product Creation}} + \underbrace{(\alpha-\delta)\Big[zV_z(z,x) + xV_x(z,x)\Big]}_{\text{Product Lifecycle}}\end{aligned} \tag{16}$$

The per-quality value for a fringe firm with unit quality with state (z, x) is:

$$\left[\rho + (\gamma - 1)\frac{\dot{\mathbf{C}}}{\mathbf{C}} + \delta - \alpha\right] v(z, x) = \underbrace{\pi(z, x)}_{\text{Flow Profit}} + \underbrace{\frac{\varsigma^2}{2} z^2 v_{zz} + \dot{v}(z, x)}_{\text{Productivity Shock and Aggregate Shift}}$$
$$+ \underbrace{\lambda\gamma\left[V_z(z, x) - V_x(z, x) - v(z, x)\right]}_{\text{Value of Selling}}$$
$$+ \underbrace{\Gamma^Z(\eta(z, x), \lambda(z, x), z, x) v_z(z, x)}_{\text{Growth in Leader}} + \underbrace{\Gamma^X(\lambda(z, x), z, x) v_x(z, x)}_{\text{Growth in Competiting Fringes}} \tag{17}$$

s.t.

$$\kappa \frac{\mathbf{w}_t}{\mathbf{C}_t} \geq \psi_0 v(z, x)$$

For a group leader (equation (16)), it optimally chooses the product creation rate and acquisition rate. Both activities increase the own quality index of leader z. In addition, the acquisition activity decreases the fringes' quality. For a group fringe (equation (17)), it gets profits according to the competition. Due to transaction, it also receives transfers from the leader according to the gains from trade. In addition, for the regions where there is no entry, the value function also changes because the leader and fringe quality changes. Free entry condition requires that for any state the value of a potential entrant is bounded by entry cost.

This intentionally simplified environment allows us to benchmark our model to literature. When we set $R(\lambda) = \infty$ and assume $\sigma \to \theta$, this model is identical to a setting in Klette and Kortum. We can thus compare our results to the literature via two margins: (1) effect of big firms and (2) effect of transactions in varieties. This baseline environment can be extended to a setting where there are more than one multi-product firms. In these settings, there is an additional strategic interactions among market leaders. We assume these strategic incentives away.

Evolution of States.- We define $h_t(z, x)$ as the density of product groups that have state (z, x). With the optimal innovation policy $\eta(z, x)$, optimal acquisition policy $\lambda(z, x)$, this distribution must solve the following differential equation within the boundary $n \leq n(z)$:

$$\dot{h}(z, x) = -\frac{\partial}{\partial z}\Gamma^Z(\eta(z, x), \lambda(z, x), z, x) h_t(z, x) - \frac{\partial}{\partial x}\Gamma^X(\lambda(z, x), z, x) h_t(z, x) + \frac{\varsigma^2}{2} z^2 \frac{\partial}{\partial z^2} h(z, x) \tag{18}$$

Aggregation.- We are now ready to write down the market clearing condition for labor. First, we define the cost-weighted markup given ferocity $\phi = z/x$ as:

$$M(\phi) = \frac{\mu(\phi)^{1-\sigma} + \phi^{-1}\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}}{\mu(\phi)^{-\sigma} + \phi^{-1}\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}} \tag{19}$$

Given our normalization of aggregate price index of markup, the wage for labor is:

$$\mathbf{w}_t = \left(\int_{z,x} P(z,x)^{1-\theta} h_t(z,x)dz\right)^{\frac{1}{\theta-1}}. \tag{20}$$

The aggregate markup in the economy is:

$$\mathbf{M}_t = \left[\int_{z,x} P(z,x)^{1-\theta} M(z/x)^{-1} h_t(z,x)dz\right]^{-1}$$

The labor productivity from equilibrium can be written as

$$\mathbf{Z}_t = \mathbf{w}_t \mathbf{M}_t \tag{21}$$

The labor market clearing condition requires that:

$$\mathbf{C}_t = \mathbf{Z}_t\left(\mathbf{L}_t - \int_{z,x}\left[R(\lambda_t(z,x))x + D(\eta_t(z,x))z\right]h_t(z,x)dz\right) \tag{22}$$

Equilibrium.- The competitive equilibrium we look for is defined as following:

Definition 3 *An equilibrium path is $\mathbf{w}_t$, $\mathbf{C}_t$, $V_t(z,x)$, $v_t(z,x)$ and $h_t(z,x)$ such that:*

1. Given $\{\mathbf{w}_t, \mathbf{C}_t\}$, $V_t(z,x)$ and $v_t(z,x)$ solve equation (16) and (17);

2. Given the policy functions, $h_t(z,x)$ solves equation (18);

3. $\mathbf{w}_t$ and $\mathbf{C}_t$ aggregates according to equation (20) and equation (22).

Proposition 4 *When $\gamma = 1$ and $\varphi = \infty$, the equilibrium is block-recursive. In other words, it can be solved in order from equation (16) to equation (22).*

Optimal Decisions.- The market leader can also innovate paying cost by itself. The optimal choice of innovation rate balances the marginal benefit of an additional product and the marginal cost of innovation:

$$\psi_0 V_z'(z,x) = D'(\eta(z,x))\frac{\mathbf{w}}{\mathbf{C}}, \tag{23}$$

The optimal search decision equalizes the extra cost to pay and the increase in matching probability:

$$(1-\gamma)\left[V_z(z,x) - V_x(z,x) - v(z,x)\right] = R'(\lambda(z,x))\frac{\mathbf{w}}{\mathbf{C}} \tag{24}$$

Combining equation (23) and (24), we derive a tight link between the innovation rate and acquisition rate from model:

$$D'(\eta^*) = \kappa + \frac{\psi_0}{1-\gamma}R'(\lambda^*). \tag{25}$$

As the group leader is always on its first-order condition, the innovation rate and acquisition rate must be linked through the marginal costs. The difference is that the marginal cost in innovation is the internal labor cost, while the cost of acquisition is both the entry cost of fringe firms and the internal search cost adjusted by new brand quality and bargaining power.

4.3 Analytical Result: Partial Equilibrium without Shocks

We consider a case that highlights the welfare impact of product reallocation analytically. To do so, we assume (1) log utility in consumption ($\gamma = 1$) and infinitely elastic labor ($\varphi = \infty$); (2) there is no productivity shock ($\varsigma = 0$).

In this simple case, all product groups would converge to a steady state where the state (z, x) stays constant. We show in the appendix that this steady state is characterized by four variables $(z^*, x^*, \eta^*, \lambda^*)$.

Proposition 5 (Steady State) *The steady state of the partial equilibrium steady state model is* $(z^*, x^*, \eta^*, \lambda^*)$ *such that:*

(optimal innovation)

$$\psi_0 \frac{\frac{\partial}{\partial z}\Pi(z^*, x^*) - D(\eta^*)}{\rho - \psi_0\eta^*} = D'(\eta^*) \tag{26}$$

(optimal acquisition)

$$(1-\gamma)\left[\frac{\frac{\partial}{\partial z}\Pi(z^*, x^*) - D(\eta^*)}{\rho - \psi_0\eta^*} - \frac{\kappa}{\psi_0}\right] = R'(\lambda^*) \tag{27}$$

(steady state)

$$0 = \left[\psi_0\eta^* + (\alpha - \delta)\right]z^* + x^*\lambda^* \tag{28}$$

(free entry)

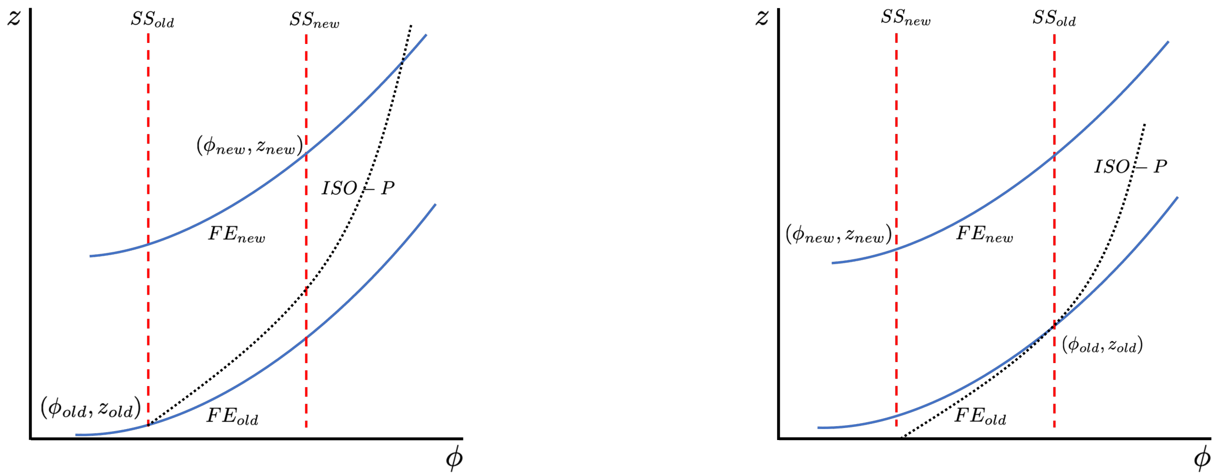
$$(\rho+\delta-\alpha)\frac{\kappa}{\psi_0}=\pi(z^*,x^*)+\frac{\gamma}{1-\gamma}\lambda^* R'(\lambda^*) \tag{29}$$

In this simplified model, the welfare of the representative household, in the steady state, can be sufficiently characterized by the price index. Notice that equation (26) and (27) allow us to directly solve η^* and λ^* as functions of (z^*, x^*). We thus get two equations in terms of leader and fringes' quality index. These two equations highlight the two mechanisms through which brand reallocation affects welfare. Equation (28) links the competition within product groups as a function of innovation rate and acquisition rate. (1) Holding innovation rate constant, a higher acquisition rates leads to less competition in the steady state (z^*/x^* increases). In terms of economics, as leaders are buying products with a faster pace, in the steady state, relative to the leaders, fringe firms must either have less products or worse products. We refer to this as the consolidation effect; Equation (29) links competition and quality of leaders to the acquisition through entry of fringes.

Using the definition $\phi^*=\frac{z^*}{x^*}$, we reach the following system of equations given any selling rate λ^*. These two equations are our takeaways from this section, it highlights all the parameters that are relevant for us to understand welfare impact of brand reallocation: for any parameter change that induces λ^* to change, what would happen to the competitiveness ϕ and quality z?

To answer this question, we plot the combination of (z^*, ϕ^*) that is consistent with equation (28) and equation (29) in figure (8), treating the selling rate λ^* as a parameter. The free-entry (FE) curve is upward sloping: when the group leader is more productive, there must be relatively less followers as they each would have less profit; The steady-state (*SS*) curve is vertical: given any selling rate there is a constant level of ferocity ϕ^* in the steady state. The steady state of the economy is thus given by the crossing of FE and SS curves. We overlay these two curves with the combination of (ϕ, z) such that the price index stays the same. In this simple steady-state model, price index is sufficient for welfare. As a result, all the points to the left of ISO-P locus leads to a higher price (for the same level of competition, leaders' quality deteriorates.) and lower welfare, and vice versa.

We ask through these graphs what is the welfare incidence of increasing transaction rates (λ increases). Through the free-entry relationship, the FE curve shifts up as the fringe firms are also compensated by option value of selling. As a result, competition increases for the same level of z. The steady-state locus could shift to the right or left, due to two countervailing forces. The direct effect of increasing transfer is that more brands are reallocated to leader. Indirectly, as the internal innovation is substitute to external acquisition. So leaders innovate less after the change. In net this could result in higher or lower concentration. In panel (a), we show a case where the concentration increases after the change. In this case, the economy ends up with a higher quality and less concentration. This new steady state point might lies above or below the ISO-P curve. As a result, the welfare could increase or decrease depending on the strength of two mechanisms. In panel (b), the concentration falls after the change. As a result the quality of leader could increase or decrease. However, because the ISO-P curve lies below FE curve when concentration falls, these changes are all welfare-improving.



(a) Case A: Concentration Increases

(b) Case B: Concentration Decreases

In both panel: curves with subscript *new* are before change while ones with *old* are after change. *FE* is the free-entry locus and *SS* is the steady state locus; *ISO-P* is the iso-price curve.

Figure 7: Impact of Increasing Transaction

4.4 Welfare and Planner's Solution

In this section we consider a social planner's solution within this environment. The planner seeks to maximize the discounted utility of representative household, subject to the innovation cost and the search frictions within the market for brand ownership. Under the assumption of equalized markup between leader and fringe firms, $\mu = \frac{\sigma}{\sigma-1}$, the two productivity indices differ by a constant ratio $\mathcal{Z}_t^* = \frac{\sigma}{\sigma-1}\mathcal{Z}_t$. With these two productivity indices, we are ready to lay out the planner's problem: (Unconstrained Planner Problem)

$$\max_{\mathcal{L}_t, b(z), a(z), \eta(z)} \int_0^\infty e^{-\rho t}\left(\frac{\mathbf{C}_t^{1-\gamma}}{1-\gamma} - \frac{\mathbf{L}_t^{1+1/\varphi}}{1+1/\varphi}\right)dt, \tag{30}$$

s.t.

$$\mathbf{C}_t = \mathbf{Z}_t^*\left(\mathbf{L}_t - \int_{z,x}\Big[R(\lambda_t(z,x))x + D(\eta_t(z,x))z\Big]h_t(z,x)dz\right) \tag{31}$$

$$\mathbf{Z}_t^* = \left(\int_0^\infty [z(1+\phi(z))]^{\frac{1-\theta}{1-\sigma}} h(z)dz \right)^{\frac{1}{\theta-1}} \tag{32}$$

$$\dot{h}(z,x) = -\frac{\partial}{\partial z}\Gamma^Z(\eta(z,x),\lambda(z,x),z,x)h_t(z,x) - \frac{\partial}{\partial x}\Gamma^X(\lambda(z,x),z,x)h_t(z,x) + \frac{\varsigma^2}{2}z^2\frac{\partial}{\partial z^2}h(z,x) \tag{33}$$

$$\text{given } h_0(z,x)$$

(Constrained Planner Problem)

$$\max_{\mathcal{L}_t, b(z), a(z), \eta(z)} \int_0^\infty e^{-\rho t}\left(\frac{\mathbf{C}_t^{1-\gamma}}{1-\gamma} - \varphi_0 \frac{\mathbf{L}_t^{1+1/\varphi}}{1+1/\varphi}\right)dt, \tag{34}$$

s.t.

$$\mathbf{C}_t = \mathbf{Z}_t\left(\mathbf{L}_t - \int_{z,x}\left[R(\lambda_t(z,x))x + D(\eta_t(z,x))z\right]h_t(z,x)dz\right) \tag{35}$$

$$\dot{h}(z,x) = -\frac{\partial}{\partial z}\Gamma^Z(\eta(z,x),\lambda(z,x),z,x)h_t(z,x) - \frac{\partial}{\partial x}\Gamma^X(\lambda(z,x),z,x)h_t(z,x) + \frac{\varsigma^2}{2}z^2\frac{\partial}{\partial z^2}h(z,x) \tag{36}$$

$$\text{given } h_0(z)$$

4.5 First Best Allocation

We first investigate the first best allocation, where the planner could both affect the innovation and acquisition decisions and the pricing and production decisions. The efficient allocation could be characterized by a set of functions that are similar to equilibrium allocation. Specifically, we define $\Lambda_t^*(z,x)$ as the marginal societal value of an industry with state (z,x) normalized by the real consumption, and $\{\mathbf{C}_t, \mathbf{L}_t, \eta_t^*(z,x), \lambda^*(z,x), V^*(z,x), v^*(z,x), h^*(z,x)\}$ as the first-best allocation. The following proposition details the first-best allocation. We omit the proof to the Appendix.

Proposition 6 *The first-best allocation* $\{\mathbf{C}_t, \mathbf{Z}_t^*, \eta_t^*(z), \alpha_t^*(z), \beta_t^*(z), \Lambda_t^*(z), h_t^*(z)\}$ *is characterized by the following conditions:*

1. Consumption and labor are optimally chosen:

$$\frac{\mathbf{L}_t^{1/\varphi}}{\mathbf{C}_t^{-\gamma}} = \mathbf{Z}_t^* \tag{37}$$

2. Innovation and reallocation are consistent with their social values:

$$
\begin{aligned}
\left[\rho + (\gamma - 1)\frac{\dot{\mathbf{C}}}{\mathbf{C}}\right] V(z,x) = & \underbrace{\frac{1}{\theta - 1}(z+x)^{\frac{\theta-1}{\sigma-1}}\mathbf{Z}^*}_{\textit{Flow Surplus}} + \underbrace{\frac{\varsigma^2}{2} z^2 V_{zz} + \dot{V}(z,x)}_{\textit{Productivity Shock and Aggregate Shift}} \\
& + \max_{\lambda} \underbrace{\lambda x \left[V_z(z,x) - V_x(z,x) - v(z,x)\right] - R(\lambda) x \frac{\mathbf{w}}{\mathbf{C}}}_{\textit{Product Acquisition}} \\
& + \max_{\eta} \underbrace{\eta z V_z(z,x) - D(\eta) z \frac{\mathbf{w}}{\mathbf{C}}}_{\textit{Product Creation}} + \underbrace{(\alpha - \delta)\left[z V_z(z,x) + x V_x(z,x)\right]}_{\textit{Product Lifecycle}} \\
& + \max_{\nu} \underbrace{\nu V_x(z,x) - \kappa \nu \frac{\mathbf{w}}{\mathbf{C}}}_{\textit{Entry}}
\end{aligned} \tag{38}
$$

3. $h_t(z,x)$ satisfies the forward equation.

4. Real consumption $\mathbf{C}_t$ and productivity $\mathbf{Z}_t^*$ follow the aggregation.

4.6 Constrained Efficient Allocation

We then investigate the first best allocation, where the planner could both affect the innovation and acquisition decisions and the pricing and production decisions. The efficient allocation could be characterized by a set of functions that are similar to equilibrium allocation. Specifically, we define $\Lambda_t^*(z)$ as the marginal societal value of an industry with state z, and $\{\mathbf{C}_t, \mathbf{Z}_t^*, \eta_t^*(z), \alpha_t^*(z), \beta_t^*(z), \Lambda_t^*(z), h_t^*(z)\}$ as the first-best allocation. In the Appendix, we show they are solution to the following set of equations.

Proposition 7 *The constrained efficient allocation is characterized by the following conditions:*

1. Innovation and reallocation are consistent with their social values:

$$
\begin{aligned}
\left[\rho + (\gamma-1)\frac{\dot{\mathbf{C}}}{\mathbf{C}}\right] V(z,x) = & \underbrace{\left[\frac{1}{\theta-1} w_t^{1-\theta} - M(z/x)^{-1}\mathbf{M}_t\right] P(z,x)^{1-\theta}}_{\textit{Flow Surplus}} + \underbrace{\frac{\varsigma^2}{2} z^2 V_{zz} + \dot{V}(z,x)}_{\textit{Productivity Shock and Aggregate Shift}} \\
& + \max_{\lambda} \underbrace{\lambda x \Big[V_z(z,x) - V_x(z,x) - v(z,x) \Big] - R(\lambda) x \frac{\mathbf{w}}{\mathbf{C}}}_{\textit{Product Acquisition}} \\
& + \max_{\eta} \underbrace{\eta z V_z(z,x) - D(\eta) z \frac{\mathbf{w}}{\mathbf{C}}}_{\textit{Product Creation}} + \underbrace{(\alpha-\delta)\Big[z V_z(z,x) + x V_x(z,x) \Big]}_{\textit{Product Lifecycle}} \\
& + \max_{\nu} \underbrace{\nu V_x(z,x) - \kappa \nu \frac{\mathbf{w}}{\mathbf{C}}}_{\textit{Entry}}
\end{aligned}
\tag{39}
$$

3. *$h_t(z,x)$ satisfies the forward equation.*

4. *Real consumption $\mathbf{C}_t$ and productivity $\mathbf{Z}_t^*$ follow the aggregation.*

4.7 Implementation

In this section, we consider the implementation of the constrained efficient allocation, in the quasi-linear case where $\varphi \to \infty$ and $\gamma \to 1$. In this case, the value function of a firm in the equilibrium can be solved without iteration on the aggregate states $\mathbf{C}_t$ and w_t. The planner's solution, however, requires iteration on the aggregates because the planner internalizes the impact of each industry on the dispersion of productivity and markup.

By comparing the planner's value function and the decentralized equilibrium, we find these two Bellman equations are otherwise identical, other than the period returns. For an individual firm, the value of having productivity z and ferocity ϕ, while choosing acquisition policy (α, β) is $\left[\frac{1}{\mu(\phi)} s(\phi) + \frac{\alpha\beta}{\phi}\frac{1}{\sigma}(1-s(\phi))\right] P(\phi) z^{\frac{1-\theta}{1-\delta}}$, while the planner's period return is $\left[\frac{1}{\theta-1} w_t^{1-\theta} - M(\alpha\beta)^{-1}\mathbf{M}_t\right] P(\alpha\beta)^{1-\theta} z^{\frac{1-\theta}{1-\sigma}}$. We consider a combination of policy instruments: a proportional tax(subsidy) on acquisition $T_b(\phi)$ and a proportional tax(subsidy) on entry cost $T_e(\phi)$. We allow the two tax rates to have dependence on the ferocity of competition in an industry. The following proposition details the tax scheme that implements the constrained efficient allocation:

Proposition 8 *The following budget balanced transfer scheme decentralizes the constrained effi-*

cient allocation:

$$T_t(\phi) = \frac{\left(\frac{1}{\sigma}(1-s(\phi)) + s(\phi)\left[1-\frac{1}{\mu(\phi)}\right]\right) - \left(\frac{1}{\theta-1}w_t^{1-\theta} - M(\phi)^{-1}\mathbf{M}_t\right)}{\left(\frac{1}{\theta-1}w_t^{1-\theta} - M(\phi)^{-1}\mathbf{M}_t\right) + \frac{\kappa\phi}{P(\phi)^{1-\theta}z^{\frac{1-\theta}{1-\sigma}}}}$$

and

$$\int_0^\infty T_t(\phi)\tau^*(\phi,z)h_t^*(z)dz = 0.$$

To better understand the shape of this function, we look for the limit of this function. In a infinitely concentrated market, $\phi \to 0$, the tax scheme sets $T_t \to \infty$. In a infinitely competitive market, $\phi \to \infty$, the tax scheme sets T_0.

5 Quantitative Results

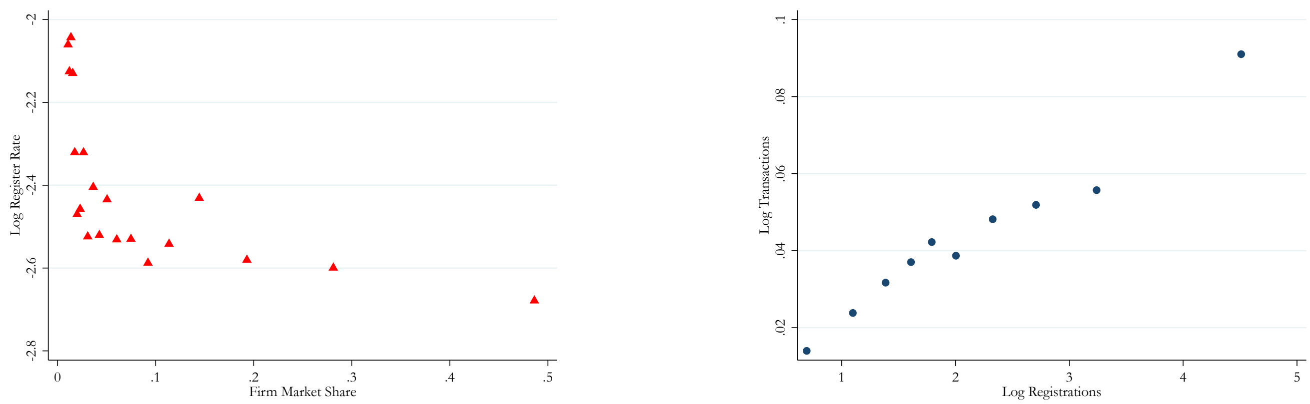
5.1 Estimation

In this section, we discuss the steps to estimate the parameters of the model. There are a set of parameters we calibrate without solving the equilibrium. We estimate the key parameters that are relevant to the discussion of reallocation and welfare to match key empirical moments. We parameterize the cost of innovation and recruiting to have constant elasticity: $R(\lambda) = \frac{\xi_0}{1+1/xi}\beta^{1+1/\xi}$ and $D(\eta) = \frac{\chi_0}{1+1/\chi}\eta^{1+1/\chi}$. In order to estimate the cost of innovation and acquisition, we have six parameters to estimate.

Recruiting and Innovation Elasticity.- To estimate the recruiting elasticity ξ and innovation elasticity χ, we utilize the firm-level correlation of innovation rate and purchase rate with firms' market shares. From the model, the firm-level innovation rate and purchase rate are correlated with each other with the following formula, assuming the entry cost is small:

$$\frac{\log \lambda\phi}{\log \eta} \approx \frac{\xi}{\chi},$$

Given our empirical analysis, the slope of a regression of purchase rate on buying rate is 0.77. Thus given any value of χ, ξ can be pinned down directly. This left us with only the innovation elasticity χ. We estimate this elasticity to match the empirical correlation between firm market share and the product creation rate. *Bargaining Power.-* Papers in literature documents that in the U.S., the average premium (calculated by the ratio of price of acquisition over the book value of bought



(a) Innovation - Market Share

(b) Innovation - Acquisition

In both panel: curves with subscript *new* are before change while ones with *old* are after change. *FE* is the free-entry locus and *SS* is the steady state locus; *ISO-P* is the iso-price curve.

Figure 8: Correlation between Innovation, Acquisition, and Market Share

firms) is 0.47. That is, the average transaction has the buyers paying 47% more than the book value of the purchased firms. We use this as a moment to identify the bargaining power. Intuitively, a higher γ means sellers take a larger share of joint surplus, and they get paid a larger fraction in the gains from trade. From the model, this premium is given by $\gamma \frac{V_z(z,x)}{\psi\pi(z,x)/(\rho+\delta-\alpha)}$. As we cannot directly link the model moment to data, we estimate γ through indrect inference.

Cost Shifters.- We are left with three cost shifters to pin down: the entry cost κ, the innovation cost shifter χ_0, and the recruiting cost shifter ξ_0. We use the average leader market share, the average innovation rate, and the average purchase rate to identify these parameters.

Calibrated Parameters.- For the rest of paramters, we directly calibrate to the values in literature or their data counterparts. We calibrate the frequency to be one year, thus set the discount rate to be 0.05. The obsolete rate of product is set to match the product exit rate in data as 0.1 and the growth rate of brand customer base to be 0.05. We calculate the variance of firm level productivity by running an AR(1) process of average firm prices, we find the variance of residual to be 0.73. We set the substitution elasticity within product group to be the median in Hottman et al. (2016) as 6.9, and the substitution elasticity across groups to be 1.2. In our quantitative result so far, we set $\gamma = 1$ and $\varphi = \infty$ but is working on cases with other parameters for a more updated draft.

Table 2: Parameter Values

Estimation	Symbol	Value	Target	Data	Model
Innovation elasticity	χ	1.68	registration slope	-0.42	-0.49
Bargaining Power	γ	0.37	average premium	0.47	0.35
Innovation shifter	χ_0	6.26×10^5	average transaction rate	0.09	0.07
Recruiting shifter	ξ_0	4.38×10^5	average registration rate	0.05	0.04
Operating Cost	κ	2.69×10^5	average leader share	0.33	0.35

Calibration	Symbol	Target	Value
Recruiting elasticity	ξ	Transaction-registration correlation	0.50
Obsolete rate	δ	Product Exit Rate	0.1
Substitution Elasticity	σ	*HRW* within Group	6.9
Substitution Elasticity	θ	-	1.2
Variance	ν	Dispersion of Price	0.73
Discount Rte	ρ	Annual Interest Rate	0.05

6 Counterfactual Analysis

An increase in search cost is isomorphic to two topical policy discussions. First, a stringent antitrust policy would induce the probability of a successful trademark transaction between firms to decline. Second, taxing or subsidizing the price of trademarks would induce a similar decline in the returns to reallocation. We explore this through shifting the cost of transacting a trademark.

In the following table, we compare the productivity in the baseline economy to a counterfactual setting where the cost of posting vacancies increase by 10%. We interpret this counterfactual either as a change in the search technology in trademark transactions, or as a stronger anti-trust policy that incurs higher cost on transactions.

7 Conclusion

Product innovation plays a central role in economic growth, but the evolution of products plays a key role in the distribution of sales and efficiency. We study the role of market power through the lens of products and consumer goodwill. We integrate product-level data to document the evolution and transfer of brands in an environment with production innovation and endogenous acquisition.

After illustrating key facts related to the dynamism of the trademark market and the importance

Table 3: Counterfactual Analysis $\xi_0(1+\tau)$

	Baseline	No Trade	50% Tax	10% Subsidy	50% Subsidy
Welfare					
Discounted	1.00	0.95	0.96	1.01	0.98
Steady State	1.00	0.94	0.94	1.03	1.01
Decomposition					
Consumption	1.00	0.89	0.89	1.08	1.08
Markup	1.00	0.90	0.91	1.07	1.15
Research Inputs	1.00	0.85	0.85	1.10	1.12

of large firms, we turn to a model of multi-product firms with pricing power and different efficiency. We use this model to study market power in brand innovation. In particular, we focus on a natural tension in brand innovation that emerges from the fact that efficient firms should have more brands, but are able to achieve market power in the process.

Our quantitative model matches the distribution of firms, entry and exit rates, the registration rate of firms and transfer rates. We use this matched model to study a relevant policy counterfactual: how does restricting brand exchange impact consumer welfare? We find that the two tensions rely importantly on how substitutable goods are to consumers and the underlying distribution of firm efficiency. Overall, the efficiency gains outweigh the potential gains from pricing power when firms market substitutable goods.

This project is a first step in unifying two key mechanisms, pricing power and efficient brand allocation, in product-variety driven growth. In doing so, we connect the framework to rich data on brands that enable a detailed study of frictional markets in brand transactions and sales. The results shed light on the importance of policymakers to understand the various forces at play when managing antitrust policies, and have awareness of the efficiency gains from product consolidation.

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Appendix

A Example of Brand-Building: Procter & Gamble

Figure A1 illustrates how many firms that rely specifically on their brand relationships are held by P&G.

Figure A1: Example of P&G Brands

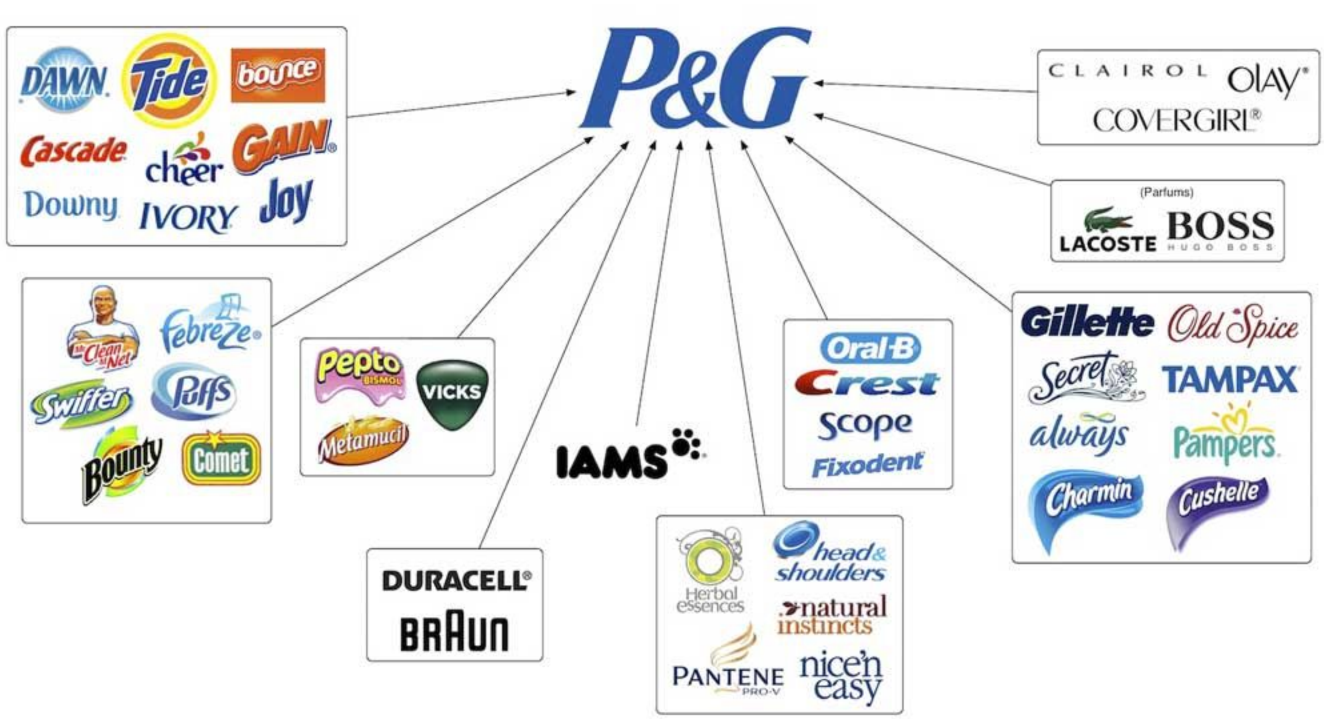


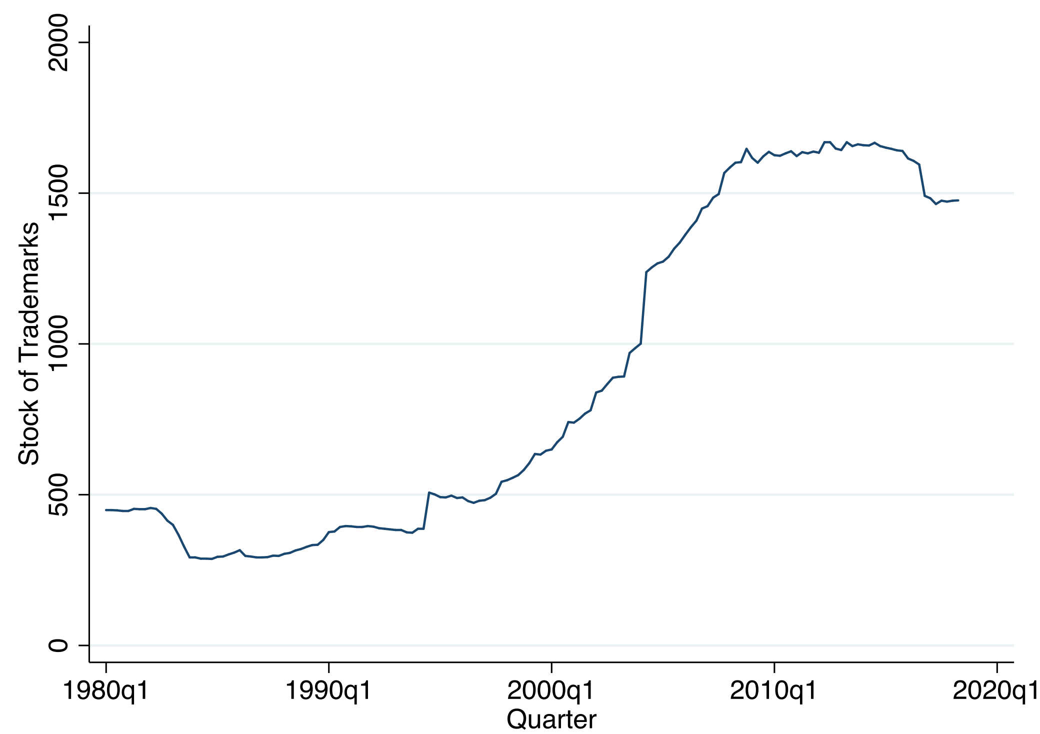
Figure A2 shows how P&G's trademark holdings have grown over time. Much of this trademark increase has come through poaching trademarks from other firms or purchasing other firms.

B Derivation for Section 4.4

B.1 Derivation of productivity indices

Equilibrium Productivity Index.- As in the main text, we define the labor productivity index in the equilibrium as $\mathbf{Z}_t$ and seek to write the real consumption as a linear function of production labor: $\mathbf{C}_t = \mathbf{Z}_t \mathbf{L}_t^p$, where $\mathbf{L}_t^p$ is the amount of labor input into production of consumption goods. Take

Figure A2: Tracing the brands of P&G over time



w

one product group with state z. The total labor cost to this product group is given by:

$$w_t l_t(z) = \Bigg[s(\phi(z)) \underbrace{\frac{1}{\mu(\phi(z))}}_{\text{Labor Share of Leader}} + (1 - s(\phi(z))) \underbrace{\frac{\sigma - 1}{\sigma}}_{\text{Labor Share of Fringe}} \Bigg] \times \underbrace{P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} \mathbf{C}_t}_{\text{Group Revenue}} \tag{40}$$

Integrating both sides of the equation above we get:

$$w_t \mathbf{L}_t^p = \int_0^\infty M(\phi(z))^{-1} P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz \mathbf{C}_t \tag{41}$$

Inverting we get the equilibrium productivity index as in the main text:

$$\mathbf{Z}_t = \frac{\left[\int_0^\infty P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz \right]^{\frac{1}{\theta - 1}}}{\int_0^\infty M(\phi(z))^{-1} P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz} \tag{42}$$

Efficient Productivity Index.- To derive the efficient productivity index, we consider a static planning problem of allocating production labor to maximize real consumption:

$$\max_{c(i,k)} \quad \left(\int_0^1 \left(\int_0^{N_t(k)} c(i,k)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}\frac{\theta-1}{\theta}} dk \right)^{\frac{\theta}{\theta-1}} \tag{43}$$

s.t.

$$c(i,k) = \left[\mathbb{D}_{i \in \mathcal{I}_k} y(k)^{\frac{1}{\sigma-1}} + \mathbb{D}_{i \notin \mathcal{I}_k} \right] l(i,k)$$

$$\int_0^1 \int_0^{N_t(k)} l(i,k) didk = \mathbf{L}^p$$

We solve this problem in two steps. First, we consider the within group optimal labor allocation:

$$\max_{c(i,k)} \quad \left(\int_0^{N_t(k)} c(i,k)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

s.t.

$$c(i,k) = \left[\mathbb{D}_{i \in \mathcal{I}_k} y(k)^{\frac{1}{\sigma-1}} + \mathbb{D}_{i \notin \mathcal{I}_k} \right] l(i,k)$$

$$\int_0^{N_t(k)} l(i,k) di = L$$

The optimal solution on labor is to allocate labor according to their productivity. Thus the labor allocated to the leader's variety and to the fringe' variety must be:

$$\frac{l(i_1,k)}{l(i_2,k)} = y(k)$$

Using this first order condition, we can write the within group production function as:

$$C(k) = Z(k)L(k),$$

where

$$Z(z) = [z(1+\phi(z))]^{\frac{1}{\sigma-1}}$$

In the second step, we solve the problem of allocating labor to each product groups:

$$\max_{C(k)} \quad \left(\int_0^1 C(z)^{\frac{\theta-1}{\theta}} h(z) dz \right)^{\frac{\theta}{\theta-1}}$$

s.t.

$$C(z) = Z(z)L(z)$$

$$\int_0^\infty L(z)h(z)di = \mathbf{L}^p$$

Similar argument to the first step implies that the labor must be allocated to their productivity $Z(z)$

and the efficient productivity index must be

$$\mathbf{Z}_t^* = \left(\int_0^\infty [z(1+\phi(z))]^{\frac{1-\theta}{1-\sigma}} h(z)dz \right)^{\frac{1}{\theta-1}} \tag{44}$$

C Necessary Condition for Planner's Optimal Solution

To derive the necessary condition for socially efficient allocation, we use the method from Nuno and Moll 2021. We first write the following Lagrangian for the planner's problem, define $\tilde{\Lambda}_t(z)$ as the discounted multiplier for the forward equation and ω_t as the multiplier for the labor resource constraint. The Lagrangian is written as:

$$\begin{aligned}\mathcal{L}(\eta,\alpha,\beta,\tilde{\Lambda},\omega) = & \int_0^\infty e^{-\rho t}\left(\frac{\mathbf{C}_t^{1-\gamma}}{1-\gamma} - \frac{\mathbf{L}_t^{1+1/\varphi}}{1+1/\varphi}\right)dt \\ & + \int_0^\infty e^{-\rho t} \int \tilde{\Lambda}_t(z)\left(-\dot{h}_t(z) - (\eta_t(z) + Q(\alpha_t(z)\beta_t(z)) - \delta)zh_t(z) + \frac{\nu^2}{2}h_t''(z)\right)dzdt \\ & + \int_0^\infty e^{-\rho t}\omega_t \left(\mathbf{Z}_t(h_t)\left[\mathbf{L}_t - \int [R(\beta_t(z)) + D(\eta_t(z)) + \kappa\alpha_t(z)\beta_t(z)]zh_t(z)dz\right] - \mathbf{C}_t\right)dt\end{aligned} \tag{45}$$

Using integration by part, w could rewrite the second line as:

$$\begin{aligned} & \int_0^\infty e^{-\rho t} \int \tilde{\Lambda}_t(z)\left(-\dot{h}_t(z) - (\eta_t(z) + Q(\alpha_t(z)\beta_t(z)) - \delta)zh_t(z) + \frac{\nu^2}{2}h_t''(z)\right)dzdt \\ = & -\lim_{T\to\infty} e^{-\rho T}\int_0^\infty \tilde{\Lambda}_T(z)h_T(z)dz + \int_0^\infty \tilde{\Lambda}_0(z)h_0(z)dz \\ & + \int_0^\infty \int e^{-\rho t}(\dot{\tilde{\Lambda}}_t(z) - \rho\tilde{\Lambda}_t(z) + (\eta_t(z) + Q(_t(z)\beta_t(z)) - \delta)z\tilde{\Lambda}_t'(z) + \frac{\nu^2}{2}z^2\tilde{\Lambda}_t''(z))h_t(z)dz \end{aligned} \tag{46}$$

The optimal allocation must be a stationary point of the Lagrangian. As the Lagrangian is a functional, to find the local extremum, we look for functions that make the gains from the following perturbation tend to zero:

$$0 = \lim_{\upsilon\to 0}\frac{\mathcal{L}|_{\mathbf{X}^*+\upsilon\mathbf{X}'} - \mathcal{L}|_{\mathbf{X}}}{\upsilon}. \tag{47}$$

In the following paragraphs, we derive the perturbation in different spaces:

$$
\begin{aligned}
0 =& \lim_{v \to 0} \frac{\mathcal{L}|_{h^*+vh'} - \mathcal{L}|_{h^*}}{v} \\
=& \int_0^\infty e^{-\rho t} \left(\frac{1}{\theta - 1} \frac{P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}}}{\int_0^\infty P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz} - \frac{M(\phi(z))^{-1} P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}}}{\int_0^\infty M(\phi(z))^{-1} P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz} \right) \omega_t \mathbf{C}_t dt \\
&- [R(\beta_t(z)) + D(\eta_t(z)) + \kappa \alpha_t \beta_t] \, z \hat{h}_t(z) dz \omega_t \mathbf{Z}_t dt \\
&+ \int_0^\infty \int_z e^{-\rho t} (\dot{\tilde{\Lambda}}_t(z) - \rho \tilde{\Lambda}_t(z) + (\eta_t(z) + Q(_t(z)\beta_t(z)) - \delta) z \tilde{\Lambda}'_t(z) + \frac{\nu^2}{2} z^2 \tilde{\Lambda}''_t(z)) h_t(z) dz dt \\
&+ e^{-\rho T} \int_z \tilde{\Lambda}_T(z) \hat{h}_T(z) dz + \int_z \tilde{\Lambda}_0(z) \hat{h}_0(z) dz
\end{aligned} \tag{48}
$$

For this equality to hold for any perturbation $\hat{h}_t(z)$, it must be the integrand are zero almost surely. Using the definition $v_t(\phi) = \frac{1}{\theta-1} \frac{1}{\int_0^\infty P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz} - M(\phi)^{-1} \frac{1}{\int_0^\infty M(\phi(z))^{-1} P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} h_t(z) dz}$, we have the following condition:

$$
\begin{aligned}
\rho \tilde{\Lambda}_t(z) =& v_t(\phi_t(z)) P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} \omega_t \mathbf{C}_t - [R(\beta_t(z)) + D(\eta_t(z)) + \kappa \alpha_t(z) \beta_t(z)] z \omega_t \mathbf{C}_t \\
&+ (\eta_t(z) + Q(_t(z)\beta_t(z)) - \delta) z \tilde{\Lambda}'_t(z) + \frac{\nu^2}{2} z^2 \tilde{\Lambda}''_t(z) + \dot{\tilde{\Lambda}}_t(z)
\end{aligned} \tag{49}
$$

The optimal choice of $\mathbf{L}_t$ and $\mathbf{C}_t$ imply:

$$
\mathbf{C}_t^{-\gamma} = \omega_t \tag{50}
$$

$$
\mathbf{L}_t^{-\gamma} = \omega_t \mathbf{Z}_t \tag{51}
$$

We define the value $\Lambda_t(z) = \frac{\tilde{\Lambda}_t(z)}{\omega_t \mathbf{C}_t}$, we have the Bellman equation in the main text:

$$
\begin{aligned}
\left(\rho + (\gamma - 1) \frac{\dot{\mathbf{C}}_t}{\mathbf{C}_t} \right) \Lambda_t(z) =& v_t(\phi_t(z)) P(\phi_t(z))^{1-\theta} z^{\frac{1-\theta}{1-\sigma}} - [R(\beta_t(z)) + D(\eta_t(z)) + \kappa \alpha_t(z) \beta_t(z)] z \frac{\mathbf{Z}_t}{\mathbf{C}_t} \\
&+ (\eta_t(z) + Q(_t(z)\beta_t(z)) - \delta) z \Lambda'_t(z) + \frac{\nu^2}{2} z^2 \Lambda''_t(z) + \dot{\Lambda}_t(z)
\end{aligned} \tag{52}
$$

The result of perturbation requires the that

$$
\Lambda'_t(z) = \frac{\mathbf{Z}_t}{\mathbf{C}_t} z R'(\beta_t(z))
$$