Speculative Bubbles and Aggregate Boom Bust Cycles: Closed and Open Economies

Robert Kollmann (*)
Université Libre de Bruxelles & CEPR

February 15, 2022

This paper constructs speculative bubbles in Real Business Cycle models of closed and open economies. Building on Blanchard's (1979) classic model of asset price bubbles, the speculative bubbles studied here arise from the absence of a transversality condition (TVC) for production capital. The lack of TVC can be due to an overlapping generations population structure. Speculative bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment, and may occur when there are no shocks to technologies and preferences. It is shown that speculative bubbles can generate bounded boom bust cycles of investment and output. Speculative bubbles are thus a novel potential driver of economic fluctuations. In a two-country model with integrated financial markets, speculative bubbles must be perfectly correlated across countries. Global speculative bubbles may, thus, help to explain the international synchronization of international business cycles.

JEL codes: E1,E3, F3,F4,C6
Keywords: speculative bubbles, boom bust cycles, business cycles in closed and open economies, dynamic general equilibrium models, RBC models, Long-Plosser model, Dellas model, international business cycle synchronization.

(*) Address: Solvay Brussels School of Economics & Management, Université Libre de Bruxelles, CP114, 50 Av. Roosevelt, 1050 Brussels, Belgium
robert_kollmann@yahoo.com www.robertkollmann.com
I thank Sumru Altug, Julien Bengui and Harald Uhlig for insightful discussions and advice. Thanks for useful comments are also due to Guido Ascari, Agustín Bénétrix, Florin Bilbiie, Nuno Coimbra, Luca Dedola, Mick Devereux, Luca Fornaro, Jordi Galí, Tom Holden, Matthew Knowles, Michael Krause, Tommaso Monacelli, Gilles Saint-Paul, Cédric Tille, Gauthier Vermandel, Raf Wouters, and to workshop participants at National Bank of Belgium, St. Andrews University, University of Navarra, Swiss National Bank, Bundesbank, Paris School of Economics, Trinity College Dublin, Central Bank of Ireland, University of Cologne, University of Rome, Peking University and at IMFC, CEPR-MMCN, CEPR-IMF-ECB, JRC-European Commission, IMAC and DYNARE conferences.

1. Introduction

This paper studies speculative bubbles that arise in Real Business Cycle (RBC) models with capital accumulation, when one assumes that there is no transversality condition (TVC) for capital. The absence of the TVC can be due to an overlapping generations (OLG) population structure. The models considered here assume rational expectations. Speculative bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment.

Except for the absence of a TVC, the models analyzed here are entirely standard. The aggregate static equations and Euler equations are *identical* to those of canonical RBC models. If a TVC is imposed, the present models have a unique equilibrium. I show how to construct speculative bubbles that feature *recurrent* boom bust cycles characterized by *bounded* investment and output expansions that are followed by abrupt contractions in real activity. Importantly, recurrent boom bust cycles may arise when there are no shocks to technologies and preferences. The model solutions considered here are globally accurate, and they thus take feasibility and non-negativity constraints for consumption, capital and output into account, even under large deviations from steady state. It is shown that, with speculative bubbles, the unconditional mean of real activity can be close to the no-bubble steady state. Speculative bubbles can generate persistent fluctuations of real activity, and capture key business cycle stylized facts. Speculative bubbles are thus a novel potential driver of economic fluctuations.

The notion of speculative bubbles, defined as multiple equilibria due to the absence of a TVC, was introduced by Blanchard (1979), in a simple linear asset pricing model. This notion has been highly influential in finance, as it provides a powerful narrative about explosive asset market booms that are followed by sudden busts.[1] However, so far, this concept has had much less impact on structural macroeconomics. To the best of my knowledge, the present paper provides the first analysis of Blanchard-type speculative bubbles, in dynamic general equilibrium business cycle models.

Both closed and open economies are analyzed. A central finding for a two-country model is that, with integrated financial markets, bounded speculative bubbles must be perfectly correlated across countries. Global bubbles may, thus, help to explain the synchronization of international business cycles.

[1] See, e.g., Mussa (1990) and Stracca (2004) for references. Google Scholar records 2918 cites (12/2021) for Blanchard (1979) and its companion paper Blanchard and Watson (1982).

Like Blanchard (1979), I assume a speculative bubble process with two states. The economy can either be in a 'boom' state or in a 'bust' (crash) state. In a boom, investment diverges positively from the no-bubble decision rule that holds under the TVC (saddle path). High investment during a boom is sustained by agents' belief that, with positive probability, investment will grow next period, thereby depressing future consumption and raising the (expected) future marginal utility-weighted return of capital. During a boom, the expansion of investment and output accelerates initially; however, due to decreasing returns, the growth of investment and output ultimately slows down, during a long-lasting boom. An uninterrupted boom has zero probability. At any time, a bust can occur; in a bust, investment drops abruptly, and reverts towards the no-bubble decision rule. Investment busts are triggered by self-fulfilling downward revisions of expected future investment. Transitions between booms and busts are prompted by a random sunspot, and occur with exogenous probabilities. While investment and output "explode" during the early phase of a boom, the global dynamics of investment and output is stable. As investment is high during booms, speculative bubbles exhibit capital over-accumulation -- the average capital stock is higher in bubbly equilibria than in the no-bubble equilibrium (that obtains when the TVC holds).

The analysis here is related to, but fundamentally different from, Ascari et al. (2019) who study *temporary* explosive equilibria, in the textbook linearized three-equation New Keynesian macro model without capital. Under rational expectations, bubbles in that model imply that the expected paths of inflation and output tend to $\pm\infty$. To rule out unbounded paths, the authors postulate *limited* rationality: once an explosive trajectory reaches a threshold, the economy is assumed to revert *permanently* to its unique stable saddle path. Under rational expectations, the future switch to the saddle path would, from the outset, rule out the emergence of bubbles. By contrast, the present paper assumes rational expectations, and capital accumulation is at the heart of the analysis here (however, the paper abstracts from nominal rigidities). In the models considered here, speculative bubbles generate *bounded* output fluctuations.

Explosive (yet bounded) output dynamics in booms distinguishes the model here from the large class of business cycle models with multiple "local" sunspot equilibria whose support is close to steady state. Local sunspot equilibria arise when the number of eigenvalues (of the linearized state-space form) outside the unit circle is less than the number of non-predetermined variables (Blanchard and Kahn (1980); Woodford (1986)). The model ingredients that may give rise to local sunspot equilibria include increasing returns, externalities, financial frictions, certain OLG structures and/or violations of the Taylor principle of monetary policy; see, e.g., Woodford (1988), Schmitt-Grohé (1997), Benhabib and Farmer (1999), Lubik and Schorfheide (2004), Holden (2016, 2021). The sunspot equilibria studied in these models generally satisfy aggregate TVCs. The model ingredients that deliver local sunspot equilibria may be debatable.[2] None of these ingredients are used in the very simple business cycle models considered in the present paper. In the models studied here, the number of eigenvalues of the linearized state-space form equals the number of non-predetermined variables. The models here have a unique solution when the TVC is imposed (as mentioned above).

Section 2 briefly reviews the speculative asset price bubble analyzed by Blanchard (1979). Sect. 3 constructs speculative bubbles in a simple Long and Plosser (1983) RBC economy, when the TVC is dropped. That model assumes log utility, a Cobb-Douglas production function and full capital depreciation. Exact closed form solutions with speculative bubbles can be derived for that model. Sect. 4 shows how speculative bubbles can be constructed in a richer, more realistic RBC economy with incomplete capital depreciation. Sects. 5 and 6 study bubbles in open economies.

2. Blanchard (1979) speculative asset price bubble

Blanchard (1979) considers a log-linear asset price model of the form $p_t = \beta \cdot E_t p_{t+1} + d_t$, where p_t is the price of a stock (in logs) at date t, while d_t is the (scaled) log dividend. $0<\beta<1$ is the investors' subjective discount factor. Assume, for simplicity that the dividend is constant, and normalized at $d_t=0$, so that $E_t p_{t+1} = \frac{1}{\beta} p_t$. As $\frac{1}{\beta}>1$, the model has a unique non-explosive solution given by $p_t=0 \ \forall t$ (Blanchard and Kahn (1980), Prop. 1). Blanchard (1979) pointed out that, if there are no transversality or boundary conditions, the model is also solved by a bubble process $\{p_t\}$ such that $p_{t+1}=0$ obtains with probability π, while $p_{t+1}=[(\frac{1}{\beta})/(1-\pi)] \cdot p_t$ obtains with probability $1-\pi$ $(0<\pi<1)$. If $p_t \neq 0$, then next period the asset price continues to diverge with probability $1-\pi$, while a 'bust' (return to the no-bubble solution $p=0$) occurs with probability π.

[2] E.g., increasing returns/externalities need to be sufficiently strong; in OLG models the steady state interest rate may not exceed the trend growth rate (r≤g) etc. Note that r>g holds in the present model.

This bubble process implies that after a bust, non-zero values of p never arise again, i.e. the bubble is 'self-ending'. Recurrent (never-ending) bubbles obtain if the bust implies a price $\Delta\neq 0$: $p_{t+1}=(\frac{1}{\beta}p_t-\Delta\pi)/(1-\pi)$ with probability $1-\pi$ and $p_{t+1}=\Delta$ with probability π. Speculative bubbles can exhibit prolonged episodes during which the asset price deviates more and more from its 'fundamental' value (p=0), before abruptly reverting towards that fundamental value.

3. Speculative bubbles in a Long-Plosser RBC economy without TVC

This Section studies speculative bubbles in a Long and Plosser (1983) RBC economy without TVC. Assume log period utility $u(C_t)=\ln(C_t)$, where $C_t\geq 0$ denotes consumption in period t. The production function is:

$$Y_t=\theta_t K_t^{\alpha},\ 0<\alpha<1, \tag{1}$$

where $Y_t, K_t, \theta_t \geq 0$ are output, capital and exogenous total factor productivity (TFP). For simplicity, labor hours are constant and normalized to unity. The resource constraint is

$$C_t+I_t=Y_t, \text{ with } I_t=K_{t+1}. \tag{2}$$

$I_t\geq 0$ is (gross) investment. A 100% capital depreciation rate is assumed, and thus investment equals next period's capital stock. (Sect. 4 considers a model with variable hours and incomplete capital depreciation.) Assume that productivity is bounded. Decreasing returns to capital $(0<\alpha<1)$ then imply that all feasible paths of capital, output and consumption are likewise bounded. The Euler equation for capital is

$$E_t\beta(C_t/C_{t+1})\alpha Y_{t+1}/K_{t+1}=1, \tag{3}$$

where $0<\beta<1$ is the subjective discount factor. Using the resource constraint $K_{t+1}=Y_t-C_t$ one can express the Euler equation (3) as a linear expectational difference equation in the output/consumption ratio $X_t\equiv Y_t/C_t \geq 1$:

$$E_t X_{t+1} = \tfrac{1}{\alpha\beta}X_t - \tfrac{1}{\alpha\beta}. \tag{4}$$

Long and Plosser (1983) assume an *infinitely-lived* representative household. That household's decision problem has a unique solution (as the problem is a well-behaved concave programming problem). The necessary and sufficient optimality conditions of that decision problem are the household's resource constraint and Euler equation (summarized by (4)) and a transversality condition (TVC) that requires that the present discounted value of the capital stock is zero, at infinity:

$$\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=0. \quad (5)$$

Note that $u'(C_{t+\tau})K_{t+\tau+1}=X_{t+\tau}-1$, so the TVC holds iff $\lim_{\tau\to\infty}\beta^{\tau}E_t X_{t+\tau}=0$. A constant output/consumption ratio $X_t=\overline{X}\equiv\frac{1}{1-\alpha\beta}>1\ \ \forall t$ satisfies (4) and the TVC (5). This is the textbook solution of the Long-Plosser model (e.g., Blanchard and Fischer (1989)). Under that solution, consumption and investment are time-invariant shares of output: $C_t=(1-\alpha\beta)Y_t,\ K_{t+1}=\alpha\beta Y_t\ \ \forall t$.

In what follows, I postulate that there is no TVC. This gives rise to multiple equilibria. I refer to a process $\{X_t\}$ that satisfies $X_t\geq 1$ and (4), but that differs from the textbook solution (derived under the TVC), as a **speculative bubble equilibrium**, or (speculative) bubble, for short. Speculative bubbles violate the TVC. [3]

The lack of TVC can be justified by the assumption that the economy has an overlapping generations (OLG) population structure. Kollmann (2020) develops an OLG structure with finitely-lived agents that has the *same* aggregate resource constraint and the *same* aggregate Euler equation as a Long-Plosser economy inhabited by an infinitely-lived representative agent. Thus equations (1)-(4) hold in such an OLG structure. At the end of her life, each agent holds zero assets. As agents have a finite horizon, the (infinite-horizon) TVC for *aggregate* capital (5) is *not* an equilibrium condition, in that OLG structure.[4]

Besides assuming an OLG structure, another potential motivation for disregarding the TVC is that detecting TVC violations may be difficult, in economies that are more complicated than the Long-Plosser economy, i.e. in models for which no closed form solution exists (see Sect. 4). TVC violations can be caused by very low-probability events in a distant future. Agents may thus lack the cognitive/computing power to detect TVC violations, so that speculative bubbles can arise (see Blanchard and Watson (1982)).

---

[3] (4) implies $E_t X_{t+\tau}=(\frac{1}{\alpha\beta})^{\tau}(X_t-\overline{X})+\overline{X}$. Thus, $\lim_{\tau\to\infty}\beta^{\tau}E_t u'(C_{t+\tau})K_{t+\tau+1}=\pm\infty$ when $X_t\neq\overline{X}$.

[4] Standard OLG models (e.g. Blanchard, 1985) assume that newborn agents have zero financial wealth at birth (and thus rely on labor income to accumulate wealth). By contrast, Kollmann (2020) assumes that each newborn is endowed with a time-invariant share of aggregate wealth. Under complete risk sharing among contemporaneous generations, this ensures that an *aggregate* Euler equation of form (3) holds. The Kollmann (2020) OLG structure thus allows to generate speculative bubbles in tractable models suitable for calibration to quarterly data. By contrast, conventional OLG models are typically much more cumbersome (due to the implied heterogeneity of generations) which makes them less useful for quantitative business cycle analysis.

3.1. Constructing speculative bubbles

(4) is satisfied for any processes

$$X_{t+1}-\overline{X}=\tfrac{1}{\alpha\beta}\cdot(X_t-\overline{X})+\varepsilon_{t+1}, \qquad (6)$$

where ε_{t+1} is a random variable whose conditional mean is zero, $E_t\varepsilon_{t+1}=0$. I will focus on bubbles processes that (i) never reach the unit lower bound of the output/consumption ratio and (ii) are not self-ending. When the unit lower bound of the output/consumption ratio is attained, all output is consumed, so that investment and next period's capital stock drop to zero; output, consumption and investment remain at zero forever thereafter. Such an "extinction" equilibrium seems empirically irrelevant. Standard business cycle models concentrate on *recurrent* fluctuations in economic activity. This is why I focus on speculative bubbles that never end. Requirements (i),(ii) necessitate that the output/consumption ratio always strictly *exceeds* the ratio in the no-bubble equilibrium: $X_t>\overline{X}$ $\forall t$. [5]

The investment/output ratio is an increasing function of the output/ consumption ratio: $K_{t+1}/Y_t=(Y_t-C_t)/Y_t=1-1/X_t$. Thus, $X_t>\overline{X}$ implies that the investment/output ratio is larger in the recurrent speculative bubbles studied here than in the text-book no-bubble equilibrium. Hence, the bubbles considered here exhibit capital over-accumulation. Note also that $X_t>\overline{X}$ implies that the *expected* path of the output/consumption ratio explodes (from (6)): $\lim_{s\to\infty}E_tX_{t+s}=\infty$. However, consumption, capital and output are *bounded,* as any feasible path for these variables is bounded, due to decreasing returns to capital (and given bounded TFP); see above.

In what follows, I construct recurrent speculative bubbles such that $X_t\geq\overline{X}+\Delta$ $\forall t$ holds for a small constant $\Delta>0$. By analogy to the recurrent (never-ending) Blanchard (1979) asset-price bubble, I consider a two-state bubble process for the output/consumption ratio $\{X_t\}_{t\geq 0}$ defined by:

(i) $X_0\geq\overline{X}+\Delta$;

(ii) $X_{t+1}=X^L_{t+1}\equiv\overline{X}+\Delta$ with probability $0<\pi<1$,

---

[5] When $X_t<\overline{X}$, then the economy hits the unit lower bound of the output/consumption ratio almost surely in a later period, as the autoregressive coefficient $1/(\alpha\beta)$ of (6) exceeds unity. When $X_t=\overline{X}$, then $X_\tau=\overline{X}$ $\forall\tau\geq t$ is the only process going forward that never hits the unit lower bound.

and $X_{t+1}=X_{t+1}^{H}\equiv\overline{X}+\{\frac{1}{\alpha\beta}[X_t-\overline{X}]-\Delta\pi\}/(1-\pi)\equiv\Psi(X_t)$ with probability 1-π for $t\geq 0$. (7)

Whether X_{t+1}^{L} or X_{t+1}^{H} is realized depends on an exogenous i.i.d. sunspot (the probability π is exogenous). Note that $X_{t+1}^{L}<X_{t+1}^{H}$. Thus, states X_{t+1}^{L} and X_{t+1}^{H} will be referred to as investment busts and booms, respectively.

Given the sequence $\{X_t\}_{t\geq 0}$, the path of capital $\{K_{t+1}\}_{t\geq 0}$ can be generated recursively (for an exogenous initial capital stock K_0) using $K_{t+1}=\{1-1/X_t\}\theta_t(K_t)^{\alpha}$ for $t\geq 0$.

An *uninterrupted* infinite sequence of investment booms (X^H) would drive the output/consumption ratio to infinity and the investment/output ratio to unity, while capital (and output) would converge towards its (finite) upper bound. Of course, an uninterrupted investment boom run has zero probability. At any time, the output/consumption ratio can drop to $\overline{X}+\Delta$, with probability π. This ensures that the investment/output ratio undergoes recurrent fluctuations. If the bust probability π is sufficiently big and if Δ is close to zero, then speculative bubbles induce fluctuations of real activity that remain most of the time near the steady state of the no-bubble economy. This is the case in the stochastic simulations reported below.

What expectations sustain the speculative bubble? In equilibrium, agents expect at date t that X_{t+1} will equal $X_{t+1}^{L}=\overline{X}+\Delta$ or $X_{t+1}^{H}=\Psi(X_t)$ with probabilities π and 1-π, respectively (the function Ψ is defined in (7)). Note that X_{t+1}^{L} and X_{t+1}^{H} are known at t. At t+1, agents are free to select a value of X_{t+1} that differs from X_{t+1}^{L} or X_{t+1}^{H}; however, in equilibrium, they chose not to do so because a choice $X_{t+1}\in\{X_{t+1}^{L},X_{t+1}^{H}\}$ is 'validated' by their date t+1 expectations about X_{t+2}. Assume that an investment **bust** occurs in t+1, so that agents choose $X_{t+1}=\overline{X}+\Delta$; in equilibrium, this choice is sustained by agents' expectation (at t+1) that X_{t+2} will equal $\overline{X}+\Delta$ or $\Psi(\overline{X}+\Delta)$ with probabilities π and 1-π, respectively. By contrast, if an investment **boom** occurs at t+1, then agents choose $X_{t+1}=X_{t+1}^{H}\equiv\Psi(X_t)$; this choice is supported by the expectation (at t+1) that X_{t+2} will equal $\overline{X}+\Delta$ or $\Psi(X_{t+1}^{H})=\Psi(\Psi(X_t))$ with probabilities π and 1-π, respectively. Note that $X_t\geq\overline{X}+\Delta$ implies that $\Psi(\overline{X}+\Delta)<\Psi(\Psi(X_t))$ holds. This shows that, in an investment boom (at t+1), agents expect a higher future investment/output ratio than in an investment bust (at t+1). Booms and busts reflect hence self-fulfilling variations in agents' expectations about the future state of the economy. An investment boom [bust] is triggered by a more [less] optimistic assessment of next period's investment/output ratio. High investment during a boom is sustained by agents' belief that, with positive probability, investment will continue to grow next period, thereby depressing future consumption and raising the (expected) future marginal utility-weighted return of capital.

3.2. Quantitative results

I next discuss stochastic model simulations. I set α=1/3 and β=0.99, as is standard in quarterly macro models. To assess whether speculative bubbles alone can generate a realistic business cycle, I assume that TFP is constant. The bust probability is set at $\pi = 0.5$. I set $\Delta = 3.8 \times 10^{-6}$ as for that value the bubble model matches the standard deviation of Hodrick-Prescott (HP) filtered historical US real GDP (see below). [6]

Figure 1 shows representative simulated paths of output (Y, continuous black line), consumption (C, red dashed line) and investment (I, blue dash-dotted line).[7] The bubble model generates sudden, but short-lived, expansions in output and investment. In a boom, the rapid rise in investment is accompanied by a contraction in consumption. At any time, a bust can occur; in a bust, investment drops abruptly.

Table 1 (Row (a)) reports model-generated standard deviations (in %) and cross-correlations of HP filtered (logged) output, consumption and investment; also shown are mean values of these variables. All model-generated business cycle statistics reported in Table 1 (and Table 2 discussed below) are based on one simulation run of T=10000 periods. The reported theoretical business cycle statistics (standard deviations, correlations) are median statistics computed across rolling windows of 200 periods.[8] Mean values (of Y,C and I) are computed using the whole simulation run (T periods) and expressed as % deviations from the no-bubble steady state.

---

[6] In the simulations, the law of motion of the bubbly output/consumption ratio (7) is initiated with a ratio $X_0 = \overline{X} + \Delta$. I set the initial capital stock K_0 at the no-bubble steady state capital stock. The effect of initial values on subsequent simulated values vanishes fast and does not noticeably affect moments over a long simulation run.

[7] The depicted simulated paths of Y,C,I are normalized by steady state no-bubble output.

[8] The HP filter is applied separately in each respective time window. 200-periods windows are used, as the historical business cycle statistics shown in Table 1 pertain to a sample of 200 quarters (see below).

To evaluate the model predictions, Table 1 also reports US historical business statistics based on HP filtered quarterly data (logged) for the period 1968-2017 (see Row (b)). The empirical standard deviations of GDP, consumption and investment are 1.47%, 1.19% and 4.96%, respectively. In the data, consumption and investment are strongly procyclical; these variables and GDP are highly serially correlated.

The model-predicted standard deviations of output, consumption and investment are 1.47%, 3.39% and 4.42%, respectively (see Row (a) of Table 1). Thus, consumption is more volatile in the model than in the data, but the model matches well the high empirical volatility of investment. In the bubble economy, consumption and investment are procyclical; output and investment are predicted to be positively serially correlated, while consumption is predicted to be negatively autocorrelated. Average output and investment are 0.5% and 2.1% higher than in the no-bubble steady state, while consumption is 0.3% lower. Thus, the mean of these endogenous variables is close to the no-bubble steady state.

Capital over-accumulation (compared to the no-bubble equilibrium) implies that the bubble economy is 'dynamically inefficient'. Abel et al. (1989) propose an empirical test of dynamic efficiency. Their key insight is that, in a dynamically efficient economy, income generated by capital (i.e. output minus the wage bill) exceeds investment. Abel et al. (1989, Table 1) show that, in US data 1929-1985, this condition is met in all years of their sample. The US historical sample average of the (capital income-investment)/GNP ratio is 13.41%.[9]

In the bubbly Long-Plosser economy, the (capital income – investment)/GDP ratio is positive in 97.01% of all quarters, but the average ratio is slightly negative, -0.09%. Note that, in the no-bubble version of the Long-Plosser economy, the (capital income – investment)/GDP ratio equals $\alpha(1-\beta)=0.33\%$, which is only slightly greater than zero, and much smaller than the empirical ratio. Thus, even modest dynamic inefficiency produces a negative mean capital income – investment gap. As shown in the next Section, an RBC economy with incomplete capital depreciation can generate speculative bubbles with sizable positive mean capital income – investment gaps.

[9] The (capital income-investment)/GNP ratio is likewise positive and sizable in post-1985 data. Mean ratio 1968-2017 (sample period used for business cycle moments reported in Table 1): 17.58%.

4. Speculative bubbles in an RBC economy (no TVC) with incomplete capital depreciation

I next show how speculative bubbles can be constructed in a richer, more realistic RBC economy with incomplete capital depreciation (and variable labor). As before, I postulate that there is no TVC for capital.

The period utility function is now assumed to be $u(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t),\ \Psi>0,$ where $0\leq L_t\leq 1$ are hours worked. The household's total time endowment (per period) is normalized to one, so $1-L_t$ is leisure.[10] The resource constraint and the output technology are

$$C_t+K_{t+1}=Y_t+(1-\delta)K_t \text{ with } Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \tag{8}$$

where $0<\delta<1$ is the capital depreciation rate. θ_t (TFP) is exogenous and follows the bounded AR(1) process $\ln(\theta_{t+1})=\rho\ln(\theta_t)+\varepsilon^{\theta}_{t+1},\ 0\leq\rho<1,$ where $\varepsilon^{\theta}_{t+1}$ is a white noise that equals $-\sigma_\theta$ or σ_θ with probability ½ $(\sigma_\theta\geq 0)$. The standard deviation of the $\varepsilon^{\theta}_{t+1}$ is thus σ_θ.[11] The economy has these efficiency conditions

$$C_t\Psi/(1-L_t)=(1-\alpha)\theta_t(K_t)^{\alpha}(L_t)^{-\alpha} \quad \text{and} \tag{9}$$

$$E_t\beta\{C_t/C_{t+1}\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(L_{t+1})^{1-\alpha}+1-\delta)=1. \tag{10}$$

(9) indicates that the household's marginal rate of substitution between leisure and consumption is equated to the marginal product of labor, while (10) is the date *t* Euler equation for capital.

(8) and (9) pin down consumption and hours worked as functions of K_{t+1},K_t,θ_t:

$$C_t=\gamma(K_{t+1},K_t,\theta_t) \text{ and } L_t=\eta(K_{t+1},K_t,\theta_t). \tag{11}$$

Substituting these expressions into the Euler equation (10) gives:

$$E_tH(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)=1, \quad \text{where} \tag{12}$$

$$H(K_{t+2},K_{t+1},K_t,\theta_{t+1},\theta_t)\equiv\beta\{\gamma(K_{t+1},K_t,\theta_t)/\gamma(K_{t+2},K_{t+1},\theta_{t+1})\}(\alpha\theta_{t+1}(K_{t+1})^{\alpha-1}(\eta(K_{t+2},K_{t+1},\theta_{t+1}))^{1-\alpha}+1-\delta).$$

The model thus boils down to an expectational difference equation in capital. The conventional no-bubble model solution, that obtains when the TVC for capital (5) is imposed, is

[10] The upper bound on labor hours implies that capital and output are bounded. Some widely used preference specifications (e.g., $u(C_t,L_t)=\ln(C_t)-\Psi\cdot(L_t)^{\mu},\ L_t\geq 0,\ \mu>1$) do not impose an upper bound on labor. Then speculative bubbles may induce unbounded growth of hours, capital and output.

[11] The discrete distribution of the TFP innovation simplifies the computation of conditional expectations in the numerical model solution.

described by a unique decision rule $K_{t+1}=\lambda(K_t,\theta_t)$ (e.g., Schmitt-Grohé and Uribe (2004)). A speculative bubble is a process $\{K_t\}$ that satisfies the Euler equation (12) but that violates the TVC and thus deviates from the no-bubble decision rule.

Recurrent speculative bubbles

By analogy to the bubble process in the Long-Plosser economy without TVC (Sect. 3), I consider bubbles in which, given date *t* information, the capital stock K_{t+1} takes one of two values: $K_{t+1}\in\{K_{t+1}^L,K_{t+1}^H\}$ with exogenous probabilities π and $1-\pi$, respectively ($0<\pi<1$), where $K_{t+1}^L\equiv\lambda(K_t,\theta_t)e^{\Delta}$, for a small constant $\Delta>0$. With probability π, the capital stock thus takes a value close to the no-bubble decision rule (as in the bubbly Long-Plosser economy). At date *t*, an exogenous i.i.d. sunspot (independent of TFP or any past endogenous variables) determines whether K_{t+1}^L or K_{t+1}^H is realized. At *t*, agents anticipate that K_{t+2} too takes one of two values $K_{t+2}\in\{K_{t+2}^L,K_{t+2}^H\}$ with probabilities π and 1-π, respectively, with $K_{t+2}^L=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$. The date *t* Euler equation (12) can thus be written as:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)+(1-\pi)\cdot E_t H(K_{t+2}^H,K_{t+1},K_t,\theta_{t+1},\theta_t)=1 \text{ for } K_{t+1}\in\{K_{t+1}^L,K_{t+1}^H\}. \quad (13)$$

The numerical simulations below consider equilibria in which, conditional on date *t* information, a TFP innovation at t+1 has an equiproportional effect on K_{t+2}^L and on K_{t+2}^H. Specifically, I construct equilibria in which $K_{t+2}^H=s_t^H\cdot K_{t+2}^L$ holds, where $s_t^H>0$ is known at *t*. Solving (13) for s_t^H (at each date) then pins down the equilibrium capital process. Computational details can be found in the Appendix (for online publication).

I set $\Delta>0$, as this is needed to generate *recurrent* bubbles (With $\Delta\leq 0$, bubbles are self-ending or ultimately hit the zero capital corner, as in the Long-Plosser economy without TVC.)

As in the bubbly Long-Plosser economy, the dynamics of capital reflects self-fulfilling variations in agents' expectations about *future* capital. Due to decreasing returns to capital and bounded TFP, the paths of capital and output are bounded. An *uninterrupted* sequence of investment booms (an infinite string of K^H realizations) would drive the capital stock towards its upper bound. However, an uninterrupted boom has zero probability. At any time, the capital stock can revert towards the no-bubble decision rule, with probability π. For values of Δ close to zero, and a sufficiently high bust probability π (as assumed in the simulations below), capital and output remain close to the range of the no-bubble equilibrium, most of the time, and the mean of capital and output is close to the no-bubble steady state.

4.1. Quantitative results

I again set $\alpha=1/3, \beta=0.99$. The capital depreciation rate is set at $\delta=0.025$. The preference parameter Ψ (utility weight on leisure) is set so that the Frisch labor supply elasticity is unity, at the no-bubble steady state.[12] Parameters in this range are conventional in quarterly macro models (e.g., King and Rebelo (1999)). I set the autocorrelation of TFP at $\rho=0.979$, while the standard deviation of TFP innovations is set at $\sigma_\theta=0.72\%$, as suggested by King and Rebelo (1999). The numerical simulations assume $\Delta=10^{-6}$ for all variants of the bubble model. That value generates standard deviations of real activity that are roughly in line with empirical statistics. I report results for two values of the bust probability: $\pi=0.2$ and $\pi=0.5$.

Table 2 reports simulated business cycle statistics (of HP filtered logged variables) for several model variants (see Cols. (1)-(10)), as well as historical US business cycle statistics (Col. (11)). Standard deviations (in %) of output (Y), consumption (C), investment (I) and hours worked (L) are shown, as well as correlations of these variables with output, autocorrelations and mean values. Also reported are means of Y,C,I,L and the mean of the (capital income-investment)/GDP ratio, as well as the fraction of periods in which this ratio is positive.

Cols. (1)-(4) of Table 2 pertain to bubble model variants with just bubble (sunspot) shocks (constant TFP assumed). Cols. (5)-(8) consider bubble model variants with joint bubble and TFP shocks. Cols. (9),(10) assume a no-bubble model (TVC imposed) with TFP shocks.[13] Cols. (1),(3),(5),(7) assume a bust probability $\pi=0.5$, while Cols. (2),(4),(6),(8) assume $\pi=0.2$. Cols. labelled 'Unit Risk Aversion' ('Unit RA') assume log utility, $u(C_t,L_t)=\ln(C_t)+\Psi\cdot\ln(1-L_t)$. Columns labelled 'High RA' assume $u(C_t,L_t)=\ln(C_t-\overline{C})+\Psi\cdot\ln(1-L_t)$, where $\overline{C}$ is a constant that is set at 0.8 times no-bubble steady state consumption. In the 'High RA' case, consumption thus

[12] (9) implies that the Frisch labor supply elasticity (LSE) with respect to the real wage (marginal product of labor) is $LSE=(1-L)/L$ at the steady state, where L are steady state hours worked. Ψ is set such that $L=0.5$, as then $LSE=1$.

[13] The no-bubble model is solved using a second-order Taylor approximation, as it is well-know that this approximation is very accurate for standard (no-bubble) RBC models (e.g., Kollmann et al. (2011a,b)).

has a strictly positive lower bound, $C_t \geq \overline{C} > 0$; the coefficient of relative risk aversion is 5, at steady state consumption.

Col. (1) of Table 2 assumes a variant of the bubble model with unit risk aversion and a bust probability π=0.5; fluctuations are just driven by bubble shocks. The predicted standard deviations of output, consumption, investment and hours worked are 0.49%, 1.08%, 4.29% and 0.74%, respectively. In line with the historical data, investment is predicted to be more volatile than output. However, the model (with unit risk aversion) predicts that consumption is more volatile than output, which is counterfactual. In the model, consumption is negatively correlated with output (a positive bubble shock raises investment; this crowds out consumption, which raises labor supply and thereby boosts output).[14] However, the model predicts that investment and hours worked are strongly procyclical, as is consistent with the data. In the model, output, consumption, investment and hours worked are positively serially correlated, but predicted autocorrelations (about 0.35) are smaller than the empirical autocorrelations (about 0.9).

A lower bust probability π=0.2 generates more persistent booms in real activity. As shown in Col. 2, for the bubble model variant with unit risk aversion, the autocorrelation of real activity is about 0.6 for π=0.2; however, consumption remains more volatile than output. Model variants with 'High Risk Aversion (RA)' generate less consumption volatility—those variants capture the fact that consumption is less volatile than output; see Cols. (3) and (4) of Table 2, where π=0.5 and π=0.2 are assumed.

In summary, the bubble model with constant TFP can generate persistent fluctuations, as well as a realistic volatility of output and aggregate demand components.

The no-bubble model driven by stochastic TFP shocks underpredicts the volatility of real activity, but it captures the fact that consumption is less volatile than output, while investment is more volatile (see Table 2, Cols. (9) and (10)). In the no-bubble model, consumption and investment are pro-cyclical; furthermore, real activity is highly serially correlated

The bubble economy with joint bubble shocks and TFP shocks generates fluctuations in real activity that are more volatile than the fluctuations exhibited by the no-bubble economy (see Table 2, Cols. (5)-(8)). The bubble equilibrium with TFP shocks is thus closer to the historical amplitude of business cycles.

[14] This is a familiar feature of flex-wage models driven by investment shocks; e.g., Coeurdacier et al. (2011).

Figure 2 shows simulated paths of output (*Y*, continuous black line), consumption (*C*, red dashed line), investment (*I*, dark blue dash-dotted line) and hours worked (*L*, light blue dotted line), for the model version with 'High Risk Aversion' and a bust probability π=0.2. Panels (1) and (2) of Fig.2 show results for the bubble economy with just bubble shocks, and for the bubble economy with joint bubble and TFP shocks, respectively. Panel (3) pertains to a no-bubble economy with TFP shocks.[15]

We see that bubble shocks induce relatively widely spaced output and investment booms (see Panel (1)). In most periods, output, consumption, investment and output remain close to the no-bubble steady state. Panels (2) and (3) of Fig. 2 show that the effect of bubbles on simulated series is clearly noticeable: the bubble economy with joint bubble and TFP shocks exhibits more rapid, short-lived, increases in investment, labor hours and output, that are followed by sharper contractions, than the no-bubble economy with TFP shocks.

In the bubble economies considered in Table 2, the mean of output, consumption and investment is again close to the no-bubble steady state (as in the bubbly Long-Plosser economy studied in Sect. 3).[16] For all bubble model variants in Table 2, the average (capital income – investment)/GDP ratio is positive and large (unlike in the Long-Plosser model); the average ratio ranges between 8.5% and 9.2%, and it is only slightly smaller than the value of that ratio in the no-bubble steady state, 9.59%.[17] Capital income exceeds investment in close to 100% of all periods. This highlights the difficulty of detecting violations of the TVC (dynamic inefficiency), as discussed above.

5. Speculative bubbles in a Dellas two-country RBC economy (no TVC)

I next study bubbles in open economies. This Section considers Dellas' (1986) two-country RBC model. The Dellas model is a two-country version of the Long and Plosser (1983) model, as it also assumes log utility, Cobb-Douglas production functions and full capital depreciation. Like the Long-Plosser model, the Dellas model has an exact closed form solution. I construct

[15] The *Y*, *C* and *I* series plotted in Fig. 2 are normalized by no-bubble steady state output; hours worked (*L*) are normalized by steady state hours. The same sequence of sunspots are used in Panels (1) and (2); the same TFP shocks are used in Panels (2) and (3). Simulated paths for the other model variants considered in Table 2 are shown in Kollmann (2020).

[16] In Table 2, mean values of *Y*,*C*,*I*,*L* are reported as % deviations from the no-bubble steady state. The mean (capital income – investment)/GDP ratio is *not* expressed as a % deviation from steady state.

[17] The no-bubble steady state (capital income – investment)/GDP ratio is $\alpha r/(\delta+r)$ with $r \equiv (1-\beta)/\beta$.

speculative bubbles that arise when there is no transversality condition (TVC), in the Dellas economy.

Assume a world with two symmetric countries, referred to as Home (H) and Foreign (F), respectively. The household of country $i=H,F$ has log preferences of the type assumed in the closed economy RBC model of Sect. 4. Thus, her period utility is: $U(C_{i,t},L_t)=\ln(C_{i,t})+\Psi\cdot\ln(1-L_{i,t})$, $\Psi>0$, where $C_{i,t}$ and $L_{i,t}$ are consumption and hours worked. Each country is specialized in the production of a distinct tradable intermediate good. Country i's intermediate good production function is $Y_{i,t}=\theta_{i,t}(K_{i,t})^{\alpha}(L_{i,t})^{1-\alpha}$, where $Y_{i,t}$, $\theta_{i,t}, K_{i,t}$ are the intermediate good output, TFP and capital in country *i*. Capital and labor are immobile internationally. TFP is exogenous and follows a bounded Markov process. The country *i* household combines local and imported intermediates into a non-tradable final good, using the Cobb-Douglas aggregator $Z_{i,t}=(y_{i,t}^{i}/\xi)^{\xi}\cdot(y_{i,t}^{j}/(1-\xi))^{1-\xi}$, $i\neq j$, where $y_{i,t}^{j}$ is the amount of intermediate j used by country *i*. There is local bias in final good production: $\frac{1}{2}<\xi<1$. The country *i* final good is used for consumption, $C_{i,t}$, and investment, $I_{i,t}$: $Z_{i,t}=C_{i,t}+I_{i,t}$. Due to full capital depreciation, the capital stock at t+1 equals investment at t: $K_{i,t+1}=I_{i,t}$. The price of country i's final good $(P_{i,t})$ equals its marginal cost: $P_{i,t}=(p_{i,t})^{\xi}\cdot(p_{j,t})^{1-\xi}$, $i\neq j$, where $p_{j,t}$ is the price of intermediate good j. Country i's demand functions for domestic and imported intermediates are: $y_{i,t}^{i}=\xi\cdot(p_{i,t}/P_{i,t})^{-1}Z_{i,t}$ and $y_{i,t}^{j}=(1-\xi)\cdot(p_{j,t}/P_{i,t})^{-1}Z_{i,t}$, for $j\neq i$. Market clearing for intermediate goods requires

$$y_{H,t}^{i}+y_{F,t}^{i}=Y_{i,t}, \text{ for } i=H,F. \tag{14}$$

Country i's terms of trade and real exchange rate are $q_{i,t}\equiv p_{i,t}/p_{j,t}$ and $RER_{i,t}\equiv P_{i,t}/P_{j,t}$, with $i\neq j$.

The model assumes complete international financial markets, so that consumption risk is efficiently shared across countries. In equilibrium, the ratio of Home to Foreign households' marginal utilities of consumption is, thus, proportional to the Home real exchange rate (Kollmann, 1991, 1995; Backus and Smith, 1993). With log utility, this implies that Home consumption spending is proportional to Foreign consumption spending: $P_{H,t}C_{H,t}=\Lambda\cdot P_{F,t}C_{F,t}$, where Λ is a date- and state-invariant term that reflects the (relative) initial wealth of the two countries. I assume that the two countries have the same initial wealth, i.e. $\Lambda=1$. Thus:

$$P_{H,t}C_{H,t} = P_{F,t}C_{F,t}. \quad (15)$$

Each household equates the marginal rate of substitution between leisure and consumption to the marginal product of labor, expressed in units of consumption, which implies

$$C_{i,t}\Psi/(1-L_{i,t}) = (p_{i,t}/P_{i,t})(1-\alpha)(Y_{i,t}/L_{i,t}). \quad (16)$$

Country i's Euler equation for domestic physical capital is:

$$E_t\beta(C_{i,t}/C_{i,t+1})[(p_{i,t+1}/P_{i,t+1})\alpha Y_{i,t+1}/K_{i,t+1}] = 1, \quad (17)$$

where the term in square brackets is country i's marginal product of capital at date t+1, expressed in units of the country i final good. Substitution of the intermediate good demand functions into the market clearing condition for intermediates (14) gives:

$$p_{i,t}Y_{i,t} = \xi\cdot(P_{i,t}C_{i,t}+P_{i,t}K_{i,t+1}) + (1-\xi)\cdot(P_{j,t}C_{j,t}+P_{j,t}K_{j,t+1}) \text{ for } i,j=H,F;\ j\neq i. \quad (18)$$

Let $\kappa_{i,t}\equiv P_{i,t}K_{i,t+1}/(P_{i,t}C_{i,t})$ denote country i's investment/consumption ratio. Using (15),(18), the labor supply and Euler equations (16),(17) can be written as

$$L_{i,t}/(1-L_{i,t}) = ((1-\alpha)/\Psi)\cdot\{1+\xi\kappa_{i,t}+(1-\xi)\kappa_{j,t}\} \text{ for } i=H,F;\ j\neq i, \quad (19)$$

$$\alpha\beta\cdot E_t(1+\xi\kappa_{H,t+1}+(1-\xi)\kappa_{F,t+1}) = \kappa_{H,t} \text{ and } \alpha\beta\cdot E_t(1+(1-\xi)\kappa_{H,t+1}+\xi\kappa_{F,t+1}) = \kappa_{F,t}. \quad (20)$$

The deterministic steady state investment/consumption ratio is $\kappa\equiv\alpha\beta/(1-\alpha\beta)$. Let $\widetilde{\kappa_{i,t}}\equiv\kappa_{i,t}-\kappa$ denote the deviation of $\kappa_{i,t}$ from its steady state value. The Euler equations (20) imply:

$$\alpha\beta\cdot E_t(\xi\widetilde{\kappa_{H,t+1}}+(1-\xi)\widetilde{\kappa_{F,t+1}}) = \widetilde{\kappa_{H,t}} \text{ and } \alpha\beta\cdot E_t((1-\xi)\widetilde{\kappa_{H,t+1}}+\xi\widetilde{\kappa_{F,t+1}}) = \widetilde{\kappa_{F,t}}. \quad (21)$$

This gives $$\begin{bmatrix} E_t\widetilde{\kappa_{H,t+1}} \\ E_t\widetilde{\kappa_{F,t+1}} \end{bmatrix} = B\cdot\begin{bmatrix} \widetilde{\kappa_{H,t}} \\ \widetilde{\kappa_{F,t}} \end{bmatrix}, \text{ with } B\equiv\frac{1}{\alpha\beta(2\xi-1)}\cdot\begin{bmatrix} \xi & -(1-\xi) \\ -(1-\xi) & \xi \end{bmatrix}. \quad (22)$$

The eigenvalues of B are $\lambda_S\equiv1/(\alpha\beta)$ and $\lambda_D\equiv1/(\alpha\beta(2\xi-1))$, with $\lambda_D>\lambda_S>1$. ($1/(2\xi-1)>1$ as $\frac{1}{2}<\xi<1$.) As both eigenvalues exceed 1, the only non-explosive solution of (22) is $\widetilde{\kappa_{i,t}}=0$ i.e. $\kappa_{i,t}=\alpha\beta/(1-\alpha\beta)$, $\forall t$, $i=H,F$. This solution satisfies Home and Foreign TVCs. Dellas (1986) focuses on the no-bubble solution.

5.1. Speculative bubbles

I now study speculative bubble equilibria with $\widetilde{\kappa_{i,t}}\neq 0$ that arise when there is no TVC. I show that the Dellas economy without TVC has bubble equilibria that feature recurrent, bounded fluctuations of capital, hours worked, output and consumption. These equilibria do not converge to zero capital or zero consumption. If the Home or Foreign capital stock ever fell to zero, then capital and output in both countries would remain stuck at zero in all subsequent periods. Such trajectories seem empirically irrelevant. The goal of the analysis here is to construct bubble equilibria with recurrent fluctuations in real activity, and thus I focus on bubbles with strictly positive capital.

As shown below, any strictly positive process for Home and Foreign capital that satisfies the Euler equations (20),(21) has to be such that

$$\widetilde{\kappa_{H,t}}=\widetilde{\kappa_{F,t}}\geq 0 \;\; \forall t. \qquad (23)$$

Thus, the bubbly investment/consumption ratio has to be always at least as large as the steady state ratio. Also, the bubble process has to be **identical** across the two countries. To see this, let $S_t\equiv\widetilde{\kappa_{H,t}}+\widetilde{\kappa_{F,t}}$ and $D_t\equiv\widetilde{\kappa_{H,t}}-\widetilde{\kappa_{F,t}}$ be the sum and the difference of the two countries' investment/ consumption ratios, expressed as deviations from steady state. (22) implies $E_tS_{t+1}=\lambda_S\cdot S_t$ and $E_tD_{t+1}=\lambda_D\cdot D_t$, where λ_S and λ_D are the eigenvalues of *B*. Note that $\widetilde{\kappa_{H,t}}=\frac{1}{2}\cdot(D_t+S_t)$ and $\widetilde{\kappa_{F,t}}=\frac{1}{2}\cdot(S_t-D_t)$. Thus, $E_t\widetilde{\kappa_{H,t+s}}=\frac{1}{2}\cdot(\lambda_S)^s\{S_t+(1/(2\xi-1))^sD_t\}$ and $E_t\widetilde{\kappa_{F,t+s}}=\frac{1}{2}(\lambda_S)^s\{S_t-(1/(2\xi-1))^sD_t\}$, where I use the fact that $\lambda_D=\lambda_S/(2\xi-1)$. As $\lambda_S>1$ and $1/(2\xi-1)>1$, a necessary condition for non-negativity of $\kappa_{H,\tau},\kappa_{F,\tau}$ in all future dates and states $\tau\geq t$ is $D_t=0$ and $S_t\geq 0$. This implies (23).[18]

Intuitively, a (positive) bubble that e.g. occurs solely in the Home country $(\widetilde{\kappa_{H,t}}>0)$ would trigger an improvement in the Home terms of trade, and a rise in the Home trade deficit, due to growing intermediate imports by Home, fueled by the bubble-induced boom in Home investment. This would put Foreign investment on a downward trajectory. If the Home bubble lasted sufficiently long, the Foreign capital stock would ultimately reach zero. Thus, a recurrent

[18] $D_t\neq 0$ would imply $\lim_{s\to\infty}E_t\widetilde{\kappa_{H,t+s}}=-\infty$ or $\lim_{s\to\infty}E_t\widetilde{\kappa_{F,t+s}}=-\infty$; with strictly positive probability, $\kappa_{H,\tau}$ or $\kappa_{F,\tau}$ would thus be **negative** at some date(s) $\tau\geq t$. Setting $D_t=0$ shows that $S_t<0$ would imply $\lim_{s\to\infty}E_t\widetilde{\kappa_{H,t+s}}=-\infty$ and $\lim_{s\to\infty}E_t\widetilde{\kappa_{F,t+s}}=-\infty$, so that $\kappa_{H,\tau}<0$ and/or $\kappa_{F,\tau}<0$ would hold with positive probability at some date(s) $\tau\geq t$.

bubble (with strictly positive capital) cannot occur just in one country.[19] Why do bubbles have to be *identical* in the two countries? The reason is that any difference between domestic and foreign investment/consumption ratios at date t ($D_t \neq 0$) would trigger a larger expected difference in period t+1; thus, the expected cross-country difference would explode, and that at a faster rate than the sum of these two-country's investment/consumption ratios (as $\lambda_D > \lambda_S$). This would drive capital to zero, in one of the countries, in future periods $\tau > t$.

In what follows, I thus assume that (23) holds. Let $\kappa_t = \kappa_{H,t} = \kappa_{F,t}$ denote the common investment/consumption ratio in both countries, and let $\widetilde{\kappa_t} \equiv \kappa_t - \kappa$ be its deviation from the steady state ratio κ. The Home and Foreign Euler equations (21) imply

$$\alpha\beta E_t \widetilde{\kappa_{t+1}} = \widetilde{\kappa_t}. \tag{24}$$

Recurrent speculative bubbles

By analogy to the bubble equilibria discussed in previous Sections, I assume that $\widetilde{\kappa_{t+1}}$ takes two values: $\widetilde{\kappa_{t+1}} \in \{\Delta, \widetilde{\kappa_{t+1}^H}\}$ with exogenous probabilities π and $1-\pi$, respectively, with $0<\pi<1$ and $\Delta>0$. $\Delta>0$ ensures that the bubble is recurrent (not self-ending) and that it does not lead to zero capital. (As in the bubbly Long-Plosser model, $\Delta=0$ would imply that bubbles are self-ending; with $\Delta<0$, the capital stock would ultimately fall to zero.)

Consider a world economy that starts in period $t=0$, with exogenous initial capital stocks $K_{H,0}, K_{F,0}$. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities π and $1-\pi$, respectively ($0<\pi<1$). Then the following process for the investment/ consumption ratio $\{\widetilde{\kappa_t}\}_{t\geq 0}$ is a recurrent speculative bubble: $\widetilde{\kappa_{t+1}} = \Delta$ if $u_{t+1}=0$ and $\widetilde{\kappa_{t+1}} = \widetilde{\kappa_{t+1}^H}$ if $u_{t+1}=1$, for $t\geq 0$, where $\widetilde{\kappa_{t+1}^H}$ solves the date t Euler equation (24). Note that (24) implies $\alpha\beta\{\pi\Delta + (1-\pi)\widetilde{\kappa_{t+1}^H}\} = \widetilde{\kappa_t}$, and so $\widetilde{\kappa_{t+1}^H} = (\widetilde{\kappa_t} - \alpha\beta\pi\Delta)/(\alpha\beta(1-\pi))$. If $\widetilde{\kappa_t} \geq \Delta$ holds, then $\widetilde{\kappa_{t+1}^H} > \widetilde{\kappa_t}$.[20]

---

[19]Note from the Foreign Euler condition shown in (21) (see second equation) that if $\widetilde{\kappa_{H,t}} > 0$ and $E_t \widetilde{\kappa_{H,t+1}} > 0$ hold, then $\widetilde{\kappa_{F,t}} = E_t \widetilde{\kappa_{F,t+1}} = 0$ is impossible. Thus a bubble cannot occur just in country H.

[20] The κ_0 ratio (initial period) is indeterminate. $\widetilde{\kappa_0} \geq \Delta$ has to hold to ensure that $\widetilde{\kappa_t} \geq \Delta$ $\forall t>0$.

Given $\{\kappa_t\}$, one can solve for hours, consumption, investment and output, using the static equilibrium conditions. $\kappa_t = \kappa_{H,t} = \kappa_{F,t}$ implies that labor hours are identical across countries (see (19)), and that investment and output, valued at market prices, are equated across countries: $P_{H,t}K_{H,t+1} = P_{F,t}K_{F,t+1}$, $p_{H,t}Y_{H,t} = p_{F,t}Y_{F,t}$. As consumption, valued at market prices, is likewise equated across countries (see (15)), net exports are zero. Country i's terms of trade equal the inverse of i's relative output: $q_{i,t} \equiv p_{i,t}/p_{j,t} = Y_{j,t}/Y_{i,t}$, $j \neq i$. Consumption and investment obey $C_{i,t} = (1/(1+\kappa_t))(p_{i,t}/P_{i,t})Y_{i,t}$ and $K_{i,t+1} = \kappa_t C_{i,t}$. As $p_{i,t}/P_{i,t} = (q_{i,t})^{1-\xi} = (Y_{j,t}/Y_{i,t})^{1-\xi}$ with $j \neq i$, we find: $K_{H,t+1} = (\kappa_t/(1+\kappa_t))(Y_{H,t})^{\xi}(Y_{F,t})^{1-\xi}$, $K_{F,t+1} = (\kappa_t/(1+\kappa_t))(Y_{H,t})^{1-\xi}(Y_{F,t})^{\xi}$. Note that the $\{\kappa_t\}$ process is unbounded. However $1/(1+\kappa_t)$ and $\kappa_t/(1+\kappa_t)$ are strictly positive and bounded; it can be seen (from preceding formulae) that this implies that capital, output and consumption are bounded.

5.2. Quantitative results

Table 3 reports simulated business statistics for the two-country Dellas model with bubbles (Cols. (1)-(3)); also shown are historical business statistics (Col. (4)). Historical standard deviations, correlations with GDP and autocorrelations are based on US data, 1968q1-2017q4; historical cross-country correlations are correlations between the US and the Euro Area, 1970q1-2017q4 Empirically, the US real exchange rate is about 2.5 times as volatile as US output; US net exports (normalized by GDP) are countercyclical. Real activity is positively correlated across the US and the Euro Area. The cross-country correlations of output and investment are close to 0.5; the cross-country correlations of consumption and employment are slightly lower (0.39).

I again set $\alpha = 1/3$, $\beta = 0.99$. The share of spending devoted to domestic intermediates is set at $\xi = 0.9$.[21] I set the bust probability at $\pi = 0.5$. Δ is set at 2.227×10^{-6}, as this parallels the calibration of the investment bust in the bubbly Long-Plosser closed economy model (Sect. 3), and generates a realistic volatility of output.[22]

[21] This is consistent with the fact that the mean US trade share (0.5×(imports+exports)/GDP) was 10% in 1968-2017.

[22] $\Delta = 2.227 \times 10^{-6}$ implies that, in a bust, the ratio of investment spending divided by nominal GDP, $Z_{i,t} \equiv P_{i,t}K_{i,t+1}/(p_{i,t}Y_{i,t}) = \kappa_t/(1+\kappa_t)$ exceeds its steady state value $\alpha\beta$ by the amount 10^{-6}, as in the closed economy.

Versions of the two-country model with TFP shocks assume that Home and Foreign TFP follow the autoregressive process that Backus et al. (1994) estimated using quarterly TFP series for the US and an aggregate of European economies:

$$\begin{bmatrix} \ln\theta_{H,t+1} \\ \ln\theta_{F,t+1} \end{bmatrix} = \begin{bmatrix} .906 & .088 \\ .088 & .906 \end{bmatrix} \cdot \begin{bmatrix} \ln\theta_{H,t} \\ \ln\theta_{F,t} \end{bmatrix} + \begin{bmatrix} \varepsilon^{\theta}_{H,t+1} \\ \varepsilon^{\theta}_{F,t+1} \end{bmatrix}, \tag{25}$$

where $\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1}$ are white noises with $Std(\varepsilon^{\theta}_{H,t+1}) = Std(\varepsilon^{\theta}_{F,t+1}) = 0.852\%$, $Corr(\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1}) = 0.258$. I assume a discrete distribution of the TFP innovations, to ensure that the TFP process is bounded.[23] (25) implies that TFP is a highly persistent process, and that there are delayed positive cross-country spillovers (positive off-diagonal elements of the autoregressive matrix).

Col. 1 of Table 3 considers a version of the bubble model with just bubble shocks (constant TFP assumed). Col. 2 assumes a bubble model with joint bubble and TFP shocks, while Col. 3 assumes a no-bubble model (TVC imposed) with TFP shocks.

The bubble model with constant TFP predicts that output, consumption, investment and hours are identical across countries, i.e. these variables are perfectly correlated across countries (see Col. 1). The dynamics of these variables corresponds, thus, to that predicted by the corresponding bubbly Long-Plosser closed economy (see Sect. 3). E.g., like its closed-economy counterpart, the Dellas economy with bubbles predicts that consumption is more volatile than output.[24] Because of the predicted perfect correlation of Home and Foreign output, the terms of trade and the real exchange rate are constant, when there are just bubble shocks.

The no-bubble Dellas model with TFP shocks generates realistic output and consumption variability (see Col. 3, Table 3); however, investment, hours worked and the real exchange rate are less volatile than in the data (hours are constant). The no-bubble model with TFP shocks generates fluctuations in output, consumption and investment that are positively correlated across countries. The predicted cross-country correlation of output (0.39) is smaller than the empirical correlation (0.53), while predicted cross-country correlations of consumption and investment (0.56) are higher than the corresponding empirical correlations (about 0.4).

---

[23] $\varepsilon^{\theta}_{H,t+1} = a \cdot \nu_{H,t+1} + b \cdot \nu_{F,t+1}$, $\varepsilon^{\theta}_{F,t+1} = b \cdot \nu_{H,t+1} + a \cdot \nu_{F,t+1}$ where $\nu_{H,t+1}$, $\nu_{F,t+1}$ are independent random variables that equal 1 or -1 with probability 0.5. I set a=0.8447%, b=0.1108% to match the stated standard dev. and corr. of $\varepsilon^{\theta}_{H,t+1}, \varepsilon^{\theta}_{F,t+1}$.

[24] The Dellas economy assumes endogenous labor. Hours worked rise in response to a positive bubble shock. This is why real activity is more volatile than in the closed economy (Long-Plosser) model with fixed labor of Sect. 2.

Note that all model variants predict a zero trade balance. The bubble economy with joint bubble shocks and TFP shocks (Col. 2) generates higher cross-country correlations of output, consumption and investment than the no-bubble economy (Col. 3). Also, the presence of TFP shocks implies that the real exchange rate shows non-negligible fluctuations (while the real exchange rate is constant in the bubble model with constant TFP, as discussed above).

6. Speculative bubbles in a two-country RBC model with incomplete capital depreciation (no TVC)

This Section discusses speculative bubbles in a more general two-country RBC model that resembles the classic International RBC model proposed by Backus et al. (1994). This model cannot be solved in closed form. It assumes incomplete capital depreciation, a CES final good aggregator, and it allows for non-unitary risk aversion. Other model features are identical to those of the Dellas model. Thus, each country is specialized in the production of a distinct tradable good. In each country, domestic and imported tradables are combined into a non-tradable final good used for consumption and investment. Complete global financial markets are assumed. The law of motion of Home and Foreign TFP is again given by (25).

As in the closed economy RBC model of Sect. 4, I assume the period utility function $U(C_{i,t},L_t)=\ln(C_{i,t}-\overline{C})+\Psi\cdot\ln(1-L_{i,t})$, $\overline{C}\geq 0$. The country i final good is generated from domestic and imported intermediates using a CES aggregator: $Z_{i,t}=[\xi^{1/\phi}\cdot(y^i_{i,t})^{(\phi-1)/\phi}+(1-\xi)^{1/\phi}\cdot(y^j_{i,t})^{(\phi-1)/\phi}]^{\phi/(\phi-1)}$, $j\neq i$, where ϕ is the substitution elasticity between domestic and imported intermediates. There is local bias in final good production: $\frac{1}{2}<\xi<1.$ The price of country i's final good $(P_{i,t})$ now is $P_{i,t}=[\xi\cdot(p_{i,t})^{1-\phi}+(1-\xi)\cdot(p_{j,t})^{1-\phi}]^{1/(1-\phi)}$, $j\neq i$, while country i's demand functions for domestic and imported inputs are $y^i_{i,t}=\xi\cdot(p_{i,t}/P_{i,t})^{-\phi}Z_{i,t}$ and $y^j_{i,t}=(1-\xi)\cdot(p_{j,t}/P_{i,t})^{-\phi}Z_{i,t}$. The law of motion of country i's capital stock is $K_{i,t+1}=(1-\delta)K_{i,t}+I_{i,t}$, where $0<\delta<1$ is the capital depreciation rate. The final good market clearing condition in country i is $Z_{i,t}=C_{i,t}+I_{i,t}$.

The *static* equilibrium conditions allow to solve for date t consumption, labor and terms of trade $C_{i,t},L_{i,t},q_{i,t}$ as functions of both countries' capital stocks in t and t+1 and of date t productivity. By substituting these functions into the two countries' capital Euler equations, one can write the Euler equations as expectational difference equations in Home and Foreign capital:

$$E_t H_i(\overrightarrow{K_{t+2}},\overrightarrow{K_{t+1}},\overrightarrow{K_t},\overrightarrow{\theta_{t+1}},\overrightarrow{\theta_t})=1 \quad \text{for } i{=}H{,}F, \tag{26}$$

where $\overrightarrow{K_t}\equiv(K_{H,t},K_{F,t})$ and $\overrightarrow{\theta_t}\equiv(\theta_{H,t},\theta_{F,t})$ are vectors of Home and Foreign capital and TFP, respectively. The function H_i maps R_+^{10} into *R*.

The no-bubble solution of the model (that obtains when TVCs are imposed) is described by decision rules $K_{i,t+1}{=}\lambda_i(\overrightarrow{K_t},\overrightarrow{\theta_t})$ that map date *t* capital and TFP into capital at date *t*+1.

Assume that there is no transversality condition (TVC) for capital, which makes speculative bubbles possible. I consider a bubble process that parallels the bubbles in previous Sections. Assume that capital $K_{i,t+1}$ takes one of two values: $K_{i,t+1}\in\{K^L_{i,t+1},K^H_{i,t+1}\}$, with probabilities *π* and 1-*π*, respectively, where $K^L_{i,t+1}{=}\lambda_i(\overrightarrow{K_t},\overrightarrow{\theta_t})\cdot e^{\Delta}$. Like in previous models, Δ>0 is required to generate *recurrent* bubbles. As in the Dellas economy with complete financial markets, the bubble has to be perfectly synchronized across countries. Hence, $K^H_{H,t+1}$ and $K^H_{F,t+1}$ are realized together (and so are $K^L_{H,t+1}$ and $K^L_{F,t+1}$). (The superscripts 'H' (boom) and 'L' (bust) refer to the state of the bubble, while the subscripts 'H' (Home) and 'F' (Foreign) refer to the country.)

Consider a world economy that starts at date *t*=0, with exogenous initial capital stocks $K_{H,0},K_{F,0}$. Let u_t be an exogenous i.i.d. sunspot that takes values 0 and 1 with probabilities *π* and 1-*π*, respectively (0<*π*<1). Then the following process for Home and Foreign capital stocks $\{K_{H,t},K_{F,t}\}_{t\geq 0}$ is a recurrent speculative bubble:

(a) $K_{i,t+2}{=}K^L_{i,t+2}\equiv\lambda_i(\overrightarrow{K_{t+1}},\overrightarrow{\theta_{t+1}})\cdot e^{\Delta}$ for *i*=*H*,*F* if $u_{t+1}{=}0$, for *t*≥0;

(b) $K_{i,t+2}{=}K^H_{i,t+2}$ for *i*=*H*,*F*, if $u_{t+1}{=}1$, for *t*≥0, where $K^H_{H,t+2},K^H_{F,t+2}$ satisfy date *t* Euler equations (26).

6.1. Quantitative results

As in Sect. 4, I set $\alpha{=}1/3,\beta{=}0.99,\delta{=}0.025$. Ψ (utility weight on leisure) is again set so that the Frisch labor supply elasticity is unity, at the steady state. As in the calibration of the Dellas model, the local spending bias parameter is set at $\xi{=}0.9$. The substitution elasticity between domestic and imported intermediates is set at $\phi{=}1.5$; that value is consistent with estimated price elasticities of aggregate trade flows and it has been widely used in International RBC models (e.g., Backus et al. (1994)). The parameters of the bubble process are the same as in the closed economy model (with incomplete capital depreciation) studied in Sect. 4; thus, Δ is again set at $\Delta=10^{-6}$, and two values of the bust probability are considered: $\pi=0.2$ and $\pi=0.5$.

Predicted business cycle statistics generated by the two-country RBC model with incomplete capital depreciation are shown in Table 4. Cols. labelled 'Unit Risk Aversion' (or 'Unit RA') assume log utility (minimum consumption set at $\overline{C}=0$). In Cols. labelled 'High RA', $\overline{C}$ is set at 0.8 times steady state consumption (implied risk aversion, at steady state: 5).

Cols. (9) and (10) of Table 4 show simulated business cycle statistics for versions of the no-bubble model (TVC imposed) driven by TFP shocks. The simulations confirm findings that are well known from the International RBC literature (e.g., Backus et al. (1994), Kollmann (1996)): a complete markets no-bubble model driven by TFP shocks can capture the historical volatility of output and investment, but it underpredicts the empirical volatility of the real exchange rate. The no-bubble model here reproduces the fact that net exports are countercyclical. However, the model-predicted cross-country correlations of output and investment are markedly lower than the corresponding historical correlations. By contrast, the model predicts that consumption is highly correlated across countries. The low predicted cross-country correlation of output reflects the fact that, with complete financial markets, a positive shock to Home productivity raises Foreign consumption, which reduces Foreign labor supply, and thus lowers Foreign output, on impact (while Home output increases).[25]

Simulated business cycle statistics for the bubble economy with just bubble shocks (constant TFP) are reported in Cols. (1)-(4) of Table 4. Standard deviations, correlations with domestic GDP, autocorrelations and mean values are *identical* to the corresponding statistics for the closed economy bubble model (with incomplete capital depreciation) studied in Sect. 4 (see Cols. (1)-(4) of Table 2). This is due to the fact that, in the two-country model with complete markets, bubbles are perfectly correlated across countries; with just bubble shocks, real activity is thus perfectly correlated across countries, the terms of trade are constant and net exports are zero. The predicted volatility of output and consumption induced by bubble shocks (Cols. (1)-(4) of Table 4) is roughly comparable to volatility in the no-bubble model with TFP shocks (Cols. (9),(10)), but the volatility of hours worked is higher in the bubble economy.

[25]The no-bubble variant of the Dellas model driven by TFP shocks generates higher cross-country output correlations (see Col. (3) of Table 3) because, in that variant, hours worked are constant.

Predicted business cycle statistics for the bubble economy, with simultaneous bubble shocks and TFP shocks, are shown in Cols. (5)-(8) of Table 4. With joint bubble shocks and TFP shocks, the predicted volatility of real activity is higher, and thus generally closer to the data, than the volatility generated by the no-bubble model with TFP shocks. The model with joint bubble and TFP shocks is especially successful at matching the positive empirical cross-country correlations of output and investment, and the counter-cyclicality of the trade balance; however the predicted cross-country consumption correlation is too high, when compared to the data.

Fig. 3 shows simulated sample paths for the model version with 'High Risk Aversion' and a bust probability π=0.2. Panels (1) and (2) of the Figure show results for the bubble economy with just bubble shocks, and for the bubble economy with joint bubble and TFP shocks, respectively. Panel (3) of Fig. 3 pertains to a no-bubble economy with TFP shocks; in that variant, the negative cross-country correlation of high-frequency output and investment fluctuations is clearly discernible. Bubble shocks induce relatively widely spaced output and investment booms that are perfectly correlated across countries (see Panel (1)). In the bubble economy with joint bubble and TFP shocks, output and investment are markedly more synchronized across countries than in the no-bubble economy with TFP shocks (see Panel (2)).

7. Conclusion

This paper studies speculative bubbles that arise in Real Business Cycle models, when one assumes that there is no transversality condition (TVC) for capital. The absence of the TVC can be due to an overlapping generations population structure. Speculative bubbles reflect self-fulfilling fluctuations in agents' expectations about future investment, and can occur when there are no shocks to technologies and preferences. It is shown that speculative bubbles can generate bounded boom bust cycles of investment and output. Speculative bubbles are thus a novel potential driver of economic fluctuations. In a two-country model with integrated financial markets, speculative bubbles must be perfectly correlated across countries. Global bubbles may, thus, help to explain the international synchronization of international business cycles.

References

Abel, Andrew, Gregory Mankiw, Lawrence Summers and Richard Zeckhauser, 1989. Assessing Dynamic Efficiency: Theory and Evidence. Review of Economic Studies 56, 1-20.

Ascari, Guido, Paolo Bonomolo and Hedibert Lopes, 2019. Walk on the Wild Side: Temporarily Unstable Paths and Multiplicative Sunspots. American Economic Review 109, 1805–1842.

Backus, David, Patrick Kehoe, and Finn Kydland, 1994. Dynamics of the Trade Balance and the Terms of Trade: The J-Curve? American Economic Review 84, 84-103.

Benhabib, Jess and Roger Farmer, 1999. Indeterminacy and Sunspots in Macroeconomics. In: Handbook of Macroeconomics (J. Taylor and M. Woodford, eds.), Elsevier, Vol. 1A, 387-448.

Blanchard, Olivier, 1979. Speculative Bubbles, Crashes and Rational Expectations. Economics Letters 3, 387-398.

Blanchard, Olivier and Charles Kahn, 1980. The Solution of Linear Difference Models under Rational Expectations. Econometrica 48, 1305-1311.

Blanchard, Olivier and Mark Watson, 1982. Bubbles, Rational Expectations and Financial Markets. NBER Working Paper 945.

Blanchard, Olivier, 1985. Debt, Deficits, and Finite Horizons. Journal of Political Economy 93, 223-247.

Blanchard, Olivier and Stanley Fischer, 1989. Lectures on Macroeconomics. Cambridge, MA: MIT Press.

Coeurdacier, Nicolas, Robert Kollmann and Philippe Martin, 2010. International Portfolios, Capital Accumulation and Foreign Assets Dynamics. Journal of International Economics 80, 100–112.

Dellas, Harris, 1986. A Real Model of the World Business Cycle. Journal of International Money and Finance 5, 381-294.

Holden, Tom, 2016. Computation of Solutions to Dynamic Models with Occasionally Binding Constraints. Working Paper, University of Surrey.

Holden, Tom, 2021. Existence and Uniqueness of Solutions to Dynamic Models with Occasionally Binding Constraints. Forthcoming: Review of Economics and Statistics.

King, Robert and Sergio Rebelo, 1999. Resuscitating Real Business Cycles. In: Handbook of Macroeconomics (J. Taylor and M. Woodford, eds.), Elsevier, Vol. 1B, 927-1007.

Kollmann, Robert, Serguei Maliar, Benjamin Malin and Paul Pichler, 2011a. Comparison of Numerical Solutions to a Suite of Multi-Country Models. Journal of Economic Dynamics and Control 35, pp.186-202.

Kollmann, Robert, Jinill Kim and Sunghyun Kim, 2011b. Solving the Multi-Country Real Business Cycle Model Using a Perturbation Method. Journal of Economic Dynamics and Control 35, 203-206.

Kollmann, Robert, 2020. Rational Bubbles in Non-Linear Business Cycle Models: Closed and Open Economies. CEPR DP 14367.

Long, John and Charles Plosser, 1983. Real Business Cycles. Journal of Political Economy 91, 39-69.

Lubik, Thomas and Frank Schorfheide, 2004. Testing for Indeterminacy: An Application to U.S. Monetary Policy. American Economic Review 94, 190-217.

Mussa, Michael, 1990. Exchange Rates in Theory and Reality. Essays in International Finance No. 179, Princeton University.

Schmitt-Grohé, Stephanie, 1997. Comparing Four Models of Aggregate Fluctuations Due to Self-Fulfilling Expectations. Journal of Economic Theory 72, 96-47.

Schmitt-Grohé, Stephanie and Martin Uribe, 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. Journal of Economic Dynamics and Control 28, 755 – 775.

Stracca, Livio, 2004. Behavioral Finance and Asset Prices: Where Do We Stand? Journal of Economic Psychology 25, 373–405.

Woodford, Michael, 1986. Stationary Sunspot Equilibria: The Case of Small Fluctuations Around a Deterministic Steady State'. Working Paper, University of Chicago.

Woodford, Michael, 1988. Expectations, Finance and Aggregate Instability. In: Finance Constraints, Expectations, and Macroeconomics (Meir Kohn and Sho-Chieh Tsiang, eds.), Oxford University Press, 229-261.

Table 1. Long-Plosser economy with speculative bubbles: business cycle statistics

	Standard dev. (%)			Corr. with Y		Autocorrelations			Mean [% deviation from SS]		
	Y	*C*	*I*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
(a) Predicted business cycle statistics											
	1.47	3.39	4.42	0.31	0.45	0.44	-0.17	0.44	0.49	-0.32	2.15
(b) Historical business cycle statistics											
	1.47	1.19	4.96	0.87	0.92	0.87	0.89	0.92			

Notes: Row (a) reports simulated business cycle statistics for a Long-Plosser economy (full capital depreciation) with speculative bubbles (no transversality condition); see Sect. 3 of paper. *Y*: output; *C*: consumption; *I*: investment.

In the simulated model, fluctuations are just driven by bubble shocks (constant TFP assumed). Bust probability π=0.5.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods. The reported simulated standard deviations, correlations with output and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Simulated series were logged and HP filtered (the HP filter was applied separately to each window of 200 periods). 'Means' (of *Y*,*C*,*I*) are sample averages over the total sample of T periods; means are expressed as % deviations from the steady state of the no-bubble economy.

Row (b) reports US historical business cycle statistics (quarterly data), 1968q1-2017q4. The empirical data are taken from BEA NIPA (Table 1.1.3). *Y*: GDP; *C*: personal consumption expenditures; *I*: fixed investment.

Table 2. RBC economy with incomplete capital depreciation: business cycle statistics

	Bubble model (no TVC): Bubble shocks; no TFP shocks: Unit Risk aversion π=0.5	Unit Risk aversion π=0.2	High RA π=0.5	High RA π=0.2	Bubble & TFP shocks: Unit RA π=0.5	Unit RA π=0.2	High RA π=0.5	High RA π=0.2	*No-bubble model*: TFP shocks: Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.27	1.60	0.98	1.57	1.14	0.72	1.47
C	1.08	2.63	0.29	0.61	1.16	2.71	0.38	0.72	0.49	0.26	1.19
I	4.29	9.38	3.22	6.51	5.38	9.85	3.86	6.72	3.33	2.20	4.96
L	0.74	1.73	1.04	2.18	0.82	1.70	1.05	2.22	0.34	0.30	1.06
Correlations with output											
C	-0.97	-0.95	-0.99	-0.98	0.04	-0.54	0.01	-0.62	0.95	0.99	0.87
I	0.98	0.96	0.99	0.99	0.89	0.86	0.97	0.98	0.99	0.99	0.92
L	0.99	0.97	0.99	0.99	0.79	0.81	0.45	0.82	0.98	-0.96	0.82
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.65	0.68	0.57	0.66	0.71	0.70	0.87
C	0.33	0.60	0.35	0.62	0.43	0.62	0.53	0.65	0.76	0.72	0.89
I	0.36	0.63	0.37	0.64	0.53	0.65	0.51	0.65	0.70	0.70	0.92
L	0.34	0.61	0.35	0.62	0.45	0.62	0.41	0.63	0.70	0.74	0.92
Means [% deviation from no-bubble steady state]											
Y	1.41	2.80	1.25	2.12	1.37	2.75	1.31	2.17	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.68	1.34	0.33	0.55	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.61	7.28	4.44	7.40	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.34	0.73	0.01	-0.03	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.16	8.78	8.92	8.53	9.58	9.58	13.41
Fraction of periods for which capital income > investment [in %]											
	99.20	96.31	99.55	97.72	99.20	96.43	99.37	97.74	100	100	100

Notes: This Table reports simulated business cycle statistics for an RBC economy with incomplete capital depreciation; see Sect. 4 of paper. *Y*: output; *C*: consumption ; *I*: investment; *L*: hours worked.

Cols. (1)-(4) pertain to bubble model variants (no transversality condition, TVC) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Cols. (5)-(8) pertain to bubble model variants, driven by joint bubble and TFP shocks. Cols. (9),(10) pertain to a no-bubble model (TVC imposed) driven by TFP shocks. 'Unit Risk Aversion': log utility; 'High Risk Aversion (RA)': consumption utility given by $\ln(C_t-\overline{C})$, with $\overline{C}>0$. π : bust probability of bubble process.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations with output and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods for which capital income > investment' likewise pertains to the whole simulation run of T periods.

Col. (11) reports US historical statistics (quarterly data). Statistics for *Y*,*C*,*I*: see Table 1. The empirical measure for '*L*' is: 'Total Employment' (Source: CPS, as reported by FRED database, series CE160V). Historical mean of (capital income – investment)/GDP ratio: based on US annual data 1929-1985 reported by Abel et al. (1989)).

Table 3. Two-country Dellas model: business cycle statistics

	Bubble model (no TVC) Bubble shocks; no TFP shocks	*Bubble model (no TVC)* Bubble & TFP shocks	*No-bubble Model* TFP shocks	Data
	(1)	*(2)*	*(3)*	*(4)*
Standard deviations [in %]				
Y	1.52	1.96	1.36	1.47
C	1.86	2.22	1.28	1.19
I	3.95	4.01	1.28	4.96
L	0.97	0.97	0.00	1.06
NX	0.00	0.00	0.00	0.43
RER	0.00	1.23	1.23	3.66
Correlations with domestic GDP				
C	0.25	0.57	0.99	0.87
I	0.76	0.88	0.99	0.92
L	0.50	0.31	--	0.82
NX	--	--	---	-0.51
RER	--	-0.41	-0.54	-0.27
Autocorrelations				
Y	0.63	0.77	0.80	0.87
C	-0.17	0.48	0.81	0.89
I	0.41	0.66	0.81	0.92
L	0.10	0.10	--	0.92
NX	--	--	--	0.78
RER	--	0.75	0.75	0.81
Cross-country correlations				
Y	1.00	0.68	0.39	0.53
C	1.00	0.84	0.56	0.39
I	1.00	0.95	0.56	0.45
L	1.00	1.00	--	0.39
Means [% deviation from no-bubble steady state]				
Y	0.95	1.18	0.22	--
C	-0.01	0.12	0.22	--
I	3.07	3.33	0.22	--
L	0.42	0.42	0.00	--
Mean (capital income – investment)/GDP [in %]				
	-0.02	-0.02	0.33	13.42
Fraction of periods with (capital income > investment) [in %]				
	97.01	97.01	100.00	100.00

Notes: This Table reports simulated business cycle statistics for a two-country RBC world (Dellas) with full capital depreciation (see Sect. 5 of paper). *Y*: GDP; *C*: consumption ; *I*: investment; *L*: labor input. *NX*: net exports/GDP; *RER*: real exchange rate. A rise in R*ER* represents an appreciation.

Table 3. (continued)

Col. (1) pertains to a version of the bubble model (no transversality condition, TVC) in which fluctuations are just driven by bubbles shocks (constant TFP assumed). Col. (2) pertains to a version of the bubble model, driven by simultaneous bubble and TFP shocks. The bubble process (Cols. 1 and 2) assumes a bust probability π=0.5. Col. (3) pertains to a no-bubble model, driven by TFP shocks.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations with domestic GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged (with exception of NX) and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods with (capital income > investment)' likewise pertains to the whole simulation run of T periods.

Col. (4) reports historical statistics. Historical standard deviations, correlations with domestic GDP and autocorrelations of GDP, consumption, investment, employment, net exports and the real exchange rate are based on quarterly US data, 1968q1-2017q4 (see Tables 2 and 3). The empirical measure of *NX* is: US nominal exports-imports (goods and services) divided by nominal GDP (from BEA NIPA Table 1.1.5). Empirical measure of the US real exchange rate: real effective exchange rate, REER (from BIS; 1968:q1-1993q4: 'narrow index'; 1994q1-2017q4: 'broad index'; a quarterly average of the monthly BIS REER series is used). Historical statistics about 'capital income – investment': based on US annual data 1929-1985 reported by Abel et al. (1989)).

Historical cross-country correlations (of Y,C,I,L) are correlations between US series and series for an aggregate of the Euro Area for 1970q1-2017q4 (logged and HP filtered quarterly series). (Euro Area data are only available from 1970q1.) Source for EA data: ECB Area-wide Model (AWM) database (version Aug. 2018). (EWM series for Y,C,I,L: YER, PCR, ITR, LNN.)

Table 4. Two-country RBC model (incomplete capital depreciation): business cycle statistics

	Bubble model (no TVC)								*No-bubble model*		
	Bubbles shocks; no TFP shocks				Bubble & TFP shocks				TFP shocks		
	Unit Risk aversion		High RA		Unit RA		High RA				
	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	π=0.5	π=0.2	Unit RA	High RA	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
Standard deviations [in %]											
Y	0.49	1.16	0.68	1.43	1.46	1.78	1.18	1.65	1.32	0.97	1.47
C	1.08	2.63	0.29	0.61	1.18	2.79	0.41	0.70	0.56	0.31	1.19
I	4.29	9.38	3.22	6.51	6.36	10.54	4.95	7.34	4.60	3.90	4.96
L	0.74	1.73	1.04	2.18	0.88	1.79	1.13	2.24	0.44	0.62	1.06
NX	0.00	0.00	0.00	0.00	0.16	0.16	0.13	0.13	0.16	0.13	0.43
RER	0.00	0.00	0.00	0.00	0.32	0.32	0.44	0.44	0.32	0.44	3.66
Correlations with domestic GDP											
C	-0.97	-0.95	-0.99	-0.98	0.09	-0.46	0.03	-0.55	0.85	0.61	0.87
I	0.98	0.96	0.99	0.99	0.90	0.88	0.97	0.98	0.95	0.96	0.92
L	0.99	0.97	0.99	0.99	0.81	0.81	0.46	0.78	0.94	-0.01	0.82
NX	--	--	--	--	-0.53	-0.46	-0.58	-0.46	-0.58	-0.68	-0.51
RER	--	--	--	--	-0.44	-0.35	-0.58	-0.39	-0.48	-0.68	-0.27
Autocorrelations											
Y	0.36	0.63	0.35	0.62	0.63	0.67	0.57	0.65	0.67	0.64	0.87
C	0.33	0.60	0.35	0.62	0.46	0.62	0.57	0.65	0.75	0.71	0.89
I	0.38	0.63	0.37	0.64	0.54	0.64	0.55	0.64	0.63	0.61	0.92
L	0.34	0.61	0.35	0.62	0.46	0.62	0.48	0.64	0.63	0.69	0.92
NX	--	--	--	--	0.61	0.61	0.66	0.66	0.61	0.66	0.78
RER	--	--	--	--	0.84	0.84	0.81	0.81	0.84	0.81	0.82
Cross-country correlations											
Y	1.00	1.00	1.00	1.00	0.29	0.54	-0.00	0.52	0.17	-0.46	0.53
C	1.00	1.00	1.00	1.00	0.96	0.99	0.98	0.99	0.84	0.96	0.39
I	1.00	1.00	1.00	1.00	0.27	0.74	-0.07	0.53	-0.35	-0.83	0.45
L	1.00	1.00	1.00	1.00	0.63	0.92	0.85	0.96	-0.35	0.46	0.39
Means [% deviation from no-bubble steady state]											
Y	1.41	2.80	1.25	2.12	1.65	3.02	1.45	2.29	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.95	1.60	0.44	0.65	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.93	7.61	4.72	7.61	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.35	0.73	0.09	0.05	0.00	0.00	--
Mean (capital income – investment)/GDP [in %]											
	9.12	8.75	8.93	8.54	9.15	8.78	8.89	8.51	9.55	9.58	13.42
Fraction of periods with (capital income > investment) [in %]											
	99.20	96.31	99.55	97.72	99.20	96.45	99.44	97.75	100	100	100

Notes: This Table reports simulated business cycle statistics for a two-country RBC model with incomplete capital depreciation (see Sect. 6 of paper). *Y*: GDP; *C*: consumption ; *I*: investment; *L*: labor input; *NX*: net exports/GDP; *RER*: real exchange rate. A rise in R*ER* represents an appreciation.

Table 4. (continued)

Cols. (1)-(4) pertain to versions of the bubble model (no transversality condition, TVC) in which fluctuations are just driven by bubbles (constant TFP assumed). Cols. (5)-(8) pertain to versions of the bubble model, driven by simultaneous bubble and TFP shocks. Cols. (9)-(10) pertain to versions of the no-bubble model, driven by TFP shocks.

'Unit Risk Aversion': log utility; 'High Risk Aversion (RA)': consumption utility given by $\ln(C_t - \overline{C})$, with $\overline{C} > 0$.

π : bust probability of bubble process.

The model-predicted business cycle statistics are based on one simulation run of T=10000 periods (for each model version). Simulated standard deviations, correlations of GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. Series were logged (with exception of NX) and HP filtered (HP filter applied separately for each window of 200 periods). 'Means' are sample averages over the total sample of T periods. The 'Fraction of periods with (capital income > investment)' likewise pertains to the whole simulation run of T periods.

Col. (11) reports historical statistics (see Table 3).

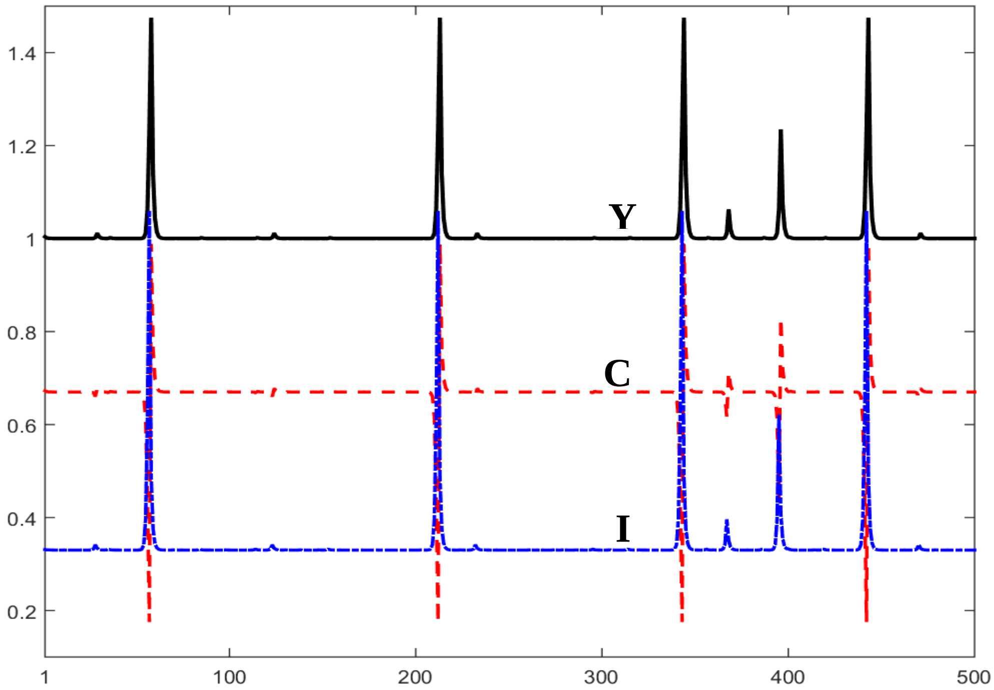

Figure 1. Long & Plosser economy with bubbles (no transversality condition): simulated paths
Simulated paths of output (Y, continuous black line), consumption (C, red dashed line) and investment (I, blue dash-dotted line) are normalized by no-bubble steady state output.

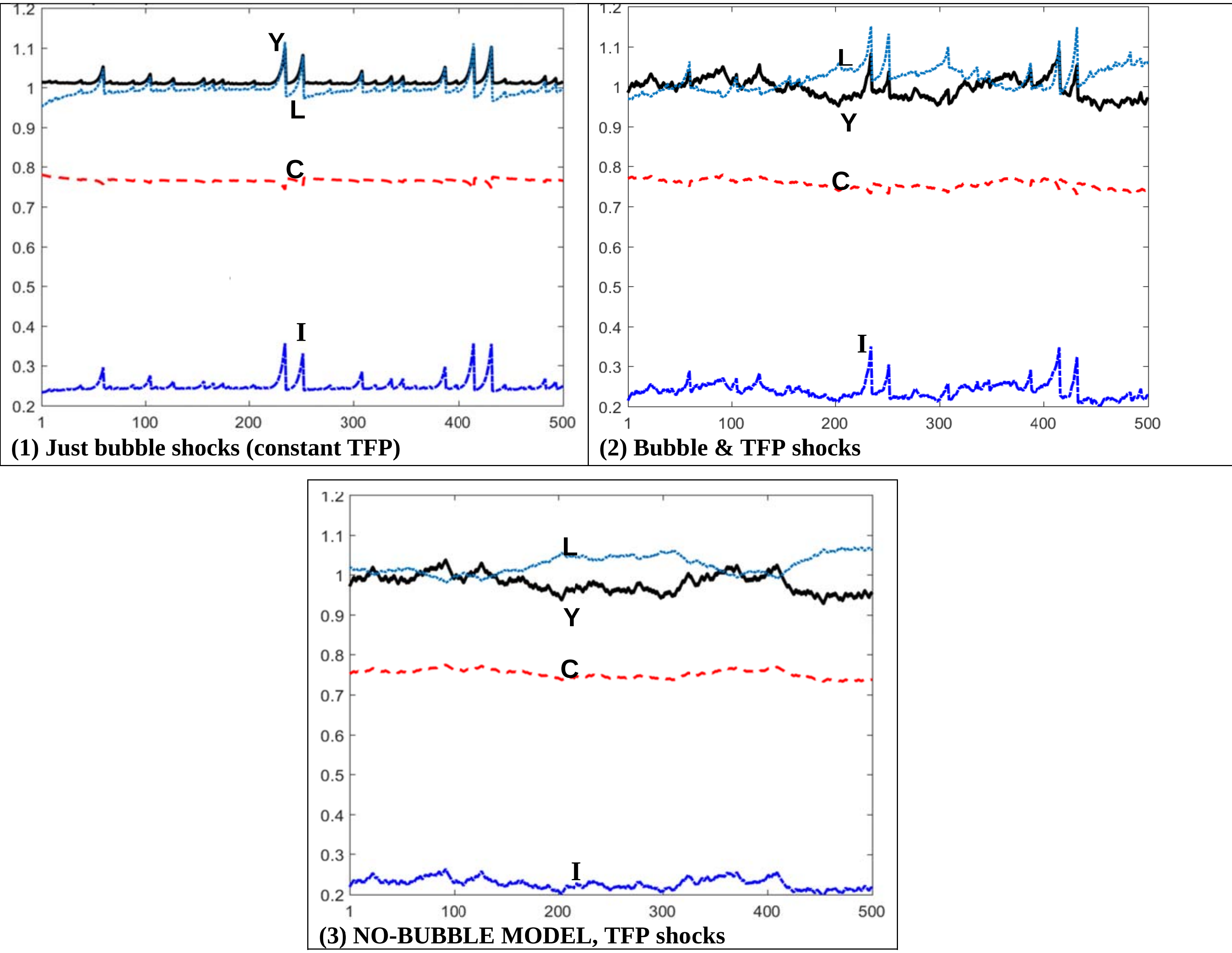

Figure 2. RBC economy with incomplete capital depreciation: simulated paths

This Figure assumes the RBC economy with incomplete capital depreciation, 'High risk aversion' and a bust probability π=0.2 described in Sect. 4. Panel (1) pertains to a bubble model (no transversality condition) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Panel (2) pertains to a bubble model driven by joint bubble and TFP shocks. Panel (3) pertains to a no-bubble model, driven by TFP shocks. Simulated paths of output (Y, continuous black line), consumption (C, red dashed line), investment (I, dark blue dash-dotted line) and hours worked (L, light blue dotted line) are shown. Y, C and I: normalized by no-bubble steady state output. L: normalized by steady state hours.

Y C I L

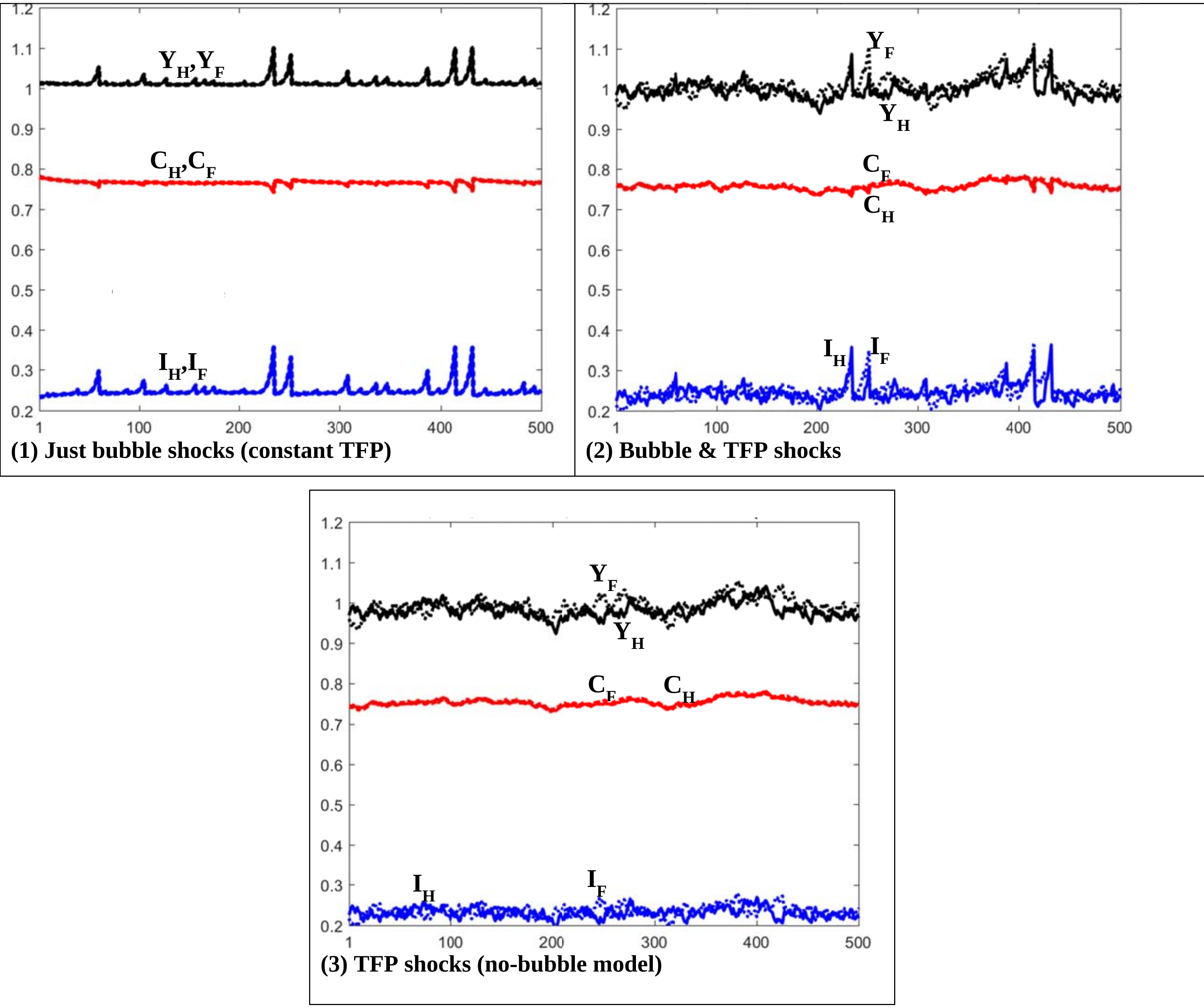

Figure 3. Two-country RBC model (incomplete capital depreciation): simulated paths

This Figure assumes the two-country RBC model with incomplete capital depreciation, 'High risk aversion' and a bust probability π=0.20 (see Sect. 6). Simulated paths of Home and Foreign GDP (Y_H, Y_F: continuous and dotted black lines), Home and Foreign consumption (C_H, C_F: continuous and dotted red lines) and investment (I_H, I_F: continuous and dotted blue lines) are shown. The plotted series are normalized by steady state GDP. Y_H Y_F C_H C_F I_H I_F

Panel (1) pertains to a bubble model (no transversality condition) in which fluctuations are just driven by bubble shocks (constant TFP assumed). Panel (2) pertains to a bubble model, driven by joint bubble and TFP shocks. Panel (3) pertains to a no-bubble model, driven by TFP shocks.

FOR ONLINE PUBLICATION

APPENDIX

Further discussion of the bubbly RBC economy (no TVC) with incomplete capital depreciation (Sect. 4)

The numerical simulations of the model developed in Sect. 4 consider bubbles for which, conditional on date t information, a TFP innovation at t+1 has an equiproportional effect on $K^L_{t+2}=\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$ and on K^H_{t+2}. Specifically, I consider equilibria in which $K^H_{t+2}=s^H_t\cdot K^L_{t+2}$ holds, where $s^H_t>0$ is in the date t information set. Thus, $K^H_{t+2}=s^H_t\cdot\lambda(K_{t+1},\theta_{t+1})e^{\Delta}$.[26] This greatly simplifies the bubble model's solution. Substituting the formula for K^H_{t+2} into the Euler equation (13) gives:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)+(1-\pi)\cdot E_t H(s^H_t\cdot\lambda(K_{t+1},\theta_{t+1})e^{\Delta},K_{t+1},K_t,\theta_{t+1},\theta_t)=1, \qquad \text{(A.1)}$$

for $K_{t+1}\in\{K^L_{t+1},K^H_{t+1}\}$, with $K^L_{t+1}=\lambda(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}=s^H_{t-1}\cdot K^L_{t+1}$. For given values of K_t, K_{t+1}, θ_t the Euler equation (A.1) only involves one unknown endogenous variables: s^H_t. Solving (A.1) for s^H_t at each date pins down the equilibrium capital process (see further discussion below). Given the equilibrium capital process, on can compute consumption, hours and output using (11).

As explained in the main text, I set $\Delta>0$, because a strictly positive Δ is needed to generate *recurrent* bubbles. As in the Long-Plosser economy without TVC, bubbles are self-ending when $\Delta=0$; by contrast, $\Delta<0$ implies that the capital stock ultimately reaches zero. [27]

Computational aspects

I. Solving for consumption and labor hours using the static equations

The static model equations can be used to solve for consumption and labor hours as functions of capital and TFP (see (11) in main text). Note that the labor supply equation (9) can be written as

$$C_t=[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}(L_t)^{-\alpha}(1-L_t). \qquad \text{(A.2)}$$

The date t resource constraint of the economy is $C_t+K_{t+1}=Y_t+(1-\delta)K_t$, where $Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}$. Substituting (A.2) into the resource constraint gives:

$$[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}(L_t)^{-\alpha}(1-L_t)=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}+(1-\delta)K_t-K_{t+1}.$$

---

[26] The AR(1) specification of TFP implies $\theta_{t+1}=(\theta_t)^{\rho}\cdot\exp(\varepsilon^{\theta}_{t+1})$, where $\varepsilon^{\theta}_{t+1}$ is the TFP innovation at t+1. The chosen specification of K^L_{t+2}, K^H_{t+2} implies that $\partial\ln(K^H_{t+2})/\partial\varepsilon^{\theta}_{t+1}=\partial\ln(K^L_{t+2})/\partial\varepsilon^{\theta}_{t+1}$; thus, an unexpected change in date t+1 productivity affects K^H_{t+2} and K^L_{t+2} by the same (relative) amount.

[27] Consider the dynamics that obtains when $\Delta=0$. Assume that, at date t, the sunspot selects (with $\Delta=0$) $K_{t+1}=K^L_{t+1}\equiv\lambda(K_t,\theta_t)$. Then Euler equation (A.1) is solved by $s^H_t=1$, so that $K^H_{t+2}=\lambda(K_{t+1},\theta_{t+1})$. This follows from the fact that $E_t H(\lambda(\lambda(K_t,\theta_t),\theta_{t+1}),\lambda(K_t,\theta_t),K_t)=1$ (Schmitt-Grohé and Uribe (2004), eqn. (4)). Thus $K_{t+2}=K^H_{t+2}=K^L_{t+2}=\lambda(K_{t+1},\theta_{t+1})$ and $K_{t+s+1}=\lambda(K_{t+s},\theta_{t+s})$ also holds $\forall\ s>1$. In all subsequent periods the dynamics of the capital stocks is hence governed by the no-bubble decision rule, i.e. the bubble has ended.

Equivalently: $1{=}A_{1,t}\cdot(L_t)^{\alpha}{+}A_2{\cdot}L_t$, with $A_{1,t}\equiv-[K_{t+1}-(1-\delta)K_t]/\{[(1-\alpha)/\Psi]\cdot\theta_t(K_t)^{\alpha}\}$, $A_2\equiv[1+\Psi/(1-\alpha)]$. For the assumed capital elasticity of output α=1/3, this (cubic) equation has a unique closed form solution for date t hours worked L_t as a function of K_{t+1},K_t,θ_t. Substitution of the formula for hours into (A.2) gives a closed form formula for consumption C_t (see (11)).

II. Euler equation

TFP is assumed to follow the AR(1) process $\ln(\theta_{t+1}){=}\rho\ln(\theta_t){+}\varepsilon^{\theta}_{t+1}$, $0{\le}\rho{<}1$, where $\varepsilon^{\theta}_{t+1}$ is a discrete innovation that equals $\varepsilon^{\theta}_{t+1}{=}-\sigma_\theta$ or $\varepsilon^{\theta}_{t+1}{=}\sigma_\theta$ with probability 1/2, respectively, where $\sigma_\theta{\ge}0$. The Euler equation (A.1) can, thus, be written as:

$$\begin{aligned}&\pi\{\tfrac{1}{2}H(\lambda(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(\lambda(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}\\ &(1-\pi)\{\tfrac{1}{2}H(s^H_t\cdot\lambda(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(s^H_t\cdot\lambda(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}=1\end{aligned} \tag{A.3}$$

for $K_{t+1}\in\{K^L_{t+1};K^H_{t+1}\}$, where $K^L_{t+1}{=}\lambda(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}{=}s^H_{t-1}K^L_{t+1}$.

In the numerical simulations, I approximate the no-bubble decision rule λ using a second-order (log-quadratic) Taylor expansion. Let $\hat{\lambda}(K_t,\theta_t)$ be the second-order Taylor expansion of the no-bubble decision rule λ. In the numerical simulations, I thus define K^L_{t+1} as $K^L_{t+1}{\equiv}\hat{\lambda}(K_t,\theta_t)$ $\forall t$. The simulations are hence based on a version of Euler equation (A.3) in which λ is replaced by $\hat{\lambda}$:

$$\begin{aligned}&\pi\{\tfrac{1}{2}H(\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}\\ &(1-\pi)\{\tfrac{1}{2}H(s^H_t\cdot\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{\sigma_\theta},\theta_t)+\tfrac{1}{2}H(s^H_t\cdot\hat{\lambda}(K_{t+1},(\theta_t)^{\rho}e^{-\sigma_\theta})e^{\Delta},K_{t+1},K_t,(\theta_t)^{\rho}e^{-\sigma_\theta},\theta_t)\}=1\end{aligned} \tag{A.4}$$

for $K_{t+1}\in\{K^L_{t+1};K^H_{t+1}\}$, where $K^L_{t+1}{=}\hat{\lambda}(K_t,\theta_t)e^{\Delta}$ and $K^H_{t+1}{=}s^H_{t-1}K^L_{t+1}$.

Conditional on K_t,K_{t+1},θ_t, this equation can be used to determine s^H_t. I employ a bisection method for that purpose.

Like the value of the bust probability π, the specification of the bust capital stock K^L is not tied down by economic theory. The only restriction is that the resulting law of motion for capital has to be bounded and strictly positive. I verified that the bubble equilibrium constructed using $\hat{\lambda}$ meets this criterion. For model variants with constant TFP, I also computed the no-bubble decision rule $K_{t+1}{=}\lambda(K_t,\theta)$ using a shooting algorithm (Judd (1998), ch.10). The second-order approximation and the shooting algorithm give no-bubble decision rules that are very close, even when capital K_t is far from the no-bubble steady state. The resulting speculative bubbles too are very similar. Computing $\hat{\lambda}$ is much faster.

III. Initial capital and equilibrium recursion

The stochastic simulations of the bubble economy start at an initial date t=0. The exogenous initial capital stock K_0 is assumed to equal the no-bubble steady state capital stock. K_1 (the period 1 capital stock set at t=0) is indeterminate, in the bubble economy. I assume

$K_1 = \lambda(K_0, \theta_0)e^{\Delta}$. The following recursion allows to simulate an equilibrium path of capital in periods t>1, for an exogenous path of TFP $\{\theta_t\}_{t\geq 0}$:

● Given K_0, K_1 and θ_0, the date t=0 Euler equation (A.4) determines s_0^H.

● At t=1, realized productivity θ_1 pins down the two possible values of the date t+2 capital stock: $K_2^H = s_0^H \cdot K_2^L$ where $K_2^L = \lambda(K_1, \theta_1)e^{\Delta}$. An exogenous random draw (sunspot) then determines at t=1 whether K_2 equals K_2^L (probability π) or K_2^H (probability 1-π). Given K_1, K_2, θ_1 the t=1 Euler equation (A.4) pins down s_1^H.

● The same process is repeated in all subsequent periods.

The effect of K_0, K_1 on endogenous variables in subsequent periods vanishes as time progresses; initial conditions do not affect moments over a long simulation run.

References

Judd, Kenneth, 1998. Numerical Methods in Economics. Cambridge, MA: MIT Press.