# A Network Foundation Of The Matching Function

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**Abstract**

The matching function is a "black box" whose satisfactory microfoundation has proven elusive for nearly forty years now. In this paper we propose a network variant of the classic balls-in-bins model that allows us to treat the problem at significant generality. The matching function that emerges endogenously depends on the whole network of connections between unemployed and firms, yet it is captured by a compact expression. We show how in special cases matching depends only on the number of unemployed and vacancies as typically assumed in the literature, and we recover existing functional forms. We also analyze how matching efficiency depends on the network structure with two main themes emerging: (a) applicant-side heterogeneity is harmful for matching efficiency, and (b) increasing search intensity, i.e. adding links to the network has an ambiguous effect on matching efficiency. Finally, we derive an "anything goes" type of result: even just within Erdös-Renyi networks, any matching function, including Cobb-Douglas and CES, can be derived assuming the mean of applicants' search intensity scales "appropriately" with the network size.

[Preliminary. Please do not circulate.]

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# 1 Introduction

The matching function is the central building block of any model that departs from the Walrasian equilibrium to capture search frictions in the market. It has proven very tractable and largely consistent with a long literature of empirical findings. The number of contexts it has been used in highlights its success: the labor market, where unemployed and vacancies coexist in equilibrium, but also credit markets, goods markets, assets trading over the counter, the new monetarist literature aiming to explain the endogenous emergence of money.

The matching function, however, is an aggregate, reduced-form construct and its microfoundation has proven elusive. Said differently, if the matching function is indeed an empirical regularity of the macroeconomy, economists have no good story as to what gives rise to it. Virtually all attempts towards such a microfoundation are based on some version of the classic balls-in-bins model of probability theory, which seems to very naturally capture the types of coordination failures hypothesized to underlie the matching function. Yet the literature has stopped short of a satisfactorily general treatment of the problem.

Early attempts gave functional forms that are empirically implausible (e.g. Petrongolo and Pissarides, 2001). More recent attempts in the directed search literature, faced with technical complications, have treated rather special instances of the problem only. For example, Burdett, Shi and Wright (2001) derive a matching function when all applicants send a single application. The literature on social networks, which has singled out the importance of personal connections in finding a job, has delivered a matching function that is increasing and concave in its two arguments, but does not exhibit constant returns to scale or resemble any of the commonly assumed specifications (e.g. Calvó-Armengol and Zenou, 2005).

In this paper we derive the matching function in a general context using tools from the theory of networks. Network theory is particularly appropriate to handle situations of granular, micro-level heterogeneity as those encountered between workers and firms in the labor market, which is what allows us to overcome pre-existing technical difficulties. Our treatment of the problem is at significantly greater generality than what has been done before: applicants can differ in the number of applications they send, and our analysis applies equally well to arbitrarily small or large economies. The model we present can be thought as a network variant of the classic balls-in-bins model, thus the key underlying friction is the coordination failure most commonly found in the literature, including all contributions mentioned above.

We adopt a simple applications-and-hiring protocol, capturing such standard coordination failures. An arbitrary bipartite network connects unemployed (applicants) and vacancies. Each unemployed applies to all vacancies they are connected to; each vacancy makes one offer at random among all applications received (if any); each unemployed chooses one offer at random among the offers received (if any). Formally, when referring to "matching" we mean the expected number of total matches over all possible realizations of scenarios.[1]

The matching function we derive is captured by a compact expression and depends generally on the whole network. In other words, matching is generally a function of the whole *sets* of connections on the two sides of the market rather than just the *sizes* of the two sets of unemployed and vacancies, as typically assumed to be the case. We have two main objectives: (a) relate this general matching function to usual matching functions depending only on the number of unemployed and vacancies; (b) analyze how matching efficiency depends on the network structure.

In a first stage, we analyze non-stochastic networks. We find applicants' heterogeneity reduces matching efficiency: holding firms' degrees fixed, link swaps which increase the dispersion in applicants' job-finding probabilities reduce overall expected matching. In the case all firms receive the same number of applications, any mean-preserving spread in applicants' degrees decreases overall matching.

A natural way to generate a standard matching function with non-stochastic networks is to impose symmetry. In particular, matching only depends on the numbers of unemployed and vacancies in a double regular network, i.e., when all applicants send the same number of applications and all firms screen the same number of applicants. Additionally this matching

[1] An example of a scenario being: applicant 1 applies to firm 1; applicant 2 also applies to firm 1. Firm 1 makes an offer to applicant 1, who also receives an offer from firm 2. Applicant 1 chooses (and gets hired by) firm 2. This outcome occurs with some probability in our model, and we take the expectation over all such possible outcomes.

function satisfies key standard properties – concavity, and constant returns to scale, and perhaps surprisingly, matching efficiency is maximized (miscoordination is minimized) when applicants send a unique application.

In a second stage, we analyze stochastic networks. We consider bipartite networks that are random conditional on the applicants' degree distribution. The degrees (search intensity) of the applicants are drawn iid from that distribution. Links then fall at random on vacancies. We show that this model nests the classic Erdös-Renyi model where every link is formed with the same probability.

We find that, again, heterogeneity reduces matching efficiency: a mean-preserving spread in the applicants' degree distribution decreases the expected matching rate. Interestingly, matching efficiency is maximized in the Erdös-Renyi model when links are formed with probability 1, i.e. the graph is complete. Contrasting this with the result on double regular graphs indicates that adding links generally has an ambiguous effect on matching efficiency.

Finally, we derive an "anything goes" type of result. Restricting ourselves to the Erdös-Renyi networks we show that any matching function, including Cobb-Douglas and CES, can be derived assuming the mean of applicants' degrees (search intensity) scales "appropriately" with the network size, i.e. the size of the two sets of unemployed and vacancies. Thus classical matching functions are consistent with a network-based matching process. Unsurprisingly, however, micro-level network features may not be identified from only observing aggregate matching.

We take our contribution to be primarily a methodological one, illustrating the use of networks to microfound a regularity that shows up in the aggregate, as has been done in other contexts (e.g. Acemoglu et al., 2012). We also consider both literatures already mentioned, directed search and social networks in the labor market, to offer complementary approaches to ours, and take part of our contribution to be in linking the two.

The directed search literature highlights the importance of information, primarily wages posted by firms, on how applicants search for jobs. Thus in our view, directed search offers models of *network creation.* A good part of that literature has focused on the efficiency of matching (e.g. Kircher, 2009; Galenianos and Kircher, 2009), which is a point of focus for us as well. The other main point of focus is wage dispersion, an issue we abstract from in the present work.[2]

Without the networks treatment however, that literature seems to have reached technical barriers concerning the setups where it can derive a matching function endogenously. Burdett, Shi and Wright (2001) derive a matching function when all applicants send a single application. Albrecht, Gautier and Vroman (2006) treat cases where applicants send multiple but the same number of applications, and they have to restrict attention only to the limiting case of a large economy. We recover the same functional forms in both special cases.

The literature on social networks in the labor market, largely stemming from Calvó-Armengol (2004), has evolved parallel and somewhat unconnected to the directed search literature.[3] This literature has singled out the importance of personal connections in finding a job. We naturally draw inspiration from this literature and are close to it conceptually, as social ties are also granular micro-level objects. For us, however, social ties and their impact are an additional layer that can be added to the analysis, not the only one.

As our model is different, the matching functional forms we get are different even in the special cases explicitly treated in this literature, as for example when all applicants have the same degree (Calvó-Armengol and Zenou, 2005). However, we do share some of the qualitative properties regarding efficiency of matching: when all applicants have the same degree, matching has an inverted-U shape as a function of that degree, echoing similar results for example in Calvó-Armengol and Zenou (2005).

The rest of the paper is structured as follows. Section 2 lays out the economic setup and introduces notation. Section 3 derives the matching function for an arbitrary given network

[2]The rich body of contributions is surveyed by Wright et al. (2021).

[3]The seminal modern contribution here can be taken to be Calvó-Armengol (2004). A series of very interesting papers have followed including Calvó-Armengol and Jackson (2004), Calvó-Armengol and Zenou (2005), Ioannides and Soetevent (2006), Galenianos (2014), Galeoti and Merlino (2014), Espinosa, Kovářík and Ruiz-Palazuelos (2021). Montgomery (1991), even though earlier, can also be added as an important and interesting contribution in this literature.

and goes over some useful special cases of networks. In section 4 we do comparative statics in the case of non-stochastic networks, i.e. taking the network as given. In section 5 we extend the analysis to stochastic networks. Section 6 concludes discussing also some avenues for future research.

## 2 The setup

We start by introducing the economic environment, some terminology and notation.

**Primitives**: The economic environment consists of two sets of agents $\mathcal{U}$ and $\mathcal{V}$, of size $U, V \in \mathbb{N}$ respectively, and a (bipartite) graph $G$ linking elements between the two sets.

We take the elements of $\mathcal{U}$ to correspond to *applicants*, i.e. workers searching for a job, and the elements of $\mathcal{V}$ to correspond to *jobs* offered by firms. In other words $\mathcal{U}$ contains the *unemployed*, while $\mathcal{V}$ contains *vacancies* in our setup.

As a convention, we will be indexing the elements of $\mathcal{U}$ by $i = 1, 2, ..., U$ and the elements of $\mathcal{V}$ by $j = 1, 2, ..., V$. Following the search and matching literature we will assume that each firm has a single vacancy to fill, thus we may be interchangeably refer to firm $j$ or vacancy $j$ as the counterparty of an applicant $i$.

The graph $G$ is represented by an adjacency matrix – denote $G = \left(g_{ij}\right)$, where $g_{ij} = 1$ if applicant $i$ is connected to firm $j$, and $g_{ij} = 0$ otherwise.[4]

We will denote by $d_i = \sum_j g_{ij}$ an applicant's *degree*, that is the number of firms the applicant connects to. Similarly a firm's degree $d_j = \sum_i g_{ij}$ corresponds to the number of applicants the firm connects to. As a matter of accounting it has to hold that $\sum_i d_i = \sum_j d_j$.

Finally we refer to an applicant's *neighborhood* as the set of firms the applicant connects to. Specifically, for an applicant $i$, define $N_i = \{j \in \mathcal{V} : g_{ij} = 1\}$. Similarly we can define the neighborhood of a firm $j$. It follows that the size of a node's neighborhood equals their degree; for applicant $i$ denote $|N_i| = d_i$.

[4] Applicants correspond to rows and firms to columns.

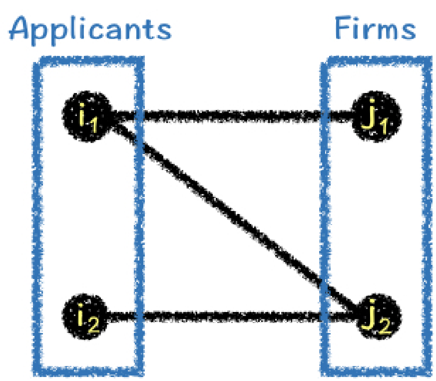
As in what follows we will be making connections to the search and matching literature, let us also define the central quantity of that literature, *market tightness*, $\theta = \frac{V}{U}$.

**An application and hiring protocol**: Taking the network of links $G$ as given, we assume applicants apply to all firms they connect to. A firm chooses one of the applicants uniformly at random to whom it makes an offer. Each applicant chooses to accept one offer uniformly at random among the offers they receive.

The key object of interest throughout our analysis, to which we now turn, is the *matching rate* defined as the expected number of matches given a particular network. We denote

$$m(G) = \mathbb{E}[\#\text{matches}|G]$$

**An example**: Let us consider the following instance[5]




Applicant $i_1$ applies both to firms $j_1$ and $j_2$, while applicant $i_2$ applies only to firm $j_2$.

Accordingly, firm $j_1$ makes an offer to applicant $i_1$, and firm $j_2$ chooses with probability 1/2 to make an offer to $i_1$ and with probability 1/2 to make an offer to $i_2$.

There are three possible outcomes:

[5]In matrix form the graph of this example is $G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For exposition, we also note that the degree of applicant $i_1$ is 2, and of applicant $i_2$ it is 1. Their corresponding neighborhoods are the sets $\{j_1, j_2\}$, and $\{j_2\}$ respectively.

(1) $j_2$ makes offer to $i_2$. $i_1$, and $i_2$ each accept their single offer.

(2) $j_2$ makes an offer to $i_1$. $i_1$ chooses $j_2$; $i_2$ has no offer.

(3) $j_2$ makes an offer to $i_1$. $i_1$ chooses $j_1$; $i_2$ has no offer.

The first outcome is the first best. Outcomes 2 and 3 are states of coexisting vacancy and unemployment, as the outcome of coordination failure, an issue we will illustrate further in the rest of the analysis.

According to our protocol, the first outcome occurs with probability 1/2; each of the other two outcomes occurs with probability 1/4. Thus the expected number of matches is given by

$$\begin{aligned} m(G) &= \frac{1}{2} \cdot 2 + 2 \cdot \frac{1}{4} \cdot 1 \\ &= 3/2 \end{aligned}$$

**A note on interpretation**: The links of the graph can correspond to social ties – as in the social networks literature (e.g. Calvó-Armengol, 2004), or skills required to apply for that job, or geographic restrictions the applicant has on where to work. In other words the graph can represent *any* relevant factors restricting the jobs an applicant can apply to, and we don't need to take a stance on it for our analysis. We will return to this point in the concluding remarks.

# 3 Matching when the network is given

What we did for the small instance in the example above, we can do in the general case for an arbitrary graph $G$.

**Proposition 1.** *For any given arbitrary graph $G$, the matching rate defined as the expected number of total matches is given by*

$$m(G) = U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \tag{1}$$

*Proof.* For any applicant $i$ the probability to receive no offer is $\prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$, and thus their probability of finding a job (= the probability of receiving at least one offer) is

$$f^i \equiv Pr\{i \text{ gets hired}\} = 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$$

Now, for each applicant define the indicator r.v. showing if they find a job, where

$$Y_i = \begin{cases} 1, & \text{w.p. } f^i \\ 0, & \text{w.p. } 1 - f^i \end{cases}$$

Then the number of matches, taking the graph as given, which by definition is the number of applicants finding a job is also a r.v., and specifically $\#\text{matches}|G = \sum_i Y_i$.

The matching rate, i.e. the expected number of matches is then

$$\begin{aligned} m(G) &= \mathbb{E}[\#\text{matches}|G] \\ &= \mathbb{E}[\sum_{i=1}^{U} Y_i] \\ &= \sum_{i=1}^{U} \mathbb{E}[Y_i] \\ &= \sum_{i=1}^{U} f^i \\ &= U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \end{aligned}$$

□

Let us pause and appreciate how compact an expression (1) is for how general it is: it gives us the expected number of matches for *any* possible graph $G$, for *any* two sets $\mathcal{U}, \mathcal{V}$.

We think it is useful to highlight that each applicant $i$ finding a match is a Bernoulli trial with probability of success $f^i$. The Bernoulli trials are not independent, they all depend on $G$, but to compute $m(G)$ independence is not required; we only use the linearity of expectation. We also note that the derivation of $f^i$ hinges on each firm deciding independently from all other firms which applicant to make an offer to.

Let us now see how (1) specializes in specific types of graphs, and specifically how it reduces in being a function only of the sizes $U, V$ of the two sets as is usual in the search and matching literature.

**Example 1**: The *complete graph* (or family of graphs) is the case where all applicants connect to all firms. In this case $N_i = \mathcal{V},\ \forall i$, and $d_j = U,\ \forall j$, thus (1) becomes

$$\begin{aligned} m(G) &= U - \sum_{i=1}^{U} \left(1 - \frac{1}{U}\right)^{V} \\ &= U\left(1 - \left(1 - \frac{1}{U}\right)^{V}\right) \end{aligned}$$

It can be seen that the matching function in this case is increasing and concave in its two arguments.[6] It can also be seen to be the matching function derived by Burdett, Shi and Wright (2001).[7]

We also note the complete graph case is precisely the classic balls-in-bins setup. Thus, it is no surprise the above is the same type of matching function derived early in the literature using the standard balls-in-bins model (e.g. Pissarides, 1979; Blanchard and Diamond, 1994).

It is broadly accepted in the literature (e.g. Petrongolo and Pissarides, 2001; Wright et al., 2021) that the empirical relevance of this functional form and its limiting version[8] is quite limited.

**Example 2**: A *double regular graph* (or family of graphs) is the case where every applicant is connected to $d_U$ firms, and each firm is connected to $d_V$ applicants.

This is a doubly-symmetric graph where all applicants and all firms search with the same "intensity," and we show it relates closely to the standard search-and-matching setup. We note that a double regular graph can be thought to correspond to a symmetric equilibrium.

[6] Shown in the appendix.

[7] Replace $U$ with $m$, and $V$ with $n$ to get their eq. 18. See also Wright et al. (2021), eq. 46.

[8] Using the result $\lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n = e^x$, the above assymptotically exhibits constant returns to scale, as for large $U$ the matching function can be taken to be approximately $m(V,U) \approx U\left(1 - e^{-\frac{V}{U}}\right)$.

For such graphs (1) gives us the matching function being

$$m(G) = U\left[1 - \left(1 - \frac{1}{d_V}\right)^{d_U}\right]$$

However, by accounting it holds that $Ud_U = Vd_V$, and utilizing this equation we can write

$$m(U, V) = U\left[1 - \left(1 - \frac{1}{d_U}\frac{V}{U}\right)^{d_U}\right]$$

where $d_U$ is taken to be a parameter, and $d_V$ is determined from $Ud_U = Vd_V$. All applicants have the same job-finding probability. Denoting $\theta = \frac{V}{U}$ this probability is[9]

$$f(\theta) = 1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

The matching function in this case can be shown to possess standard properties assumed in the literature: it is constant returns to scale, increasing, and concave in both $U$, and $V$. The matching function can also be shown to be approximated at a first-order by a Cobb-Douglas function.[10] We show these results formally in the appendix.

We think the following result is interesting to compare with the more general results that follow on efficiency.

**Proposition 2.** *For the double regular network the matching rate is maximized when $d_u = 1$.*

*Proof.* See appendix. □

[9] Naturally, not *any* choice of a $d_U$ will do; $d_U$ has to be an integer and it has to be such that $d_V$ is also an integer. This points to the limitation of this model (or special case) we are considering here. Having said that, there are a couple remarks to make which we think make this model useful. Also the standard "relaxation" of integer constraints applies when we are talking about thousands or millions of objects.

[10] We note that the 1st-order approximation result is not specific to this family of graphs: *any* matching function that exhibits CRS, to a 1st-order approximation is Cobb-Douglas.

# 4 Comparative statics when the network is given

We start again from the matching function formula (1) in the general case

$$m(G) = U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$$

Define $p_i \equiv \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$, and $x_i \equiv \ln(p_i)$. $p_i$ is the probability applicant $i$ receives no offer (and thus the complement of the probability applicant $i$ is hired). $x_i$ is an increasing transformation of that. We will denote by $x$ the $U-$dimensional vector $(x_i)$.

**Lemma 1.** *Assume $x'$ is a mean-preserving spread (MPS)[11] of $x$, and $x$ and $x'$ correspond to two networks $G, G'$ respectively with the same number of applicants $U$. Then*

$$m(G') < m(G)$$

*Proof.* A direct application of the fact that when $x'$ is a MPS of $x$, then $\sum_i e^{x'_i} > \sum_i e^{x_i}$. □

This is a rather general result, but for it to not just be an abstract statement, we want to be able to say what changes in networks create MPS in $x$'s. We show cases where the result applies and illustrate its policy relevance. What we can already see from the result is that matching efficiency relates intricately to the distribution of applicants' probability to (not) be hired.

**Proposition 3.** *Suppose $d_j = d_V, \forall j$. If $(d'_i)$ is a MPS of $(d_i)$, then $m(G') < m(G)$.*

*Proof.* In this case

$$\begin{aligned} x_i &= \ln \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \\ &= \ln \left(1 - \frac{1}{d_V}\right)^{d_i} \end{aligned}$$

[11] The Appendix of this section provides some background material on mean-preserving spreads over arbitrary vectors which are not necessarily a probability distribution.

$$= d_i \ln\left(1 - \frac{1}{d_V}\right)$$

Thus if $(d_i')$ is a MPS of $(d_i)$, $x'$ is a MPS of $x$ and thus lemma 1 applies. $\square$

**Corollary 1.** *Suppose $d_j = d_V, \forall j$, and let $G_R$ denote the corresponding doubly regular graph, if that exists. Then for any graph $G$, $m(G) \leq m(G_R)$.*

Thus, when the situation is homogeneous on the firms' side, matching efficiency increases with homogeneity on the applicants' side as well. Conversely, any increase in the spread in applicants' degrees will reduce matching efficiency.

An applicant's degree is 1-1 related to the probability of the applicant receiving no offers $(p_i)$ and thus the probability the applicant gets hired $(1 - p_i)$. By equalizing the applicants' degrees (spreading their degrees as evenly as possible) we equalize their probability of being hired. This result is generalizable beyond the case where firms' degrees are equal.

In fact both proposition 3 and its corollary have more general analogs, where we hold any arbitrary distribution of $(d_j)$'s fixed.

**Theorem 1.** *Take an arbitrary network $G$, and let $G'$ denote the network resulting from swapping a link $ij$ with link $i'j$, where $i' \neq i$. Then $m(G') < m(G)$, if $|\ln(p_i) - \ln(p_{i'})| < \ln(p_i') - \ln(p_{i'}')$.*

*Proof.* We start by showing that a link swap between two applicants is mean-preserving in $x$'s. Indeed, we have

$$\begin{aligned}
x_i &= \ln \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \\
&= \sum_{j \in N_i} \ln\left(1 - \frac{1}{d_j}\right) \\
&= \sum_{j} \ln\left(1 - \frac{1}{d_j}\right)^{g_{ij}} \\
&= \sum_{j} g_{ij} \ln\left(1 - \frac{1}{d_j}\right)
\end{aligned}$$

Then swapping a link $ij$ with link $i'j$ means $g_{ij} = 1 \to g'_{ij} = 0$, and $g_{i'j} = 0 \to g'_{i'j} = 1$, or

$$x'_i = x_i - \ln\left(1 - \frac{1}{d_j}\right)$$
$$x'_{i'} = x_{i'} + \ln\left(1 - \frac{1}{d_j}\right)$$

Thus $x'_i + x'_{i'} = x_i + x_{i'}$.

What remains to be shown is that if in addition $\ln(p'_i) - \ln(p'_{i'}) > |\ln(p_i) - \ln(p_{i'})|$, then $x' \equiv \ln(p')$ is a MPS of $x \equiv \ln(p)$, and thus lemma 1 will apply.

Indeed

$$x'_i > x_i$$
$$x'_{i'} < x_{i'}$$

And as shown in the appendix, $x'_i - x'_{i'} > |x_i - x_{i'}|$ is a sufficient condition for $x'$ being a MPS of $x$. □

This is a useful proposition for policy purposes as it provides a (greedy) algorithm of how to improve a network:[12]

**Corollary 2.** *Fix the firms' degree distribution* $(d_j)$*, and let* $G_{\tilde{R}}$ *denote the corresponding graph from exhausting all link swaps that reduce inequality in applicant's probability of being hired. Then* $m(G) \leq m(G_{\tilde{R}})$.

So what we have shown is that holding the firms' side fixed – in the precise sense of holding the firms' degree distribution fixed, equalizing the probabilities of applicants be hired increases the efficiency of the system. This is a quite clear-cut result stating that inequality in probability to be hired is *never* (constrained) optimal for the aggregate economy.

[12] We have no reason to believe this algorithm gives a unique $G^R$ independent of the order of swaps; it does give us though a direction towards a "local optimum."

# 5 Matching under a model of network generation

Up to now we have taken the underlying network – the bipartite graph, as given and analyzed the implications for matching. We now introduce a model of a (bipartite) random graph as a model of how the graph is generated. As before we will be indexing applicants with $i = \{1, 2, .., U\}$, and firms with $j = \{1, 2, ..., V\}$.

**A random graph model**: Take applicant degrees to be i.i.d. draws from a given distribution $\vec{p} = (p_0, p_1, ...p_V)$, where $p_k \equiv Pr\{d_i = k\}$. For each applicant, conditional on a given draw from that distribution, the links are assumed to fall at random on an equal number of distinct firms among the $V$.[13]

The applicant-degree distribution $\vec{p}$ can be any arbitrary distribution over the non-negative integers. The random graph model induces a distribution of degrees on the firm side, which we will show is a binomial distribution.

We will denote the firm-degree distribution by $\vec{z} = (z_0, z_1, ...z_U)$, where $z_k \equiv Pr\{d_j = k\}$. We will also denote the mean degree on the applicant and firm sides by $\bar{d}_U, \bar{d}_V$ respectively, i.e. $\bar{d}_U = \sum_k kp_k$ and $\bar{d}_V = \sum_k kz_k$.

**Lemma 2.** *Conditional on an applicant-degree distribution $\vec{p}$, the degrees on the firm side follow a binomial distribution, denote $d_j \sim Bin(\lambda, U)$, where $\lambda = \frac{\bar{d}_U}{V}$.*

*Proof.* We have

$$z_k = Pr\left\{\sum_{i=1}^{U} X_{ij} = k\right\}$$

where $X_{ij}$ is an indicator, being 1 if applicant $i$ links to (has applied to) firm $j$.

[13] More formally that is random sampling of $d_i$ elements from a population of $V$ without replacement.

Since all $i$ are ex-ante i.i.d, $X_{ij}$ are also i.i.d with probability[14]

$$\begin{aligned} Pr\{X_{ij}=1\} &= \sum_{k=1}^{V} Pr\{X_{ij}=1|d_i=k\}p_k \\ &= \sum_{k=1}^{V} \frac{\binom{V-1}{k-1}}{\binom{V}{k}} p_k \end{aligned}$$

Define $\lambda \equiv \sum_{k=1}^{V} \frac{\binom{V-1}{k-1}}{\binom{V}{k}} p_k$. Now, by noticing that $\frac{\binom{V-1}{k-1}}{\binom{V}{k}} = \frac{k}{V}$, it follows that

$$\lambda = \frac{\bar{d}_U}{V} \tag{*}$$

Then $X_{ij}$ are Bernoulli with probability of success $\lambda$, and thus $d_j \sim Bin(\lambda, U)$. □

We note that $\lambda$ is a function of $\vec{p}, V$ but for notational simplicity we are not denoting this explicitly.

**Remark**: Since $d_j \sim Bin(\lambda, U)$, it follows that $\bar{d}_V = \lambda U$, and thus[15]

$$\frac{\bar{d}_V}{U} = \frac{\bar{d}_U}{V}$$

This is a useful relationship we will invoke again in our analysis later.

**Corollary 3.** *A mean-preserving spread in the distribution of $d_i$'s leaves the distribution of $d_j$'s unchanged.*

*Proof.* This follows from $d_j$'s following a binomial distribution, and its parameter $\lambda$ depending only on $\bar{d}_U, V$, which stay constant with a mean-preserving spread. □

We will return to this result when we do comparative statics.

[14] The second line follows from a standard combinatorial argument: we want to find how many choices include a particular element $i$, among all the $\binom{V}{k}$ possible choices. We fix element $i$, and are free to choose the remaining $k-1$ elements from the remaining $V-1$ elements of the pool: these are precisely $\binom{V-1}{k-1}$.

[15] In fact it can be shown this is an accounting identity that has to hold for any bipartite random graph.

## 5.1 Moving to matching

We now derive the matching rate in the stochastic network case. The matching rate now is a *double* expectation, over who makes an offer to whom (as before), but also over the realized network $G$. In other words, the matching rate now is

$$m = \mathbb{E}_G[m(G)]$$

We first prove a lemma we will need regarding the *excess degree* of a firm an applicant connects to. The excess degree – denote by $\tilde{d}$, refers to the number of edges leaving the firm *other than* the edge of the said applicant.[16]

**Lemma 3.** *The excess degrees of all firms an applicant connects to (a) are i.i.d, and (b) it holds that* $Pr\{\tilde{d} = k\} = \frac{(1+k)z_{1+k}}{\bar{d}_v}$.

*Proof.* The degrees of firms are i.i.d following $Bin(\lambda, U)$. Thus the excess degrees of a firm an applicant connects to are also i.i.d and $\tilde{d} \sim Bin(\lambda, U-1)$, since $U-1$ only of the firm's degree Bernoulli trials remain to be determined. It follows that

$$\begin{aligned} Pr\{\tilde{d} = k\} &= \binom{U-1}{k}\lambda^k(1-\lambda)^{U-1-k} \\ &= \frac{(U-1)!}{k!(U-1-k)!}\lambda^k(1-\lambda)^{U-1-k} \\ &= \frac{1+k}{\lambda U}\frac{U!}{(1+k)!(U-1-k)!}\lambda^{1+k}(1-\lambda)^{U-1-k} \\ &= \frac{(1+k)z_{1+k}}{\bar{d}_V} \end{aligned}$$

□

**Theorem 2.** *The matching rate in our stochastic network model, defined as* $m = \mathbb{E}_G[m(G)]$, *is given by*

$$m = U\left(1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U}\right) \tag{2}$$

[16] We note that the result $Pr\{\tilde{d} = k\} = \frac{(1+k)z_{1+k}}{\bar{d}_v}$ is a special case of a more general result known for the configuration model (e.g. Newman (2003), Jackson (2010)). The result is exact in our case, while in the configuration model it is approximate and holds asymptotically for a large number of nodes.

*where* $\phi = \frac{1-z_0}{\bar{d}_V}$. $z_0 = (1-\lambda)^U$ *is the probability a firm receives no applications.*

*Proof.* We have that

$$Pr\{i \text{ finds job} \mid N_i, d_j \forall j \in N_i\} = 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$$

Therefore the (ex-ante) probability an applicant finds a job is given by

$$\begin{aligned}
f &= 1 - \mathbb{E}_{N_i} \left\{ \mathbb{E}_{(d_j)_{j \in N_i}} \left\{ \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \middle| N_i \right\} \right\} \\
&= 1 - \sum_{N_i} p_{N_i} \mathbb{E}_{(d_j)_{j \in N_i}} \left\{ \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \middle| N_i \right\} \\
&= 1 - \sum_{N_i} p_{N_i} \prod_{j \in N_i} \left(1 - \mathbb{E}_{d_j} \left\{ \frac{1}{d_j} \middle| N_i \right\}\right) \\
&= 1 - \sum_{N_i} p_{N_i} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{|N_i|} \\
&= 1 - \sum_{d_u} \sum_{N_i : |N_i| = d_u} p_{d_u} \frac{1}{\binom{V}{d_u}} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{d_u} \\
&= 1 - \sum_{d_u} p_{d_u} \frac{1}{\binom{V}{d_u}} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{d_u} \sum_{N_i : |N_i| = d_u} 1 \\
&= 1 - \sum_{d_u} p_{d_u} \left(1 - \mathbb{E}_{\tilde{d}} \left\{ \frac{1}{1+\tilde{d}} \right\}\right)^{d_u} \\
&= 1 - \sum_{d_u} p_{d_u} \left(1 - \sum_{k=0}^{U-1} \frac{1}{1+k} \frac{(1+k) z_{1+k}}{\bar{d}_V}\right)^{d_u} \\
&= 1 - \sum_{d_u} p_{d_u} \left(1 - \frac{1}{\bar{d}_V} \sum_{k=1}^{U} z_k\right)^{d_u} \\
&= 1 - \sum_{d_u} p_{d_u} \left(1 - \frac{1-z_0}{\bar{d}_V}\right)^{d_u}
\end{aligned}$$

Since this it the probability of each applicant finding a job, the expected number of matches is given by $m = \sum_i f = Uf$.[17] □

[17] Note that $\phi < 1$: $\bar{d}_v = \sum_{k=1}^{U} k z_k = \sum_{k=1}^{U} z_k + \sum_{k=1}^{U} (k-1) z_k > \sum_{k=1}^{U} z_k = 1 - z_0$.

In the derivation of $f$, to go from the 2nd to the 3rd line we used the fact that $d_j$'s are independently distributed within $i$'s neighborhood (part (a) of lemma), and to go from the 3rd to the 4th line we used that $d_j$'s are also identically distributed within any neighborhood (also part (a) of lemma). To go to the 5th line, we enumerate the neighborhoods by their size and use the fact that the probability to generate a particular neighborhood $N_i$ of size $d_u$ is to draw a degree of $d_u$ and then choose the one among the $\binom{V}{d_u}$ neighborhoods of such size. To go to the 6th line we notice that nothing depends on the exact neighborhood $N_i$, only its size, thus we factor everything out of the second sum. To go to the 7th line we use the fact that the number of neighborhoods with $d_u$ members is precisely $\binom{V}{d_u}$, hence this term cancels. To go to the 8th line we use part (b) of the lemma.

As the key object determining the matching rate is $f$, we will be working directly with it when convenient.

## 5.2 The special case of Erdös-Renyi

Given its popularity in the literature of random graphs, we derive the matching properties of the Erdös-Renyi model, which we show is a special case of our model.

**Lemma 4.** *When $d_i \sim Bin(\mu, V)$, our stochastic network model becomes the Erdös-Renyi model, that is the network can be created drawing each link with the same probability $\mu$.*

*Proof.* We have already shown the firm degree distribution is binomial with parameter $\lambda = \frac{\bar{d}_U}{V}$. But since $d_i \sim Bin(\mu, V)$, $\bar{d}_U = \mu V$. Thus $\lambda = \mu$, and $d_j \sim Bin(\mu, U)$.

It follows that in this case the model can be constructed drawing each link with probability $\mu$ as this process amounts to precisely $V$ Bernoulli trials for each applicant, and $U$ Bernoulli trials for each vacancy all with probability of success $\mu$. $\square$

**Corollary 4.** *In the case of the Erdös-Renyi model the matching function is given by*

$$m = U\left(1 - \left[1 - \frac{1-(1-\mu)^U}{U}\right]^V\right)$$

*Proof.* The matching function is $m = U \cdot f$. We will work with the job-finding probability $f$. Theorem 2 specializes in this case as

$$
\begin{aligned}
f &= 1 - \sum_{d_U=0}^{V} p_{d_U} (1-\phi)^{d_U} \\
&= 1 - \sum_{d_U=0}^{V} \binom{V}{d_u} \mu^{d_U} (1-\mu)^{V-d_U} (1-\phi)^{d_U} \\
&= 1 - \sum_{d_U=0}^{V} \binom{V}{d_u} [\mu(1-\phi)]^{d_U} (1-\mu)^{V-d_U} \\
&= 1 - [1 - \mu + \mu(1-\phi)]^V \\
&= 1 - [1 - \mu\phi]^V \\
&= 1 - \left[1 - \mu \frac{1 - z_0}{\mu U}\right]^V \\
&= 1 - \left(1 - \frac{1 - (1-\mu)^U}{U}\right)^V
\end{aligned}
$$

□

Two polar cases are readily verifiable: As we would expect, for $\mu = 0$, we have the empty graph, and $f = 0$; For $\mu = 1$, we have the complete graph, and $f = 1 - \left[1 - \frac{1}{U}\right]^V$.

**Corollary 5.** *The matching function in the Erdös-Renyi model is increasing in $\mu$, and thus it is maximized when $\mu = 1$ (the complete graph).*

*Proof.* It follows directly from the expression for $f$. □

Contrasting this with the result on double regular graphs, it indicates that a higher search intensity has generally an ambiguous effect on matching efficiency. We will see another variant of this finding in the more general analysis that follows.

## 5.3 Comparative statics in a random network

The applicants' job-finding probability is given generally from the expression

$$f = 1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U}$$

where $\phi = \frac{1-z_0}{\bar{d}_V}$.

We have already established that a mean-preserving spread in $d_i$'s leaves the distribution of $d_j$'s unchanged (corollary 3), and thus $\phi$ is left unchanged as well. We can further show the following proposition

**Proposition 4.** *A mean-preserving spread*[18] *in the distribution of $d_i$'s reduces the applicants' job-finding probability $f$.*

*Proof.* Denote by $\vec{p}\,'$ a mean-preserving spread of $\vec{p}$. Equivalently, the two distributions have the same mean, and $\vec{p}$ second-order stochastically dominates (SOSD) $\vec{p}\,'$ (MCWG proposition 6.D.2). From the definition of SOSD it holds that for every non-decreasing concave functions $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ it holds that

$$\sum_{d_U} p'_{d_U} u(d_U) \leq \sum_{d_U} p_{d_U} u(d_U)$$

Now, $-(1-\phi)^{d_U}$ is an increasing and (strictly) concave function, and then from the definition of SOSD we have

$$\sum_{d_U} p'_{d_U}[-(1-\phi)^{d_U}] \leq \sum_{d_U} p_{d_U}[-(1-\phi)^{d_U}] \Rightarrow$$
$$1 - \sum_{d_U} p'_{d_U}(1-\phi)^{d_U} \leq 1 - \sum_{d_U} p_{d_U}(1-\phi)^{d_U} \Rightarrow$$

[18] A mean-preserving spread is defined as a compound lottery, say $d'_U = d_U + Y$, where $\mathbb{E}[Y|d_U] = 0$ (Rothschild and Stiglitz, 1970). For example define $Y = 0$, if $d_U = 0$, and $Y = \begin{cases} +1, \text{w.p. } 1/2 \\ -1, \text{w.p. } 1/2 \end{cases}$ if $d_U \geq 1$.

$$f' \leq f$$

with equality holding iff $\phi = 0$ or $\phi = 1$. □

**Corollary 6.** *Given $\bar{d}_U$, the matching rate is maximized when everyone sends the same number of applications, $\bar{d}_U$.*

*Proof.* We have

$$1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U} < 1 - (1-\phi)^{\bar{d}_U}$$

following from Jensen's inequality. □

We note that proposition 4 and its corollary echo the results on the impact of heterogeneity on matching efficiency of section 4.

## 5.4 Specific functional forms as special cases

We look at the Erdös-Renyi model, where it holds that

$$f = 1 - \left[1 - \frac{1-z_0}{U}\right]^V$$

When $U \to +\infty$ we get

$$f \to 1 - e^{-(1-z_0)V/U}$$

Now assume it holds that

$$\frac{V}{U}(1-z_0) = -\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right)$$

where $\tilde{m}(U,V)$ is some functional form that can serve as a matching function.[19]

[19] We notice we need to restrict ourselves to cases where $\tilde{m}(U,V) < U$ so that the $\ln(\cdot)$ term is well defined. Cases of matching functions where $\tilde{m}(U,V) \leq U$, e.g. $\tilde{m}(U,V) = \min\{U,V\}$ cannot be handled here. We notice however the analytic tractability of CES, which does satisfy $\tilde{m}(U,V) < U,V$, can give us the Leontief form as limit case when $\gamma \to \infty$.

It will then be the case that

$$f \to \frac{\tilde{m}(U,V)}{U}$$

With a few manipulations we can get an equation explicitly defining an "appropriate" $\bar{d}_U$ as a function of $U, V$ that gives the desired matching function. More specifically, we use that for $U \to +\infty$, the binomial firm-degree distribution becomes approximately a Poisson and hence $z_0 \to e^{-\bar{d}_V}$ to get

$$\begin{aligned}
\frac{V}{U}(1 - z_0) &= -\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
\frac{V}{U}(1 - e^{-U\bar{d}_U/V}) &= -\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
1 - e^{-U\bar{d}_U/V} &= -\frac{U}{V}\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
e^{-U\bar{d}_U/V} &= 1 + \frac{U}{V}\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
\bar{d}_U &= -\frac{V}{U}\ln\left(1 + \frac{U}{V}\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right)\right) \qquad (**)
\end{aligned}$$

**The special case of CES**: $\tilde{m}(U,V) = (U^{-\gamma} + V^{-\gamma})^{-\frac{1}{\gamma}}$, $\gamma > 0$.[20]

**The special case of Cobb-Douglas**: $\tilde{m}(U,V) = m_0 V^{\eta} U^{1-\eta}$, where $\eta, m_0 \in (0,1)$.[21]

The question remains if the above equation defines a feasible $\bar{d}_U$, i.e. if $\bar{d}_U > 0$. As long as $\tilde{m}(U,V) < U$ sufficiently, so that $\ln\left(1 - \frac{\tilde{m}(U,V)}{U}\right) \approx -\frac{\tilde{m}(U,V)}{U}$ is a good approximation, from the 3rd line above we get $1 - e^{-U\bar{d}_U/V} = \frac{\tilde{m}(U,V)}{V}$, and as long as $\frac{\tilde{m}(U,V)}{V} < 1$ (which we do impose) this equation pins down a unique $\bar{d}_U > 0$.

[20] For this functional form it can be checked that $\tilde{m}(U,V) < U, V$.

[21] We restrict $U, V$ in the regions where $\tilde{m}(U,V) < U, V$, which are those $V, U$ s.t. $m_0^{1/(1-\eta)} < \frac{V}{U} < m_0^{-1/\eta}$. The lower $m_0$, the greater this region, while for $m_0 = 1$ the region is empty. The fact that Cobb-Douglas is less tractable than the CES in the discrete case is known in the literature. Cobb-Douglas can also be derived as the 1st-order approximation to the CES.

## 5.5 Random network model discussion

Our result on being able to generate specific matching functions can be taken to illustrate "how much" or rather "what type" of a knife-edge case the Cobb-Douglas, the CES, or in fact any specification of the aggregate matching function are. Through the lens of our model, to get one of these specifications amounts to a particular type of *scaling* of the applicant-degree distribution: For any pair of $U, V$, $\bar{d}_U$ has to scale "appropriately" – as given by (**), for the matching function to be of the respective functional form.

We wish to highlight this is a novel feature of our analysis, illustrating that the applicants' search effort, i.e. the applicant degrees, are elevated to first-tier citizens in our analysis: search intensity and a matching technology of certain properties are not two separate things, they are one and the same. More concretely, it is only under a specific distribution of applicant-degrees (search intensity) that certain functional forms emerge.

We note this type of analysis is to some extent analogous to the asymptotic analysis of the the Erdös-Renyi model, where the key parameter of the distribution, $p$, is parameterized as a function of the size, and thus we study the properties of the graph for different specifications of $p(n)$ (e.g. Jackson 4.2.2, p. 89).

More broadly, let us note that our random nets treatment may appear to have a somewhat atypical feature, namely that applicants with $d_i = 0$ are included in the set of "unemployed."[22] While this is indeed worth pointing out, it can be handled by our setup without affecting any of the analysis by limiting attention to distributions with $Pr\{d_i = 0\} = 0$. Then no applicant has zero degree, that is only people who exert some search effort to find a job, sending at least one application are included among the unemployed, as typically done in both the empirical and theoretical literature. In other words, our random graph approach treats in a unified way the decision to enter the labor market (send $\geq 1$ applications – extensive margin of search) and the search effort an applicant puts (number of applications – intensive margin.)

[22] In the Erdös-Renyi model for example, some nodes in the set of $\mathcal{U}$ will indeed have zero degree. Applying twice the 1st-order approx. $\ln(1+x) \approx x$ in (**), we get $\bar{d}_U \approx \frac{\tilde{m}(U,V)}{U} < 1$.

At the same time it is worth highlighting that some firms *will* have zero degree, but this is perfectly normal: a firm can find no match either because nobody applied to its vacancy, or because it made an offer to someone who took another offer.

# 6 Concluding remarks

We conclude with some overall remarks on our setup and then with some directions of future research we think are pertinent.

We note that essentially our random network model assumes firms have no limit on how many applications they want to screen; they accept as many applications as applicants send. This assumption can be relaxed assuming each firm has a randomly drawn degree, and "meeting" amounts to connecting the stubs of applicant and firm degrees. That model is the bipartite configuration model analyzed by Newman, Strogatz, Watts (2001); asymptotically it will have the same behavior as ours, with the only difference being that $z_k$ will not be binomial, but an arbitrary distribution, and a primitive. Firms having a capacity of how many applications they accept sounds empirically somewhat counterfactual, and in fact the search-and-matching literature has considered firm search intensity as a not very important margin (Pissarides, 2000). For all these reasons we abstract from it.

Going back to the discussion on the interpretation of what the graph is capturing of section 2, we think it is important to highlight one issue. In our stochastic network model we can still pretend we are agnostic as to what the network represents: links form at random on firms for some reason we are not modeling explicitly. However, for any plausible interpretation, one might expect correlation patterns to emerge which are not present in our analysis.

Let us take for example links to represent either skills required for specific jobs, or social ties between workers searching for jobs and friends employed at firms. Then if applicant $i$ connects to two jobs – say $j, j'$, conditional of applicant $i'$ connecting to job $j$, they may have a higher than average probability to also connect to job $j'$. This relates to the fundamental notion of *clustering* in the networks literature, and it is absent from our model, for which the probabilities of each link are independent.

We think this is an issue of first-order interest to be refined in future theoretical work, and be tested empirically. Apart from its realism, clustering is expected to matter quantitatively, as we would expect coordination failures be (potentially significantly) exacerbated in its presence: to put it simply, clustering implies that the same people compete for the same jobs.

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# Appendices

## A Section 3, Matching when the network is given

**Proposition.** *The matching function in the case of the complete graph is increasing and concave in its two arguments*

*Proof.* The matching function in this case is

$$m(U,V) = U\left(1 - \left(1 - \frac{1}{U}\right)^V\right)$$

Its derivatives are of the respective signs:

$$\frac{\partial m}{\partial V} = -U\big(1 - \frac{1}{U}\big)^V \ln\big(1 - \frac{1}{U}\big) > 0, \text{ and } \frac{\partial^2 m}{\partial V^2} = -U\big(1 - \frac{1}{U}\big)^V \big[\ln\big(1 - \frac{1}{U}\big)\big]^2 < 0$$
$$\frac{\partial m}{\partial U} = 1 - \big(1 - \frac{1}{U}\big)^{V-1}\big(\frac{U-(1+V)}{U}\big) > 0, \text{ and } \frac{\partial^2 m}{\partial U^2} = -\frac{V^2}{U^3}\big(1 - \frac{1}{U}\big)^{V-2} < 0$$

□

**Proposition.** *The matching function $m$ for a double regular graph exhibits constant returns to scale, and it is increasing and concave in each of its arguments.*

*Proof.* The matching function is

$$m(U,V;d_U) = U\left[1 - \left(1 - \frac{1}{d_U}\frac{V}{U}\right)^{d_U}\right]$$

Constant returns to scale follow from the definition, as $\forall\lambda > 0$

$$\begin{aligned} m(\lambda U,\lambda V;d_U) &= \lambda U\left[1 - \left(1 - \frac{1}{d_U}\frac{\lambda V}{\lambda U}\right)^{d_U}\right] \\ &= \lambda m(U,V;d_U) \end{aligned}$$

For the rest it helps to express $m$ in terms of the job-finding probability $m(U,V) = Uf(\theta)$, where $f(\theta) = \left[1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}\right]$, and we dropped the parameter $d_U$ as an argument of the functions for notational convenience.

So for monotonicity and concavity we check the derivatives:

$$\frac{\partial m}{\partial V} = f'(\theta) \tag{i}$$
$$\frac{\partial^2 m}{\partial V^2} = f''(\theta)\frac{1}{U} \tag{ii}$$
$$\frac{\partial m}{\partial U} = f(\theta) - f'(\theta)\theta \tag{iii}$$
$$\frac{\partial^2 m}{\partial U^2} = f''(\theta)\theta^2 U^{-1} \tag{iv}$$

We first show that $f$ is increasing and concave, signing conditions (i), (ii), (iv):

$$f'(\theta) = (1-\theta)^{d_U-1} \geq 0$$
$$f''(\theta) = -(d_U-1)(1-\theta)^{d_U-2} \leq 0$$

We also note that $f(0) = 0$, and $f(1) = 1 - \left(1 - \frac{1}{d_U}\right)^{d_U} \leq 1$.

To get the sign of (iii) we show that $f(\theta) - f'(\theta)\theta \geq 0$: Define $Q(\theta) = f(\theta) - f'(\theta)\theta$. But $Q' = -\theta f'' \geq 0$. And since $Q(0) = 0$, $Q(\theta) \geq 0$. $\square$

We note the elasticity of $m(\cdot)$ is not constant.[23] Specifically, denote $\eta(\theta) \equiv \frac{\partial m}{\partial V}\frac{V}{m}$, then

$$\eta(\theta) = \frac{f'(\theta)\theta}{f(\theta)}$$

Of course, from CRS we have that $\frac{\partial m}{\partial U}\frac{U}{m} = 1 - \eta(\theta)$. It follows from the concavity of $m$ w.r.t $U$ that $\eta(\theta) < 1$, as we showed above that $f'(\theta)\theta \leq f(\theta)$.

**Proposition.** *For tightness $\theta = \frac{V}{U}$ around 1, the matching function $m$ for a double regular graph is equal to a Cobb-Douglas function up to 1st-order. Specifically, one can write*

$$m(U, V) \approx m_0 V^{\tilde{\eta}} U^{1-\tilde{\eta}},$$

*where $m_0 = f(1) \leq 1$, and $\tilde{\eta} = \eta(1) < 1$.*

[23]Constant elasticity is not considered one of the characteristic properties of the matching function. For example the Cobb-Douglas has constant elasticity, while the specification $m(V, U) = \left[V^{-\gamma} + U^{-\gamma}\right]^{-\frac{1}{\gamma}}, \gamma > 0$ does not.

*Proof.* We take logs of the matching function

$$\ln(m) = \ln(U) + \ln\left(1 - \left[1 - \frac{1}{d_U} e^{\ln\left(\frac{V}{U}\right)}\right]^{d_U}\right)$$

Define $L(x) \equiv \ln\left(1 - \left[1 - \frac{1}{d_U} e^{x}\right]^{d_U}\right)$, where $x \equiv \ln\left(\frac{V}{U}\right)$. We can take the Taylor expansion of $L(x)$ around any $x_0 \in (-\infty, \ln(d_U))$; we choose to do so around $x_0 = 0$:

$$L(x) = \sum_{n=0}^{\infty} \frac{L^{(n)}(0)}{n!} x^n$$

The 1st-order approximation yields

$$L(x) \approx L(0) + L'(0)x$$

And hence the matching function is (approximately) of the Cobb-Douglas form:

$$\ln(m) \approx L(0) + (1 - L'(0)) \ln(U) + L'(0) \ln(V)$$

where $L(0) = \ln\left(1 - \left(1 - \frac{1}{d_U}\right)^{d_U}\right)$, $L'(0) = \frac{\left(1 - \frac{1}{d_U}\right)^{d_U - 1}}{1 - \left(1 - \frac{1}{d_U}\right)^{d_U}}$. $\square$

We also notice that $L'(0) = \frac{f'(1) \cdot 1}{f(1)} = \eta(1)$, and $L(0) = \ln(f(1))$. Thus we have shown that at a 1st-order, for cases when $V \approx U$, and all applicants and firms are symmetric we can write

$$m(U, V) \approx m_0 V^{\tilde{\eta}} U^{1-\tilde{\eta}},$$

where $m_0 = f(1) \leq 1$, and $\tilde{\eta} = \eta(1) < 1$.

**Proposition.** *For the double regular network the matching rate is maximized when* $d_u = 1$.

*Proof.* The job-finding probability in the double regular graph case is

$$f = 1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

To study its monotonicity holding $\theta$ fixed and varying $d_U$, let us define and study the monotonicity of the auxiliary function

$$h(d_U) = \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

where naturally $d_U \geq \theta$ for the function to be well-defined.[24] From now on we drop the subscript $U$ to simplify notation.

Define

$$\begin{aligned}\tilde{h}(d) &= \ln(h(d)) \\ &= d\ln\left(1 - \frac{1}{d}\theta\right)\end{aligned}$$

Now,

$$\begin{aligned}\tilde{h}'(d) &= \ln\left(1 - \frac{1}{d}\theta\right) + \frac{d\frac{1}{d^2}\theta}{1 - \frac{1}{d}\theta} \\ &= \ln\left(1 - \frac{1}{d}\theta\right) + \frac{\frac{1}{d}\theta}{1 - \frac{1}{d}\theta}\end{aligned}$$

We can show this is always $> 0$. Raise both sides to the power of $e$ to get

$$e^{\tilde{h}'} = \left(1 - \frac{1}{d}\theta\right) e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}}$$

But we know $e^x \geq 1 + x, \forall x \geq 0$, thus

$$\begin{aligned}e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq 1 + \frac{\frac{1}{d}\theta}{1 - \frac{1}{d}\theta} \Rightarrow \\ e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq \frac{1}{1 - \frac{1}{d}\theta} \Rightarrow \\ \left(1 - \frac{1}{d}\theta\right) e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq 1\end{aligned}$$

Since $e^{\tilde{h}'} \geq 1$, it follows that $\tilde{h}' \geq 0$, thus $\tilde{h}$ is increasing in $d$, thus $h$ is increasing in $d$, and hence $f$ is decreasing in $d$. □

# B Section 4, Comparative statics when network is given

In this section we provide some background material on mean-preserving spreads over arbitrary vectors which are not necessarily a probability distribution.

[24] This constraint is imposed in our model from $Ud_U = Vd_V$, and $d_V \geq 1$.

**Definition** *A vector $x'$ is **a mean preserving spread** (MPS) of vector $x$ if they have the same mean $\sum_i x_i = \sum_i x'_i$ and if $x$ can be obtained from $x'$ by a series of **Pigou-Dalton transfers**, ignoring the identities of the agents.*

**Definition** *A transfer $t > 0$ from one agent to another when the two agents are endowed with $x_1, x_2$ respectively of some quantity, is **a Pigou-Dalton transfer** if $x_1 > x_2$ AND $x_1 - t \geq x_2 + t$.*

That is a Pigou-Dalton transfer between two agents is one such that an amount is transferred from the richer to the poorer agent preserving their relative positions. The quantity under consideration can be anything, e.g. wealth, number of friends etc.

We make the following observations following straight from the definitions.

**Remark 1**: Any sequence of Pigou-Dalton transfers is mean-preserving.

**Remark 2**: It can be helpful to think of a mean-preserving spread (MPS) $x'$ of a vector $x$, as created from $x$ doing "inverse" Pigou-Dalton transfers. "Inverse" Pigou-Dalton transfers are transfers where the rich become richer and the poor poorer.

**Remark 3**: A mean-preserving spread (MPS) increases inequality in the outcomes, while a Pigou-Dalton transfer decreases it.

**Remark 4**: The identities of the agents do not matter for a MPS, as long as inequality increases between the initial and final outcomes. It is worth clarifying a bit further this qualification of the definition.

Take two agents for which $x'_1 > x'_2$. Transfer $t > 0$ from 1 to 2 such that $x_1 \equiv x'_1 - t \geq x'_2 + t \equiv x_2$. Then according to the definition $x'$ is a MPS of $x$. Note also that

$$x'_1 > x'_1 - t = x_1$$
$$x'_2 > x'_2 + t = x_2$$

Thus

$$x'_1 - x'_2 > x_1 - x_2 \tag{*}$$

Now, take two agents for which $x_1' > x_2'$. Transfer again $t > 0$ from 1 to 2 such that $x_1 < x_2$, but $x_1' \geq x_2$, and $x_1 \geq x_2'$. Then $x'$ is again a MPS of $x$ because we can just switch the labels of $x_1$ and $x_2$, and be in the previous situation.[25]

Alternatively, note that

$$x_1' > x_2' + t = x_2$$
$$x_2' > x_1' - t = x_1$$

Thus

$$x_1' - x_2' > x_2 - x_1 \tag{**}$$

Combining (*) and (**) we see that $x'$ is a MPS of $x$, if

$$x_1' - x_2' > |x_1 - x_2|$$

This is an operationally easy to apply sufficient condition of a MPS, which will be useful for us, so we state it as a lemma:

**Lemma** (MPS sufficiency condition). *$x'$ is a MPS of $x$, if the two differ in only two dimensions, 1, and 2, and for these dimensions the following holds:*

$$x_1' - x_2' > |x_1 - x_2|$$

*where $x_1 \equiv x_1' - t, x_2 \equiv x_2' + t$ for some $t > 0$.*

For a MPS we can show the following basic result:

**Proposition.** *For a strictly convex function $h(\cdot)$, if $x'$ is a MPS of $x$, then it holds that*

$$\sum_i h(x_i') > \sum_i h(x_i)$$

*Proof.* It suffices to show the inequality holds for a single Pigou-Dalton transfer (up to relabeling). Then by applying it repetitively, we can show it holds for any sequence of such transfers. Let us assume a transfer occurs between agents 1 and 2.

[25] That is denote $x_1 \equiv \tilde{x}_2$, and $x_2 \equiv \tilde{x}_1$. Then we are in the case where $x_1' > \tilde{x}_1 \geq \tilde{x}_2 > x_2'$ which is identical to the previous case, only we have $\tilde{x}_1, \tilde{x}_2$ instead of $x_1, x_2$ respectively.

Assume $x'_2 = x_2 + t, x'_1 = x_1 - t$, and $x_2 \geq x_1$, where $t > 0$. Then

$$\sum_i h(x'_i) > \sum_i h(x_i) \Leftrightarrow$$
$$h(x_1 - t) + h(x_2 + t) > h(x_1) + h(x_2) \Leftrightarrow$$
$$h(x_2 + t) - h(x_2) > h(x_1) - h(x_1 - t) \Leftrightarrow$$
$$\frac{h(x_2 + t) - h(x_2)}{t} > \frac{h(x_1) - h(x_1 - t)}{t}$$

But we know from the mean value theorem there exist $\tilde{c} \in (x_2, x_2 + t)$ and $\tilde{\tilde{c}} \in (x_1 - t, x_1)$ s.t.

$$h'(\tilde{c}) = \frac{h(x_2 + t) - h(x_2)}{t}$$
$$h'(\tilde{\tilde{c}}) = \frac{h(x_1) - h(x_1 - t)}{t}$$

We also know that since $h(\cdot)$ is convex, $h'(\cdot)$ is increasing thus $h'(\tilde{c}) > h'(\tilde{\tilde{c}})$, completing the proof. $\square$

Note: Even though typically the outcome vectors $x, x'$ are taken to be positive in applications (e.g. income redistribution), this is not a requirement. The result holds equally well for positive and negative outcome vectors.