Preference Conditions for Linear Demand Functions

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Abstract

Taking consumer preferences as the primitive and a linear demand system as the desideratum, I investigate how the two are related via a novel approach to demand integrability that relies on some recent results in Diasakos and Gerasimou (2022). The methodology applies irrespectively of whether prices are normalized with respect to a numeraire or income, leading to a complete characterization of linear demand systems in terms of the underlying rational preference relation and analytical solutions for the direct utility function. The results provide a proper microfoundation for linear demand systems that fills some potentially misleading gaps in the extant literature.

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1 Introduction

Linear demand functions have been used extensively in economics as a convenient modelling tool to showcase important properties of market systems. The reliance on linear demand has been long standing in the modern theory of industrial organization (see Amir et al. (2017) or Kopel et al. (2017) for insightful overviews) and important in the empirical estimation of consumer demand (see Deaton (1974b)-(1978) but also Deaton (1974a) for aggregate demand) as well as of labour supply (see Stern (1984) for an overview). Linear demand functions are also common in economic textbooks to demonstrate various properties of consumer or market demand. Given these observations, it is somewhat surprising that incomplete progress has been made with respect to a proper characterization of the underlying preferences which can rationalize linear demand systems.

It is well known that linear demand is not easily generated by rational preferences or market structures. With respect to preferences, the existing literature has looked at the problem from the classical perspective on demand integrability: the (Marshallian) demand function of interest is assumed to satisfy enough regularity conditions (e.g., smoothness, the Law of Demand, injectivity, or the Slutsky matrix being symmetric and negative semi-definite) for the corresponding system of PDEs to be solved by an appropriate expenditure function, which leads to a utility function via duality (see Houthakker (1960), Epstein (1981), or Jackson (1986); see also Epstein (1982), LaFrance (1990) or Nocke and Schutz (2017) for incomplete demand systems). In this spirit, LaFrance (1985) established that individual linear demand places strong restrictions on the underlying preference: it requires a quadratic or Leontief quasi-direct utility function. In a similar spirit, Alperovich and Weksler (1996) solve for the underlying direct utility in the two-commodities case with income-normalized prices. With respect to market structures, Jaffe and Weyl (2010) but also Jaffe and Kominers (2012) have shown that multi-product aggregate linear demand cannot easily result from smooth rational discrete-choice models.[1] More recently, Amir et al. (2017) investigated the required properties for a quasi-linear/quadratic utility function to generate a linear demand function satisfying the Law of Demand; as it turns out, these properties have important implications for some widely used theoretical frameworks in industrial organization.

Their important contributions notwithstanding, these studies fail to assign unambiguously the key desirable properties of linear demand to the requisite characteristics

[1] More precisely, as follows also from the analysis in Armstrong and Vickers (2015), smooth discrete-choice models are not compatible with linear demand if the support of the underlying valuation density function is to include an open set of full dimensionality.

of rational choice. As a result, they fall short from actually characterizing the microfoundations of linear demand - an important desideratum as linear demand models are deployed mainly to obtain basic economic intuition to facilitate predictions and policy making (see, for instance, Berry and Haile (2021) for a discussion on the advantages of preference-based demand estimation).

In contrast to the existing literature, the present study takes consumer preferences as the primitive and a most general formulation of a linear demand system as the desideratum. To analyse how the two are related, I take a novel approach to demand integrability that relies on some recent results in Diasakos and Gerasimou (2022). They refer to a weak notion of smooth preferences which admits geometric interpretation via the concept of a preference gradient and the associated property of preference differentiability. Diasakos and Gerasimou (2022) establish that this is fundamentally linked to the invertibility of the resulting demand function.

The present study begins by showing that this notion of smooth preferences provides also theoretical underpinnings for a ubiquitous assumption in the literature on the microfoundations of linear demand; namely, that we refer to incomplete demand systems. More precisely, there are $k \in \mathbb{N} \setminus \{0\}$ commodities whose demand levels are observed, but also $m + 1$ commodities ($m \in \mathbb{N}$) with unobserved demands. The observed demand function is linear (i.e., exhibits constant coefficients) with respect to the prices of the k commodities.

Another feature that sets the present approach apart from the extant literature is that it applies for either of the two possible price-normalization regimes (with respect to the price of a numeraire commodity or income). I proceed to establish that, under either price-normalization regime, a linear demand function is generated by a differentiable preference relation if and only if (i) the unobserved part of the demand system comprises but *one* commodity (i.e., $m = 0$) while (ii) the matrix of constant coefficients on the prices of the observed commodities is non-singular (see Theorems 1 and 3 below). Combining preference differentiability with properties (i)-(ii) facilitates a straightforward integrability exercise via the inverse demand function. This leads to analytical solutions for the underlying utility function (see Theorems 2 and 4 below).

Somewhat unexpectedly perhaps, when prices are normalized with respect to a numeraire, the combination of preference differentiability and properties (i)-(ii) above dictates that the linear demand *cannot* depend on income. As to be expected, on the other hand, the utility function takes the quasi-linear/quadratic form; hence, by well-known arguments, the Slutsky matrix of the total demand system is symmetric and negative semi-definite. Given these observations, it follows that the matrix of constant coefficients is symmetric and negative definite; thus, the linear demand obeys the strict Law of Demand.

This translates into important messages with respect to the quest for micro-foundations of linear demand systems (such as multi-variate linear demand functions for differentiated products in oligopolistic markets). Linear demand systems that do not satisfy the strict Law of Demand or are income dependent are *not* rationalizable by smooth preferences. By contrast, linear demand systems that are income independent and satisfy the strict Law of Demand are fully consistent with continuous, strictly monotonic, strictly convex, and weakly smooth rationalizing preferences. Yet we should note not only that such demand systems are incomplete, but more importantly that their unobserved part plays an integral role for the underlying preference relation: it depicts a numeraire commodity whose marginal utility is constant.

The next section introduces the notational and theoretical backdrop. Section 3 presents the main analysis itself along with the underlying intuition. In Section 4, we compare our results with those in the relevant literature and discuss their implications for micro-founding linear demand functions. Section 5 concludes. The proofs that have been omitted from the main analysis are presented in Section 6 while some additional supporting results can be found in the Appendix.

2 The theoretical framework

As our consumption set, we consider an open and convex $X \subseteq \mathbb{R}^n_{++}$ where $n \in \mathbb{N}$: $n \geq 2$. The consumer's preferences are captured by a continuous weak order $\succsim$ on X (i.e. by a complete and transitive binary relation whose graph is a closed subset of $X \times X$). For $A \subseteq X$, we let

$$\max_{\succsim} A := \{x \in A : x \succsim y \text{ for all } y \in A\}$$

denote the set of all $\succsim$-greatest elements in A. Given some set $Y \subseteq \mathbb{R}^n_{++}$ of income-normalized strictly positive prices, the budget correspondence $B : Y \twoheadrightarrow X$ is defined by[2]

$$B(p) := \{x \in X : px \leq 1\}$$

We will say that $\succsim$ *generates* the demand function $\xi : Y \twoheadrightarrow X$ if the latter is defined by

$$\xi(p) := \max_{\succsim} B(p)$$

[2]Throughout the paper, for any $x, y \in \mathbb{R}^k$ and $1 < k \leq n$ the dot-product $p^\intercal x$ will be denoted simply by px.

We will refer to such a demand correspondence as *rational*. A rational demand correspondence is *onto* if, for all $x \in X$ there exists $p \in Y$ such that $x \in \xi(p)$. If $\xi(\cdot)$ is single-valued (hence a demand *function*), it is said to be *injective* if for all $p, p' \in Y$, $p \neq p'$ implies $\xi(p) \neq \xi(p')$. A demand function $\xi : Y \to X$ that is both injective and onto is *invertible*. If $\xi(\cdot)$ has this property, then the inverse demand given by

$$p(x) := \{p \in Y : \ x = \xi(p)\}$$

is itself a well-defined bijective function $p : X \to Y$.

Proposition 1 in Diasakos and Gerasimou (2022) establishes that, within the realm of continuous preferences, a rational demand function $\xi : Y \to X$ requires that the generating preference relation $\succsim$ is strictly convex and strictly monotone on X.[3] Their analysis proceeds to show that, within the realm of strictly convex, strictly monotone and continuous preferences, the generated demand function is invertible (in fact, an homeomorphism) if and only if the underlying preference relation satisfies a particular notion of smoothness, *weak smoothness*.

The first notion of smooth preferences in the literature was proposed in Debreu (1972), where a preference relation $\succsim$ on a consumption set X was defined to be *smooth of order r* (C^r for short) if the graph of the indifference relation (i.e., the set $\{(x, y) \in X \times X : x \sim y\} \subset X \times X$) is a C^r-manifold on $X \times X$.[4] A monotonic preference relation on X is C^r if and only if it is representable by a C^r (i.e., r-times continuously differentiable) utility function. Generalizing Debreu's notion, Neilson (1991) defined a preference relation on X as *weakly smooth of order r* if each of its *indifference sets* ($\mathcal{I}_x := \{z \in X : z \sim x\}$, $x \in X$) is a C^r-manifold on X. In Diasakos and Gerasimou (2022), a preference relation that is weakly smooth of order 1 is referred to simply as *weakly smooth*.

More recently Rubinstein (2006) defined the preference relation $\succsim$ on X to be *differentiable* if for every $x \in X$ there exists $p_x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$\{z \in \mathbb{R}^n : p_x \cdot z > 0\} = \{z \in \mathbb{R}^n : \exists \lambda_z^* > 0, x + \lambda z \succ x \ \forall \lambda \in (0, \lambda_z^*)\} \tag{1}$$

[3]For two distinct vectors $x, y \in \mathbb{R}^n$, we write $x > y$ [resp. $x \gg y$] whenever $x_i \geq y_i$ [resp. $x_i > y_i$] for all $i \in \{1, \ldots, n\}$. The preferences are said to be *convex* if, for all $x, y \in X$ and any $\alpha \in [0, 1]$, $x \succsim y$ implies $\alpha x + (1 - \alpha) y \succsim y$, and *monotonic* if $x \gg y$ implies $x \succ y$. They are *strictly convex* if, for all $x, y \in X$ and $\alpha \in (0, 1)$, $x \succsim y$ implies $\alpha x + (1 - \alpha) y \succ y$, and *strictly monotonic* if $x > y$ implies $x \succ y$.

[4]Let $A \subseteq \mathbb{R}^n$. A function $f : A \to \mathbb{R}^n$ is an *homeomorphism* if it is injective, continuous, and its inverse function is continuous on $f(A)$. Letting A be in addition open, a C^r function $f : A \to \mathbb{R}^n$ is a C^r *diffeomorphism* if it is an homeomorphism with a C^r inverse function. A set $M \subseteq \mathbb{R}^n$ is a C^r *k-dimensional* ($k \leq n$) *manifold* if for every $x \in M$ there is a C^r diffeomorphism $f : A \to \mathbb{R}^n$ ($A \subseteq \mathbb{R}^n$ open) which carries the open set $A \cap (\mathbb{R}^k \times \{\mathbf{0}^{n-k}\})$ onto an open neighborhood of x in M. For more details and some economic-theoretic examples, see Chapter 1.H in Mas-Colell (1985).

To interpret this geometrically, for distinct bundles x and z in X, call z an *improvement direction at* x if there exists $\lambda^* > 0$ such that $x + \lambda z \succ x$ for all $\lambda \in (0, \lambda^*)$, assuming $(x + \lambda z) \in X$. In light of this definition, the right-hand side of (1) defines the set of all improvement directions at x. The left-hand side of (1) defines the set of all directions that get strictly positive valuation by some vector p_x. Preference differentiability means that there exists some p_x such that the set of all directions that receive strictly positive valuations coincides with the set of all improvement directions of $\succsim$ at x. Such a vector p_x will be referred to as a *preference gradient* at x.[5]

To relate these notions to the present investigation, for any $x \in X$ consider the projection of $\mathcal{I}_x$ along the ith dimension of $\mathbb{R}^n_+$,

$$\mathcal{I}^i_x := \{z_i \in \mathbb{R}_+ : \text{ there exists } z_{-i} \in \mathbb{R}^{n-1}_+ \text{ such that } z \in \mathcal{I}_x\},$$

and define the set

$$\mathcal{I}^{-i}_x := \{z_{-i} \in \mathbb{R}^{n-1}_+ : \text{ there exists } z_i \in \mathbb{R}_+ \text{ such that } z \in \mathcal{I}_x\}$$

analogously, as the projection of $\mathcal{I}_x$ on $\mathbb{R}^{n-1}_+$ (the resulting subspace when the ith dimension is removed from $\mathbb{R}^n_+$). We can construct then the *indifference-projection correspondence* $l_i(\cdot|x) : \mathcal{I}^{-i}_x \twoheadrightarrow \mathcal{I}^i_x$ for good i by requiring

$$z_i \in l_i(z_{-i}|x) \iff z \in \mathcal{I}_x$$

whose graph is the indifference set $\mathcal{I}_x$. As established in Diasakos and Gerasimou (2022), for $\succsim$ continuous, strictly convex and strictly monotonic, the mapping $l_i(\cdot|x)$ is a locally convex and thus also continuous function. As a result, its local subdifferential $\partial l_i(z_{-i}|x)$, which comprises the collection of the function's local subgradients at z_{-i}, is non-empty and fundamentally linked to its smoothness: $l_i(\cdot|x)$ is differentiable at z_{-i} if and only if $\partial l_i(z_{-i}|x)$ is a singleton, in which case the unique local subgradient coincides with the gradient.

With regard to economic interpretation, when $l_i(\cdot|x)$ is differentiable at z_{-i} the jth entry $\partial l_i(z_{-i}|x)/\partial z_j$ of the gradient $\nabla l_i(z_{-i}|x)$ defines the marginal rate of substitution of good i for good $j \neq i$. Indeed, if $\succsim$ is representable by a utility function $u : X \to \mathbb{R}$

[5] An intuitive interpretation for the entries of p_x is that they represent the consumer's "subjective values" of the different goods relative to the reference bundle x: *"Starting from x, any small move in a direction that is evaluated by this vector as positive is an improvement"* (Rubinstein, 2006 p. 71). The notion of preference gradient can also be viewed as a generalization of the notion of *valuation equilibrium* in Radner (1993).

that is continuously differentiable at z, we have

$$\frac{\partial l_i\left(z_{-i}|x\right)}{\partial z_j} = -\frac{\dfrac{\partial u(z)}{\partial z_j}}{\dfrac{\partial u(z)}{\partial z_i}} \tag{2}$$

The right-hand side of this equation depicts the textbook definition of the marginal rate of substitution of good i for good j. The definition rests upon invoking the Implicit Function Theorem; thus, upon assuming that $u(\cdot)$ is a C^1 function (equivalently, that $\succsim$ is itself C^1). By contrast, the left-hand side of (2) exists and is continuous in a more general environment: when $\succsim$ is differentiable - see Proposition 2 in Diasakos and Gerasimou (2022). And given that it is continuous, strictly convex and strictly monotonic, $\succsim$ being differentiable is equivalent to $\succsim$ being weakly smooth - see Theorem 1 in Diasakos and Gerasimou (2022).

More importantly for our purposes, $\succsim$ being differentiable is equivalent to $\succsim$ generating a unique, homeomorphic demand function $\xi : Y \to X$ with Y an open subset of R^n_{++} - see Proposition 3 in Diasakos and Gerasimou (2022). Specifically, letting $q_{-i}(x)$ denote the negative of the gradient $l_i(\cdot|x)$ at x, the preference gradient p_x coincides with $p(x)$, the value of the inverse demand at this bundle. Formally, we have

$$q_{-i}(x) \;:=\; -\nabla l_i(x_{-i}|x) \tag{3}$$

$$q_i(x) \;=\; \frac{1}{x_i + q_{-i}(x) \cdot x_{-i}} \tag{4}$$

$$p(x) \;=\; q_i(x)\big(1, q_{-i}(x)\big) \tag{5}$$

where $q_{-i}(x) \in \mathbb{R}^{n-1}_{++}$, $q_i(x) > 0$, and $p(x) \in \mathbb{R}^n_{++}$. Notice finally that, although taking distinct index goods i and j in the above system leads to distinct vectors $(q_i(x), q_{-i}(x))$ and $(q_j(x), q_{-j}(x))$, the preference gradient, $p(x)$, is invariant with respect to the choice of the index good. Moreover, that $q_i(x) = p_i(x)$ for the index good i is due to the fact that we normalize prices with respect to income.

3 Linear demand

The preceding overview of the key theoretical concepts was given in terms of prices that are normalized with respect to income. Yet most of the literature on linear demand concerns itself with the case when prices are normalized with respect to a numeraire commodity.

3.1 When prices are normalized with respect to a numeraire

Taking the nth commodity as the numeraire, we can deploy (5) above to define the functions $w : Y_n \to \mathbb{R}_{++}$ and $q_{-n} : Y \to \mathbb{R}^{n-1}_{++}$, respectively, by $w(p_n) := 1/p_n$ and $q_{-n}(p) = p_{-n}/p_n$. We then have a mapping between the income-normalized prices $p \in Y$ from the preceding section and the corresponding vector of numeraire-normalized prices and income, $(q_{-n}, w) \in Q \times W$ - where $q_n := q_{-n}(p)$ while $W := w(Y_n)$ and $Q := q_{-n}(Y)$. This mapping gives also the numeraire-normalized (i.e., Marshallian) demand $\widetilde{\xi} : Q \times W \to X$ as $\widetilde{\xi}(q_{-n}, w) := \xi((1, q_{-n}(p)/w)$. Clearly, since $w(\cdot)$ is an homeomorphism, if $\succsim$ is continuous, strictly convex, strictly monotonic and differentiable on X then $\widetilde{\xi}(\cdot)$ is itself an homeomoprhism and thus $Q \times W$ is open in $\mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++}$.

We will restrict attention to demand functions $\widetilde{\xi} : Q \times W \to X$ that satisfy both of the following conditions.

(A) The domain $Q \times W$ has non-empty interior:[6]

$$\exists (q, \varepsilon) \in (Q \times W) \times \mathbb{R}_{++} : \mathcal{B}_q(\varepsilon) \subset Q \times W$$

(B) For at least one of the non-numeraire commodities its quantity demanded responds to a change in its own relative price, other things being equal:

$$\exists\, (j, q, \delta) \in \{1, \ldots, n-1\} \times (Q \times W) \times \mathbb{R} \setminus \{0\} :\ q + \delta e_j \in Q \times W \,\wedge\, \widetilde{\xi}_j\left(q + \delta e_j\right) \neq \widetilde{\xi}_j\left(q\right)$$

Together conditions (A)-(B) above provide the theoretical underpinnings (see Claim 2 and Remark (ii) in Appendix A) for a key assumption in the literature on linear demand: namely, that the observed linear demand system is incomplete. Specifically, linear demand models always assume that, for some $k \in \mathbb{N} : 1 \leq k < n$, the demands of the commodities indexed by $M := \{k+1, \ldots, n\}$ are unobserved. The observed linear form depicts the demands of the commodities indexed by $K := \{1, \ldots, k\}$; their demand exhibits constant coefficients with respect to the prices $q_1, \cdots, q_k$.

We will depict the unobserved demands by the vector $\mathbf{z} \in X_M$ and the observed ones by $x \in X_K$. Moreover, for $M_0 := M \setminus \{n\}$ we denote the respective relative prices by $q_{M_0} \in Q_{M_0}$ and $q_K \in Q_K$.[7] Letting then $x(\cdot)$ denote the observed components of

[6] For $y \in \mathbb{R}^n$ and $\varepsilon > 0$, $\mathcal{B}_\varepsilon(y)$ denotes the open ball in $\mathbb{R}^n$ with center y and radius ε. For $i \in \mathcal{N} := \{1, \ldots, n\}$, e_i denotes the vector in $\mathbb{R}^n$ with 1 as its ith entry and zeroes everywhere else.

[7] Take $\mathcal{A} \subset \mathcal{N}$. For $y \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, we let $y_\mathcal{A}$ and $S_\mathcal{A}$ denote, respectively, the projections of y and S on the subspace that results from $\mathbb{R}^n$ when the dimensions in $\mathcal{N} \setminus \mathcal{A}$ are removed.

$\tilde{\xi}(\cdot)$, a linear demand system is given by

$$x\left(q_K, q_{M_0}, w\right) := \alpha\left(q_{M_0}, w\right) + Bq_K \tag{6}$$

where B is a $k \times k$ matrix of constants while $a : Q_{M_0} \times W \to R^k$ is a continuous function.

A theoretical justification for the formulation in (6) is given by the assumption that the total demand system $\tilde{\xi}(\cdot)$ satisfies conditions (A)-(B) above simultaneously. With respect to (B), given condition (A), it suffices for (**??**) that B has a non-zero diagonal element or a symmetric principal minor (see Remarks (iii)-(iv) in Appendix A). With respect to condition (A), if the demand system is generated by a continuous, strictly monotonic, strictly convex, and differentiable preference relation then $Q \times W$ itself is open. In fact, differentiability of the underlying preference relation places additional restrictions not only on the formulation for the observed linear demand but also on the total commodity system itself.

Theorem 1 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the observed demand function in (6). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular, $M_0 = \varnothing$, and $\alpha(\cdot)$ is a constant.

Proof. That (i) $\Rightarrow$ (ii) is due to the following results (see Section 6 for the corresponding proofs).

Lemma 3.1 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\tilde{\xi} : Q \to X$ whose projection on the dimensions in K, $x : Q_{M_0} \times Q_K \times W \to X_K$, is given by $x\left(q_{M_0}, q_K, w\right) := \alpha\left(q_{M_0}, w\right) - Bq_K$ for some function $a : Q_{M_0} \times W \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if B is non-singular.*

Lemma 3.2 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\tilde{\xi} : Q \to X$ whose projection on the dimensions in K, $x : Q_{M_0} \times Q_K \times W \to X_K$, is given by $x\left(q_{M_0}, q_K, w\right) := \alpha\left(q_{M_0}, w\right) - Bq_K$ for some function $a : Q_{M_0} \times W \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if $M_0 = \varnothing$.*

Lemma 3.3 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\tilde{\xi} : Q \to X$ whose projection on the dimensions in K, $x : Q_{M_0} \times Q_K \times W \to X_K$, is given by $x\left(q_{M_0}, q_K, w\right) := \alpha\left(q_{M_0}, w\right) - Bq_K$ for some function $a : Q_{M_0} \times W \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if $\alpha\left(\cdot\right)$ is a constant.*

To show that (ii) $\Rightarrow$ (i), observe first that, by Theorem 1 in Diasakos and Gerasimou (2022), $\succsim$ is differentiable if the total demand $\widetilde{\xi}(\cdot)$ is injective. To see that the latter property does hold under the hypotheses in (ii), let $\alpha(\cdot) := \alpha$ and take $(q^1, w_1), (q'', w_2) \in Q_K \times W$ with $(q', w_1) \neq (q'', w_2)$. There are two cases to consider. If $q' \neq q''$, we cannot have $\alpha + Bq' = x(q') = x(q'') = \alpha + Bq''$ given that B is non-singular; clearly, we must have $\widetilde{\xi}(q', w_1) \neq \widetilde{\xi}(q'', w_2)$. If $q' = q = q''$ and $w_1 \neq w_2$, notice that $\widetilde{\xi}(q, w_1) = (z, x(q)) = \widetilde{\xi}(q, w_2)$ only if $w_1 - qx(q) = z = w_2 - qx(q)$; i.e., only if $w_1 = w_2$. ■

The involved arguments concern the "only if" direction of the theorem. Lemma 3.1 shows that $\succsim$ is differentiable only if B is non-singular; it does so by an argument ad absurdum which can be outlined intuitively as follows. If B is singular, there exists $v \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $Bv = \mathbf{0}$; hence, such that $x(q^0_{M_0}, q^0_K + \lambda v, w_0) = x(q^0_{M_0}, q^0_K, w_0)$ for some $(q^0_{M_0}, q^0_K, w_0) \in Q \times W$ and $\lambda \in \mathbb{R} \setminus \{0\}$ sufficiently small. It is straightforward to show that this leads to a violation of the WARP when $vx(q^0_{M_0}, q^0_K, w_0) = 0$ or $vx(q^0_{M_0}, q^0_K, w_0) = vx(q^0_{M_0}, q^0_K, w')$ for some $w' \in W$ with $w' \neq w_0$. If $vx(q_{M_0}, q_K, w) \neq vx(q_{M_0}, q_K, q, w')$ for all $(q_{M_0}, q_K) \in Q$ and all $w, w' \in W$ with $w' \neq w$, we fix the unobserved part of the demand at the bundle $z^0 := z(q^0_{M_0}, q^0_K, w_0)$ and restrict attention to the relationship between the n-dimensional price-income space $Q \times W$ and the k-dimensional space of observed demand bundles $\{(z^0, x), x \in X_K\}$. The latter is open in $\mathbb{R}^k$, and thus can be covered by a collection of hyperplanes $\{x \in X_K : vx = \rho,\ \rho \in L\}$ from some interval $L \subseteq \mathbb{R}$. Letting $x^0 := x(q^0_{M_0}, q^0_K, w_0)$, we show that the hyperplane $\{x \in X_K : vx = vx^0\}$ embeds the set $X_{(z^0, x^0)} := \{x \in X_K : x = x(q^0_{M_0}, q_K, w_0), q_K \in \mathcal{B}_{q^0_K}\}$ for some neighbourhood $\mathcal{B}_{q^0_K}$ of q^0_K in Q_K. But this is absurd as $\succsim$ is differentiable only if the demand system is an homeomorphism; being the image of $\mathcal{B}_{q^0_K}$ under an homeomorphic demand, $X_{(\mathbf{z}^0, x^0)}$ must be open in $\mathbb{R}^k$.

Given this result, Lemma 3.2 establishes that $\succsim$ is differentiable only if the set of unobserved commodities is a singleton. To do so we exploit the fact that preference differentiability allows for direct demand integrability along each indifference set via the function $q_K(\cdot)$ - recall equation (3). Under the functional form in (6) and as B is invertible, this leads to a quasi-indirect utility function which is quasi-linear in the unobserved demands. To complete the argument we show that the linear part cannot admit a multi-dimensional consumption vector.

Finally, Lemma 3.3 shows that $\succsim$ is differentiable only if the function $\alpha(\cdot)$ is independent of income - its only possible argument since Q_{M_0} is empty (Lemma 3.2). The argument is once again ad absurdum; it exploits the functional form in (6) and that B is invertible. Dropping the subscript K from our notation, we fix again the unobserved part at $z^0 := z(q^0, w_0)$ and look at the relationship between the $(k+1)$-dimensional space $Q_K \times W$ and the k-dimensional space $\{(z^0, x), x \in X_K\}$. A contradiction obtains now by considering the set $X_{(z^0,x^0)} := \{x \in X_K : x = x(q,w), (q,w) \in \mathcal{B}_{(q^0,w_0)}\}$ for some neighbourhood $\mathcal{B}_{(q^0,w_0)}$ of (q^0, w_0) in $Q_K \times W$. As $\mathcal{B}_{(q^0,w_0)}$ is open in $\mathbb{R}^{k+1}$, so should be its image under the homeomorphic demand. Yet the latter lies in $X_{(z^0,x^0)} \subset \mathbb{R}^k$.

In light of these results, under preference differentiability (6) reads as follows

$$x(q) := \alpha + Bq, \qquad q \in Q_K \tag{7}$$

where $\alpha \in \mathbb{R}^k$ is a constant and B is non-singular, while $M = \{n\}$. Moreover, integrability of the preference gradient function traces out now the indifference sets analytically. This leads to a complete characterization of the linear demand function in terms of the generating preference relation.

Theorem 2 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the observed demand function in (7). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular.
(iii). $\succsim$ is represented by the utility function $u : X \to \mathbb{R}$ given by

$$u(z, x) := z - xB^{-1}\alpha + xB^{-1}x/2 \tag{8}$$

(iv). B is symmetric and negative definite.
(v). $x(\cdot)$ satisfies the strict Law of Demand:

$$(q' - q'')(x(q') - x(q'')) < 0 \qquad \forall q', q'' \in Q_K : \; q' \neq q''$$

Proof. That (i) $\Leftrightarrow$ (ii) is due to Theorem 1 while (iv) $\Rightarrow$ (v) holds trivially. Moreover, Q_K being open, (v) necessitates that B is non-singular.[8] It remains to show that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
<u>(ii) $\Rightarrow$ (iii)</u>. Let B be non-singular (and, thus, $\succsim$ be differentiable). We can write then $q = B^{-1}(x - \alpha)$ where $q = -\nabla_x l_z(x|(z, x))$ - recall equation (3). For any $(z^0, x^0) \in X$, therefore, we must obey the system of differential equations

$$\partial z/\partial x_i = \left(B^{-1}(\alpha - x)\right)_i, \qquad i = 1, \ldots, k \tag{9}$$

[8]Let $q' \in Q_K$. If B is singular, there exists $v \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $Bv = \mathbf{0}$; i.e., such that $x(q' + \lambda v) = x(q')$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ sufficiently small to give $q' + \lambda v \in Q_K$. This contradicts (v).

along the indifference curve $\mathcal{I}_{(z^0,x^0)}$. Integrating along this curve gives

$$z = xB^{-1}\alpha - xB^{-1}x/2 + c, \qquad (z,x) \in \mathcal{I}_{(z^0,x^0)}$$

where c remains constant along $\mathcal{I}_{(z^0,x^0)}$. To get the claim set $u(z,x) := c$.

(iii) $\Rightarrow$ (iv). The UMP for the objective in (8) results in the inverse demand $q(x) = B^{-1}(\alpha - x)$. Observe also that, being represented by the C^1 utility function in (8), $\succsim$ is itself C^1 and thus differentiable. As a result, by Proposition 2 in Diasakos and Gerasimou (2022), $q(\cdot)$ must be injective; hence, B^{-1} must be non-singular. Clearly, the total demand is $\widetilde{\xi}(\cdot) := (z(\cdot), x(\cdot))$ where $x(\cdot)$ is given by (7) while $z(q,w) := w - qx(q)$. It is trivial to check now that $\widetilde{\xi}(\cdot)$ satisfies the hypotheses of Theorem 1 in Hurwicz and Uzawa (1971). As a result, the Slutsky matrix of $\widetilde{\xi}(\cdot)$ must be symmetric and negative semi-definite. And as its kth principal minor, the Slutsky matrix for $x(\cdot)$, coincides with B, the latter is also symmetric and negative semi-definite; more precisely, symmetric and negative definite given that it is non-singular.[9] ■

Remark. Within the realm of Theorem 2, the requirement that $\succsim$ be monotonic imposes the following restriction on its domain:

$$X \subseteq \mathbb{R}_{++} \times \left\{ x \in \mathbb{R}^{n-1}_{++} : \mathbf{0} \gg B^{-1}(\alpha - x) \right\}$$

3.2 When prices are normalized with respect to income

Turning now to the case of income-normalized prices, the underlying intuition is essentially the same as before. We will restrict attention to demand functions $\xi : Y \to X$ that satisfy both of the following conditions.

(A*) The domain Y has non-empty interior:

$$\exists (p, \varepsilon) \in Y \times \mathbb{R}_{++} : \mathcal{B}_p(\varepsilon) \subset Y$$

(B*) For at least one of the commodities its quantity demanded responds to a change in its own relative price, other things being equal:

$$\exists\, (i, p, \delta) \in \{1, \ldots, n\} \times Y \times \mathbb{R} \setminus \{0\} : \; p + \delta e_i \in Y \;\wedge\; \xi_i(p + \delta e_i) \neq \xi_i(p)$$

[9] Recall that a symmetric (square) matrix is positive semidefinite [resp. positive definite] if and only if all of its eigenvalues are nonnegative [resp. strictly positive], while a (square) matrix is non-singular if and only if all of its eigenvalues are non-zero.

Similarly to the case where prices were normalized relative to a numeraire, conditions (A*)-(B*) above provide the theoretical underpinnings for the premise that the observed linear demand system is incomplete (see Claim 1 and Remark (iii) in Appendix A). The relative prices will be depicted now by $p_M \in Y_M$ and $p_K \in Y_K$ while the observed linear demand system is given by

$$x\left(p_M, p_K\right) := \alpha\left(p_M\right) + Bp_K \qquad \left(p_M, p_K\right) \in Y \tag{10}$$

where B is a $k \times k$ matrix of constants while $a : Y_M \to R^k$ is a continuous function.

A theoretical justification for the formulation in (10) is given by the assumption that $\xi(\cdot)$ satisfies conditions (A*)-(B*). Given condition (A*), it suffices for condition (B*) that B has a non-zero diagonal element or a symmetric principal minor (see Remarks (iii)-(iv) in Appendix A). With respect to condition (A*), if the demand system is generated by a strictly monotonic, strictly convex and differentiable preference relation then Y is necessarily open. And as before, differentiability of the underlying preference relation places additional restrictions not only on the formulation for the observed linear demand but also on the total demand system itself.

Theorem 3 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the observed demand function in (10). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular and $M = \{n\}$.

Proof. That (i) $\Rightarrow$ (ii) is due to the following results (see Section 6 for the corresponding proofs).

Corollary 3.1 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\xi : Y \to X$ whose projection on the dimensions in K, $x : P_M \times P_K \to X_K$ given by $x\left(p_M, p_K\right) := \alpha\left(p_M\right) - Bp$ for some function $\alpha : P_M \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if B is non-singular.*

Corollary 3.2 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\xi : Y \to X$ whose projection on the dimensions in K, $x : P_M \times P_K \to X_K$ given by $x\left(p_M, p_K\right) := \alpha\left(p_M\right) - Bp$ for some function $\alpha : P_M \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if $M_0 = \varnothing$.*

The argument for (ii) $\Rightarrow$ (i) is trivially similar to the respective part in the proof of Theorem 1. ■

Given these results, under preference differentiability and assuming that $\alpha(\cdot)$ is also a linear function, the expression in (10) above reduces to the following

$$x\left(p_n, p_K\right) := \alpha + \gamma p_n + B p_K, \qquad \left(p_n, p_K\right) \in Y \tag{11}$$

where $\alpha, \gamma \in \mathbb{R}^k$ are constants while $M = \{n\}$. In light of Theorem 3, preference differentiability allows for direct integrability of the formulation in (11) along the indifference sets via the preference gradient function. As before, this leads to a complete characterization of the linear demand function in terms of the properties of the underlying generating preference.

Theorem 4 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the demand function in (11). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular.
(iii). $\succsim$ is represented by the utility function $u : X \to \mathbb{R}$ given by

$$u\left(z, x\right) := \begin{cases} \left(z - xB^{-1}\gamma\right) \exp\left(\int_{X_K^0} \frac{B^{-1}(x-\alpha)}{1 - xB^{-1}(x-\alpha)} dx\right) & x \in X_K^0 \\ 0 & x \in X_K \setminus X_K^0 \end{cases} \tag{12}$$

where $X_K^0 =: \{x \in X_K : xB^{-1}(x - \alpha) \neq 1\}$.

Proof. That (i) $\Leftrightarrow$ (ii) is due to Theorem 3. Moreover, since $u(\cdot)$ is C^1 so must be $\succsim$. Hence, the preference is weakly smooth and that (iii) $\Rightarrow$ (i) is due to Proposition 2 in Diasakos and Gerasimou (2022). It remains to show that (ii) $\Rightarrow$ (iii).
(ii) $\Rightarrow$ (iii). Let B be non-singular (and, thus, $\succsim$ be differentiable). Recall also equations (3)-(5). We have $p = p_n q_K$ with $q_K = -\nabla_x l_z(x|(z, x))$ and $p_n = 1/(z + q_K x)$. The given demand schedule can be written therefore as follows

$$x = \alpha + p_n\left(Bq_K + \gamma\right)$$

or

$$\begin{aligned} q_K = B^{-1}\left(p_n^{-1}\left(x - \alpha\right) - \gamma\right) &= B^{-1}\left(\left(z + q_K x\right)\left(x - \alpha\right) - \gamma\right) \\ &= B^{-1}\left(x - \alpha\right) q_K x + B^{-1}\left(z\left(x - \alpha\right) - \gamma\right) \end{aligned}$$

which implies

$$\left(1 - x^\intercal B^{-1}\left(x - \alpha\right)\right) q_K x - x^\intercal B^{-1}\left(x - \alpha\right) z = -x^\intercal B^{-1}\gamma \tag{13}$$

For any given $(z^0, x^0) \in X$, therefore, we must obey the differential equation

$$(1 - f(x)) \sum_{j=1}^{k} x_j \partial z / \partial x_j + f(x) z = g(x) \gamma \tag{14}$$

along $\mathcal{I}_{(z^0, x^0)}$, where $f : X_K \to \mathbb{R}$ and $g : X_K \to \mathbb{R}^k$ are given by $f(x) := xB^{-1}(x - \alpha)$ and $g(x) := xB^{-1}$.

Define the functions $h_1 : X_1 \to S$ and $h_j : S \times X_j \to \mathbb{R}_{++}$ for $j \in K \setminus \{1\}$, respectively, by $h_1(x_1) := \ln x_1$ and $h_j(\tau, x_j) := x_j / h_1(\tau)$ where $S := \{\tau \in \mathbb{R}_{++} : e^\tau \in X_1\}$. For any $x_1 \in X_1$, letting $\tau = h_1(x_1)$ we have now the following parametrization: $x_1 = e^\tau$ and $x_j = e^\tau h_j(\tau, x_j)$ for $j \in K \setminus \{1\}$.

Let then the vector-valued function $h : S \times X_{-1} \to \mathbb{R}_{++}$ be given by $h(\tau, x_{-1})_j := h_j(\tau, x_j)$ $j \in K \setminus \{1\}$ and fix an arbitrary $x_{-1} \in X_{-1}$. As we have $x_i = \partial x_i / \partial \tau$ for any $i \in K$, we can transform the PDE in (14) to the following ODE

$$(1 - f(\tau, h(\tau, x_{-1}))) \mathrm{d}z/\mathrm{d}\tau + f(\tau, h(\tau, x_{-1})) z = g(\tau, h(\tau, x_{-1})) \gamma$$

Restricting attention to the set X_K^0, this equation can be re-written as follows

$$\mathrm{d}z/\mathrm{d}\tau + \frac{f(\tau, h(\tau, x_{-1})) z}{1 - f(\tau, h(\tau, x_{-1}))} = \frac{g(\tau, h(\tau, x_{-1})) \gamma}{1 - f(\tau, h(\tau, x_{-1}))}$$

with the solution

$$z = \frac{1}{\mu(x_{-1})} \left(\int \frac{\mu(x_{-1}) g(\tau, h(\tau, x_{-1})) \gamma}{1 - f(\tau, h(\tau, x_{-1}))} \mathrm{d}\tau + c \right) \tag{15}$$

where

$$\mu(x_{-1}) := \exp \left(\int \frac{f(\tau, h(\tau, x_{-1}))}{1 - f(\tau, h(\tau, x_{-1}))} \mathrm{d}\tau \right)$$

Moreover, we also have

$$\begin{aligned} \Delta(e^\tau \mu(x_{-1})) &= \mu(x_{-1}) \Delta e^\tau + e^\tau \Delta \mu(x_{-1}) \\ &= e^\tau \left(1 + \frac{f(\tau, h(\tau, x_{-1}))}{1 - f(\tau, h(\tau, x_{-1}))} \right) \mu(x_{-1}) \Delta\tau = \frac{e^\tau \mu(x_{-1})}{1 - f(\tau, h(\tau, x_{-1}))} \Delta\tau \end{aligned}$$

and thus

$$
\begin{aligned}
\Delta\left(\mu\left(x_{-1}\right) g\left(\tau, h\left(\tau, x_{-1}\right)\right)\right) &= \Delta\left(\mu\left(x_{-1}\right) \sum_{j=1}^{k} B_j^{-1} h_j\left(\tau, x_j\right)\right) \\
&= \Delta\left(\mu\left(x_{-1}\right) \sum_{j=1}^{k} B_j^{-1} e^{\tau} x_j\right) \\
&= \Delta\left(e^{\tau} \mu\left(x_{-1}\right)\right) \sum_{j=1}^{k} B_j^{-1} x_j \\
&= \frac{e^{\tau} \mu\left(x_{-1}\right)}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \sum_{j=1}^{k} B_j^{-1} x_j \Delta\tau \\
&= \frac{\mu\left(x_{-1}\right)}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \sum_{j=1}^{k} B_j^{-1} h_j\left(\tau, x_j\right) \Delta\tau \\
&= \frac{\mu\left(x_{-1}\right) g\left(\tau, h\left(\tau, x_{-1}\right)\right)}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \Delta\tau
\end{aligned}
$$

That is,

$$\int \frac{\mu\left(x_{-1}\right) g\left(\tau, h\left(\tau, x_{-1}\right)\right)}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \mathrm{d}\tau = \mu\left(x_{-1}\right) g\left(\tau, h\left(\tau, x_{-1}\right)\right)$$

and (15) reads

$$z = \left(c + \mu\left(x_{-1}\right) g\left(\tau, h\left(\tau, x_{-1}\right)\right) \gamma\right) / \mu\left(x_{-1}\right)$$

To complete the argument observe also that

$$\begin{aligned}
\Delta \ln \mu\left(x_{-1}\right) &= \frac{\Delta \mu\left(x_{-1}\right)}{\mu\left(x_{-1}\right)} \\
&= \frac{f\left(\tau, h\left(\tau, x_{-1}\right)\right) \Delta \tau}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \\
&= \frac{x B^{-1}(x-\alpha) \Delta \tau}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \\
&= \frac{\left(\sum_{i \in K} \sum_{j \in K} x_i B_{ij}^{-1}\left(x_j-\alpha_j\right)\right) \Delta \tau}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \\
&= \frac{\sum_{i \in K}\left(\sum_{j \in K} B_{ij}^{-1}\left(x_j-\alpha_j\right)\right) x_i \Delta \tau}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \\
&= \frac{\sum_{i \in K}\left(B_i^{-1}(x-\alpha)\right) \Delta x_i}{1-f(x)}=\frac{B_i^{-1}(x-\alpha) \Delta x}{1-x B_i^{-1}(x-\alpha)}
\end{aligned}$$

Hence, we have

$$\int \frac{f\left(\tau, h\left(\tau, x_{-1}\right)\right)}{1-f\left(\tau, h\left(\tau, x_{-1}\right)\right)} \mathrm{d} \tau = int \frac{B^{-1}(x-\alpha)}{1-x B^{-1}(x-\alpha)} \mathrm{d} x$$

The claim now follows by setting $u\left(z^0, x^0\right) := c$. On the set $\{(z, x) \in X : x \notin X_K^0\}$ we have $f(x) = 1$ and $z = x^{\intercal} B^{-1} \gamma$. Let $u(z, x) = 0$ along the indifference curve $\left\{(z, x) \in X : z = x^{\intercal} B^{-1} \gamma \wedge f(x) = 1\right\}$.
Finally, to verify our solution on the set X_K^0, using (13) we get that

$$p_n^{-1} = q_K x + z = \frac{x B^{-1}(x-\alpha) z - x B^{-1} \gamma}{1-x B^{-1}(x-\alpha)} + z = \frac{z - x B^{-1} \gamma}{1-x B^{-1}(x-\alpha)}$$

Thus, (12) gives

$$q_K = \frac{\nabla_x u(z, x)}{\partial u(z, x) / \partial z} = -B^{-1} \gamma + \left(z - x B^{-1} \gamma\right) \frac{B^{-1}(x-\alpha)}{1-x B^{-1}(x-\alpha)} = -B^{-1} \gamma + p_n^{-1} B^{-1}(x-\alpha)$$

as required. ■

Remark. Within the realm of Theorem 4, the requirement that $\succsim$ be monotonic imposes the following restriction on its domain:

$$X \subseteq \left\{(z, x) \in \mathbb{R}_{++}^n : \left(z - x B^{-1} \gamma\right) \frac{B^{-1}(x-\alpha)}{1-x B^{-1}(x-\alpha)} \gg B^{-1} \gamma\right\}$$

To conclude our analysis, comparisons between Theorems 1 and 3, but also between Theorems 2 and 4 above, are noteworthy. With respect to the former pair, both theorems establish that $\succsim$ being differentiable is fundamentally related to B being non-singular and M being singleton. A key feature in the proofs of Lemmas 3.1-3.2 renders the argument supporting Theorem 3 for its most part a corollary of that supporting Theorem 1. Whenever we appeal to the linearity of (6) in the proofs of Lemmas 3.1-3.2, we do so while holding income constant. We can do this also in the realm of (10): $x(\cdot)$ is linear in q_K for a given p_n. Yet, in sharp contrast to Theorem 1, Theorem 3 above does not claim that $\alpha(\cdot)$ must be a constant. In the proof of Lemma 3.3 we use that the linear part of $x(\cdot)$ in (6) is *independent* of income.[10] This does not obtain under the formulation in (10).

The discrepancy between Theorems 1 and 3 accounts in turn for the difference in scope between Theorems 2 and 4. The fact that $\alpha(\cdot)$ is constant in the realm of the former theorem ensures that the kth principal minor of the Slutsky matrix for $\widetilde{\xi}(\cdot)$ coincides with B. Being also non-singular, the matrix must be symmetric and negative definite; as a result, $x(\cdot)$ must also obey the strict Law of Demand. By contrast, under (10) there is no immediate mapping between the Slutsky matrix for $\xi(\cdot)$ and B.

Finally, we should note that Alperovich and Weksler (1996) investigate the demand formulation in (11) when $n = 2$. In this case, $k = 1$ and (14) reads

$$(\beta - x(x - \alpha))\, \mathrm{d}z/\mathrm{d}x + (x - \alpha)\, z = \gamma \tag{16}$$

where α, γ, and β are all scalars. For this special case, Alperovich and Weksler (1996) obtain a closed-form solution for the utility function in (12).[11] Notice also that restricting attention to the incomplete demand system in (11) can be justified by the conjunction of conditions (A*)-(B*) for the complete demand system. With respect to condition (A*), it suffices that the domain Y is open - an assumption to be found in Alperovich and Weksler (1996). Regarding condition (B*), given condition (A*), it suffices that the matrix B has a non-zero diagonal element (see Remark (iii) in Appendix A). B is a non-zero scalar in Alperovich and Weksler (1996).

[10] As the linear part of $x(\cdot)$ in (6) is separated from the part that varies with income, $\epsilon(\cdot)$ as defined by (30) is independent of q_K.

[11] (16) above is equivalent to (5) in Alperovich and Weksler (1996) - once a typo in their second term has been corrected.

4 Discussion and related literature

The studies most relevant for the preceding analysis are LaFrance (1985) and Amir et al. (2017). LaFrance (1985) examines the demand formulation in (6) distinguishing between two cases: whether or not $a(\cdot)$ is function of income. For the latter case, he takes B to be symmetric and negative semidefinite and establishes that the underlying quasi-direct (conditional upon the relative prices of the unobserved commodities) utility function must be quasi-linear/quadratic. For the case where $a(\cdot)$ does vary with income, LaFrance (1985) shows that the quasi-direct (conditional also upon income in this case) utility function must be Leontief. Amir et al. (2017) take the set of unobserved commodities to be a singleton and the demand formulation to be given by (7). They show that this can be generated by the utility function in (8) if B^{-1} is a symmetric, negative definite matrix with non-zero diagonal entries.

As we have seen already, restricting attention to the incomplete demand systems in (6)-(7) can be justified by the conjunction of conditions (A)-(B) for the complete demand system. With respect to condition (A), it suffices that the domain $Q \times W$ is open - an assumption to be found in both aforementioned studies. Regarding condition (B), given condition (A), it suffices that B has a non-zero diagonal element or a symmetric principal minor (see Remarks (iii)-(iv) in Appendix A). The former restriction is assumed in Amir et al. (2017), where the diagonal elements of B are all non-zero. The latter restriction can be found in LaFrance (1985) where B itself is symmetric.[12]

With respect to the restrictions placed on B, under the formulation in (7), the matrix being symmetric and negative semidefinite can be justified by assuming that the Slutsky matrix of the complete demand system itself is symmetric and negative semidefinite. Yet our analysis facilitates a direct connection with the underlying rationalizing preference. By Theorem 1, as long as the preference relation is weakly smooth, B must also be non-singular; hence, B being negative semi-definite is equivalent to B being negative definite. Moreover, the possibility of more than one unobserved commodities in LaFrance (1985) is a vacuous generalization while his argument for the case where $a(\cdot)$ does vary with income should be read as ad absurdum - that a weakly smooth preference must be Leontief is absurd. As for the analysis in Amir et al. (2017), Theorem 1 provides underpinnings for the theoretical framework itself. Their starting point is a continuously differentiable utility function; hence, a utility representation for preferences that are weakly smooth. A linear demand system generated by such preferences can only have the form in (7).

[12]It should be noted that condition (B) cannot admit the case where the complete demand system $\widetilde{\xi}_j(\cdot)$ is constant everywhere. The underlying intuition is the same as that in Jaffe and Weyl (2010) which shows that a complete demand system cannot be linear under discrete choice.

Our analysis relates also to the study in Nocke and Schutz (2017). This investigates the integrability of demand systems of the form $x(q)$ - i.e., the observed demands are independent of income - that satisfy the Law of Demand and for which there exists a function $v(\cdot)$ such that $\nabla_q v(q) = -x(q)$. Nocke and Schutz (2017) establish the existence of a rationalizing objective function, $z + \phi_x(q)$, where the convex function $\phi_x(q) := \inf_{q>>0}\{qx + v(q)\}$ is minimized at q_x: $x = x(q_x)$. Yet their objective lends itself to a direct utility function if and only if $x(\cdot)$ is invertible; for we need to be able to define the inverse demand function $q(\cdot)$ before proceeding to get $u(z+x) = z + \phi_x(q(x))$. When $x(\cdot)$ is in particular linear, the demand system investigated in Nocke and Schutz (2017) coincides with that in (7). In this case, $x(\cdot)$ is invertible if and only if B is non-singular (Theorem 1). And as the latter property requires that B is also symmetric (Theorem 2), we get that $v(q) = -\alpha q - q^{\intercal} Bq/2$ while $\phi_x(q) = q_x^{\intercal} x - q_x B q_x/2$ with q_x: $x = \alpha + Bq_x$. That is, $q_x = B^{-1}(x-\alpha)$ and $u(\cdot)$ takes the form in (8).

The present results bear also implications regarding the quest for microfoundations of demand estimation. Theorems 1-2 place strong restrictions on the quadratic utility the applied economist may appeal to. For instance, the form $(x-\alpha)A(x-\alpha)$ - see Deaton (1978) - is valid if and only if the $(n-1)$-th principal minor of A is symmetric and negative definite while $A_{nn} = 0 = A_{jn} + A_{nj}$ for $j = 1, \cdots, n-1$. Similarly, an additive utility function - see Houthakker (1960) - is consistent with linear demand if and only if it is of the form $u(z,x) = z + \sum_{j=1}^{n-1}(\alpha_j x_j + b_j x_j^2)$ while, for all j, $\alpha_j = 0$ implies $b_j = 0$; the matrix B of the corresponding linear demand is diagonal.

Most importantly perhaps, our results shed new light on the quest for microfoundations of linear demand systems especially in the context of applications in theoretical industrial organization. Amir et al. (2017) suggest that models of multi-variate linear demand functions for differentiated products ought to be regarded with some suspicion when the demand functions in question do not satisfy the Law of Demand. Their tone is understandably cautious given their key hypothesis of an underlying strictly concave quasi-linear/quadratic utility function. By contrast, based on a complete characterization of linear demand functions in terms of the underlying rationalizing preferences, our approach leads to far more commanding conclusions. Multivariate linear demand functions for differentiated products that do not satisfy the (strict) Law of Demand or are income dependent are *not* rationalizable - at least not by rational preferences smooth enough to allow for tractable utility functions.

By contrast, linear demand systems that do satisfy the strict Law of Demand and are income independent are fully consistent with continuous, strictly monotonic, strictly convex, and weakly smooth rationalizing preferences. Furthermore, the scope of this rationalizability allows linear aggregate demand but also linear market demand that results from horizontally or vertically differentiated products to be microfounded on continuous rational preferences.

Consider for instance a market where individuals indexed by some finite set H exhibit the observed demands $x^h(p) = \alpha^h + B^h p$ for $h \in H$. By Theorem 2, the individual demands are rationalizable only if B^h is symmetric and negative definite for all $h \in H$. In this case, the matrix $\sum_{h\in H} B^h$ will also be symmetric and negative definite; hence, the aggregate demand, $x(p) = \sum_{h\in H} \alpha^h + (\sum_{h\in H} B^h)p$, will itself be rationalizable by a single hypothetical agent. Her preferences are continuous, strictly convex, strictly monotonic, differentiable and represented by the utility function in (8) with $\alpha = \sum_{h\in H} \alpha^h$ and $B = \sum_{h\in H} B^h$.[13]

Alternatively, we can consider the model of vertically differentiated products in Amir et al. (2016) where the typical product j has quality q_j with $q_j < q_{j+1}$ for $1 \geq j < k$ ($k \in \mathbb{N}$: $k \geq 2$). There is a continuum of consumers uniformly distributed on $[0,1]$ with each consumer purchasing at most one good. If she buys good i at price p_i, she obtains a surplus of $\theta q_j - p_j$. Each consumer chooses the product with the highest surplus, provided it is nonnegative. The resulting market demand is given by

$$x_1 = \frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1},\ x_k = 1 - \frac{p_k - p_{k-1}}{q_k - q_{k-1}},\ x_j = \frac{p_{j+1} - p_j}{q_{j+1} - q_j} - \frac{p_j - p_{j-1}}{q_j - q_{j-1}},\ 1 < j < k$$

Amir et al. (2016) present also a model of spatially differentiated products on a star-shaped city with k selling locations. The city has $k-1$ roads radiating from a center and stretching indefinitely into suburbs. There is one shop at the center and one branch along each road at a distance of one unit from the center. The central shop offers consumers a value v_1 at price p_1 while the typical branch offers value v_j at price p_j. Along each road, there is a continuum of consumers uniformly distributed with each consumer incurring travel costs of τ per unit of distance. Each consumer seeks to maximize her surplus $v_j - p_j - \tau s$ where s is the distance of the jth shop from her location. The resulting market demand is given by

$$x_1 = \frac{(k-1)(\tau - v_1 - p_1) - \sum_{j=2}^{k} (v_j - p_j)}{2\tau},\ x_j = \frac{\tau + 3v_j - 3p_j - v_1 + p_1}{2\tau},\ j > 1$$

As we show in Appendix B, in either case the demand system for these differentiated products can be written as $x(p) = \alpha + Bp$ with B symmetric and negative definite. By

[13] Of course, this hypothetical agent is *not* a representative agent. Her demand, $x(p) = \alpha + Bp$, depicts the market demand for the economy consisting of the individuals in H under the restriction that her aggregate consumption bundle $x(p)$ gets allocated in a particular way, as $\{x^h\}_{h\in H}$ where $x^h = \alpha^h + B^h p$.

Theorem 2, either market demand schedule can be generated by a single agent whose preferences are represented by the utility function in (8).

Of course, rationalizing linear demand systems such as the above via continuous, strictly monotonic, strictly convex, and weakly smooth preferences entails two caveat properties. On the one hand, there is an unobserved part in the complete demand system consisting solely of the numeraire commodity for which the marginal utility is always constant. On the other hand, the range of the observed linear demand may correspond to only a subset of the domain of the rationalizing preference relation; beyond the observed range, the preference will exhibit a bliss point with respect to the observed commodities.

Both properties have an important role to play in the justification of using linear demand models for economic applications. They point towards situations where the observed choice bundles correspond to quantities and expenditure that are relatively small. For instance, when shopping in supermarkets, most people seldom face strictly binding budget constraints - especially if we allow also for lending and borrowing in which case the real budget constraint should be lifetime earnings. Moreover, most people would certainly reach a bliss point if they were to face unlimited quantities of supermarket products. Similarly, most firms seldom face strictly binding budget constraints when purchasing inputs. And under most models of industrial organization, processing unlimited amounts of inputs would eventually bring a firm to a loss-making position. It is common situations of this type that could be described by linear demand models of consumption behaviour or industrial organization. Being far from exhausting the total bank account, the welfare-maximizing consumer's or the cost-minimizing manager's estimate of how much an additional pound is worth, the marginal utility of money, remains constant in the background. What really matters then for the optimization problem at hand is the relative prices of the choice variables.

It should be noted finally that similar results obtain also in the case where prices are normalized with respect to income - instead of a numeraire. In this case, linear demand systems of the form $x(p_n, p_K) = \alpha(p_n) + Bp_K$ where $(p_n, p_K) \in Y$ are consistent with continuous, strictly monotonic, strictly convex, and weakly smooth rationalizing preferences as long as B is non-singular. The unobserved part of the complete demand system consists solely of a single commodity while beyond the observed range the preference will exhibit a bliss point with respect to the observed commodities. However, the marginal utility of the unobserved commodity is no longer constant while $x(\cdot)$ does not necessarily obey the Law of Demand.

5 Concluding remarks

Taking rational consumer preferences as the primitive and a linear demand system as the desideratum, we obtain a complete characterization in terms of the properties for the underlying rationalizing preference relation and analytical solution for the corresponding direct utility function. Yet the desideratum of microfounding linear demand on rational preferences that are smooth in the least sense to be representable by tractable utility functions demands that we are not agnostic about the unobserved part of the demand system. The latter cannot be hidden under the "ceteris paribus" carpet, nor behind the curtain of sufficiently high income remaining unspent in the background. Rationalizing linear demand systems with smooth preferences renders the unobserved part of the total system an integral component of the underlying preference relation.

In the case where the prices are normalized with respect to a numeraire, an observed linear demand system implies that the unobserved part comprises solely the numeraire commodity for which the marginal utility is always constant. If we want to interpret this as some basket of goods and services, a Hicksian composite commodity, we have to accept that there can be no substitution effects within the basket, nor between the basket and the observed commodities.[14] When the prices are instead income-normalized, preference characterization allows more leeway in terms of the interpretation of the (again single) commodity that comprises the unobserved part of a linear demand system. As its marginal utility is no longer constant, we can view it now as a Hicksian composite commodity for which there can be substitution effects within the basket as well as between the basket and the observed commodities. Alas this comes at the expense of a cumbersome utility representation.

6 Proofs

Proof of Lemma 3.1

To establish the contrapositive statement, let B be singular; that is, let there be $v \in \mathbb{R}^n \setminus \{0\}$ such that $Bv = 0$. Take an arbitrary $x^0 \in X_K$. Since the complete demand system $\widetilde{\xi}(\cdot)$ generated by $\succsim$ is onto - see Proposition 1 in Diasakos and Gerasimou (2022) - there exists $(q^0_{M_0}, q^0_K, w_0) \in Q_{M_0} \times Q_K \times W$ such that $x^0 = x(q^0_{M_0}, q^0_K, w_0)$. To argue ad absurdum, suppose also that $\succsim$ is differentiable. As this implies that $Q_{M_0} \times Q_K \times W$ is open - see Theorem 1 in Diasakos and Gerasimou (2022) - we may

[14]For details on the concept of "Hicksian composite commodity," see Woods (1979).

take $\lambda_0 \in \mathbb{R}_{++}$ sufficiently small so that $(q^0_{M_0}, q^0_K + \lambda v, w_0) \in Q_{M_0} \times Q_K \times W$ for all $\lambda \in (-\lambda_0, \lambda_0)$. Define then the function $q : (-\lambda_0, \lambda_0) \to Q_K$ by $q(\lambda) := q^0_K + \lambda v$. This gives $x(q^0_{M_0}, q(\cdot), w0) = x^0$. Moreover, letting $z^0 := z(q^0_{M_0}, q^0_K, w_0)$, we have

$$z^0_n + q^0_{M_0} z^0 + q\left(\lambda\right) x^0 = w_0 + \left(q\left(\lambda\right) - q^0_K\right) x^0 = w_0 + \lambda v x^0 \tag{17}$$

and

$$\begin{aligned} & z_n\left(q^0_{M_0}, q\left(\lambda\right)\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q\left(\lambda\right)\right) + q^0_K x\left(q^0_{M_0}, q\left(\lambda\right)\right) \\ = \; & w_0 + \left(q^0_K - q\left(\lambda\right)\right) x\left(q^0_{M_0}, q\left(\lambda\right)\right) = w_0 - \lambda v x^0 \end{aligned} \tag{18}$$

If $vx^0 = 0$, the desired contradiction obtains immediately. For, on the one hand, the bundle (z^0, x^0) is affordable at the price vector $(q^0_{M_0}, q(\lambda), w_0)$ while at the same time $(z(q^0_{M_0}, q(\lambda)), x(q^0_{M_0}, q(\lambda)))$ is affordable at $(q^0_{M_0}, q^0_K, w_0)$. Yet, on the other hand, $\succsim$ is differentiable only if the demand system is injective - see again Theorem 1 in Diasakos and Gerasimou (2022). The two bundles being thus distinct, we have a violation of the WARP.[15]

Suppose next that $vx(q_{M_0}, q_K, w) \neq 0$ for all $(q_{M_0}, q_K, w) \in Q_{M_0} \times Q_K \times W$. We must consider the following cases.

<u>Case I</u>: There exists $w' \in W \setminus \{w_0\}$ such that $vx(q^0_{M_0}, q^0_K, w') = vx^0$.

Observe first that, letting $\Delta\alpha(q^0_{M_0}, w_0) := \alpha(q^0_{M_0}, w') - \alpha(q^0_{M_0}, w_0)$, we have

$$x\left(q^0_{M_0}, q^0_K, w'\right) - \Delta\alpha\left(q^0_{M_0}, w_0\right) = x\left(q^0_{M_0}, q^0_K, w_0\right) = x^0 = x\left(q^0_{M_0}, q^0_K + \lambda v, w_0\right) \tag{19}$$

Letting also $\Delta w := w' - w_0$ and $\lambda := -\Delta w / vx^0$ we get that

$$\begin{aligned} w_0 - \lambda v x^0 = w' &= z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) + q^0_K x\left(q^0_{M_0}, q^0_K, w'\right) \\ &= z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) + q^0_K\left(x^0 + \Delta\alpha\left(q^0_{M_0}, w_0\right)\right) \\ &= z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) \\ &\quad + w_0 - z^0_n - q^0_{M_0} z^0 + q^0_K \Delta a\left(q^0_{M_0}, w_0\right) \end{aligned}$$

and thus

$$z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) = z^0_n + q^0_{M_0} z^0 - \lambda v x^0 - q^0_K \Delta a\left(q^0_{M_0}, w_0\right) \tag{20}$$

[15]The demand system $\widetilde{\xi}(\cdot)$ results from the maximization of the rational and strictly convex preference $\succsim$. It is well known that $\widetilde{\xi}(\cdot)$ must satisfy the Weak Axiom of Revealed Preference.

However, (19) and (20) together imply that

$$\begin{aligned}
& z_n\left(q^0_{M_0},q^0_K,w'\right)+q^0_{M_0}z_{-n}\left(q^0_{M_0},q^0_K,w'\right)+\left(q^0_K+\lambda v\right)x\left(q^0_{M_0},q^0_K,w'\right)\\
= & z^0_n+q^0_{M_0}z^0-\lambda v x^0-q^0_K\Delta\alpha\left(q^0_{M_0},w_0\right)+\left(q^0_K+\lambda v\right)\left(x^0+\Delta\alpha\left(q^0_{M_0},w_0\right)\right)\\
= & z^0_n+q^0_{M_0}z^0+q^0_Kx^0+\lambda v\Delta\alpha\left(q^0_{M_0},w_0\right)\\
= & w_0+\lambda v\left(x\left(q^0_{M_0},q^0_K,w'\right)-x\left(q^0_{M_0},q^0_K,w_0\right)\right)=w_0
\end{aligned}$$

as well as that

$$\begin{aligned}
& z_n\left(q^0_{M_0},q^0_K+\lambda v,w_0\right)+q^0_{M_0}z_{-n}\left(q^0_{M_0},q^0_K+\lambda v,w_0\right)+q^0_Kx\left(q^0_{M_0},q^0_K+\lambda v,w_0\right)\\
= & w_0+\left(q^0_K-\left(q^0_K+\lambda v\right)\right)x\left(q^0_{M_0},q^0_K+\lambda v,w_0\right)=w_0-\lambda v x^0=w'
\end{aligned}$$

another violation of the WARP.
<u>Case II</u>: $vx(q_{M_0},q_K,w')\neq vx(q_{M_0},q_K,w)$ for all $(q_{M_0},q_K,w),(q_{M_0},q_K,w')\in Q_{M_0}\times Q_K\times W$.
Consider the sets

$$\begin{aligned}
Q^0 &:= \left\{(q_{M_0},q_K,w)\in Q_{M_0}\times Q_K\times W:\ z\left(q_{M_0},q_K,w\right)=z^0\right\}\\
X^0_K &:= \left\{(z,x)\in X:\ z=z^0\right\}
\end{aligned}$$

Since X is open in $\mathbb{R}^n_{++}$, the set X^0_K is open in $\mathbb{R}^k_{++}$. Since the total demand is an homeomorphism so is its restriction $x:Y_{z^0}\to X^0_K$; hence, Y_{z^0} is also open in $\mathbb{R}^k_{++}$. Moreover, the hyperplane

$$X^*_K:=\left\{x\in X^0_K:\ vx=vx^0\right\}$$

being open in $\mathbb{R}^{k-1}_{++}$, so is the preimage

$$Q^*:=\left\{(q_{M_0},q_K,w)\in Q^0:\ x(q_{M_0},q_K,w)\in X^*_K\right\}$$

Observe now that, for any $x(q^1_{M_0},q^1_K,w_1),x(q^2_{M_0},q^2_K,w_2)\in X^*_K$, we have

$$\begin{aligned}
0 &= vx\left(q^1_{M_0},q^1_K,w_1\right)-vx\left(q^2_{M_0},q^2_K,w_2\right)\\
&= vx\left(q^1_{M_0},q^1_K,w_1\right)-vx\left(q^2_{M_0},q^2_K,w_1\right)\\
&\quad +vx\left(q^2_{M_0},q^2_K,w_1\right)-vx\left(q^2_{M_0},q^2_K,w_2\right) \qquad (21)
\end{aligned}$$

As a result, $x(q^2_{M_0}, q^2_K, w_1) \in X^*_K$ renders the first difference on the right-hand side of (21) above zero, necessitating in turn that $vx(q^2_{M_0}, q^2_K, w_1) = vx(q^2_{M_0}, q^2_K, w_2)$. Yet the latter equality contradicts the very hypothesis that defines the case under consideration. Clearly, for any $w_1 \neq w_2$, we have $x(q^2_{M_0}, q^2_K, w_1) \notin X^*_K$ if $x(q^2_{M_0}, q^2_K, w_2) \in X^*_K$. Similarly, we have that

$$\begin{aligned} 0 &= vx\left(q^1_{M_0}, q^1_K, w_1\right) - vx\left(q^2_{M_0}, q^2_K, w_2\right) \\ &= vx\left(q^1_{M_0}, q^1_K, w_1\right) - vx\left(q^1_{M_0}, q^1_K, w_2\right) \\ &\quad + vx\left(q^1_{M_0}, q^1_K, w_2\right) - vx\left(q^2_{M_0}, q^2_K, w_2\right) \end{aligned} \tag{22}$$

And as the first difference on the right-hand side of (22) cannot be zero, for any $(q^1_{M_0}, q^1_K) \neq (q^2_{M_0}, q^2_K)$, we must have $x(q^1_{M_0}, q^1_K, w_2) \notin X^*_K$ if $x(q^2_{M_0}, q^2_K, w_2) \in X^*_K$.

Let now Q^*_n and $Q^*_{M_0 \cup K}$ be, respectively, the projections of Q^* along the income and the remaining $n-1$ price dimensions. The preceding argument means that there must exist a bijection $f : Q^*_n \to Q^*_{M_0 \cup K}$ such that any $x(q_{M_0}, q_K, w) \in X^*_K$ can be written as $x(f(w), w)$. However, since $Q^*_{M_0 \cup K}$ is open in $\mathbb{R}^{k-2}_{++}$ while X^*_K is open in $\mathbb{R}^{k-1}_{++}$, this is absurd. For, on the one hand, the homeomorphism $f(\cdot)$ necessitates that $k-2 = 1$. On the other hand, the graph of $f(\cdot)$ being open in $\mathbb{R}_{++}$, the homeomorphism $x(\text{graph} f(\cdot))$ on X^*_K necessitates that $k-1=1$.

Given the preceding contradiction, we conclude that income remains constant (at w_0) along the hyperplane X^*_K. We will show now that, for any $q^1_K, q^2_K \in Q_K$ with $q^1_K \neq q^2_K$, we cannot have $vx(q^0_{M_0}, q^1_K, w_0) \neq vx(q^0_{M_0}, q^2_K, w_0)$. To argue by contradiction, let

$$vB\left(q^2_K - q^1_K\right) = vx\left(q^0_{M_0}, q^2_K, w_0\right) - vx\left(q^0_{M_0}, q^1_K, w_0\right) \neq 0$$

Choose $(\kappa, w) \in (0,1) \times \mathcal{B}_{w_0}$ such that $v\alpha(\tilde{q}^0, w) = v\alpha(\tilde{q}^0, w_0) - \kappa vB(q^2_K - q^1_K)$.[16] Letting $q^3_K := \kappa q^2_K + (1-\kappa) q^1_K$, we now have

$$\begin{aligned} v\left(x\left(q^0_{M_0}, q^1_K, w_0\right) - x\left(q^0_{M_0}, q^3_K, w\right)\right) &= v\left(B\left(q^1_K - q^3_K\right) + \alpha\left(\tilde{q}^0, w_0\right) - \alpha\left(\tilde{q}^0, w\right)\right) \\ &= \kappa vB\left(q^1_K - q^2_K\right) + v\left(\alpha\left(\tilde{q}^0, w_0\right) - \alpha\left(\tilde{q}^0, w\right)\right) \\ &= 0 \end{aligned}$$

[16] By hypothesis, in this case, $\alpha(\tilde{q}^0, \cdot)$ is a non-constant function; hence, by continuity, $v\alpha(\tilde{q}^0, \cdot)$ is one-to-one on a sufficiently small neighbourhood of w_0. Observe also that, choosing κ sufficiently small, brings w arbitrarily close to w_0. The existence of w follows from the continuity of $\alpha(\tilde{q}^0, \cdot)$.

This contradicts though that income remains constant along X_K^*.
Clearly, we have

$$\left\{x \in X_K^0 :\ x = x\left(q_{M_0}^0, q_K, w_0\right),\ q_K \in Q_K\right\} \subseteq X_K^* \tag{23}$$

Take now $\varepsilon \in \mathbb{R}_{++}$ sufficiently small so that $\mathcal{B}_{q_K^0}(\varepsilon) \subset Q_K^0$ - where Q_K^0 is the projection of Q^0 on Q_K. Consider also the budget sets

$$B(q_K) := \left\{x \in X :\ q_K x = w_0 - q_{M_0}^0 z^0\right\}, \qquad q_K \in Q_K$$

The restriction of the preference relation $\succsim$ on $X \times \{z^0\}$ being strictly convex, strictly monotonic, and continuous, we obtain an homeomorphic demand function $\widetilde{x} : Q_K^* \to X$ where Q_K^* is an open subset of the set $(w_0 - q_{M_0}^0 z^0)^{-1} Q_K$. Letting now

$$\kappa_1 = \min\{1, 1/(w_0 - q_{M_0}^0 z^0)\}$$

and comparing $\widetilde{x}(\cdot)$ with $x^0(\cdot) := x(q_{M_0}^0, \cdot, w_0)$ on $\mathcal{B}_{q_K^0}(\kappa_1 \varepsilon)$ reveals the desired contradiction. For we must have $(z^0, x^0(q_K)) \succsim (z^0, \widetilde{x}(q_K))$ everywhere on $\mathcal{B}_{q_K^0}(\kappa_1 \varepsilon)$. Yet, $\widetilde{x}(\cdot)$ being an homeomorphism, the image set $\widetilde{x}(\mathcal{B}_{q_K^0}(\kappa_1 \varepsilon))$ is an open neighbourhood of x^0 in $\mathbb{R}_{++}^k$ while (23) necessitates that $x^0(\mathcal{B}_{q_K^0}(\kappa_1 \varepsilon)) \subseteq X_K^*$, which is open in $\mathbb{R}_{++}^{k-1}$. The contradiction is due to the monotonicity of $\succsim$. ■

Proof of Corollary 3.1

Recall how the two sets of normalized prices are related: $(p_M, p_K) = p_n((1, q_{M_0}), q_K)$ and $p_n = 1/w$. The argument in the preceding proof remains valid once we replace w_0, w', w, w_1, and w_2, respectively, by $1/p_n^0$, $1/p_n'$, $1/p_n$, $1/p_n^1$, and $1/p_n^2$.
A slight adjustment must be in Case I. Letting now $\Delta\alpha(q_{M_0}^0, p_n^0) := \alpha(q_{M_0}^0, p_n') - \alpha(q_{M_0}^0, p_n^0)$, we have

$$\begin{aligned} x\left(q_{M_0}^0, q_K^0, p_n'\right) - \Delta\alpha\left(q_{M_0}^0, p_n^0\right) - \Delta p B q_K^0 &= x\left(q_{M_0}^0, q_K^0, p_n^0\right) \\ &= x^0 \\ &= x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) \end{aligned} \tag{24}$$

where $\Delta p_n^0 := p_n' - p_n^0$ and $\lambda := \Delta p_n^0 / (p_n^0 p_n' v x^0)$. That is,

$$\begin{aligned}
1/p_n - \lambda v x^0 &= 1/p_n' \\
&= z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + q_K^0 x\left(q_{M_0}^0, q_K^0, p_n'\right) \\
&= z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) \\
&\quad + q_K^0\left(x^0 + \Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \\
&= z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) \\
&\quad + 1/p_n^0 - q_{M_0}^0 z^0 + q_K^0\left(\Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right)
\end{aligned}$$

and thus

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p'\right) \\
= \; & z_n^0 + q_{M_0}^0 z_{-n}^0 - \lambda v x^0 - q_K^0\left(\Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \qquad (25)
\end{aligned}$$

Now (24)-(25) imply that

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + \left(q_K^0 + \lambda v\right) x\left(q_{M_0}^0, q_K^0, p_n'\right) \\
= \; & z_n^0 + q_{M_0}^0 z_{-n}^0 - \lambda v x^0 - q_K^0\left(\Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \\
& + \left(q_K^0 + \lambda v\right)\left(x^0 + \Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p B q_K^0\right) \\
= \; & z_n^0 + q_{M_0}^0 z_{-n}^0 + q_K^0 x^0 + \lambda v\left(\Delta a\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \\
= \; & 1/p_n^0 + \lambda v\left(x\left(q_{M_0}^0, q_K^0, p_n'\right) - x^0\right) = 1/p_n^0
\end{aligned}$$

as well as

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) + q_K^0 x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) \\
= \; & 1/p_n^0 + \left(q_K^0 - \left(q_K^0 + \lambda v\right)\right) x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) \\
= \; & 1/p_n^0 - \lambda v x^0 = 1/p_n'
\end{aligned}$$

Yet

$$\begin{aligned}
& p_n^0 z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + p_n^0 q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + p_n^0\left(q_K^0 + \lambda v\right) x\left(q_{M_0}^0, q_K^0, p_n'\right) \\
= \; & 1 \\
= \; & p_n' z_n\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) + p_n' q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) + p_n' q_K^0 x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right)
\end{aligned}$$

is a violation of the WARP. ■

Proof of Lemma 3.2

To argue ad absurdum, let $j \in M_0 \neq \varnothing$. Recall first that $\succsim$ is differentiable at (z,x) if and only if the vector of relative prices (q_{M_0}, q_K) is the unique subgradient of $l_n(\cdot|(z,x))$ at (z_{-n}, x). Hence, $\succsim$ being differentiable, we have

$$l_n\left((z_{-n},x)\,|\,(z,x)\right) + q_{M_0} z_{-n} + q_K x \leq l_n\left((\widetilde{z}_{-n},\widetilde{x})\,|\,(z,x)\right) + q_{M_0}\widetilde{z}_{-n} + q_K \widetilde{x}$$

for any $(\widetilde{z}_{-n},\widetilde{x}) \in \mathcal{I}^{-i}_{(z,x)}$. And as $l_n(\cdot|(z,x))$ is differentiable everywhere along the latter set, this necessitates in fact that

$$\begin{aligned} 0 &= \nabla_{z_{-n}} l_n\left((z_{-n},x)\,|\,(z,x)\right) + q_{M_0} && (26)\\ 0 &= \nabla_{x} l_n\left((z_{-n},x)\,|\,(z,x)\right) + q_K && (27)\end{aligned}$$

Take now $(q^0_{M_0}, q^0_K, w_0) \in Q_{M_0} \times Q_K \times W$, and let $z^0 := z(q^0_{M_0}, q^0_K, w_0)$ and $x^0 := x(q^0_{M_0}, q^0_K, w_0)$. Obviously, the system (26)-(27) must hold everywhere on the indifference set $\mathcal{I}_{(z^0,x^0)}$. Given this, if we restrict attention to relative price changes in the set $\left\{q^0_{M_0}\right\} \times Q_K$, we move along $\mathcal{I}_{(z^0,x^0)}$ as long as we obey the following system of partial differential equations

$$\begin{aligned} \partial z_n/\partial z_j &= -q^0_j, \qquad j \in M_0\\ \partial z_n/\partial x_j &= -q_j\left(w,x\right) = \left(B^{-1}\left(\alpha\left(q^0_{M_0},w\right) - x\right)\right)_j, \qquad j \in K\end{aligned}$$

where the last equality above uses the fact that B is non-singular (recall Lemma 3.1). Integrating then along $\mathcal{I}_{(z^0,x^0)}$, we have

$$z_n = xB^{-1}\alpha\left(q^0_{M_0},w\right)B^{-1} - xB^{-1}x/2 - q^0_{M_0} z_{-n} + c_0, \qquad (z,x) \in \mathcal{I}_{(z^0,x^0)}$$

where c_0 remains constant along $\mathcal{I}_{(z^0,x^0)}$. We can define thus a quasi-indirect utility function $v : X \times M \to \mathbb{R}$ by setting $v(z^0, x^0, q^0_{M_0}) := c_0$; that is, by letting

$$\begin{aligned} v\left(z,x,q_{M_0},w\right) &:= -xB^{-1}\alpha\left(q_{M_0},w\right)B^{-1} + xB^{-1}x/2 + q_{M_0} z_{-n} + z_n\\ &= -xB^{-1}x/2 + q\left(x,q_{M_0},w\right)x + q_{M_0} z_{-n} + z_n\end{aligned}$$

Notice now that, as X is open in $\mathbb{R}^n_{++}$, taking $\varepsilon_0 > 0$ sufficiently small, the hyperplane

$$X^*_M := \left\{(z,x^0) \in X:\ z := (z^0_n + q^0_j\varepsilon, z^0_j - \varepsilon, z^0_{-(n,j)}),\ \varepsilon \in (0,\varepsilon_0)\right\}$$

lies in X and is open in $\mathbb{R}^{n-k-1}_{++}$. Consider also renormalizing the prices relative to income. As $p^0_n := 1/w_0$ and $p^0_j = q^0_j/w_0$, we get that

$$p^0_M z = p^0_M z^0 + \left(p^0_n q^0_j - p^0_j\right)\varepsilon = p^0_M z^0 \tag{28}$$

or equivalently $p^0_M z^0 + p^0_K x^0 = 1 = p^0_M z + p^0_K x^0$. Clearly, $(z^0, x^0) \succ (z, x^0)$ for any $(z, x^0) \in X^*_M$.

Observe next that the hyperplane

$$Y^*_{-(n,j)} := \left\{(p_{M_0\setminus j}, p_K) \in Y_{M_0\setminus\{j\}} \times Y_K :\ p_{M_0\setminus j} z^0_{M_0\setminus j} + p_K x^0 = 1 - (p^0_n z^0_n + p^0_j z^0_j)\right\}$$

is open in $\mathbb{R}^{n-3}_{++}$, and restrict the homeomorphic total demand $\xi(\cdot)$ to the domain $Y_j \times Y_n \times Y^*_{-(n,j)}$. The restriction itself being homeomorphic, the image set $\xi(Y_j \times Y_n \times Y^*_{-(n,j)})$ must be also open in $X_j \times X_n \times \mathbb{R}^{n-3}_{++}$. Moreover, since $(p^0_M, p^0_K) \in Y_j \times Y_n \times Y^*_{-(n,j)}$, $\xi(Y_j \times Y_n \times Y^*_{-(n,j)})$ must include a neighbourhood of (z^0, x^0) in $X_n \times X_j \times \mathbb{R}^{n-3}_{++}$. That is, $\xi(Y_j \times Y_n \times Y^*_{-(n,j)}) \cap X^*_M \neq \varnothing$.

Choosing, therefore, a sufficiently small $\varepsilon_1 \in (0, \varepsilon_0)$, we can find $(p^1_M, p^1_K) \in Y_j \times Y_n \times Y^*_{-(n,j)}$ such that $z^1 = z(p^1_M, p^1_K)$ and $x^0 = x(p^1_M, p^1_K)$ where $z^1 := (z^0_n + q^0_j\varepsilon_1, z^0_j - \varepsilon_1, z^0_{-(n,j)})$. Now, since $(z^1, x^0) \in X^*_M$, we must have $(z^0, x^0) \succ (z^1, x^0)$. Taking $w_1 := 1/p^1_n$ and $q^1_j = w - 1p^1_j$, this necessitates that

$$0 < p^1_M\left(z^0 - z^1\right) + p^1_K\left(x^0 - x^0\right) = -\left(p^1_n q^0_j - p^1_j\right)\varepsilon_1 = -\left(q^0_j - q^1_j\right)\varepsilon_1/w_1$$

i.e., that $q^1_j > q^0_j$. Yet as we also have

$$\begin{aligned}
p^0_n z^0_n + p^0_j z^0_j = 1 - \left(p^1_{M\setminus\{j\}} z^0_{-(n,j)} + p^1_K x^0\right) &= p^1_n z^1_n + p^1_j z^1_j \\
&= \left(z^1_n + q^1_j z^1_j\right)/w_1 \\
&> \left(z^1_n + q^0_j z^1_j\right)/w_1 \\
&= \frac{w_0}{w_1}\left(p^0_n z^1_n + p^0_j z^1_j\right) = \frac{w_0}{w_1}\left(p^0_n z^0_n + p^0_j z^0_j\right)
\end{aligned}$$

we get in fact that $w_0 < w_1$. This implies in turn that

$$\begin{aligned}
z_n^0 + q_{M_0}^0 z_{-n}^0 + q_K^0 x^0 &= w_0 \\
&< w_1 \\
&= z_n^1 + q_{M_0}^1 z_{-n}^1 + q_K^1 x^0 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + q_{M_0}^0 z_{-n}^1 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + w_0 p_{M\setminus\{n\}}^0 z_{-n}^1 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 \\
&\quad + w_0 \left(p_{M\setminus\{n\}}^0 z_{-n}^0 + p_n^0 \left(z_n^0 - z_n^1\right)\right) \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 \\
&\quad + w_0 \left(p_{M\setminus\{n\}}^0 z_{-n}^0 - p_n^0 \varepsilon_1\right) \\
&< z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + w_0 p_{M\setminus\{n\}}^0 z_{-n}^0 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + q_{M_0}^0 z_{-n}^0
\end{aligned}$$

where the penultimate equality above follows from (28). Clearly, we have that

$$z_n^1 - z_n^0 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 > 0 \tag{29}$$

But then we must have

$$\begin{aligned}
v\left(z^1, x^0, q_{M_0}^1, w_1\right) &= -xB^{-1}x/2 + q\left(x^0, q_{M_0}^1, w_1\right) x^0 + q_{M_0}^1 z_{-n}^1 + z_n^1 \\
&= -xB^{-1}x/2 + q_K^1 x^0 + q_{M_0}^1 z_{-n}^1 + z_n^1 \\
&> -xB^{-1}x/2 + q_K^0 x^0 + q_{M_0}^0 z_{-n}^0 + z_n^0 \\
&= -xB^{-1}x/2 + q\left(x^0, q_{M_0}^0, w_0\right) x^0 + q_{M_0}^0 z_{-n}^0 + z_n^0 = v\left(z^0, x^0, q_{M_0}^0, w_0\right)
\end{aligned}$$

the inequality above due to (29). And as this means that $(z^1, x^0) \succ (z^0, x^0)$, the desired contradiction follows from the absurdity $(z^1, x^0) \succ (z^0, x^0) \succ (z^1, x^0)$. ■

Proof of Corollary 3.2

Recall again how the two sets of normalized prices are related: $(p_M, p_K) = p_n((1, q_{M_0}), q_K)$. The preceding proof applies as is - with the slight adjustment that B above should be replaced by $p_n B$. ■

Proof of Lemma 3.3

Let $\succsim$ be differentiable. As $M_0 = \varnothing$ (Lemma 3.2), $\alpha(\cdot)$ can be a function only of income. In what follows, we will drop the subscript K from the members of Q_K and write (6) as $x(q,w) := \alpha(w) + Bq$. To argue ad absurdum, suppose that $\alpha(\cdot)$ is not constant around the arbitrary point $w_0 \in W$. Letting then $\lambda_0 \in \mathbb{R}_{++}$ be sufficiently small, we must have $\alpha(w) \neq \alpha(w_0)$ for all $w \in (w_0 - \lambda_0, w_0 + \lambda_0) \setminus \{w_0\}$. Take also an arbitrary $q^0 \in Q_K$ and let $x^0 := x(q^0, w_0)$ and $z_0 := z(q^0, w_0)$. Consider also the sets

$$\begin{aligned} Q_{Kz_0} &:= \{q \in Q_K : \ z(q, w_0) = z_0\} \\ X_{z_0} &:= \{(z, x) \in X : \ z = z_0\} \end{aligned}$$

Since X is open in $\mathbb{R}^n_{++}$, the set X_{z_0} is open in $\mathbb{R}^{n-1}_{++}$. Since the total demand is an homeomorphism so is the mapping $x : Y_{z_0} \to X_{z_0}$; hence, Q_{Kz_0} is also open in $\mathbb{R}^{n-1}_{++}$. And as $(z_0, x^0) \in X_{z_0}$, taking $\varepsilon_0, \rho_0 \in \mathbb{R}_{++}$ both sufficiently small ensures that $\mathcal{B}_{x^0}(\varepsilon_0) \subset X_{z_0}$ and $\mathcal{B}_{q^0}(\rho_0) \subset Q_{Kz_0}$.
Recall now that, $\succsim$ being differentiable, B is non-singular (Lemma 3.1). As a result, the function

$$x^0(q) := \alpha(w_0) + Bq$$

defines an homeomorphism $x^0 : \mathcal{B}_{q^0}(\rho_0) \to \mathcal{B}_{x^0}(\varepsilon_0)$. Moreover, since $\alpha(w_0 + \lambda) \neq \alpha(w_0)$, we have $x(q, w_0) \neq x(q, w_0 + \lambda)$ for all $(\lambda, q) \in (-\lambda_0, \lambda_0) \times \mathcal{B}_{q^0}(\rho_0)$. In fact, letting $\lambda_1 \in (0, \lambda_0)$ and $\rho_1 \in (0, \rho_0)$ both sufficiently small so that $||\alpha(w_0 + \lambda) - \alpha(w_0)|| < \varepsilon_0/2$ for all $\lambda \in (-\lambda_1, \lambda_1)$ and $x(q, w_0) \in \mathcal{B}_{x^0}(\varepsilon/2)$ for all $q \in \mathcal{B}_{q^0}(\rho_1)$, we have

$$\begin{aligned} ||x(q, w_0 + \lambda) - x^0|| &\leq ||x(q, w_0 + \lambda) - x(q, w_0)|| + ||x(q, w_0) - x^0|| \\ &= ||\alpha(w_0 + \lambda) - \alpha(w_0)|| + ||x(q, w_0 + \lambda) - x^0|| < \varepsilon_0 \end{aligned}$$

That is, $x(q, w_0 + \lambda) \in \mathcal{B}_{x^0}(\varepsilon_0)$ for all $(\lambda, q) \in (-\lambda_1, \lambda_1) \times \mathcal{B}_{q^0}(\rho_1)$. And as $x^0(\cdot)$ is an homeomorphism, we have that

$$\exists! \ q^\lambda \in \mathcal{B}_{q^0}(\rho_0) : \qquad x(q, w_0 + \lambda) = x^0(q^\lambda)$$

Define then the $(-\lambda_1, \lambda_1) \to \mathcal{B}_{q^0}(\rho_0)$ function $\epsilon(q, \lambda) := q^\lambda - q$, and observe that the last relation above can be also written as

$$x(q, w_0 + \lambda) = x(q + \epsilon(q, \lambda), w_0)$$

Clearly, for all $(\lambda, q) \in (-\lambda_1, \lambda_1) \times \mathcal{B}_{q^0}(\rho_1)$, we have

$$\epsilon(q, \lambda) = \epsilon(\lambda) := B^{-1}(\alpha(w_0 + \lambda) - \alpha(w_0)) \tag{30}$$

This implies in turn that

$$\begin{aligned} x(q - \epsilon(\lambda), w_0 + \lambda) &= B(q - \epsilon(\lambda)) + \alpha(w_0 + \lambda) \\ &= Bq - (\alpha(w_0 + \lambda) - \alpha(w_0)) + \alpha(w_0 + \lambda) = x(q, w_0) \end{aligned}$$

and thus

$$\begin{aligned} w_0 + \lambda &= z(q - \epsilon(\lambda), w_0 + \lambda) + (q - \epsilon(\lambda))\, x(q - \epsilon(\lambda), w_0 + \lambda) \\ &= z(q - \epsilon(\lambda), w_0 + \lambda) + (q - \epsilon(\lambda))\, x(q, w_0) \\ &= z(q - \epsilon(\lambda), w_0 + \lambda) + w_0 - z(q, w_0) - \epsilon(\lambda)\, x(q, w_0) \end{aligned}$$

or, equivalently,

$$z(q - \epsilon(\lambda), w_0 + \lambda) = z(q, w_0) + \lambda + \epsilon(\lambda)\, x(q, w_0) \tag{31}$$

Recall now the quasi-indirect utility function we obtained in the proof of Lemma 3.2. As $M_0 = \varnothing$, this reads here

$$v(z, x, w) = xB^{-1}x/2 + q(x, w)\, x + z$$

That is,

$$\begin{aligned} v(q - \epsilon(\lambda), w_0 + \lambda) &= x(q - \epsilon(\lambda), w_0 + \lambda)\, B^{-1} x(q - \epsilon(\lambda), w_0 + \lambda)/2 \\ &\quad + (q - \epsilon)\, x(q - \epsilon(\lambda), w_0 + \lambda) + z(q - \epsilon(\lambda), w_0 + \lambda) \\ &= x(q, w_0)\, B^{-1} x(q, w_0)/2 + (q - \epsilon)\, x(q, w_0) \\ &\quad + z(q, w_0) + \lambda + \epsilon(\lambda)\, x(q, w_0) \\ &= v(q, w_0) + \lambda \end{aligned} \tag{32}$$

which implies in turn that $\lambda \mapsto z(q, \lambda) := z(q - \epsilon(\lambda), w_0 + \lambda)$ is an injective function.[17] Hence, for any $q \in \mathcal{B}_{q^0}(\rho_1)$, the image of $z(q, \lambda)$ on $(-\lambda_1, \lambda_1)$ is an open neighbourhood around the point $z(q, w_0)$.

Take now $\delta_0 \in \mathbb{R}_{++}$ such that the neighbourhood $\mathcal{B}_{z^0}(\delta_0)$ lies within the domain. Let

[17] To see first that $z(q, \cdot)$ is a function, notice that we cannot have $z', z'' \in z(q, \lambda)$ with $z' \neq z''$. For (32) would imply then that $(z', x(q, w_0)) \sim (z'', x(q, w_0))$, an absurdity under monotonicity. To see now that $z(q, \cdot)$ must be injective, observe that we cannot have $z(q, \lambda') = z(q, \lambda'')$ with $\lambda' \neq \lambda''$. For, letting $z' := z(q, \lambda_1)$, (32) would imply now that $(z', x(q, w_0)) \succ (z', x(q, w_0))$.

also $z^0(\lambda) := z(q^0, \lambda)$. By the preceding argument, taking $\lambda_2 \in (0, \lambda_1)$ sufficiently small, $z^0(\cdot)$ on $(-\lambda_2, \lambda_2)$ maps onto $\mathcal{B}_{\mathbf{z}^0}(\delta_1)$ for some $\delta_1 \in (0, \delta_0)$.
Fix now some $\lambda \in (-\lambda_2, \lambda_2)$ and consider the sets

$$\begin{aligned} Q_{Kz^0(\lambda)} &:= \left\{(q, w_0 + \lambda) \in Q_K : \; z\left(q - \epsilon\left(\lambda\right), w_0 + \lambda\right) = z^0\left(\lambda\right)\right\} \\ X_{z^0(\lambda)} &:= \left\{(z, x) \in X : \; z = z^0\left(\lambda\right)\right\} \end{aligned}$$

By the same argument as in the opening paragraph above, $X_{z^0(\lambda)}$ and $Q_{Kz^0(\lambda)}$ are open in $\mathbb{R}^{n-1}_{++}$. And as $q^0 \in Q_{Kz^0(\lambda)}$, choosing $\rho_\lambda \in (0, \rho_1)$ sufficiently small, we get $\mathcal{B}_{q^0}(\rho_\lambda) \subset Q_{Kz^0(\lambda)}$. Moreover, using (31) above, we have that

$$\begin{aligned} 0 &= z(q - \epsilon\left(\lambda\right), w_0 + \lambda) - z^0\left(\lambda\right) \\ &= \left(z\left(q, w_0\right) + \lambda + \epsilon\left(\lambda\right) x\left(q, w_0\right) - z\left(q^0, w_0\right) - \lambda - \epsilon\left(\lambda\right) x\left(q^0, w_0\right)\right) \\ &= \left(z^0 + \lambda + \epsilon\left(\lambda\right) x\left(q, w_0\right) - \mathbf{z}^0 - \lambda - \epsilon\left(\lambda\right) x^0\right) \\ &= \epsilon\left(\lambda\right)\left(x^0\left(q\right) - x^0\right) \qquad \forall q \in \mathcal{B}_{q^0}\left(\rho_\lambda\right) \end{aligned}$$

As though $x^0(\cdot)$ is an homeomorphism, it maps $\mathcal{B}_{q^0}(\rho_\lambda)$ onto $\mathcal{B}_{x^0}(\varepsilon_\lambda)$ for some $\varepsilon_\lambda \in (0, \varepsilon_0)$. We have established thus that $\epsilon(\lambda)(x - x^0) = 0$ for every $x \in \mathcal{B}_{x^0}(\varepsilon_\lambda)$; equivalently, that $\epsilon(\lambda) = 0$. To complete the argument, recall (30). Since B^{-1} is non-singular, $\epsilon(\lambda) = 0$ implies that $\alpha(w_0 + \lambda) = \alpha(w_0)$. And as λ above was chosen arbitrarily, $\alpha(\cdot)$ must remain constant on $(w_0 - \lambda_2, w_0 + \lambda_2)$. ■

Appendices

A An incomplete demand system

To make the exposition in this section less cumbersome, for $y \in \mathbb{R}^n$ and $i \in \mathcal{N}$ we will use the notation y_i and y_{-i} in lieu of $y_{\{i\}}$ and $y_{\mathcal{N}\setminus\{i\}}$, respectively. That is, y_i and y_{-i} will denote, respectively, the projections of y on the ith dimension of $\mathbb{R}^n$ and on the subspace that results from $\mathbb{R}^n$ when the ith dimension is removed. Taking also $j \in \mathcal{N} \setminus \{i\}$, we will use the notation $y_{-(i,j)}$ in lieu of $y_{\mathcal{N}\setminus\{i,j\}}$; i.e., $y_{-(i,j)}$ will denote the projection of y on the subspace that results from $\mathbb{R}^n$ when both the ith and jth dimensions are removed. Finally, $||y||$ denotes the Euclidean norm of y.

Claim 1 *Let the demand system $\xi : Y \to X$ be given by $\xi(p) := \alpha + Ap$, where α and A are, respectively, a constant n-dimensional real vector and an $n \times n$ real matrix. Suppose also that $\xi(\cdot)$ satisfies Walras' law. Then at least one of conditions (A*) in the main text and*

(C)* $\exists \epsilon \in \mathbb{R}^n \setminus \{0\}$ *such that* $\epsilon A \epsilon \neq 0$*,*

cannot hold.

Proof. To establish the claim arguing ad absurdum, suppose that both conditions hold simultaneously. Letting $(p, \varepsilon) \in Y \times \mathbb{R}_{++}$ be as in (A*) and $\epsilon \in \mathbb{R}^n \setminus \{0\}$ be as in (C*), take $\lambda \in (0,1)$ sufficiently small so that $\lambda ||\epsilon|| \leq 1$ and define the $(-\varepsilon, \varepsilon) \to Y$ function $p(\delta) := p + \delta\lambda\epsilon$. By Walras' law we ought to have

$$\begin{aligned} p\alpha + pAp = p\xi(p) = 1 &= p(\delta)\,\xi(p(\delta)) \\ &= p(\delta)\,\alpha + p(\delta)\,Ap(\delta) \\ &= p\alpha + pAp + \delta\lambda\epsilon\alpha + \delta\lambda\epsilon Ap + \delta^2\lambda^2\epsilon A\epsilon + \delta\lambda pA\epsilon \end{aligned}$$

As this implies in turn that

$$\delta = -\frac{\epsilon\alpha + \epsilon Ap + pA\epsilon}{\lambda\epsilon A\epsilon} \qquad \forall \delta \in (-\varepsilon, 0) \cup (0, \varepsilon) \tag{33}$$

the desired contradiction obtains immediately. ■

Claim 2 *Let the demand system* $\widetilde{\xi} : Q \times W \to X$ *be given by* $\widetilde{\xi}(q) := \alpha + Aq$*, where* α *and* A *are, respectively, a constant n-dimensional real vector and an* $n \times n$ *real matrix. Suppose also that* $\widetilde{\xi}(\cdot)$ *satisfies Walra's law. Then at least one of conditions (A) in the main text and*

(C) $\exists \epsilon \in \mathbb{R}^n \setminus \{0\}$ *such that* $\epsilon_{-i}A_{-i}\epsilon \neq 0$ *- where* A_{-i} *denotes the* $(n-1) \times n$ *matrix that results from* A *when its ith row is removed, -*

cannot hold.

Proof. Letting $(q, \varepsilon) \in Y \times \mathbb{R}_{++}$ be as in (A) and $\epsilon \in \mathbb{R}^n \setminus \{0\}$ be as in (C), take $\lambda \in (0,1)$ sufficiently small so that $\lambda ||\epsilon|| \leq 1$ and define the $(-\varepsilon, \varepsilon) \to Y$ function $q(\delta) := q + \delta\lambda\epsilon$. Using Walras' law again we now have

$$\begin{aligned} \delta\lambda &= w + \delta\lambda - w \\ &= \xi_i(q + \delta\lambda\epsilon) + (q_{-i} + \delta\lambda\epsilon_{-i})\,\xi_{-i}(q + \delta\lambda\epsilon) - (\xi_i(q) + q_{-i}\xi_{-i}(q)) \\ &= \xi_i(q + \delta\lambda\epsilon) - \xi_i(q) + q_{-i}(\xi_{-i}(q + \delta\lambda\epsilon) - \xi_{-i}(q)) + \delta\lambda\epsilon_{-i}\xi_{-i}(q + \delta\lambda\epsilon) \\ &= \delta\lambda\,(A_i^r\epsilon + q_{-i}A_{-i}\epsilon + \epsilon_{-i}\alpha_{-i} + \epsilon_{-i}A_{-i}q + \delta\lambda\epsilon_{-i}A_{-i}\epsilon) \end{aligned}$$

where A_i^r denotes the ith row of A. As the last equality above means that

$$\delta = \frac{1 - \left(A_i^r\epsilon + q_{-i}A_{-i}\epsilon + \epsilon_{-i}\alpha_{-i} + \epsilon_{-i}A_{-i}q\right)}{\lambda\epsilon_{-i}A_{-i}\epsilon} \quad \forall \delta \in (-\varepsilon, 0) \cup (0, \varepsilon)$$

the claim follows. ■

Remarks

(i). Notice that (B*) in the main text is a sufficient condition for hypothesis (C*) in Claim 1. To see this, suppose that condition (C*) above does not hold. We have then $\epsilon A \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting $\epsilon := e_i/2$ we get that $A_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. But then (B*) cannot hold.

(ii) Similarly, (B) in the main text is a sufficient conditon for hypothesis (C) in Claim 2. To see this, suppose that (C) above does not hold. We have then $\epsilon_{-i} A \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting $\epsilon := e_j/2$ we get that $A_{jj} = 0$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. But then (B) cannot obtain.

(iii). Condition (B*) [resp. (B)] in the main text is equivalent to the requirement that *one of the diagonal elements of A [resp. A_{-i}] is not zero*.

(iv). For hypothesis (C*) [resp. (C)] to hold, it suffices that *one of the principal minors of A [resp. A_{-i}] is symmetric*.
To see this for hypothesis (C*), suppose again that $\epsilon A \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting now $\epsilon := e_i + e_j$ for arbitrary $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we get that $A_{ii} + A_{ij} + A_{ji} + A_{jj} = 0$; i.e., that $A_{ij} + A_{ji} = 0$ (for, as observed above, we also have $A_{ii} = 0 = A_{jj}$).
For hypothesis (C), suppose again that $\epsilon_{-i} A_{-i} \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting now $\epsilon := e_j + e_k$ for arbitrary $j, k \in \{1, \ldots, n\} \setminus \{i\}$ with $j \neq k$, we get that $A_{jj} + A_{jk} + A_{kj} + A_{kk} = 0$; i.e., that $A_{jk} + A_{kj} = 0$ (for we also have $A_{jj} = 0 = A_{kk}$).

B Market demand for differentiated products

Vertically differentiated products

Letting $q_0 := 0$ and $\Delta q_{j-1} := q_j - q_{j-1}$, the market demand for vertically differentiated products in Amir et al. (2016) is given by $x(p) = \alpha + Bp$ where $\alpha^\intercal = (0, \ldots, 0, 1)$ and

$$B = -\begin{pmatrix}
\frac{1}{\Delta q_0} + \frac{1}{\Delta q_1} & \frac{-1}{\Delta q_1} & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\frac{-1}{\Delta q_1} & \frac{1}{\Delta q_1} + \frac{1}{\Delta q_2} & \frac{-1}{\Delta q_2} & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \frac{-1}{\Delta q_{j-1}} & \frac{1}{\Delta q_{j-1}} + \frac{1}{\Delta q_j} & \frac{-1}{\Delta q_j} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{-1}{\Delta q_{k-2}} & \frac{1}{\Delta q_{k-2}} + \frac{1}{\Delta q_{k-1}} & \frac{-1}{\Delta q_{k-1}} \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \frac{-1}{\Delta q_{k-1}} & \frac{1}{\Delta q_{k-1}}
\end{pmatrix}$$

This matrix is symmetric. To see that it is also negative definite, observe that any $x \in \mathbb{R}^k \setminus \{0\}$ gives

$$Bx = \begin{pmatrix} -\left(\frac{1}{\Delta q_0} + \frac{1}{\Delta q_1}\right) x_1 + \frac{x_2}{\Delta q_1} \\ \frac{x_1}{\Delta q_1} - \left(\frac{1}{\Delta q_1} + \frac{1}{\Delta q_2}\right) x_2 + \frac{x_3}{\Delta q_2} \\ \vdots \\ \frac{x_{k-2}}{\Delta q_{k-2}} - \left(\frac{1}{\Delta q_{k-2}} + \frac{1}{\Delta q_{k-1}}\right) x_{k-1} + \frac{x_k}{\Delta q_{k-1}} \\ \frac{x_{k-1}}{\Delta q_{k-1}} - \frac{x_k}{\Delta q_{k-1}} \end{pmatrix}$$

Thus, we have

$$xBx = -\sum_{j=1}^{k-1} \left(\frac{1}{\Delta q_{j-1}} + \frac{1}{\Delta q_j}\right) x_j^2 + 2\sum_{j=2}^{k} \frac{x_{j-1}x_j}{\Delta q_{j-1}} - \frac{x_k^2}{\Delta q_{k-1}} = -\sum_{j=2}^{k} \frac{\left(x_{j-1} - x_j\right)^2}{\Delta q_{j-1}} - \sum_{j=1}^{k-1} \frac{x_j^2}{\Delta q_j} < 0$$

as required.

Horizontally differentiated products

The market demand for horizontally differentiated products in Amir et al. (2016) is given by $x\left(p\right) = \alpha + Bp$ where

$$\alpha = \frac{1}{2\tau} \begin{pmatrix} (k-1)\left(\tau + v_1\right) - \sum_{j=2}^{k} v_j \\ \tau - v_1 + 3v_2 \\ \vdots \\ \tau - v_1 + 3v_k \end{pmatrix}$$

and

$$B = \frac{1}{2\tau} \begin{pmatrix} -\left(k-1\right) & 1 & \dots & \dots & \dots & \dots & \dots & 1 \\ 1 & -3 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & -3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & \dots & \dots & \dots & 0 & -3 \end{pmatrix}$$

This matrix is symmetric. To see that it is also negative definite, observe that any $x \in \mathbb{R}^k \setminus \{0\}$ gives

$$Bx = \begin{pmatrix} -(k-1)\,x_1 + \sum_{j \in K\setminus\{1\}} x_j \\ x_1 - 3x_2 \\ \vdots \\ x_1 - 3x_k \end{pmatrix}$$

Thus, we have

$$xBx = -(k-1)\,x_1^2 + 2\sum_{j\in K\setminus\{1\}} x_1 x_j - 3\sum_{j\in K\setminus\{1\}} x_j^2 = -2\sum_{j\in K\setminus\{1\}} x_j^2 - \sum_{j\in K\setminus\{1\}} \left(x_1 - x_j\right)^2 < 0$$

as required.

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