Should we increase or decrease public debt? Optimal fiscal policy with heterogeneous agents*

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Abstract

We analyze optimal fiscal policy in a heterogeneous-agent model with capital accumulation and aggregate shocks, where the government uses public debt, capital tax and non-linear labor tax to finance public spending. First, we prove that the existence of a steady-state equilibrium depends on three conditions, which have different economic interpretations: a Laffer condition, a Blanchard-Kahn condition and a Straub-Werning condition. We identify two new results in a simplified version of the model. First, we show that the equilibrium can feature both a positive level of public debt and a positive capital tax at the steady state. Second, we prove that optimal public debt increases if persistence of a positive public spending shock is low, whereas it decreases when the persistence is high. We show that our results still hold in a quantitative version of the model, where the optimal dynamics of the whole set of fiscal tools is analyzed. The quantitative model also provides new results on optimal tax progressivity and on the size of the fiscal multiplier.

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1 Introduction

What is the optimal level of public debt? Should it increase or decrease when public spending is increasing? Should the government increase temporarily capital tax or other distorting taxes, affecting the progressivity of the tax system? These old questions are likely to stay relevant in the coming years in many countries. Heterogeneous-agent models are a useful laboratory to explore such questions, as they include both necessary general equilibrium considerations and detailed redistributive effects, as in the Bewley-Huggett-Imohoroglu-Aiyagari literature (Bewley, 1983; Imrohoroğlu, 1989; Huggett, 1993; Aiyagari, 1994; Krusell and Smith, 1998). We thus study the optimal time-varying fiscal policy after a public spending shock in an heterogeneous-agent model with capital and labor tax, public debt and where the tax progressivity can be time-varying.

This analysis first requires some clarifications about optimal fiscal policy in heterogeneous-agent models, both for the steady state and for the dynamics. Indeed, some contributions, reviewed below, have questioned the ability of such model to deliver relevant insights regarding fiscal policies, in the context of Ramsey program with commitment. Hence, we first solve a tractable model, where optimal policies can be analytically derived, and then we show that the results are preserved in a quantitative model. The tractable model relies on deterministic income fluctuations and possibly occasionally-binding credit constraints, in the spirit of Woodford (1990), and on a utility function exhibiting no wealth effect of labor supply as in Diamond (1998).[1] The Ricardian equivalence does not hold when the planner cannot use lump-sum taxes, consistently with Bhandari et al. (2017), and thus public debt is uniquely determined. We prove three results. First, we show that a steady state equilibrium exists if three conditions are fulfilled: A Laffer condition, a Straub-Werning condition, and a Blanchard Kahn condition. These three conditions have different economic interpretation: The Laffer condition states that public spending should not be too high, otherwise distorting taxes cannot levy enough resources. The Straub-Werning condition, elaborating on Straub and Werning (2020) states that the public spending must be low enough, otherwise the planner wants to deviate from the steady state by decreasing the capital stock (although it could levy enough resources at the steady state). The Blanchard-Kahn condition is a stability condition, that requires the planner not to deviate from the steady state by increasing the capital stock. In addition, we identify two new thresholds. The first one is a cut-off level of public spending below which both the optimal level of capital tax and public debt are positive in a stable steady-state equilibrium. This result confirms the claim of Aiyagari (1995) and Aiyagari and McGrattan (1998) that heterogeneous-agent model can deliver positive capital tax and public debt as an optimal outcome. Interestingly, this result only depends on occasionally-binding credit constraints, and not on market incompleteness (as income fluctuations are deterministic). The second threshold concerns the persistence of the public spending shock.

[1] This environment allows studying the concavity of the problem and the qualification of the constraints, and the dynamics stability of the steady state, when it exists.

We show that for when the shock has a low persistence, public debt increases on impact, whereas it decreases when the persistence is high. Importantly, in both cases, the initial positive shock is identical. The reason for this result is that when the persistence is low, an increase in public debt allows the planner to smooth tax. A small increase in taxes finances the public debt reduction after its initial increase. When the persistence is high, the planner wants to decrease public debt, and related future interest repayment. This mitigates the long-lasting increase in public spending.

We then show that these properties remain valid in a quantitative heterogeneous-agents model with aggregate spending shock, where utility function is general, and where the planner can use a non-linear labor tax, a capital tax, and public debt to finance a temporary increase in public debt. We use and improve the methodology developed in LeGrand and Ragot (2022) to derive the first-order conditions of the planner and simulate the dynamics of the optimal allocation. We follow Heathcote et al. (2017) representation of the US tax system. In the same vein as Heathcote and Tsujiyama (2021), we estimate the social welfare function to reproduce a realistic steady-state fiscal system, calibrated on the US economy. Starting from this economy, we implement a temporary increase in public spending. Public debt is found to increase when the low persistence of public spending shock is low, while its increases for higher persistence. We find that the cut-off value for persistence is approximately 0.81, when the public spending shock follow a first-order autoregressive process.[2] In addition, we show that the size of the cumulative multiplier on public spending depends on the persistence, where higher persistence implies higher cumulative multiplier.

The paper is related to the recent literature on optimal policies in heterogeneous-agents model. It is first related to tractable models, allowing to derive optimal policies, (Bilbiie, 2008, Gottardi et al., 2014 Heathcote et al., 2017, Bilbiie and Ragot, 2020, Acharya et al., 2020, Heathcote and Tsujiyama, 2021 among many others). In this literature, we find that the framework of Woodford (1990) is particularly useful to study optimal fiscal policy.

Second, there is a recent, and relatively thin, quantitative literature studying optimal Ramsey policies in heterogeneous-agents models considering transitions (e.g., Conesa et al., 2009, Açikgöz et al., 2018, Dyrda and Pedroni, 2018, Nuño and Thomas, 2020, Bhandari et al., 2020). In this literature, we use LeGrand and Ragot (2022) who use a Lagrangian approach (taken from Marcet and Marimon, 2019) to derive the first-order conditions of the planner. Following, the literature we assume that the solution is interior and follow a first-order approach. We then use a truncation procedure to simulate the model. In LeGrand et al. (2021), we consider optimal monetary-fiscal policy in a nominal framework featuring price rigidities, and show that monetary tools are redundant when a rich set of fiscal tools are available. In the current paper, we derive new results on equilibrium properties and on the optimal dynamics of public debt.

[2]The annual persistence of a standard public spending shock is 0.89 on US data – see Chari et al. (1994) or Farhi (2010).

Finally, the paper is related to the literature on optimal capital taxation (Chari et al., 1994, Farhi, 2010, Chari et al., 2016, or Straub and Werning, 2020 among other), optimal tax redistribution and progressivity (e.g., Bassetto, 2014 or Heathcote et al., 2017), and the size of the fiscal multiplier in heterogeneous-agent models (Ferriere and Navarro, 2020).

The rest of the paper is organized as follows. In Section 2, we present the general environment. We solve the tractable model in Section 3. The general model is solved in Section 4. Section 5 concludes.

2 The environment

Time is discrete and indexed by $t = 0, 1, \ldots$ The economy is populated a continuum of mass two of agents distributed along a set I with measure ℓ. We follow Green (1994) and assume that the law of large numbers holds. The economy features production and a benevolent government that raises discretionary taxes to finance exogenous public spending.

2.1 Risks

The economy is plagued by two risks: an aggregate risk and an idiosyncratic risk. The aggregate shock solely affects public spending, denoted by $(G_t)_{t\geq 0}$ and is therefore assimilated to public spending. Furthermore, we assume that the whole path of public spending $(G_t)_{t\geq 0}$ becomes known to all agents in period 0. We will solve for the optimal adjustment of economy after such a shock, also known as a MIT shock.[3]

In addition to aggregate risk, agents face an uninsurable productivity risk, denoted by y. Individual productivity levels follow independent first-order Markov chains, whose state-space is the finite set $\{y_1, \ldots, y_K\}$ and the transition matrix is denoted by Π. We assume that the Markov chain admits a stationary distribution that is denoted by the K-dimensional vector n^y, verifying $n^y = \Pi n^y$.[4] When an agent is endowed with productivity y, she will earn a before-tax labor wage $\tilde{w}yl$, where l denotes her labor supply and $\tilde{w}$ is the before-tax hourly wage.

2.2 Production

The production sector is standard. The unique consumption good of the economy is produced by a profit-maximizing representative firm. At any date t, the firm production function combines labor L_t and capital K_{t-1} – that needs to be installed one period in advance – to produce Y_t units of the consumption good. The production function is assumed to be of the Cobb-Douglas

[3]It is known that one can derive a first-order approximation of the dynamics of the model in the presence of aggregate shocks, using the information obtained from MIT shocks (Boppart et al., 2018, Auclert et al., 2019)

[4]In the quantitative analysis of Section 4, the Markov chain can be shown to be irreducible and aperiodic – hence n^y exists and is unique. In the theoretical investigation of Section 3, the matrix is anti-diagonal.

type featuring constant returns to scale and capital depreciation. The TFP is normalized to one. Formally, the production is defined as:

$$Y_t = F(K_{t-1}, L_t) = K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1},$$

where $\alpha \in (0,1)$ is the capital share and $\delta \in (0,1)$ the capital depreciation rate.

The firm rents labor and capital at respective factor prices $\tilde{w}_t$ and $\tilde{r}_t$. The profit maximization conditions of the firm implies the following expressions for factor prices:

$$\tilde{w}_t = F_L(K_{t-1}, L_t) \text{ and } \tilde{r}_t = F_K(K_{t-1}, L_t). \tag{1}$$

2.3 Assets

In addition to capital, the economy also features public debt, whose size is denoted by B_t in period t. Public debt consists of one-period bonds issued by a benevolent government, that are assumed default-free. We assume the existence of a risk-neutral financial intermediary that collects the whole stock of public debt and capital. This intermediary issues shares that are the sole tradable assets for agents. This market arrangement allows to consider two different asset classes, without a portfolio choice (e.g., Gornemann et al., 2016, Bhandari et al., 2020). We will denote by a_t the agents' holdings in fund shares. We assume that agents are prevented from borrowing more than the exogenous amount $\overline{a}$.

Finally, the absence of arbitrage of the fund no-profit condition imply that the fund shares and public debt must pay the same interest rate as capital. There is therefore a unique (before-tax) interest rate $\tilde{r}_t$ at date t in the economy.

2.4 Government

A benevolent government has to finance the exogenous stream of public spending $(G_t)_{t\geq 0}$, by levying distortionary taxes on capital and labor and issuing public debt. The tax on capital is linear, with a rate $(\tau_t^K)_{t\geq 0}$ and levied on fund shares holdings of agents. There is no distinction between public debt bonds or capital shares for taxation.[5] The tax on labor income is assumed to be non-linear, and possibly time-varying. We denote by $T_t(\tilde{w}yl)$ the amount of labor tax paid by an agent earning the labor income $\tilde{w}yl$ by supplying l hours at a wage rate $\tilde{w}$ and a productivity y. We follow Heathcote et al. (2017) (henceforth, HSV) and consider the following functional form:

$$T_t(\tilde{w}yl) := \tilde{w}yl - \kappa_t(\tilde{w}yl)^{1-\tau_t}, \tag{2}$$

where κ captures the level of labor taxation and τ the progressivity. Both parameters are assumed to be time-varying and will be planner's instruments. When $\tau_t = 0$, labor tax is linear with rate

[5] The financial intermediary is not taxed and simply considered as a market arrangement.

$1-\kappa_t$. Oppositely, the case $\tau_t = 1$ corresponds to full income redistribution, where all agents earn the same post-tax income κ_t. Functional form (2), combined with the linear capital tax, allows one to realistically reproduce the actual US system and its progressivity (see Ferriere and Navarro, 2020).[6]

Using the public debt description of Section 2.3, the government budget constraint can thus be written as:

$$G_t + (1+\tilde{r}_t)B_{t-1} = \int T_t(\tilde{w}_t y^i l_t^i)\ell(di) + \tau_t^K \tilde{r}_t(B_{t-1} + K_{t-1}) + B_t. \tag{3}$$

To simplify the government budget constraint, we introduce in the spirit of Chamley (1986), generalized post-tax factor prices, that are denoted without a tilde. We define the gross and net interest rates r_t and R_t, as well as the wage rate w_t, as follows:

$$w_t := \kappa_t(\tilde{w}_t)^{1-\tau_t}, \tag{4}$$

$$R_t := 1 + r_t = 1 + (1-\tau_t^K)\tilde{r}_t. \tag{5}$$

The model can analytically be expressed using the pair of post-tax rates (R_t, w_t) rather than pre-tax ones $(\tilde{r}_t, \tilde{w}_t)$. This considerably simplifies the model exposition and its tractability. The values of the fiscal instruments τ_t^K, κ_t, and τ_t can then be recovered from the allocation.

With the post-tax notation and taking advantage of the property of homogeneity of the production function, we deduce that the governmental budget constraint (3) can also be written as follows:

$$G_t + R_t B_{t-1} + (R_t - 1)K_{t-1} + w_t \int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di) = F(K_{t-1}, L_t) + B_t. \tag{6}$$

2.5 Agents' program and resource constraints

At each date t, agents consume a unique good in quantity c_t and supply labor in quantity l_t. They derive an instantaneous utility from consumption and labor supply denoted by $U(c_t, l_t)$. The utility function will be specified later on. Agents are expected utility maximizers with standard additive intertemporal preferences. The discount factor is constant and denoted $\beta \in (0,1)$. Agents maximize at date 0 the expected discounted value of future utilities, equal to $\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t, l_t)\right]$, where $\mathbb{E}_0$ is the unconditional expectation over the aggregate risk and over the agent's own idiosyncratic risk.

Agents can save When choosing their plans for consumption $(c_t)_{t\geq 0}$, labor supply $(l_t)_{t\geq 0}$, and savings (in fund shares) $(a_t)_{t\geq 0}$ to maximize their expected utility, agents face two constraints: (i) a budget constraint, and (ii) a credit constraint. Their budget constraints states that

[6] The literature uses either the combination of a linear tax and of a lump-sum transfer (e.g., Dyrda and Pedroni, 2018, Açikgöz et al., 2018) or the HSV structure. Heathcote and Tsujiyama (2021) show that the HSV structure is quantitatively more relevant. Opting for the HSV tax structure enables us to discuss t he dynamics of optimal tax progressivity, following a public spending shock.

agents' consumption and savings should be solely financed out of net labor income and net capital income. Using the post-tax rate definition (4), the post-tax labor income amounts to $\tilde{w}_t y_t^i l_t^i - T_t(\tilde{w}_t y_t^i l_t^i) = w_t (y_t^i l_t^i)^{1-\tau_t}$ for an agent supplying labor l_t^i with productivity y_t^i. The post-tax capital income is equal to $R_t a_{t-1}^i$ for an agent with beginning-of-period wealth a_{t-1}^i. Formally, the program of an agent i can be expressed as:

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t^i, l_t^i), \tag{7}$$

$$c_t^i + a_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \tag{8}$$

$$a_t^i \geq -\underline{a},\ c_t^i > 0,\ l_t^i > 0. \tag{9}$$

Denoting by $\beta^t \nu_t^i \geq 0$ the Lagrange multiplier on the agent's credit constraint, the consumption Euler equation can be written as:

$$U_c(c_t^i, l_t^i) = \beta \mathbb{E}_t \left[R_t U_c(c_{t+1}^i, l_{t+1}^i) \right] + \nu_t^i, \tag{10}$$

where U_c and U_l denote the derivatives of U with respect to the first and second variables, respectively. Note that, because of our assumption of MIT shocks, the expectation operator in (10) as well as in the rest solely concerns idiosyncratic shocks.

The labor Euler equation yields:

$$-U_l(c_t^i, l_t^i) = (1-\tau_t) w_t (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i). \tag{11}$$

We now express economy-wide constraints. The clearing of financial and labor markets implies the following relationships:

$$A_t = K_t + B_t \text{ and } \int y_t^i l_t^i \ell(di) = L_t. \tag{12}$$

The clearing of the goods market reflects that the sum of aggregate consumption, public spending and new capital stock balances the output production and past capital:

$$\int_i c_t^i \ell(di) + G_t + K_t = K_{t-1} + F(A_t, K_{t-1}, L_t). \tag{13}$$

We can now formulate our definition of a sequential equilibrium in this economy.

Definition 1 (Competitive equilibrium) *A competitive equilibrium is a collection of individual variables $(c_t^i, l_t^i, a_t^i)_{t\geq 0, i\in\mathcal{I}}$, of aggregate quantities $(K_t, L_t, Y_t)_{t\geq 0}$, of price processes $(\tilde{w}_t, \tilde{r}_t)_{t\geq 0}$, of fiscal policy $(\tau_t^K, \kappa_t, \tau_t, B_t)_{t\geq 0}$ and of public spending $(G_t)_{t\geq 0}$ such that, for an initial distribution of wealth and productivity $(a_{-1}^i, y_0^i)_{i\in\mathcal{I}}$, and for initial values of the aggregate shock z_0 and of capital stock and public debt verifying $K_{-1} + B_{-1} = \int_i a_{-1}^i \ell(di)$, we have:*

1. *given prices, individual strategies $(c_t^i, l_t^i, a_t^i)_{t\geq 0, i\in\mathcal{I}}$ solve the agent's optimization program in*

equations (7)–(9);

2. *financial, labor, and goods markets clear: for any $t \geq 0$, equations (12) and (13) hold;*

3. *the government budget is balanced: equation (3) holds for all $t \geq 0$;*

4. *pre-tax factor prices $(\tilde{w}_t, \tilde{r}_t)_{t\geq 0}$ are consistent with the firm's program (1);*

2.6 The Ramsey equilibrium

The Ramsey program consists in characterizing the fiscal policy that corresponds to the competitive equilibrium with the highest aggregate welfare. This problem is difficult. The labor tax directly affects the labor supply, the capital tax directly affects the saving incentives, public debt directly affects the capital stock for a given total private saving. All these instruments have general equilibrium effect on prices and thus on the welfare of heterogeneous agents. As mentioned in the introduction, the existence of stationary equilibria with strictly positive values for the instrument is an open question. We first provide a characterization in a simple environment, before presenting quantitative investigation.

We now turn to the formal expression of the Ramsey program. The first step is thus to define an aggregate welfare criterion. We assume that the aggregate welfare is the weighted sum of individual intertemporal utilities. The weight attached to a given agent i at date t is assumed to depend on their productivity at date t: $\omega_t^i := \omega(y_t^i)$, as in Heathcote and Tsujiyama (2021). In consequence, two agents sharing the same productivity will have the same weight. Formally, the planner's aggregate welfare criterion can be expressed as:

$$W_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \int_i \omega_t^i U(c_t^i, l_t^i) \ell(di) \right]. \tag{14}$$

The Ramsey problem thus consists in choosing the fiscal instruments $(\tau_t^K, \kappa_t, \tau_t, B_t)_{t\geq 0}$ (as a function of the realization of the aggregate shock and of the initial distribution of the state variables of agents) which correspond to the competitive equilibrium with the highest aggregate welfare. Formally, the Ramsey program ca, be written as follows:

$$\max_{\left(r_t,w_t,B_t,K_t,L_t,(a_t^i,c_t^i,l_t^i,\nu_t^i)_i\right)_{t\geq 0}} \sum_{t=0}^{\infty}\beta^t \int_i \omega_t^i(u(c_t^i)-v(l_t^i))\ell(di), \tag{15}$$

$$\tag{16}$$

$$G_t + R_tB_{t-1} + (R_t-1)K_{t-1} + w_t\int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di) = F(K_{t-1},L_t) + B_t \tag{17}$$

$$\text{for all } i\in\mathcal{I}\text{: } a_t^i + c_t^i = R_t a_{t-1}^i + w_t(y_t^i l_t^i)^{1-\tau_t}, \tag{18}$$

$$a_t^i \geq -\bar{a},\ \nu_t^i(a_t^i+\bar{a}) = 0,\ \nu_t^i \geq 0, \tag{19}$$

$$U_c(c_t^i,l_t^i) = \beta\mathbb{E}_t\left[R_tU_c(c_{t+1}^i,l_{t+1}^i)\right] + \nu_t^i, \tag{20}$$

$$-U_l(c_t^i,l_t^i) = (1-\tau_t)w_t y_t^i(y_t^i l_t^i)^{-\tau_t}U_c(c_t^i,l_t^i), \tag{21}$$

$$K_t + B_t = \int_i a_t^i\ell(di),\ L_t = \int_i y_t^i l_t^i \ell\left(di\right), \tag{22}$$

Formally, the Ramsey program consists for the planner to maximize aggregate welfare W_0 subject to the governmental budget constraint (16) and to the constraints characterizing the competitive equilibrium: individual budget constraints (8), individual Euler equations (10) and (11), individual credit and positivity constraints (9), market clearing conditions (12) and factor price definitions (1), (4), and (5). We solve this program using a Lagrangian approach, presented in LeGrand and Ragot (2022).[7]

We denote as $\beta^t\lambda_{c,t}^i$ the Lagrange multiplier on the period t Euler equation of agents i, equation (20). When the credit constraints of agents i is binding $a_t^i = -\bar{a}$, and $\lambda_{c,t}^i = 0$, as the Euler equation is not a constraint. It is shown in LeGrand and Ragot (2022) that (when the credit constraint does not bind), the equilibrium can feature either $\lambda_{c,t}^i > 0$ or $\lambda_{c,t}^i < 0$ depending on whether the agents save too much or too little *seen from* the planner perspective. Similarly, we denote by $\beta^t\lambda_{l,t}^i$, the Lagrange multiplier on the labor supply (21), and by $\beta^t\mu_t$ the Lagrange multiplier on the government budget constraint (16)

To save some place, we derive the first-order conditions of the planner in Appendix A. Note that we follow the literature and assume the solution are interiors and first-order conditions of the planner are sufficient to characterize the optimal allocation, as Straub and Werning (2020) among many others. We provide some quantitative checks below.

To simplify the interpretation of the first-order conditions of the Ramsey program, we

[7] In LeGrand and Ragot (2022), we show that this method can be used with occasionally binding credit constraints, taking limits of penalty functions. See also Açikgöz et al. (2018) to solve for policies with a utilitarian social welfare function.

introduce the marginal social valuation of liquidity for agent i, defined as:

$$\psi_t^i := \omega_t^i U_c(c_t^i, l_t^i) - \left(\lambda_{c,t}^i - (1+r_t)\lambda_{c,t-1}^i\right) U_{cc}(c_t^i, l_t^i) \\ + \lambda_{l,t}^i \left(U_{cl}(c_t^i, l_t^i) - (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} U_{cc}(c_t^i, l_t^i)\right). \tag{23}$$

This complex expression has a simple interpretation. It is the net value for the planner of transferring one unit of resources to agents i (if it could). First, the gain for the planner would be to increase marginal utility, bedighted with the relevant weight $\omega_t^i U_c(c_t^i, l_t^i)$. Second, one additional unit of resources to agent i changes the incentive to save from period $t-1$ to period t, captured by the term with $\lambda_{c,t-1}^i$. Third, it also affects the incentive to save from period t to period $t+1$, captured by the term with $\lambda_{c,t}^i$. Fourth, it affects the incentive to work , captured by the terms in $\lambda_{l,t}^i$. For these last three terms, the effect is multiplied by the marginal change in the marginal utility of consumption, which is the term $U_{cc}(c_t^i, l_t^i)$.

From (23), we also define the net social valuation of liquidity than accounts for the opportunity cost of liquidity, measured by the Lagrange multiplier :

$$\hat{\psi}_t^i := \psi_t^i - \mu_t. \tag{24}$$

With this notation, the first order conditions of the planner can be easily interpreted. First, for an unconstrained agent i, the planner implements a liquidity smoothing condition:

$$\hat{\psi}_t^i = \beta \mathbb{E}_t \left[R_{t+1} \hat{\psi}_{t+1}^i\right]. \tag{25}$$

Equation (25) is a generalized version of the Euler equation (10) (and it is actually the same equation, when all Lagrange multipliers are 0), in which the planner internalizes in the definition of $\hat{\psi}_t^i$ the general equilibrium externalities when setting individual savings.

The first-order condition with respect to labor can be written as:

$$\psi_{l,t}^i = (1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i \\ + \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t) \tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i, \tag{26}$$

where we have defined:

$$\psi_{l,t}^i := -\omega_t^i U_l(c_t^i, l_t^i) - \lambda_{l,t}^i U_{ll}(c_t^i, l_t^i) \\ + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}) U_{cl}(c_t^i, l_t^i). \tag{27}$$

Similarly to ψ_t^i for consumption, the quantity $\psi_{l,t}^i$ is the social marginal value of labor supply by agent i. The Ramsey first-order condition (26) is a generalized version of the labor Euler equation (11).

The first-order condition with respect to public debt can be written as:

$$\mu_t = \beta(1 + \tilde{r}_{t+1})\mu_{t+1}, \tag{28}$$

without expectation operator thanks to the MIT shock assumption. Equation (28) shows that the planner aims at smoothing the shadow cost of the government budget constraint through time.

The other first-order conditions with respect to R_t, w_t, and τ_t can respectively be written as:

$$0 = \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j)\right) \ell(dj), \tag{29}$$

$$0 = \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ell(dj), \tag{30}$$

$$\begin{aligned} 0 = &\int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ln(y_t^j l_t^j)(dj) \\ &+ \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j) \ell(dj). \end{aligned} \tag{31}$$

All these equations have a similar interpretation. They involve equalizing the net valuation of liquidity weighted aggregated over the whole population with the relevant weight (e.g., $\int_j \hat{\psi}_t^j a_{t-1}^j \ell(dj)$ in the case of the interest rate) to the general-equilibrium distortion of the instrument (e.g., distortion of savings incentives for the interest rate).

The analytical characterization of the dynamics is a first step to determine the optimal policy. However, standard recursive techniques cannot be used to compute the policy. The problem of the planner could be written recursively, but in this case the state space would include the joint distribution of beginning-of-period wealth and Lagrange multipliers on consumption Euler equations (i.e., the joint distribution of $(a_{t-1}^i, \lambda_{c,t-1}^i)_i$). Indeed, beginning-of-period wealth a_{t-1}^i and past value of the Lagrange multiplier λ_{t-1}^i both appear in the first-order conditions of the Ramsey program. To compute the solution, we again follow LeGrand and Ragot (2022) and we consider a truncated representation of this problem. We provide the details of the computational implementation – including all the required analytical developments –in Appendix E. This numerical solution can be of independent interest as the solution of this type of Ramsey problem is both new and not straightforward.

3 Analytical results

In this section, we derive analytical results regarding the Ramsey equilibrium of Section 2.6.

3.1 Model specification

To obtain a tractable framework we specify a number of model aspects. These specifications are only valid in the analytical analysis of this section and a more general framework will be considered in the quantitative exercise of Section 4. The first assumption concerns the functional form of labor taxes that are assumed to be linear.

Assumption A *We assume that the labor tax is linear. Formally, we set in (2)* $\tau_t = 0$ *and denote* $\tau_t^L := 1 - \kappa_t$*, such that:*

$$T_t(wyl) := \tau_t^L wyl.$$

Our second assumption is about the specification utility function.

Assumption B *We assume that the instantaneous utility function* U *is of the GHH-type:*

$$U(c,l) := \ln\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}\right),$$

where $\varphi > 0$ *is the Frisch elasticity of labor supply, and* $\chi > 0$ *scales labor disutility.*

Assumption B simplifies the algebra for the Ramsey program by avoiding wealth effects for the labor supply. The log function implies that income and substitution effects exactly compensate each other.

The third assumption is about the productivity process.

Assumption C *We assume that there are only two productivity levels, equal to zero and one respectively:* $\mathcal{Y} = \{0, 1\}$*. Furthermore, the transition matrix is anti-diagonal:*

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{32}$$

while the initial distribution is such that: (i) a mass one of agents have productivity 1 with an identical beginning-of-period wealth; and (ii) a mass one of agents have productivity 0 with an identical beginning-of-period wealth (but possibly different from the one of employed agents).

The main implication of Assumption C is to simplify the equilibrium wealth distribution. First, there are only two productivity levels. The first one corresponds to a null productivity, and hence to a null labor supply. This zero productivity state will be called unemployment. The other productivity level is normalized to one and will correspond to employment. Second, equation (32) implies that the transitions to and out of unemployment are deterministic. Currently unemployed agents become employed in the next period and the other way around. Coupled with the assumption regarding the initial wealth distribution, Assumption C implies that at any date, the equilibrium only features two types of agents and two wealth levels. Our setup is thus similar to the one of Woodford (1990), in which 2 agents switch deterministically between employment and unemployment. For the sake of simplicity, the two types of agents will be called according to their current employment status: "employed" (subscript e) and unemployed (subscript u).

The fourth and last assumption is about the credit constraint.

Assumption D *The credit-constraint is normalized to zero:* $\underline{a} = 0$.

3.2 The Ramsey program

Taking advantage of Assumptions B–D, we specify further the environment. Using the peculiar equilibrium structure, the individual budget constraints (8) become:

$$c_{e,t} + a_{e,t} = R_t a_{u,t-1} + w_t l_{e,t}, \tag{33}$$

$$c_{u,t} + a_{u,t} = R_t a_{e,t-1}, \tag{34}$$

for employed (subscript e) and unemployed (subscript u), respectively. Note that the definitions (4) and (5) of the post-tax rates R_t and w_t are still valid (with $\tau_t = 0$ and $\kappa_t = 1 - \tau_t^L$). We can already state a first result regarding employed agents.

Result 1. *In any equilibrium, employed agents cannot be credit-constrained at any date.*

This is a direct consequence of budget constraint (34) with $c_{u,t} > 0$. Should we have $a_{e,t} = 0$ at some date t, we would have $c_{u,t+1} = -a_{u,t+1} \leq 0$, which would contradict the consumption positivity constraint. A consequence of Result 1. is that we only have two possible types of (steady-state) equilibria: one in which unemployed agents are not constrained, and one in which they are.

Taking advantage of the GHH property of the utility function and of the linearity of labor taxes, the labor Euler equation (11) for employed agents simplifies into:

$$l_{e,t} = (\chi w_t)^{\varphi}, \tag{35}$$

which only depends on the hourly wage w_t. The labor and financial market clearing conditions become in this set-up:

$$L_t = l_{e,t} \text{ and } B_t + K_t = a_{e,t} + a_{u,t}. \tag{36}$$

The governmental budget constraint (6) can be simplified using (35) and (36) as follows:

$$\begin{aligned} G_t + B_{t-1} + (R_t - 1)(a_{e,t-1} + a_{u,t-1}) + w_t(\chi w_t)^{\varphi} = \\ B_t + F(A_t, a_{e,t-1} + a_{u,t-1} - B_{t-1}, (\chi w_t)^{\varphi}). \end{aligned} \tag{37}$$

Finally, using budget constraints (33) and (34) and labor Euler equation (35), we deduce that Euler equations for consumption (10) can be expressed as:

$$\beta\left(R_t a_{u,t-1} - a_{e,t} + \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi}\right) = a_{e,t} - a_{u,t+1}/R_{t+1}, \tag{38}$$

$$R_{t+1}a_{u,t} - a_{e,t+1} + \frac{w_{t+1}(\chi w_{t+1})^{\varphi}}{1+\varphi} \geq \beta R_{t+1}(R_t a_{e,t-1} - a_{u,t}), \tag{39}$$

where the first Euler equation holds with equality at all dates as a consequence of Result 1.. Expectations have been dropped from Euler equations due to MIT shocks and the deterministic transition between income levels.

We now investigate the two possible types of Ramsey equilibria: one in which unemployed agents are not credit-constrained and that will correspond to the first best allocation, the other one in which unemployed agents are constrained.

3.3 The first-best equilibrium

The first-best allocation is characterized by perfect risk-sharing and no tax. In other words, no agent is credit-constrained and the Euler equation (39) of unemployed agent holds with equality. The public debt will be negative and will therefore consist of governmental asset holdings, whose payoffs will finance the governmental public spending. We will focus here on the steady state allocation only and we will use a FB subscript to denote first-best quantities. We already know that $\tau^K_{FB} = \tau^L_{FB} = 0$.

The first observation is that since the unemployed agent is no credit-constrained, her Euler equation (39) holds with equality. By combining the two Euler equations (38) and (39), we obtain: $(c_{u,FB})^{-1} = (\beta R_{FB})^2(c_{u,FB})^{-1}$, where $(c_{u,FB})^{-1} = Ra_{e,FB} - a_{u,FB} > 0$. This implies that we have:

$$\beta R_{FB} = 1 = \beta(1+\tilde{r}_{FB}), \tag{40}$$

where the second inequality comes from $\tau^K_{FB} = 0$. Since taxes are null, $\beta R_{FB} = 1$ first allows us to compute the public debt from the governmental budget constraint (3):

$$B_{FB} = -\frac{\beta}{1-\beta}G < 0, \tag{41}$$

reflecting that public debt is always negative. The government actually owns assets that enables it to finance public spending out of asset holding payoffs – at the gross rate β. Second, equality (40) allows us deduce from the definitions (1) of $\tilde{r}_t$ and $\tilde{w}_t$ he capital-to-labor ratio $k_{FB} := K_{FB}/L_{FB}$, the post-tax wage rate w_{FB}, and the GDP $Y_{FB} := K^{\alpha}_{FB}L^{1-\alpha}_{FB}$:

$$k_{FB} = \left(\alpha(\frac{1}{\beta}+\delta-1)^{-1}\right)^{\frac{1}{1-\alpha}},\ w_{FB} = (1-\alpha)k^{\alpha}_{FB},\ Y_{FB} = (\chi(1-\alpha))^{\varphi}k^{\alpha(1+\varphi)}_{FB}. \tag{42}$$

Note that in absence of tax, output, capital, and real wage are independent of public spending. They only depend on production and preference parameters. Combining equations in (42), with financial market clearing condition and Euler equation (38) allows us to compute agents' asset holdings:

$$a_{u,FB} = \frac{\beta(\overline{g}_1 - g_{FB})}{2(1-\beta)} Y_{FB}, \tag{43}$$

$$a_{e,FB} = a_{u,FB} + \frac{\beta}{1+\beta} \frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}, \tag{44}$$

where we have denoted:

$$\overline{g}_1 = \frac{1-\beta}{\beta} \frac{\alpha}{1/\beta + (\delta - 1)} - \frac{1-\beta}{1+\beta} \frac{1-\alpha}{\varphi+1}, \tag{45}$$

$$g_{FB} = \frac{G}{Y_{FB}}. \tag{46}$$

The quantity g_{FB} can be interpreted as the public spending-to-GDP ratio. Equation (44) shows that the gap in asset holdings between employed and unemployed agents is always positive – reflecting that employed save more than unemployed – and independent of public spending. Equation (43) implies that the savings of unemployed agents is a share of aggregate output Y_{FB} and that this share diminishes with public spending. The reason is that higher public spending implies a more negative public debt – or equivalently a larger public asset holding – which crowds out private savings. This crowding-out effect has two consequences. First, it harms aggregate welfare. Indeed, computing individual allocations from asset holdings (43) and (44) yields:

$$c_e - \frac{1}{\chi} \frac{l_e^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} = c_{u,FB} = \frac{1-\beta}{\beta} a_{u,FB} + \frac{1}{1+\beta} \frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}, \tag{47}$$

where perfect risk-sharing is reflected in the equal period utility levels for unemployed and employed agents. Equation (47) makes it clear that individual allocations – and hence aggregate welfare – diminishes with public spending, resulting from the crowding-out of private savings.

A second consequence of the crowding-out of private savings by public savings is that when public spending becomes too large (in particular when $g_{FB} > \overline{g}_1$), sustaining the first-best allocation would require unemployed agents to provide private liquidity to the government to allow it to hold a sufficiently large asset position to finance public spending. However, private borrowing being prevented by Assumption D, the first-best equilibrium can only exist for not-too-large levels of public spending. It stops existing when financing public spending requires the government to borrow from agents.

The following proposition summarizes these results.

Proposition 1 *If $g_{FB} \leq \overline{g}_1$, the Ramsey problem of Section 3.2 admits a first-best steady-state equilibrium characterized by equations (40)–(47) and that features: (i) zero taxes: $\tau^L_{FB} = \tau^K_{FB} = 0$,* *and (ii) a perfect risk-sharing between the two agent types.*

The proof can be found in Appendix B. A corollary of Proposition 1 is that a necessary condition for the first-best equilibrium does not exist when $\overline{g}_1 < 0$ – which is for instance always the case when α sufficiently small.

Corollary 1 *When $\overline{g}_1 < 0$, no first-best equilibrium exists, independently of the level of public spending.*

3.4 The equilibrium with binding credit constraints

We now turn to the only other equilibrium that admits a interior steady state.[8] To rule out the possibility of a first-best equilibrium, we make the following assumption.

Assumption E *We assume:*

$$g_{FB} > \overline{g}_1.$$

This equilibrium features binding credit constraint for unemployed agents. In that case, unemployed agents hold no asset at any date: $a_{u,t} = 0$. The Euler equation (38) of employed agents implies:

$$a_{e,t} = \frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} > 0, \tag{48}$$

which is positive whenever $w_t > 0$. Substituting the expression (48) of $a_{e,t}$ and using $a_{u,t} = 0$, the financial market clearing condition becomes:

$$B_t + K_t = \frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi}. \tag{49}$$

3.4.1 The Ramsey program

We now turn to the expression of the Ramsey program of Section 2.6. The first option would be to directly use the first-order conditions (25)–(31) that we derived in the general case and use them with the specifications of Assumptions A–D. One of the difficulty raised with this general approach is that the solution is subject to a number of caveats – notably because the optimization program includes non-linear constraints. To circumvent this difficulty, we take another route that takes advantage of the tractability of our analytical setup. This direct approach allows us to prove that some constraint qualifications hold, such that the optimum of the Ramsey program indeed solves the Karush–Kuhn–Tucker conditions – i.e., the FOC of the Lagrangian (see Section 3.4.2). A second benefit is that we can formally check that the solutions to the first-order conditions (computed in Section 3.4.3) are an actual maximum, since we can formally

[8]We explain in Section C.5 below that no other equilibrium with interior steady state exists.

check the second-order conditions (Section 3.4.4). We show in Appendix C.2 that the first-order conditions we derive with this new approach are identical to those derived in the general case.

To simplify the Ramsey program of Section 2.6, we proceed in two steps. First, we use individual budget constraints (33) and (34) and the Euler labor equation (35) to express the Ramsey program in terms of savings choices and of the three instruments of fiscal policy $(w_t, R_t, B_t)_{t\geq 0}$. Second, we use the savings expression (48) of employed agents (that reflects employed agents' Euler equation) and $a_{u,t} = 0$ to express the Ramsey solely as a function of the fiscal policy $(B_t, w_t, R_t)_{t\geq 0}$ and with a unique constraint (the government budget constraint):

$$\max_{(B_t,w_t,R_t)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log\left(\frac{1}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{\varphi+1}\right) + \log(R_t \frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi})\right), \tag{50}$$

$$\text{s.t. } G + B_{t-1} + (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} + w_t(\chi w_t)^{\varphi} = \tag{51}$$

$$F(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}) + B_t,$$

with furthermore the Euler inequality (39) stating than unemployed are actually credit-constrained. At the steady-state, this condition is equivalent to $\beta R < 1$ – which will always hold in this equilibrium. Two other constraints are implicit in the above program: (i) $w_t > 0$, and (ii) $R_t > 0$, which correspond to the positivity of consumption levels for employed and unemployed agents.

Before deriving first-order conditions of the Lagrangian problem, we show that the Karush–Kuhn–Tucker conditions can apply to our problem. To do so, we verify that the so-called constraint qualification hold in our set-up

3.4.2 Constraint qualification

In our problem, even though the objective function is concave, the equality constraints are not linear and the standard Slater (1950)'s conditions do not apply. However, we can check that the linear independence constraint qualification (LICQ) holds in our problem. This constraint qualification requires the gradients of equality constraints to be linearly independent at the optimum (or equivalently that the gradient is locally surjective). At any date t, two constraints matter for the instruments of date t. These are the constraints at dates t and $t+1$. We can check that their gradient can be written as:

$$\begin{pmatrix} 1 & \varphi(\chi w_t)^{\varphi}\frac{\tilde{w}_t}{w_t} - (\varphi+1)(\chi w_t)^{\varphi} & -\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} \\ -\tilde{r}_{t+1} - 1 & \frac{\beta}{1+\beta}(\chi w_t)^{\varphi}\tilde{r}_{t+1} - (R_{t+1}-1)\frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} & 0 \end{pmatrix}, \tag{52}$$

which forms a matrix of rank 2. Indeed, looking at first and third columns of the matrix in (52) makes it clear that a sufficient condition is $(1+\tilde{r}_{t+1})w_{t-1} \neq 0$. This condition must hold at the optimum, since: (i) equation (1) implies $\tilde{r}_{t+1} \geq 0$, and (ii) we must have $w_{t-1} > 0$.

3.4.3 First-order conditions

The FOCs associated to the Ramsey program (50)–(51) can be written as (for $t \geq 0$):

$$(1+\beta)(1+\varphi) = \left(1 - (1+\beta)\varphi \frac{\tau_t^L}{1-\tau_t^L}\right)\mu_t w_t (\chi w_t)^\varphi, \tag{53}$$

$$\mu_t = \beta(1+\tilde{r}_{t+1})\mu_{t+1}, \tag{54}$$

$$1 = R_t \mu_t \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi}, \tag{55}$$

where we still denote by $\beta^t \mu_t$ the Lagrange multiplier on the governmental budget constraint and also define w_{-1} as the solution of $a_{-1} = \frac{\beta}{1+\beta}\frac{w_{-1}(\chi w_{-1})^\varphi}{1+\varphi}$.

Equation (53) characterizes the labor tax, while (55) characterizes the capital tax. Equation (54) is an Euler-like equation for the Lagrange multiplier on the governmental budget constraint – and does not feature any expectation operator because of MIT shocks. We show in Appendix C.2 that the first-order conditions (53)–(55) are implied by those derived in the in the general case of Section 2.6.

An implication of the above FOCs and of $w_t, R_t > 0$ is that we must have at all dates $\mu_t > 0$, and $\tau_t^L < \frac{1}{1+(1+\beta)\varphi}$. We summarize this in the following result.

Result 2. *The equilibrium with a binding credit constraint features the following restrictions:*

$$w_t, R_t, \mu_t > 0 \text{ and } \tau_t^L < \overline{\tau}_{SW}^L, \tag{56}$$

$$\text{where: } \overline{\tau}_{SW}^L = \frac{1}{1+(1+\beta)\varphi} < 1. \tag{57}$$

The restriction on the labor tax implies that the labor tax cannot be too large for the Ramsey equilibrium with binding credit constraint to exist. This constraint is actually connected to the positivity of the Lagrange multiplier μ_t. Indeed, the FOC (53) makes it clear that we have:

$$\mu_t = \frac{(1+\beta)(1+\varphi)(1-\tau_t^L)}{\overline{\tau}_{SW}^L - \tau_t^L} \frac{\overline{\tau}_{SW}^L}{w_t(\chi w_t)^\varphi},$$

which becomes negative when $\overline{\tau}_{SW}^L < \tau_t^L < 1$. As we will make it clearer in later (see the discussion of Proposition 2), the mechanism when $\tau_t^L > \overline{\tau}_{SW}^L$ is similar to the one at play in Straub and Werning (2020). We will therefore refer to the constraint $\tau_t^L < \overline{\tau}_{SW}^L$ as the Straub-Werning (henceforth, SW) constraint.

FOC (55) implies that the SW constraint also relates to the positivity of the post-tax gross rate. Indeed, when the SW constraint does not hold and $\tau_t^L > \overline{\tau}_{SW}^L$, the gross interest rate is negative, which in turns implies a negative consumption for unemployed agents. The intuition is that when public spending becomes higher than a given threshold, their financing requires a capital tax that is so high that the gross post-tax interest rate becomes negative.

Before turning to the steady-state analysis, let us say a word about second-order conditions and the fact that our first-order conditions are actually picking up a local maximum.

3.4.4 Second-order conditions

In the program (50)–(51), we can use the constraint (51) to substitute for the expression of R_t. We can further use financial market constraint (49) to express the public debt B_t as a function of capital K_t and wage post-tax wage w_t. The planner's program (104)–(105) can be equivalently rewritten as a function of K_t and w_t:

$$\max_{(K_t,w_t)_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big(\log\left(w_t(\chi w_t)^{\varphi}\right) + \log(K_{t-1} + F(K_{t-1}, (\chi w_t)^{\varphi}) + \frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} - K_t - G_t - w_t(\chi w_t)^{\varphi})\Big).$$

We can further modify this program by defining $W_t = w_t(\chi w_t)^{\varphi}$ and dropping constants:

$$\max_{(K_t,w_t)_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg(\log(W_t) \tag{58}$$

$$+\log\left(K_{t-1} + F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}}) - \frac{1+\varphi+\varphi\beta}{(1+\beta)(1+\varphi)} W_t - K_t - G_t\right)\Bigg). \tag{59}$$

The function $(W_t, K_{t-1}) \mapsto F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}})$ is a concave as the the composition of concave and increasing functions. We thus deduce that the mapping defined by $(W_t, K_{t-1}, K_t) \mapsto \log(W_t) + \log\left(K_{t-1} + F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}}) - \frac{1+\varphi+\varphi\beta}{(1+\beta)(1+\varphi)} W_t - K_t - G_t\right)$ is concave. Any interior optimum characterized by first-order conditions must be a maximum.

3.4.5 Steady-state analysis

Steady-state characterization. We will denote steady-state quantities with no subscript. For instance, R will be the steady-state gross post-tax interest rate. First, note that the restrictions of Result 2. still holds at the steady state. In particular, $\mu > 0$ implies from FOC (54) that:

$$\tilde{r} = \frac{1-\beta}{\beta}, \tag{60}$$

as in the first-best equilibrium. We also have, as in the first-best $K/L = K_{FB}/L_{FB}$, as well as $\tilde{w} = w_{FB}$, $w = (1-\tau^L)w_{FB}$, and $Y = (1-\tau^L)^{\varphi} Y_{FB}$. Using $R = 1 + (1-\tau^K)\tilde{r}$ and (60), we obtain the following expression for the capital tax:

$$\tau^K = \varphi \frac{1+\beta}{1-\beta} \frac{\tau^L}{1-\tau^L}, \tag{61}$$

which is an increasing function of the labor tax. Note that the capital tax is positive whenever the labor tax is and even though the capital tax expression is unbounded from above when the labor tax approaches 1 ($\tau^K \to_{\tau^L \to 1} \infty$), the steady-state version of Result 2. actually provides an upper bound: $\tau^K < \frac{1}{1-\beta}$. Reaching the upper bound on the capital tax $\overline{\tau}^K_{SW} := \frac{1}{1-\beta}$ is equivalent to reaching the SW upper bound $\overline{\tau}^L_{SW}$ on the labor tax. Both correspond to the public spending $\overline{g}_{SW}$, defined by:

$$\overline{g}_{SW} = \overline{g}_1 + (1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)(1-\overline{\tau}^K_{SW})^{\varphi}. \tag{62}$$

The SW constraint can thus also be expressed as:

$$g_{FB} < \overline{g}_{SW}. \tag{63}$$

As explained before, a public spending higher than the SW bound would imply a high capital tax that is so high that the post-tax return on agents' assets is negative and hence that the consumption of unemployed agents is negative.

Determining the labor tax. The labor tax is determined as the tax that enables the government to balance its budget constraint. At the steady state, the governmental budget constraint (51) implies after some algebra that τ^L is a solution of the following equation:

$$\mathcal{T}(\tau^L) = 0, \tag{64}$$

$$\text{where: } \mathcal{T} : \tau \in (-\infty, 1) \mapsto \tau - \frac{1}{1-\alpha}\frac{g_{FB}(1-\tau)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}. \tag{65}$$

The mapping $\tau \mapsto \mathcal{T}(\tau)$ is akin to a Laffer curve. Indeed, we can check that $\mathcal{T}$ is continuously differentiable, strictly concave, with a unique maximum over $(-\infty, 1)$. In consequence, the function $\mathcal{T}$ admits either zero, one, or two solutions. The number of solutions depends on the level of public spending through g_{FB} in (65). When public spending are too high, there is no level of labor tax that make this public spending sustainable: $T(\tau) < 0$ for all $\tau \in (-\infty, 1)$. When the public spending is sustainable, T typically admits two roots, which correspond to the typical Laffer trade-off between tax rate and tax base. The smaller root corresponds to a low tax and a high labor supply, while the higher root corresponds to a high tax and a low labor supply. There is a third case that is the limit between sustainability and no sustainability. In this situation, there is a unique tax rate that enables public spending to be financed. We plot the three possibilities in Figure 1.

The limit case of the Laffer curve happens when the extremum point of the Laffer curve is the only root of the function. It can be checked that this corresponds to the tax level $\overline{\tau}^L_{La}$ that

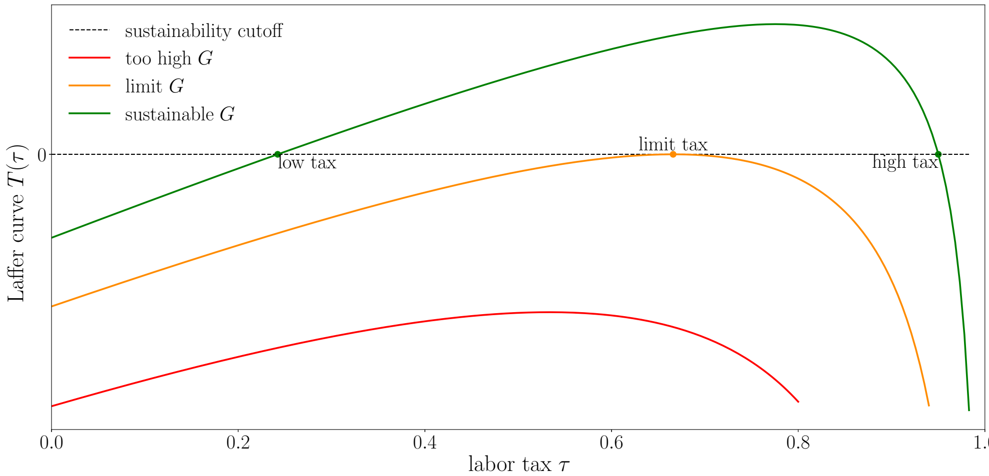

Figure 1: Examples of three Laffer curves $\tau \mapsto \mathcal{T}(\tau)$ of equation (65), for three different values of g_{FB}. The three cases correspond to: (i) two admissible tax values; (ii) a unique limit tax value; (iii) no admissible tax. The parametrization is: $\beta = 0.97$, $\alpha = 0.3$, $\phi = 0.5$, $\delta = 1.0$, and g_{FB} takes one of the three values in $[0.2, 0.3631, 0.6]$.

verifies $\mathcal{T}(\overline{\tau}^L_{La}) = \mathcal{T}'(\overline{\tau}^L_{La}) = 0$, or equivalently to:

$$\overline{\tau}^L_{La} = \frac{1}{1+\varphi} - \frac{1}{1-\alpha}\frac{\varphi}{1+\varphi}\frac{\overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}. \tag{66}$$

This corresponds to a public spending $\overline{g}_{La}$, defined as:

$$\overline{g}_{La} = \frac{1-\alpha}{\varphi}\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)(1-\overline{\tau}^L_{La})^{1+\varphi}. \tag{67}$$

So, any public spending such that $g_{FB} > \overline{g}_{La}$ is not sustainable and cannot be financed by any tax system. Oppositely, when $g_{FB} < \overline{g}_{La}$, two different tax levels enable the government to finance public spending and the planner will always opt for the lowest tax rate. Indeed, taxes have an unambiguously negative impact on consumption levels, since they can be written as:

$$c_e = \frac{1}{1+\beta}(1-\tau^L)^{\varphi+1}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{1+\varphi}, \; c_u = (1-(1-\beta)\tau^K)c_e. \tag{68}$$

So larger taxes decrease consumption and hence individual welfare.

We will henceforth refer to the restriction $g_{FB} < \overline{g}_{La}$ as the Laffer constraint. Note that whether the Laffer constraint is more stringent than the SW constraint depends on parameters and in general both restrictions must be considered.

The steady-state equilibrium existence. The following proposition summarizes our findings regarding equilibrium existence.

Proposition 2 *When $\overline{g}_1 \leq g_{FB}$, $g_{FB} \leq \overline{g}_{SW}$, and $g_{FB} < \overline{g}_{La}$, there exists a steady-state equilibrium with binding credit constraint for unemployed agents. The tax rate τ^L is determined as:*

$$\tau^L = \min\{\tau \in (-\infty, 1) : \mathcal{T}(\tau) = 0\}, \tag{69}$$

where $\mathcal{T}$ is defined in (65). The equilibrium allocation is then characterized by:

1. *positive taxes τ^L and τ^K whose expressions are given in (61) and (64);*

2. *positive consumption allocations (68);*

3. *a positive gross interest rate R and a positive long-run multiplier μ.*

We can verify that $\overline{g}_1 \leq \overline{g}_{SW}$ and $\overline{g}_1 \leq \overline{g}_{La}$. The former inequality is proved in Appendix C, while the later is a direct implication of the definition (62). Therefore, the credit-constrained equilibrium always exists for some values of public spending. It can also be observed that when $g_{FB} = \overline{g}_1$, equations (61) and (64) implies $\tau^L = \tau^K = 0$, as in the perfect risk-sharing equilibrium. In consequence, there is no discontinuity between the first-best and the credit-constraint equilibria around $g_{FB} = \overline{g}_1$.

We will conclude by two remarks regarding equilibrium existence. The first one is related to fact that we have only mentioned two equilibria so far: the first-best and the credit-constrained one. The latter features sizable inequalities. For instance, the inequality in consumption, measured by the ratio $\frac{c_e}{c_u}$, is an increasing function of taxes. Formally:

$$\frac{c_e}{c_u} = \frac{1}{1-(1-\beta)\tau^K} = \frac{1}{1-(1+\beta)\varphi\frac{\tau^L}{1-\tau^L}},$$

which can become infinitely large when $\tau^L \to \overline{\tau}^L_{SW}$. We could thus wonder whether a full risk-sharing equilibrium with positive labor tax (and hence null capital tax) would not sometimes exist. The answer is twofold: yes such an equilibrium can exist, but it always dominated (in terms of aggregate welfare) by the credit-constrained equilibrium of Proposition 2. The intuition is that the full risk-sharing arrangement imposes a zero capital tax, which means that public spending should be solely financed out of the labor tax. The distortions implied by this high labor tax involve a high burden on agents and make the aggregate welfare lower than in an equilibrium where public spending financing relies on both capital and labor taxes. In other words, for any level of public spending, financing this public spending through a combination of capital and labor taxes generates smaller distortions than a financing relying solely on labor tax. This is proved formally in Appendix C.5.

The second and last remark concerns situations where $\overline{g}_{SW} < g_{FB} \leq \overline{g}_{La}$. The public spending level is smaller than the Laffer bound, implying that the public spending can be financed. However, because the public spending is higher than the SW bound, the financing of this public spending implies negative values at the steady state for Lagrange multiplier, gross interest, and unemployed consumption. As previously explained, such situations cannot be ruled out and are possible for some parametrization. However, even though a steady-state equilibrium does not exist in these situations, we can check that there exists a non-stationary equilibrium, where the Lagrange multiplier on the governmental budget constraint diverges to infinity: $\mu_t \to_t \infty$ and the gross interest rate converges to $R_t \to_t 0$. In other words, this situation is similar to the one in Straub and Werning (2020), where the stationary equilibrium does not exist but a non-stationary one does. It is noteworthy that a similar pattern emerges despite the differences between our set-ups. We have time-varying agents' types (although the switch deterministic), endogenous credit constraint, distorting tax on endogenous labor supply, and public debt.

A key difference with Straub and Werning (2020) or with Lansing (1999) is that a steady-state equilibrium (including a finite Lagrange multiplier) exists for some public spending levels, even for log utilities and equal weight between agents. Proposition 2 shows that a crucial determinant for the existence of a steady-state equilibrium is the level of public spending.

Public debt. We now turn to the expression of public debt in the credit-constrained equilibrium. The financial market clearing condition implies that the steady-state public debt B can be written as follows:

$$B = \frac{\beta}{1-\beta}\left(-\frac{1-\beta}{1+\beta}\frac{1-\alpha}{1+\varphi}\tau_l - \overline{g}_1\right)(\chi w)^{\varphi}\left(\frac{K}{L}\right)^{\alpha}, \tag{70}$$

where the labor tax is defined in equation (69) of Proposition 2. We deduce that the equilibrium features a positive public debt is positive iff:

$$\tau^L < \frac{1+\varphi}{1-\alpha}\frac{1+\beta}{1-\beta}(-\overline{g}_1). \tag{71}$$

A necessary condition for the public debt to be positive is $\overline{g}_1$ to be negative – which precludes from Proposition 1 the existence of a first-best equilibrium. An equivalent condition to condition (71) is that public spending g_{FB} is not too large. Indeed, it can be seen from (70) that public debt decreases with labor tax and hence with public spending. The higher public spending, the smaller the public debt. We know from equation (41) that in the first-best equilibrium, the public debt is negative and decreases – becomes more negative – with public spending. A positive public debt with the credit-constrained equilibrium is thus not compatible with the continuity of equilibrium around $g_{FB} = \overline{g}_1$ since this would require a negative public debt for $g_{FB} < \overline{g}_1$ and a positive one for $g_{FB} > \overline{g}_1$. In consequence, positive public debt imposes the non-existence of the first-best equilibrium. This is summarized in the following result.

Result 3. *Steady-state public is positive:* $B \geq 0$ *iff* $\overline{g}_1 \leq 0$ *and* $g_{FB} \leq \overline{g}_{pos}$*, where:*

$$\overline{g}_{pos} = (-\overline{g}_1)\frac{(1+2\varphi)(1+\beta)}{1-\beta}\left(\frac{(1+2\varphi)(1+\beta)}{1-\alpha}\frac{\alpha}{1+\beta(\delta-1)}\right)^{\varphi}. \tag{72}$$

The proof is in Appendix C.4.

In the credit-constrained equilibrium, a positive public debt enables the planner to provide public liquidity to agents. This enables agents to smooth out the unemployment shock through private savings. However, when public spending increases, the planner needs to raise higher taxes.

A positive public debt only exists in our economy when the first-best equilibrium does not exist. Furthermore, public debt is decreasing with public spending: the higher public spending, the lower public debt (in absolute value). The reason is that an increase in public spending leads the planner to increase labor and capital taxes, which are both distortionary. This crowds out private savings and hence diminishes the room for public debt.

3.4.6 Dynamic analysis

The first-order dynamic system. After the thorough analysis of the steady-state equilibrium, we investigate the dynamic in this equilibrium. We focus on a special case, with full capital depreciation: $\delta = 1$. We will denote with a hat the relative deviation to the steady-state value: $\widehat{x}_t = \frac{x_t - x}{x}$ for generic variable x_t with steady-state value x. The public spending shock is assumed to be defined as follows:

$$\widehat{G}_t = \begin{cases} \sigma_G \varepsilon_{G,0} & \text{if } t = 0, \\ \rho_G \widehat{G}_{t-1} & \text{if } t > 0, \end{cases} \tag{73}$$

$$\text{where: } \varepsilon_{G,0} \sim \mathcal{N}(0,1),$$

and $\sigma_G > 0$ and $\rho_G \in (-1, 1)$. The shock only happens at date $t = 0$ and then persists with parameter ρ_G – as is consistent with our assumption of MIT shock. The dynamic of the economy can be summarized by the capital as a unique state variable and the public spending shock. It can be computed thanks to a first-order development around the steady-state allocation. The outcome is gathered in the following result.

Result 4. *The dynamics of the capital stock and of the shadow cost of the governmental budget constraint is given by the following system:*

$$\widehat{K}_t = \rho_K \widehat{K}_{t-1} + \sigma_K \widehat{G}_t, \tag{74}$$

$$\widehat{\mu}_t = \rho_\mu \widehat{K}_{t-1} + \sigma_\mu \widehat{G}_t, \tag{75}$$

where the expressions of coefficients are given in Appendix D.2. These coefficients solely depend on model parameters and not on those of the dynamics of $(\widehat{G}_t)_{t\geq 0}$.

The dynamic system (74)–(75) is stable when the autoregressive coefficient ρ_K is smaller than one in absolute value. In our setup, this is equivalent to verifying Blanchard-Kahn conditions. The result regarding system stability is summarized in the following proposition.

Proposition 3 *The system (74)–(75) is stable –* $|\rho_K| < 1$ *– iff:*

$$\alpha \leq \frac{1}{1+(1-\beta)(1+\varphi)}. \tag{76}$$

We furthermore have:

$$\sigma_\mu > 0 > \sigma_K. \tag{77}$$

The dynamic system is stable under the condition (76), which imposes an upper bound on α. Note that this upper bound is always strictly smaller than one and hence can be binding. This condition on α always holds when public debt is positive, i.e., when $\overline{g}_1 < 0$. A second result of Proposition 3 is that $\sigma_\mu > 0 > \sigma_K$. Equation (75) implies thus that at impact, an increase in public spending increases the shadow cost of governmental budget constraint, while it diminishes capital. This reflects the fact that the governmental budget constraint becomes more binding, while it crowds out some resources out of capital.

Role of the persistence of public spending shock ρ_G. The analysis of the role of the persistence ρ_G is split into three parts: (i) the role at impact on capital and governmental budget shadow cost, (ii) the role on the dynamics of capital; (iii) the role on public debt. We assume here that we consider a positive public spending shock: $\widehat{G}_0 > 0$.

Regarding the first aspect we have the following result.

Result 5. *We have:*

$$\frac{\partial \sigma_\mu}{\partial \rho_G} > 0 \text{ and } \frac{\partial \sigma_K}{\partial \rho_G} > 0.$$

In words, a higher persistence of the public spending shock strengthens the variation of $\widehat{\mu}_t$ at impact: the higher ρ_G,the more the Lagrange multiplier μ_t increases at impact. This reflects that a higher persistence of the public spending shock means greater public funding requirements in the subsequent periods, and hence raises the shadow cost of governmental budget constraint. The result is opposite for the capital. A higher persistence dampens the decrease of capital at impact. This is related to the effect of persistence on public debt described in Proposition 4. Regarding capital, we can obtain additional results. By induction, we can derive from (73) and (74) the closed-form expression of the capital IRF:

$$\widehat{K}_t = \sigma_K \widehat{G}_0 \frac{\rho_K^{t+1} - \rho_G^{t+1}}{\rho_K - \rho_G}, \tag{78}$$

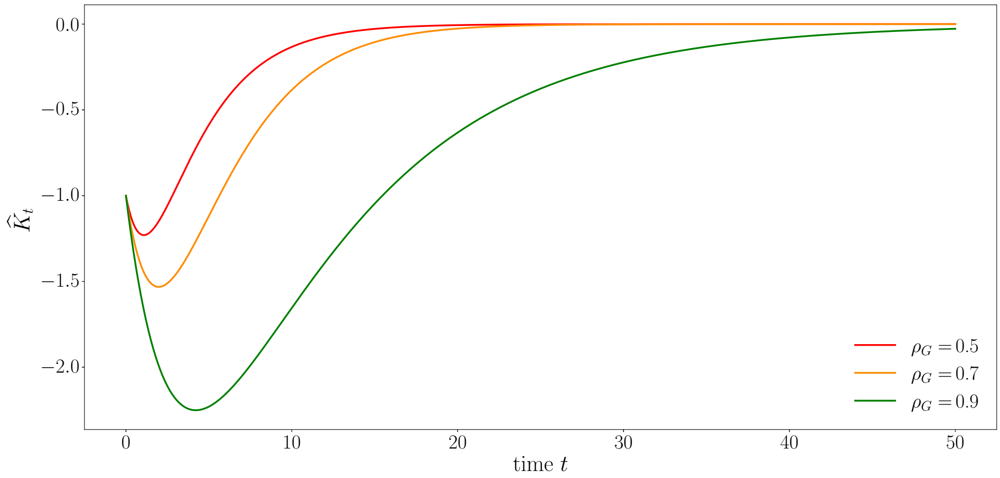

Figure 2: Examples of three IRFs $t \mapsto \widehat{K_t}$ of equation (78), for three different values of the persistence ρ_G of the public spending shock. The shock at impact is normalized to -1 (i.e., $\sigma_K \widehat{G}_0 = -1$). The parametrization is the same as in Figure 1: $\beta = 0.97$, $\alpha = 0.3$, $\phi = 0.5$, $\delta = 1.0$, but for g_{FB} set to 0.3. This corresponds to $\rho_K \approx 0.7290$.

which allows us to completely characterize the capital path following a public spending shock. At impact, the relative variation of capital is negative by a quantity $\sigma_K \widehat{G}_0 < 0$. Then, the profile is the capital variation is humped-shaped: it starts decreasing further, before increasing and reverting back to zero. The length of the capital depreciation (during which $\widehat{K_t}$ diminishes following the initial shock) can be shown to be an increasing function of the persistence ρ_G: the higher ρ_G, the longer the recession. The impact on the depth of recessions is in general ambiguous, but it can be shown that (i) when the persistence is sufficiently high, it increases recession depth; (ii) the threshold value decreases with ρ_K. This is illustrated on Figure 2.

Finally, regrading public debt, we have the following result.

Proposition 4 *Denoting by $\widehat{B}_0$ the public debt variation at impact, we have:*

$$\frac{\partial \widehat{B}_0}{\partial \rho_G} < 0.$$

Proposition 4 states that the variation of public debt at impact is dampened for larger persistence of the public debt shock. The intuition is rather straightforward. A very transitory shock will be recovered very quickly and can be smoothed out by public debt, whose increase will also be transitory. Conversely, a very persistent shock will require steady increase in taxes. Diminishing the variation of public debt (typically, a decrease for very persistent process) enables the planner to limit the crowding-out of impact and thus the overall impact of the shock on the economy.

In general, the sign of $\widehat{B}_0$ is ambiguous and can be positive or negative, depending in particular on the magnitude of the persistence ρ_G of the public spending shock.

4 Quantitative analysis

We relax the assumptions of Section 3 to simulate the dynamics of capital and public debt in a quantitatively relevant environment. The quantitative strategy is as follows. First to calibrate standard parameters to obtain a realistic steady-state allocation with the actual US fiscal policy. Second, following the inverse taxation problem (Bourguignon and Amadeo, 2015, Heathcote and Tsujiyama, 2021, Chang et al., 2018), we identify an "empirically motivated" social welfare function, such that this steady-state allocation is optimal for the planner. The gain of this methodology is to observe the dynamics of the tax system, considering a quantitatively realistic initial (and final) equilibrium. Starting from this allocation, we implement a period-0 shock on public spending to observe the dynamics of fiscal instruments after the public spending shock.

4.1 Calibration

The period is a quarter.

Preferences. The utility function is now assumed to be separable in labor, which is a quantitatively more relevant option than the GHH utility function for incomplete-market economies (see Auclert et al., 2021):

$$U(c,l) = u(c) - v(l),$$
$$\text{with: } u(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \text{ and } v(l) = \frac{1}{\chi}\frac{l^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}.$$

We set the inverse of intertemporal elasticity of substitution to $\sigma = 2$, which is a standard value used in the literature. For the disutility of labor, we choose $\phi = 0.5$ to match a Frisch elasticity for labor supply of 0.5, which is the value recommended by Chetty et al. (2011) for the intensive margin in heterogeneous-agent models. The scaling parameter is set to $\chi = 0.05$, which implies normalizing the aggregate labor supply to 1/3. Finally, the discount factor is $\beta = 0.99$.

Second, households productivity levels can be arbitrary high, where the transition probabilities are calibrated to match the actual labor market dynamics in the US. Third, the tax on labor is non-linear and it has the form used by Heathcote et al. (2017) (henceforth, the HSV tax system) to reproduce the progressivity of the actual US system.

The period utility function over consumption is $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$,

Idiosyncratic risk. We calibrate the productivity process to match the actual labor market dynamics in the US. We focus on a standard AR(1) process:

$$\log y_t = \rho_y \log y_{t-1} + \varepsilon_t^y,$$
$$\text{where: } \varepsilon_t^y \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_y^2).$$

Following the strategy of Castaeneda et al. (2003), we choose the parameters (ρ_y, σ_y) to target some key moments.[9] We choose three targets. The first one is the variance of the logarithm of consumption, that enables us to capture consumption inequality. Heathcote and Tsujiyama (2021) report a value of $\text{Var}(\log c) = 0.23$. We also target the log-variance of wages to match income inequality, which is found to be $\text{Var}(\log w) = 0.47$ by Heathcote and Tsujiyama (2021). The third target is the debt-to-GDP ratio which allows us to replicate a realistic financial market equilibrium. We target a value of $B/Y = 61.5\%$, which is the mean ratio over the period (Dyrda and Pedroni, 2018). Calibrating these three moments yields $\rho_y = 0.993$ and $\sigma_y = 0.082$. These parameters are close to those from a direct estimation of the productivity process on PSID data, which corresponds to $\rho_y = 0.9923$ and $\sigma_y = 0.0983$ (see Boppart et al., 2018 and Krueger et al., 2018). The data targets and their model counterparts are reported in Table 1. This simple

	Data	Model
Variance of log consumption Var(log$c$)	0.23	0.20
Variance of log income Var(log$y$)	0.47	0.49
Debt-to-GDP ratio B/Y	61.5%	61.4%

Table 1: Model calibration: targets and model counterparts.

representation is doing a good job in matching the three targeted moments. Furthermore, we can check that this calibration generates a reasonable wealth distribution, even though we do not calibrate it explicitly.[10] Indeed, the calibrated model implies a Gini coefficient of wealth equal to 0.66, which is close, even though below, its empirical counterpart of 0.77. It is known that additional model features must be introduced to match the high wealth inequality in the US, such as heterogeneous discount rates (see Krusell and Smith, 1998), or entrepreneurship (Quadrini, 1999), or stochastic financial returns, which are not considered here.

Finally, we discretize the productivity process using the Rouwenhorst (1995) procedure with 7 idiosyncratic states.

[9] More precisely, we minimize the quadratic difference between the model-generated moments and their empirical counterpart, following the Simulated Method of Moments. In the current environment, we see this procedure as a "sophisticated" calibration, rather than an actual SMM – as we equally weight the three moments.

[10] For the problem under consideration, we consider matching the dispersion of consumption may be more important than the distribution of wealth, which motivates the exclusion of this moment from our calibration strategy.

Technology . The production function is Cobb-Douglas: $F(K, L) = K^{\alpha}L^{1-\alpha} - \delta K$. The capital share is set to $\alpha = 36\%$ and the depreciation rate to $\delta = 2.5\%$, as in Krueger et al. (2018) among others.

Taxes and government budget constraint. The capital tax is taken from Trabandt and Uhlig (2011), who use the methodology of Mendoza et al. (1994) on public finance data prior to 2008. Their estimation for the US in 2007 (before the financial crisis) yields a capital tax (including both personal and corporate taxes) of $\tau^K = 36\%$. For the labor we consider the HSV functional form of equation (2). The progressivity of the labor tax is taken from Heathcote et al. (2017), who report an estimate $\tau = 0.181$. We choose κ to match a public-spending-to-GDP ratio equal to 19%, as in Heathcote and Tsujiyama (2021).

Summary. Table 2 provides a summary of the model parameters.

Parameter	Description	Value
	Preference and technology	
β	Discount factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
$\bar{a}$	Credit limit	0
χ	Scaling param. labor supply	0.05
φ	Frisch elasticity labor supply	0.5
	Shock process	
ρ_y	Autocorrelation idio. income	0.993
σ_y	Standard dev. idio. income	0.082
	Tax system	
τ^K	Capital tax	36%
κ	Sacaling of Labor tax	0.75
τ	Progressivity of tax	0.181

Table 2: Parameter values in the baseline calibration. See text for descriptions and targets.

4.2 Truncation and estimating Pareto weights

Standard recursive techniques cannot be used to compute the optimal Ramsey policy. The problem of the planner, as derived analytically in Section 2.6, could be written recursively, but the state space would include the joint distribution of beginning-of-period savings and Lagrange multipliers on consumption Euler equations (i.e., (a, λ_c)), as past values of Lagrange multipliers

($\lambda_{c,t-1}$) appear in the first-order condition of the planner. To compute the solution, we follow LeGrand and Ragot (2022) and we consider a truncated representation of this problem. We provide a detailed account of the computational implementation that can be of independent interest as solving such Ramsey problems is not straightforward.

More precisely, to investigate the optimal dynamics of the instruments after a shock, we start with providing an exact truncated aggregation of the steady-state model, and we then follow the dynamics of the truncated representation using perturbation methods. The algebra is provided in Appendix E.

The truncation length is set to $N = 3$, which is shown to be provide a good representation of the dynamics. We have to to estimate the weights of the social welfare function, such that the first-order conditions of the planner at the steady state are consistent with actual US tax system (as described in Section 4.1). However, the problem is in general under-identified, since we have only two constraints (for the capital and labor tax) but seven different weights (one per productivity level). Following Heathcote and Tsujiyama (2021), we introduce productivity weights which depend on the productivity level and define a parametric quadratic representation of weights, as follows:

$$\log \omega_y := \theta_1 \log y + \theta_2 (\log y)^2 .$$

As explained in Appendix E, matching capital and labor tax yields $\theta_1 = 0.603597$ and $\theta_2 = 0.325546$. In an environment without saving, Heathcote and Tsujiyama (2021) estimate the relationship $\log \omega_y = \theta \log y$ and find $\theta = 0.517$. Our estimate is pretty close from theirs and the quantitative difference mostly comes from the additional instruments we use.[11]

4.3 Model dynamics

We now simulate the optimal dynamics of the four fiscal tools $(\tau_t^\kappa, B_t, \kappa_t, \tau_t)$ after a public spending shock occurring in period $t = 0$. The dynamics of the shock is the same as in equation (73) of the analytical section. After a initial shock in period 0, public spending reverts back to equilibrium at a rate ρ_G. We present a first-order approximation of the dynamics, implying that the initial shock should be considered to be not too large.

We first plot the dynamics of the model for two values of the persistence of the shock $\rho_G = 0.99$ and $\rho_G = 0.90$. Figure 3 plots the dynamics of public debt B, labor income tax (level κ and progressivity τ), capital tax τ^K, output Y, capital K, aggregate labor L, and consumption C in both cases.

Panel 1 represents the dynamics of public spending over GDP, it increases by 1% and goes back to equilibrium at a rate $\rho_G = 0.90$ (black solid line) or $\rho_G = 0.99$ (blue dashed line). Panel 2 plots the dynamics of public debt-to-GDP ratio. It can be observed that public debt increases

[11] We cannot strictly reproduce the specification of Heathcote and Tsujiyama (2021) within our framework, as we need two parameters for matching planner's first-order conditions.

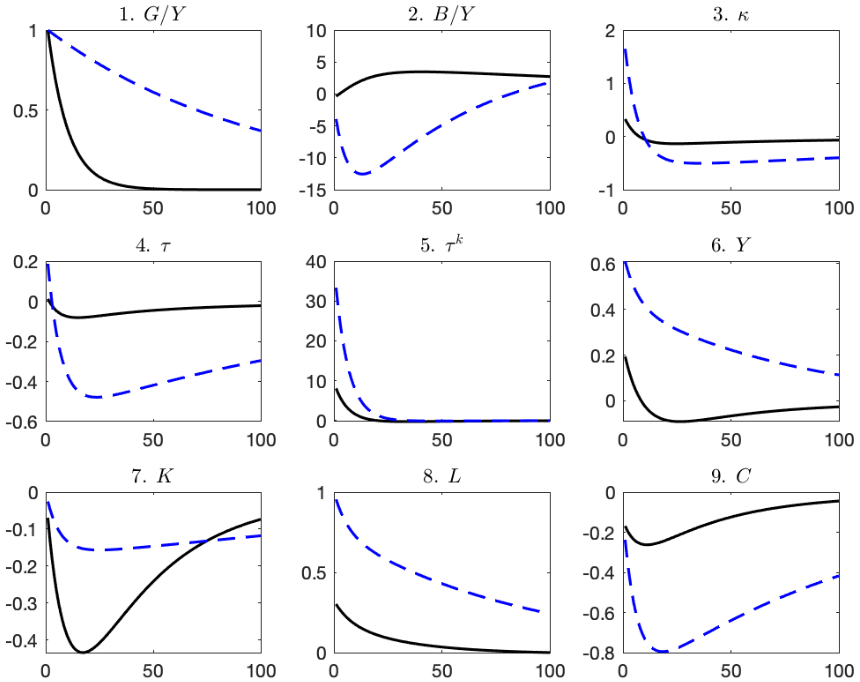

Figure 3: Dynamics of selected variables. G/Y, $B/Y, \kappa, \tau, \tau^K$ are absolute deviation (%). Y, K, L, C are proportional deviation (%). The black solid line is for $\rho_G = 0.90$. The blue dashed line is for $\rho_G = 0.99$

for the low value of the persistence ρ_G, whereas it decreases for the high value. Panel 3 plots the dynamics of the level of labor tax κ. It increases in both cases but more when the shock is more persistent. When the level persistence is high, progressively increase on impact by 0.2% and then decreases. it is more stable when the persistence is low. Note that the deviation of progressively is low compared to the deviation of the level of tax κ. Overall, as shown below, the overall return on the tax on labor income fluctuates less than the capital tax. Indeed, Panel 5 show that the capital tax increase by 30% when persistence is high, and by 10% when persistence is low. The increase in the resources of the state mainly comes from the change of capital tax, which is more volatile than labor tax. GDP increases in both cases, but less when the persistence is low. We discuss below the implications for the multiplier of public spending. When the persistence is high, capital decreases less and aggregate labor increases more compared to the case where persistence is low (Panel 7 and 8). This generates the higher output when persistence is high, which is necessary to finance the increase in G. Finally, the fall in consumption is higher when persistence is high.

Dynamics of tax return. We now plots the overall return on labor tax and capital tax as a percentage of steady-state GDP to observe how the state finances it spending.

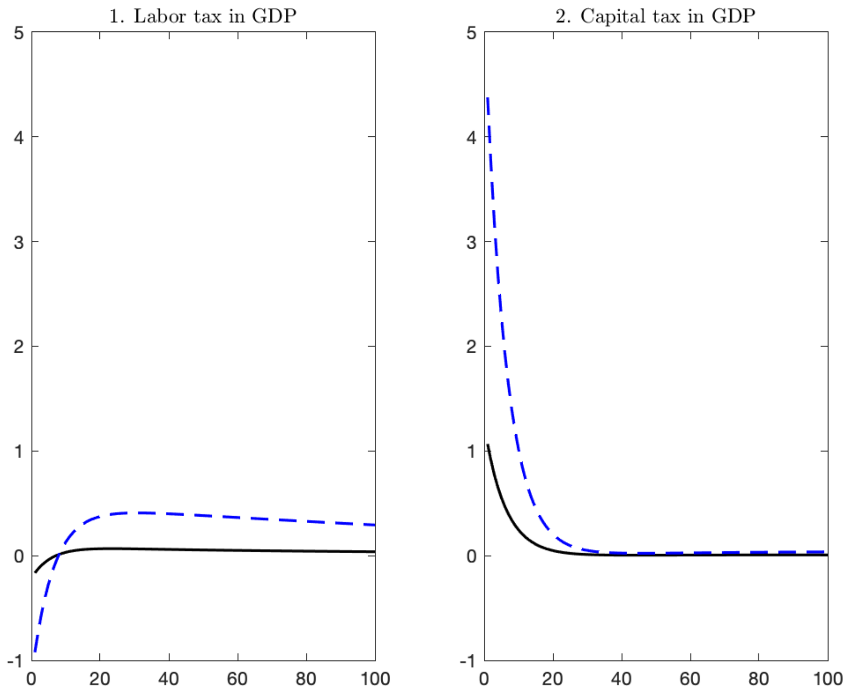

Figure 4: Overall return on labor tax (Panel 1) and capital tax (Panel 2), as a percentage of steady-state GDP.

One can observe that the capital tax over GDP (Panel 2) is more volatile than the labor tax return (Panel 1). The increase in capital tax is used to finance the increase in G but also a decrease in labor tax return.

Cumulative multipliers. As typically done in the empirical literature (see Ramey and Zubairy, 2018 for instance), we compute the spending multiplier m_h at horizon h as follows:

$$m_h = \frac{\sum_{t=0}^{h} (Y_t - Y)}{\sum_{t=0}^{h} (G_t - G)}.$$

Note that m_1 is the impact multiplier. Figure 5 plots the multiplier as a function of the time t after the shock.

One can observe that the multiplier is higher the more persistent the shock. Indeed, the output is higher when the persistence of the shock is higher because of various effect. First, the overall tax on labor decreases, which increases labor supply. Second the consumption is lower when the persistence is higher what generates an increase in labor supply to compensate for the

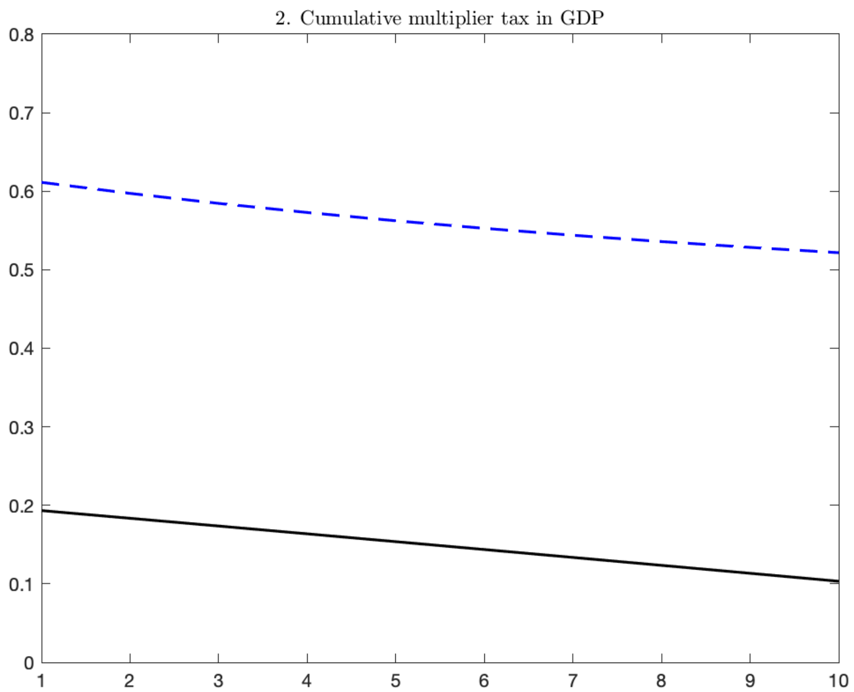

Figure 5: Cumulative multiplier as a function of the time t after the shock.

fall in consumption.

5 Conclusion

We investigate the optimal dynamics of fiscal system after a public spending shock in an heterogeneous agent model. We first contribute to the clarification of the conditions for relevant equilibria to exist. The key friction for equilibrium existence is occasionally-binding credit constraint, which provide a rationale for both positive capital tax and public debt. The second contribution of this paper is to show that the dynamics of public debt depends on the persistence of the public spending shock. For low persistence, public debt is pro-cyclical. For high persistence; the public debt is countercyclical. In addition, we find that capital tax increases, and more so when persistence is high. We show that these properties are qualitatively robust in a model where the actual US tax system is implemented at the steady state thanks to an inverse optimal taxation approach. The simulation of the quantitative model relies on the Lagrangian-Truncation approach developed in LeGrand and Ragot (2022).

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Appendix

A First-order conditions of the individual Ramsey program

The Ramsey problem can be written as follows:

$$\max_{\left(r_t,\tilde{w}_t,\tilde{r}_t,\tau_t^K,\tau_t,\kappa_t,B_t,K_t,L_t,\Pi_t,(a_t^i,c_t^i,l_t^i,\nu_t^i)_i\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\int_i \omega_t^i U(c_t^i,l_t^i)\ell(di)\right], \tag{79}$$

$$\tag{80}$$

$$G_t + R_t B_{t-1} + (R_t-1)K_{t-1} + w_t\int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di) = K_{t-1}^{\alpha}L_t^{1-\alpha} - \delta K_{t-1} + B_t \tag{81}$$

$$\text{for all } i\in\mathcal{I}\text{: } a_t^i + c_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \tag{82}$$

$$a_t^i \geq -\bar{a},\ \nu_t^i(a_t^i+\bar{a}) = 0,\ \nu_t^i \geq 0, \tag{83}$$

$$U_c(c_t^i,l_t^i) = \beta\mathbb{E}_t\left[R_{t+1}U_c(c_{t+1}^i,l_{t+1}^i)\right] + \nu_t^i, \tag{84}$$

$$-U_l(c_t^i,l_t^i) = (1-\tau_t)w_t(y_t^i)^{1-\tau_t}(l_t^i)^{-\tau_t}U_c(c_t^i,l_t^i), \tag{85}$$

$$K_t + B_t = \int_i a_t^i\ell(di),\ L_t = \int_i y_t^i l_t^i \ell\,(di)\,. \tag{86}$$

The Lagrangian can be written as:

$$\mathcal{L} = \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int_i \omega_t^i U(c_t^i,l_t^i)\ell(di) \tag{87}$$

$$\begin{aligned}
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int_i\left(\lambda_{c,t}^i - R_t\lambda_{c,t-1}^i\right)U_c(c_t^i,l_t^i)\ell(di)\\
&+ \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int_i\lambda_{l,t}^i\left(U_l(c_t^i,l_t^i) + (1-\tau_t)w_t(y_t^i)^{1-\tau_t}(l_t^i)^{-\tau_t}U_c(c_t^i,l_t^i)\right)\ell(di)\\
&- \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mu_t\Bigg(G_t + (1-\delta)B_{t-1} + (R_t-1+\delta)\int_i a_{t-1}^i\ell(di) + w_t\int_i (y_t^i l_t^i)^{1-\tau_t}\ell(di)\\
&\qquad - (\int_i a_{t-1}^i\ell(di) - B_{t-1})^{\alpha}(\int_i y_t^i l_t^i\ell(di))^{1-\alpha} - B_t\Bigg).
\end{aligned} \tag{88}$$

where:

$$c_t^i = -a_t^i + R_t a_{t-1}^i + w_t(y_t^i l_t^i)^{1-\tau_t} \tag{89}$$

FOC with respect to savings choices. Deriving (87) with respect to a_t^i yields:

$$\begin{aligned}
0 = &\beta^t \int_j \omega_t^j U_c(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) \\
&- \beta^t \int_j \left(\lambda_{c,t}^j - R_t \lambda_{c,t-1}^j\right) U_{cc}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \\
&+ \beta^t \int_j \lambda_{l,t}^j U_{cl}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) \\
&+ \beta^t (1-\tau_t) w_t \int_j \lambda_{l,t}^j (y_t^j)^{1-\tau_t} (l_t^j)^{-\tau_t} U_{cc}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) \\
&+ \beta^{t+1} \mathbb{E}_t \left[\int_j \omega_{t+1}^j U_c(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \right] \\
&- \beta^{t+1} \mathbb{E}_t \left[\int_j \left(\lambda_{c,t+1}^j - R_{t+1} \lambda_{c,t}^j\right) U_{cc}(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right] \\
&+ \beta^t \mathbb{E}_t \left[\int_j \lambda_{l,t+1}^j U_{cl}(c_{t+1}^i, l_{t+1}^i) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right] \\
&+ \beta^{t+1} (1-\tau_{t+1}) w_{t+1} \mathbb{E}_t \left[\int_j \lambda_{l,t+1}^j (y_{t+1}^j)^{1-\tau_{t+1}} (l_{t+1}^j)^{-\tau_{t+1}} U_{cc}(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right] \\
&+ \beta^{t+1} \mathbb{E}_t \left[\mu_{t+1} \left(\alpha K_t^{\alpha-1} L_{t+1}^{1-\alpha} - (r_{t+1} + \delta) \right) \right]
\end{aligned}$$

We also denote:

$$\begin{aligned}
\psi_t^i = &\omega_t^i U_c(c_t^i, l_t^i) + \lambda_{l,t}^i U_{cl}(c_t^i, l_t^i) \\
&- \left(\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}\right) U_{cc}(c_t^i, l_t^i).
\end{aligned} \tag{90}$$

and get using $\tilde{r}_{t+1} = \alpha K_t^{\alpha-1} L_{t+1}^{1-\alpha} - \delta$:

$$\begin{aligned}
0 = &\int_j \psi_t^j \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) + \beta \mathbb{E}_t \left[\int_j \psi_{t+1}^j \frac{\partial c_{t+1}^j}{\partial a_t^i} \right] \\
&+ \beta \mathbb{E}_t \left[\mu_{t+1} (\tilde{r}_{t+1} - R_{t+1} + 1) \right].
\end{aligned}$$

Using (89), we obtain $\frac{\partial c_t^j}{\partial a_t^i} = -1_{i=j}$ and $\frac{\partial c_{t+1}^j}{\partial a_t^i} = R_{t+1} 1_{i=j}$, from which we deduce:

$$\psi_t^i = \beta \mathbb{E}_t \left[R_{t+1} \psi_{t+1}^i \right] + \beta \mathbb{E}_t \left[\mu_{t+1} (1 + \tilde{r}_{t+1} - R_{t+1}) \right].$$

FOC with respect to labor supply. Deriving (87) with respect to l_t^i yields:

$$0 = \int_j \psi_t^j \frac{\partial c_t^j}{\partial l_t^i} \ell(dj) - \psi_{l,t}^j$$
$$- \lambda_{l,t}^i (1-\tau_t)\tau_t w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t - 1} U_c(c_t^i, l_t^i) - \mu_t (w_t (1-\tau_t)(y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} - F_{L,t} y_t^i),$$

where we have defined:

$$\psi_{l,t}^i = -\omega_t^i U_l(c_t^i, l_t^i) - \lambda_{l,t}^i U_{ll}(c_t^i, l_t^i) + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}) U_{cl}(c_t^i, l_t^i).$$

Using (89), we obtain $\frac{\partial c_t^j}{\partial l_t^i} = (1-\tau_t) w_t (y_t^j)^{1-\tau_t} (l_t^i)^{-\tau_t} 1_{i=j}$,which implies:

$$\psi_{l,t}^i = (1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i$$
$$+ \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t)\tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i.$$

FOC with respect to the interest rate. Deriving (87) with respect to R_t yields:

$$0 = \int_j \left(\psi_t^j \frac{\partial c_t^j}{\partial R_t} + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj) - \mu_t \int_j a_{t-1}^j \ell(dj).$$

From (89), we obtain $\frac{\partial c_t^j}{\partial R_t} = a_{t-1}^j$, which yields:

$$0 = \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj).$$

FOC with respect to the wage rate. Deriving (87) with respect to w_t yields:

$$0 = \int_j \left(\psi_t^j \frac{\partial c_t^j}{\partial w_t} + \lambda_{l,t}^j (1-\tau_t)(y_t^j)^{1-\tau_t} (l_t^j)^{-\tau_t} U_c(c_t^j, l_t^j) \right) \ell(dj)$$
$$- \mu_t \int_j (y_t^j l_t^j)^{1-\tau_t} \ell(dj)$$

From (89), we get $\frac{\partial c_t^j}{\partial w_t} = (y_t^j l_t^j)^{1-\tau_t}$ and:

$$0 = \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \ell(dj).$$

FOC with respect to public debt. Deriving (87) with respect to B_t yields:

$$0 = \mu_t - \beta \left[(1 - \delta - \alpha K_{t-1}^{\alpha} L_t^{1-\alpha} \mu_{t+1} \right],$$

or using the definition of $\tilde{r}_{t+1}$:

$$\mu_t = \beta(1 + \tilde{r}_{t+1})\mu_{t+1}.$$

FOC with respect to progressivity. Deriving (87) with respect to τ_t yields:

$$\begin{aligned}
0 = &\int_j \psi_t^j \frac{\partial c_t^j}{\partial \tau_t} \ell(dj) \\
&+ w_t \int_j \lambda_{l,t}^j \frac{\partial}{\partial \tau_t} \left((1-\tau_t)(y_t^j l_t^j)^{1-\tau_t}\right) (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj) \\
&- \mu_t w_t \int_j \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t}\right) \ell(dj).
\end{aligned}$$

From (89), we have $\frac{\partial c_t^j}{\partial \tau_t} = (y_t^j l_t^j)^{1-\tau_t}$ and:

$$\begin{aligned}
0 = &\int_j \hat{\psi}_t^j \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t}\right) (dj) \\
&+ \int_j \lambda_{l,t}^j \left(-(y_t^j l_t^j)^{1-\tau_t} + (1-\tau_t)\frac{\partial}{\partial \tau_t}(y_t^j l_t^j)^{1-\tau_t}\right) (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}$$

and

$$\begin{aligned}
0 = &\int_j \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t}\right) (dj) \\
&- \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}$$

Using $\frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t}\right) = -\ln(y_t^j l_t^j)(y_t^j l_t^j)^{1-\tau_t}$, we finally deduce:

$$\begin{aligned}
0 = &\int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ln(y_t^j l_t^j)(dj) \\
&+ \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}$$

Summary of FOCs.

$$\begin{aligned}
\hat{\psi}^i_t &= \beta\mathbb{E}_t\left[(1+r_{t+1})\hat{\psi}^i_{t+1}\right],\\
\psi^i_{l,t} &= (1-\tau_t)w_t y^i_t (y^i_t l^i_t)^{-\tau_t}\hat{\psi}^i_t\\
&\quad + \mu_t F_{L,t} y^i_t - \lambda^i_{l,t}(1-\tau_t)\tau_t w_t y^i_t (y^i_t l^i_t)^{-\tau_t} U_c(c^i_t, l^i_t)/l^i_t,\\
\mu_t &= \beta(1+\tilde{r}_{t+1})\mu_{t+1}\\
0 &= \int_j \left(\hat{\psi}^j_t a^j_{t-1} + \lambda^j_{c,t-1} U_c(c^j_t, l^j_t)\right)\ell(dj),\\
0 &= \int_j (y^j_t l^j_t)^{1-\tau_t}\left(\hat{\psi}^j_t + \lambda^j_{l,t}(1-\tau_t)U_c(c^j_t, l^j_t)/l^j_t\right)\ell(dj),\\
0 &= \int_j (y^j_t l^j_t)^{1-\tau_t}\left(\hat{\psi}^j_t + \lambda^j_{l,t}(1-\tau_t)U_c(c^j_t, l^j_t)/l^j_t\right)\ln(y^j_t l^j_t)(dj)\\
&\quad + \int_j \lambda^j_{l,t}(y^j_t l^j_t)^{1-\tau_t}(U_c(c^j_t, l^j_t)/l^j_t)\ell(dj).
\end{aligned}$$

B Proof of Proposition 1

The first-best equilibrium is characterized by optimal consumption smoothing and no inefficient distortions. As consequence agents are unconstrained and taxes are $\tau^K = \tau^L = 0$. We focus on the steady state allocation. Since both agents are unconstrained, the combination of Euler equations (38) and (39) yields $u'\left(c_e - \frac{1}{\chi}\frac{l_e^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right) = (\beta R)^2 u'\left(c_e - \frac{1}{\chi}\frac{l_e^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right)$, and hence:

$$\beta R_{FB} = 1, \tag{91}$$

while the government budget constraint (37) implies that the public debt verifies:

$$B_{FB} = -\frac{\beta}{1-\beta}G < 0.$$

The previous condition is necessary but not sufficient to ensure that the first-best allocation can be implementer. Indeed, the additional constraint is that no agents are constrained. We now derive this additional condition.

Factor prices definitions (1) with (91) and $L_{FB} = l_e = (\chi w_{FB})^{\varphi}$ yield:

$$\frac{K_{FB}}{L_{FB}} = \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}, \tag{92}$$

from which we easily deduce:

$$w_{FB} = (1-\alpha)\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha}{1-\alpha}}, \tag{93}$$

$$Y_{FB} = K_{FB}^{\alpha}L_{FB}^{1-\alpha} = (\chi(1-\alpha))^{\varphi}\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}, \tag{94}$$

$$K_{FB} = \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}(\chi w_{FB})^{\varphi}. \tag{95}$$

Furthermore, since agents are unconstrained, Euler equations imply $c_{u,FB} = c_{e,FB} - \frac{1}{\chi}\frac{l_{e,FB}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$, or after substituting by budget constraints: $R_{FB}a_{u,FB} - a_{e,FB} + \frac{w(\chi w)^{\varphi}}{\varphi+1} = R_{FB}a_{e,FB} - a_{u,FB}$. With (91), this yields:

$$a_{e,FB} - a_{u,FB} = \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}, \tag{96}$$

$$a_{u,FB} + a_{e,FB} = K_{FB} - \frac{\beta}{1-\beta}G, \tag{97}$$

where the second equality is the financial market clearing condition. The combination of both previous equations implies:

$$2\frac{1-\beta}{\beta}\frac{a_{u,FB}}{Y_{FB}} = \overline{g}_1 - g_{FB}, \tag{98}$$

$$\text{with: } \overline{g}_1 = \frac{1-\beta}{\beta}\frac{\alpha}{1/\beta+(\delta-1)} - \frac{1-\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}, \tag{99}$$

$$g_{FB} = \frac{G}{Y_{FB}}. \tag{100}$$

Due to the the credit constraint $a_{u,FB} \geq 0$, if the first-best equilibrium exists, equation (98) implies that $g_{FB} \leq \overline{g}_1$. We can then deduce $a_{e,FB}$ from (96):

$$a_{e,FB} = a_{u,FB} + \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1},$$

which verifies $a_{e,FB} \geq a_{u,FB} \geq 0$.

C Characterizing the steady-state equilibrium with positive capital taxes

C.1 FOCs derivation

We focus on the case where unemployed agents are credit-constrained. Note that the situation where both unemployed agents are credit-constrained is not optimal whenever $u'(0) = \infty$. Indeed, when both agents are credit-constrained, deviating and having employed agents to save a small amount yields an finite increase in unemployed agents utility.

Using individual budget constraints, Euler equations (38) and (39) become:

$$u'\left(\frac{w_t(\chi w_t)^\varphi}{\varphi+1} - a_{e,t}\right) = \beta \mathbb{E}_t\left[R_{t+1}u'(R_{t+1}a_{e,t})\right], \tag{101}$$

$$u'(R_t a_{e,t-1}) > \beta \mathbb{E}_t\left[R_{t+1}u'\left(\frac{w_{t+1}(\chi w_{t+1})^\varphi}{\varphi+1} - a_{e,t+1}\right)\right].$$

Using log preferences, we deduce from Euler equation (101):

$$a_{e,t} = \frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^\varphi}{1+\varphi} \geq 0. \tag{102}$$

After some simplification, the Ramsey program can then be written as:

$$\max_{\{B_t,w_t,R_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log\left(\frac{1}{1+\beta}\frac{w_t(\chi w_t)^\varphi}{\varphi+1}\right) + \log(R_t \frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi})\right), \tag{103}$$

$$w_{t+1}(\chi w_{t+1})^\varphi > \beta^2 R_{t+1}R_t w_t(\chi w_t)^\varphi, \tag{104}$$

$$G + B_{t-1} + (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} + w_t(\chi w_t)^\varphi = B_t \tag{105}$$
$$+ F(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} - B_{t-1}, (\chi w_t)^\varphi).$$

Note that the Euler inequality for unemployed agents (104) is equivalent at the steady state to $\beta R < 1$, which will always hold in equilibrium.

The Lagrangian associated to program (103)–(105) can be written (up to some constants independent of policies):

$$\mathcal{L} = (1+\beta)(\varphi+1)\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(w_t) + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(R_t) + \log(a_{e,-1}) \tag{106}$$

$$+ \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \mu_t \Big(F(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} - B_{t-1}, (\chi w_t)^\varphi) + B_t - G_t - B_{t-1}$$
$$- (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} - w_t(\chi w_t)^\varphi\Big) \tag{107}$$
$$+ \mu_0\left(F(K_{-1}, (\chi w_0)^\varphi) + B_0 - G_0 - B_{-1} - (R_0 - 1)a_{-1} - w_0(\chi w_0)^\varphi\right).$$

Defining by convention w_{-1} as $\frac{\beta}{1+\beta}\frac{w_{-1}(\chi w_{-1})^{\varphi}}{1+\varphi} = a_{-1}$, FOCs associated to the Lagrangian (106) can be summarized as (for $t \geq 0$):

$$0 = (1+\beta)(\varphi+1)\frac{1}{w_t} + \beta(\chi w_t)^{\varphi}\frac{\beta}{1+\beta}\mathbb{E}_t\left[\mu_{t+1}(F_{K,t+1} - R_{t+1} + 1)\right] \tag{108}$$
$$+ \chi\mu_t(\chi w_t)^{\varphi-1}\left(\varphi F_{L,t} - (\varphi+1)w_t\right),$$
$$\mu_t = \beta\mathbb{E}_t\left[(1+F_{K,t+1})\mu_{t+1}\right], \tag{109}$$
$$1 = R_t\mu_t\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}. \tag{110}$$

We can take advantage of FOCs (109) and (110) to simplify FOC (108) as follows:

$$\mu_t w_t(\chi w_t)^{\varphi}\left(1 - (1+\beta)\varphi\frac{\tau_t^L}{1-\tau_t^L}\right) = (1+\varphi)(1+\beta), \tag{111}$$

which is a time-t equation only and does not raise convergence issues. The only dynamic FOC is the forward-looking equation (109). We will check that the system is well-defined and does not raise convergence issues.

C.2 Checking that FOCs are identical

We check here that the first-order conditions of the Ramsey program derived in the general case of Section 2.6 exactly simplify to the first-order conditions derived in the specific case of Section 3.4.3. We start with expressing ψ_t^i and $\psi_{l,t}^i$ (equations (23) and (27)) in the context of the GHH utility function. We denote by $C = c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}$. Since $U(c,l) = \ln\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}\right)$, we compute:

$$U_c(c,l) = \frac{1}{C},\ U_{cc}(c,l) = -\frac{1}{C^2},\ U_l(c,l) = -\chi^{-1}l^{1/\varphi}\frac{1}{C},$$
$$U_{ll}(c,l) = -\frac{\chi^{-1}l^{1/\varphi-1}}{C}\left(\frac{1}{\varphi} + \frac{\chi^{-1}l^{1/\varphi}}{C}\right),\ U_{cl}(c,l) = \frac{\chi^{-1}l^{1/\varphi}}{C^2}.$$

Plugging this into equations (23) and (27) and using the labor Euler equation (11) stating that $\chi^{-1}l_t^{i,1/\varphi} = y_t^i w_t$, we deduce that the expressions of ψ_t^i and $\psi_{l,t}^i$ become:

$$\psi_t^i C_t^i = 1 + \left(\lambda_{c,t}^i - R_t\lambda_{c,t-1}^i\right)\frac{1}{C_t^i}, \tag{112}$$
$$\psi_{l,t}^i C_t^i = y_t^i w_t\left(1 + \frac{\lambda_{l,t}^i}{\varphi l_t^i} + (\lambda_{c,t}^i - R_t\lambda_{c,t-1}^i)\frac{1}{C_t^i}\right). \tag{113}$$

We now turn to the FOCs. Note that FOC (28) is exactly the same as FOC (54), while FOC (31) has no equivalent in the simplified version since the progressivity parameter τ_t is set to zero. FOC (26) can also be written with $\tau_t = 0$: $\psi_{l,t}^i = w_t y_t^i \psi_t^i + \mu_t(F_{L,t} - w_t)y_t^i$. Plugging (112) and

(113) yields:

$$\frac{\lambda_{l,t}^i}{\varphi l_t^i}\frac{y_t^i w_t}{C_t^i} = \mu_t(F_{L,t} - w_t)y_t^i,$$

which is equivalent to $0 = 0$ for unemployed agents since their productivity is null. For employed agent with a productivity normalized to one, it becomes:

$$\lambda_{e,l,t} = \varphi \mu_t l_{e,t} C_{e,t} \frac{\tau_t^L}{1 - \tau_t^L}. \tag{114}$$

The three remaining FOCs are equations (25), (29), and (30). Taking advantage of the deterministic transitions between employment and unemployment, as well as the fact that unemployed agents are credit-constrained (implying $a_{u,t-1} = \lambda_{u,c,t-1} = 0$) with null productivity, these three FOCs can also be written as follows ($a_{e,t-1}, l_{e,t} > 0$):

$$\psi_{e,t} - \mu_t = \beta R_{t+1}(\psi_{u,t+1} - \mu_{t+1}), \tag{115}$$

$$\mu_t C_{u,t} = \psi_{u,t} C_{u,t} + \frac{\lambda_{e,c,t-1}}{a_{e,t-1}}, \tag{116}$$

$$\mu_t C_{e,t} = \psi_{e,t} C_{e,t} + \frac{\lambda_{e,l,t}}{l_{e,t}}, \tag{117}$$

while similarly expressions of ψ_t^i in (112) can further be specified as:

$$\psi_{e,t} C_{e,t} = 1 + \frac{\lambda_{e,c,t}}{C_{e,t}}, \tag{118}$$

$$\psi_{u,t} C_{u,t} = 1 - R_t \lambda_{e,c,t-1} \frac{1}{C_{u,t}}. \tag{119}$$

Combining (116) and (119) with $a_{e,t-1} = \frac{C_{u,t}}{R_t}$ (which is unemployed agents' budget constraint (34)) implies:

$$\mu_t C_{u,t} = 1, \tag{120}$$

with the expression of $C_{u,t} = R_t \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}$ is identical to FOC (55).

Using the consumption Euler equation (38) stating that $\frac{1}{C_{e,t}} = \beta R_{t+1} \frac{1}{C_{u,t+1}}$, the budget constraints (33) and (34) implying that $C_{u,t} = \beta R_t C_{e,t-1}$, and (120) meaning that $1 = \beta \mu_{t+1} R_{t+1} C_{e,t}$, we deduce from (115) and (118):

$$\frac{\lambda_{e,c,t}}{C_{e,t}} = \frac{\beta}{1+\beta}(\mu_t C_{e,t} - 1). \tag{121}$$

Finally, we turn to FOC (117). Combined with the expressions of $\lambda_{e,l,t}$ in (114), $\psi_{e,t}$ in (118), and of $\lambda_{e,c,t}$ in (121), it becomes:

$$C_{e,t}\mu_t \left(1 - (1+\beta)\varphi \frac{\tau_t^L}{1-\tau_t^L}\right) = 1. \tag{122}$$

Using the budget constraint (33) stating that $C_{e,t} = \frac{w_t(\chi w_t)^\varphi}{(1+\beta)(1+\varphi)}$, equation (122) becomes FOC (53). This completes the proof that the generic FOCs of Section 2.6 exactly imply the specific FOCs of Section 3.4.3.

C.3 Steady state

Note that because of the FOC (110), $\mu = 0$ or $R = 0$ is not possible at the steady state. FOCs (108)–(110) and governmental budget constraint (105) become at the steady state, where we denote variable without subscripts:

$$\frac{1}{1+\beta}\mu w(\chi w)^\varphi = \varphi + 1 + \mu(\chi w)^\varphi \varphi(F_L - w), \tag{123}$$

$$1 = \beta(1 + F_K) \tag{124}$$

$$1 = R\mu \frac{\beta}{1+\beta}\frac{w(\chi w)^\varphi}{1+\varphi} \tag{125}$$

$$F(\frac{\beta}{1+\beta}\frac{w(\chi w)^\varphi}{1+\varphi} - B, (\chi w)^\varphi) = G + (R-1)\frac{\beta}{1+\beta}\frac{w(\chi w)^\varphi}{1+\varphi} + w(\chi w)^\varphi \tag{126}$$

Using (125) and $w = (1-\tau^L)F_L$, equation (123) becomes:

$$\frac{1}{\beta} - R = \varphi\frac{1+\beta}{\beta}\left(\frac{F_L}{w} - 1\right). \tag{127}$$

Using $w = (1-\tau^L)F_L$, and $R - 1 = (1-\tau^K)F_K = (1-\tau^K)(\beta^{-1} - 1)$, (127) yields:

$$\tau^K = \varphi\frac{1+\beta}{1-\beta}\frac{\tau^L}{1-\tau^L}. \tag{128}$$

After several manipulations and using (124) and (128), as well as the properties of F, the governmental budget constraint (126) implies that τ^L is a solution of the following equation:

$$\tau^L = \frac{1}{1-\alpha}\frac{g_{FB}(1-\tau^L)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}, \tag{129}$$

where $\overline{g}_1$ and g_{FB} are defined in (99) and (100) respectively. Equation (129) can admit zero, one, or two solutions (as the right hand-side is convex). When two solutions are available, the planner unambiguously chooses the lowest tax (since it is associated to higher post-tax wages and hence higher consumption).

Regarding allocation, we have:

$$c_e = \frac{1}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}, \tag{130}$$

$$c_u = \frac{1-(1-\beta)\tau^K}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}. \tag{131}$$

Finally, a condition for the $\tau^K > 0$-equilibrium to exist is $c_u > 0$, or equivalently, using (128) that the solution of (129) must verify:

$$(1+(1+\beta)\varphi)\tau^L < 1. \tag{132}$$

C.4 Characterization of positive public debt

The financial market clearing condition (36) implies using (102) and the definition of w:

$$B = (\chi w)^{\varphi}\left(\frac{\beta}{1+\beta}\frac{1-\tau^L}{1+\varphi}F_L - \frac{K}{L}\right),$$

which is positive iff: $\frac{\beta}{1+\beta}\frac{1-\tau^L}{1+\varphi} > \frac{1}{F_L}\frac{K}{L}$. Using FOC (190) and the definitions of F and $\overline{g}_1$, we can simplify $\frac{1}{F_L}\frac{K}{L}$ and obtain that $B > 0$ iff:

$$\tau^L < -\frac{1+\varphi}{1-\alpha}\frac{1+\beta}{1-\beta}\overline{g}_1. \tag{133}$$

Using the expression (129) of τ^L, we get an equivalent condition to (133):

$$g_{FB}(1-\tau^L)^{-\varphi} < \overline{g}_{\text{pos}},$$

$$\text{where: } \overline{g}_{\text{pos}} = \frac{1+\beta}{1-\beta}(1+2\varphi)(-\overline{g}_1). \tag{134}$$

C.5 Non-existence of the $\tau^K = 0$-equilibrium

We prove here that, at the steady state, the equilibrium featuring full risk-sharing and $\tau^K = 0$ does not exist. More precisely, we show that it is always dominated by the equilibrium featuring binding credit constraint and $\tau^K > 0$ (Sections C.1 and C.3). We recall that the 0-subscript relates to the equilibrium with $\tau^K = 0$, and no subscript to the equilibrium with $\tau^K > 0$. The proof is split into two parts: (i) when the $\tau_k > 0$-equilibrium exists, i.e., when condition (132) holds (Section C.5.2); and (ii) when the $\tau_k > 0$-equilibrium does not exist, i.e., when condition (132) does not holds (Section C.5.3).

C.5.1 Characterization of the $\tau^K = 0$-equilibrium

We focus on the full insurance equilibrium with zero capital tax. We use a 0 subscript to denote quantities in this case: $\tau_0^K = 0$. With the same steps as in Section B, we have:

$$w_0 = (1-\tau^L)w_{FB}, \tag{135}$$

$$K_0 = (1-\tau^L)^{\varphi}K_{FB}, \tag{136}$$

$$Y_0 = (1-\tau^L)^{\varphi}Y_{FB}, \tag{137}$$

Governmental budget constraint (37) becomes:

$$B_0 = -\frac{\beta}{1-\beta}G + \frac{\beta}{1-\beta}\tau_0^L(1-\tau_0^L)^{\varphi}w_{FB}(\chi w_{FB})^{\varphi}.$$

Perfect risk sharing (i.e., $c_{u,0} = c_{e,0} - \frac{1}{\chi}\frac{l_{e,0}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$) and financial market clearing (i.e., $A_0 = K_0 + B_0$) imply (as in (96) and (96)), after proper substitution:

$$a_{e,0} - a_{u,0} = \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}(1-\tau_0^L)^{\varphi+1}, \tag{138}$$

$$a_{u,0} + a_{e,0} = (1-\tau^L)^{\varphi}K_{FB} - \frac{\beta}{1-\beta}G + \frac{\beta}{1-\beta}\tau_0^L(1-\tau_0^L)^{\varphi}w_{FB}(\chi w_{FB})^{\varphi}. \tag{139}$$

We deduce by combination of the two previous equations:

$$\begin{aligned} 2a_{u,0} = {} & (1-\tau_0^L)^{\varphi}K_{FB} - \frac{\beta}{1-\beta}G - \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}(1-\tau_0^L)^{\varphi+1} \\ & + \frac{\beta}{1-\beta}\tau_0^L(1-\tau_0^L)^{\varphi}w_{FB}(\chi w_{FB})^{\varphi}. \end{aligned}$$

Dividing by Y_0 of (137) and using notation (93)–(95) and (45), we obtain:

$$2\frac{a_{u,0}}{Y_0} = \frac{\beta}{1-\beta}(\overline{g}_1 - g_{FB}(1-\tau_0^L)^{-\varphi}) + \left(\frac{1}{1-\beta} + \frac{1}{1+\beta}\frac{1}{\varphi+1}\right)\beta\tau_0^L(1-\alpha). \tag{140}$$

We turn to the computation of $a_{e,0}$. Using (138) and (139), we get:

$$2\frac{a_{e,0}}{Y} = 2\frac{a_{u,0}}{Y} + 2\frac{\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}(1-\tau_0^L),$$

implying that $a_{e,0} \geq a_{u,0}$ for all values of $\tau_0^L \leq 1$. We compute the consumption level $c_{u,0}$ from individual budget constraint (34):

$$2\frac{c_{u,0}}{Y_{FB}} = (1-\tau_0^L)^{\varphi}\overline{g}_1 - \frac{G}{Y_{FB}} + \frac{2}{1+\beta}\frac{1-\alpha}{\varphi+1}(1-\tau_0^L)^{\varphi} + \frac{\varphi}{\varphi+1}(1-\alpha)\tau_0^L(1-\tau_0^L)^{\varphi}. \tag{141}$$

Computing the derivative of $2\frac{c_{u,0}}{Y_{FB}}$ with respect to the labor tax τ_0^L yields:

$$\frac{1}{\varphi(1-\tau_0^L)^{\varphi-1}}\frac{\partial}{\partial\tau_0^L}2\frac{c_{u,0}}{Y_{FB}} = -\frac{(1-\beta)\alpha}{1+\beta(\delta-1)} - (1-\alpha)\tau_0^L < 0, \tag{142}$$

whenever $\tau_0^L \geq 0$. We deduce from the last inequality that $c_{u,0}$ (and hence aggregate welfare since $c_{u,0} = c_{e,0} - \frac{1}{\chi}\frac{l_0^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$) is decreasing with τ_0^L. Since $a_{e,0} \geq a_{u,0}$ for all values of τ_0^L, the value of τ_0^L is chosen as small as possible for credit constraints to hold and hence such that $a_{u,0} = 0$. From (140), τ_0^L is the solution of:

$$\tau_0^L = \frac{1}{1+\frac{1-\beta}{1+\beta}\frac{1}{\varphi+1}}\frac{g_{FB}(1-\tau_0^L)^{-\varphi} - \overline{g}_1}{1-\alpha}. \tag{143}$$

In words, the planner chooses the lowest possible labor tax to reduce distortions. Finally, regarding allocation, we compute:

$$c_{u,0} = c_{e,0} - \chi^{-1}\frac{l_0^{1+1/\varphi}}{1+1/\varphi} = \frac{1}{1+\beta}\frac{w_0(\chi w_0)^{\varphi}}{1+\varphi}. \tag{144}$$

Laffer curve. Equation (143) admits 0, 1 or 2 solutions, and reflects some form of Laffer curve. The case with zero solution appears when no equilibrium exists: the public spending G is too high to be financed and no level of labor tax allows the governmental budget to hold. The case with 2 solutions is the standard case when the equilibrium exists: it features either a low tax/high labor supply or a high tax/low labor supply combination. The planner (since inequality (142) holds) unambiguously opts for the lowest tax. Finally the 1-solution case is a limit case that only occurs for a unique value of public spending.

C.5.2 Case where the $\tau_k > 0$-equilibrium exists

We will show that the allocations of the $\tau^K = 0$ and $\tau^K > 0$ equilibria are the outcomes of two optimization programs, where the first one is identical to the other one, up to an additional constraint.

The proof rely on the expression of the problem, for which bot cases $\tau^K > 0$ and $\tau^K = 0$ can be solution. More formally, we consider the following program:

$$\max_{\{B_t,w_t,R_t\}} \sum_{t=0}^{\infty} \beta^t \left((1+\beta) \log \left(\frac{1}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{\varphi+1} \right) + \log(\beta R_t) \right) \tag{145}$$

$$G + B_{t-1} + (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} + w_t(\chi w_t)^{\varphi} = B_t \tag{146}$$

$$+ F(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}),$$

$$R_t \geq 1 + \tilde{r}_t \tag{147}$$

where the interest rate $\tilde{r}_t$ in the constraint (147) is taken as exogenous with $\tilde{r}_t = F_{K,t}$. We now show that the previous program has the desired properties.

We start with the case $\tau^K = 0$. Denoting by $\beta^t \mu_t$ the Lagrange multiplier associated to the constraint (146), the maximization with respect to B_t yields: $\mu_t = \beta(1 + F_{K,t+1})\mu_{t+1}$,or at the steady state: $\beta(1 + F_K) = 1$.The constraint (146) implies then at the steady state, using (92)–(95) that the labor tax, denoted $\hat{\tau}_0^l$ verifies:

$$(1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)\hat{\tau}_0^l = \frac{g_{FB}}{(1-\hat{\tau}_0^l)^{\varphi}} - \overline{g}_1, \tag{148}$$

which is the equation as (143) for τ_0^L. Since the planner will also choose the lowest solution to (148), we deduce that $\hat{\tau}_0^l = \tau_0^L$. Consumption levels then mechanically verify equation (144), which proves that the steady-state equilibrium with $\tau^K = 0$ is a steady-state solution of the program (145)–(146) where we impose $\tau_t^K = 0$ at all dates.

We now turn to the unconstrained case ($\tau^K \neq 0$). In that case, the FOCs of the program (145)–(146), with respect to B_t, R_t, and w_t, respectively are:

$$\mu_t = \mu_{t+1}\beta(1 + F_{K,t}),$$

$$1 = R_t\mu_t \frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi},$$

$$\frac{(1+\beta)(1+\varphi)}{w_t} = \frac{\mu_t}{w_t}\left((\varphi+1)w_t(\chi w_t)^{\varphi} - \varphi F_{L,t}(\chi w_t)^{\varphi}\right)$$

$$+ \frac{\beta\mu_{t+1}}{w_t}(R_{t+1} - 1 - F_{K,t+1})\frac{\beta}{1+\beta}w_t(\chi w_t)^{\varphi}.$$

At the steady-state, we obtain:

$$1 = \beta(1 + F_K), \tag{149}$$

$$1 = R\mu\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}, \tag{150}$$

$$\frac{(1+\beta)(1+\varphi)}{\mu(\chi w)^{\varphi}} = (\varphi+1)w - \varphi F_L + \beta(R - 1 - F_K)\frac{\beta}{1+\beta}w. \tag{151}$$

With (149) and (150), equation (151) yields, after some manipulation that taxes $\hat{\tau}^k$ and $\hat{\tau}^l$ verify:

$$\hat{\tau}^k = \varphi \frac{1+\beta}{1-\beta} \frac{\hat{\tau}^l}{1-\hat{\tau}^l},$$

which is the same relationship as (128) for τ^K. As we did in the constrained case, the constraint (146) of the program at the steady state yields for $\hat{\tau}^l$ the same definition as equation (129) for τ^L. We deduce that $\hat{\tau}^l = \tau^L$ and $\hat{\tau}^k = \tau^K$, when τ^L satisfies condition (132). Consumption levels (130) and (131) then easily follow. It is also easy to check that $\tau^K, \tau^L > 0$.

We therefore deduce that the allocation with $\tau^K = 0$ is the solution of a constrained program and is hence dominated by the allocation $\tau_k \neq 0$ – when ever the later exist.[12]

C.5.3 Case where the $\tau_k > 0$-equilibrium does not exist

For the sake of completeness, we now show that an equilibrium with $\tau^K = 0$ does not exist even when the equilibrium where $\tau^K > 0$ does not exist. Assume now that the solution of (129) does not verify condition (132). We will show that in that case the $\tau_k = 0$-equilibrium does not exist either.

To do so, we focus on the limit case when condition (132) does not hold, implying that the solution, denoted τ_m^L, to (129) is:

$$\tau_m^L = \frac{1}{1+(1+\beta)\varphi}. \tag{152}$$

The argument easily extends to any value $\tau^L \geq \tau_m^L$ (see explanation after equation (154)). Equation (129) implies that it corresponds to a public spending $g_{FB,0}$ verifying:

$$g_{FB,0}(1-\tau_m^L)^{-\varphi} = (1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)\tau_m^L + \overline{g}_1. \tag{153}$$

To show that the $\tau_k = 0$-equilibrium does not exist, we show that there is no solution to (143), and more precisely that, for all τ_0^L:

$$\tau_0^L < \frac{g_{FB,0}(1-\tau_0^L)^{-\varphi} - \overline{g}_1}{(1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}. \tag{154}$$

Note that the argument we develop would easily extend to any solution τ^L to (129), such that $\tau^L \geq \tau_m^L$. Indeed, these cases would imply public spending levels higher than $g_{FB,0}$. The equilibrium non-existence would then be implied by inequality (154).

To show that inequality (154), notice that $\tau_0 \in (-\infty, 1) \mapsto g_{FB,0}(1-\tau_0^L)^{-\varphi} - \overline{g}_1 - (1-$

[12]Note that the argument could not be applied right away from the initial program formulation of Section 3 because with $\tau_k \neq 0$, the constraint $a_{u,t} = 0$ was binding – which is not present anymore with the modified program (145)–(146).

$\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)\tau_0^L$ is convex admits a global minimum denoted $\tau_{0,\min}^L$, defined as:

$$1-\tau_{0,\min}^L=\left(\frac{\varphi g_{FB,0}}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}\right)^{\frac{1}{\varphi+1}} \tag{155}$$

To prove inequality (154), we only need to show that:

$$\Delta>0, \tag{156}$$

$$\text{where: } \Delta=\frac{g_{FB,0}(1-\tau_{0,\min}^L)^{-\varphi}-\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}-\tau_{0,\min}^L. \tag{157}$$

Using (155), the expression (157) of Δ becomes:

$$\Delta=(\varphi^{-1}+1)\left(\frac{\varphi g_{FB,0}}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}\right)^{\frac{1}{\varphi+1}}-\frac{\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}-1. \tag{158}$$

The definition (153) of $g_{FB,0}$ implies:

$$\begin{aligned}
&\frac{\varphi g_{FB,0}}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}\\
&=(1-\tau_m^L)^{\varphi+1}\left(\frac{\varphi\tau^L}{1-\tau^L}+\frac{\varphi(1-\alpha)\frac{\varphi}{1+\varphi}\frac{\tau_m^L}{1-\tau_m^L}}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}+\frac{\varphi\overline{g}_1\frac{1}{1-\tau_m^L}}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}\right),\\
&=(1-\tau^L)^{\varphi+1}\left(\frac{1}{1+\beta}+\frac{\varphi}{(1+\beta)(1+\varphi)+1-\beta}+\frac{\varphi\overline{g}_1\frac{1}{1-\tau^L}}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}\right),
\end{aligned}$$

where the second inequality comes from the definition (152) of τ_m^L. Plugging this value into (158) yields:

$$\begin{aligned}
\Delta=&\frac{(1+\beta)(\varphi+1)}{1+(1+\beta)\varphi}\left(\frac{2(1+(1+\beta)\varphi)}{(1+\beta)((1+\beta)(1+\varphi)+1-\beta)}+\frac{\frac{1+(1+\beta)\varphi}{1+\beta}\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}\right)^{\frac{1}{\varphi+1}}\\
&-\frac{\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}-1.
\end{aligned} \tag{159}$$

In (159), Δ can be seen as a function of $\tilde{g}_1=\frac{\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)}$, defined on $(-\frac{2}{(1+\beta)(1+\varphi)+1-\beta},\infty)$. It is straightforward to check that this function is concave, admits a unique maximum equal to $\frac{(1+\beta)\varphi}{(1+\beta)(1+\varphi)+1-\beta}>0$ that is reached in $\tilde{g}_1^*=\frac{-2\varphi(1+\beta)}{(1+(1+\beta)\varphi)((1+\beta)(1+\varphi)+1-\beta)}$. Thus, there exist two (mathematical) bounds denoted $\tilde{g}_1^{\inf}<\tilde{g}_1^*<\tilde{g}_1^{\sup}$, such that $\Delta(\tilde{g}_1)>0$ iff $\tilde{g}_1\in(\tilde{g}_1^{\inf},\tilde{g}_1^{\sup})$. The rest of the proof consists in finding two economical bounds on $\tilde{g}_1$, denoted by $\tilde{g}_1^{\min}$ and $\tilde{g}_1^{\max}$ and to prove that $\Delta(\tilde{g}_1^{\min})>0$ and $\Delta(\tilde{g}_1^{\max})>0$. We can then deduce from the properties of the function Δ that $\Delta(\tilde{g}_1) > 0$ for all economically acceptable $\tilde{g}_1$.

Lower bound on $\tilde{g}_1$. The definition (99) of $\overline{g}_1 = \frac{1-\beta}{\beta}\frac{\alpha}{1/\beta+\delta-1} - \frac{1-\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}$ readily implies:

$$\frac{\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)} \geq -\frac{1-\beta}{(1+\beta)(1+\varphi)+1-\beta} = \tilde{g}_1^{\min}.$$

From (159), we deduce:

$$\begin{aligned}\Delta(\tilde{g}_1^{\min}) &\geq \frac{(1+\beta)(1+\varphi)}{(1+\beta)(1+\varphi)+1-\beta}\left(\left(1+\frac{1}{1+(1+\beta)\varphi}\right)^{\frac{\varphi}{\varphi+1}}-1\right),\\ &> 0,\end{aligned}$$

where the second inequality is a direct implication of $\beta \in (0,1)$ and $\varphi > 0$.

Upper bound on $\tilde{g}_1$. The upper bound on $\tilde{g}_1$ is less straightforward. Equation (153) – seen as an equation in τ_m^L for a given $g_{FB,0}$ – admits one or two roots (since by construction the no-root case is excluded). To guarantee that the smallest solution is chosen, the derivative of the $\tau \mapsto (1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}\right)\tau + \overline{g}_1 - g_{FB,0}(1-\tau)^{-\varphi}$ must be positive in τ_m^L (the function being concave, it has to intercept 0 before it reaches its maximum). Or equivalently:

$$\varphi g_{FB,0}(1-\tau_m^L)^{-\varphi-1} \leq (1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}\right).$$

Using the definition (153) of $g_{FB,0}$, we obtain that this condition is equivalent to:

$$\frac{\overline{g}_1}{(1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}\right)} \leq \frac{2\beta}{(1+\beta)(1+\varphi)+1-\beta} = \tilde{g}_1^{\max}.$$

From (159), we obtain, after some manipulations:

$$\frac{\Delta(\tilde{g}_1^{\max})}{\tau_m^L} \geq (1+\beta)(1+\varphi)\left(\left(1+\frac{\varphi(1+\beta)}{1+(1+\varphi(1+\beta))}\right)^{\frac{1}{\varphi+1}}-1\right) - \beta\frac{\varphi(1+\beta)}{(1+(1+\varphi(1+\beta)))},$$

whose left-handside can be seen as a function of $\frac{\varphi(1+\beta)}{1+(1+\varphi(1+\beta))}$ (that lies in $(0,1)$). We denote:

$$\tilde{\Delta} : x \in (0,1) \mapsto (1+\beta)(\varphi+1)\left((1+x)^{\frac{1}{\varphi+1}}-1\right) - \beta x.$$

Using a second-order Taylor development, we have for $x \in (0,1)$:

$$\frac{\tilde{\Delta}(x)}{x} \geq 1 - \frac{\varphi}{\varphi+1}\frac{1+\beta}{2}x > 0,$$

where the second inequality comes from $x < 1$, $\beta < 1$, and $\varphi > 0$. This implies $\Delta(\tilde{g}_1^{\max}) > 0$ and concludes the proof.

C.6 A non-interior steady-state equilibrium

Here we investigate the case when (129) admits a solution, but when this solution does not verify condition (132). We have:

$$\left(1-(1+\varphi(1+\beta))\tau_t^L\right)(1-\tau_t^L)^{\varphi}\mu_t\tilde{w}_t(\chi\tilde{w}_t)^{\varphi} = (1+\beta)(1+\varphi), \tag{160}$$

$$\frac{\mu_{t+1}}{\mu_t} = \frac{1}{\beta(1+F_{K,t+1})}, \tag{161}$$

$$(1+(1-\tau_t^K)F_{K,t})\mu_t(1-\tau_{t-1}^L)^{\varphi+1}\tilde{w}_{t-1}(\chi\tilde{w}_{t-1})^{\varphi} = \frac{(1+\beta)(1+\varphi)}{\beta} \tag{162}$$

Equation (160) implies that for all t:

$$\tau_t^L \leq \frac{1}{1+\varphi(1+\beta)}.$$

In particular, $\tau^L = \lim_{t\to\infty}\tau_t^L \leq \frac{1}{1+\varphi(1+\beta)}$. From (160), we also understand that there are possibly non-interior steady states, featuring $\lim_t \mu_t = \infty$ or $\lim_t \tilde{w}_t = \infty$.

First case: $\lim w_t = w^* < \infty$.

- The case $w^* = 0$ is not possible. Otherwise there are no resources to pay G.
- Assume that $\lim \mu_t = \infty$, then equation (160) implies $\lim_t \tau_t^L = (1+\varphi(1+\beta))^{-1}$. Equation (162) then yields $\lim_t(1+(1-\tau_t^K)F_{K,t}) = \lim_t R_t = 0$.

Second case: $\lim_t w_t = \infty$. We thus have $\lim_t \tilde{w}_t = \infty$. We also have from factor price definitions:

$$\chi\tilde{w}_t = \left(\frac{\chi(1-\alpha)}{(1-\tau_t^L)^{\alpha\varphi}}\right)^{\frac{1}{1+\varphi\alpha}} K_{t-1}^{\frac{\alpha}{1+\varphi\alpha}},$$

which yields $\lim K_t = \infty$ and $\lim_t \frac{K_{t-1}}{(\chi w_t)^{\varphi}} = \infty$. We deduce $\lim_t F_{K,t} = -\delta$. We then deduce $\lim_t \mu_t = \infty$, $\lim_t \tau_t^L = (1+\varphi(1+\beta))^{-1}$, and $\lim_t R_t = 0$.

These two non-stationary equilibria feature $\lim_t \mu_t = \infty$ and $\lim_t R_t = 0$.

D Model dynamics in the presence of aggregate shocks

D.1 Model linearization

Defining:

$$\theta = \frac{1}{1+\varphi}\frac{\beta}{1+\beta}, \tag{163}$$

FOCs (108) and (109) and governmental budget constraint (105) become:

$$\mu_t = \beta(1 + \alpha Z_{t+1} K_t^{\alpha-1} \chi^{(1-\alpha)\varphi} w_{t+1}^{(1-\alpha)\varphi} - \delta)\mu_{t+1}, \tag{164}$$

$$0 = 1 - \mu_t w_t (\chi w_t)^{\varphi} (1-\theta) + \frac{\varphi}{1+\varphi}\mu_t (1-\alpha) K_{t-1}^{\alpha} (\chi w_t)^{\varphi(1-\alpha)}, \tag{165}$$

$$K_{t-1}^{\alpha}(\chi w_t)^{\varphi(1-\alpha)} = G_t + K_t - (1-\delta)K_{t-1} + \frac{1}{\mu_t} + (1-\theta) w_t (\chi w_t)^{\varphi}. \tag{166}$$

We deduce R_t from $1 = R_t \mu_t \theta w_{t-1} (\chi w_{t-1})^{\varphi}$ (i.e., FOC (110)) and B_t from $B_t = \theta w_t (\chi w_t)^{\varphi} - K_t$ (i.e., financial market clearing).

We denote by a hat the proportional deviation to the steady state value. Formally, for a generic variable x: $\widehat{x} = \frac{x_t - x}{x}$. The linearization of equations (164)–(166) yield after some manipulation:

$$\widehat{\mu}_t - E_t \widehat{\mu}_{t+1} = (1 - \beta(1-\delta))(\widehat{Z}_{t+1} + (\alpha-1)\widehat{K}_t + (1-\alpha)\varphi E_t \widehat{w}_{t+1}), \tag{167}$$

$$0 = -\alpha \widehat{K}_{t-1} + (A-1)\widehat{\mu}_t + ((\varphi+1)(A-1) + 1 + \varphi\alpha)\widehat{w}_t, \tag{168}$$

$$0 = \frac{G}{Y}\widehat{G}_t + \frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\left(\widehat{K}_t - \beta^{-1}\widehat{K}_{t-1}\right) - (A-1)\varphi\frac{1-\alpha}{1+\varphi}\widehat{\mu}_t \tag{169}$$
$$+ (A-1)\,\varphi(1-\alpha)\widehat{w}_t,$$

where τ^L is defined in (129) and where:

$$A = (1 + \frac{1}{\varphi(1+\beta)})(1-\tau^L) > 1, \tag{170}$$

where the inequality comes from condition (132) for the existence of the equilibrium.

D.2 Public debt spending shock

In the remainder, we will focus on full capital depreciation: $\delta = 1$.

Dynamic system. In that case, we can show that, when setting:

$$r_\mu = \frac{(1+\varphi)(A-1)+1+\alpha\varphi}{(1+\alpha\varphi)A}, \tag{171}$$

$$t_\mu = (1-\alpha)\frac{(1+\varphi)(A-1)+1}{(1+\alpha\varphi)A}, \tag{172}$$

$$r_K = \frac{1-\alpha}{\alpha\beta}(A-1)\frac{\varphi}{1+\varphi}\left(1+\frac{(1+\varphi)(A-1)}{(1+\varphi)(A-1)+1+\varphi\alpha}\right), \tag{173}$$

$$t_K = \frac{1}{\beta}\frac{(1+\varphi\alpha)A}{(1+\varphi)(A-1)+1+\varphi\alpha}, \tag{174}$$

$$s_K = -\frac{G}{\alpha\beta Y}, \tag{175}$$

we obtain from (167)–(169):

$$E_t\left[\widehat{\mu}_{t+1}\right] = r_\mu\widehat{\mu}_t + t_\mu\widehat{K}_t, \tag{176}$$

$$\widehat{K}_t = r_K\widehat{\mu}_t + t_K\widehat{K}_{t-1} + s_K\widehat{G}_t, \tag{177}$$

where the dynamics of $\widehat{G}_t$ is given by:

$$\widehat{G}_t = \rho_G\widehat{G}_{t-1} + \sigma_G\varepsilon_{G,t}, \tag{178}$$

$$\text{where: } \varepsilon_{G,t} \sim_{\text{IID}} \mathcal{N}(0,1),$$

and $\sigma_G > 0$ and $\rho_G \in (-1,1)$.

Since $A > 1$, it can be checked that the coefficients t_K, r_K, t_μ are positive, while $r_\mu > 1$.

Deriving a simplified dynamic system. We look for coefficients coefficients ρ_K, σ_K, ρ_μ, σ_μ, such that:

$$\widehat{K}_t = \rho_K\widehat{K}_{t-1} + \sigma_K\widehat{G}_t \tag{179}$$

$$\widehat{\mu}_t = \rho_\mu\widehat{K}_{t-1} + \sigma_\mu\widehat{G}_t \tag{180}$$

Combining (176)–(177) yields:

$$\begin{aligned} E_t\widehat{K}_{t+1} &= r_\mu(\widehat{K}_t - t_K\widehat{K}_{t-1} - s_K\widehat{G}_t) + r_K t_\mu\widehat{K}_t + t_K\widehat{K}_t + s_K\rho_G\widehat{G}_t \\ &= -r_\mu t_K\widehat{K}_{t-1} - s_K r_\mu\widehat{G}_t + (r_K t_\mu + r_\mu + t_K)\widehat{K}_t + s_K\rho_G\widehat{G}_t \end{aligned}$$

$$E_t\widehat{K}_{t+1} - (t_K + r_\mu + r_K t_\mu)\widehat{K}_t + r_\mu t_K\widehat{K}_{t-1} = (s_K\rho_G - r_\mu s_K)\widehat{G}_t.$$

Using (179), we obtain that ρ_K must verify solve the following equation:

$$\rho_K^2 - (t_K + r_\mu + r_K t_\mu)\rho_K + r_\mu t_K = 0, \tag{181}$$

whose discriminant is:

$$D = (t_K + r_\mu + r_K t_\mu)^2 - 4 r_\mu t_K. \tag{182}$$

Since $t_K, r_\mu, r_K, t_\mu \geq 0$, we have $D \geq (t_K + r_\mu)^2 - 4r_\mu t_K = (t_K - r_\mu)^2 > 0$, where the strict inequality comes from $t_K = \frac{1}{\beta r_\mu} > 0$. Equation (181) thus admits two distinct roots, which are:

$$\rho_{K,1} = \frac{t_K + r_\mu + r_K t_\mu + \sqrt{D}}{2} \text{ and } \rho_{K,2} = \frac{t_K + r_\mu + r_K t_\mu - \sqrt{D}}{2}. \tag{183}$$

Since $(t_K + r_\mu + r_K t_\mu)^2 > D > 0$, we deduce that $0 < \rho_{K,2} < \rho_{K,1}$. Furthermore, we can check that a necessary and sufficient condition for the equilibrium to be stable is:

$$\alpha \leq \frac{1}{1 + (1-\beta)(1+\varphi)} < 1, \tag{184}$$

where the second inequality comes from $\beta \in (0,1)$. Note that a sufficient condition for the stability is $\overline{g}_1 < 0$ – which is equivalent to $\alpha \leq \frac{1}{1+(1+\beta)(1+\varphi)}$ and hence implies (184).

Let us prove it. The condition $\rho_{K,2} < 1$ is equivalent to $J := t_K + r_\mu + r_K t_\mu - r_\mu t_K - 1 > 0$. Using equations (171)–(174), we can show that:

$$\begin{aligned}
\frac{J}{J_0} &= (\beta(1+\varphi)(A-1) + (1+\alpha\varphi)(\beta - A)) \\
&\quad + \frac{1-\alpha}{\alpha(1+\varphi)}((1+\varphi)(A-1)+1)\left(2(1+\varphi)(A-1)+1+\varphi\alpha\right), \\
\text{where: } J_0 &= \frac{\varphi(1-\alpha)(A-1)}{\beta(1+\alpha\varphi)A((1+\varphi)(A-1)+1+\varphi\alpha)}.
\end{aligned}$$

Since $A > 1$, $J_0 > 0$ and the sign of J is the one of:

$$\begin{aligned}
\beta(1+\varphi)(A-1) + (1+\alpha\varphi)(\beta - 1 - (A-1)) + \\
\frac{1-\alpha}{\alpha(1+\varphi)}((1+\varphi)(A-1)+1)\left(2(1+\varphi)(A-1)+1+\varphi\alpha\right),
\end{aligned}$$

which can be seen as a quadratic polynomial in $A-1$, that we denote $P(\cdot)$. After some rearrangement, we obtain:

$$\begin{aligned}
P(A-1) = {} & \frac{1+\alpha\varphi}{1+\varphi}(-(1-\beta)(1+\varphi) + \frac{1-\alpha}{\alpha}) + \\
& + (A-1)\left(-(1-\beta)(1+\varphi) + \frac{1-\alpha}{\alpha} + 2(1+\alpha\varphi)\frac{1-\alpha}{\alpha}\right) \\
& + (A-1)^2\frac{1-\alpha}{\alpha}2(1+\varphi).
\end{aligned}$$

A necessary condition for $P(A-1) > 0$ for all $A > 1$ is $P(0) \geq 0$. However, $P(0) \geq 0 \Rightarrow P'(0) > 0$

(since $\beta \in (0,1)$). So, since $P''(0) \geq 0$, $P(0) \geq 0$ is a necessary and sufficient condition for $P(A-1) > 0$ for $A > 1$. The condition $P(0) \geq 0$ is equivalent to condition (184), which concludes the proof regarding equilibrium stability.

Stability and characterization of the system (179)–(180). The Blanchard-Kahn conditions involve checking that $\rho_K < 1$.

Since $0 < \rho_{K,2} < \rho_{K,1}$ and $\rho_{K,2}\rho_{K,1} = \beta^{-1} > 1$, we must have $\rho_{K,1} > 1$, which imposes that $\rho_K = \rho_{K,2}$. The stability Blanchard-Kahn condition requires $\rho_{K,2} < 1$. Note that in the limit case when the equilibrium does not exist (i.e., condition (132) holds with equality), and which corresponds to $A = 1$, it is straightforward to check that $\rho_{K,2} = 1$ and that the dynamic system is not stable.

To characterize further the dynamic system (179)–(180), we deduce from (176)–(177) that ρ_μ is connected through ρ_K with:

$$(r_\mu - \rho_K)\rho_\mu = -t_\mu \rho_K. \tag{185}$$

Since $r_\mu > 1$, $t_\mu > 0$, and $\rho_K \in (0,1)$, we deduce that $\rho_\mu < 0$.

Regarding parameters σ_K and σ_μ, we have from (176)–(177):

$$\sigma_K = r_K \sigma_\mu + s_K, \tag{186}$$

$$r_\mu \sigma_\mu = (\rho_\mu - t_\mu)\sigma_K + \sigma_\mu \rho_G. \tag{187}$$

Equation (187) implies:

$$(r_\mu - \rho_G)\sigma_\mu = (\rho_\mu - t_\mu)\sigma_K. \tag{188}$$

Using $r_\mu > 1 > \rho_G$ and (185) implying that $\rho_\mu - t_\mu = r_\mu \rho_\mu / \rho_K < 0$, we deduce that σ_μ and σ_K have opposite signs. Using $r_K > 0$ and $s_K < 0$ in equation (186), we deduce that $\sigma_\mu > 0 > \sigma_K$.

The role of the shock persistence ρ_G. Combining (186) and (187) yields:

$$(r_\mu + (t_\mu - \rho_\mu)r_K)\sigma_\mu = (\rho_\mu - t_\mu)s_K + \sigma_\mu \rho_G,$$

which yields, by the implicit function theorem:

$$(r_\mu - \rho_G + (t_\mu - \rho_\mu)r_K)\frac{\partial \sigma_\mu}{\partial \rho_G} = \sigma_\mu,$$

since only σ_μ (and σ_K) depend on ρ_G. Since $r_\mu > 1 > \rho_G$, and $\sigma_\mu, t_\mu, r_K > 0 > \rho_\mu$, we deduce using the previous equation and (186) that:

$$\frac{\partial \sigma_\mu}{\partial \rho_G} > 0 \text{ and } \frac{\partial \sigma_K}{\partial \rho_G} > 0.$$

Dynamic of the capital stock. By induction we can then prove that the dynamics (178) and (179) of $\widehat{G}_t$ and $\widehat{K}_t$ can be written as:

$$\widehat{G}_t = \rho_G^t \sigma_G \varepsilon_{G,0},$$
$$\widehat{K}_t = \sigma_K \sigma_G \frac{\rho_G^{t+1} - \rho_K^{t+1}}{\rho_G - \rho_K} \varepsilon_{G,0},$$

where by assumption we have $\widehat{K}_{-1} = \widehat{G}_{-1} = 0$ (no deviation from the steady state).

Let define:

$$\phi(t) = \begin{cases} \frac{\rho_K^{t+1} - \rho_G^{t+1}}{\rho_K - \rho_G} & \text{if } \rho_K \neq \rho_G, \\ (t+1)\rho_G^t & \text{if } \rho_K = \rho_G, \end{cases}$$

with $\phi(0) = 1$, $\phi(\infty) = 0$, and:

$$(\rho_K - \rho_G)\phi'(t) = \ln(\rho_K)\rho_K^{t+1} - \ln(\rho_G)\rho_G^{t+1}.$$

We have $\phi'(t_m) = 0$ iff:

$$t_m + 1 = \begin{cases} \frac{\ln(-\ln(\rho_K)) - \ln(-\ln(\rho_G))}{\ln(\rho_G) - \ln(\rho_K)} > 0 & \text{if } \rho_K \neq \rho_G, \\ -\frac{1}{\ln(\rho_G)} > 0 & \text{if } \rho_K = \rho_G, \end{cases}$$

It is direct to check that $\phi'(t) > 0$ iff $t < t_m$. The capital response is procyclical (it has the sign of $\widehat{G}_0$). When $\widehat{G}_0 > 0$, capital increases until date t_m before decreasing and converging back to its steady-state value.

We now investigate the impact of ρ_G on t_m. Defining $r_G := -\ln(\rho_G)$ and $r_K := -\ln(\rho_K)$, we obtain:

$$\frac{\partial t_m}{\partial r_G} = \frac{\frac{r_G - r_K}{r_G} - (\ln(r_G) - \ln(r_K))}{(r_G - r_K)^2} \text{ if } \rho_K \neq \rho_G.$$

By Taylor-Lagrange theorem, there exists $r \in (r_K, r_G)$, such that:

$$\ln(r_K) - \ln(r_G) = \frac{r_K - r_G}{r_G} - \frac{(r_K - r_G)^2}{2r^2},$$

from which we deduce:

$$\frac{\partial t_m}{\partial r_G} = \begin{cases} \frac{-\frac{(r_K - r_G)^2}{2r^2}}{(r_G - r_K)^2} < 0 & \text{if } \rho_K \neq \rho_G, \\ -\frac{1}{r_G^2} < 0 & \text{if } \rho_K = \rho_G, \end{cases}$$

So t_m decreases with r_G and increases with ρ_G: the more persistent ρ_G, the longer the impact of capital dynamics.

We now study the impact of ρ_G on the $\phi(t_m)$, the maximal value of ϕ (which corresponds to the maximal variation of capital stock following the public spending shock.

$$\phi(t_m) = \begin{cases} \frac{e^{-\frac{\ln(r_G)-\ln(r_K)}{r_G-r_K}r_K} - e^{-\frac{\ln(r_G)-\ln(r_K)}{r_G-r_K}r_G}}{e^{-r_K}-e^{-r_G}} & \text{if } \rho_K \neq \rho_G, \\ r_G^{-1}e^{-r_G(r_G^{-1}-1)} > 0 & \text{if } \rho_K = \rho_G, \end{cases}$$

We focus on the case where $\rho_K \neq \rho_G$ and $\rho_K < 1$. Note that we have:

$$\phi(t_m) \to_{\rho_G \to 1} \frac{1}{1-\rho_K}.$$

and

$$\begin{aligned} \phi(t_m) &= \frac{\left(\frac{r_K}{r_G}\right)^{\frac{r_K}{r_G-r_K}} - \left(\frac{r_K}{r_G}\right)^{\frac{r_G}{r_G-r_K}}}{e^{-r_K} - e^{-r_G}} \\ &= \frac{\left(\frac{r_G}{r_K}\right)^{-\frac{1}{r_G/r_K-1}} - \left(\frac{r_G}{r_K}\right)^{-\frac{1}{r_G/r_K-1}-1}}{e^{-r_K}\left(1-e^{-r_K(r_G/r_K-1)}\right)} \end{aligned}$$

We define $x := r_G/r_K - 1$, such that $\frac{r_G}{r_K} = 1+x$, $\frac{r_K}{r_G-r_K} = \frac{1}{r_G/r_K-1} = \frac{1}{x}$, and $\frac{r_G}{r_G-r_K} = 1+\frac{1}{x}$, and we define $f(x) := \phi(t_m) = \frac{(1+x)^{-\frac{1}{x}}-(1+x)^{-\frac{1}{x}-1}}{1-e^{-r_K x}}$, such that:

$$(1+x)^{\frac{1}{x}+1}f'(x) = \frac{\frac{\ln(1+x)}{x}(1-e^{-r_K x}) - x r_K e^{-r_K x}}{(1-e^{-r_K x})^2}.$$

Note that:

$$(1+x)^{\frac{1}{x}+1}f'(x) \sim_{x\to -1} \frac{\ln(1+x)}{e^{r_K}-1},$$

which is negative whenever x is sufficiently close to -1. In other words, f decreases with $x = r_G/r_K - 1$, and hence increases with ρ_G.

Dynamic of public debt. Regarding public debt, the financial market clearing implies that $B_t = \frac{\beta}{1+\beta}\frac{\chi^\varphi}{1+\varphi}w_t^{1+\varphi} - K_t$ and thus that the dynamics is given by:

$$B\widehat{B}_t = \frac{\beta}{1+\beta}\chi^\varphi w^{1+\varphi}\widehat{w}_t - K\widehat{K}_t.$$

At impact ($t = 0$), we have:

$$B\widehat{B}_0 = -\left(\frac{\beta}{1+\beta}\chi^\varphi w^{1+\varphi}\frac{A-1}{(\varphi+1)(A-1)+1+\varphi\alpha}\sigma_\mu + \sigma_K K\right)\sigma_G\varepsilon_{G,0} \tag{189}$$

As a consequence, if the public debt is positive at the steady state ($B > 0$ equivalent to $\bar{g}_1 < 0$ – see Section C.4, then for a positive initial shock, $\varepsilon_{G,0} > 0$, $\frac{\partial\sigma_K}{\partial\rho_G} > 0$ implies $\frac{\partial\widehat{B}_0}{\partial\rho_G} < 0$. The higher the shock persistence, the variation of public debt at impact decreases.

$$\frac{\partial \widehat{B}_0}{\partial \rho_G} < 0.$$

Using (188) with (189) and FOC (190) to simplify $\frac{K}{LF_L}$ into $\frac{\beta\alpha}{1-\alpha}$, we obtain:

$$B\widehat{B}_0 = -\sigma_\mu \sigma_G (\chi w)^\varphi F_L \left(\frac{\beta}{1+\beta} \frac{(1-\tau^L)(A-1)}{(\varphi+1)(A-1)+1+\varphi\alpha} + \frac{r_\mu - \rho_G}{\rho_\mu - t_\mu} \frac{\beta\alpha}{1-\alpha} \right) \varepsilon_{G,0},$$

and finally using the relationship (170) between A and $1-\tau^L$:

$$B\widehat{B}_0 = -\frac{\sigma_\mu \sigma_G (\chi w)^\varphi F_L \beta}{\rho_\mu - t_\mu} \left(\frac{\varphi}{1+\varphi(1+\beta)} \frac{A(A-1)(\rho_\mu - t_\mu)}{(\varphi+1)(A-1)+1+\varphi\alpha} + (r_\mu - \rho_G) \frac{\alpha}{1-\alpha} \right) \varepsilon_{G,0}, \quad (190)$$

Even if public debt is positive at the steady state ($B > 0$), the sign of $\widehat{B}_0$ is ambiguous, since $\rho_\mu - t_\mu < 0$. It is the same for the quantity between brackets in (190) that can be positive or negative, depending in particular on the magnitude of the persistence ρ_G of the public spending shock. We illustrate it below in a particular tractable case.

E The Ramsey program on the truncated model

E.1 Formulation

We define the set of $(\xi^{u,0}_{y^N})_{y^N}$ such that:

$$\sum_{y^t \in \mathcal{Y}^t | (y^t_{t-N+1}, \ldots, y^t_t) = y^N} u(c_t(y^t)) = \xi^{u,0}_{y^N} u\Big(\sum_{y^t \in \mathcal{Y}^t | (y^t_{t-N+1}, \ldots, y^t_t) = y^N} c_t(y^t) \Big),$$

or compactly:

$$\xi^{u,0}_{y^N} u(c_{t,y^N}) := \sum_{y^N} u(c^i_t).$$

Similarly, we define $(\xi^{v,0}_{y^N})$, $(\xi^{u,1}_{y^N})$, $(\xi^{\tau}_{y^N})$, and $(\xi^{v,1}_{y^N})$ such that:

$$\begin{aligned}
\xi^{v,0}_{y^N} v(l_{t,y^N}) &:= \sum_{y^N} v(l^i_t), \\
\xi^{u,1}_{y^N} u'(c_{t,y^N}) &:= \sum_{y^N} u'(c^i_t), \\
\xi^{\tau}_{y^N} \sum_{y^N} (l_{t,y^N})^{1-\tau_t} &:= \sum_{y^N} (l^i_t)^{1-\tau_t}, \\
\xi^{v,1}_{t,s^N} v'(l_{t,y^N}) &:= \tau_t w_t \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}).
\end{aligned}$$

The Ramsey problem can then be written as:

$$\max_{\left(r_t,\tilde{w}_t,\tilde{r}_t,\tau_t^K,\tau_t,\kappa_t,B_t,K_t,L_t,\Pi_t,(a_t^i,c_t^i,l_t^i,\nu_t^i)_i\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\sum_{y^N}S_{t,y^N}\omega_t^i(\xi_{y^N}^{u,0}u(c_{t,y^N})-\xi_{y^N}^{v,0}v(l_{t,y^N}))\right],$$

$$G_t+T_t+(1+r_t)B_{t-1}+r_tK_{t-1}+w_t\xi_{y^N}^{y}\sum_{y^N}(l_{t,y^N}y_{y^N})^{\tau_t}\ell(di)=F(K_{t-1},L_t,z_t)+B_t,$$

$$\begin{aligned}
\text{for all } y^N\in\mathcal{Y}:\ ,\ & c_{t,y^N}+a_{t,y^N}=w_t\xi_{y^N}^{y}(l_{t,y^N}y_{y^N})^{\tau_t}+(1+r_t)\tilde{a}_{t,y^N}+T_t,\\
& a_{t,y^N}\geq-\bar{a},\ \nu_{t,y^N}(a_{t,y^N}+\bar{a})=0,\ \nu_{t,y^N}\geq 0,\\
& \xi_{y^N}^{u,E}u'(c_{t,y^N})=\beta\mathbb{E}_t\left[\sum_{\tilde{y}^N\in\mathcal{Y}^N}\Pi_{t,y^N\tilde{y}^N}\xi_{\tilde{y}^N}^{u,E}u'(c_{t+1,\tilde{y}^N})(1+r_{t+1})\right]+\nu_{t,y^N},\\
& \xi_{t,s^N}^{v,1}v'(l_{t,y^N})\equiv\tau_tw_t\xi_{y^N}^{y}(l_{t,y^N}y_{y^N})^{\tau_t}\xi_{y^N}^{u,1}(u'(c_{t,y^N})/l_{t,y^N}),\\
& K_t+B_t=\sum_{y^N}S_{t,y^N}a_{t,y^N},\ L_t=\sum_{y^N}S_{t,y^N}y_{t,y^N}^{i}l_{t,y^N}.
\end{aligned}$$

E.2 Factorization

We now factorize the Ramsey problem of Section E.1. We define:

$$\begin{aligned}
J=&\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{y^N\in\mathcal{Y}}\Big[S_{t,y^N}\Big(\Big(\omega_{y^N}\xi_{y^N}^{u,0}u(c_{t,y^N})-\xi_{y^N}^{v,0}v(l_{t,y^N})\Big)\\
&-\Big(\lambda_{c,t,y^N}-\tilde{\lambda}_{c,t,y^N}(1+r_t)\Big)\xi_{y^N}^{u,1}U_c(c_{t,y^N},l_{t,y^N})\Big),\\
&-\lambda_{l,t,y^N}\Big(v'(l_{t,y^N})-\tau_tw_t(y_{t,y^N})^{\tau_t}\xi_{y^N}^{y}(l_{t,y^N})^{\tau_t-1}\xi_{y^N}^{u,1}u'(c_{t,y^N})\Big)\Big].
\end{aligned}$$

The Ramsey program becomes maximizing J subject to the following constraints:

$$G_t+T_t+(1+r_t)B_{t-1}+r_tK_{t-1}+w_t\xi_{y^N}^{\tau}\sum_{y^N}(l_{t,y^N}y_{y^N})^{\tau_t}\ell(di)=F(K_{t-1},L_t,z_t)+B_t$$

$$\begin{aligned}
\text{for all } y^N\in\mathcal{Y}:\ ,\ & c_{t,y^N}+a_{t,y^N}=w_t\xi_{y^N}^{y}(l_{t,y^N}y_{y^N})^{\tau_t}+(1+r_t)\tilde{a}_{t,y^N}+T_t,\\
& a_{t,y^N}\geq-\bar{a},\ \nu_{t,y^N}(a_{t,y^N}+\bar{a})=0,\ \nu_{t,y^N}\geq 0,\\
& K_t+B_t=\sum_{y^N}S_{t,y^N}a_{t,y^N},\ L_t=\sum_{y^N}S_{t,y^N}y_{t,y^N}^{i}l_{t,y^N}.
\end{aligned}$$

E.3 FOCs of the planner

Before expressing the FOCs of the Ramsey program, we define:

$$\begin{aligned}
\hat{\psi}_{t,y^N}:=&\,\omega_{y^N}\xi_{y^N}^{u,0}u'(c_{t,y^N})-\mu_t\\
&-\Big(\lambda_{c,t,y^N}\xi_{y^N}^{u,E}-(1+r_t)\tilde{\lambda}_{c,t,y^N}\xi_{y^N}^{u,E}-\lambda_{l,t,y^N}\xi_{y^N}^{y}\tau_tw_t(y_0^N)^{\tau_t}l_{t,y^N}^{\tau_t-1}\xi_{y^N}^{u,1}\Big)u''(c_{t,y^N}).
\end{aligned}$$

The two Euler equations can be written as follows:

$$\xi^{u,E}_{y^N} u'(c_{t,y^N}) = \beta \mathbb{E}_t \Big[\sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \xi^{u,E}_{\tilde{y}^N} u'(c_{t+1,\tilde{y}^N})(1+r_{t+1}) \Big] + \nu_{t,y^N},$$

$$\xi^{v,1}_{t,s^N} v'(l_{t,y^N}) = \tau_t w_t \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N})$$

while the constraints of the Ramsey program become:

$$\begin{aligned} B_t + K^{\alpha}_{t-1} L^{1-\alpha}_t &= G_t + T_t + (1-\delta) B_{t-1} + (r_t + \delta) A_{t-1} \\ &\quad + w_t \sum_{y^N \in \mathcal{Y}^N} S_{t,y^N} \xi^{\tau}_{y^N} (y_{y^N} l_{t,y^N})^{\tau_t}, \\ \tilde{\lambda}_{t,y^N} &= \frac{1}{S_{t,y^N}} \sum_{\tilde{y}^N \in \mathcal{Y}^N} S_{t-1,\tilde{y}^N} \lambda_{t-1,\tilde{y}^N} \Pi_{t,\tilde{y}^N,y^N}, \\ c_{t,y^N} + a_{t,y^N} &= w_t (l_{t,y^N} y_{y^N})^{\tau_t} + (1+r_t) \tilde{a}_{t,y^N} + T_t, \\ a_{t,y^N} \geq 0 \text{ and } \tilde{a}_{t,y^N} &= \sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{\tilde{y}^N y^N,t} \frac{S_{t-1,\tilde{y}^N}}{S_{t,y^N}} a_{t-1,\tilde{y}^N}. \end{aligned}$$

The FOCs of the Ramsey program can finally be written as follows:

$$\begin{aligned} \hat{\psi}_{t,y^N} &= \beta \mathbb{E}_t \Big[(1+r_{t+1}) \sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \hat{\psi}_{t+1,\tilde{y}^N} \Big] \text{ if } \nu_{y^N} = 0 \text{ and } \lambda_{t,y^N} = 0 \text{ otherwise}, \\ \hat{\psi}_{t,y^N} &= \frac{1}{\tau_t w_t \xi^{\tau}_{y^N} (y^N_0)^{\tau_t} l^{\tau_t - 1}_{t,y^N}} (\omega_{y^N} \xi^{v,0}_{y^N} v'(l_{t,y^N}) + \lambda_{l,t,y^N} \xi^{v,1}_{y^N} v''(l_{t,y^N})) \\ &\quad - \lambda_{l,t,y^N} (\tau_t - 1) \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \\ &\quad - \mu_t (1-\alpha) \frac{Y_t}{\tau_t w_t \xi^{\tau}_{y^N} (y^N_0)^{\tau_t - 1} l^{\tau_t - 1}_{t,y^N} L_t}, \\ 0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \left(\hat{\psi}_{t,y^N} \tilde{a}_{t,y^N} + \tilde{\lambda}_{c,t,y^N} \xi^{u,E}_{y^N} u'(c_{t,y^N}) \right), \\ 0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} \left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N} \tau_t \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \right), \\ 0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \hat{\psi}_{t,y^N}, \\ \mu_t &= \beta \mathbb{E} \left[\mu_{t+1} \left(1 + \alpha \frac{Y_{t+1}}{K_t} - \delta \right) \right], \\ 0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \lambda_{l,t,y^N} \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \\ &\quad + \sum_{y^N \in \mathcal{Y}} S_{y^N} \left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N} \tau_t \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \right) \ln \left(l_{t,y^N} y_{y^N} \right) \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t}. \end{aligned}$$