Undergraduate Course Allocation through Pseudo-Markets*

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Abstract

We consider a many-to-many matching problem with a priority structure such as the one in undergraduate course allocation. In order to incorporate course priorities, we develop a deterministic pseudo-market mechanism with priority-specific prices that is based on the approximate competitive equilibrium from equal incomes. This novel mechanism, the Pseudo-Market with Priorities mechanism, prevents justified course envy, prevents Pareto improvements among students respecting the priority structure, is strategy-proof in the large, bounds envy by a single course among students at the same level of priority, and maintains a small upper bound on the market-clearing error. In a simulated environment, we show that this mechanism increases student utility and outcome fairness when compared to the commonly used in practice Random Serial Dictatorship.

JEL classification: D47, D63, D82, C63, C78, I21

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1 Introduction

Every academic term, over 6500 post secondary institutions across America assign course schedules to a total of nearly twenty million students.[1] Based on timing and prerequisites, students have a limited number of schedules they can possibly be assigned in a term. Students also have quite heterogeneous preferences over courses. In the face of room size and teaching constraints, university registrars prioritize giving course seats to some students over others based on factors such as student seniority and major. As a result, university registrars face the challenging task of deciding how to allocate seats in overdemanded courses to undergraduate students.

In this paper, we approach the task of allocating course schedules to students in the context of a many-to-many matching problem with heterogeneous student preferences and course priorities. We propose a *deterministic* allocation mechanism, the *Pseudo-Market with Priorities (PMP) mechanism*, that is based on the seminal Budish (2011)'s approximate competitive equilibrium from equal incomes (A-CEEI). This mechanism elicits student preferences over courses schedules, assigns almost equal budgets to students, and computes an approximate competitive equilibrium allocation and priority-specific prices. We require the equilibrium prices to respect the priority structure such that there is a "cutoff" priority level for each course, where students at the cutoff pay a non-negative price, students below the cutoff pay zero price, and students above the cutoff are not able to afford the course. We show that the PMP mechanism *prevents Pareto improvements respecting the priority structure* and *prevents justified course envy*, in that each student does not want to change her course assignment if she is given an opportunity to choose courses among courses in her assigned schedule and the assigned schedule of any student of lower priority. In addition, it *bounds envy by a single course among students at the same level of priority*, is *strategy-proof in the large*, and maintains the same upper bound for market-clearing error as in the case without priorities.

We highlight how university registrars would benefit by adopting the Pseudo-Market with Priorities mechanism for undergraduate course allocation. A typical university registration system uses a variant of the random serial dictatorship (RSD) mechanism, where students select courses in order of seniority and a priority structure based on student majors is enforced by setting aside seats in each course to be made available only to students in certain majors.[2] The PMP mechanism has two distinct advantages over the RSD mechanism. First, it endogenously computes course set-asides. In particular, the exact number of set-asides is typically hard to determine ex ante due to fluctuations in student demand for courses. On one hand, large set-asides lead to a large number of available seats demanded by students outside the major that the university registrar needs to assign manually with some ad hoc procedure after the main allocation process is over. On the other, small set-asides might lead to many students being rejected seats in courses that are required for their majors. The PMP mechanism relieves university registrars from having to correctly estimate the number of set-asides by computing them endogenously for each set of

[1] Based on National Center for Education Statistics, 2017-2018, https://nces.ed.gov/.

[2] Instead of a serial dictatorship, University of California system schools use an "enrollment pass" mechanism (see also Budish and Cantillon (2012)). The enrollment pass mechanism has the same problem in determining the exact number of set-asides.

submitted student preferences and available seats in courses. This ensures that all students who require a course for their major are assigned these courses and limits remaining seats that need to be assigned manually. Second, the PMP mechanism allocates courses more fairly among students, as it bounds envy by a single course among students with at the same level of priority.[3]

We also analyze environments where students have the same priorities for all courses, such as when priorities are based on student seniority. For these environments, we present two alternative mechanisms. The *Iterated Pseudo-Market mechanism* computes Budish (2011)'s A-CEEI's sequentially, starting with only the students at the highest level of priority and moving downwards. We show that when courses have common priorities, the set of possible allocations of the PMP and Iterated Pseudo-Market mechanisms are the same. It follows that this mechanism has all the same properties as those of the PMP mechanism. In addition, the mechanism *prevents Pareto improvements among students*, a stronger result than in the case without common priorities.

An alternative to the Iterated Pseudo-Market mechanism for environments with common priorities is the *Budget-Adjusted Pseudo-Market mechanism*, which assigns different budgets to students based on priorities, but maintains the same course prices for all students. This mechanism *prevents Pareto improvements among students*, is *strategy-proof in the large*, and *bounds envy by a single course among students at the same level of priority*. The main difference between this mechanism and the Iterated Pseudo-Market mechanism is that it allows students from different levels of priority to compete for the same seats in courses. This leads to a more equal allocation of courses across levels of priority, and may cause justified course envy. Still, the Budget-Adjusted Pseudo-Market mechanism prevents *justified schedule envy*, a weaker fairness concept that guarantees each student does not prefer the assigned schedule of a lower priority student to her own schedule assignment.

To provide additional support for our arguments, we compare the performance of the PMP and RSD mechanisms in a university-sized simulated environment. We conduct simulations with 5000 students and 1000 courses, where each course has 26 available seats and each student demands a schedule containing up to 10 courses, and investigate both major-specific course priority structures and common course priorities. We compare the PMP and RSD mechanism based on the mean and standard deviation of students' utility across 100 simulations. We also calculate the number of beneficial adjustments to the final assignments as explained below.

For major-specific priorities, students and courses are randomly and evenly distributed across fifty majors, and a binary priority structure in each course gives priority to students in a specific major. The PMP mechanism accounts for these priorities through priority-specific prices, while the RSD mechanism sets aside a number of seats in each course that can only be taken by students sharing the course's major. While the average student utility is only slightly larger for the PMP mechanism over the RSD mechanism, the standard deviation is significantly smaller. This suggests that the PMP mechanism can allocate courses more equally across students without sacrificing student utilities. We also account for two types

[3] A more equal distribution of courses was pointed out by Budish (2011) as one of benefits of the A-CEEI in the context of business school course allocation. The computation of course set-asides is a novel feature of undergraduate course allocation.

of inefficiencies that arise in the RSD mechanism due to ex ante chosen set-asides. First, there may be unclaimed set-aside seats and students outside the course's major that want to receive these seats. Second, there may be students in a course's major who are not able to receive a seat in a course that a student outside the major is able to take. While the first situation is more prevalent for large set-asides, the second one is more prevalent for smaller set-asides. For an optimized choice of set-asides in the RSD mechanism, still nearly 18% of students on average would gain from an adjustment in the number of set-asides. In turn, the PMP mechanism endogenously finds set-asides with only a small number of possible beneficial adjustments. These beneficial adjustments may arise only because market clears approximately.

For common priorities, students are evenly distributed across four years of study, and are prioritized based on seniority across all courses. The RSD mechanism does not assign seat asides for courses as there is a common ranking among students after the mechanism tie-breaking is realized. Our simulation results suggest that high priority students, i.e., juniors and seniors, enjoy approximately the same mean and standard deviation of students' utility for the PMP and RSD mechanisms. However, the PMP mechanism leads to an increase in mean utility for freshmen and sophomores. These students also enjoy a smaller standard deviation of utility suggesting again that the PMP mechanism allocates courses more equally across these cohorts.

Literature review. This paper contributes to the market design literature analyzing the assignment of courses to students. In environments where each student requires only one course, Diebold, Aziz, Bichler, Matthes, and Schneider (2014) provide an excellent survey on stable matching mechanisms.[4] The assignment of multiple courses to students is a more difficult many-to-many matching problem, where students value the bundles of courses (course schedules) instead of individual courses. In this setting, Echenique and Oviedo (2006) develop a general theory of stability, and Papai (2001) and Ehlers and Klaus (2003) show that the only strategy-proof and Pareto efficient mechanisms are variants of a serial dictatorship mechanism. Unfortunately, serial dictatorships lead to highly unequal outcomes among agents. Budish and Cantillon (2012) use theory and field data to argue why a non-strategy proof mechanism used at Harvard Business School to allocate courses to students has superior properties to serial dictatorships. Nguyen, Peivandi, and Vohra (2016) relaxes the requirement of strategy-proofness and designs a random allocation mechanism that is ordinally efficient, envy-free, and weakly strategy-proof. The mechanism might also violate each course's capacity constraint, but by no more than the size of one student course schedule. These appealing theoretical properties lead to the implementation of the mechanism at the Technical University of Munich (see Bichler and Merting, 2021).

Allocation with priorities has also been explored in the setting of cadet-branch matching in Sönmez and Switzer (2013) and Sönmez (2014). Cadet-branch matching concerns the matching of USMA and ROTC cadets to branches and terms (required years of service). Their model considers a priority ranking of cadets for each branch, which is often the same across branches and referred to as an order-of-merit list. They introduce the cadet-optimal stable mechanism, which is fair (in that no student prefers the branch-term pair of a student with lower priority on the order-of-merit list), stable, and strategy-proof.

[4] Diebold and Bichler (2017) also presents the results of several field experiments that support theoretical findings.

Our analysis has three key differences. First, we consider a many-to-many matching problem without contracts, where each student demands course schedules containing up to a number of courses k, and do not impose any restrictions on students' preferences. Second, in contrast to our common priorities setting (see Section 4), the order-of-merit list is a strict ranking of cadets. Third, we consider pseudo-market mechanisms, and satisfy approximate properties for stability and strategy-proofness.[5]

Hylland and Zeckhauser (1979) first proposed to use fake money and competitive equilibrium to randomly allocate objects to agents without transfers through so-called pseudo-market mechanisms. An important advantage of pseudo-markets is that they elicit the participants' cardinal preferences, allowing them to allocate objects more efficiently than most ordinal mechanisms (e.g., deferred acceptance mechanism). He, Miralles, Pycia, and Yan (2018) incorporate a priority structure into pseudo-markets. Though, our paper differs in that we consider deterministic mechanisms and our main emphasis is many-to-many matching problems such as undergraduate course allocation. In contrast, He, Miralles, Pycia, and Yan (2018) analyze random mechanisms with an emphasis on unit-demand settings such as school choice.[6] Miralles and Pycia (2021a) establish versions of the First and Second Welfare Theorems for random allocations through pseudo-markets. Also, Ashlagi and Shi (2016) show that in continuum economies, any efficient, symmetric, and strategy-proof random assignment can be expressed as the result of the equal budget pseudo-market mechanism.[7] Echenique, Miralles, and Zhang (forthcoming) study pseudo-market solution to random allocation problems with constraints. Pycia (2021) present an excellent survey of papers that employ pseudo-markets for random allocations in environments without transfers.

In contrast to the above papers, we analyze mechanisms that deliver a *deterministic* assignment of course schedules to students. For deterministic assignments, Budish (2011) shows how to adopt the idea of pseudo-markets that might not exactly satisfy the market clearing condition. Budish proposes the A-CEEI mechanism, which is strategy-proof in the large, finds an allocation that bounds student envy by one course and has no Pareto improvements among students, and approximately clears the market. This mechanism was successfully implemented in at Wharton Business School in the 2014 academic year (see Budish, Cachon, Kessler, and Othman, 2017; Budish and Kessler, 2021). The main contribution of this paper is to extend Budish's A-CEEI mechanism to many-to-many matching settings with course priorities, an important feature of the undergraduate course allocation problem.

The paper is organized as follows. Section 2 introduces the environment of course allocation with priorities and presents some definitions. Section 3 develops the Pseudo-Market with Priorities mechanism

[5] Sönmez and Switzer (2013) also shows that the cadet-optimal stable mechanism respects improvements. That is, if π_1 and π_2 are two lists of base priority rankings, where cadet i is higher ranked under some branch in π_1, at least as highly ranked in all branches in π_1, and the relative priority between all other cadets stays the same between π_1 and π_2, then cadet i does not receive a strictly worse assignment from the cadet-optimal stable mechanism under π_1 instead of π_2. Because we do not specify which Pseudo-Market Equilibrium with Priorities our mechanism selects, our mechanism does not necessarily respect improvements. Though, this property is less relevant in our analysis as it is in the case of cadet-branch matching. We base priorities on factors such as year of study and major, and if a student changes their major or moves to a later year of study (i.e., if the student's levels of priority were to change), their preferences over course schedules will change too.

[6] He, Miralles, Pycia, and Yan (2018) also explain how their results can be extended to many-to-many settings with additive utility, but still random allocations.

[7] Miralles and Pycia (2021b) show that continuum economies might have very different qualitative properties than large finite markets.

and investigates its properties. Section 4 considers a more restrictive environment with common course priorities and analyzes two alternative mechanisms. Finally, Section 6 concludes. Most of the proofs are postponed to the Appendix.

2 Environment

Course allocation is a many-to-many matching problem described by the tuple $(\mathcal{S}, \mathcal{C}, Q, V, \mathcal{R})$.

- $\mathcal{S} = \{1, ..., S\}$ is a set of students; in reference to students, we use feminine pronouns.
- $\mathcal{C} = \{1, ..., M\}$ is a set of courses.
- $Q = (q_1, ..., q_M)$ is a vector of course capacities; each course c can enroll at most q_c students.
- $V = (\succsim_1, ..., \succsim_S)$ is a vector of student preferences over course schedules. Students are typically restricted to a set of permissible course schedules due to factors such as course meeting times and prerequisites. Each student can take at most one seat in any course, and at most k courses in total. We assume that these restrictions are incorporated into student preferences, and, for simplicity of exposition, that $1 \leq k \leq M/2$. We also assume that student preferences over permissible course schedules are *strict*. Our results allow for the general case of substitutes and complements among courses.
- $\mathcal{R} = \{r_{s,c}\}_{s \in \mathcal{S}, c \in \mathcal{C}}$ is a course priority structure, where $r_{s,c} \in \{1, ..., R\}$ is the priority of student s for course c, with a smaller number meaning a higher level of priority. The priority levels need not to be distinct and could be the same for several students.

We consider deterministic allocations of courses to students. An allocation $x = (x_s)_{s \in \mathcal{S}}$ assigns a course schedule to each student, where $x_s \in 2^{\mathcal{C}}$ for each $s \in \mathcal{S}$. Allocation x is *feasible* if $\sum_{s \in \mathcal{S}} x_{s,c} \leq q_c$ for each course $c \in \mathcal{C}$. For ease of notation, we view the schedule x_s as both a set of courses assigned to student s and a vector from the set $\{0, 1\}^M$.

We evaluate allocations based on fairness and efficiency. The seminal measure of fairness is envy-freeness, as introduced by Foley (1967): an allocation x *prevents envy* if there are no students $s, s' \in S$ such that $x_{s'} \succ_s x_s$. The presence of priorities might allow some amount of envy among students that is justified.

Definition 1. *An allocation x* ***prevents justified course envy*** *if there are no students $s, s' \in \mathcal{S}$ and course $c \in C$ such that $r_{s,c} < r_{s',c}$, $c \notin x_s$, $c \in x'_s$ and $c \in \arg\max_{\succsim_s} x_s \cup c$.*

That is, if a student is willing to add a new course to her current schedule or replace one of her assigned courses with the new course, there should be no student of a lower level of priority for the new course who has a seat in the new course. The concept of preventing justified course envy directly corresponds to the absence of course-student blocking pairs (see Roth and Sotomayor, 1990).[8] This concept also coincides

[8]See He, Miralles, Pycia, and Yan (2018) and Kesten and Ünver (2015) for related envy concepts for random assignments.

with the concept of (pairwise) stability if no course seats remain unassigned. In the case where courses have common priorities (i.e., each course has the same priorities over students), we also consider a weaker concept of an allocation *preventing justified schedule envy,* for which no student prefers the schedule of a lower priority student to her own schedule. We will also investigate allocations that satisfy *envy bounded by a single course among students at the same level of priority*, an approximate notion of envy-freeness among students that share a priority level (see Section 4).

Additionally, we evaluate course allocations based on Pareto efficiency. An allocation y *Pareto dominates* an allocation x if there is at least one student who strictly prefers her course schedule in y and all other students weakly prefer their course schedules in y. As we explain below, we analyze approximate market equilibria, for which the market-clearing condition is satisfied with a small error. As a result, some unfilled course seats could be present. Hence, we distinguish between Pareto improvements among students and Pareto improvements between students and the university registrar that accounts for the possibility of unfilled seats. An allocation *prevents Pareto improvements among students* if there is no reallocation of the assigned seats in courses among students to obtain more preferred course schedules. An allocation *prevents Pareto improvements between students and the university registrar* if the university registrar cannot improve the course allocation by assigning students seats in courses with unfilled seats.

To account for the priority structure, we also introduce "course rankings" over allocations. Following Schlegel and Mamageishvili (2020), we define course rankings based on first-order stochastic dominance.[9] We say that y dominates x for course c, and write $y_c \succeq_c x_c$, if for all $r = 1, ..., R$ we have that $\sum_{s:r_{s,c} \leq r} y_{s,c} \geq \sum_{s:r_{s,c} \leq r} x_{s,c}$. Using this definition, we consider the following criterion of efficiency.

Definition 2. *An allocation x* **prevents Pareto improvements among students respecting the priority structure** *if for each assignment y that Pareto dominates x and has the same total number of assigned seats for each course, i.e., $\sum_{s \in S} y_{s,c} = \sum_{s \in S} x_{s,c}$ for all $c \in \mathcal{C}$, there is a course $c' \in \mathcal{C}$ for which $y_{c'} \not\succeq_{c'} x_{c'}$.*

The main difference between our concept of preventing Pareto improvements among students respecting the priority structure and priority-constrained efficiency, as introduced by Schlegel and Mamageishvili (2020), is that our concept requires any potential Pareto improvement to have the same total number of assigned seats in each course as currently assigned.[10] This again accounts for the possibility of unfilled seats. Pareto improvements are possible between students and the university registrar when students can improve on their assigned schedule by adding courses with unfilled seats.[11]

Allocations are found through mechanisms, which systematically elicit student preferences $(\succsim_s)_{s \in \mathcal{S}}$ over course schedules. Evidence from business schools demonstrates that mechanisms requiring strategic play on behalf of students can lead to large complications with efficiency (Budish and Cantillon 2012, Budish and Kessler 2021, and Sönmez and Ünver 2010).

[9] Note that course rankings coincide with course priorities in single-unit demand environments.

[10] We also consider only deterministic allocations.

[11] Note Definition 2 reduces to Budish's approximate Pareto efficiency in environments without course priorities.

Definition 3. *A course allocation mechanism is* ***strategy-proof*** *if there is no student s, who, by reporting manipulated preferences $\succsim'_s$, receives an allocation she strictly prefers to the course schedule she would get by reporting $\succsim_s$.*

We are mainly interested in mechanisms that are *strategy-proof in the large*; that is, strategy-proof in a limit market in which each student regards the "prices" in pseudo-market mechanisms, as introduced in the following section, as exogenous to her report (see Azevedo and Budish, 2019; Budish, 2011).[12] We also assume that strategic play is only a concern for the students, as course priorities are typically set based on commonly observable factors such as student seniority or major.

3 Pseudo-Market with Priorities

In this section, we present our novel mechanism, which allocates courses to students by extending the concept of approximate competitive equilibrium from equal incomes (see Budish, 2011) to settings with course priorities. For this purpose, we allocate to each student s a budget of fake money b^*_s. To obtain good fairness properties, we assume that budgets are almost the same, with $1 \leq \min_s b^*_s \leq \max_s b^*_s \leq 1+\beta$ for some small $\beta > 0$. The parameter β is interpreted as the maximum allowable budget inequality across students. We denote a positive number strictly greater than the largest allowed budget as $\bar{b} = 1 + \beta + \varepsilon$ for some $\varepsilon > 0$. We also allow for a slack in the market-clearing condition, which is bounded by $\alpha \geq 0$. To respect the priority structure, we require a special structure on priority-specific prices as defined below.

Definition 4. *An allocation x^*, prices p^*, and budgets b^* constitute an (α, β)-Pseudo-Market Equilibrium with Priorities if*

1. $x^*_s \in \max_{\succsim_s} \left\{ x' \in 2^{\mathcal{C}} : \sum_{c \in \mathcal{C}} p^*_{c,r_{s,c}} x'_c \leq b^*_s \right\}$ *for each student $s \in \mathcal{S}$.*

2. *For each course $c \in \mathcal{C}$, there exists a cut-off priority level r^*_c such that $\sum_{\{s \in \mathcal{S}: r_{s,c} < r^*_c\}} x^*_{s,c} < q_c$ and*

$$p^*_{c,r} \in \begin{cases} \{0\} & r < r^*_c \\ [0, \bar{b}) & r = r^*_c \\ [\bar{b}, +\infty) & r > r^*_c \end{cases}. \tag{1}$$

3. $||z^*||_2 \leq \alpha$, *where $z^* = (z^*_1, ..., z^*_M)$ and*

   (a) $z^*_c = \sum_s x^*_{s,c} - q_c$ *if* $p^*_{c,R} > 0$,

   (b) $z^*_c = \max(\sum_s x^*_{s,c} - q_c, 0)$ *if* $p^*_{c,R} = 0$.

4. $1 \leq \min_s b^*_s \leq \max_s b^*_s \leq 1 + \beta$.

[12]See He, Miralles, Pycia, and Yan (2018) for asymptotic incentive compatibility in random assignment matching models.

Unlike Budish (2011), the above definition of pseudo-market equilibrium allows for course prices to depend on priority levels, so that the vector of prices is $p^* = \left\{p^*_{c,r}\right\}_{c\in\mathcal{C},r\in\mathcal{R}} \in \mathbb{R}^{MR}$. In the case of $R = 2$, there are only two levels of priority: students with priority and all others. In undergraduate course allocation, major-specific priority structure is an example of such a priority structure.

Condition (1) ensures that the equilibrium allocation satisfies the priority structure, i.e., high priority students can obtain the course seats for free, the students at the cutoff level of priority face non-negative prices, and lower priority students cannot afford the course. A similar condition on prices first appeared in He, Miralles, Pycia, and Yan (2018) in the context of random allocation with priorities. In their setting, the pseudo-market equilibrium requires the market to clear exactly for courses with positive prices. In our setting, the market clears with some error. The market-clearing error for a course depends on its price for the lowest priority R, so that under-demand is not counted as an error if the price for the lowest priority group is zero. For all other cases, under-demand is counted towards the market-clearing error.[13] Condition $\sum_{\{s\in\mathcal{S}:r_{s,c}<r^*_c\}} x^*_{s,c} < q_c$ also ensures that the cutoff is accurately defined, i.e., the cutoff cannot be lowered without causing under-demand for the course.

3.1 The Existence of Pseudo-Market Equilibrium with Priorities

Each course has up to R levels of priority for students. Hence, any price vector p will have MR components. This could potentially be problematic, as Budish (2011) establishes that the worst-case bound on the market-clearing error is proportional to the square root of the dimension of the price space. The main insight of the theorem presented below is that condition (1) allows us to reduce the effective dimension of the price space. In particular, all but one price for a given course are either zero or above the maximum allowable budget. Hence, there is only one relevant price component for each course. The theorem below shows there always exists a Pseudo-Market Equilibrium with Priorities with the same market-clearing error as in the case without priorities.[14]

Theorem 1. *For any $\beta > 0$, there exists a $(\sqrt{kM/2}, \beta)$-Pseudo-Market Equilibrium with Priorities.*

Proof. Consider an economy $(\mathcal{S}, \mathcal{C}, Q, \mathcal{V}, \mathcal{R})$ and consider a budget vector $b = (b_1, ..., b_M)$ that satisfies $1 \leq \min_s(b_s) \leq \max_s(b_s) \leq 1 + \beta$ for some $\beta > 0$. Also, set $\bar{b} = 1 + \beta + \varepsilon$ for some $\varepsilon > 0$.

We consider the M-dimensional set $\mathcal{T} = [0, R\bar{b}]^M$, which allows to conveniently parameterize priority-specific prices and look for a competitive market equilibrium in a lower dimensional space. In particular,

[13] We want to mention that this requirement is a non-trivial extension of Budish (2011)'s condition to settings with course priorities. One consider an alternative definition where the market-clearing error for under-demanded courses with zero price for each level of priority is not counted as an error. In this case, one could use the cutoff level to artificially lower the market-clearing error. For example, consider the situation of a course with a price of zero at or below the cutoff level of priority and a price of $\bar{b}$ above the cutoff level. In this case, the market-clearing error is zero according to the alternative definition. At the same time, the unfilled course seats should be counted towards the market-clearing error as they could be potentially eliminated if the price above the cutoff level was decreased.

[14] Budish (2011) shows the existence of an A-CEEI with market-clearing error at most $\sqrt{\sigma M}/2$, where $\sigma = \min\{2k, M\}$. Because we assume $k \leq M/2$, the worst-case bound on the market-clearing error is $\sqrt{kM/2}$.

for each vector $t \in \mathcal{T}$, course $c \in \mathcal{C}$, and level of priority $r \in \{1, ..., R\}$, we define priority-specific prices as

$$p_{c,r}(t) = \max(t_c - (R - r)\bar{b}, 0). \tag{2}$$

For each $t \in \mathcal{T}$ and each $c \in \mathcal{C}$ there is a unique cutoff level of priority $r_c^*(t) \in \mathcal{R}$ such that for any $r \in \{1, ..., R\}$, $p_{c,r}(t)$ satisfies

$$p_{c,r}(t) \in \begin{cases} \{0\} & r < r_c^*(t) \\ [0, \bar{b}) & r = r_c^*(t) \\ [\bar{b}, +\infty) & r > r_c^*(t) \end{cases}. \tag{3}$$

We will also consider an auxiliary enlargement of this set, $\tilde{\mathcal{T}} = [-1, R\bar{b} + 1]^M$, and similarly define $p_{c,r}(\tilde{t})$ for $\tilde{t} \in \tilde{\mathcal{T}}$. We also define demand function $d_s : \tilde{\mathcal{T}} \to 2^{\mathcal{C}}$ as

$$d_s(\tilde{t}) = \max_{(\succsim_s)} \left\{ x' \in 2^{\mathcal{C}} : \sum_{c \in \mathcal{C}} x'_{s,c} \max(\tilde{t}_c - (R - r_{s,c})\bar{b}, 0) \leq b_s + \tau_{s,x} \right\},$$

where the $\tau_{s,x}$ are agent- and schedule-specific taxes chosen to ensure that the demand is single-valued (similarly to Budish (2011)).For each course $c \in \mathcal{C}$, excess demand $z_c : \tilde{\mathcal{T}} \to \mathbb{Z}$ is defined by

$$z_c(\tilde{t}) = \sum_{s \in \mathcal{S}} x_{s,c}^* - q_c,$$

where $x_s^* = d_s(t^*)$ for all $s \in \mathcal{S}$. The excess demand is bounded because $-S \leq z_c \leq S - 1$ for all $c \in \mathcal{C}$. We also define a budget surface for each student $s \in \mathcal{S}$ and each schedule $x \in 2^{\mathcal{C}}$ as

$$H(s, x) = \left\{ \tilde{t} \in \tilde{\mathcal{T}} : \sum_{c \in \mathcal{C}} x_{s,c} \max(\tilde{t}_c - (R - r_{s,c})\bar{b}, 0) = b_s + \tau_{s,x} \right\}.$$

Note that budget surface $H(s, x)$ may not be a hyperplane as in the case without priorities (see Budish, 2011). Lemma A1 in the Appendix shows that it is possible to choose b_s and $\tau_{s,x}$ such that at most M budget constraints intersect for any $\tilde{t} \in \tilde{\mathcal{T}}$.

Next, we define a truncation function trunc: $\tilde{\mathcal{T}} \to \mathcal{T}$, where truncation is defined for each $c \in \mathcal{C}$ as

$$(\text{trunc}(\tilde{t}))_c = \min\{R\bar{b}, \max\{0, \tilde{t}_c\}\}.$$

Also, we introduce a tâttonnement price adjustment function $f : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$ by

$$f(\tilde{t}) = \text{trunc}(\tilde{t}) + \gamma z(\text{trunc}(\tilde{t})),$$

where $\gamma \in (0, 1/S)$. Suppose that f has a fixed point $\tilde{t}^* = f(\tilde{t}^*)$ and denote its truncation by $t^* = \text{trunc}(\tilde{t}^*)$. We show that prices $\{p_{c,r_{s,c}}(t^*)\}_{c \in \mathcal{C}, r \in \mathcal{R}}$ defined by equation (2), allocation $x_s^* = d_s(t^*)$, and budgets $b_s^* = b_s + \tau_{s,x_s^*}$ for all $s \in \mathcal{S}$ constitute an exact competitive equilibrium (or $(0, \beta)$-Pseudo-Market

Equilibrium with Priorities as in Definition 4). We might only need to slightly adjust t^* to find another fixed point that satisfies $\sum_{\{s\in\mathcal{S}:r_{s,c}<r_c^*(t^*)\}} x^*_{s,c} < q_c$.

- Prices $\{p_{c,r}(t^*)\}_{c\in\mathcal{C},r\in\mathcal{R}}$ and cutoffs defined by (2) and (3) ensure that condition (1) is satisfied.

- The definition of demand function implies that any course schedule that student s prefers to $x^*_s = d_s(t^*)$ must cost strictly more than $b^*_s = b_s + \tau_{s,x^*_s}$.

- $p_{c,R}(t^*) > 0$ implies $z_c(t^*) = 0$. To see this, note that equation (2) implies $\tilde{t}^*_c > 0$. In addition, we must have $\tilde{t}^*_c < R\bar{b}$; otherwise $(\text{trunc}(\tilde{t}^*))_c = R\bar{b}$ and $z_c(\tilde{t}^*) < 0$, which contradicts the fixed point equation. Hence, $\tilde{t}^*_c \in (0, R\bar{b})$ and the fixed point equation ensures $z_c(t^*) = 0$.

- $p_{c,R}(t^*) = 0$ implies $z_c(t^*) \leq 0$. To see this, consider two cases. If $\tilde{t}^*_c \in (0, R\bar{b})$, the fixed point equation implies $z_c(t^*) = 0$. If $\tilde{t}^*_c \in [-1, 0]$, we have $t^*_c \equiv \text{trunc}(\tilde{t}^*_c) = 0$, $p_{c,R}(t^*) = 0$, and the fixed point ensures $z_c(t^*) \leq 0$.

- To make sure condition $\sum_{\{s\in\mathcal{S}:r_{s,c}<r_c^*(t^*)\}} x^*_{s,c} < q_c$ is satisfied (see condition (1)), assume from the contrary that the demand for a course c across priority levels smaller than $r^*_c(t^*)$ is greater than its number of seats, i.e., $\sum_{\{s\in\mathcal{S}:r_{s,c}<r_c^*(t^*)\}} x^*_{s,c} \geq q_c$. The above two bullet points establish that $z_c(t^*) \leq 0$. Hence, our assumption implies $\sum_{\{s\in\mathcal{S}:r_{s,c}<r_c^*(t^*)\}} x^*_{s,c} - q_c = \sum_{s\in\mathcal{S}} x^*_{s,c} - q_c = 0$ or there is no demand for course c from students at level of priority $r^*_c(t^*)$. So, we can consider $\hat{t}^*_c = t^*_c + \bar{b} - p_{c,r^*_c(t^*)}$, where the cutoff priority group faces price $\bar{b}$ and the prices of smaller priority levels do not change. With $\hat{t}^* = (\hat{t}^*_c, t^*_{-c})$, we obtain that $p_{c,r^*_c(\hat{t}^*)} = 0$ and $p_{c,r^*_c(\hat{t}^*)} = \bar{b}$ and the adjustment does not change the demand and excess demand for course c for all students, but it decreases the cutoff level $r^*_c(\hat{t}^*) = r^*_c(t^*) - 1$. If we still have $\sum_{\{s\in\mathcal{S}:r_{s,c}<r_c^*(\hat{t}^*)\}} x^*_{s,c} \geq q_c$, we repeat the price adjustment until the condition is satisfied. Finally, $z_c(\hat{t}^*) = z_c(t^*) = 0$ implies $\hat{t}^*_c < R\bar{b}$ and $\hat{t}^*_c > t^*_c \geq 0$. Therefore, $\hat{t}^*_c = \text{trunc}(\hat{t}^*_c) + z_c(\hat{t}^*)$, and, hence, vector $\hat{t}^*$ satisfies the fixed point equation and is such that $\sum_{\{s\in\mathcal{S}:r_{s,c}<r_c^*(\hat{t}^*)\}} x^*_{s,c} < q_c$.

Overall, if f has a fixed point $\tilde{t}^* = f(\tilde{t}^*)$, its truncation $t^* = \text{trunc}(\tilde{t}^*)$ (or its adjustment $\hat{t}^*$) is an exact competitive equilibrium price vector for allocation $x^*_s = d_s(t^*)$ and budgets b^*_s for all $s \in \mathcal{S}$.

Though, the fixed point of operator f might fail to exist. Following Budish (2011), we define a "convexification" of f, $F : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$, by

$$F(\tilde{t}) = co\{y : \exists \text{ a sequence } \tilde{t}^w \to \tilde{t}, \tilde{t} \neq \tilde{t}^w \in \tilde{\mathcal{T}} \text{ such that } f(\tilde{t}^w) \to y\},$$

where co denotes the convex hull of the set. F is nonempty, $\tilde{\mathcal{T}}$ is compact and convex, and $F(t)$ is convex. From Lemma 2.4 of Cromme and Diener (1991), F is an upper hemicontinuous correspondence and hence has a fixed point by Kakutani's fixed point theorem. We denote a fixed point by $\tilde{t}^* \in F(\tilde{t}^*)$, and let again $t^* = \text{trunc}(\tilde{t}^*)$ be its truncation.

Note that the reduction of the MR-dimensional space of prices $\{p_{c,r}\}_{c\in\mathcal{C},r\in\mathcal{R}}$ to M-dimensional space $\tilde{\mathcal{T}} = [-1, R\bar{b} + 1]^M$ allows us to use the original steps of Budish (2011) to establish the existence of a

$(\sqrt{kM/2}, \beta)$-Pseudo-Market Equilibrium with Priorities for any $\beta > 0$. For completeness, we provide adapted steps in the Appendix. □

Overall, Theorem 1 establishes the existence of the Pseudo-Market Equilibrium with Priorities, which is an extension of Theorem 1 in Budish (2011) to environments with priorities. The equilibrium has the same worst-case bound on the market-clearing error as the A-CEEI in the same environment without priorities.

3.2 Pseudo-Market with Priorities Mechanism

Using the Pseudo-Market Equilibrium with Priorities, we introduce the Pseudo-Market with Priorities (PMP) mechanism and investigate its properties in terms of fairness, efficiency, and the ability of students to manipulate its outcome.

Definition 5 (Pseudo-Market with Priorities Mechanism)**.** *The Pseudo-Market with Priorities mechanism with market-clearing error α and budget inequality β is defined through the following steps:*

1. *Each student s reports preferences $\succsim_s$ over permissible course schedules.*
2. *Each student s is assigned a budget b_s^* that is a uniform random draw from $[1, 1+\beta]$.*
3. *Compute prices $p^* = \{p_{c,r}^*\}_{c\in\mathcal{C}, r\in\mathcal{R}}$ and allocation $x^* = (x_1^*, ..., x_S^*)$ such that*
   - *for each student $s \in \mathcal{S}$, allocation x_s^* maximizes her utility:*
   $$x_s^* = \max_{\succsim_s} \left\{ x' \in 2^{\mathcal{C}} : \sum_{c\in\mathcal{C}} p_{c,r_{s,c}}^* x_c' \leq b_s^* \right\}$$
   - *for each $c \in \mathcal{C}$, there exists a cutoff priority level r_c^* such that $\sum_{\{s\in\mathcal{S}: r_{s,c} < r_c^*\}} x_{s,c}^* < q_c$ and*
   $$p_{c,r}^* \in \begin{cases} \{0\} & r < r_c^* \\ [0, \bar{b}) & r = r_c^* \\ [\bar{b}, +\infty) & r > r_c^* \end{cases}.$$
   - *the market-clearing error is smaller than α, i.e., $||z^*||_2 \leq \alpha$, where $z^* = (z_c^*)_{c\in\mathcal{C}}$, and*
   $$z_c^* = \begin{cases} \sum_{s\in\mathcal{S}} x_{s,c}^* - q_c & \text{if } p_{c,R}^* > 0 \\ \max\left[\left(\sum_{s\in\mathcal{S}} x_{s,c}^* - q_c\right), 0\right] & \text{if } p_{c,R}^* = 0 \end{cases}$$

The PMP mechanism depends on the level of allowable market clearing error α and the level of budget inequality β. We avoid this dependence for the results that hold for all non-negative α and β, and we will be more specific when restrictions are necessary. The PMP mechanism extends the A-CEEI mechanism introduced by Budish (2011) to settings with priority-specific prices. We first establish that the condition on priority-specific prices (1) ensures a form of fairness among students. In particular, we show that the outcome of the PMP mechanism prevents justified course envy.

Theorem 2. *The outcome of the Pseudo-Market Equilibrium with Priorities mechanism prevents justified course envy.*

Proof. Let (x, p, b) be an outcome of the Pseudo-Market Equilibrium with Priorities mechanism. Consider a student s and a course c such that $c \notin x_s$, but $c \in \arg\max_{\succsim_s} x_s \cup c$. The definition of the Pseudo-Market Equilibrium with Priorities (Definition 4) implies that we must then have $p_{c,r_{s,c}} > 0$. Condition (1) on equilibrium prices p then implies that for any student s' with a lower priority for course c, i.e., $r_{s',c} > r_{s,c}$, we must have $p_{c,r_{s',c}} \geq \bar{b}$. Therefore, $c \notin x_{s'}$, as an allocation x cannot possibly assign a student s' a seat in course c. $\square$

This property is closely related to (pairwise) stability in two-sided matching markets (see Echenique and Oviedo, 2006). The main difference is the possibility of blocking course-student pairs and individual courses due to the presence of under-subscribed and over-subscribed courses for the outcome of the PMP mechanism. In particular, there may be students who want to take courses with unfilled seats. Also, the oversubscribed courses may want to drop some students from their assignments. However, as Theorem 1 shows, these instances are rare. At the same time, there is no student left wanting to get a course for which she has a higher priority than some student who is assigned a seat in the course.

However, the priorities in undergraduate course allocation are often weak. For example, seniority-based priorities lead to almost two thousand students at the same year of study at Carnegie Mellon University. Hence, a more equal distribution of courses among students at the same level of priority might be also desirable. This property is achieved in the settings without priorities by A-CEEI mechanism: the outcome of A-CEEI mechanism bounds student's envy of students by a single course among students when the budget inequality β is sufficiently small. In Section 4, we show that the PMP mechanism bounds envy by a single course among students at the same level of priority, with the same restriction on the size of β.

The PMP mechanism is not Pareto efficient. This is because it also needs to satisfy course priorities that might be in conflict with student preferences, similarly to how stable allocations in one-sided matching markets might not be Pareto efficient (see Roth and Sotomayor, 1990). We illustrate this in the following example:

Example 1. *Let us consider an economy with two students $\mathcal{S} = \{1, 2\}$ and two courses $\mathcal{C} = \{A, B\}$, each of capacity one. Student preferences are as follows:*

$$1 : A \succ B \tag{4}$$

$$2 : B \succ A \tag{5}$$

Student budgets are $b^1 = 1$ and $b^2 = 1 + \beta$ for some $0 < \beta < 1$.[15] *We assume that course priorities are the opposite of student preferences, with $r_{1,B} = r_{2,A} = 1$ and $r_{1,A} = r_{2,B} = 2$. The following set of prices*

Student	$p^*_{s,A}$	$p^*_{s,B}$
1:	*2*	*1*
2:	*1*	*2*

with priority cutoffs $r^*_A = r^*_B = 1$ and allocation

Student	$x^*_{s,A}$	$x^*_{s,B}$
1:	*0*	*1*
2:	*1*	*0*

consistute an exact market equilibrium with course priorities. However, the allocation is not Pareto efficient, as the students prefer to exchange their assigned seats.

However, the next result shows that the outcome of the PMP mechanism satisfies a form of constrained Pareto efficiency, i.e., students cannot trade courses among each other to improve their outcome without violating the priority structure.

Theorem 3. *The outcome of the Pseudo-Market Equilibrium with Priorities mechanism prevents Pareto improvements among students respecting the priority structure.*

The proof of the above result closely follows the proof of Schlegel and Mamageishvili (2020) for single-unit settings with random allocations, and is hence it is postponed to Appendix. Note that we assume each agent has strict preferences over course schedules. Hence, for the above result we do not require an additional condition that an agent chooses the cheapest course schedule when multiple course schedules are optimal as in Miralles and Pycia (2021a) and Schlegel and Mamageishvili (2020).

Finally, we establish that students have almost no incentive to manipulate the PMP mechanism for large populations, which is typically the case for undergraduate course allocation.

Theorem 4. *The Pseudo-Market with Priorities mechanism is strategy-proof in the large.*

To prove this result, we show that the PMP mechanism is a semi-anonymous mechanism that is envy-free but for tie breaking (see Azevedo and Budish, 2019; Kalai, 2004). Supplementary Appendix C of Azevedo and Budish (2019) extends their Theorem 1 to semi-anonymous mechanism, proving that any semi-anonymous mechanism that is envy-free or envy-free but for tie breaking is strategy-proof in the large. Our Theorem 4 follows from this result. [16]

[15] This example could be similarly extended to the environment where there is more than two courses and each student requires to be enrolled in at most $k > 2$ courses.

[16] One can extend Budish (2010)'s definition of a continuum replication of an economy to account for object (course) priorities. By leveraging on the price structure as in Theorem 1 (i.e., defining $\mathcal{T} = [0, R\bar{b}]^M$ and prices as in equation 2), the steps of Theorem 4 of Budish (2010) can be adapted to obtain the result.

4 Common Priorities

In this section, we consider a restrictive environment, requiring that each student is at the same level of priority in every course. We denote the priority of student s as $r_s \equiv r_{s,c} = r_{s,c'}$ for all $c, c' \in \mathcal{C}$ and the corresponding partition of the set of students as $\mathcal{S} = \cup_{r=1}^{R} \mathcal{S}_r$, where $\mathcal{S}_r = \{s \in \mathcal{S} : r_s = r\}$. This setting has been considered in the context of random assignments in Miralles (2017). A typical example of such a priority structure is seniority in undergraduate course allocation. For these environments, we show that the outcome of the PMP mechanism satisfies additional fairness and efficiency properties. We also propose two alternative mechanisms, establish their properties, and compare them to the PMP mechanism.

In environments with common priorities, there can be many students at the same level of priority. Hence, the criterion of preventing justified course envy is not very powerful (see Theorem 2). At the same time, we cannot hope to assign courses to students in many environments in an envy-free way without using lotteries, especially when students have similar preferences. An innovation proposed by Budish (2011) is to consider envy bounded by one course. We apply this measure of fairness to students at the same level of priority.

Definition 6. *An allocation satisfies* **envy bounded by a single course among students at the same level of priority** *if, for any $s, s' \in \mathcal{S}_r, r = 1, ..., R$, either $x_s \succsim_s x_{s'}$ or there exists some course c such that $x_s \succsim_s (x_{s'} \setminus \{c\})$.*

Using this concept of fairness we establish the following result.

Theorem 5. *For budget inequality $\beta \leq \frac{1}{k-1}$, the outcome of the Pseudo-Market with Priorities mechanism satisfies envy bounded by a single course among students at the same level of priority.*[17]

Next, we consider two alternative mechanisms that assign course seats to students when courses have common priorities.

4.1 Iterated Pseudo-Market Mechanism

The presence of a common priority structure suggests that the equilibrium allocation can be implemented sequentially, starting from the highest priority students and proceeding to the lowest priority students. We execute this idea using the Iterated Pseudo-Market mechanism, as defined below. We note already now that the sequential way of allocating courses to students has a computational advantage, which is an important factor in undergraduate course allocation as university registrars typically need to allocate hundreds of courses to thousands of students.

Definition 7 (Iterated Pseudo-Market Mechanism)**.** *The Iterated Pseudo-Market mechanism with market-clearing error α and budget inequality β is defined through the following steps:*

[17] More general priority structures may assign students different levels of priority in different courses. The result of Theorem 5 can be extended to these settings with a small change in Definition 6. An allocation satisfies *envy bounded by a single course with respect to the priority structure* $\mathcal{R}$ if, for any $s, s' \in \mathcal{S}$ such that $r_{s,c} \leq r_{s',c}$ for all $c \in \mathcal{C}$, either $x_s \succsim_s x_{s'}$ or there exists some course c^* such that $x_s \succsim_s (x_{s'} \setminus \{c^*\})$.

1. *Each student s reports preferences $\succsim_s$ over permissible schedules.*

2. *Each student s is assigned a budget b_s^* that is a uniform random draw from $[1, 1+\beta]$.*

3. *Consider a vector of non-negative market-clearing errors $(\alpha_1, ..., \alpha_R)$ such that $\sqrt{\sum_{r=1}^R \alpha_r^2} = \alpha$. For $r = 1$ denote the set of available courses by $\mathcal{C}_1 \equiv \mathcal{C}$ and the number of available seats as $q_{c,1} \equiv q_c$ for all $c \in \mathcal{C}_r$.*

4. *For from $r = 1$ through $r = R$, do the following:*

   - *for courses $c \in \mathcal{C}_r$ compute prices $p_r^* = (p_{c,r}^*)_{c \in \mathcal{C}_r}$ and allocations $x_r^* = (x_s^*)_{s \in \mathcal{S}_r}$ such that:*
     - *for each student $s \in \mathcal{S}_r$, allocation x_s^* maximizes her utility:*

$$x_s^* = \max_{\succsim_s} \{x' \in 2^{\mathcal{C}_r} : \sum_{c \in \mathcal{C}_r} p_{c,r}^* x_c' \leq b_s^*\};$$

     - *the market-clearing error is smaller than α_r, i.e., $||z_r^*||_2 \leq \alpha_r$, where $z_r^* = (z_{c,r}^*)_{c \in \mathcal{C}_r}$, and*

$$z_{c,r}^* = \begin{cases} \sum_{s \in \mathcal{S}_r} x_{s,c}^* - q_{c,r} & \text{if } p_{c,r}^* > 0 \\ \max\left[(\sum_{s \in \mathcal{S}_r} x_{s,c}^* - q_{c,r}), 0\right] & \text{if } p_{c,r}^* = 0 \end{cases};$$

   - *for courses $c \in \mathcal{C} \backslash \mathcal{C}_r$, set prices $p_{c,r}^* \geq \bar{b}$ and excess demand $z_{c,r}^* = 0$;*
   - *Denote $\mathcal{C}_{r+1} \subseteq \mathcal{C}_r$ by the set of courses c with $p_{c,r}^* = 0$ and $\sum_{s \in \mathcal{S}_r} x_{s,c}^* < q_{c,r}$, and update $q_{c,r+1} = q_{c,r} - \sum_{s \in \mathcal{S}_r} x_{s,c}^*$ for each $c \in \mathcal{C}_{r+1}$.*

The Iterated Pseudo-Market mechanism employs a common priority order by computing an A-CEEI allocation for each group of students with equal priority. When $R = S$, the Iterated Pseudo-Market mechanism is equivalent to a serial dictatorship. When $R = 1$, no student has priority over any other student, and the mechanism is equivalent to the A-CEEI mechanism.

For courses with a price of zero and no excess demand, the quotas of each course are updated by deducting the number of assigned seats. For courses with positive prices, all seats are removed from the assignment procedure.[18] This allows us to replicate the requirements on market equilibrium prices with priorities (see condition (1) in Definition 4): if the price for the course is positive for some priority level, there are no seats assigned to students with a lower priority. The procedure also removes any courses with zero price if all seats are filled by students. Overall, we obtain the following result, which relates the Iterated Pseudo-Market mechanism to the Pseudo-Market with Priorities mechanism.

[18]The idea of a sequential pseudo-market mechanism was also considered by Miralles (2017) for economies with random assignments. The method of removing course seats between rounds is the key difference between the two mechanisms. This allows us to relate the Iterated Pseudo-Market mechanism to the Pseudo-Market with Priorities mechanism. This also ensures that Pareto improvements among students are not possible if students have strict preferences over course schedules.

Theorem 6. *Consider an environment with common priorities. (x^*, b^*, p^*) is the outcome of a Pseudo-Market with Priorities mechanism if and only if it is the outcome of an Iterated Pseudo-Market Mechanism with the same market-clearing error.*

The above result and Theorem 1 imply that there always exists an Iterated Pseudo-Market mechanism with market-clearing error $\alpha = \sqrt{kM/2}$ and budget inequality $\beta > 0$. Also, the relationship between the two mechanisms allows us to transmit many results established above to the Iterated Pseudo-Market mechanism. In addition, we establish that the outcome of the Iterated Pseudo-Market mechanism prevents Pareto improvements among students, which strengthens the result of Theorem 3.

Corollary 1. *The Iterated Pseudo-Market Mechanism:*

(i) prevents Pareto improvements among students;

(ii) prevents justified course envy;

(iii) bounds envy by a single course among students at the same level of priority if $\beta \leq \frac{1}{k-1}$;

(iv) is strategy-proof in the large.

Lastly, we want to discuss a design choice in the implementation of the Iterated Pseudo-Market mechanism. Removing all seats in courses with positive prices in the Iterated Pseudo-Market mechanism has an obvious disadvantage, as the students at a lower level of priority might be willing to take these seats. The mechanism could be modified to allow these seats to be available to lower priority students. However, this would lead to possible justified course envy and the existence of Pareto improvements among students at different levels of priority. This would also invalidate the result of Theorem 6, as it could potentially lead to prices in $(0, \bar{b})$ at several priority levels for one course.

4.2 Budget-Adjusted Pseudo-Market Mechanism

The Iterated Pseudo-Market mechanism differentiates among students through priority-specific prices. These priority-specific prices make sure that the final allocation respects student priorities, so that there is no justified course envy among students. This may be too demanding for some environments, and a weaker distinction among students based on budgets might be preferable. We propose the Budget-Adjusted Pseudo-Market (BAPM) mechanism, which is a variant of Budish's A-CEEI mechanism that assigns budgets to students based on priorities such that any student $s \in \mathcal{S}_r$ could afford k courses even if the price for each course equals to the entire budget of any student $s' \in \mathcal{S}_{r-1}$; that is, we assume $k \cdot \max_{s \in \mathcal{S}_{r-1}} b^*_s \leq \min_{s' \in \mathcal{S}_r} b^*_{s'}$.

Definition 8 (Budget-Adjusted Pseudo-Market Mechanism)**.** *The Budget-Adjusted Pseudo-Market (BAPM) mechanism is defined through the following steps:*

1. *Each student s reports preferences $\succsim_s$ over permissible schedules.*

2. *For $r = 1, ..., R$, assign every $s \in \mathcal{S}_r$ a budget b_s^* that is a uniform draw from $[k^{R-r}(1+\beta)^{R-r},\ k^{R-r}(1+\beta)^{R-r+1}]$;*

3. *Compute a set of prices $p^* = (p_1^*, ..., p_M^*)$ and allocations $x^* = (x_1^*, ..., x_S^*)$ such that*

   - *for each student $s \in \mathcal{S}$, allocation x_s^* maximizes her utility:*

$$x_s^* = \max_{\succsim_s} \{x' \in 2^{\mathcal{C}} : \sum_{c \in \mathcal{C}} p_c^* x_c' \leq b_s^*\};$$

   - *the market-clearing error is smaller than α, i.e., $||z^*||_2 \leq \alpha$, where $z^* = (z_c^*)_{c \in \mathcal{C}}$, and*

$$z_c^* = \begin{cases} \sum_s x_{s,c}^* - q_c & \text{if } p_c^* > 0 \\ \max\left[(\sum_s x_{s,c}^* - q_c), 0\right] & \text{if } p_c^* = 0 \end{cases}$$

This mechanism modifies the A-CEEI mechanism of Budish (2011) only in how the budgets are assigned to students. In the typical case of undergraduate course allocation, there are $R = 4$ student cohorts. Here, the BAPM mechanism determines budgets for each $s \in \mathcal{S}$ as follows:

- If $s \in \mathcal{S}_4$, b_s^* is a random draw from $[1, 1 + \beta]$.
- If $s \in \mathcal{S}_3$, b_s^* is a random draw from $[k(1 + \beta), k(1 + \beta)^2]$.
- If $s \in \mathcal{S}_2$, b_s^* is a random draw from $[k^2(1 + \beta)^2, k^2(1 + \beta)^3]$.
- If $s \in \mathcal{S}_1$, b_s^* is a random draw from $[k^3(1 + \beta)^3, k^3(1 + \beta)^4]$.

Designing budgets in this manner ensures the allocation of the BAPM mechanism bounds envy by a single course among students at the same level of priority. Also, Theorem 1 in Budish (2011) guarantees that there exists a BAPM mechanism for any budget inequality $\beta > 0$ with a market-clearing error at most $\alpha = \sqrt{kM/2}$. The choice of budgets also ensures that the outcome of the BAPM mechanism prevents justified schedule envy, as defined below.

Definition 9. *An allocation x* ***prevents justified schedule envy*** *if there are no students $s, s' \in \mathcal{S}_r$, $r < r'$, such that $x_{s'} \succ_s x_s$.*

An allocation that prevents justified course envy will also prevent justified schedule envy. Though, it is possible that an allocation prevents justified schedule envy but does not prevent justified course envy; this measure is less restrictive and permits a student to have seats in courses that would improve the course schedules of students with higher priority. In addition to the above property, the BAPM mechanism also inherits properties from the original A-CEEI mechanism.

Theorem 7. *The Budget-Adjusted Pseudo-Market mechanism:*

(i) prevents Pareto improvements among students;

(ii) prevents justified schedule envy;

(iii) bounds envy by a single course among students at the same priority level if $\beta \leq \frac{1}{k-1}$*;*

(iv) is strategy-proof in the large.

The BAPM mechanism ensures that a student can always afford the course schedule of any student with lower priority. This prevents justified *schedule* envy. However, the BAPM mechanism could cause students at varying levels of priority to compete for the same course seats, leading to justified *course* envy. This point is illustrated with the following example.

Example 2. *Let us consider an economy with three students* $\mathcal{S} = \{1, 2, 3\}$ *and six courses* $\mathcal{C} = \{A, B, C, D, E, F\}$*, each with one seat. Each student has additively separable utility over courses, where the utility a student gets from taking course* A *is* 1*,* B *is* 2*, ...,* F *is* 6*. Suppose that* $\mathcal{S}_1 = \{1, 2\}$ *and* $\mathcal{S}_2 = \{3\}$*. Note that in the Iterated Pseudo-Market mechanism, students* 1 *and* 2 *would divide courses* C *through* F *and leave student* 3 *with courses* A *and* B*, preventing justified course envy.*[19] *Consider the BAPM mechanism with budgets* $b^* = \{b_1^*, b_2^*, b_3^*\} = \{2.11, 2.1, 1\}$*. For the following prices and allocation, the market exactly clears.*

$$\text{Prices: } p^* = \{p_A^*,\ p_B^*,\ p_C^*,\ p_D^*,\ p_E^*,\ p_F^*\} = \{0,\ 0.01,\ 0.8,\ 1,\ 1.3,\ 2.1\}$$

$$\text{Allocation: } x^* = \{x_1^*, x_2^*, x_3^*\} = \{\{0,0,1,0,1,0\}, \{0,1,0,0,0,1\}, \{1,0,0,1,0,0\}\}$$

When competition for popular courses raises prices high, students with a higher level of priority who get seats in popular courses may no longer afford some courses that students with lower level of priority can. Note that student 1 *and* 2*'s demand for courses* C *through* F *raise course prices such that student* 3 *is able to get course* D *due to the large payments students* 1 *and* 2 *respectively make for courses* E *and* F*. Though, student* 1 *prefers course* D *to course* C *and student* 2 *prefers course* D *to course* B*. As a result, while the BAPM mechanism prevents justified schedule envy, it does not prevent justified course envy.*

5 Simulations

We compare the performance of the Pseudo-Market with Priorities and Random Serial Dictatorship mechanisms through simulations of the course allocation process in a US college-sized environment. Our environment consists of 5000 students and 1000 courses, where each student demands a seat in five courses and each course has 26 seats.

First, we consider a setting with major-specific priorities, where each student is assigned a major, and a binary priority structure is used to prioritize students of a certain major in each course. The PMP mechanism utilizes this priority structure through priority-specific prices, which are computed endogenously

[19] It is also possible that the market would clear with some error. In this case, student 1 or student 2 could be assigned course A or course B, but this would require the unfilled course(s) from C through F to have a positive price, making them unavailable to student 3.

during the assignment process. In contrast, the RSD mechanism sets aside seats in each course that can only be taken by students with priority, where the number of set-asides in each course is set exogenously based on course assignment in the previous years. We show that the PMP mechanism increases average student utility, is more fair, and can endogenously compute the approximate number of set-aside seats necessary in each course.

Second, we consider a setting with common priorities, where students are each assigned to one of four levels of priority referred to as "years of study". We find a Pseudo-Market Equilibrium with Priorities through the Iterated Pseudo-Market Mechanism (see Theorem 6), sequentially computing an A-CEEI for students at each year of study. To find an RSD allocation, we randomize the order of students within each year of study and assign each student their most-preferred available course schedule, beginning with those with those at the highest level of priority (fourth-year students) and ending with those at the lowest level of priority (first-year students). We show that the PMP mechanism increases student's average utility and is more fair among students within a year of study.

5.1 Major-Specific Priorities

We first consider a setting with major-specific priorities. To create a priority structure, students and courses are randomly and evenly distributed across fifty majors. Each course gives priority to students that share its major, setting $r = 1$ for students with the same major (majors) and $r = 2$ for those who do not (non-majors).

Students are given a nonzero utility for ten out of the 1000 courses, including five courses within their major and five randomly selected others. This is a realistic assumption; even though a university may run hundreds of courses in a semester, any given student will only consider taking a small subset of the courses. Moreover, it allows our program to accommodate a large number of courses. For each student-course pair, this utility is made up of a non-random, random, and major-specific component. For course $c = 1, ..., 1000$, if student s gives course c a nonzero utility, student s's utility for taking course c is $c + \varepsilon_{s,c} + 250 \cdot [r_{s,c} = 1]$, where $\varepsilon_{s,c}$ is a random integer between -500 and 500 assigned to each student-course pair with positive utility, and $[r_{s,c} = 1]$ is an indicator variable which adds 250 utility if c is in s's major. This utility design emphasizes that while some courses may be popular or unpopular among students, individual preferences vary. Preferences over course schedules are additively separable with respect to individual course utilities.

To find a Pseudo-Market Equilibrium with Priorities, we use a Tabu Search process similar to that of Othman, Sandholm, and Budish (2010) and Budish, Cachon, Kessler, and Othman (2017). Starting with a random vector of prices, each student's utility-maximizing schedule is found, and neighboring vectors are produced that adjust course prices based on over and under-demand. The neighboring vector with the largest reduction in market-clearing error is selected as a new starting point. The mechanism terminates after reaching the worst-case bound on the market-clearing error of $\alpha = \sqrt{kM/2} = 50$ and failing to improve significantly on the market-clearing error within five choices of a neighboring vector.

The RSD mechanism is conducted based on a randomly determined priority order among students.

Mechanism	Mean Student Utility	St. Dev. Student Utility	Beneficial Adjustments
PMP	4069	808	79
RSD	3848	1202	876

Table 1: The mean and standard deviation of students' utility, and the number of beneficial adjustments for the PMP and RSD mechanisms for major-specific priorities.

According to this order, students are assigned their most-preferred available course schedule. When a student selects a course within their major that has not filled all of its set-aside seats, the student will be assigned a set-aside seat. When no set-asides remain, students in the course's major take the remaining seats. Students outside the course major can only take seats outside set-asides. As a result, even if there are unfilled seats in a course, the course may not available to a non-major student. In practice, this is used to aid the entry of students to courses within their majors, but can lead to inefficiency when too many or too few seats are reserved.

University registrars rely on information from previous years to estimate how many seats to reserve for majors in each course. To emulate this process, we randomly generate 25 environments, each consisting of student utilities, student and course majors, and a ranking across students. We utilize the deferred acceptance algorithm to determine an optimized number of set-asides for each course. This ensures that no non-major can receive a seat in a course when a major cannot, and no seat in a course is left empty when a student could benefit by taking it. Assigning each course set-asides equal to the number of seats it assigns to majors, it cannot be that a course has too many or too few set-asides; no non-major can receive a seat in a course when a major can, and no unclaimed set-aside seats prevent a non-major from taking the course. We take the average across the 25 groups (with each value rounded to the nearest integer) to be the set-asides we use in our simulations.

In turn, the PMP mechanism endogenously and optimally determines the number of set-asides. The PMP still allows for a possibility of a limited number of beneficial adjustments. When the market clears approximately, if there is under-demand when course prices are positive, there could exist unfilled seats that students at or above the cutoff level could benefit from taking. To provide a measure of beneficial adjustments in the PMP mechanism, we count the number of under-demanded seats when course prices are positive. Note that the under-demand will lead to beneficial adjustments only if there are students above the cutoff who would have demanded the course.

Table 1 presents results averaged across 100 simulations. As measures of student satisfaction and outcome fairness, we consider the mean and standard deviation of student total utility. We also count the number of possible beneficial adjustments in the RSD and PMP mechanisms as described above. The PMP mechanism is an improvement over the RSD in terms of students' average utility and leads to a more fair outcome by reducing the standard deviation of students' utilities. Additionally, the PMP mechanism leads to far less beneficial adjustments than the RSD mechanism. The 2734 beneficial adjustments in the

RSD come from an average of 1450 different students, making up nearly 29% of the student body. In the PMP mechanism, under-demand is an upper bound on the true number of possible beneficial adjustments in the PMP mechanism; under-demand will only lead to beneficial adjustments if there are students above the cutoff would have demanded the course.

This suggests that reserving seats for students can lead to inefficiency even if the university registrar bases its set-asides on information from previous years. Because the PMP mechanism endogenously determines its cutoff groups, it can approximately determine the number of set-asides necessary. In doing this, the PMP mechanism allows for a university to prioritize the entry of students into courses with less of a sacrifice to efficiency as the RSD.

5.2 Common Priorities

Next, we consider a setting where courses have common priorities. Students and courses are evenly distributed across four levels of priority referred to as "years of study", where all courses prioritize students based on seniority. To do this, we set $r = 1$ for fourth-year students ("Seniors"), $r = 2$ for third-year students ("Juniors"), $r = 3$ for second-year students ("Sophomores"), and $r = 4$ for first-year students ("Freshmen").

As in our simulations with major-specific priorities, preferences over course schedules are additively separable with respect to individual course utilities. Students are given a nonzero utility for ten out of the 1000 courses. For each student-course pair with a nonzero utility, utility is made up of a non-random, random, and year-specific component. For course $c = 1, ..., 1000$, student s's utility for taking course c is $c + \varepsilon_{s,c} + 250 \cdot [y_{s,c} = 1]$, where $\varepsilon_{s,c}$ is a random integer between -500 and 500 assigned to each student-course pair with positive utility and $[y_{s,c} = 1]$ is an indicator variable which adds 250 utility if c is assigned to s' year of study.

To find a Pseudo-Market Equilibrium with Priorities, we use the Iterated Pseudo-Market mechanism. To do this, we compute an A-CEEI for each year of study, beginning with seniors and ending with freshmen, and remove seats between rounds according to step 4 of Definition 7. In order to guarantee that the error across rounds falls beneath the market-clearing error, we require that each round r has a market-clearing error of at most $\sqrt{kM_r/2}$, where M_r is the number of courses with seats assigned to students in round r. In order to conduct the RSD mechanism, students are assigned their most-preferred available course schedule, beginning with seniors and ending with freshmen. To break ties, students are randomly ordered within their year of study.

Our results are recorded in Table 2. For each mechanism, we report the mean and standard deviation of students' utility at each year of study. Among seniors, in all but one round, both the PMP and RSD mechanisms assign each student their most-preferred course schedule. At less-prioritized years of study, the PMP mechanism has higher mean student utilities and lower standard deviations. The difference is most prevalent for freshmen students, where the PMP mechanism has a 30% higher mean utility than the RSD mechanism. This exhibits the ability of the PMP mechanism to increase student satisfaction, while

Mechanism	Freshman	Sophomore	Junior	Senior
PMP	2559 (825)	3926 (845)	4723 (749)	4742 (750)
RSD	1962 (1004)	3871 (974)	4704 (758)	4742 (750)

Table 2: The mean (the standard deviation) of students' utilities at each year of study for the PMP and RSD mechanisms when courses have common priorities.

providing for a more equal distribution of courses among students at each year of study.

6 Conclusion

In this paper, we explore a many-to-many matching problem that typically arises in undergraduate course allocation. We allow courses to prioritize students based on factors such as seniority and major, and design a deterministic allocation mechanism, the Pseudo-Market with Priorities mechanism, that respects this priority structure. The PMP mechanism is an extension of the A-CEEI mechanism to settings with course priorities and maintains its upper bound on the outcome's market-clearing error. This mechanism prevents justified course envy, prevents Pareto improvements between students respecting the priority structure, is strategy-proof in the large, and bounds envy by a single course among students at the same level of priority. For environments with common priorities, we also explore two alternative mechanisms.

We highlight how university registrars could benefit from adopting the PMP mechanism through computer simulations. First, the PMP mechanism respects course priorities and computes course set-asides endogenously, which relieves the burden of estimating them correctly from university registrars. In doing this, the PMP mechanism also ensures that students who require courses for their majors are assigned these courses and leaves almost no seats that need to be assigned manually. Second, the PMP mechanism leads to a more equal distribution of courses among students at the same level of priority than the serial dictatorship mechanism.

There are several concerns that university registrars would need to address prior to implementing the PMP mechanism for undergraduate course allocation. For one, it is necessary to have a preference reporting language that allows for students to express how they compare different course schedules and account for what substitutabilities and complementarities may exist among courses. Software developed by Budish, Cachon, Kessler, and Othman (2017) to implement the A-CEEI mechanism in Wharton Business School shows this can be done in a manner that is easily accessible to students (see also Bichler and Merting (2021)). Moreover, a lot of computational power is needed to compute a market equilibrium with a small market-clearing error for an undergraduate population with thousands of students and hundreds of courses. We show that the computational problem is manageable if students consider only taking a subset of the available courses and students' utility is additive. The computational issues should also be manageable if one allows for limited course complementarities as in Budish, Cachon, Kessler, and Othman

(2017).

Finally, adopting the PMP mechanism would provide valuable data to university registrars on student demand. As students have almost no incentive to misrepresent their preferences, the market-clearing prices of these mechanisms can serve as indicators of student demand and help universities distinguish popular courses from unpopular ones. Using this information, universities can adjust class sizes, timing, and sections to further increase student satisfaction.

A Appendix

Proof of Theorem 1. We have shown in the main text how the arguments of Steps 1-3 of Theorem 1 of Budish (2011) need to be modified for the setting with priority-specific course prices. The adaptation of Steps 4-9 follows closely the original proof. However, we first establish that it is possible to choose budgets b_s and $\tau_{s,x}$ such that at most M budget constraints intersect for any $\tilde{t} \in \tilde{\mathcal{T}}$.

Lemma A1. *One can choose taxes $\{\tau_{s,x}\}_{s\in\mathcal{S},x\in 2^{\mathcal{C}}}$ that satisfy the following conditions:*

(i) Taxes are small ($-\varepsilon < \tau_{s,x} < \varepsilon$);

(ii) Taxes favor more preferred bundles ($\tau_{s,x} > \tau_{s,x'}$ for $x' \succ_s x$);

(iii) The inequality bounds are preserved ($-1 \leq \min_{s,x}(b_s + \tau_{s,x}) \leq \max_{s,x}(b_s + \tau_{s,x}) \leq 1 + \beta$);

(iv) No perturbed budgets are equal ($b_s + \tau_{s,x} \neq b_{s'} + \tau_{s',x}$);

(v) There is no auxiliary price vector $\tilde{t} \in \tilde{\mathcal{T}}$ at which more than M budget constraints $H(s,x)$ intersect.

Proof. Let us fix $x \in 2^{\mathcal{C}}$ for each $s \in \mathcal{S}$. Budish (2011) establishes the possibility to choose $\{\tau_{s,x}\}_{s\in\mathcal{S},x\in 2^{\mathcal{C}}}$ that satisfy the first four conditions. We now establish that it is always possible to slightly change taxes such that condition (v) is also satisfied.

For this purpose, let us assume that more than M budget constraints $H(s,x)$ intersect and denote

$$\mathcal{I} = \{s \in S : \cap_s H(s,x) \neq \varnothing\},$$

with $|\mathcal{I}| > M$. For each $\tilde{t} \in \tilde{\mathcal{T}}$ consider prices $\{p_{c,r}(\tilde{t})\}_{c\in\mathcal{C},r\in\{1,\dots,R\}}$ defined by equation (2) and cutoffs defined by (3). The definition of cutoffs $r_c^*(\tilde{t})$ implies that

$$\forall s \in \mathcal{I}, c \in \mathcal{C} : r_{s,c} < r_c^*(\tilde{t}), x_{s,c} \cdot p_{c,r_{s,c}}(\tilde{t}) = 0.$$

In addition, the definition of cutoffs implies $p_{c,r_{s,c}} \geq \bar{b}$ for $R \geq r_{s,c} > r_c^*(\tilde{t})$. Therefore, agent s's budget constraint is satisfied with equality if a seat in course c is not allocated to agent s for $r_{s,c} > r_c^*(\tilde{t})$ or $x_{s,c} = 0$. Hence, we have also

$$\forall s \in \mathcal{I}, c \in \mathcal{C} : r_{s,c} > r_c^*(\tilde{t}), x_{s,c} \cdot p_{c,r_{s,c}}(\tilde{t}) = 0.$$

Therefore, we obtain that $x_{s,c} \cdot p_{c,r_{s,c}}(\tilde{t}) \neq 0$ for $r_{s,c} = r_c^*(\tilde{t})$. That is, entries in agent s's budget constraint, $s \in \mathcal{I}$, are non-zero only if $r_{s,c} = r_c^*(\tilde{t})$. Hence, we can write

$$x_{s,c} \cdot p_{c,r_{s,c}}(\tilde{t}) \equiv x_{s,c} \cdot p_c(\tilde{t}).$$

Overall, we obtain that the set of equations $\{s \in \mathcal{I} : H(s,x) = 0\}$ is the set of linear equations with coefficients $x_{s,c} \in \{0,1\}$:

$$\begin{cases} x_{s,1} \cdot p_1(\tilde{t}) + x_{s,2} \cdot p_2(\tilde{t}) + ... + x_{s,M} \cdot p_M(\tilde{t}) & = b_s + \tau_{s,x} \\ .... & .... \\ x_{s',1} \cdot p_1(\tilde{t}) + x_{s',2} \cdot p_2(\tilde{t}) + ... + x_{s',M} \cdot p_M(\tilde{t}) & = b_{s'} + \tau_{s'x} \end{cases}$$

for $s, s' \in \mathcal{I}$. Since $|\mathcal{I}| > M$ for any $\tilde{t} \in \mathcal{T}$, there are at most M linear independent linear equations of prices. So, the Rouché–Capelli theorem implies that we can choose $\tau_{s,x}$ such that only at most M equations are satisfied for any $\tilde{t}$.[20] $\square$

Step 4. Similarly to Theorem 1 of Budish (2011), if the price vector t^* is not on any budget constraint, then t^* is an exact competitive equilibrium price vector. Suppose that t^* is not on any budget constraint. Then, there is a neighborhood around t^* where each agent's demand is unchanging in price. At price t^*, f is continuous, and as a result, $F(t^*) = f(t^*)$.

- If $t^* = \tilde{t}^*$, then $F(\tilde{t}^*) = F(t^*) = f(t^*)$, and thus $t^* = \tilde{t}^* \in F(\tilde{t}^*) = f(\tilde{t}^*)$. Therefore, t^* is a fixed point. Hence, as we established earlier, it is an exact competitive equilibrium price vector.
- If $t^* \neq \tilde{t}^*$, we establish the following lemma.

  Lemma A2. *For any $\tilde{t} \in \tilde{\mathcal{T}} \setminus \mathcal{T}$, (i) $f(\tilde{t}) = f(trunc(\tilde{t}))$ and (ii) $F(\tilde{t}) \subseteq F(trunc(\tilde{t}))$.*

  Proof. Statement (i) follows from the definition of f. For statement (ii), consider a point y for which there exists a sequence $\tilde{t}^w \to \tilde{t}$, $\tilde{t}^w \neq \tilde{t}$ such that $f(\tilde{t}^w) \to y$. Consider $\text{trunc}(\tilde{t}^w)$. As $\text{trunc}(\cdot)$ is continuous, this sequence will converge to $\text{trunc}(\tilde{t})$. Statement (i) implies $f(\text{trunc}(\tilde{t}^w))$ converges to y. As a result, $y \in F(\tilde{t})$ implies that $y \in F(\text{trunc}(\tilde{t}))$, and thus $F(\tilde{t}) \subseteq F(\text{trunc}(\tilde{t}))$. $\square$

  As a result, since $F(t^*) = f(t^*)$ and $\tilde{t}^* \in F(\tilde{t}^*)$, $\tilde{t}^* \in F(t^*) = f(t^*) = f(\tilde{t}^*)$, and thus $\tilde{t}^* = f(\tilde{t}^*)$. Therefore, t^* is an exact competitive equilibrium price vector.

Step 5. Next, suppose that t^* is on $1 \leq L \leq M$ budget constraints. We denote $\Phi = \{0,1\}^L$, and construct a set of 2^L price vectors $\{p^\phi\}_{\phi \in \Phi}$ that satisfy the following conditions:

1. Each t^ϕ is close enough to t^* that there is a path from t^ϕ to t^* that does not cross any budget constraint.
2. Each t^ϕ is on the "affordable" side of the ℓth budget constraint if $\phi_\ell = 0$ and is on the "unaffordable" side if $\phi_\ell = 1$.

[20] Note that when we change $\tau_{s,x}$, some other budget constraints might start intersecting. Since the set of possible intersecting budget constraints is finite and $\tau_{s,x}$ varies continuously, we can always choose $\tau_{s,x}$ without influencing the intersection property of the other budget constraints.

To construct price vectors $\{p^\phi\}_{\phi\in\Phi}$, we note that each of the L intersecting budget constraints defines two sets:

$$H^0_\ell = \left\{ \tilde{t} \in \tilde{\mathcal{T}} : \sum_{c\in\mathcal{C}} x_{s_\ell c} \max(\tilde{t}_c - (R - r_{s_\ell,c})\bar{b}, 0) \leq b_{s_\ell} + \tau_{s_\ell, x_{s_\ell}} \right\}$$

$$H^1_\ell = \left\{ \tilde{t} \in \tilde{\mathcal{T}} : \sum_{c\in\mathcal{C}} x_{s_\ell c} \max(\tilde{t}_c - (R - r_{s_\ell,c})\bar{b}, 0) > b_{s_\ell} + \tau_{s_\ell, x_{s_\ell}} \right\}$$

The first set delineates the set of prices for which agent s_ℓ can afford schedule x_{s_ℓ}, whereas the second set delineates the set of prices for which agent s_ℓ can't afford x_{s_ℓ}. Let $\phi = (\phi_1, ..., \phi_L) \in \Phi$ be an L-dimensional vector of zeros and ones, and the polytope $\pi(\phi) := \cap_{\ell=1}^L H_\ell^{\phi_\ell}$ be the set of points in $\mathcal{T}$ that belong to the intersection of sets indexed by ϕ. Let $H = \{H(s,x)\}_{s\in\mathcal{I}, x\in 2^{\mathcal{C}}}$ be the finite set of all budget constraints formed by any student-course schedule pair (s,x). We then define

$$\delta < \inf_{\tilde{t}'' \in \mathcal{T}, H \in \mathcal{H}} \{||(t^* - \tilde{t}'')||_2 : \tilde{t}'' \in H, t^* \notin H\},$$

which denotes the distance such that any budget constraint that t^* does not belong to is further than δ away from t^*. Let $B_\delta(t^*)$ be a δ-ball of t^*. Now, for each $\phi \in \Phi$ we define $\tilde{t}^\phi$ to an arbitrary element of $\pi(\phi) \cap B_\delta(t^*)$. Such price vectors satisfy the two requirements outlined above.

Step 6. We now show that a perfect market-clearing excess demand vector lies in the convex hull of $\{z(p^\phi)\}_{\phi\in\Phi}$. For this purpose, we first show that for any $y \in F(t^*)$, we must have $y = t^* + \sum_{\phi\in\Phi} \lambda^\phi z(p^\phi)$, where $\sum_{\phi\in\Phi} \lambda^\phi = 1$. Take some $y \in F(t^*)$. Consider a sequence $t^w \to t^*, t^* \neq t^w \in \tilde{\mathcal{T}}$ such that $f(t^w) \to y'$. Note that the sequence t^w consists of a finite number of subsequences $t^{w,\phi} \in \pi(\phi) \cap B_\delta(t^*)$ for some $\phi \in \Phi$.[21] Since all elements of the subsequence $t^{w,\phi}$ are on the same side of set H^ϕ_ℓ, every agent has the same choice at every point. Hence, if the subsequence has infinite number of elements, we must have $t^{w,\phi} \to t^*$ and

$$f(t^{w,\phi}) \to t^* + \gamma z(t^\phi).$$

Therefore, the limit of $f(t^w)$ for the original sequence t^w must be also $t^* + \gamma z(t^{\phi'})$ for some $\phi' \in \Phi$. Therefore, $y' = t^* + \gamma z(t^{\phi'})$. Since, by definition, y is a convex combinations of such y', we must have $y = t^* + \sum_{\phi\in\Phi} \lambda^\phi z(t^\phi)$, where $\sum_{\phi\in\Phi} \lambda^\phi = 1$.

Lemma A2 implies that $\tilde{t}^* \in F(\tilde{t}^*) \subseteq F(t^*)$. Hence, we must have

$$\tilde{t}^* = t^* + \sum_{\phi\in\Phi} \lambda^\phi z(t^\phi).$$

[21] Some subsequences can have only a finite number of elements.

for some $\{\lambda_\phi\}_{\phi\in\Phi}$ with $\sum_{\phi\in\Phi}\lambda^\phi = 1$. We also denote

$$\zeta = \sum_{\phi\in\Phi} \lambda^\phi z(t^\phi) = \frac{\tilde{t}^* - t^*}{\gamma}.$$

We note that $\zeta \leq 0$ and $\zeta_c < 0$ implies $t_c^* = 0$. Hence, vector ζ is a perfect market-clearing excess demand vector and it lies in the convex hull of $\{z(p^\phi)\}_{\phi\in\Phi}$.

Steps 7-9. The structure of excess demand has the same geometric structure as in Budish (2011). In particular, denote L' be the number of agents whose budget constraints intersect at price t^*. We rename agents such that $s = 1, ..., L'$. We denote the number of budgets of student s that intersect at t^* as w_s. Since at most M budgets constraints can intersect, we must have $L \equiv \sum_{s=1}^{L'} w_s \leq M$. We also denote the bundles pertaining to s's the budget constraints as $x_s^1 \succ ... \succ x_s^{w_s}$.

Similarly to Budish (2011), we consider bundles that s demands at prices near t^*. In the set $H^0(s, x_s^1)$, agent s can purchase her favorite bundle x_s^1. Hence, one does not need to know whether prices belong to sets $H^0(s, x_2), ..., H^0(s, x_{w_s})$. Let us denote the demand for prices at $H^0(s, x_s^1) \cap B_\delta(t^*) \cap \tilde{\mathcal{T}}$ as d_s^0. Similarly, we consider prices in $H^1(s, x^m) \cap H^0(s, x^{m+1})$ and denote the corresponding demands as d_s^m for $m = 1, ..., w_s$. Overall, agent $s = 1, ..., L'$ purchase $w_s + 1$ distinct bundles at prices near to p^*.

Let us denote the excess demand of the remaining agents as

$$z_{S\setminus\{1,...,L'\}}(t^*) = \sum_{s=L'+1}^{S} d_s(t^*) - q.$$

Hence, a perfect market-clearing excess demand vector lies in the convex hull of $\{z(p^\phi)\}_{\phi\in\Phi}$ with the elements

$$z_{S\setminus\{1,...,L'\}}(t^*) + \sum_{s=1}^{L'}\sum_{f=1}^{w_s} a_s^f d_s^f$$

where $0 \leq a_s^f \leq 1, s = 1, ..., L', f = 1, ..., w_s$ and $\sum_{f=1}^{w_s} a_s^f = 1$, $s = 1, ..., L'$. Budish (2011) shows in Step 8 of Theorem 1 that there exists a vertex of the above geometric structure that is within $\sqrt{kM/2}$ distance from the perfect market-clearing excess demand vector. He also explains how one should adjust agent budgets to find an approximate competitive equilibrium. These purely mathematical arguments remain unchanged from his original paper. □

Proof of Theorem 3. Let (x, p, b) be a (α, β)-Pseudo-Market Equilibrium with Priorities. Suppose that allocation y Pareto dominates allocation x and the number of course seats allocated to students for both allocations y and x are the same. Hence, there is a student s' who strictly prefer allocation y to x. In addition,

$$\sum_{c\in C} p_{c,r_{s',c}} y_{s',c} > b_{s'} \geq \sum_{c\in C} p_{c,r_{s',c}} x_{s',c}.$$

for all other students we must have

$$\sum_{c\in C} p_{c,r_{s,c}} y_{s,c} \geq b_s \geq \sum_{c\in C} p_{c,r_{s,c}} x_{s,c}.$$

as we assume that students have strict preferences among course schedules. Summing the inequalities over all agents, we obtain

$$\sum_{s\in S}\sum_{c\in C} p_{c,r_{s,c}} y_{s,c} > \sum_{s\in S}\sum_{c\in C} p_{c,r_{s,c}} x_{s,c}.$$

We can further rearrange

$$\sum_{c\in C}\sum_{r=1}^{R} p_{c,r} \sum_{\{s:r_{s,c}=r\}} y_{s,c} > \sum_{c\in C}\sum_{r=1}^{R} p_{c,r} \sum_{\{s:r_{s,c}=r\}} x_{s,c}.$$

Hence, we obtain

$$0 < \sum_{c\in C}\sum_{r=1}^{R} p_{c,r} \sum_{\{s:r_{s,c}=r\}} (y_{s,c} - x_{s,c}) = \sum_{c\in C}\sum_{r=1}^{R-1} (p_{c,r} - p_{c,r+1}) \sum_{\{s:r_{s,c}\leq r\}} (y_{s,c} - x_{s,c}) + p_{c,R} \sum_{\{s:r_{s,c}\leq R\}} (y_{s,c} - x_{s,c}).$$

As the number of course seats allocated to students for allocations y and x are the same, the last term equals zero. Also, $p_{c,r} - p_{c,r+1} \leq 0$. Hence, we must have $\sum_{\{s:r_{s,c}\leq r\}}(y_{s,c} - x_{s,c}) < 0$ for at least some course c and rank r. Therefore, we must have $y_c \not\succeq_c x_c$. Hence, allocation x prevents Pareto improvements among students respecting priority structure. □

Proof of Theorem 4. To show that the PMP mechanism is strategy-proof in the large, we show that it is a semi-anonymous mechanism that is envy-free but for tie-breaking (EF-TB). Theorem 1 of Azevedo and Budish (2019) notes that an anonymous mechanism that is envy-free or EF-TB is also strategy-proof in the large, and Supplementary Appendix C notes that this can be extended to the case of semi-anonymous mechanisms. Here, the definition of the EF-TB condition is generalized to semi-anonymous mechanisms: a semi-anonymous mechanism satisfies the EF-TB condition if no agent envies another agent in the same group, and with lower lottery number.

We first define the PMP mechanism as a semi-anonymous mechanism in the setting of Azevedo and Budish (2019). Students each have a type $t_s \in T$, which fully describes their reported preferences and level of priority in each course. We assign to each student is assigned a lottery number l_s as a random draw from $[0, 1]$, and a budget b^*_s equal to $1 + l_s * \beta$, where β is chosen to be the maximum allowable budget difference. Given budgets $b^* = \{b^*_s\}_{s\in\mathcal{S}}$, students' reported preferences, and course priorities $\mathcal{R}$, a Pseudo-Market Equilibrium with Priorities is determined, and each student is assigned their equilibrium course schedule. Therefore, we can define the PMP mechanism as

$$\Phi^n(t) = \int_{l\in[0,1]^S} x^S(t,l)dl,$$

where t is a vector of student types, l is a vector of lottery draws, and the function $x^S(t,l)$ assigns a course schedule to each student $s \in \mathcal{S} = \{1, ..., S\}$.[22]

We can partition the set of students $\mathcal{S}$ into groups of the form $S_{r_{s,1},r_{s,2},...,r_{s,M}}$, where $r_{s,c} \in \{1,...,R\}$ for each $c \in \mathcal{C}$. This groups students with others who have the same level of priority in each course, as these students are treated the same by the PMP mechanism (i.e., they face the same prices). To see that the PMP mechanism is EF-TB, note that any student has a larger budget than a student with a lower lottery number than her. It follows from condition 1 of Definition 4 that any student prefers their assigned course schedule to any student with a lower budget at the same level of priority in every course. As a result, no student will envy another student in the same group with a lower lottery number, and the PMP mechanism is hence EF-TB. Then, by the extension of Theorem 1 of Azevedo and Budish (2019) to semi-anonymous mechanisms, the PMP mechanism is strategy-proof in the large.

Proof of Theorem 5. Assume that $\beta \leq \frac{1}{k-1}$. We establish that the outcome of the Pseudo-Market with Priorities mechanism satisfies envy bounded by a single course among students the same level of priority. Let us consider two students $s, s' \in \mathcal{S}_r, r = 1, ..., R$. We denote the course schedule assigned in market equilibrium to student s' as $x_{s'} = (c_{j_1}, ..., c_{j_{k'}})$ for $j_1, ..., j_{k'} \in \{1, ..., M\}$ and $k' \leq k$. Suppose that s envies student s' and that this envy is not bounded by one course. Therefore, student s is not able to afford course schedules of student s even if one course from schedule $x_{s'}$ is dropped. That is,

$$p_{c,r} \cdot x_{s'} \backslash \{c_{j_\ell}\} > b_s,$$

for $\ell = 1, ..., k'$. Since s' has the same priority, we also have

$$p_{c,r} \cdot x_{s'} \backslash \{c_{j_\ell}\} > b_s,$$

for $\ell = 1, ..., k'$. Summing these inequalities over ℓ and taking $b_{s'} \geq p_{c,r} \cdot x_{s'}$ into account, we obtain

$$(k-1)b_{s'} \geq (k-1)p_{c,r} \cdot x_{s'} \geq (k'-1)p_{c,r} \cdot x_{s'} > kb_s.$$

The latter implies $\frac{b_{s'}}{b_s} > \frac{k}{k-1} \geq 1 + \beta$, which is a contradiction to how budgets are allocated. □

Proof of Theorem 6. Let (x^*, b^*, p^*) be the outcome of a Pseudo-Market with Priorities mechanism that has market-clearing error $\alpha \geq 0$. We show that the outcome can be supported with some Iterated Pseudo-Market mechanism with the same market-clearing error. To accomplish this, we first show that for each student $s \in S_r$ in each round $r = 1, ..., R$, demand x^*_s is optimal. Second, we consider excess demand $\tilde{z}_{c,r}$ for each $c \in \mathcal{C}_r$ and round $r = 1, ..., R$ in the Iterated Pseudo-Market mechanism. We show that $\tilde{z}_{c,r}$ equals zero except for one priority level, for which it equals the excess demand of the Pseudo-Market with

[22] This definition of the PMP mechanism follows the same structure as that of the A-CEEI mechanism in Supplementary Appendix D of Azevedo and Budish (2019)

Priorities mechanism z_c^*.

- *Rounds* $r = 1$. The set of available courses $\mathcal{C}_r \equiv \mathcal{C}$ and $q_{c,r} = q_c$ for each $c \in \mathcal{C}_r$. Then, allocation x_s^* for each $s \in \mathcal{S}_r$ maximizes student's utility among courses $\mathcal{C}_r$, i.e.,

$$x_s^* = \max_{\succsim_s} \{x' \in 2^{\mathcal{C}_r} : \sum_{c\in\mathcal{C}_r} p_{c,r}^* x_c' \leq b_s^*\}.$$

We have two possible cases for excess demand $\tilde{z}_{c,r}$.

1. If $r < r_c^*$, then $\sum_{\{s\in\mathcal{S}_{r'}:r'\leq r\}} x_{s,c}^* < q_c$ and $\tilde{z}_{c,r} = \max(\sum_{\{s\in\mathcal{S}_{r'}:r'\leq r\}} x_{s,c}^* - q_c, 0) = 0$.
2. If $r = r_c^*$, then price $p_{c,r'} \geq \bar{b}$ for each level $r' = r+1, ..., R$, and each $s \in S_{r'}$ cannot afford course c, i.e., $x_{s,c} = 0$. Therefore, if $p_{c,r}^* > 0$ we also must have $p_{c,R}^* > 0$ and

$$\tilde{z}_{c,r} = \sum_{r'\leq r} \sum_{s\in\mathcal{S}_{r'}} x_{s,c}^* - q_c = \sum_{s\in\mathcal{S}} x_{s,c}^* - q_c = z_c^*.$$

If $p_{c,r}^* = 0$, there are two cases. If this is the last round $r = R$ or $\sum_{s\in\mathcal{S}_r} x_{s,c}^* \geq q_{c,r}$ we have

$$\tilde{z}_{c,r} = \max(\sum_{s\in\mathcal{S}_r} x_{s,c}^* - q_{c,r}, 0) = \max(\sum_{s\in\mathcal{S}} x_{s,c}^* - q_c, 0) = z_c^*.$$

If $\sum_{s\in\mathcal{S}_r} x_{s,c}^* < q_{c,r}$ and $r < R$, excess demand $\tilde{z}_{c,r} = 0$ differs from $z_c^* = \sum_{s\in\mathcal{S}} x_{s,c}^* - q_c < 0$. In this case, the excess demand is pushed to period $r+1$ as explained in 3) for rounds $r = 2, ..., R$.

- *Rounds* $r = 2, ..., R$. The set of available courses $\mathcal{C}_r$ consists only of those c with either i) $r < r_c^* + 1$ or ii) $r = r_c^* + 1, p_{c,r_c^*} = 0$, and $\sum_{s\in\mathcal{S}_{r_c^*}} x_{s,c}^* < q_{c,r_c^*}$, where $q_{c,r} = q_c - \sum_{r'<r} \sum_{s\in\mathcal{S}_{r'}} x_{s,c}^*$. Condition (1) implies that for every course $c \in \mathcal{C}\backslash\mathcal{C}_r$ we have $p_{c,r}^* \geq \bar{b}$. Hence, allocation x_s^* for each student $s \in \mathcal{S}_r$ maximizes student's utility among courses $\mathcal{C}_r$, i.e.,

$$x_s^* \equiv \max_{\succsim_s} \{x' \in 2^{\mathcal{C}} : \sum_{c\in C} p_{c,r}^* x_c' \leq b_s^*\} = \max_{\succsim_s} \{x' \in 2^{\mathcal{C}_r} : \sum_{c\in C_r} p_{c,r}^* x_c' \leq b_s^*\}.$$

We have three possible cases for excess demand $\tilde{z}_{c,r}$.

1) If $r < r_c^*$, then $\sum_{\{s\in\mathcal{S}_{r'}:r'\leq r\}} x_{s,c}^* < q_c$ and $\tilde{z}_{c,r} = \max(\sum_{\{s\in\mathcal{S}_{r'}:r'\leq r\}} x_{s,c}^* - q_c, 0) = 0$.
2) If $r = r_c^*$, then price $p_{c,r'} \geq \bar{b}$ for each level $r' = r+1, ..., R$, and each $s \in S_{r'}$ cannot afford course c, i.e., $x_{s,c} = 0$. Therefore, if $p_{c,r}^* > 0$ we also must have $p_{c,R}^* > 0$ and

$$\tilde{z}_{c,r} = \sum_{r'\leq r} \sum_{s\in\mathcal{S}_{r'}} x_{s,c}^* - q_c = \sum_{s\in\mathcal{S}} x_{s,c}^* - q_c = z_c^*.$$

If $p_{c,r}^* = 0$, there are two cases. If this is the last round $r = R$ or $\sum_{s\in\mathcal{S}_r} x_{s,c}^* \geq q_{c,r}$ we have

$$\tilde{z}_{c,r} = \max(\sum_{s\in\mathcal{S}_r} x_{s,c}^* - q_{c,r}, 0) = \max(\sum_{s\in\mathcal{S}} x_{s,c}^* - q_c, 0) = z_c^*.$$

If $\sum_{s\in\mathcal{S}_r} x^*_{s,c} < q_{c,r}$ and $r < R$, excess demand $\tilde{z}_{c,r} = 0$ differs from $z^*_c = \sum_{s\in\mathcal{S}} x^*_{s,c} - q_c < 0$. In this case, the excess demand is pushed to the next period $r+1$ as explained below.

3) If $r = r^*_c + 1, p_{c,r^*_c} = 0$, and $\sum_{s\in\mathcal{S}_r} x^*_{s,c} < q_{c,r^*_c}$, then price $p_{c,r'} \geq \bar{b}$ for each level $r' = r, ..., R$ including $p_{c,R} \geq \bar{b} > 0$; hence, each $s \in S_{r'}$ cannot afford course c, i.e., $x_{s,c} = 0$, and

$$\tilde{z}_{c,r} = \sum_{s\in\mathcal{S}_{r-1}} x^*_{s,c} - q_{c,r-1} = \sum_{s\in\mathcal{S}} x^*_{s,c} - q_c = z^*_c.$$

Overall, we obtain that $\tilde{z}_{c,r^*_c} = z^*_c$ if $p^*_{c,r^*_c} > 0$ or $p_{c,r^*_c} = 0$ and $\sum_{s\in\mathcal{S}_r} x^*_{s,c} \geq q_{c,r*_c}$, and $\tilde{z}_{c,r^*_c+1} = z^*_c$ if $p^*_{c,r^*_c} = 0$ and $\sum_{s\in\mathcal{S}_r} x^*_{s,c} \geq q_{c,r^*_c}$. Also, once the course is removed at round r, we have $\tilde{z}_{c,r'} = 0$ for $r' = r+1, ..., R$. Therefore, if we denote $\sum_{c\in\mathcal{C}} \tilde{z}^2_{c,r} = \alpha^2_r$ for each $r = 1, ..., R$, we obtain

$$\sum_{r=1}^{R} \alpha^2_r = \sum_{r=1}^{R}\sum_{c\in\mathcal{C}} \tilde{z}^2_{c,r} = \sum_{c\in\mathcal{C}} (z^*_c)^2 = \alpha^2.$$

Hence, tuple (x^*, p^*, b^*) is the outcome of an Iterated Pseudo-Market mechanism with the same total market-clearing error.

Let us now assume that (x^*, p^*, b^*) is the outcome of an Iterated Pseudo-Market Mechanism with the vector of market-clearing errors $(\alpha_1, ..., \alpha_R)$ such that $\sqrt{\sum_{r=1}^{R} \alpha^2_r} = \alpha$. To show that the outcome can be supported with some Pseudo-Market with Priorities mechanism with the same market-clearing error, we establish the following steps.

- *Prices p^* satisfy the cut-off pricing rule.* To see why the Iterated Pseudo-Market mechanism results in cut-off prices that satisfy condition (1), notice that once the price for a course becomes positive or the price is zero and all course seats are assigned, the course is removed from the mechanism. Once the course is removed its price is set above or equal to $\bar{b}$.

  In particular, for each $c \in C$ we set $r^*_c = r$ if either there is round r such that $\bar{b} > p^*_{c,r} > 0$ or $p^*_{c,r} = 0$ and $\sum_{s\in\mathcal{S}_r} x^*_{s,c} \geq q_{c,r}$; otherwise, we set $r^*_c = R$. Definition 7 then implies that $p^*_{c,r} \geq \bar{b}$ for any $r > r^*_c$. In addition, for all $r < r^*_c$ we must have $p_{c,r} = 0$ and $\sum_{s\in\mathcal{S}_r} x^*_{s,c} - q_{c,r} = \sum_{\{s\in\mathcal{S}_{r'}, r'\leq r\}} x^*_{s,c} - q_c < 0$. Therefore, $\sum_{\{s\in\mathcal{S}_r, r<r^*_c\}} x^*_{s,c} < q_c$ as required by the Pseudo-Market Equilibrium with Priorities.

- *Student's demand is optimal, i.e.,* $x^*_s \in \max_{\succsim_s}\{x' \in 2^{\mathcal{C}} : \sum_{c\in\mathcal{C}} p^*_{c,r} x'_c \leq b^*_s\}$, $r = r_s$, $s \in \mathcal{S}$. Using the price cut-offs introduced above, the definition of the Iterated Pseudo-Market mechanism implies that the set of available courses C_r consists only of those courses for which $r \leq r^*_c$. As $p^*_{c,r} \geq \bar{b}$ for any $r > r^*_c$, we obtain

$$x^*_s = \max_{\succsim_s} \{x' \in 2^{\mathcal{C}_r} : \sum_{c\in\mathcal{C}_r} p^*_{c,r} x'_c \leq b^*_s\} = \max_{\succsim_s}\{x' \in 2^{\mathcal{C}} : \sum_{c\in\mathcal{C}} p^*_{c,r} x'_c \leq b^*_s\}.$$

- *Market clearing error satisfies* $\sum_{c\in\mathcal{C}}(z_c^*)^2 \leq \alpha^2$, *where* $z_c^* = \begin{cases} \sum_s x_{s,c}^* - q_c \text{ if } p_{c,R}^* > 0, \\ \max(\sum_s x_{s,c}^* - q_c, 0) \text{ if } p_{c,R}^* = 0 \end{cases}$.

  The definition of the Iterated Pseudo-Market mechanism implies that $z_{c,r}$ could be different from zero only for rounds when $r = r_c^*$. For all other rounds, the excess demand equals zero. Therefore, if $p_{c,R}^* > 0$ there are two possibilities.

  - In case $p_{c,r_c^*} > 0$, we have $z_c^* = \sum_{s\in S} x_{s,c}^* - q_c = \sum_{s\in S_{r^*}} x_{s,c}^* - q_{c,r} = z_{c,r_c^*}^*$.
  - In case $p_{c,r_c^*} = 0$ and $\sum_{s\in S_{r_c^*}} x_{s,c}^* \geq q_{c,r_c^*}$, we have $z_c^* = \sum_{s\in S} x_{s,c}^* - q_c = \max(\sum_{s\in S_{r_c^*}} x_{s,c}^* - q_{c,r_c^*}, 0) = z_{c,r_c^*}^*$.

  If $p_{c,R}^* = 0$ we also have $z_c^* = \max(\sum_{s\in S} x_{s,c}^* - q_c, 0) = \max(\sum_{s\in S_R} x_{s,c}^* - q_{c,R}, 0) = z_{c,R}^*$.

  Overall, excess demand for each course c is different from zero at most for one round in the Iterated Pseudo-Market mechanism. For this round, it coincides with the excess demand for the Pseudo-Market with Priorities mechanism. Hence, $\sum_{c\in C}(z_c^*)^2 = \sum_{r=1}^{R}\sum_{c\in C_r}(z_{c,r}^*)^2 = \alpha^2$.

The three steps presented above imply that (x^*, p^*, b^*) is an outcome of a Pseudo-Market with Priorities mechanism with the same market-clearing error. □

Proof of Corollary 1. Results (ii), (iii), (iv) follow from Theorem 6 that establishes the relationship between Iterated Pseudo-Market mechanisms and the Pseudo-Market with Priorities mechanisms and the results of Theorems 2, 4, and 5. For (i), note that no Pareto improvements are possible among students at the same level of priority because the iterated market mechanism finds an A-CEEI among students at each level of priority, and, thus, the result follows from Proposition 2 in Budish (2011). Moreover, no Pareto improving trades are possible among students at different levels of priority. Consider students $s, s' \in \mathcal{S}$, where $s \in \mathcal{S}_r$ and $s' \in \mathcal{S}_{r'}$, with $r < r'$. Then, s is assigned her most-preferred course schedule she can afford. Hence, s can only improve her course schedule by getting seats in courses she cannot afford. Though, if s cannot afford a seat in the course c, then $p_{c,r} > 0$, which results in $q_{c,r+1} = 0$. As a result, s can never benefit from a trade with s', and therefore, no Pareto improving trades are possible among students at different levels of priority. □

Proof of Theorem 7. Note that the BAPM mechanism finds an (α, β)-CEEI. Hence, it inherits properties (i) and (iv) of the A-CEEI (see Proposition 2 in Budish (2011) and Theorem 4 in Budish (2010)). To see why the property (ii) is true, note any student of a higher priority is assigned budget that she can afford any course schedule of a student of a lower priority. Student's utility maximization 3a) in Definition 8 then implies property (ii).

To prove property (iii), we show that the allocation of course seats to students of the same priority level is itself an A-CEEI allocation. Let b^*, p^*, and x^* represent the budgets, course prices, and allocation of course schedules to students from the BAPM mechanism. Recall that $\mathcal{S}_r$ is the set of students that have priority $r = 1, ..., R$. We also denote the set of courses that fill at least one seat with students from $\mathcal{S}_r$ as $\mathcal{C}_r$. Similarly, $Q_r = (q_{c,r}^*)_{c\in\mathcal{C}_r}$ denotes the number of seats filled by students from $\mathcal{S}_r$. Also, $V_r = \{\succsim_s\}_{s\in\mathcal{S}_r}$.

Consider the combinatorial assignment problem $(\mathcal{S}_r, \mathcal{C}_r, Q_r, V_r)$ and assign students budgets $(b_s^*)_{s\in\mathcal{S}_r}$. For this assignment problem, the budgets $(b_s^*)_{s\in\mathcal{S}_r}$, course prices $(p_c^*)_{c\in\mathcal{C}_r}$, and allocation $(x_s^*)_{s\in\mathcal{S}_r}$ form an A-CEEI. For each student $s \in \mathcal{S}_r$, b_s^* is a random draw from the interval $[k^{R-r}(1+\beta)^{R-r},\ k^{R-r}(1+\beta)^{R-r+1}]$. So, the price of each course $c \in \mathcal{C}_r$ is in the interval $[0, k^{R-r}(1+\beta)^r]$. When we consider scaled down student budgets $(\frac{1}{k^{R-r}(1+\beta)^{R-r}} \cdot b_s^*)_{s\in\mathcal{S}_r}$ and course prices $(\frac{1}{k^{R-r}(1+\beta)^{R-r}} \cdot p_c^*)_{c\in\mathcal{C}_r}$, budgets are random draws from $[1, 1+\beta]$ and course prices are set in the interval $[0, 1+\beta]$. As the proportional decrease in student budgets and course prices does not change the course schedule that maximizes student's utility, $(x_s^*)_{s\in\mathcal{S}_r}$ is still an A-CEEI for $(\mathcal{S}_r, \mathcal{C}_r, Q_r, V_r)$. As a result, by Theorem 3 of Budish (2011), the allocation $(x_s^*)_{s\in\mathcal{S}_r}$ bounds envy by a single course among students at the same priority level if $\beta \leq \frac{1}{k-1}$. $\square$

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