Monopoly, Product Quality, and Flexible Learning *

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Abstract

A seller offers a buyer a schedule of transfers and associated product qualities, as in Mussa and Rosen (1978). After observing this schedule, the buyer chooses a flexible costly signal about his type. We show it is without loss to focus on a class of mechanisms that compensate the buyer for his learning costs. Using these mechanisms, we prove the quality always lies strictly below the efficient level. This strict downward distortion holds even if the buyer acquires no information or when the buyer's posterior type is the highest possible given his signal, reversing the "no distortion at the top" feature that holds when information is exogenous.

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1. Introduction

The technological advancements of the last few decades have made it easier for consumers to learn about products before trading. When choosing what information to acquire, it is natural for buyers to rely on the set of available products and trade terms. Consider a consumer shopping for a mobile-phone subscription, for example. To evaluate a pay-per-minute plan, such a consumer would have to obtain a finer estimate of their expected phone usage than they would need for evaluating a plan with unlimited calls. Because the buyer's willingness to pay depends on her information, the seller will likely consider the impact her menu has on the buyer's learning decisions when choosing what contracts to offer. For instance, there may be no point in adding novel feature to one's products if consumers never invest in learning what these features are before purchasing. In this paper, we study how the need to guide the buyer's learning impacts the menu offered by a multiproduct monopolist.

We study a model in which a seller of vertically differentiated products decides what menu to offer. Unlike the classical model of Mussa and Rosen (1978) and Maskin and Riley (1984), we do not assume the buyer possess private information when they first see the monopolists' menu. Instead, the buyer gets to choose what to learn about his type after observing the menu. The buyer's information choice is flexible: he can use any signal structure he wants about type. Signals come at a cost which is affine, smooth, strictly increasing in informativeness, and have infinite slopes at the boundaries—we expand on these assumptions below. The monopolist's menu designates a schedule of qualities and associated transfers, where the monopolist's marginal costs are strictly increasing with quality. Our main interest is in the structure of this menu and the efficiency of the resulting allocation with respect to the buyer's chosen information.

We now describe the buyer's preferences, signal choice, and cost of information. We assume the same buyer preferences as in Mussa and Rosen (1978). Specifically, we postulate the buyer's preferences are quasilinear in money, and that his marginal utility from quality is constant, and equals to his type, $\theta \in \Theta = \left[\underline{\theta}, \bar{\theta}\right]$. Combined with expected-payoff maximization, this preference specification implies that the buyer's posterior type estimate pins down his payoffs from any quality-transfer pair, and through it, his selection from any menu. Consequently, for any fixed menu, the distribution of the buyer's posterior type estimate fully determines trade outcomes. We therefore let the buyer choose any distribution for her posterior type estimate that is consistent with some signal structure. Following Ravid, Roesler, and Szentes (2022), we define the cost of information acquisition directly as a function of this distribution. We assume this function is affine and increasing in informativeness, which we show is equivalent to the cost of each distribution being equal to its integral against a convex function, c. Finally, we further require c to be a smooth function that admits infinite slopes at the boundaries of Θ.

Our modeling assumptions imply the buyer's optimal learning program is a special case of the more general mean-measurable information design problem (Gentzkow and Kamenica, 2016; Dworczak and Martini, 2019; Arieli et al., 2020; Kleiner, Moldovanu, and Strack, 2021). Specifically, the buyer chooses a cummulative distribution function (CDF) for his posterior-type estimate in order to maximize the integral of a function of her posterior expected type. In our case, this function equals the buyer's net utility, which is his payoff from truthfully reporting his realized type θ to the monopolist's mechanism, minus $c(\theta)$. A CDF is feasible if one can attain the true type distribution via mean-preserving spreads. Intuitively, spreading any mass a CDF puts on θ in a mean-preserving manner corresponds to obtaining a more informative signal that better discriminates between types above and below θ.

Our main result shows that the buyer's chosen quality always lies strictly below the efficient level conditional on his signal realization. This strict downward distortion of quality holds even when the buyer's posterior type is the highest possible given his signal, a feature that stands in contrast to the case in which the buyer's information is fixed. In that case, it is well known that the monopolist's optimal allocation involves "no distortion at the top": the type with the highest value in the distribution receives the efficient quality level.

Our result is driven by the fact that the monopolist must leave the buyer with moral-hazard rents due to her inability to contract on the buyer's learning decision. For intuition, consider the problem of maximizing the monopolist's profits across all menus that induce the buyer to obtain no information. With exogenous information, the buyer must remain ignorant, and so it is optimal for the monopolist to propose a menu that extracts the buyer's ex-ante surplus. Specifically, the monopolist offers a menu consisting only of the ex-ante efficient quality in exchange for the buyer's ex-ante willingness to pay. This offer, however, can never dissuade the buyer from learning when information is endogenous: because the buyer's ex-post optimal decision depends on whether his type is above or below average, the buyer's net utility has a convex kink at the prior mean, and so the buyer would strictly benefit from obtaining additional information. Hence, to incentivize the buyer to remain ignorant, the monopolist must given him a positive surplus, which, in turn, induces the monopolist to decrease the quality she offers to the buyer.

The above moral hazard is reminiscent of the moral hazard identified in Mensch (2021), who studies the optimal way to auction an indivisable good to buyers who flexibly acquire information about their value after observing the mechanism. Mensch (2021) shows that in the single-buyer case of his model, this moral hazard results in a reduction of the auctioneer's revenue. However, in the indivisible goods model, this revenue reduction does not translate to inefficient trade with the buyer with highest posterior expected value, because the seller can always convert an increase in the probability of sale into additional revenue. By contrast, the convex cost of quality in our model means the monopolist determines the quality she provides using a marginal cost vs. marginal revenue calculation. Consequently, the monopolist in our model responds to a reduction in the marginal revenue she obtains from the buyer's highest type by reducing the quality she provides that type below the efficient level.

In addition to the difference in the monopolist's problem, we also differ from Mensch (2021) in the way we model the buyer's information acqusition. More specifically, Mensch (2021) models the buyer's signal structure via its induced distribution over the buyer's posterior beliefs. These distributions come at a cost that is posterior separable and admits infinite slopes at the boundaries of the simplex. By contrast, we assume the buyer's learning costs depend only on the distribution of her posterior type-estimate. Whereas our approaches are equivalent when types are binary, with more types the two frameworks are incomparable, since our buyer's learning costs cannot have infinite slope at posteriors whose expectations lies strictly between the highest and lowest types.

In a concurrent paper, Thereze (2022) analyzes a variant of the problem studied in our paper but in which information acquisition is modeled using Mensch (2021)'s approach. Like us, he concludes there is downward distortion of quality for all types, including at the top. He also derives several comparative statics results which we do not prove on the costs of information acquisition.

Our approach for modeling the buyer's learning problem admits several advantages over the framework used by Mensch (2021) and Thereze (2022). First, our approach allows us to accommodate discrete and continuous type distributions, meaning that our model is more comparable to the fixed information models studied in the literature, such as Mussa and Rosen (1978) and Maskin and Riley (1984). Second, and more importantly, our approach allows us to solve our problem using tools developed for mean-measurable information design problems (Gentzkow and Kamenica, 2016, Dworczak and Martini, 2019, Arieli et al., 2020, Kleiner, Moldovanu, and Strack, 2021). Indeed, to solve our model, we prove a variation on Dworczak and Martini's (2019) duality theorem that applies to our setting.[1] This theorem delivers a shadow price function, that gives the maximal benefit the buyer can ob-

[1] In particular, we prove a variant of the theorem that allows the net utility to have non-bounded slopes at the edges of the interval $\left[\underline{\theta}, \bar{\theta}\right]$.

tain from splitting any potential type-estimate via mean-preserving spreads. Using this price function, we show it is without loss to focus on a particular class of mechanisms, which we call information-cost canceling mechanisms. These mechanisms decompose the buyer's rents into two parts: one part which cancels out the buyer's costs of learning, and another part comes from the derivative of the buyer's shadow price function. Using the structure of these mechanisms, we identify different classes of perturbations that we use to prove our main result.

Related Literature. Our paper lies in the literature studying the interaction between flexible information acquisition and trade. In addition to the papers of Mensch (2021) and Thereze (2022) mentioned above, the closest papers to ours are Condorelli and Szentes (2020) and Ravid, Roesler, and Szentes (2022), both of which study models of bilateral trade with a single indivisible good. In Condorelli and Szentes (2020), the buyer publicly chooses the distribution of his valuation at a cost before the seller designs his mechanism, whereas Ravid, Roesler, and Szentes (2022) study a model in which the buyer selects a costly signal at the same time that the monopolist picks her mechanism. As mentioned previously, we follow Ravid, Roesler, and Szentes (2022) in assuming the buyer's learning costs are a function of the distribution of his posterior expectation. Compared to Ravid, Roesler, and Szentes (2022), we impose stronger assumptions on the shape of the buyer's costs, in that we require it to be affine.

We also contribute to the large and growing literature on information design (Aumann and Maschler, 1995, Kamenica and Gentzkow, 2011, Bergemann and Morris, 2013). Within this literature, our work most closely relates to papers who study the interaction between information design and trade. Several papers study the set of possible outcomes in bilateral trade settings with indivisable goods as one varies each party's information—see Bergemann, Brooks, and Morris (2015), Roesler and Szentes (2017), Kartik and Zhong (2019). Haghpanah and Siegel (forthcoming) study the set of attainable buyer-seller surplus pairs when the seller has multiple products in his disposal. They show the first-best consumer surplus is not attainable whenever the seller finds it optimal to offer multiple products. In a related paper (Haghpanah and Siegel, 2022), the same authors show that a binary market segmentation can create a Pareto improvement in most markets that are inefficiently served by a multi-product monopolist. [2]

This paper also contributes to the burgeoning literature on rational inattention, started by

[2] Armstrong and Zhou (2022) studies the effect of information on profits and consumer surplus in oligopolistic competition. In addition, several papers use information design tools to study information provision in markets. For example, see Hwang, Kim, and Boleslavsky (2019), Smolin (2020), and Yang (forthcoming).

the seminal papers of Sims (1998; 2003), and developed into models of flexible information acquisition by Caplin and Dean (2013; 2015), Matějka and McKay (2015), and Caplin, Dean, and Leahy (2021) using a posterior-separable approach to modeling information costs. Since then, there have been a number of applications of rational inattention to various economic problems, such as global games (Yang, 2015, Morris and Yang, 2021, Denti, 2022), bargaining (Ravid, 2020), and attention management (Lipnowski, Mathevet, and Wei 2020). Our work expands this list of applications by showing that rational inattention can create a new inefficiency in monopolistic screening.

Several papers use more structured learning models to explore how the buyer's incentives to acquire information depends on the selling mechanism in the context of auctions. Persico (2000) shows buyers learn less information in a second price auction than in a first-price one, provided that their signals are affiliated. Compte and Jehiel (2007) claim simultaneous auctions generate lower revenue than dynamic ones when buyer's have an opportunity to learn. Shi (2012) characterizes the revenue-maximizing auction in private-value settings. In addition to their focus on auctions, these models differ from ours in that they required the buyer to choose among a set of signal structures that can be linearly ordered in their informativeness.[3]

2. Model

There is a monopolist (she) and a buyer (he). The game begins with the monopolist offering the buyer a contract, which is a compact set of pairs, $M \subseteq [0, \bar{q}] \times \mathbb{R}$. Each menu item $(q, t) \in M$ corresponds to a transfer of t to be paid to the monopolist by the buyer, and the quality q of the product the buyer gets in exchange. The buyer's utility from (q, t) depends on her type, $\boldsymbol{\theta}$, a random variable distributed over $\Theta = [\underline{\theta}, \bar{\theta}] \subseteq \mathbb{R}_+$ according to a CDF F_0. We denote the prior-expected type by $\theta_0 := \int \theta F(\mathrm{d}\theta)$, and assume F_0 includes $\underline{\theta}$ and $\bar{\theta}$ in its support. Given θ, the buyer's utility from (q, t) is

$$u(\theta, q, t) = \theta q - t.$$

[3] Another strand of the literature studies the seller's benefits from revealing information about the buyers' valuations prior to participating in an auction; see, for example, Milgrom and Weber (1982), Ganuza (2004), Bergemann and Pesendorfer (2007), Ganuza and Penalva (2010), and Li and Shi (2017).

The monopolist's payoff from the buyer's choosen menu item (q, t) is

$$\pi(q, t) = t - \kappa(q),$$

where $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing, continuously differentiable, and strictly convex function satisfying $\kappa(0) = 0$, $\kappa'(0) \leq \underline{\theta}$ and $\kappa'(\bar{q}) > \bar{\theta}$. We require the monopolist to give the buyer the option of not buying the monopolist's product, which is equivalent to requiring M to include the option $(0, 0)$. Both the monopolist and the buyer wish to maximize their expected utility.

Neither the monopolist nor the buyer know $\boldsymbol{\theta}$, but the buyer can choose to learn about it after observing the monopolist's menu. The buyer's information acquisition is flexible, meaning he can use any signal s in order to learn about $\boldsymbol{\theta}$. Our assumption on the buyer's utility mean his expected payoff from any menu item depends on her posterior mean, $\mathbb{E}[\boldsymbol{\theta}|\mathrm{s}]$. Therefore, the marginal distribution of $\mathbb{E}[\boldsymbol{\theta}|\mathrm{s}]$ pins down the buyer's expected trade surplus from any menu. This distribution also determines the probability the buyer purchases any menu item, which, in turn, is sufficient for calculating the monopolist's profits and optimal menu. In other words, trade outcomes depend only on the marginal distribution of the buyer's posterior mean, and so we identify each signal with the CDF of this marginal.[4] More precisely, letting $\mathcal{F}$ be the set of all CDFs over Θ, we let the buyer choose any element of $\mathcal{F}$ that can arise as the marginal CDF of $\mathbb{E}[\boldsymbol{\theta}|\mathrm{s}]$ for some s. We denote this set by $\mathcal{I}$ and describe it formally below.

As observed by Gentzkow and Kamenica (2016), F is the CDF of the marginal distribution of the buyer's posterior mean for some signal if and only if it is a mean-preserving contraction of the prior, F_0. Recall that $F \in \mathcal{F}$ is a **mean-preserving spread** of $F' \in \mathcal{F}$ (denoted by $F \succeq F'$) if and only if

$$\int_{\tilde{\theta} \leq \theta} (F - G)\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} \geq 0 \, \forall \theta \in \Theta, \text{ with equality at } \theta = \bar{\theta}.$$

The CDF F is a **strict mean-preserving spread** of F' (denoted by $F \succ F'$) if both $F \succeq F'$

[4] This method of modeling flexible information is common in the information-design literature---see, for example, Gentzkow and Kamenica (2016), Roesler and Szentes (2017), Kolotilin (2018), and Dworczak and Martini (2019).

and $F' \neq F$.[5] Letting

$$I_F : \Theta \to \mathbb{R},$$
$$\theta \mapsto \int_{[\underline{\theta},\theta]} (F_0 - F)\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta},$$

one can then define the set of feasible posterior-mean distributions $\mathcal{I}$ as

$$\mathcal{I} = \left\{F \in \mathcal{F} : I_F(\theta) \geq 0 \text{ for all } \theta, \text{ and } I_F\left(\bar{\theta}\right) = 0\right\}.$$

In what follows, we refer to CDFs in $\mathcal{I}$ as signals.

Information acquisition comes at a cost. Our model of the buyer's costs of learning follows Ravid, Roesler, and Szentes (2020). In general, different information structures generating the same distribution of posterior expectations might come at different costs. However, because the buyer's expected payoff from trade depends only on the distribution of this posterior expectation, F, she would always use the least expensive signal structure that leads to F. In fact, the buyer may even randomize to get F. Thus, we can evaluate the cost of F by the expected cost of the cheapest randomization that generates it, resulting in a *convex* indirect cost function,

$$C : \mathcal{I} \to \mathbb{R}_+.$$

We follow Ravid, Roesler, and Szentes (2020), and state our assumptions directly in terms of this C. We assume C is continuous, affine, and strictly increasing in informativeness—i.e., $C(F) > C(F')$ whenever F is a strict mean-preserving of F'. In the appendix, we prove these properties imply the existence of some continuous, strictly convex function $c : \Theta \to \mathbb{R}_+$ such that

$$C(F) = \int c(s)\, F(\mathrm{d}s).$$

Moreover, we show it is without loss for c to attain its minimum at θ_0. In addition, we require c to be a twice continuously differentiable function that admits infinite slope at the boundaries—that is, [6]

$$\lim_{\theta \to \bar{\theta}} c'(\theta) = -\lim_{\theta \to \underline{\theta}} c'(\theta) = \infty.$$

After choosing F, the buyer gets to see its realization, $\theta \in \Theta$, and chooses whether to participate in the mechanism, and if so, what item to select from the menu to maximize her

[5] Notice $\succeq$ is reflexive and anti-symmetric, meaning $F \succeq F'$ and $F' \succeq F$ if and only if $F = F'$.

[6] In an extension, we relax this assumption, allowing for additional cost functions, e.g. quadratic, as long as the slope at the boundaries is sufficiently steep.

expected utility.

To summarize, the game begins with the monopolist choosing a menu. Next, the buyer observes the menu, and chooses what signal $F \in \mathcal{I}$ to acquire. The buyer then sees her signal realization $\theta \in \Theta$, and chooses an item from the monopolist's menu. We are interested in finding the menu that maximizes the monopolist's expected profits.

Theorem 1. *A monopolist optimal menu exists.*

Our timing assumptions mean the buyer's interim expected payoff is fully determined by her posterior-value estimate. Hence, the revelation monopolist implies it is sufficient to focus on direct revelation mechanisms. Such mechanisms can be described with two maps,

$$Q : \Theta \to [0, \bar{q}]\,, \; T : \Theta \to \mathbb{R}_+,$$

where $Q(\theta)$ and $T(\theta)$ correspond to the quality and transfer pair chosen by a buyer with signal realization θ. These mappings must satisfy the standard incentive compatibility and individual rationality constraints,

$$\theta Q(\theta) - T(\theta) \geq \theta Q(\theta') - T(\theta') \quad \forall \theta, \theta' \in \Theta, \tag{1}$$

$$\theta Q(\theta) - T(\theta) \geq 0 \quad \forall \theta \in \Theta. \tag{2}$$

Usual envelope style reasoning (Myerson, 1981) delivers that a Q and T satisfy the above two conditions if and only if Q is increasing and

$$T(\theta) = \theta Q(\theta) - \int_{\underline{\theta}}^{\theta} Q\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} - \underline{u}, \tag{3}$$

where $\underline{u}$ is the utility granted to the lowest possible type,

$$\underline{u} = \underline{\theta} Q(\underline{\theta}) - T(\underline{\theta})\,.$$

It follows that $\underline{u}$ and Q are sufficient for pinning down every feasible IC and IR mechanism. Letting $\mathcal{Q}$ be the set of all increasing functions from Θ to $[0, \bar{q}]$, we refer to a $Q \in \mathcal{Q}$ as an **allocation**, to $(Q, \underline{u}) \in \mathcal{Q} \times \mathbb{R}_+$ as a **mechanism**, and let $T_{Q,\underline{u}}$ denote the transfer implied by (3). Observe this implies that a type θ buyer's utility from truthful reporting under $(Q, \underline{u})$ is

$$V_{Q,\underline{u}}(\theta) := \underline{u} + \theta Q(\theta) - T(\theta) = \underline{u} + \int_{\underline{\theta}}^{\theta} Q\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta}.$$

Given an allocation Q, take

$$\begin{aligned}\bar{\theta}_Q &:= \inf\{\theta : Q(\theta) = Q(\bar{\theta}) \text{ and} \\ \underline{\theta}_Q &:= \sup\{\theta : Q(\theta) = Q(\underline{\theta})\end{aligned}$$

to be last a first types at which Q changes, respectively.

We now state the monopolist's problem of choosing a profit maximizing $(Q, \underline{u})$. Given a mechanism $(Q, \underline{u})$, the buyer's utility from using signal $F \in \mathcal{I}$ is

$$U(Q, \underline{u}, F) = \int (V_{Q,\underline{u}}[(\theta) - c(\theta)]\, F(\mathrm{d}\theta).$$

We refer to a mechanism-signal tuple $(Q, \underline{u}, F)$ as **an outcome**, and say the outcome is **incentive compatible** (IC) if F maximizes the buyer's utility given $(Q, \underline{u})$,

$$F \in \operatorname*{argmax}_{F' \in \mathcal{I}} U(Q, \underline{u}, F').$$

Consistent with this terminology, whenever $(Q, \underline{u}, F)$ is IC, we say $(Q, \underline{u})$ is F**-incentive compatible** (F-IC) and F is $(Q, \underline{u})$**-incentive compatible** ($(Q, \underline{u})$-IC). Denote the monopolist's payoff when the buyer reports a signal realization of θ by

$$\pi_{Q,\underline{u}}(\theta) := T_{Q,\underline{u}}(\theta) - \kappa(Q(\theta)) = \theta Q(\theta) - V_{Q,\underline{u}}(\theta) - \kappa(Q(\theta)).$$

Then, we can write the monopolist's profit from using offering $(Q, \underline{u})$ when the buyer uses F as

$$\Pi(Q, \underline{u}, F) = \int \pi_{Q,\underline{u}}\left(\tilde{\theta}\right) F\left(\mathrm{d}\tilde{\theta}\right),$$

and so the monopolist's problem is given by

$$\max_{(Q,\underline{u},F)} \int \pi_{Q,\underline{u}}\left(\tilde{\theta}\right) F\left(\mathrm{d}\tilde{\theta}\right) \text{ s.t. } (Q, \underline{u}, F) \text{ is IC.}$$

3. Cost-Canceling Mechanisms

In this section, we show it is without loss to restrict the monopolist to a special class of mechanisms. We begin by characterizing the solution to the buyer's optimal learning problem. To obtain our characterization, we first prove a variant of Dworczak and Martini's (2019) duality theorem that can be applied to our setting. To state the theorem, we require a few additional definitions. We say a function $P:\Theta\to\mathbb{R}$ is an F**-shadow price** if it is Lipschitz continuous, convex, and affine on any interval overwhich F's mean-preserving-spread constraint is slack) that is, over any interval $\left(\underline{\theta}_0,\bar{\theta}_0\right)\subseteq\{\theta: I_F(\theta)>0\}$. Given an upper semicontinuous function $\phi:\Theta\to\mathbb{R}$, we say P is an F**-shadow price for** ϕ (or, equivalently, that ϕ admits P as an F-shadow price) if P is an F-shadow price that majorizes ϕ, and equals to it for all θ over F's support. Let

$$\mathcal{I}_0=\left\{F\in\mathcal{F}:\int\theta F(\mathrm{d}\theta)=\theta_0\right\}$$

to be the set of CDFs over Θ with a mean of θ_0. Say the function ϕ satisfies **edge-irrelevance** if an $\tilde{F}\in\operatorname{argmax}_{F\in\mathcal{I}_0}\int\phi(\theta)F(\mathrm{d}\theta)$ exists such that $\operatorname{supp}\tilde{F}\subset\left(\underline{\theta},\bar{\theta}\right)$. Geometrically, edge irrelevance is equivalent the existence of a line that lies everywhere above ϕ, and touches it at two points, one from each side of θ_0.

Theorem 2. *Fix some upper semicontinuous $\phi:\Theta\to\mathbb{R}$ and $F^*\in\mathcal{I}$. If ϕ admits an F^*-shadow price, then*

$$F^*\in\underset{F\in\mathcal{I}}{\operatorname{argmax}}\int\phi(\theta)F(\mathrm{d}\theta). \tag{4}$$

Moreover, if ϕ satisfies edge-irrelevance, the converse also holds—i.e., F^* satisfies (4) only if ϕ admits an F-shadow price.

The first part of the theorem is just a restatement of Theorem 1 from Dworczak and Martini (2019).[7] The theorem's second part is a variation on Dworczak and Martini's (2019) Theorem 2. In particular, their theorem (as well as the generalization by Dizdar and Kováč (2020)) requires ϕ to have a bounded slope at the vacinity of $\bar{\theta}$ and $\underline{\theta}$. This requirement makes the theorem unapplicable to our setting, because the slope of the buyer's objective, $V_{Q,\underline{u}}-c$, explodes the edges of the interval $\left[\underline{\theta},\bar{\theta}\right]$. To accommodate the buyer's objective, our theorem replaces the bounded slope condition with edge irrelevance. We now show $V_{Q,\underline{u}}-c$ satisfies edge irrelevance for all mechanisms the monopolist can offer.

Lemma 1. *For any mechanism $(Q,\underline{u})$, the function $\phi=V_{Q,\underline{u}}-c$ satisfies edge irrelevance.*

The intuition for the lemma is straightforward. Because the slope of c explodes as θ approaches $\underline{\theta}$ and $\bar{\theta}$, the buyer's objective $V_{Q,\underline{u}}-c$ must be strictly concave on the edges of Θ. As such, one can improve upon any distribution in $\mathcal{I}_0$ that puts positive mass around any one of Θ's edges.

[7] Dworczak and Martini (2019) replace the requirement that P is affine on any open interval over which $I_F(\theta)>0$ with the condition that $\int P\mathrm{d}F=\int P\mathrm{d}F_0$. Since P is convex, one can show the two conditions are equivalent.

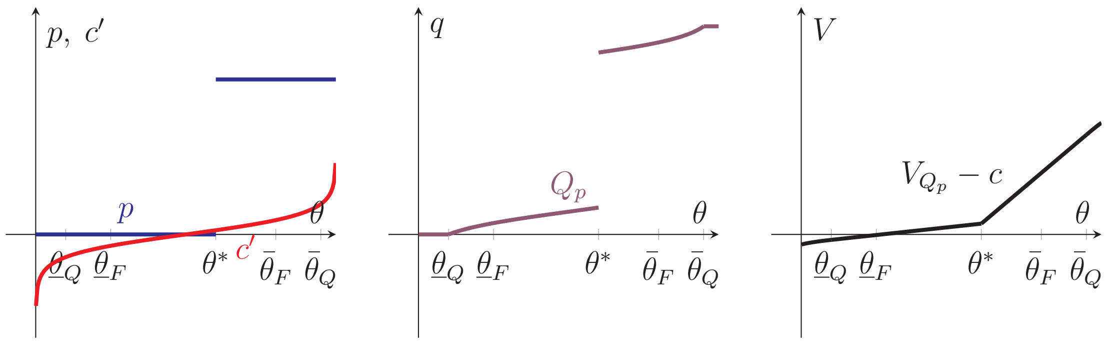

Figure 1: Construction of a F-ICC allocation for an F that satisfies $I_F(\theta^*) = 0$.

An immediate implication of the above results is that, F can be optimal given $(Q, \underline{u})$ only if F is supported on the interior of Θ. Moreover, the mean-preserving spread constraint must be slack at the edges of F's support. To state this result, let

$$\underline{\theta}_F := \min \operatorname{supp} F \text{ and } \bar{\theta}_F := \max \operatorname{supp} F$$

be the lowest and highest realizations in the support of F, respectively.

Corollary 1. *Suppose* $(Q, \underline{u})$ *is* F*-IC. Then,* $[\underline{\theta}_F, \bar{\theta}_F] \subseteq \left(\underline{\theta}, \bar{\theta}\right)$*. Moreover, an* $\epsilon > 0$ *exists such that* $I_F(\theta) > 0$ *holds for* $\theta \in B_\epsilon(\underline{\theta}_F) \cup B_\epsilon\left(\bar{\theta}_F\right)$*.*[8]

Next, we use the above results to show it is without loss to focus on the class of information-cost canceling mechanisms which we now define. A function $p : \Theta \to \mathbb{R}$ is an F**-shadow derivative** if it is bounded, increasing, constant on any interval $(\theta', \theta'') \subseteq \{x : I_F(x) > 0\}$, and satisfies $p(\underline{\theta}_F) \geq -c'(\underline{\theta}_F)$ and $p\left(\bar{\theta}_F\right) \leq \bar{q} - c'\left(\bar{\theta}_F\right)$. Given an F-shadow derivative p, define the induced allocation Q^p via

$$Q^p(\theta) = \min\left\{(p(\theta) + c'(\theta))_+, \bar{q}\right\} = \begin{cases} p(\theta) + c'(\theta), & \theta \in [\underline{\theta}_F, \bar{\theta}_F] \\ \max\{p(\underline{\theta}_F) + c'(\theta), 0\}, & \theta < \underline{\theta}_F \\ \min\{p(\bar{\theta}_F) + c'(\theta), \bar{q}\}, & \theta > \bar{\theta}_F. \end{cases} \tag{5}$$

We say an allocation Q is F**-information cost-canceling** (F-ICC) if $Q = Q^p$ for some F-shadow derivative p. We say p F-verifies $(Q, \underline{u})$ whenever $Q = Q^p$, and let p_Q be the function that F-verifies Q. A mechanism $(Q, \underline{u})$ is an F**-information cost canceling** mechanism if Q is an $F-$ICC allocation.

[8]Here, $B_\epsilon(\theta) := (\theta - \epsilon, \theta + \epsilon)$ refers to the $\epsilon > 0$ ball around θ.

Our next result shows every F-ICC is F-IC. Moreover, every F-IC mechanism admits an equivalent F-ICC mechanism.

Theorem 3. *Every F-ICC mechanism is F-IC. Moreover, if $\left(\tilde{Q},\underline{\tilde{u}}\right)$ is an F-IC mechanism, then an F-ICC mechanism $(Q,\underline{u})$ exists such that $\underline{u}\geq\underline{\tilde{u}}$, and for all $\theta\in\operatorname{supp}F$, both $Q(\theta)=\tilde{Q}(\theta)$ and $V_{Q,\underline{u}}(\theta)=V_{\tilde{Q},\underline{\tilde{u}}}(\theta)$ hold.*

We first sketch the argument establishing that every F-ICC mechanism $(Q,\underline{u})$ is F-IC. The argument relies on showing that the function

$$P_{Q,\underline{u}}(\theta)=V_{Q,\underline{u}}(\underline{\theta}_F)-c(\underline{\theta}_F)+\int_{\underline{\theta}_F}^{\theta}p\left(\tilde{\theta}\right)\mathrm{d}\tilde{\theta}. \tag{6}$$

is an F-shadow price for $V_{Q,\underline{u}}-c$. For a sketch, observe $P_{Q,\underline{u}}$ admits p as a derivative almost everywhere by the fundamental theorem of calculus. Thus, $P_{Q,\underline{u}}$ is convex and Lipschitz because p is increasing and bounded, and whenever F's mean-preserving spread constraint is slack, P is affine because p is constant. It follows $P_{Q,\underline{u}}$ is an F-shadow price. It is also easy to see that $P_{Q,\underline{u}}(\theta)=(V_{Q,\underline{u}}-c)(\theta)$ for all θ in F's support: by definition, $P_{Q,\underline{u}}$ and $V_{Q,\underline{u}}-c$ are equal at $\underline{\theta}_F$ and admit the same derivative everywhere over the interval $\left[\underline{\theta}_F,\bar{\theta}_F\right]$, which includes $\operatorname{supp}F$. In the appendix, we use the structure of Q and p outside of $\left[\underline{\theta}_F,\bar{\theta}_F\right]$ to show $P_{Q,\underline{u}}\geq V_{Q,\underline{u}}-c$ holds for all θ, and so establish $P_{Q,\underline{u}}$ is a F-shadow price for $V_{Q,\underline{u}}-c$.

For the converse direction, the theorem's proof transforms an F-IC mechanism $\left(\tilde{Q},\underline{\tilde{u}}\right)$ into a payoff-equivalent F-ICC mechanism $(Q,\underline{u})$. To do so, we observe that, since the buyer's objective $V_{\tilde{Q},\underline{\tilde{u}}}-c$ satisfies edge-irrelevance, Theorem 2 delivers a shadow price P for $V_{\tilde{Q},\underline{\tilde{u}}}-c$. Noting P is differentiable almost everywhere by convexity, we show one can find a version p of the derivative of P that equals $\tilde{Q}-c'$ over F's support. Using this p, we then construct an F-ICC allocation Q as in equation (5). Our choice of p guarantees $Q=\tilde{Q}$ on the support of F. We also show one can choose $\underline{u}$ so that P equals the shadow price $P_{Q,\underline{u}}$, which then implies $V_{Q,\underline{u}}$ equals $V_{\tilde{Q},\underline{\tilde{u}}}$ over the desired range. We then show this $\underline{u}$ is larger than $\underline{\tilde{u}}$.

4. Quality Under-Provision

In this section, we prove our main theorem: the monopolist provides quality strictly below the efficient level to all types in the support of the buyer's information structure. As a preliminary step, observe setting $\underline{u}=0$ is always optimal for the monopolist. Thus, from here on we abuse notation, writing $\pi_Q := \pi_{(Q,0)}$ and $V_Q := V_{Q,0}$, and use (Q, F) to refer to the outcome $(Q, 0, F)$.

We begin the analysis by introducing two classes of perturbation to the buyer's information, holding the allocation fixed. Clearly, whenever (Q^*, F^*) is monopolist-optimal, the monopolist cannot benefit from having the buyer change her information to any other signal F—so long as F is Q^*-IC. Thus, to meaningfully perturb the buyer's information, we must identify other signals that optimal for the buyer given a fixed allocation. As we now explain, finding such signals is particularly easy whenever Q^* is F^*-ICC. The reason is that, the F^*-shadow derivative p_{Q^*} is also an F-shadow derivative for any F that is supported in $\left[\underline{\theta}_{Q^*}, \bar{\theta}_{Q^*}\right]$ and whose mean-preserving spread constraint is slack over intervals where p is constant—that is, $I_F(\theta) > 0$ only if p is constant in the neighborhood of θ. It follows any such F is also Q^*-IC. Hence, the monopolist cannot benefit from the buyer switching her information to any such F. In the appendix we use this observation to obtain the following lemma.

Lemma 2. *Let (Q^*, F^*) be a monopolist optimal pair in which Q^* is F^*-ICC with associated F^*-shadow derivative p_{Q^*}. Suppose p_{Q^*} is constant over $[\theta_*, \theta^*] \subseteq \left[\underline{\theta}_Q, \bar{\theta}_Q\right]$, and $(\theta_*, \theta^*) \cap \left(\underline{\theta}_F, \bar{\theta}_F\right) \neq \varnothing$. Then,*

1. *If $\theta_1, \theta_2 \in \operatorname{supp} F^*(\cdot|\theta \in [\theta_*, \theta^*])$, then*

$$\pi_{Q^*}(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha\pi_{Q^*}(\theta_1) + (1-\alpha)\pi_{Q^*}(\theta_2) \tag{7}$$

   for all $\alpha \in [0, 1]$.

2. *If $I_F(\theta) > 0$ for all $\theta \in [\theta_1, \theta_2] \subseteq [\theta_*, \theta^*]$, then*

$$\pi_{Q^*}(\alpha\theta_1 + (1-\alpha)\theta_2) \geq \alpha\pi_{Q^*}(\theta_1) + (1-\alpha)\pi_{Q^*}(\theta_2) \tag{8}$$

   for all $\alpha \in [0, 1]$ such that $\alpha\theta_1 + (1-\alpha)\theta_2 \in \operatorname{supp} F^*(\cdot|\theta \in [\theta_*, \theta^*])$.

For intuition, consider the lemma's part 1, and suppose first F^* has atoms at θ_1 and θ_2. As explained above, that p_{Q^*} is constant on $[\theta_*, \theta^*]$, means that one can pool together some mass from θ_1 and θ_2 without violating the buyer's incentive constraints. It follows such pooling cannot benefit the monopolist—that is, (7) must hold. To prove the result without atoms, we approximate each θ with a shrinking neighborhood.[9] The intuition for part 2 of the lemma is

[9] This argument is similar to the proof of Proposition 1 in Ravid, Roesler, and Szentes (2022).

similar: if equation (8) did not hold, the monopolist would strictly benefit from having the buyer spread the mass he puts on (a small neighbrohood around) $\alpha\theta_1 + (1-\alpha)\,\theta_2$ across θ_1 and θ_2, thereby violating optimality of F^*. The lemma follows.

Next, we turn to our paper's main result: the quality that the monopolist allocates to any *interim type* θ is below the efficient level. We do this by showing that for any other mechanism, we can construct a perturbation that generates a strict improvement. This contrasts with the standard results with exogenous information, in which there is "no distortion at the top:" the highest type in the support, $\bar{\theta}_F$, receives the efficient quality level (Mussa and Rosen (1978), Maskin and Riley (1984)). Our theorem also shows that the monopolists expected marginal cost is strictly below its fixed information whenever the lowest type receives positive quality.

Theorem 4. *Every monopolist optimal outcome* (Q^*, F^*) *admits an allocation* Q *such that* $Q = Q^*$ F^**-almost surely,* (Q, F^*) *is also monopolist optimal, and*

$$\theta > \kappa'\left(Q\left(\theta\right)\right) \text{ for all } \theta \in \operatorname{supp} F^*.$$

Moreover, if $Q(\underline{\theta}_F) > 0$*, then*

$$\int \kappa'(Q(\theta))F(\mathrm{d}\theta) = \underline{\theta}_Q < \underline{\theta}_F.$$

To prove the theorem, we begin by replacing Q^* with an F-almost surely equal F-ICC mechanism Q. Next, we use the following observation of Mussa and Rosen (1978):[10] the quality sold to any type $\theta^* < \bar{\theta}_F$ must be inefficiently low whenever the monopolist's average cost conditional on $\theta \geq \theta^*$ is below θ^*. Formally, for any $\theta^* \in [\underline{\theta}_F, \bar{\theta}_F)$, $\kappa'\left(Q\left(\theta^*\right)\right) < \theta^*$ must hold whenever

$$\int_{\theta \geq \theta^*} \kappa'\left(Q\left(\theta\right)\right) F^*\left(\mathrm{d}\theta | \theta \geq \theta^*\right) \leq \theta^*. \tag{9}$$

To see why this inequality implies the quality provided to θ^* is inefficiently low, note that, because Q is F-ICC, it is is strictly increasing over $[\underline{\theta}_F, \bar{\theta}_F)$. Since the marginal cost for quality provision is strictly increasing as well, equation (9) implies that

$$\kappa'\left(Q\left(\theta^*\right)\right) < \int_{\theta \geq \theta^*} \kappa'\left(Q\left(\theta\right)\right) F^*\left(\mathrm{d}\theta | \theta \geq \theta^*\right) < \theta^*,$$

that is, $Q\left(\theta^*\right)$ lies strictly below its efficient level.

[10] See equation (8) in Mussa and Rosen (1978) and the subsequent discussion.

Mussa and Rosen (1978) establish equation (9) for any θ^* at which Q is strictly increasing by slightly reducing the quality provided to all types above $\theta^* - \varepsilon$ for a sequence of shrinking $\varepsilon > 0$. As ε vanishes, the monopolist's cost savings converge to the left hand side of the above equation, whereas the reduction in the monopolist's revenue converges to the inequality's right hand side. Intuitively, while the monopolist charges a price he θ^* for the marginal quality increment provided to type θ^*, the buyer's incentive constraint prevent the monopolist from charging a higher price for this increment from higher types. Equation (9) says that, at the optimum, the total revenue generated by the marginal quality increment must be weakly larger than the associated costs, which equals the monopolist's average marginal cost across all types above θ^*.

Because changing the buyer's allocation may cause the buyer to change his signal, one cannot directly apply Mussa and Rosen (1978)'s argument in our environment. As a result, equation (9) need not hold at the optimum for all θ^*. One can, however, adapt Mussa and Rosen (1978)'s argument to establish (9) for θ^* at which Q's F^*-shadow derivative p_Q is strictly increasing. To do so, we show that, whenever p_Q strictly increases at θ^*, one can obtain a new F^*-shadow derivative by slightly decreasing quality for all types above $\theta^* - \epsilon$ for some $\epsilon \geq 0$. Using this new derivative, one can construct a new F^*-ICC that equals Q^* for types below $\theta^* - \epsilon$, and slightly reduces quality for all types above $\theta^* - \epsilon$. In fact, we prove one can construct these allocations for a vanishing sequence of ϵ. Using this sequence, we can then follow Mussa and Rosen (1978)'s argument to establish (9) holds for all θ^* at which p is strictly monotone. Since reducing all of p_Q by a constant also leads to a new F^*-shadow derivative, similar reasoning shows quality must also be inefficiently low at $\underline{\theta}_F$.

Next, we sketch the argument showing quality is inefficiently low when p_Q is constant in the neighborhood of $\theta^* > \underline{\theta}_F$. Note $\theta^* = \bar{\theta}_F$ falls within this case, because I_{F^*} is continuous and $I_{F^*}\left(\bar{\theta}_F\right) > 0$ by Corollary 1. The key to our argument is the observation that the slope of π_Q is positive at some θ if and only if $Q(\theta)$ lies below the efficient level. To gain some intuition for this equivalence, suppose p_Q is differentiable at θ. Then a simple application of the envelope theorem reveals that

$$\pi_Q'(\theta) = (\theta - \kappa'(Q(\theta)))\left(c''(\theta) + p_Q'(\theta)\right).$$

Because $c'' > 0$ and $p' \geq 0$, the above implies $\pi_Q'(\theta) > 0$ if and only of $\theta > \kappa'(Q(\theta))$. For the more general case, we show quality being inefficiently low at θ is equivalent to the slope of π being positive just to the right of θ. To see why this equivalence is useful, observe first that π_Q is differentiable at θ^*, because p_Q is constant in the neighborhood of θ^*. Let

θ_* be the lowest value of θ in $\left[\underline{\theta}_F, \bar{\theta}_F\right]$ such that $p_Q(\theta) = p_Q(\theta^*)$. Since p must strictly increase just to left of θ_*, $Q(\theta_*)$ must be inefficiently low. Moreover, because p_Q is constant over the interval $[\theta_*, \theta^*]$, Lemma 2 part 1 implies that the line connecting $(\theta_*, \pi_Q(\theta_*))$ with $(\theta^*, \pi_Q(\theta^*))$ lies weakly above π_Q over the interval $[\theta_*, \theta^*]$. It follows $\pi_Q'(\theta^*)$ must be weakly higher than the slope of this line, which in turn, must be larger than the slope of π_Q when θ_* is approached from the right. But since quality is under provided at θ_*, this latter slope is strictly positive, meaning $\pi_Q'(\theta^*)$ is strictly positive, too. Hence $Q(\theta^*)$ is inefficiently low—that is, $\kappa'(Q(\theta^*)) < \theta^*$.

5. An Example

In this section, we illustrate how to use our tools to solve a simple binary state example. As we show, this example results in the buyer acquiring no information. Suppose the buyer's learning costs are given by

$$c(\theta) = \theta \ln \theta + (1-\theta)\ln(1-\theta) - \ln 0.5,$$

the seller's production costs are

$$\kappa(q) = e^q - q - 1,$$

and that the buyer's type equals $\bar{\theta} = 1$ or $\underline{\theta} = 0$ with equal probability, meaning $\theta_0 = 0.5$. Since the state space is binary, and all posteriors $\theta \in \operatorname{supp} F$ are interior, the information constraint never ever binds on $\operatorname{supp} F$. It follows that p is an F-shadow price if and only if equals some constant $p_0 \in \mathbb{R}$ for all θ. Abusing notation, we let

$$Q_{p_0}(\theta) = \min\left\{(p_0 + c'(\theta))_+, \bar{q}\right\}$$

be the implied F-ICC mechanism. Observe the highest θ to which Q assigns zero quality, $\underline{\theta}_{Q_{p_0}}$, solves the equation

$$p_0 + c'\left(\underline{\theta}_{Q_{p_0}}\right) = 0.$$

Hence, one can parameterize the set of information-cost canceling allocations by the highest type that does not participate in the mechanism, $\underline{\theta}_Q$. Fixing $\underline{\theta}_Q$, one can explicitly solve for the value of the associated information-cost canceling allocation within the interval $\left[\underline{\theta}_Q, \bar{\theta}_Q\right]$,

$$Q(\theta) = c'(\theta) - c'(\underline{\theta}_Q) = \ln\left(\frac{\theta}{1-\theta}\right) - \ln\left(\frac{\underline{\theta}_Q}{1-\underline{\theta}_Q}\right).$$

As such, the buyer's indirect utility is given by

$$\begin{aligned} V_Q(\theta) &= \int_{\underline{\theta}_Q}^{\theta} [c'(\tilde{\theta}) - c'(\underline{\theta}_Q)]\mathrm{d}\tilde{\theta} \\ &= c\left(\theta\right) - c\left(\underline{\theta}_Q\right) - c'\left(\underline{\theta}_Q\right)\left(\theta - \underline{\theta}_Q\right). \end{aligned}$$

Thus, conditional on the buyer's type realization being θ, the monopolist's profit is given by

$$\begin{aligned} \pi_Q(\theta) &= Q(\theta)\theta - V_Q(\theta) - \kappa(Q(\theta)) \\ &= \ln\left(\frac{\theta}{\underline{\theta}_Q}\right) + 2\ln\left(\frac{1-\underline{\theta}_Q}{1-\theta}\right) - \frac{\theta(1-\underline{\theta}_Q)}{\underline{\theta}_Q(1-\theta)}. \end{aligned}$$

We now witness that π_Q is concave for every feasible $\underline{\theta}_Q$. To do so, observe the second derivative of π_Q is given by

$$\pi_Q''(\theta) = \frac{2}{(1-\theta)^2} - \frac{1}{\theta^2} - \left(\frac{1-\underline{\theta}_Q}{\underline{\theta}_Q}\right)\left(\frac{2}{(1-\theta)^3}\right).$$

Now, because p is an F-shadow derivative, $c'\left(\underline{\theta}_F\right) \geq p\left(\underline{\theta}_F\right) = p_0 = c'\left(\underline{\theta}_Q\right)$. It follows $\underline{\theta}_Q \leq \underline{\theta}_F \leq \theta_0 = 0.5$, meaning

$$\begin{aligned} \pi_Q''(\theta) &\leq \frac{2}{(1-\theta)^2} - \frac{1}{\theta^2} - \frac{2}{(1-\theta)^3} \\ &\leq 2\left((1-\theta)^{-2} - (1-\theta)^{-3}\right) \leq 0, \end{aligned}$$

where the last inequality is strict for all $\theta < 1$. It follows $\pi_Q\left(\theta\right)$ is strictly concave for any information cancelling allocation Q.

We now use the above concavity to argue the buyer must obtain no information in the monopolist's optimal outcome, (Q^*, F^*). To see why, notice if the support of F^* includes two distinct signal realizations $\theta \neq \theta'$, then

$$\pi_{Q^*}\left(0.5\theta_1 + 0.5\theta_2\right) \leq 0.5\pi_{Q^*}\left(\theta_1\right) + 0.5\pi_{Q^*}\left(\theta_2\right) < \pi_{Q^*}\left(0.5\theta_1 + 0.5\theta_2\right),$$

where the first inequality follows from Lemma 2, and the second from π_{Q^*} being strictly concave. Hence, $\operatorname{supp} F^*$ must be a singleton—that is, F^* is uninformative.

It remains only to find the monopolist optimal allocation. Since the buyer learns nothing at the monopolist's optimal outcome, the optimal allocation is determined by the $\underline{\theta}_Q$ that solves

$$\max_{\underline{\theta}_Q \in [0,0.5]} \ln 2 + 2 \ln \left(1 - \underline{\theta}_Q\right) - \ln \left(\underline{\theta}_Q\right) - \left(\frac{1 - \underline{\theta}_Q}{\underline{\theta}_Q}\right).$$

One can show the solution to the above problem is unique, and given by $\underline{\theta}_Q = \sqrt{2} - 1$. The resulting profit for the monopolist is $\Pi\left(Q, \mathbf{1}_{[0.5,\infty)}\right) \approx 0.09$, and the buyer's utility is $V_Q(0.5) \approx 0.01$.

To conclude, we demonstrate that quality is distorted downwards. Given the buyer's decision not to learn, efficiency requires the buyer to get the quality q^* that solves $e^{q^*} - 1 = 0.5$—i.e., $q^* = \ln 1.5 \approx 0.41$. By contrast, the monopolist provides the buyer with quality

$$c'(0.5) - c'\left(\sqrt{2} - 1\right) \approx 0.35.$$

6. Concluding Remarks

We conclude our paper with a few brief remarks regarding our assumptions and results.

Support vs. positive probability. Theorem 4 implies that with endogenous information, the monopolist may shade downward the quality she provides to all buyer types, including the one whose valuation is maximal. This result stands in contrast to the conclusion one obtains when information is exogenous, where highest buyer type is allocated the efficient quality. As such, our paper suggests that an analyst who examines the market under the assumption that information is exogenous may come to erroneous conclusions regarding the efficiency of the market's allocation. However, one might wonder whether this error actually occurs: since the buyer's type distribution is endogenous, it is possible that the buyer chooses an F that assigns zero probability to the top of its support. It turns out, however, that in our model, it is without loss for F to put positive probability on $\bar{\theta}_F$. To see why, suppose the optimal F does not generates $\bar{\theta}_F$ with positive probability. Then F must assign positive probability to the interval $\left(\bar{\theta}_F - \epsilon, \bar{\theta}_F\right)$ for all sufficiently small $\epsilon > 0$. Since $I_F\left(\bar{\theta}_F\right) > 0$, one can choose $\epsilon > 0$ so that the mean-preserving spread constraint is slack over $\left[\bar{\theta}_F - \epsilon, \bar{\theta}_F\right]$. Consider now what happens if we alter F by splitting all the mass F puts on interval $\left(\bar{\theta}_F - \epsilon, \bar{\theta}_F\right)$ to the interval's edges, $\bar{\theta}_F - \epsilon$ and $\bar{\theta}_F$. Because the seller is using an

$F-$ICC mechanism, the strict positivity of I_F means $V_Q - c$ is affine over $\left[\bar{\theta}_F - \epsilon, \bar{\theta}_F\right]$, and so this spread is IC for the buyer. Moreover, this spread cannot hurt the seller by Lemma 2. It follows there is always an optimal F that assigns a positive probability to the top of its support.

Infinite slopes. Throughout the paper, we assumed c admits infinite slopes at the edges of Θ. We use this assumption to prove that the support of the buyer's signal always lies in the interior of Θ. An important consequence is that the mean-preserving spread constraint is always slack at $\bar{\theta}_F$. This slack enables us to obtain restrictions on $Q\left(\bar{\theta}_F\right)$ using perturbations to the buyer's information. One can show this logic continues to hold even when the slope of c is bounded, but sufficiently high around $\bar{\theta}$. However, if $c'\left(\bar{\theta}\right)$ is sufficiently low, our results no longer hold, because it is possible that $\bar{\theta}_F = \bar{\theta}$. Whenever this equality holds, $I_F(\bar{\theta}_F) = 0$, and so the monopolist can freely increase the quality she provides to $\bar{\theta}_F$ without influencing the buyer's information. In this case, one can apply the usual reasoning of Mussa and Rosen (1978) to obtain quality must be efficient at $\bar{\theta}_F$, thereby reversing our result.

Quality upper bound. We also limited the monopolist to offering qualities that lie below an upper bound $\bar{q}$ that lies above the efficient quality for the highest possible type, $\bar{\theta}$. Since this latter quality lies strictly above the efficient quality for any type below $\bar{\theta}$, Theorem 4 implies this upper bound never binds. It follows the exact value of $\bar{q}$ has no impact on the monopolist's optimal menu.

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A. Proofs

A.1. Cost Function Characterization

In this section we show a continuous cost function $C : \mathcal{I} \to \mathbb{R}$ is affine and strictly increasing in informativeness if and only if a strictly convex continuous function $c : \Theta \to \mathbb{R}$ exists such that $C(F) = \int c(\theta)\,\mathrm{d}F(\theta)$. To prove this result, note the Riesz representation theorem implies C is continuous and affine if and only if $C(F) = \int \tilde{c}(\theta)\,\mathrm{d}F(\theta)$ for some continuous $\tilde{c} : \Theta \to \mathbb{R}$. All that remains is to show $\tilde{c}$ must be strictly convex. For this purpose, fix any $x, y, z \in \left(\underline{\theta}, \bar{\theta}\right)$ such that $y = \beta x + (1-\beta)z$ for some $\beta \in (0,1)$. By Lemma 6 in Ravid, Roesler, and Szentes (2020), one can find $F', F'' \in \mathcal{I}$ such that $F' \succ F''$, and

$$F' - F'' = \gamma\left(\beta \mathbf{1}_{[x,\bar{\theta}]} + (1-\beta)\mathbf{1}_{[z,\bar{\theta}]} - \mathbf{1}_{[y,\bar{\theta}]}\right).$$

Since C is strictly increasing in $\succeq$, it follows

$$0 < C(F') - C(F'') = \gamma\left(\beta\tilde{c}(x) + (1-\beta)\tilde{c}(z) - \tilde{c}(y)\right).$$

The claim follows.

A.2. Proof of Theorem 1

We begin by formally defining the buyer's maximization problem holding the monopolist's menu fixed. Let $X = [0, \bar{q}] \times \mathbb{R}_+$, and endow the set of Borel measures over $X \times \Theta$, $\Delta(X \times \Theta)$, with the weak* topology. Given a menu M, the buyer's program can be written as

$$\begin{aligned}\max_{\xi \in \Delta(X\times\Theta)} &\int (\theta q - t)\,\mathrm{d}\xi(q,t,\theta) - C\left(\mathrm{marg}_\Theta\, \mu\right)\\ \text{s.t.} &\operatorname{supp}\mu \subseteq M \times \Theta,\\ &\mathrm{marg}_\Theta \mu \preceq F_0.\end{aligned}$$

Observe the above program involves the maximization of a continuous objective over a compact constraint set, and so the set of solution, $\Xi(M)$, is non-empty for every compact M. Letting $\mathcal{M}$ be the collection of compact subsets of X that contain $(0,0)$, the monopolist's program can be written as

$$\max_{(\xi,M)\in\Delta(X\times\Theta)\times\mathcal{M}}\int\left(t-\kappa\left(q\right)\right)\mathrm{d}\xi\left(q,t,\theta\right)$$
$$\text{s.t.}\ \xi\in\Xi\left(M\right).$$

Notice it is without loss to assume $M\subseteq\bar{X}=[0,\bar{q}]\times\left[0,\bar{\theta}\bar{q}\right]$,because the buyer strictly prefers $(0,0)$ over any menu item that includes a transfer strictly above $\bar{\theta}\bar{q}$. Letting $\mathcal{K}\left(\bar{X}\right)$ be the set of all compact non-empty subsets of $\bar{X}$ endowed with the Hausdorff metric,

$$d\left(A,B\right)=\max\left\{\max_{b\in B}\min_{a\in A}d\left(b,a\right),\max_{a\in A}\min_{b\in B}d\left(a,b\right)\right\},$$

and take $\bar{\mathcal{M}}$ to be the elements of $\mathcal{K}\left(\bar{X}\right)$ that contain $(0,0)$. Taking $\bar{\Xi}$ to be the restriction of Ξ to $\bar{\mathcal{M}}$, it follows the monopolist's problem can be rewritten as

$$\max_{(M,\xi)\in\mathrm{Gr}\,\bar{\Xi}}\int\left(t-\kappa\left(q\right)\right)\mathrm{d}\xi\left(q,t,\theta\right).\tag{10}$$

Observe $\bar{\mathcal{M}}$ is a closed subset of $\mathcal{K}\left(\bar{X}\right)$, and so because $\mathcal{K}\left(\bar{X}\right)$ is compact (Aliprantis and Border (2006), Theorem 3.85), $\bar{\mathcal{M}}$ must be compact as well. It follows, by Berge's theorem of the maximum, that $\bar{\Xi}$ is upper-hemicontinuous, and that $\bar{\Xi}$ has a closed graph (Aliprantis and Border (2006), Theorem 17.10). Hence, this graph must be compact because it is a subset of $\bar{\mathcal{M}}\times\Delta\left(\bar{X}\times\Theta\right)$, which is compact. That (10) admits a solution follows.

A.3. Proof of Implementability

In what follows, denote the set of all Lipschitz functions from Θ to $\mathbb{R}$ by $\mathrm{Lip}\left(\Theta\right)$. As the forward direction of 2 is the same as in Dworczak and Martini (2019), it remains to prove the converse direction of the theorem. Take $\mathrm{ca}_{+}\Theta$ to be the set of (countably additive) positive Borel measures over Θ. For any $[\theta',\theta'']\subseteq\Theta$, define the set $\mathcal{I}_{\theta',\theta''}\subseteq\mathcal{F}$ as the set of all CDFs for which $I_F\left(\theta\right)\geq 0$ holds for all $\theta\in\Theta\setminus[\theta',\theta'']$,and such that $I_F\left(\bar{\theta}\right)=1$. Observe this set is convex. The following lemma readily follows from Luenberger (1997), Theorem 1 in Section 8.3.

Lemma 3. *Suppose F^* satisfies (4). Then for every $[\theta',\theta'']\subset\left(\underline{\theta},\bar{\theta}\right)$, a convex $\Lambda\in\mathrm{Lip}\left(\Theta\right)$ exists such that*

(i) $F^*\in\mathrm{argmax}_{F\in\mathcal{I}_{\theta',\theta''}}\int\left(\phi-\Lambda\right)\left(\cdot\right)\mathrm{d}F$.

(ii) Λ *is affine on any convex subset of* $\{\theta : I_{F^*} > 0\} \cup [\underline{\theta}, \theta') \cup (\theta'', \bar{\theta}]$.

Proof. Fix any $[\theta', \theta''] \subset \left(\underline{\theta}, \bar{\theta}\right)$, and observe $F \in \mathcal{I}$ if and only if $F \in \mathcal{I}_{\theta',\theta''}$ and $I_F(\theta) \geq 0$ for all $\theta \in [\theta', \theta'']$. Therefore, F^* satisfies (4) if and only if it solves the following constrained concave optimization problem,

$$\max_{F \in \mathcal{I}_{\theta',\theta''}} \int \phi(\theta) F(\mathrm{d}\theta)$$
$$\text{s.t. } I_F|_{[\theta',\theta'']} \geq \mathbf{0},$$

where $\mathbf{0}$ is defined as the zero function from $[\theta', \theta'']$ to $\mathbb{R}$. Viewing $I_F|_{[\theta',\theta'']}$ as a mapping from $\mathcal{F}$ to $\mathcal{C}[\theta', \theta'']$ (where $\mathcal{C}[\theta', \theta'']$ is equipped with the supnorm) and observing that $I_{\mathbf{1}_{[\theta_0,\infty)}}(\theta)$ is strictly positive for all $\theta \in [\theta', \theta'']$ (due to $[\theta', \theta''] \subset \left(\underline{\theta}, \bar{\theta}\right)$), one can apply Theorem 1 in section 8.3 of Luenberger (1997) to obtain an element $\lambda^* \in (\mathcal{C}[\theta', \theta''])^*$ such that

$$F^* \in \underset{F \in \mathcal{I}_{\theta',\theta''}}{\operatorname{argmax}} \int \phi(\cdot)\,\mathrm{d}F + \langle \lambda^*, I_F \rangle,$$

and $\langle \lambda^*, I_F \rangle = 0$. Appealing to the Riesz representation theorem, we get a $\lambda \in \mathrm{ca}_+[\theta', \theta'']$ such that $\langle \lambda^*, \varphi \rangle = \int \varphi \mathrm{d}\lambda$ for all $\varphi \in \mathcal{C}[\theta', \theta'']$. Extending λ to Θ by setting $\lambda\left(\tilde{\Theta}\right) = \lambda\left(\tilde{\Theta} \cap [\theta', \theta'']\right)$ then delivers a measure with the following properties:

1. $F^* \in \operatorname{argmax}_{F \in \mathcal{I}_{\theta',\theta''}} \int \phi(\theta) F(\mathrm{d}\theta) + \int I_F(\theta) \lambda(\mathrm{d}\theta)$
2. $\lambda\left(\{\theta : I_{F^*} > 0\} \cup [0, \theta') \cup (\theta'', \bar{\theta}]\right) = 0.$

Now, define $\tilde{\lambda} : \Theta \to \mathbb{R}$ via $\tilde{\lambda}(\theta) = \lambda[\underline{\theta}, \theta]$, and let

$$\Lambda : \Theta \to \mathbb{R}$$
$$\theta \mapsto \int_{\underline{\theta}}^{\theta} \tilde{\lambda}(\cdot)\, \mathrm{d}\tilde{\theta}.$$

Property 2 above implies $\tilde{\lambda}$ is constant on $\{\theta : I_{F^*} > 0\} \cup [0, \theta') \cup (\theta'', \bar{\theta}]$, meaning Λ satisfies (ii). To see that Property 1 implies (i), observe Λ is increasing, Lipschitz, and right continuous, and that

$$\begin{aligned}\int I_F(\cdot)\,\mathrm{d}\lambda &= \int I_F(\cdot)\,\mathrm{d}\tilde{\lambda} = -\int \tilde{\lambda}(\cdot)\,\mathrm{d}I_F \\ &= \int \tilde{\lambda}(\cdot)(F - F_0)(\cdot)\,\mathrm{d}\theta = \int (F - F_0)(\cdot)\,\mathrm{d}\Lambda = \int \Lambda(\cdot)\,\mathrm{d}F,\end{aligned}$$

where the second and last equalities follow from integration by parts, and the third (fourth) equality from $F - F_0$ ($\tilde{\lambda}$) being the almost everywhere derivative of the absolutely continuous function I_F (Λ). Since $\int \Lambda(\cdot)\,\mathrm{d}F_0$ is independent of F, it can be dropped from the maximization. The proof is now complete. $\square$

The previous lemma notes that the usual Lagrange relaxation can be applied to any completely interior interval of Θ. The following lemma uses the theorem's regularity condition to extend this relaxation to the interval's edges.

Lemma 4. *Suppose F^* satisfies (4), and that an $\tilde{F} \in \operatorname{argmax}_{F\in\mathcal{I}_0} \int \phi \mathrm{d}F$ exists such that $\operatorname{supp}\tilde{F} \subset \left(\underline{\theta}, \bar{\theta}\right)$. Then a convex $\Lambda \in \mathrm{Lip}\,(\Theta)$ exists such that*

(i) $F^* \in \operatorname{argmax}_{F\in\mathcal{I}_0} \int (\phi - \Lambda)(\cdot)\,\mathrm{d}F$.

(ii) Λ *is affine on any convex subset of* $\{\theta : I_{F^*} > 0\}$.

Proof. Let $\tilde{\theta}_1 = \min \operatorname{supp} \tilde{F}$ and $\tilde{\theta}_2 = \max \operatorname{supp} \tilde{F}$, and observe $\underline{\theta} < \tilde{\theta}_1 \leq \theta_0 \leq \tilde{\theta}_2 < \bar{\theta}$. Take some $a \in \left(\underline{\theta}, \tilde{\theta}_1\right)$ and some $b \in \left(\tilde{\theta}_2, \bar{\theta}\right)$, and let $\Lambda \in \mathrm{Lip}\,(\Theta)$ be the function from Lemma 3 applied for $[a, b]$. Notice Λ is affine on any convex subset of $\{\theta : I_{F^*} > 0\}$, and so proving (i) is all that remains, which we do by contradiction. Thus, suppose (i) is not satisfied. Since $(\mathcal{I}_0, \succeq)$ forms a lattice, and because $\succeq$ is continuous, the set

$$\operatorname*{argmax}_{F\in\mathcal{I}_0} \int (\phi - \Lambda)(\theta)\,\mathrm{d}F(\mathrm{d}\theta)$$

admits a $\succeq$-minimal element, which we denote by $\hat{F}$. Being $\succeq$ $-$minimal, $\hat{F}$ has at most two elements in its support. Thus, we can write $\operatorname{supp} \hat{F} = \left\{\hat{\theta}_1, \hat{\theta}_2\right\}$, where $\hat{\theta}_1 \leq \theta_0 \leq \hat{\theta}_2$. If $\hat{F} = F^*$, we are done. Thus, suppose $\hat{F} \neq F^*$. By Lemma 3, $\hat{F} \notin \mathcal{I}_{a,b}$, meaning a $\theta^* \in [\underline{\theta}, a) \cup (b, \bar{\theta}]$ exists such that $I_{\hat{F}}(\theta^*) < 0$. Suppose without loss of generality that $\theta^* < a$. Since $I_F(\theta) = 0$ for all $\theta \leq \hat{\theta}_1$, it follows $\hat{\theta}_1 < \theta^* < a < \tilde{\theta}_1 \leq \theta_0$, and so $\hat{\theta}_2 > \theta_0$. Thus, we can write $\hat{F} = \hat{p}\mathbf{1}_{[\hat{\theta}_1,\infty)} + (1-\hat{p})\,\mathbf{1}_{[\hat{\theta}_2,\infty)}$ for some $\hat{p} \in (0, 1)$. Moreover, $\left(\hat{\theta}_1, \hat{\theta}_2\right) \supset (a, \theta_0) \ni \tilde{\theta}_1$, and so $\tilde{\theta}_1 = q\hat{\theta}_1 + (1-q)\,\hat{\theta}_2$ must hold for some $q \in (0, 1)$. Take any $\epsilon < \min\{\hat{p}, 1-\hat{p}\}$,and observe

$$\begin{aligned}\hat{F} &= \hat{p}\mathbf{1}_{[\hat{\theta}_1,\infty)} + (1-\hat{p})\,\mathbf{1}_{[\hat{\theta}_2,\infty)} \\ &\succ (\hat{p} - q\epsilon)\,\mathbf{1}_{[\hat{\theta}_1,\infty)} + (1 - \hat{p} - \epsilon\,(1-q))\,\mathbf{1}_{[\hat{\theta}_2,\infty)} + \epsilon\mathbf{1}_{[\tilde{\theta}_1,\infty)} =: F_\epsilon.\end{aligned}$$

Next, we establish $\int (\phi - \Lambda)(\cdot)\, \mathrm{d}\hat{F} \leq \int (\phi - \Lambda)(\cdot)\, \mathrm{d}F_\epsilon$, implying

$$F_\epsilon \in \underset{F \in \mathcal{I}_0}{\operatorname{argmax}} \int (\phi - \Lambda)(\theta)\, F(\mathrm{d}\theta)$$

, a contradiction to $\succeq$-minimality of $\hat{F}$. To obtain this contradiction, let $\hat{\phi}$ be the concave envelope of ϕ—that is, the lowest concave and upper semicontinuous function that majorizes ϕ. Since $\tilde{F} \in \operatorname{argmax}_{F \in \mathcal{I}_0} \int \phi(\cdot)\, \mathrm{d}F$, ϕ must coincide with its concave envelope over the support of $\tilde{F}$. Thus,

$$\phi\left(\tilde{\theta}_1\right) = \hat{\phi}\left(\tilde{\theta}_1\right) \geq q\hat{\phi}\left(\hat{\theta}_1\right) + (1-q)\,\hat{\phi}\left(\hat{\theta}_2\right) \geq q\phi\left(\hat{\theta}_1\right) + (1-q)\,\phi\left(\hat{\theta}_2\right),$$

where the first inequality follows from Jensen, and the second from $\hat{\phi}$ majorizing ϕ. In addition, convexity of Λ delivers

$$\Lambda\left(\tilde{\theta}_1\right) \leq q\Lambda\left(\hat{\theta}_1\right) + (1-q)\,\Lambda\left(\hat{\theta}_2\right).$$

Therefore,

$$\begin{aligned}\int (\phi - \Lambda)(\cdot)\, \mathrm{d}\left(F_\epsilon - \hat{F}\right) =& \epsilon\left[\hat{\phi}\left(\tilde{\theta}_1\right) - \left(q\phi\left(\hat{\theta}_1\right) + (1-q)\,\phi\left(\hat{\theta}_2\right)\right)\right] \\ &- \epsilon\left[\Lambda\left(\tilde{\theta}_1\right) - \left(q\Lambda\left(\hat{\theta}_1\right) + (1-q)\,\Lambda\left(\hat{\theta}_2\right)\right)\right] \geq 0,\end{aligned}$$

as required. The proof is now complete. □

Next, we prove a simple multiplier result regarding the auxiliary problem

$$\max_{F \in \mathcal{I}_0} \int \varphi(\theta)\, \mathrm{d}F(\theta), \tag{11}$$

for some upper-semicontinuous $\varphi : \Theta \to \mathbb{R}$.

Lemma 5. *The CDF F^* solves the program (11) if and only if a $\gamma \in \mathbb{R}$ exists such that*

$$F^* \in \underset{F \in \mathcal{F}}{\operatorname{argmax}} \int \varphi(\theta) + \gamma\theta \mathrm{d}F(\theta).$$

Proof. Suppose a γ as above exists. Then for every $F \in \mathcal{I}_0$,

$$\begin{aligned}
0 \geq \int \varphi(\theta) + \gamma\theta \mathrm{d}(F - F^*)(\theta) &= \int \varphi(\theta) \mathrm{d}(F(\theta) - F^*(\theta)) + \gamma \int \theta \mathrm{d}(F - F^*)(\theta) \\
&= \int \varphi(\theta) \mathrm{d}(F(\theta) - F^*(\theta)) + \gamma(\theta_0 - \theta_0) \\
&\geq \int \varphi(\theta) \mathrm{d}(F(\theta) - F^*(\theta)),
\end{aligned}$$

that is, F^* solves (11). For the converse, write first the program (11) as

$$\max_{F \in \mathcal{F}_0} \int \varphi(\theta) \mathrm{d}F(\theta) \text{ s.t. } \int \theta - \theta_0 \mathrm{d}F \geq 0 \text{ and } \int \theta_0 - \theta \mathrm{d}F \geq 0.$$

The Convex-Multiplier rule (Pourciau, 1983) delivers a $(\gamma_0, \gamma_1, \gamma_2) \in \mathbb{R}^3_+ \setminus \{\mathbf{0}\}$such that $\gamma_1 \left(\int \theta - \theta_0 \mathrm{d}F\right) = 0$, $\gamma_2 \left(\int \theta_0 - \theta \mathrm{d}F\right) = 0$, and

$$F^* \in \operatorname*{argmax}_{F \in \mathcal{F}} \int \gamma_0 \varphi(\theta) + (\gamma_1 - \gamma_2)(\theta - \theta_0) \mathrm{d}F(\theta).$$

We now argue $\gamma_0 > 0$. To do so, note that if $\gamma_0 = 0$, then either γ_1 or γ_2 are strictly positive, but not both. Suppose $\gamma_1 > 0$ (the argument for $\gamma_2 > 0$ is symmetric). Then

$$F^* \in \operatorname*{argmax}_{F \in \mathcal{F}} \int \gamma_1 (\theta - \theta_0) \mathrm{d}F(\theta) = \left\{ \mathbf{1}_{[\bar{\theta}, \infty)} \right\},$$

a contradiction to $F^* \in \mathcal{I}_0$. Thus, $\lambda_0 > 0$, and so

$$\operatorname*{argmax}_{F \in \mathcal{F}} \int \gamma_0 \varphi(\theta) + (\gamma_1 - \gamma_2)(\theta - \theta_0) \mathrm{d}F(\theta) = \operatorname*{argmax}_{F \in \mathcal{F}} \int \varphi(\theta) + \left(\frac{\gamma_1 - \gamma_2}{\gamma_0}\right) \theta \mathrm{d}F(\theta).$$

Setting $\gamma = \left(\frac{\gamma_1 - \gamma_2}{\gamma_0}\right)$ completes the proof. $\square$

We now show that one can modify the multiplier Λ in such a way that preserves (ii) and extends (i) to the entire set of CDFs over Θ.

Lemma 6. *Suppose F^* satisfies (4), and that an $\tilde{F} \in \operatorname{argmax}_{F \in \mathcal{I}_0} \int \phi \mathrm{d}F$ exists such that $\operatorname{supp}\tilde{F} \subset \left(\underline{\theta}, \bar{\theta}\right)$. Then a convex $\Lambda \in \operatorname{Lip}(\Theta)$ exists such that*

(i) $F^* \in \operatorname{argmax}_{F \in \mathcal{F}} \int (\phi - \Lambda)(\cdot) \mathrm{d}F$.

(ii) Λ *is affine on any convex subset of* $\{\theta : I_{F^*} > 0\}$.

Proof. Begin by applying Lemma 4 to obtain a convex $\tilde{\Lambda} \in \text{Lip}(\Theta)$ satisfying (ii) above and such that F^* solves

$$\max_{F \in \mathcal{I}_0} \int \left(\phi - \tilde{\Lambda}\right)(\theta)\, F(\mathrm{d}\theta) = \max_{F \in \mathcal{F}} \int \left(\phi - \tilde{\Lambda}\right)(\theta)\, F(\mathrm{d}\theta)$$
$$\text{s.t.} \quad \int \theta F\,(\mathrm{d}\theta) = \theta_0.$$

By Lemma 5, a $\lambda \in \mathbb{R}$ exists such that

$$F \in \underset{F \in \mathcal{F}}{\operatorname{argmax}} \int (\phi - \Lambda)(\theta) + \lambda\theta \mathrm{d}F(\theta)\,.$$

Thus, defining $\Lambda(\theta) = \tilde{\Lambda}(\theta) + \lambda(\theta - \theta_0)$ completes the proof. □

Given a function from a convex set $X \subseteq \mathbb{R}$ into the reals, $\varphi : X \to \mathbb{R}$, we use the following notational conventions. If ϕ is increasing, we let $\varphi_-(x) = \sup_{y<x} \varphi(y)$ and $\varphi_+(y) = \inf_{y>x} f(y)$. If φ is convex, we let φ'_- and φ'_+ denote its left and right derivatives, respectively, whenever those exist. We now proceed to prove Theorem 2.

Proof of Theorem 2. Suppose first $F^* \in \mathcal{I}$ is such that a P exists for which the theorem's condition (i) and (ii) hold. Observe

$$\begin{aligned}
\int P(\theta)\, \mathrm{d}(F^* - F_0)(\theta) &= \int (F_0 - F^*)(\theta)\, P(\mathrm{d}\theta) \\
&= \int (F_0 - F^*)(\theta)\, P'_+(\theta)\, \mathrm{d}\theta \\
&= \int P'_+(\theta)\, I_{F^*}(\mathrm{d}\theta) \\
&= -\int I_{F^*}(\theta)\, P'_+(\mathrm{d}\theta) = 0,
\end{aligned}$$

where the first and penultimate equalities follow from integration by parts, the second (third) equality following from $P'_+(F_0 - F^*)$ being an almost everywhere derivative of the absolutely continuous function P (I_{F^*}), and the last equality from P being affine on any interval over which $I_F > 0$. That F^* solves (4) then follows from Theorem 1 in Dworczak and Martini (2019).

Next, suppose $F^* \in \mathcal{I}$ solves (4) and that $\tilde{F} \in \operatorname{argmax}_{F \in \mathcal{I}_0} \int \phi \mathrm{d}F$ exists such that $\text{supp}\tilde{F} \subset \left(\underline{\theta}, \bar{\theta}\right)$. By Lemma 6, a $\Lambda \in \text{Lip}(\Theta)$ exists satisfying condition (i) and (ii) of the lemma. Define $P(\theta) := \Lambda(\theta) + \max(\phi - \Lambda)(\Theta)$, and observe that the theorem's condition (i) obviously holds. To see the theorem's condition (ii) holds, note first $\text{supp}\, F^* \subseteq$

$\operatorname{argmax}_{\theta\in\Theta} \phi(\theta) - \Lambda(\theta)$, meaning $P = \phi$ for all $\theta \in \operatorname{supp} F^*$. Second, observe

$$\phi(\theta) = \Lambda(\theta) + \phi(\theta) - \Lambda(\theta) \le \Lambda(\theta) + \max(\phi - \Lambda)(\Theta) = P(\theta).$$

Thus, P also satisfies condition (i). The proof is now complete. $\square$

Next, we prove the regularity condition required by Theorem 2 applies for any bounded mechanism the seller may offer the buyer.

Proof of 1. We prove the lemma by contradiction. Suppose (the argument for $\underline{\theta}$ is symmetric) that $\bar{\theta} \in \operatorname{supp} F^*$. By Lemma 5, a $\lambda \in \mathbb{R}$ exists such that $\operatorname{supp} F^* \subseteq V_Q(\theta) - c(\theta) + \lambda\theta =: \varphi(\theta)$. Since φ is continuous, a strictly increasing sequence $\{\theta_n\}$ exists such that $\theta_n \nearrow \bar{\theta}$, $\theta_n < \bar{\theta}$, and $\varphi(\theta_{n+1}) \ge \varphi(\theta_n)$ for all n. Therefore,

$$\begin{aligned}
0 &\le \frac{\varphi(\theta_{n+1}) - \varphi(\theta_n)}{\theta_{n+1} - \theta_n} \\
&= (\theta_{n+1} - \theta_n)^{-1} \left[V_Q(\theta_{n+1}) - V_Q(\theta_n) + \lambda(\theta_{n+1} - \theta_n) - (c(\theta_{n+1}) - c(\theta_n))\right] \\
&\le Q(\theta_{n+1}) + \lambda + \left(\frac{c(\theta_n) - c(\theta_{n+1})}{\theta_{n+1} - \theta_n}\right) \\
&\le \bar{q} + \lambda + \left(\frac{c(\theta_n) - c(\theta_{n+1})}{\theta_{n+1} - \theta_n}\right) \\
&\le \bar{q} + \lambda + c'(\theta_n)\left(\frac{\theta_n - \theta_{n+1}}{\theta_{n+1} - \theta_n}\right) = \bar{q} + \lambda - c'(\theta_n) \to -\infty,
\end{aligned}$$

where the second and fourth inequality follow from V_Q and c being convex, the third inequality from $Q \le \bar{q}$, and convergence from $c'(\theta) \to \infty$ as $\theta \nearrow \bar{\theta}$.

[Proof of Theorem 3]As a preliminary step, suppose F is IC for some mechanism, and that we have some F-ICC allocation $\hat{Q}$ that is F-verified by $\hat{p}$. We make two observations. First, since $\underline{\theta}_F > \underline{\theta}$ (Corollary 1), and because c' is continuous, strictly increasing and satisfies $\lim_{\theta\searrow\underline{\theta}} c'(\theta) = -\infty$, $\underline{\theta}_{\hat{Q}} \in (\underline{\theta}, \underline{\theta}_F]$ is the unique solution to $\hat{p}(\underline{\theta}_F) + c'\left(\underline{\theta}_{\hat{Q}}\right) = 0$. Second, that $\lim_{\theta\nearrow\bar{\theta}} c'(\theta) = \infty$ together with $\bar{\theta}_F < \bar{\theta}$ implies $\bar{\theta}_{\hat{Q}}$ is the unique element in $[\bar{\theta}_F, \bar{\theta})$ such that $\hat{p}(\bar{\theta}_F) + c'(\bar{\theta}_{\hat{Q}}) = \bar{q}$. Below we use these observations to prove the theorem.

Next, we show every F-ICC mechanism is F-IC. Let Q be an F-ICC allocation, and consider the shadow price $P_{Q,\underline{u}}$ defined in equation (6). As explained after Theorem 3, $P_{Q,\underline{u}} = V_{Q,\underline{u}} - c$ for every $\theta \in \left[\underline{\theta}_F, \bar{\theta}_F\right]$. It remains to show $P_{Q,\underline{u}} \ge V_{Q,\underline{u}} - c$ for all $\theta \in \Theta \setminus \left[\underline{\theta}_F, \bar{\theta}_F\right]$. In the next paragraph we claim $V_{Q,\underline{u}} - c$ is concave on $[\underline{\theta}, \underline{\theta}_F]$ and on $\left[\bar{\theta}_F, \bar{\theta}\right]$. Using this claim, one can deduce that $P_{Q,\underline{u}} \ge V_{Q,\underline{u}} - c$ using the following inequality chain,

$$\begin{aligned}
P_{Q,\underline{u}}(\theta) &= P_{Q,\underline{u}}(\underline{\theta}_F) - \int_{\theta}^{\underline{\theta}_F} p\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} \\
&= P_{Q,\underline{u}}(\underline{\theta}_F) - \int_{\theta}^{\underline{\theta}_F} p(\underline{\theta}_F) \mathrm{d}\tilde{\theta} \\
&= \left(V_{Q,\underline{u}} - c\right)(\underline{\theta}_F) - \int_{\theta}^{\underline{\theta}_F} Q(\underline{\theta}_F) - c'(\underline{\theta}_F) \mathrm{d}\tilde{\theta} \\
&\geq \left(V_{Q,\underline{u}} - c\right)(\underline{\theta}_F) - \int_{\theta}^{\underline{\theta}_F} Q\left(\tilde{\theta}\right) - c'\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} = V_{Q,\underline{u}}(\theta) - c(\theta),
\end{aligned}$$

where the second equality follows from observing that F's mean preserving spread constraint is slack on $\theta \in \left(\underline{\theta}, \bar{\theta}\right) \setminus \left[\underline{\theta}_F, \bar{\theta}_F\right]$ meaning p is constant over this set, the third and fourth equality from $P_{Q,\underline{u}}(\theta) = \left(V_{Q,\underline{u}} - c\right)(\theta)$ and $p(\theta) = Q(\theta) - c'(\theta)$ holding at $\theta = \underline{\theta}_F$, and the inequality from the claim that $V_{Q,\underline{u}} - c$ being concave on $[\underline{\theta}, \underline{\theta}_F]$. A similar inequality chain delivers $P_{Q,\underline{u}} \geq V_{Q,\underline{u}} - c$ for the range $\left[\bar{\theta}_F, \bar{\theta}\right]$.

To conclude the proof that every F-ICC mechanism is F-IC, we now argue $V_{Q,\underline{u}} - c$ is concave over $[\underline{\theta}, \underline{\theta}_F]$ (the argument for $\left[\bar{\theta}_F, \bar{\theta}\right]$ is similar). For this, it is sufficient to show $Q - c'$ is decreasing over said interval. Thus, pick any $\theta < \theta'$ in $[\underline{\theta}, \underline{\theta}_F]$. We show $(Q - c')(\theta) \geq (Q - c')(\theta')$. The inequality obviously holds if $\theta, \theta' \in \left[\underline{\theta}, \underline{\theta}_Q\right]$, because c' is strictly increasing and Q is constant over said interval. The inequality also holds if $\theta, \theta' \in \left[\underline{\theta}_Q, \underline{\theta}_F\right]$, because then we have $(Q - c')(\theta) = p(\underline{\theta}_F) = (Q - c')(\theta')$. Finally, suppose $\theta \leq \underline{\theta}_Q \leq \theta'$. Then,

$$\begin{aligned}
(Q - c')(\theta) &\geq (Q - c')\left(\underline{\theta}_Q\right) \\
&= p(\underline{\theta}_F) + c'\left(\underline{\theta}_Q\right) - c'\left(\underline{\theta}_Q\right) = p(\underline{\theta}_F) = (Q - c')(\theta'),
\end{aligned}$$

where the first equality follows from $p\left(\underline{\theta}_Q\right) = p(\underline{\theta}_F)$, and the last equality from the definition of Q. This concludes the argument that every F-ICC mechanism is F-IC.

Next, we argue every F-IC mechanism admits an equivalent F-ICC mechanism. By Theorem 2, a mechanism $\tilde{Q}$ is F-IC if and only if a shadow price $P : \Theta \to \mathbb{R}$ exists such that $P(\theta) \geq V_{\tilde{Q},\underline{u}} - c$, with equality holding for all $\theta \in \operatorname{supp} F$. Since P is convex and Lipschitz, an increasing $p : \Theta \to \mathbb{R}$ exists such that for all θ, both $p(\theta) \in [P'_-(\theta), P'_+(\theta)]$ and

$$P(\theta) = P(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} p(s)\, \mathrm{d}s \tag{12}$$

hold. Moreover the above holds for every $p : \Theta \to \mathbb{R}$ such that $p(\theta) \in [P'_-(\theta), P'_+(\theta)]$, and every such p is bounded and increasing. We now argue we can choose p so that $p(\theta) = \tilde{Q}(\theta) - c'(\theta)$ for all $\theta \in \operatorname{supp} F$. To do so, let $\tilde{v} = V_{\tilde{Q}} - c$, and observe that it is left and right differentiable, because both $V_{\tilde{Q},\underline{u}}$ and c are. Moreover, for every $\theta \in \operatorname{supp} F$,

$$\begin{aligned}\tilde{v}'_{-}(\theta) = \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[\tilde{v}(\theta) - \tilde{v}(\theta-\epsilon)\right] &= \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[P(\theta) - \underline{u}_{\tilde{Q}} - \tilde{v}(\theta-\epsilon)\right] \\ &\geq \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[P(\theta) - P(\theta-\epsilon)\right] = P'_{-}(\theta),\end{aligned}$$

and

$$\begin{aligned}\tilde{v}'_{+}(\theta) = \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[\tilde{v}(\theta+\epsilon) - \tilde{v}(\theta)\right] &\leq \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[P(\theta+\epsilon) - \underline{u}_{\tilde{Q}} - \tilde{v}(\theta)\right] \\ &= \lim_{\epsilon\searrow 0}\frac{1}{\epsilon}\left[P(\theta+\epsilon) - P(\theta)\right] = P'_{+}(\theta).\end{aligned}$$

Because $\tilde{Q}(\theta) - c'(\theta) \in \left[\tilde{v}'_{-}(\theta), \tilde{v}'_{+}(\theta)\right]$ for all θ, it follows

$$\tilde{Q}(\theta) - c'(\theta) + \underline{u}_{\tilde{Q}} \in \left[P'_{-}(\theta), P'_{+}(\theta)\right].$$

Hence, we can set $p(\theta) = \tilde{Q}(\theta) - c'(\theta)$ for all $\theta \in \operatorname{supp} F$ in a way that satisfies (12).

Define the $F-$ICC allocation Q from p as in equation (5), and set

$$\underline{u} := P(\underline{\theta}_F) + c(\underline{\theta}_F) - V_Q(\underline{\theta}_F).$$

Note that

$$\begin{aligned}\underline{u} &= P(\underline{\theta}_F) + c(\underline{\theta}_F) - V_Q(\underline{\theta}_F) \\ &= P\left(\underline{\theta}_Q\right) + c\left(\underline{\theta}_Q\right) + \int_{\underline{\theta}_Q}^{\underline{\theta}_F} p(\theta) + c'(\theta) - Q(\theta)\,\mathrm{d}\theta \\ &= P\left(\underline{\theta}_Q\right) + c\left(\underline{\theta}_Q\right) + \int_{\underline{\theta}_Q}^{\underline{\theta}_F} p(\theta) - \tilde{Q}(\underline{\theta}_F) + c'(\underline{\theta}_F)\mathrm{d}\theta \\ &= P\left(\underline{\theta}_Q\right) + c\left(\underline{\theta}_Q\right) + \int_{\underline{\theta}_Q}^{\underline{\theta}_F} p(\underline{\theta}_F) - \tilde{Q}(\underline{\theta}_F) + c'(\underline{\theta}_F)\mathrm{d}\theta = P\left(\underline{\theta}_Q\right) + c\left(\underline{\theta}_Q\right).\end{aligned}$$

where the second equality follows from the first fundamental theorem of calculus, the third from the definition of Q and the fact that $p(\underline{\theta}_F) = \tilde{Q}(\underline{\theta}_F) + c'(\underline{\theta}_F)$ for all $\theta \in \left(\underline{\theta}_Q, \underline{\theta}_F\right)$, the fourth from $I_F(\theta) > 0$ for all $\theta \in \left[\underline{\theta}_Q, \underline{\theta}_F\right]$ (and therefore p is constant on $\left[\underline{\theta}_Q, \underline{\theta}_F + \epsilon\right]$ for some $\epsilon > 0$), and the last equality from $p(\underline{\theta}_F) = \tilde{Q}(\underline{\theta}_F) - c'(\underline{\theta}_F)$.

We now argue $P = P_{Q,\underline{u}}$, where $P_{Q,\underline{u}}$ is defined as in equation (6),

$$P_{Q,\underline{u}}(\theta) = V_{Q,\underline{u}}(\underline{\theta}_F) - c(\underline{\theta}_F) + \int_{\underline{\theta}_F}^{\theta} p\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta}.$$

Towards this goal, observe first that

$$\begin{aligned}
P_{Q,\underline{u}}(\underline{\theta}_F) &= V_{Q,\underline{u}}(\underline{\theta}_F) - c(\underline{\theta}_F) \\
&= \underline{u} - c\left(\underline{\theta}_Q\right) + \int_{\underline{\theta}_Q}^{\underline{\theta}_F} Q(\theta) - c'(\theta)\,\mathrm{d}\theta \\
&= P\left(\underline{\theta}_Q\right) + \int_{\underline{\theta}_Q}^{\underline{\theta}_F} p(\underline{\theta}_F)\,\mathrm{d}\theta \\
&= P\left(\underline{\theta}_Q\right) + \int_{\underline{\theta}_Q}^{\underline{\theta}_F} p(\theta)\,\mathrm{d}\theta = P(\underline{\theta}_F),
\end{aligned}$$

where the third equality follows $\underline{u} = P\left(\underline{\theta}_Q\right) + c\left(\underline{\theta}_Q\right)$, and the fourth equality from $I_F(\theta) > 0$ for all $\theta \in \left[\underline{\theta}_Q, \underline{\theta}_F\right]$ (and therefore $p(\theta)$ is constant and equal to $p(\underline{\theta}_F)$ on $\left[\underline{\theta}_Q, \underline{\theta}_F + \epsilon\right]$ for some $\epsilon > 0$). We therefore get the following equality chain

$$P_{Q,\underline{u}}(\theta) = V_{Q,\underline{u}}(\underline{\theta}_F) - c(\underline{\theta}_F) + \int_{\underline{\theta}_F}^{\theta} p\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} = P(\underline{\theta}_F) + \int_{\underline{\theta}_F}^{\theta} p\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} = P(\theta).$$

It follows $P(\theta) \geq V_{Q,\underline{u}}(\theta) - c(\theta)$ for all θ, with equality whenever $\theta \in \operatorname{supp} F$. Therefore, for every $\theta \in \operatorname{supp} F$ we have

$$V_{Q,\underline{u}}(\theta) - c(\theta) = P(\theta) = V_{\tilde{Q},\underline{\tilde{u}}}(\theta) - c(\theta),$$

meaning $V_{Q,\underline{u}}(\theta) = V_{\tilde{Q},\underline{\tilde{u}}}(\theta)$ holds for all such θ, as required. All that remains is to show that $\underline{u} \geq \underline{\tilde{u}}$, which follows from observing that

$$\underline{u} = P\left(\underline{\theta}_Q\right) + c\left(\underline{\theta}_Q\right) \geq V_{\tilde{Q},\underline{\tilde{u}}}\left(\underline{\theta}_Q\right) \geq \underline{\tilde{u}}.$$

The proof is now complete. □

A.4. Proof of Corollary 1

Let us first argue that if $(Q, \underline{u})$ is F-IC, then $\operatorname{supp} F \subseteq \left(\underline{\theta}, \bar{\theta}\right)$. For this goal, let P be some $\left(F, V_{Q,\underline{u}} - c\right)$-shadow price, and note

$$M \geq \frac{1}{\epsilon}\left(P\left(\underline{\theta}_F + \epsilon\right) - P\left(\underline{\theta}_F\right)\right) \geq \frac{1}{\epsilon}\left[\left(V_{Q,\underline{u}} - c\right)\left(\underline{\theta}_F + \epsilon\right) - \left(V_{Q,\underline{u}} - c\right)\left(\underline{\theta}_F\right)\right],$$

where M is the P's Lipchitz constant. Since $\lim_{\theta \searrow \underline{\theta}} c'(\theta) = -\infty$, if $\underline{\theta}_F = \underline{\theta}$, the above equation's right hand side would go to ∞, which is impossible. A similar argument implies $\bar{\theta}_F < \underline{\theta}$.

We now claim I_F must be strictly positive in the neighborhood of $\underline{\theta}_F$ and $\bar{\theta}_F$. Since I_F is continuous, to prove the claim it is enough to show $I_F\left(\underline{\theta}_F\right) > 0$ and $I_F\left(\bar{\theta}_F\right) > 0$ both hold. For this purpose, note that, because $\left\{\underline{\theta}, \bar{\theta}\right\} \subseteq \operatorname{supp} F_0$, F_0 is strictly positive on $(\underline{\theta}, \underline{\theta}_F)$, and strictly below 1 on $\left(\bar{\theta}_F, \bar{\theta}\right)$. Therefore,

$$I_F\left(\underline{\theta}_F\right) = \int_{\underline{\theta}}^{\underline{\theta}_F} \left(F_0 - F\right)(\theta)\,\mathrm{d}\theta = \int_{\underline{\theta}}^{\underline{\theta}_F} F_0(\theta)\,\mathrm{d}\theta > 0,$$

and

$$\begin{aligned} I_F\left(\bar{\theta}_F\right) &= \int_{\underline{\theta}}^{\bar{\theta}_F} \left(F_0 - F\right)(\theta)\,\mathrm{d}\theta \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \left(F_0 - F\right)(\theta)\,\mathrm{d}\theta - \int_{\bar{\theta}_F}^{\bar{\theta}} \left(F_0 - F\right)(\theta)\,\mathrm{d}\theta = \int_{\bar{\theta}_F}^{\bar{\theta}} \left(1 - F_0\right)(\theta)\,\mathrm{d}\theta > 0. \end{aligned}$$

The corollary follows.

A.5. Proofs of Feasible Perturbations

A.5.1. Proof of Lemma 2

Observe first that both parts trivially hold when $\theta_1 = \theta_2$ or when $\alpha \in \{0, 1\}$. Therefore, suppose (without loss of generality) that $\theta_1 < \theta_2$. The proof of both of the Lemma's parts proceeds as follows. Using that p_Q is constant on $[\theta_*, \theta^*]$, we construct a family of informational deviations which are incentive-compatible for the buyer and that are indexed by $\epsilon > 0$. As ϵ vanishes, the difference between these deviations and F^* converges to the difference between an atom at $\alpha\theta_1 + (1-\alpha)\theta_2$ and a split of that atom's mass between an atom on θ_1 and an atom on θ_2 for the first part, and vice-versa for the second part. Then, we show the desired inequality using optimality of (Q^*, F^*) and continuity of $\pi_{Q^*}|_{[\theta_*,\theta^*]}$ (where the latter is implied by continuity of c' and p_Q being constant over $[\theta_*, \theta^*]$).

Proof. As a preliminary step, let $G = F^*(\cdot|\theta \in [\theta_*, \theta^*])$, $H = F^*(\cdot|\theta \notin [\theta_*, \theta^*])$, and $\beta = F^*(\theta^*) - F^*_-(\theta_*)$, and observe $F^* = \beta G + (1-\beta) H$. In addition, notice that $(V - c)$ is affine on $[\theta_*, \theta^*]$, because for any $\theta \in [\theta_*, \theta^*] \subseteq \left[\underline{\theta}_Q, \bar{\theta}_Q\right]$,

$$\begin{aligned}
V_{Q^*}(\theta) - c(\theta) &= V_{Q^*}(\theta_*) - c(\theta_*) + \int_{\theta_*}^{\theta} (Q^* - c')(\theta)\,\mathrm{d}\theta \\
&= V_{Q^*}(\theta_*) - c(\theta_*) + \int_{\theta_*}^{\theta} p_{Q^*}(\theta)\,\mathrm{d}\theta = V_{Q^*}(\theta_*) - c(\theta_*) + p_Q(\theta_*)(\theta - \theta_*),
\end{aligned}$$

where the last equality follows from p_Q being constant on $[\theta_*, \theta^*] \subseteq \left[\underline{\theta}_Q, \bar{\theta}_Q\right]$.

Proof of Part 1. We begin by constructing the above-mentioned class of informational deviations. Take any $\epsilon \in \left(0, \frac{1}{2}(\theta_2 - \theta_1)\right)$ (which implies $[\theta_1 - \epsilon, \theta_1 + \epsilon] \cap [\theta_2 - \epsilon, \theta_2 + \epsilon] = \varnothing$), and define the following objects:

$$\begin{aligned}
G_{0,\epsilon} &= G(\cdot|\theta \notin [\theta_1 - \epsilon, \theta_1 + \epsilon] \cup [\theta_2 - \epsilon, \theta_2 + \epsilon]), \\
G_{1,\epsilon} &= G(\cdot|\theta \in [\theta_1 - \epsilon, \theta_1 + \epsilon]), \\
G_{2,\epsilon} &= G(\cdot|\theta \in [\theta_2 - \epsilon, \theta_2 + \epsilon]), \\
\gamma_{1,\epsilon} &= G(\theta_1 + \epsilon) - G_-(\theta_2 - \epsilon), \\
\gamma_{2,\epsilon} &= G(\theta_2 + \epsilon) - G_-(\theta_2 - \epsilon), \\
\gamma_{0,\epsilon} &= 1 - \gamma_{1,\epsilon} - \gamma_{2,\epsilon}.
\end{aligned}$$

Clearly, $G = \sum_{i=0}^{2} \gamma_{i,\epsilon} G_{i,\epsilon}$. Moreover, since $\theta_1, \theta_2 \in \operatorname{supp} G$, both $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ are strictly positive for all $\epsilon > 0$. For any $\epsilon \in \left(0, \frac{1}{2}(\theta_2 - \theta_1)\right)$, define

$$\begin{aligned}
\theta_\epsilon &= \int \theta\,\mathrm{d}(\alpha G_{1,\epsilon} + (1-\alpha) G_{2,\epsilon}), \\
\tilde{\gamma}_\epsilon &= \min\{\gamma_{1,\epsilon}, \gamma_{2,\epsilon}\} > 0, \text{ and} \\
G_\epsilon &= \gamma_{0,\epsilon} G_{0,\epsilon} + \eta_\epsilon \mathbf{1}_{[\theta_\epsilon,\infty)} + (\gamma_{1,\epsilon} - \alpha\eta_\epsilon) G_{1,\epsilon} + (\gamma_{2,\epsilon} - (1-\alpha)\eta_\epsilon) G_{2,\epsilon}.
\end{aligned}$$

In words, G_ϵ alters G by pooling $\alpha\tilde{\gamma}_\epsilon$ mass from the ϵ-ball around θ_1 and $(1-\alpha)\tilde{\gamma}_\epsilon$ mass from the ϵ-ball around θ_2 and pooling them to create an $\tilde{\gamma}_\epsilon > 0$ mass on θ_ϵ—that is,

$$G_\epsilon - G = \tilde{\gamma}_\epsilon \left(\mathbf{1}_{[\theta_\epsilon,\infty)} - (\alpha G_{1,\epsilon} + (1-\alpha) G_{2,\epsilon})\right).$$

With the above in hand, we can finally define our informational pertubation—specifically, take $F_\epsilon = \beta G_\epsilon + (1-\beta) H$.

Next, we argue $F_\epsilon \in \mathcal{I}$ and that Q^* is $F_\epsilon-$IC. For the first claim, observe that, because $\alpha G_{1,\epsilon} + (1-\alpha) G_{2,\epsilon} \succ \mathbf{1}_{[\theta_\epsilon,\infty)}$, G_ϵ is less informative than G, and so $F_\epsilon \preceq F^* \preceq F_0$. That $F_\epsilon \in \mathcal{I}$ follows from $\preceq$ being transitive. To see Q^* is F_ϵ-IC for all $\epsilon \in \left(0, \frac{1}{2}(\theta_2 - \theta_1)\right)$, observe

$$\begin{aligned}\int (V_{Q^*} - c)\,\mathrm{d}\,(F_\epsilon - F^*) &= \beta \int (V_{Q^*} - c)\,\mathrm{d}\,(G_\epsilon - G) \\ &= \beta \tilde{\gamma}_\epsilon \left[\int (V_{Q^*} - c)\,\mathrm{d}\left(\mathbf{1}_{[\theta_\epsilon,\infty)} - (\alpha G_{1,\epsilon} + (1-\alpha) G_{2,\epsilon})\right)\right] = 0,\end{aligned}$$

where the last equality follows from $\alpha G_{1,\epsilon} + (1-\alpha) G_{2,\epsilon} \succ \mathbf{1}_{[\theta_\epsilon,\infty)}$, the support of $\alpha G_{1,\epsilon} + (1-\alpha) G_{2,\epsilon}$ being contained in $[\theta_*, \theta^*] \subseteq \left[\underline{\theta}_Q, \bar{\theta}_Q\right]$, and $V_{Q^*} - c$ being affine on $[\theta_*, \theta^*]$.

Now, because (Q^*, F^*) is monopolist-optimal, that Q^* is $F_\epsilon-$IC all small $\epsilon > 0$ means that $\int \pi_{Q^*} \mathrm{d}F_\epsilon \leq \int \pi_{Q^*} \mathrm{d}F$. Rearranging this inequality, dividing by $\beta\tilde{\gamma}_\epsilon$, and taking ϵ to zero delivers

$$\begin{aligned}0 \leq \frac{1}{\beta\tilde{\gamma}_\epsilon} \int \pi_{Q^*}\,\mathrm{d}(F^* - F_\epsilon) &= \frac{1}{\tilde{\gamma}_\epsilon} \int \pi_{Q^*} \mathrm{d}\,(G - G_\epsilon) \\ &= \left[\alpha \int \pi_{Q^*}\,\mathrm{d}G_{1,\epsilon} + (1-\alpha) \int \pi_{Q^*}\,\mathrm{d}G_{2,\epsilon}\right] - \pi_{Q^*}(\theta_\epsilon) \\ &\to (\alpha \pi_{Q^*}(\theta_1) + (1-\alpha)\pi_{Q^*}(\theta_2)) - \pi_{Q^*}\left(\alpha\theta_1 + (1-\alpha)\theta_2\right),\end{aligned}$$

where convergence follows from continuity of $\pi_{Q^*}|_{[\theta_*,\theta^*]}$, convergence of $F_{1,\epsilon}$ and $F_{2,\epsilon}$ to $\mathbf{1}_{[\theta_1,\infty)}$ and $\mathbf{1}_{[\theta_2,\infty)}$ respectively, and $\theta_\epsilon \to \alpha\theta_1 + (1-\alpha)\theta_2$. $\square$

Proof of Part 2. Suppose now $[\theta_1, \theta_2] \subseteq [\theta_*, \theta^*]$ is such that $I_{F^*}(\theta') > 0$ holds for all $\theta' \in [\theta_1, \theta_2]$, and that $\alpha \in (0,1)$ is such that $\theta_\alpha := \alpha\theta_1 + (1-\alpha)\theta_2 \in \operatorname{supp} G$. We begin by defining the above-mentioned family of deviations. For any strictly positive $\epsilon < \min\{\theta - \theta_1, \theta_2 - \theta\}$, define

$$\begin{aligned}G_{0,\epsilon}(\cdot) &:= G(\cdot|\theta \notin [\theta_\alpha - \epsilon, \theta_\alpha + \epsilon]), \\ G_{1,\epsilon}(\cdot) &:= G(\cdot|\theta \in [\theta_\alpha - \epsilon, \theta_\alpha + \epsilon]), \\ \theta_\epsilon &:= \int \theta\,\mathrm{d}G_{1,\epsilon}(\theta) \\ \gamma_\epsilon &:= G(\theta_\alpha + \epsilon) - G_-(\theta_\alpha - \epsilon).\end{aligned}$$

Clearly, $G = (1-\gamma_\epsilon) G_{0,\epsilon} + \gamma_\epsilon G_{1,\epsilon}$. Observe $\gamma_\epsilon > 0$, because $\theta_\alpha \in \operatorname{supp} G$, and that an

$\alpha_\epsilon \in (0,1)$ exists such that

$$\theta_\epsilon = \alpha_\epsilon \theta_1 + (1-\alpha_\epsilon)\theta_2,$$

by our choice of ϵ. Obviously, $\theta_\epsilon \to \theta_\alpha$, and $\alpha_\epsilon \to \alpha$. For a given $\tilde{\gamma} \in (0, \gamma_\epsilon)$, define

$$G_{\eta,\epsilon} = (1-\gamma_\epsilon)G_{0,\epsilon} + (\gamma_\epsilon - \tilde{\gamma})G_{1,\epsilon} + \tilde{\gamma}\left(\alpha_\epsilon \mathbf{1}_{[\theta_1,\infty)} + (1-\alpha_\epsilon)\mathbf{1}_{[\theta_2,\infty)}\right).$$

Clearly, $G_{\tilde{\gamma},\epsilon}$ is a CDF.

We now construct our informational deviation: set $F_{\tilde{\gamma},\epsilon} := \beta G_{\tilde{\gamma},\epsilon} + (1-\beta) H$ for all ϵ and $\tilde{\gamma}$ satisfying the above conditions. We begin by arguing that this deviation is a signal—that is, $F_{\tilde{\gamma},\epsilon} \in \mathcal{I}$—whenever $\tilde{\gamma}$ is sufficiently small (holding ϵ fixed). To do so, observe that the function $F \mapsto I_F(\theta)$ is affine for all θ, meaning that

$$I_{F^*} - I_{F_{\tilde{\gamma},\epsilon}} = \tilde{\gamma}\beta\left(\alpha_\epsilon I_{\mathbf{1}_{[\theta_1,\infty)}} + (1-\alpha_\epsilon) I_{\mathbf{1}_{[\theta_2,\infty)}} - I_{G_{1,\epsilon}}\right) < 0, \tag{13}$$

where the inequality follows from $G_{1,\epsilon} \prec \alpha_\epsilon \mathbf{1}_{[\theta_1,\infty)} + (1-\alpha_\epsilon)\mathbf{1}_{[\theta_2,\infty)}$. Since the support of $G_{1,\epsilon}$, $\mathbf{1}_{[\theta_1,\infty)}$, and $\mathbf{1}_{[\theta_2,\infty)}$ is contained in $[\theta_1,\infty)$, it follows $I_{F_{\tilde{\gamma},\epsilon}}(\theta) = I_{F^*}(\theta) \geq 0$ for all $\theta \leq \theta_1$. Next, observe that for any $F \in \mathcal{F}$ and any $\theta \geq \max(\operatorname{supp} F)$, $\int_{\theta' \leq \theta} F(\theta')\,\mathrm{d}\theta' = \theta - \int \theta' \mathrm{d}F(\theta')$, $I_{G_{1,\epsilon}}(\theta) = \alpha_\epsilon I_{\mathbf{1}_{[\theta_1,\infty)}}(\theta) - (1-\alpha_\epsilon) I_{\mathbf{1}_{[\theta_2,\infty)}}(\theta)$ for all $\theta \geq \theta_2$, meaning that $I_{F_{\tilde{\gamma},\epsilon}}(\theta) = I_{F^*}(\theta) \geq 0$ holds for all such θ. Consider now the case $\theta \in (\theta_1, \theta_2)$. That I_F is continuous for all F, combined with I_{F^*} being strictly positive over $[\theta_1,\theta_2]$ implies a $\zeta := \min I_{F^*}([\theta_1,\theta_2]) > 0$ and that

$$\xi_\epsilon := \min_{\theta \in [\theta_1,\theta_2]} \left(\alpha_\epsilon I_{\mathbf{1}_{[\theta_1,\infty)}} + (1-\alpha_\epsilon) I_{\mathbf{1}_{[\theta_2,\infty)}} - I_{G_{1,\epsilon}}\right) > -\infty.$$

Recalling that $\xi_\epsilon \leq 0$, one can see that whenever $\tilde{\gamma} < -\zeta/\beta\xi_\epsilon$, $\theta \in [\theta_1,\theta_2]$ implies

$$I_{F_{\tilde{\gamma},\epsilon}}(\theta) \geq I_{F^*}(\theta) + \tilde{\gamma}\beta\xi_\epsilon \geq \zeta + \tilde{\gamma}\beta\xi_\epsilon \geq 0.$$

Thus, we have shown $F_{\tilde{\gamma},\epsilon} \in \mathcal{I}$ for all $\tilde{\gamma} < -\zeta/\beta\xi_\epsilon$.

We now argue Q^* is $F_{\tilde{\gamma},\epsilon}$-IC for all above-mentioned ϵ and all $\tilde{\gamma} < -\zeta/\beta\xi_\epsilon$. To see this, observe

$$\begin{aligned}\int (V_{Q^*} - c)\,\mathrm{d}(F_{\tilde{\gamma},\epsilon} - F^*) &= \tilde{\gamma}\beta \int (V_{Q^*} - c)\,\mathrm{d}\left(\alpha_\epsilon \mathbf{1}_{[\theta_1,\infty)} + (1-\alpha_\epsilon)\mathbf{1}_{[\theta_2,\infty)} - G_{1,\epsilon}\right) \\ &= \tilde{\gamma}\beta\left(\alpha_\epsilon (V_{Q^*} - c)(\theta_1) + (1-\alpha_\epsilon)(V_{Q^*} - c)(\theta_2) - (V_{Q^*} - c)(\theta_\epsilon)\right) = 0,\end{aligned}$$

where the last equality follows from $\alpha_\epsilon \mathbf{1}_{[\theta_1,\infty)} + (1-\alpha_\epsilon)\mathbf{1}_{[\theta_2,\infty)} \succeq G_{1,\epsilon}$, the support of $\alpha_\epsilon \mathbf{1}_{[\theta_1,\infty)} + (1-\alpha_\epsilon)\mathbf{1}_{[\theta_2,\infty)}$ and $G_{1,\epsilon}$ being contained in $[\theta_*, \theta^*]$, and $V_{Q^*} - c$ being affine on $[\theta_*, \theta^*]$.

For the proof's last step, observe that, because Q^* is $F_{\tilde{\gamma},\epsilon}$-IC for the buyer for all small ϵ and $\tilde{\gamma}$, monopolist optimality of (Q^*, F^*) implies

$$
\begin{aligned}
0 \geq \frac{1}{\tilde{\gamma}} \int \pi_{Q^*} \mathrm{d}\left(F_{\tilde{\gamma},\epsilon} - F^*\right) &= \alpha_\epsilon \pi_{Q^*}\left(\theta_1\right) + \left(1-\alpha_\epsilon\right)\pi_{Q^*}\left(\theta_2\right) - \pi\left(\theta_\epsilon\right) \\
&\xrightarrow{\epsilon \to 0} \alpha \pi_{Q^*}\left(\theta_1\right) + \left(1-\alpha\right)\pi_{Q^*}\left(\theta_2\right) - \pi\left(\theta_\alpha\right),
\end{aligned}
$$

where convergence follows from $\theta_\epsilon \to \theta_\alpha$, $\alpha_\epsilon \to \alpha$, and π_{Q^*} being continuous on $[\theta_*, \theta^*]$. The desired inequality follows.

A.5.2. Allocation Perturbations

In this section, we prove two lemmas. The first result shows the set of allocations that are F-IC is convex.

Lemma 7. *Suppose Q and $\tilde{Q}$ are both F-IC. Then, $(1-\beta)\,Q + \beta\tilde{Q}$ is also F-IC for all $\beta \in [0,1]$.*

Proof. Note that for any two allocations Q, $\tilde{Q}$, and any $\beta \in [0,1]$,

$$
\begin{aligned}
V_{(1-\beta)Q+\beta\tilde{Q}}\left(\theta\right) &= \int_{\underline{\theta}}^{\theta} \left(\left(1-\beta\right)Q\left(\tilde{\theta}\right) + \beta\tilde{Q}\left(\tilde{\theta}\right)\right)\mathrm{d}\tilde{\theta} \\
&= \left(1-\beta\right)\int_{\underline{\theta}}^{\theta} Q\left(\tilde{\theta}\right)\mathrm{d}\tilde{\theta} + \beta\int_{\underline{\theta}}^{\theta}\tilde{Q}\left(\tilde{\theta}\right)\mathrm{d}\tilde{\theta} = \left(1-\beta\right)V_Q\left(\theta\right) + \beta V_{\tilde{Q}}\left(\theta\right).
\end{aligned}
$$

Therefore, if both Q, $\tilde{Q}$ are F-IC, one obtains the following inequality for all $\tilde{F}$,

$$
\begin{aligned}
\int \left(V_{(1-\beta)Q+\beta\tilde{Q}} - c\right)\left(\theta\right)\mathrm{d}F\left(\theta\right) &= \left(1-\beta\right)\int\left(V_Q - c\right)\left(\theta\right)\mathrm{d}F\left(\theta\right) + \beta\int\left(V_{\tilde{Q}} - c\right)\left(\theta\right)\mathrm{d}F\left(\theta\right) \\
&\geq \left(1-\beta\right)\int\left(V_Q - c\right)\left(\theta\right)\mathrm{d}\tilde{F}\left(\theta\right) + \beta\int\left(V_{\tilde{Q}} - c\right)\left(\theta\right)\mathrm{d}\tilde{F}\left(\theta\right) \\
&= \int\left(V_{(1-\beta)Q+\beta\tilde{Q}} - c\right)\left(\theta\right)\mathrm{d}\tilde{F}\left(\theta\right),
\end{aligned}
$$

meaning $(1-\beta)\,Q + \beta\tilde{Q}$ is also F-IC. □

Next, we obtain a first order condition for the monopolist's optimal outcome by perturbing the allocation while keeping the buyer's information fixed.

Lemma 8. *Let (Q^*, F^*) be monopolist optimal. Suppose Q also incentivizes F^*. Then,*

$$\int (\theta - \kappa' (Q^* (\theta))) (Q - Q^*) (\theta) - (V_Q - V_{Q^*}) (\theta) \, \mathrm{d}F^* (\theta) \leq 0.$$

Proof. Suppose (Q^*, F^*) is monopolist optimal, and let Q be any other $F-$IC allocation. Defining the allocation $Q_\varepsilon := Q^* + \epsilon (Q - Q^*)$ for every $\epsilon \in (0, 1)$, it follows from the previous lemma that Q_ϵ is also F-IC. Therefore, it must be that (Q_ϵ, F^*) must be weakly worse for the monopolist than (Q^*, F^*) . In other words, we must have

$$\int (\pi_{Q_\epsilon} (\theta) - \pi_Q (\theta)) \, \mathrm{d}F^* (\theta) \leq 0$$

for all ϵ. Dividing this inequality by $\epsilon > 0$, and taking the limit as $\epsilon \searrow 0$ gives

$$\begin{aligned}
0 &\geq \frac{1}{\epsilon} \int (\pi_{Q_\epsilon} (\theta) - \pi_Q (\theta)) \, F^*(\mathrm{d}\theta) \\
&= \int \theta (Q - Q^*) (\theta) - (V_Q - V_{Q^*}) (\theta) \, F^*(\mathrm{d}\theta) \\
&\quad - \int \frac{1}{\epsilon} (\kappa (Q^* (\theta) + \epsilon (Q - Q^*) (\theta)) - \kappa (Q^* (\theta))) \, F^*(\mathrm{d}\theta) \\
&\to \int \theta (Q - Q^*) (\theta) - (V_Q - V_{Q^*}) (\theta) \, F^*(\mathrm{d}\theta) \\
&\quad - \int \kappa' (Q^* (\theta)) (Q - Q^*) (\theta) \, F^*(\mathrm{d}\theta),
\end{aligned}$$

where convergence follows from Beppo Levi's Theorem (e.g., Aliprantis and Border (2006) Theorem 11.18).[11] The lemma follows. □

A.6. Proof of Theorem 4

Before proving the theorem, we recall a few of our notational conventions. Given a function from a convex set $X \subseteq \mathbb{R}$ into the reals, $\varphi : X \to \mathbb{R}$, we use the following notational conventions. If ϕ is increasing, we let $\varphi_- (x) = \sup_{y<x} \varphi (y)$ and $\varphi_+ (y) = \inf_{y>x} f (y)$. If φ is convex, we let φ'_- and φ'_+ denote its left and right derivatives, respectively, whenever those exist.

We begin our proof by showing one can find an F-almost surely equal version of Q that is left or right continuous whenever $\kappa' \circ Q_+ (\theta) < \theta$ and $\kappa' \circ Q_- (\theta) > \theta$, respectively.

Lemma 9. *A mechanism $\tilde{Q}$ exists that is F-almost surely equal to Q such that $\left(\tilde{Q}, F\right)$ is*

[11]Because κ is convex, the function $\epsilon \mapsto \frac{1}{\epsilon} (\kappa (q + \epsilon (\tilde{q} - q)) - \kappa (q))$ is decreasing in ϵ for all $\tilde{q}$ and q.

monopolist optimal, and for which $\tilde{Q}_+(\theta) = \tilde{Q}(\theta)$ *whenever* $\kappa' \circ \tilde{Q}_+(\theta) < \theta$ *and* $\tilde{Q}_-(\theta) = \tilde{Q}(\theta)$ *whenever* $\kappa' \circ \tilde{Q}_-(\theta) > \theta$*.*

Proof. Define the mechanism $\tilde{Q}$ via

$$\tilde{Q}(\theta) := \underset{q \in [Q_-(\theta), Q_+(\theta)]}{\operatorname{argmax}} \; \theta q - \kappa(q),$$

which is well-defined because the objective is strictly concave and so admits a unique maximizer. Obviously,

$$\int (\theta Q(\theta) - \kappa(Q(\theta))) - \left(\theta \tilde{Q}(\theta) - \kappa\left(\tilde{Q}(\theta)\right)\right) F(\mathrm{d}\theta) \leq 0,$$

with equality holding if and only if $\tilde{Q}$ equals Q F-almost surely. Observe $\tilde{Q}$ satisfies the desired properties and is equal to Q at any θ at which Q is continuous, and so $V_Q = V_{\tilde{Q}}$, because Q can only have countably many discontinuities. It follows $\left(\tilde{Q}, F\right)$ is incentive compatible for B. Since (Q, F) is monopolist optimal,

$$0 \leq \int \pi_Q(\theta) - \pi_{\tilde{Q}}(\theta) \, \mathrm{d}F(\theta) = \int (\theta Q(\theta) - \kappa(Q(\theta))) - \left(\theta \tilde{Q}(\theta) - \kappa\left(\tilde{Q}(\theta)\right)\right) \mathrm{d}F(\theta) \leq 0.$$

Hence, $\tilde{Q}$ equals Q F-almost surely, as desired. □

From now on, we assume Q is an F-ICC mechanism satisfying the conditions of Lemma 9. We now obtain a sufficient condition for inefficiently low quality.

Lemma 10. *Fix any* $\theta^* \in [\underline{\theta}_F, \bar{\theta}_F)$*, and suppose one of the following two conditions hold:*

$$\int_{\theta > \theta^*} \kappa' \circ Q(\theta) \, \mathrm{d}F(\theta) \leq (1 - F(\theta^*)) \, \theta^* \tag{14}$$

$$\int_{\theta \geq \theta^*} \kappa' \circ Q(\theta) \, \mathrm{d}F(\theta) \leq (1 - F_-(\theta^*)) \, \theta^*. \tag{15}$$

Then $\kappa' \circ Q(\theta^*) < \theta^*$*. Morever, (14) implies (15).*

Proof. We first show (14) implies $\kappa' \circ Q(\theta^*) < \theta^*$, and then show the same strict inequality follows from (15). Thus, assume (14) holds, and suppose $\kappa' \circ Q(\theta^*) \geq \theta^*$ for a contradiction. Because Q is F-ICC, it is strictly increasing on $\left[\underline{\theta}_F, \bar{\theta}_F\right] \supseteq \left[\theta^*, \bar{\theta}_F\right]$, and so $\kappa' \circ Q(\theta) > \kappa' \circ Q(\theta^*)$ for all $\theta > \theta^*$, because κ' is strictly increasing. Therefore,

$$\theta^* < \int \kappa' \circ Q(\theta) \, \mathrm{d}F\left(\theta | \theta \in (\theta^*, \bar{\theta}_F]\right) = \int \kappa' \circ Q(\theta) \, \mathrm{d}F(\theta | \theta > \theta^*) = \frac{\int_{\theta > \theta^*} \kappa' \circ Q(\theta) \, \mathrm{d}F(\theta)}{1 - F(\theta^*)}, \tag{16}$$

contradicting (14). We now show (15) also implies $\kappa' \circ Q(\theta^*) < \theta^*$. If $F_-(\theta^*) = F(\theta^*)$, then (15) implies (14), so we are done. Assume then $F_-(\theta^*) < F(\theta^*)$, and suppose $\kappa' \circ Q(\theta^*) \geq \theta^*$ for a contradiction. Then $\kappa' \circ Q(\theta) > \theta^*$ for all $\theta > \theta^*$, because κ' is strictly increasing. Therefore, equation (16) holds, meaning that

$$\begin{aligned}\int_{\theta \geq \theta^*} \kappa' \circ Q(\theta)\, \mathrm{d}F(\theta) &= \int_{\theta > \theta^*} \kappa' \circ Q(\theta)\, \mathrm{d}F(\theta) + (F(\theta^*) - F_-(\theta^*))\, \kappa' \circ Q(\theta^*) \\ &> (1 - F(\theta^*))\, \theta^* + (F(\theta^*) - F_-(\theta^*))\, \kappa' \circ Q(\theta^*) \\ &\geq (1 - F(\theta^*))\, \theta^* + (F(\theta^*) - F_-(\theta^*))\, \theta^* = (1 - F_-(\theta^*))\, \theta^*,\end{aligned}$$

where the first inequality follows from (16), and the second from the contradiction assumption. As the above inequality contradicts (15), we are done. Finally, we show (14) implies (15). To do so, observe (14) delivers

$$\begin{aligned}\int_{\theta \geq \theta^*} \kappa' \circ Q(\theta)\, \mathrm{d}F(\theta) &= \int_{\theta > \theta^*} \kappa' \circ Q(\theta)\, \mathrm{d}F(\theta) + (F(\theta^*) - F_-(\theta^*))\, \kappa' \circ Q(\theta^*) \\ &\leq (1 - F(\theta^*))\, \theta^* + (F(\theta^*) - F_-(\theta^*))\, \theta^* = (1 - F_-(\theta^*))\, \theta^*.\end{aligned}$$

The proof is now complete. □

We now proceed to show quality is inefficient low for any θ^* that is at the bottom of the support of F. As already remarked, in this case, p_Q is constant in the neighborhood of θ^* (due to Corollary 1).

Lemma 11. *Quality is inefficiently low at $\underline{\theta}_F$—i.e., $\kappa' \circ Q(\underline{\theta}_F) < \underline{\theta}_F$. Moreover, equation (15) holds at $\underline{\theta}_F$ whenever $Q(\underline{\theta}_F) > 0$.*

Proof. Because $\underline{\theta}_F > \underline{\theta}$, the lemma is obvious if $Q(\underline{\theta}_F) = 0$. Suppose then that $Q(\underline{\theta}_F) > 0$. For any $\varepsilon \in (0, Q(\underline{\theta}_F))$, let $\theta_\varepsilon = \inf\{\theta : Q(\theta) \geq \varepsilon\}$, and define $p_\varepsilon(\cdot) := p(\cdot) - \varepsilon$. Observe p_ε is an F-shadow derivative, because p is. Let Q_ε be the F-ICC allocation defined by p_ε as in (5) (note Q_ε is well-defined because $Q(\underline{\theta}_F) > \varepsilon$). Obviously, $\theta_\varepsilon \leq \underline{\theta}_F$, and

$$Q_\varepsilon(\theta) = p_Q(\theta) + c'(\theta) - \varepsilon = Q(\theta)$$

for all $\theta \in \left[\theta_\varepsilon, \bar{\theta}_F\right]$. Noting that for all $\theta \geq \underline{\theta}_F$,

$$V_{Q_\varepsilon}(\theta) - V_Q(\theta) = \int_{\theta_\varepsilon}^{\theta} -\varepsilon\, \mathrm{d}\tilde{\theta} + \int_{\underline{\theta}}^{\theta_\varepsilon} -Q\left(\tilde{\theta}\right) \mathrm{d}\tilde{\theta} = \varepsilon(\theta_\varepsilon - \theta) - V_Q(\theta_\varepsilon),$$

and so by Lemma 8 we have

$$
\begin{aligned}
0 &\geq \int \left(\theta - \kappa' \circ Q\left(\theta\right)\right)\left(-\varepsilon\right) - \varepsilon\left(\theta_\varepsilon - \theta\right) + V_Q\left(\theta_\varepsilon\right) \mathrm{d}F\left(\theta\right) \\
&= \int \left(\kappa' \circ Q\left(\theta\right) - \theta_\varepsilon\right)\varepsilon + V_Q\left(\theta_\varepsilon\right) \mathrm{d}F\left(\theta\right).
\end{aligned}
$$

This inequality, however, implies that

$$
\begin{aligned}
\varepsilon \int_{\theta \geq \underline{\theta}_F} \kappa' \circ Q\left(\theta\right) \mathrm{d}F\left(\theta\right) = \varepsilon \int \kappa' \circ Q\left(\theta\right) \mathrm{d}F\left(\theta\right) & \\
\leq \int \varepsilon\theta_\varepsilon - V_Q\left(\theta_\varepsilon\right) \mathrm{d}F \leq \varepsilon\theta_\varepsilon \leq \varepsilon\underline{\theta}_F &= \varepsilon\underline{\theta}_F\left(1 - F_-\left(\underline{\theta}_F\right)\right),
\end{aligned}
$$

where the second inequality follows from $V_Q \geq 0$. Dividing both sides of the above inequality by $\varepsilon > 0$ gives equation (15), and so $\kappa' \circ Q\left(\underline{\theta}_F\right) < \underline{\theta}_F$ holds by Lemma (10). □

Next, we show quality is inefficiently low whenever p_Q jumps at θ^*.

Lemma 12. *Suppose $\theta^* \in \left[\underline{\theta}_F, \bar{\theta}_F\right]$ be such that $p_{Q-}\left(\theta^*\right) < p_{Q+}\left(\theta^*\right)$. Then, (15) holds at θ^* and $\kappa' \circ Q\left(\theta^*\right) < \theta^*$.*

Proof. Observe Q being F-ICC and $p_{Q-}\left(\theta^*\right) < p_{Q+}\left(\theta^*\right)$ means $I\left(\theta^*\right) = 0$, and so $\theta^* > \underline{\theta}_F$. For any $\varepsilon \in \left(0, p_{Q+}\left(\theta^*\right) - p_{Q-}\left(\theta^*\right)\right)$, define

$$
p_\varepsilon\left(\theta\right) = \begin{cases} p_Q\left(\theta\right) & \text{if } \theta < \theta^*, \\ p_Q\left(\theta^*\right) \wedge \left(p_{Q+}\left(\theta^*\right) - \varepsilon\right) & \text{if } \theta = \theta^*, \\ p_Q\left(\theta\right) - \varepsilon & \text{if } \theta > \theta^*. \end{cases}
$$

It is easy to verify that p_ε is an F-shadow price derivative because p_Q is and $I\left(\theta^*\right) = 0$. Let Q_ε be the F-ICC mechanism associated with p_ε. It follows Q_ε is F-IC, and so can apply Lemma 8 to get the following inequality for every $\varepsilon \in \left(0, p_{Q+}\left(\theta^*\right) - p_{Q-}\left(\theta^*\right)\right)$,

$$
\begin{aligned}
0 \geq& \int_{\theta > \theta^*} \left(\theta - \kappa' \circ Q\left(\theta\right)\right)\left(-\varepsilon\right) - \varepsilon\left(\theta^* - \theta\right) \mathrm{d}F\left(\theta\right) \\
&+ \left(F\left(\theta^*\right) - F_-\left(\theta^*\right)\right)\left(\theta^* - \kappa' \circ Q\left(\theta^*\right)\right)\left(p_\varepsilon\left(\theta^*\right) - p_Q\left(\theta^*\right)\right) \\
=& \int_{\theta > \theta^*} \left(\kappa' \circ Q\left(\theta\right) - \theta^*\right)\varepsilon \mathrm{d}F\left(\theta\right) \\
&+ \left(F\left(\theta^*\right) - F_-\left(\theta^*\right)\right)\left(\theta^* - \kappa' \circ Q\left(\theta^*\right)\right)\left(p_\varepsilon\left(\theta^*\right) - p_Q\left(\theta^*\right)\right).
\end{aligned}
$$

Rearranging gives

$$\int_{\theta>\theta^*} \kappa' \circ Q(\theta)\,\mathrm{d}F(\theta) \leq (1-F(\theta^*))\,\theta^* + (F(\theta^*)-F_-(\theta^*))\,(\theta^*-\kappa'\circ Q(\theta^*))\left(\frac{p_\varepsilon(\theta^*)-p_Q(\theta^*)}{\varepsilon}\right). \tag{17}$$

We now distinguish between two cases. Suppose first $p_Q(\theta^*) < p_{Q+}(\theta^*)$. Then for all small enough $\varepsilon > 0$, $p_\varepsilon(\theta^*) - p_Q(\theta^*) = 0$, and so equation (17) is equivalent to (14). The lemma then follows from Lemma 10. Suppose then $p_Q(\theta^*) = p_{Q+}(\theta^*)$. Then $p_\varepsilon(\theta^*) - p_Q(\theta^*) = \varepsilon$. Substituting into (17) and rearranging gives (15), and so again Lemma 10 delivers the desired conclusion. □

Our next task is to show quality is inefficiently low at θ^* when p_Q is non-constant just below θ^* or just above θ^*. Towards this goal, we prove the following lemma that enables us to move from one F-ICC mechanism to another.

Lemma 13. *Suppose $\theta_1, \theta_2 \in (\underline{\theta}_F, \bar{\theta}_F]$ are such that $\theta_1 < \theta_2$ and $I_F(\theta_1) = I_F(\theta_2) = 0$ and that Q is F-ICC. Define $\delta = p_Q(\theta_2) - p_Q(\theta_1)$*

$$\tilde{p}(\theta) = \begin{cases} p_Q(\theta) & \text{if } \theta \leq \theta_1 \\ p_Q(\theta) - \delta & \text{if } \theta \geq \theta_2 \\ p_Q(\theta_1) & \text{if } \theta \in [\theta_1, \theta_2]. \end{cases}$$

Then $\tilde{p}$ is an F-shadow derivative. Moreover, the allocation $\tilde{Q}$ defined from $\tilde{p}$ using equation (5) is well-defined.

Proof. The result follows immediately from p_Q being an F-shadow derivative, and Q being a well-defined F-ICC allocation. □

We now prove quality is inefficiently low at θ^* whenever p_Q is non-constant just below it.

Lemma 14. *Suppose $\theta^* \in (\underline{\theta}_F, \bar{\theta}_F]$ satisfies $p_{Q-}(\theta^*) = p_{Q+}(\theta^*)$ and $p_{Q-}(\theta^*) > p_Q(\theta)$ holds for all $\theta < \theta^*$. Then (15) holds at θ^* and $\kappa'(\theta^*) < \theta^*$.*

Proof. Suppose θ^* satisfies the lemma's premise. We begin by arguing that we can find a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ in $\left[\underline{\theta}_F, \bar{\theta}_F\right]$ such that $\theta_n \nearrow \theta^*$, $I_F(\theta_n) = 0$ for all n, and $p_Q(\theta_n) < p_Q(\theta_{n+1})$ for all n. We then use this sequence to construct a sequence of allocations that keep

F incentive compatible for B. This allocation sequence, combined with Lemma 8 delivers a sequence of first-order conditions whose limit delivers (15) at θ^*. That $\kappa'\left(Q(\theta^*)\right) < \theta^*$ then follows.

Let us find the sequence $\{\theta_n\}_{n\in\mathbb{N}}$. For every $\delta > 0$, p_Q is non-constant on $[\theta^* - \delta, \theta^*]$, for if it was, $p_{Q-}(\theta^*) = p_Q(\theta^* - \delta) < p_{Q-}(\theta^*)$. It follows we can find a sequence $\left\{\tilde{\theta}_n\right\}_{n\in\mathbb{N}}$ in $(\underline{\theta}_F, \theta^*)$ with $\tilde{\theta}_n \nearrow \theta^*$ such that $p_Q\left(\tilde{\theta}_n\right) < p_Q\left(\tilde{\theta}_{n+1}\right)$ for all n. It follows p_Q is non-constant on $\left[\tilde{\theta}_m, \tilde{\theta}_n\right]$ for any $m < n$, and so every $m < n$ admits some $\theta_{m,n} \in \left[\tilde{\theta}_m, \tilde{\theta}_n\right]$ for which $I_F(\theta_{m,n}) = 0$. Choosing $\theta_n := \theta_{2n,2n+1}$, we have $\theta_n \nearrow \theta^*$, and

$$p_Q(\theta_n) = p_Q(\theta_{2n,2n+1}) \le p_Q\left(\tilde{\theta}_{2n+1}\right) < p_Q\left(\tilde{\theta}_{2n+2}\right) \le p_Q\left(\theta_{2(n+1),2(n+1)+1}\right) = p_Q(\theta_{n+1}),$$

meaning $\{\theta_n\}_{n\in\mathbb{N}}$ is as desired.

We now construct an F-ICC mechanism for every θ_n in the above sequence. For this purpose, let $\delta_n = p_Q(\theta^*) - p_Q(\theta_n) > 0$,

$$p_n(\theta) = \begin{cases} p_Q(\theta) & \text{if } \theta \le \theta_n \\ p_Q(\theta) - \delta_n & \text{if } \theta \ge \theta^* \\ p_Q(\theta_n) & \text{if } \theta \in [\theta_n, \theta^*], \end{cases}$$

and let Q_n be the allocation induced by p_n via (5). In view of Lemma 13, to argue Q_n is F-ICC, it is sufficient to argue $I_F(\theta^*) = 0$, because $I_F(\theta_n) = 0$. But $I_F(\theta^*) = 0$ is obvious, since $\{\theta : I_F(\theta) = 0\}$ is closed (because I_F is continuous), $I_F(\theta_m) = 0$ holds for all m, and $\theta_n \nearrow \theta$. Thus, Q_n is F-ICC.

Our next goal is to apply Lemma 8 to get a first-order condition indexed by n. For this purpose, observe

$$V_{Q_n}(\theta) - V_Q(\theta) = \int_{\theta_n \wedge \theta}^{\theta^* \wedge \theta} \left(p_Q(\theta_n) - p_Q\left(\tilde{\theta}\right)\right) \mathrm{d}\tilde{\theta} - \delta_n\left(\theta - \theta^* \wedge \theta\right).$$

Therefore, Lemma 8 delivers the following inequality for all n,

$$\begin{aligned}0 &\geq \int_{\theta\geq\theta^*} (\theta - \kappa' \circ Q(\theta))(-\delta_n)\,\mathrm{d}F(\theta) + \int_{\theta\in[\theta_n,\theta^*)} (\theta - \kappa' \circ Q(\theta))(p_Q(\theta_n) - p_Q(\theta))\,\mathrm{d}F(\theta) \\ &\quad - \int_{\theta\geq\theta_n}\int_{\theta_n}^{\theta^*\wedge\theta} \left(p_Q(\theta_n) - p_Q\left(\tilde{\theta}\right)\right)\mathrm{d}\tilde{\theta}\mathrm{d}F(\theta) - \int_{\theta\geq\theta^*} -\delta_n(\theta-\theta^*)\,\mathrm{d}F(\theta) \\ &= \int_{\theta\geq\theta^*} (\kappa' \circ Q(\theta) - \theta^*)\,\delta_n\mathrm{d}F(\theta) + \int_{\theta\in[\theta_n,\theta^*)} (\theta - \kappa' \circ Q(\theta))(p_Q(\theta_n) - p_Q(\theta))\,\mathrm{d}F(\theta) \\ &\quad - \int_{\theta\geq\theta_n}\int_{\theta_n}^{\theta^*\wedge\theta} \left(p_Q(\theta_n) - p_Q\left(\tilde{\theta}\right)\right)\mathrm{d}\tilde{\theta}\mathrm{d}F(\theta).\end{aligned}$$

Rearranging and noting that $p_Q(\theta_n) \leq p_Q(\theta)$ for all $\theta \geq \theta_n$ delivers

$$\begin{aligned}\int_{\theta\geq\theta^*} \kappa' \circ Q(\theta)\,\mathrm{d}F(\theta) &\leq \int_{\theta\geq\theta^*} \theta^*\mathrm{d}F(\theta) - \int_{\theta\in[\theta_n,\theta^*)} (\theta - \kappa' \circ Q(\theta))\frac{1}{\delta_n}(p_Q(\theta_n) - p_Q(\theta))\,\mathrm{d}F(\theta) \\ &\quad + \int_{\theta\geq\theta_n}\int_{\theta_n}^{\theta^*\wedge\theta} \frac{1}{\delta_n}\left(p_Q(\theta_n) - p_Q\left(\tilde{\theta}\right)\right)\mathrm{d}\tilde{\theta}\mathrm{d}F(\theta) \\ &\leq \int_{\theta\geq\theta^*} \theta^*\mathrm{d}F(\theta) - \int_{\theta\in[\theta_n,\theta^*)} (\theta - \kappa' \circ Q(\theta))\frac{1}{\delta_n}(p_Q(\theta_n) - p_Q(\theta))\,\mathrm{d}F(\theta) \\ &\leq \int_{\theta\geq\theta^*} \theta^*\mathrm{d}F(\theta) + \int_{\theta\in[\theta_n,\theta^*)} |\theta - \kappa' \circ Q(\theta)|\left|\frac{1}{\delta_n}(p_Q(\theta_n) - p_Q(\theta))\right|\mathrm{d}F(\theta)\end{aligned} \tag{18}$$

We now show taking the limit of equation (18) as $n \to \infty$ delivers equation (15). To do so, observe $|p_Q(\theta_n) - p_Q(\theta)| \leq \delta_n$ for all $\theta \in [\theta_n, \theta^*)$, and that $\theta - \kappa' \circ Q(\theta) \leq \bar{\theta}_F + \kappa' \circ Q\left(\bar{\theta}_F\right)$ for all $\theta \in [\theta_n, \theta^*)$. Therefore, an M exists such that $|\theta - \kappa' \circ Q(\theta)|\left|\frac{1}{\delta_n}(p_Q(\theta_n) - p_Q(\theta))\right| \leq M$ for all n. Substituting back into (18) and taking limit with n delivers

$$\int_{\theta\geq\theta^*} \kappa' \circ Q(\theta)\,\mathrm{d}F(\theta) \leq \int_{\theta\geq\theta^*} \theta^*\mathrm{d}F(\theta) + M(F_-(\theta^*) - F(\theta_n)) \to \int_{\theta\geq\theta^*} \theta^*\mathrm{d}F(\theta).$$

Hence (15) holds at θ^*, completing the proof in view of Lemma 10. □

We now replicate the argument behind Lemma 14, with some minor adjustments, to show quality is inefficiently low at any θ^* above which p_Q is non-constant.

Lemma 15. *Suppose $\theta^* \in (\underline{\theta}_F, \bar{\theta}_F]$ satisfies $p_{Q-}(\theta^*) = p_{Q+}(\theta^*)$ and $p_{Q+}(\theta^*) < p_Q(\theta)$ holds for all $\theta > \theta^*$. Then $\kappa' \circ Q(\theta^*) < \theta^*$ and (14) (and a fortiori (15)) holds at θ^*.*

Proof. We begin by finding a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ in $\left[\underline{\theta}_F, \bar{\theta}_F\right]$ such that $\theta_n \searrow \theta^*$, $I_F(\theta_n) = 0$ for all n, and $p_Q(\theta_n) > p_Q(\theta_{n+1})$ for all n. We then construct a corresponding sequence of allocations that keep F incentive compatible for the buyer. This allocation sequence, combined with Lemma 8 delivers a sequence of first-order conditions whose limit delivers (15) at θ^*. That $\kappa' \circ Q(\theta^*) < \theta^*$ follows from Lemma 10.

Let us find the sequence $\{\theta_n\}_{n\in\mathbb{N}}$. Observe that for every $\delta > 0$, p_Q is non-constant on $[\theta^*, \theta^* + \delta]$, for if it was constant, $p_{Q+}(\theta^*) = p_Q(\theta^* + \delta) > p_{Q+}(\theta^*)$. It follows we can find a sequence $\left\{\tilde{\theta}_n\right\}_{n\in\mathbb{N}}$ in $\left(\theta^*, \bar{\theta}\right)$ with $\tilde{\theta}_n \searrow \theta^*$ such that $p_Q\left(\tilde{\theta}_n\right) > p_Q\left(\tilde{\theta}_{n+1}\right)$ for all n. To define $\{\theta_n\}_{n\in\mathbb{N}}$, observe p_Q is non-constant on $\left[\tilde{\theta}_m, \tilde{\theta}_n\right]$ for any $m < n$, and so every $m < n$ admits some $\theta_{m,n} \in \left[\tilde{\theta}_m, \tilde{\theta}_n\right]$ for which $I_F(\theta_{m,n}) = 0$. Choosing $\theta_n := \theta_{2n,2n+1}$, we have $\theta_n \searrow \theta^*$, and

$$p_Q(\theta_n) = p_Q(\theta_{2n,2n+1}) \geq p_Q\left(\tilde{\theta}_{2n+1}\right) > p_Q\left(\tilde{\theta}_{2n+2}\right) \geq p_Q\left(\theta_{2(n+1),2(n+1)+1}\right) = p_Q(\theta_{n+1}).$$

Finally, observe $I_F(\theta) = 0$ and $\theta < \bar{\theta}$ implies $\theta < \bar{\theta}_F$. Hence, because θ_n is strictly decreasing, it has at most one element weakly above $\bar{\theta}_F$, and so it is without loss to take $\{\theta_n\}_{n\in\mathbb{N}}$ to be strictly below $\bar{\theta}_F$, as desired.

We now construct an F-ICC mechanism for every θ_n in the above sequence. Let $\delta_n := p_Q(\theta_n) - p_Q(\theta^*) > 0$. Define

$$p_n(\theta) = \begin{cases} p_Q(\theta) & \text{if } \theta \leq \theta^* \\ p_Q(\theta) - \delta_n & \text{if } \theta \geq \theta_n \\ p_Q(\theta^*) & \text{if } \theta \in [\theta^*, \theta_n], \end{cases}$$

and let Q_n be the allocation induced by p_n via equation (5). We now argue Q_n is F-ICC. In view of Lemma 13, it is sufficient to show $I_F(\theta^*) = 0$, because $I_F(\theta_n) = 0$. But $I_F(\theta^*) = 0$ is obvious, since $\{\theta : I_F(\theta) = 0\}$ is closed (because I_F is continuous), $I_F(\theta_m) = 0$ holds for all m, and $\theta_n \searrow \theta$. Thus, Q_n is F-ICC.

Our next goal is to apply Lemma 8 to get a first-order condition indexed by n. For this purpose, observe

$$V_{Q_n}(\theta) - V_Q(\theta) = \int_{\theta^* \wedge \theta}^{\theta_n \wedge \theta} \left(p_Q(\theta^*) - p_Q\left(\tilde{\theta}\right)\right) \mathrm{d}\tilde{\theta} - \delta_n (\theta - \theta_n \wedge \theta).$$

Therefore, Lemma 8 delivers the following inequality for all n:

$$
\begin{aligned}
0 \geq & \int_{\theta \geq \theta_n} (\theta - \kappa' \circ Q(\theta))(-\delta_n) \mathrm{d}F(\theta) + \int_{\theta \in [\theta^*, \theta_n)} (\theta - \kappa' \circ Q(\theta))(p_Q(\theta^*) - p_Q(\theta)) \mathrm{d}F(\theta) \\
& - \int_{\theta \geq \theta^*} \int_{\theta^*}^{\theta_n \wedge \theta} \left(p_Q(\theta^*) - p_Q\left(\tilde{\theta}\right)\right) \mathrm{d}\tilde{\theta} \mathrm{d}F(\theta) - \int_{\theta \geq \theta_n} -\delta_n (\theta - \theta_n) \mathrm{d}F(\theta) \\
= & \int_{\theta \geq \theta_n} (\kappa' \circ Q(\theta) - \theta_n) \delta_n \mathrm{d}F(\theta) + \int_{\theta \in [\theta^*, \theta_n)} (\theta - \kappa' \circ Q(\theta))(p_Q(\theta^*) - p_Q(\theta)) \mathrm{d}F(\theta) \\
& - \int_{\theta \geq \theta^*} \int_{\theta^*}^{\theta_n \wedge \theta} \left(p_Q(\theta^*) - p_Q\left(\tilde{\theta}\right)\right) \mathrm{d}\tilde{\theta} \mathrm{d}F(\theta).
\end{aligned}
$$

Dividing both sides by δ_n and noting that $p_Q(\theta^*) \leq p_Q(\theta)$ for all $\theta \geq \theta^*$ delivers

$$
\begin{aligned}
0 \geq & \int_{\theta \geq \theta_n} (\kappa' \circ Q(\theta) - \theta_n) \mathrm{d}F(\theta) - \int_{\theta \in [\theta^*, \theta_n)} (\kappa' \circ Q(\theta) - \theta) \left(\frac{p_Q(\theta^*) - p_Q(\theta)}{\delta_n}\right) \mathrm{d}F(\theta) \\
\geq & \int_{\theta \geq \theta_n} (\kappa' \circ Q(\theta) - \theta_n) \mathrm{d}F(\theta) - \int_{\theta \in [\theta^*, \theta_n)} |\kappa' \circ Q(\theta) - \theta| \left|\frac{p_Q(\theta^*) - p_Q(\theta)}{\delta_n}\right| \mathrm{d}F(\theta).
\end{aligned} \tag{19}
$$

We now show taking the limit of equation (19) as $n \to \infty$ delivers equation (14). To do so, observe first $\mathbf{1}_{[\theta_n,\infty)}(\theta)(\kappa' \circ Q(\theta) - \theta_n)$ converges pointwise to $\mathbf{1}_{(\theta^*,\infty)}(\theta)(\kappa' \circ Q(\theta) - \theta^*)$. Second, notice $|p_Q(\theta^*) - p_Q(\theta)| \leq \delta_n$ for all $\theta \in [\theta_n, \theta^*)$, and that $\theta - \kappa' \circ Q(\theta) \leq \bar{\theta}_F + \kappa' \circ Q\left(\bar{\theta}_F\right)$ for all $\theta \in [\theta_n, \theta^*)$. Therefore, an M exists such that $|\theta - \kappa' \circ Q(\theta)| \left|\frac{1}{\delta_n}(p_Q(\theta_n) - p_Q(\theta))\right| \leq M$ for all n. Substituting these facts back into (19) gives

$$
\begin{aligned}
0 \geq & \int_{\theta \geq \theta_n} (\kappa' \circ Q(\theta) - \theta_n) \mathrm{d}F(\theta) - \int_{\theta \in [\theta^*, \theta_n)} |\kappa' \circ Q(\theta) - \theta| \left|\frac{p_Q(\theta^*) - p_Q(\theta)}{\delta_n}\right| \mathrm{d}F(\theta) \\
\geq & \int \mathbf{1}_{[\theta_n,\infty)}(\theta)(\kappa' \circ Q(\theta) - \theta_n) \mathrm{d}F(\theta) - M(F_-(\theta_n) - F(\theta^*)) \\
\to & \int \mathbf{1}_{(\theta^*,\infty)}(\theta)(\kappa' \circ Q(\theta) - \theta^*) \mathrm{d}F(\theta) = \int_{\theta > \theta^*} \kappa' \circ Q(\theta) \mathrm{d}F(\theta) - \theta^*(1 - F(\theta^*)).
\end{aligned}
$$

where convergence follows from right continuity of F and Lebesgue dominated convergence theorem. Hence, (14) holds. Appealing to Lemma 10 therefore completes the proof. □

We now complete the proof by considering the last remaining case: p_Q is constant around θ^*.

Lemma 16. *Suppose $\theta^* \in \operatorname{supp} F$ is such that p_Q is constant on $[\theta^* - \delta, \theta^* + \delta]$ for some $\delta > 0$. Then $\kappa' \circ Q(\theta^*) < \theta^*$.*

Proof. Let $\theta_* = \inf\left\{\theta \geq \underline{\theta}_F : p_{Q+}(\theta) = p_Q(\theta^*)\right\}$. We now argue (15) holds at θ_*. There are three cases to consider. If $\theta_* = \underline{\theta}_F$, the desired inequality follows from Lemma 11. Suppose $\theta_* > \underline{\theta}_F$. Then $p_{Q+}(\theta) < p_Q(\theta_*)$ for all $\theta < \theta_*$, and so either $p_{Q-}(\theta_*) < p_{Q+}(\theta_*)$—in which case (15) follows from Lemma 12—or $p_{Q-}(\theta_*) = p_{Q+}(\theta_*)$ and $p_{Q-}(\theta_*) > p_Q(\theta)$ for all $\theta < \theta_*$, and so (15) follows from Lemma 14. Either way, (15) holds at θ_*.

Taking $\bar{\theta}^* = (\theta^* + \delta) \wedge \bar{\theta}_F$, let $G := F\left(\cdot|\theta \in \left[\theta_*, \bar{\theta}^*\right]\right)$. We claim (15) holds for $\theta' = \min(\operatorname{supp} G)$. Clearly we are done if $\theta' = \theta_*$. If $\theta' > \theta_*$, then $\theta_* \notin \operatorname{supp} G$, and so $F_-(\theta_*) = F(\theta_*) = F_-(\theta')$. We therefore have the following inequality chain:

$$\begin{aligned}(1 - F_-(\theta'))\,\theta' > (1 - F_-(\theta'))\,\theta_* &= (1 - F(\theta_*))\,\theta_* \\ &\geq \int_{\theta > \theta_*} \kappa' \circ Q(\theta)\,\mathrm{d}F(\theta) = \int_{\theta \geq \theta'} \kappa' \circ Q(\theta)\,\mathrm{d}F(\theta),\end{aligned}$$

where the weak inequality follows from (15) holding at θ_*— that is, (14) holds at θ', and so (15) holds as well (see Lemma 10).

If $\theta^* = \theta'$, Lemma 10 delivers $\kappa' \circ Q(\theta^*) < \theta^*$, and so there is nothing left to prove. Thus, we suppose $\theta^* \neq \theta'$ from here on . Since $\theta^* \in \operatorname{supp} G$, we must have $\theta^* > \theta'$.

We now argue p_Q is constant on $\left[\theta', \bar{\theta}^*\right]$. To do so, notice $\kappa' \circ Q(\theta') < \theta'$ implies $Q(\theta') = Q_+(\theta')$ in view of Q being selected via Lemma 9. Hence,

$$p_Q(\theta') = Q(\theta') - c'(\theta') = Q_+(\theta') - c'(\theta') = p_{Q+}(\theta') = p_Q(\theta^*),$$

where the last equality follows from $\theta' \geq \theta_*$. It follows p_Q is constant on $[\theta', \theta^*] \cup [\theta^* - \delta, \bar{\theta}^*) = [\theta', \bar{\theta}^*)$. Recalling $\bar{\theta}^* = \min\left\{\bar{\theta}_F, \theta^* + \delta\right\}$, it follows

$$p_Q(\theta^*) \leq p_Q\left(\bar{\theta}^*\right) \leq p_{Q+}\left(\bar{\theta}^*\right) = p_Q(\theta^*).$$

In other words, p_Q is constant on $\left[\theta', \bar{\theta}^*\right]$.

Consider now the line segment connecting $(\theta', \pi_Q(\theta'))$ with $(\theta^*, \pi_Q(\theta^*))$,

$$\begin{aligned}&\varphi : [\theta', \theta^*] \to \mathbb{R}, \\ &\qquad \theta \mapsto \pi_Q(\theta') + \left(\frac{\pi_Q(\theta^*) - \pi_Q(\theta')}{\theta^* - \theta'}\right)(\theta - \theta').\end{aligned}$$

We claim $\varphi(\theta) \geq \pi_Q(\theta)$ for all $\theta \in [\theta', \theta^*]$. Obviously, $\varphi(\theta) = \pi_Q(\theta)$ whenever $\theta \in$

$\{\theta', \theta^*\}$. For $\theta \in (\theta', \theta^*)$, we get the following inequality,

$$\begin{aligned}\varphi(\theta) &= \left(\frac{\theta^* - \theta}{\theta^* - \theta'}\right)\varphi(\theta') + \left(\frac{\theta - \theta'}{\theta^* - \theta'}\right)\varphi(\theta^*) \\ &= \left(\frac{\theta^* - \theta}{\theta^* - \theta'}\right)\pi_Q(\theta') + \left(\frac{\theta - \theta'}{\theta^* - \theta'}\right)\pi_Q(\theta^*) \geq \pi_Q(\theta),\end{aligned}$$

where the inequality follows from Lemma 2 part 1, which applies because p_Q is constant on $\left[\theta', \bar{\theta}^*\right]$.

Next, we show $\left(\frac{\pi_Q(\theta^*) - \pi_Q(\theta')}{\theta^* - \theta'}\right)$ is strictly positive. For this purpose, fix any $\epsilon \in (0, \theta^* - \theta')$. Observe

$$Q(\theta' + \epsilon) - Q_+(\theta') = Q(\theta' + \epsilon) - Q(\theta') = c'(\theta' + \epsilon) - c'(\theta'),$$

because p_Q is constant on $\left[\theta', \bar{\theta}^*\right]$. It follows $Q'_+(\theta') = c''(\theta')$, delivering the following inequality chain,

$$\begin{aligned}\left(\frac{\pi_Q(\theta^*) - \pi_Q(\theta')}{\theta^* - \theta'}\right) &= \frac{1}{\epsilon}\left[\varphi(\theta' + \epsilon) - \varphi(\theta')\right] \\ &\geq \frac{1}{\epsilon}\left[\pi_Q(\theta' + \epsilon) - \pi_Q(\theta')\right] \\ &= \frac{1}{\epsilon}\theta'\left(Q(\theta' + \epsilon) - Q(\theta')\right) - \frac{1}{\epsilon}\left[\kappa \circ Q(\theta' + \epsilon) - \kappa \circ Q(\theta')\right] \\ &\quad + Q(\theta' + \epsilon) - \frac{1}{\epsilon}\left[V_Q(\theta' + \epsilon) - V_Q(\theta')\right] \\ &\to \left(\theta' - \kappa' \circ Q(\theta')\right)c''(\theta') > 0,\end{aligned}$$

where convergence follows from the chain rule and $V'_{Q+}(\theta') = Q_+(\theta')$, and the strict inequality from c being strictly convex.

We now turn to establishing $\kappa' \circ Q(\theta^*) < \theta^*$, thereby concluding the proof. Towards this goal, notice again that for any $\epsilon \in (0, \theta^* - \theta')$,

$$Q(\theta^*) - Q(\theta^* - \epsilon) = c'(\theta^*) - c'(\theta^* - \epsilon).$$

because p_Q is constant on $\left[\theta', \bar{\theta}^*\right]$. Therefore, $Q'_-(\theta^*) = c''(\theta')$. We therefore obtain the following inequality chain,

$$
\begin{aligned}
0<\left(\frac{\pi_Q\left(\theta^*\right)-\pi_Q\left(\theta'\right)}{\theta^*-\theta'}\right) &= \frac{1}{\epsilon}\left[\varphi\left(\theta^*\right)-\varphi\left(\theta^*-\epsilon\right)\right] \\
&\leq \frac{1}{\epsilon}\left[\pi_Q\left(\theta^*\right)-\pi_Q\left(\theta^*-\epsilon\right)\right] \\
&= \frac{1}{\epsilon}\theta^*\left(Q\left(\theta^*\right)-Q\left(\theta^*-\epsilon\right)\right)-\frac{1}{\epsilon}\left[\kappa\circ Q\left(\theta^*\right)-\kappa\circ Q\left(\theta^*-\epsilon\right)\right] \\
&\quad +Q\left(\theta^*-\epsilon\right)-\frac{1}{\epsilon}\left[V_Q\left(\theta^*\right)-V_Q\left(\theta^*-\epsilon\right)\right] \\
&\to \left(\theta^*-\kappa'\circ Q\left(\theta^*\right)\right)c''\left(\theta^*\right),
\end{aligned}
$$

where convergence follows from the chain rule and $V'_{Q-}\left(\theta^*\right)=Q_-\left(\theta^*\right)$. Since $c''\left(\theta^*\right)>0$, the above inequality implies $\kappa'\circ Q\left(\theta^*\right)<\theta^*$, as required. $\square$