# Information Aggregation with Delegation of Votes 

Amrita Dhillon*<br>amrita.dhillon@kcl.ac.uk<br>Dilip Ravindran ${ }^{\ddagger}$<br>dilip.ravindran@hu-berlin.de

Grammateia Kotsialou ${ }^{\dagger}$<br>g.m.kotsialou@lse.ac.uk

Dimitrios Xefteris ${ }^{\S}$<br>xefteris.dimitrios@ucy.ac.cy

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#### Abstract

Recent developments in blockchain technology have made possible greater progress on secure electronic voting, opening the way to better ways of democratic decision making. In this paper we formalise the features of "liquid democracy" which allows voters to delegate their votes to other voters, and we explore whether it improves information aggregation as compared to direct voting. We consider a two-alternative setup with truth-seeking voters (informed and uninformed) and partisan ones (leftists and rightists), and we show that delegation improves information aggregation in finite elections. We also propose a mechanism that further improves the information aggregation properties of delegation in private information settings, by guaranteeing that all vote transfers are from uninformed to informed truth-seeking voters. Delegation offers effective ways for truth-seeking uninformed voters to boost the vote-share of the alternative that matches the state of the world in all considered setups and hence deserves policy makers' attention.


Keywords: Liquid democracy, delegation, experts, abstention, majority voting, information aggregation, Condorcet Jury theorem.

JEL Codes: D72

[^0]
## 1 Introduction

Recent progress on distributed ledger technologies and, in particular, in blockchain based voting has opened up the possibilities of new and improved ways of voting ${ }^{1}$ (see e.g. Dhillon et al., 2021). Liquid Democracy, which was first suggested by Miller (1969) and discussed by Shubik (1970), is one such electoral rule that combines direct democracy with representative democracy. Under liquid democracy, voters choose whether to vote themselves or delegate their vote to another voter who may be more knowledgeable on a particular issue, and they can delegate differently with respect to different issues. Several versions of liquid democracy have been proposed in the literature (e.g. where only one round of delegation is allowed or where voters who receive delegated votes may further delegate those votes), some of which have already been used in settings of applied interest (e.g. by certain political parties in Europe). Liquid Democracy may be particularly well suited for referenda when there are usually binary issues involved. Figure 1 below illustrates the features of liquid democracy contrasted with direct and representative democracy.


Figure 1: Design differences in three types of democratic models: representative, direct and liquid democracy. In representative democracy, citizens are represented by elected officials who then vote on behalf of the first. In direct democracy, citizens vote directly. In liquid democracy, every citizen either votes directly or delegates to a representative of her choice.

[^1]Blum and Zuber (2016) present two main normative arguments in favour of liquid democracy. First, delegating to more informed people allows better decisions when there is an objectively "correct" decision. Second, delegation allows greater equality: instead of creating two types of citizens with unequal power to influence policy (the representatives who vote on policy and ordinary voters who can only vote for representatives), liquid democracy allows voters to vote directly on policy or delegate votes to other agents who can vote directly on policy.

In this paper we focus on the first issue: We study the information aggregation properties of delegation in elections with two candidates and two states of nature, where truth-seeking and partisan voters co-exist. Truth-seeking voters have state dependent preferences over candidates, while partisan voters have fixed state-independent preferences over the two alternatives. Truth-seeking voters are either informed (i.e. they know the state of the world) or uninformed, and partisans are divided into two (potentially unequal) groups depending on which candidate they support. We ask if delegation improves on direct voting in simple majority elections i.e. if the likelihood of choosing the welfare maximising candidate/alternative is higher with delegation than with direct voting.

A priori, it is not obvious that delegation will improve outcomes when the identity of informed truth seeking voters is not known - in which case it is possible that partisans get extra votes. Indeed, even with complete information, when informed voters are imperfectly and heterogeneously informed, it is possible that a few voters get all the extra votes which may end up in loss of information overall.

Our main contribution is to show that delegation can improve information aggregation in finite elections of any size, both when the identities of "experts" (informed truth-seeking voters) are known, and when they are not. When voter types are common knowledge, we show that the game with delegation is dominance solvable, while the game without delegation is not. It is well known that multiple Nash equilibria are pervasive in voting games due to many situations where a single voter is not pivotal. Voting games admit Nash equilibria which are based on unreasonable beliefs on the part of voters. Ruling out weakly dominated strategies helps to rule out such implausible Nash equilibria. In this context, using Iterated Elimination of Weakly Dominated Strategies (IEWDS) is a powerful tool for predicting outcomes in voting games (Dhillon and Lockwood, 2004). ${ }^{1}$

It is worth noting that the outcome of the IEWDS is the efficient one and requires only two rounds of iterated elimination, i.e. it is cognitively easy. Voters can more easily coordinate (using symmetric strategies) on the efficient equilibria, in contrast to the game without delegation which is not dominance solvable ${ }^{2}$. Indeed, without delegation, reaching an efficient equilibrium is hard in the sense that voters of the same type (namely, the uninformed truth-seeking ones) need to employ asymmetric strategies. That is, in absence of a coordination mechanism electoral accidents are likely to occur. Things are much

[^2]safer with delegation: the game is both dominance solvable and efficiency does not require that different voters of the same type behave in a different way. Finally, participation is greater when delegation is allowed than when it is not. Indeed, all votes are cast either directly or indirectly (i.e. via delegates) when delegation is allowed. When delegation is not allowed, substantial abstention takes place in every efficient equilibrium even when voting is costless (Feddersen and Pesendorfer (1996)).

What happens when the types of voters are their private information? Then delegation introduces trade-offs and might potentially lead to worse outcomes. On the one hand it increases the vote-share of the efficient alternative, conditional on the vote transfers being from uninformed to informed truthseeking voters; but it may also harm the electoral prospects of the efficient alternative if uninformed voters delegate their votes inadvertently to partisan voters. Indeed, since the players types are unobservable, mistakes in vote transfers are highly likely. We prove that the best undominated equilibrium of the game with delegation (i.e. in terms of the probability with which it leads to the efficient outcome) is always at least as good as the best undominated equilibrium of the game without delegation. This is a very strong result, since it holds for every possible type distribution and society size, and establishes that despite the trade-off, delegation is welfare improving.

Interestingly, this result generalises to the case in which truth-seeking voters are partially and heterogeneously informed (i.e. when the quality of information held by each truth-seeking voter is allowed to differ). In such a setting delegation introduces an additional dilemma: when less informed -yet, not completely uninformed- voters delegate their votes to more informed ones (even if they know who they are), then the electoral impact of more informed voters increases (which is desirable), but pieces of valuable information are left out of the aggregation process (which is undesirable). Whether the net effect of delegation is positive or negative, depends on the exact behavior employed by the voters (Kahng et al., 2018). We prove that the game with delegation admits an equilibrium that is weakly better compared to all equilibria of the game without delegation, even in the presence of such an additional complication. This reinforces the case for liquid democracy. To our knowledge, this paper is the first to establish that rational truth-seeking voters can always exploit delegation to improve their welfare, when partisan voters also exist in the electorate.

Finally, we note that even if delegation improves expected welfare, mistaken transfers from uninformed truth-seeking voters to partisans cannot be ruled out when types are private information. We ask if we can design a mechanism with delegation that guarantees efficient transfers of votes even with private information. We show that when the number of informed truth-seeking, uninformed truthseeking, and partisan voters is known (yet, the number of voters of each type of partisan is not), then we can design a mechanism with this desirable feature. This mechanism always admits an equilibrium such that all vote transfers are from uninformed voters to informed truth-seeking ones.

The mechanism is based on voters being able to drop or pick up ballots anonymously into a common receptacle. ${ }^{1}$ If anyone picks up a vote, a central entity keeps a record of how they voted and if the candidate they voted for loses the election, the voter has to pay a penalty. A secure electronic voting technology (such as blockchain) is key in order for such a process to be possible and to enjoy the

[^3]required legitimacy. We show that when the number of voters is sufficiently large, there exists an equilibrium in the game with delegation where uninformed voters drop their votes in the receptacle and only informed truth-seeking voters pick up an extra vote, while partisans do not. The assumption that the voting records of delegates can be seen in order to impose the penalty may seem strong, but even with representative democracy accountability requires transparency on voting records. Moreover, as we show in section 6 , the penalty needs to be infinitesimally small to generate the result.

Overall, our main findings combine and strengthen the case for liquid democracy: if delegation of votes is allowed, the aggregation of information can be improved both when the types of the voters is known, and when the preferences and the information held by voters is their private information. Of course, the possibility of designing ways to enhance the quality of vote transfers in blockchain systems is not limited to the mechanism that we propose. But the identification of an incentive compatible protocol of transfers that eliminates, in theory, the possibility that delegated votes end up to partisans, indicates that alternative -and, potentially, more empirically relevant- ways of secure transfers are plausible.

The rest of the paper is organised as follows: Section 2 discusses the related literature, Section 3 presents the definitions and notation we use, Section 4 deals with the case when information on voter types is common knowledge, Section 5 analyses the case when voter types are private information, Section 6 describes an incentive compatible delegation mechanism that does better in terms of aggregating information as compared to direct voting, even when voter types are private information; Section 7 provides some robustness analysis and, finally, Section 8 points to potential avenues for future research and concludes.

## 2 Related Literature

There is a vast literature on information aggregation in two candidate elections starting with the seminal work on the Condorcet Jury theorem (1785), which showed that, with two alternatives, two states of the world and common values, if each individual voted for the correct alternative with probability strictly greater than half, then the probability that a majority would choose the correct alternative converges to one as the society grows large. The theorem assumed sincere voting. Austen-Smith and Banks (1996) showed that sincere voting was not rational in such a setting. McLennan (1998) (for common value elections) and Feddersen and Pesendorfer (1997) (with heterogeneous voters and private information on voter types) show that, even for two candidate elections, there is always an equilibrium that aggregates information efficiently asymptotically as the size of the electorate goes to infinity.

Closest to our paper is Feddersen and Pesendorfer (1996), which allows for heterogeneous voters and private information. In their setting where the size of the population is unknown, yet large, and truth-seeking and partisan voters co-exist; they show that elections aggregate information efficiently i.e. the equilibrium outcome is asymptotically the same as though information on the state were common knowledge. In equilibrium, informed truth-seeking voters and partisan voters vote for the alternative that they support, while uninformed truth-seeking voters employ a non-trivial behavior.

More specifically, they are subject to a "swing voters curse": a substantial fraction of voters abstain strategically to allow informed truth-seeking voters to be decisive, even when voting is costless. For their main result, they do need large elections. ${ }^{1}$ In contrast, we show that delegation can improve on simple majority voting with abstention in elections of any size, with and without private information on voter types.

Christoff and Grossi (2017) and Kahng et al. (2018) also study information aggregation with delegated voting, but do not employ an equilibrium approach. Christoff and Grossi (2017) focuses on the aggregation of individual choices to social choice and the unintended effects of delegation on the rationality postulates satisfied by direct voting. Kahng et al. (2018) study the case where voters have different levels of information, complete information and a network setting. They show that when voters delegate only to more informed voters who are within their local network, then delegation can lead to worse outcomes than simple majority voting, due to the concentration of power. They argue that if alternative behavioral assumptions are imposed then delegation can lead to better outcomes. The key point of the paper is that too much delegation to the same voters (given the network structure) risks losing out on valuable information. In this paper, we complement their work by showing that when behavior rules are not fixed but rather endogenously determined by rational voters, when voting is simultaneous, then delegation can lead to welfare improvements in equilibrium. Bloembergen et al. (2019), study equilibrium behaviour in a network setting. Voters know which of two outcomes they would prefer with a probability between 0.5 and 1 (voter accuracy). They focus on the decision problem of voters, when voters have a choice of direct voting, which incurs a cost, and delegated voting, when they know the accuracy type of the other voters and the probability that they have a similar preference, but they do not know the true type. The authors show existence of Nash equilibria and average accuracy achieved in a network setting. Recently, Armstrong and Larson (2021) focus on a basic theoretical model, without the use of partisan voters, and show that delegation always reaches an equilibrium with weakly higher group accuracy at identifying the ground truth outcome. However, their experiments show that neither optimal delegations nor efficiently computable delegation strategies significantly improve accuracy in small or realistically sized electorates, respectively. These conclusions are based on comparisons of outcomes on fixed voting settings using five different delegation mechanisms.

Beyond the specific issue of delegation, our work also relates to studies which try to assess the information aggregation properties of different electoral systems. Bhattacharya (2013) and Barelli et al. (2017) extend the Condorcet's Jury theorem to heterogeneous state-dependent preferences and to general state and signal spaces respectively, and show that information aggregation depends on the complexity of the preference and information structure. Goertz and Maniquet (2011), Bouton et al. (2016) and Ahn and Oliveros (2016) study the properties of approval voting and other scoring rules, for any electorate size, when a divided majority occurs due to disagreements among the truth-seeking voters regarding the most likely state of the world. These papers focus on the case where the majority needs to coordinate in order to eliminate the possibility of inefficient outcomes, and show that approval voting performs better. As far as runoff systems with two (or more) voting rounds are concerned,

[^4]Piketty (2000) observes that due to the existence of multiple voting rounds, the majority group should have additional opportunities to coordinate and aggregate their information. Martinelli (2002) shows that efficient aggregation of information is feasible in equilibrium under a two-round runoff rule, in the setting of a divided majority with three alternatives. Tsakas and Xefteris (2021) extend this possibility result to more general settings. Herrera et al. (2019) assess theoretically the information aggregation properties of more "proportional" systems. That is, when the a small change in the voteshare distribution affects the outcome, even if it does not affect the winner of the election. They find that in large societies, relatively uninformed voters abstain, and information is aggregated efficiently. Finally, McMurray (2017) and Prato and Wolton (2017) show that voting might be less efficient in aggregating information, when the policy alternatives are proposed by self-interested candidates.

We conclude this section with a discussion on using blockchains for the implementation of complex voting mechanisms such as liquid democracy. Blockchain technology, which was created through the introduction of Bitcoin (Nakamoto, 2009), provides a unique solution to the problem of coming to agreement on what data is valid, shared and then saved. One of the major advantages of this technology is the ability to incorporate smart contracts. These are pieces of publicly readable code placed on a blockchain to enforce mechanisms and protocols without requiring to trust a third party. Since the creation of the Ethereum blockchain (Wood, 2014), Turing-complete ${ }^{1}$ smart contracts can be placed on a blockchain. This means that any mechanism design can be enforced via smart contracts, including the vote transfer mechanism that we describe in this paper and, of course, other delegation mechanisms of votes in the context of liquid democracy (see Kotsialou and Riley, 2020, Colley et al., 2020, Escoffier et al., 2020, Brill and Talmon, 2018, Gölz et al., 2018, Boldi et al., 2011). Note that blockchain experts have already started experimenting by building voting mechanisms on blockchains, with some of the first examples including the following: McCorry et al. (2017) uses smart contracts to avoid using any trusted authority to either complete the tally or protect the voters' privacy, Riley et al. (2019) show how smart contracts are used to keep track of company shareholdings, allowing for real-time elections on company matters in a decentralised manner on the blockchain. Another recent implementation is the integration of the e-voting protocol Selene (Ryan et al. (2016)) with blockchain technology (see Sallal et al., 2020), which, in this case, acts as a publicly available bulletin board to post anonymised vote information and verification evidence. Therefore, the actual implementation of the liquid democracy system studied here is technologically feasible.

## 3 Preliminaries

We build on the model of Feddersen and Pesendorfer (1996). Consider a set of voters $N=\{1, \ldots, n\}$ who are voting over a binary issue according to a simple majority rule (ties are resolved with equiprobable draws). The set of the two alternatives is given by $C=\{a, b\}$. There are two states of nature, $S=s_{a}, s_{b}$.

[^5]Priors: State $s_{a}$ is drawn by nature with a probability $p \in[0,1]$ and state $s_{b}$ with probability $1-p$.

Types of voters: Each voter $i \in N$ has a type $t_{i}$ from the set $T=\{A, B, I, U\}$, where the types are described as follows:

- $t_{i}=A$ (or $B$ ): voters of type $A$ (respectively, $B$ ) are partisans of alternative $a$ (or alternative $b$ ) who prefer their own candidate winning regardless of the state of nature. We consider only these two types of partisans.
- $t_{i}=I$ : voters of type $I$ are the informed truth-seeking voters. Informed voters observe the state of nature and prefer candidate $a(b)$ when the state is $s_{a}$ (respectively $s_{b}$ ).
- $t_{i}=U$ : voters with type $U$ are the uninformed truth-seeking voters. They do not observe the state of nature and have the same values as informed truth-seeking voters.

To simplify, we will be using notation $A$ voters, $B$ voters, $I$ voters, $U$ voters, respectively, and when we consider a partisan voter it is implied that this voter could be either an $A$ voter or a $B$ voter. We denote as $p_{t_{i}}^{i}$ the probability that voter $i \in N$ is a voter of type $t_{i} \in T$ and as $t=\left(t_{1}, \ldots, t_{n}\right)$ a (type) profile that indicates the types of all voters in the electorate $N$. The probabilities $p_{t_{i}}^{i}$ are common knowledge, however, the realisation of the type for each voter can be either public or private information to the voter.

Strategies. When voters select what to do after they observe their type, we say that they use an interim strategy, and when they choose an action for every possible type they may be assigned before the assignment of types, we say that they use an ex-ante strategy. For our purposes it is more convenient to use the former whenever types are publicly observed, and the latter in the case of incomplete information. Notice that, given that we have a finite game, every ex-ante Bayesian Nash equilibrium is guaranteed to be an interim Bayesian Nash equilibrium -the arguably, standard equilibrium notion for such games- and hence, our choice to proceed with an ex-ante formulation does have any drawback in that respect.

Interim Strategies: When delegation is not allowed, each voter $i \in N$ with any type $t_{i} \in\{A, B, U\}$ chooses a strategy $P_{i}$ from the available strategies set $\mathcal{P}=\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$, where $\mathbf{a}$ indicates the strategy of voting for candidate $a, \mathbf{b}$ indicates the strategy of voting for candidate $b$ and $\mathbf{x}$ the strategy of abstaining. When delegation of votes is allowed, then the strategy set of these types of players is given by $\mathcal{P}_{\mathbf{d}}=\left\{\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{d}_{j: j \in N-\{i\}}\right\}$.

For completeness of our model, we make the following assumption: when a voter $i$ delegates her vote to another voter $j$, who votes for an alternative, say $a$, then $i$ 's vote also goes in favour of the same alternative, $a$. In any other case, e.g. if $j$ abstains or delegates, $i$ 's vote is cancelled (not counted). ${ }^{1}$

[^6]No result relies on this assumption. Furthermore, if voter $j \in N$ decides to vote and holds more than her own vote, we assume that she uses the same (voting) strategy for all the votes she possesses, e.g. if ten other voters have delegated to voter $j$, then $j$ casts all the eleven votes in favour of her chosen alternative. This assumption is made for simplicity, and our results do not depend on it.

When delegation is not allowed, a voter $i$ with type $t_{i}=I$ has a strategy $P_{i} \in \mathcal{P}_{I}=\mathcal{P}^{2}=$ $\{(\mathbf{a}, \mathbf{a}),(\mathbf{a}, \mathbf{b}),(\mathbf{a}, \mathbf{x}), \ldots\}$, where the chosen strategy of $i$ is conditioned to the state of nature, i.e. strategy ( $\mathbf{a}, \mathbf{b}$ ) implies that $i$ votes for $a$ in state $s_{a}$ and for $b$ in state $s_{b}$. When delegation is allowed, a voter $i$ with type $t_{i}=I$ has a strategy $P_{i} \in \mathcal{P}_{I, \mathbf{d}}=\mathcal{P}_{\mathbf{d}}{ }^{2}$. Essentially, what we do is to assume that informed voters can condition their action on the actual state of the world. This is to all effects equivalent to assuming that they observe the state of the world and choose an action afterwards, but it further allows us to conduct the formal analysis in a more efficient manner.

Ex-ante Strategies: When players need to choose a course of action for each possible type they are assigned before the types are drawn, the strategy for a voter $i$ is a four-component vector where the first, second, third and fourth component indicate the action of voter $i$ if her drawn type is $A, B, U$ or $I$, and this is denoted by

$$
\begin{equation*}
Q_{i} \in \mathcal{P}^{3} \times \mathcal{P}_{I} \tag{1}
\end{equation*}
$$

Recall that if the option of delegation is available, then $Q_{i} \in \mathcal{P}_{\mathbf{d}}^{3} \times \mathcal{P}_{I, \mathbf{d}}$.

Strategy profiles. After types are drawn, and provided they are publicly observed, we define as $P=\left(P_{1}, \ldots, P_{n}\right)$ an interim strategy profile of voters and as $\left(P_{i}^{\prime}, P_{-i}\right)$ the interim strategy profile that results when voter $i$ unilaterally deviates from $P_{i}$ to another interim strategy $P_{i}^{\prime}$. Similarly, for every $i \in N$, we define as $Q=\left(Q_{1}, \ldots, Q_{n}\right)$ an ex-ante strategy profile of voters and as ( $Q_{i}^{\prime}, Q_{-i}$ ) the ex-ante strategy profile that results when voter $i$ unilaterally deviates from $Q_{i}$ to another ex-ante strategy $Q_{i}^{\prime}$.

Payoffs: Depending on the state of nature, the utility for each type of voter is given by

$$
\begin{gather*}
u_{A}\left(a, s_{c}\right)= \begin{cases}1, & c \in\{a, b\} \\
0, & \text { otherwise. }\end{cases}  \tag{2}\\
u_{B}\left(b, s_{c}\right)= \begin{cases}1, & c \in\{a, b\}, \\
0, & \text { otherwise. }\end{cases}  \tag{3}\\
u_{I}\left(j, s_{c}\right)=u_{U}\left(j, s_{c}\right)= \begin{cases}1, & \text { if } j=c, \text { where } c \in\{a, b\}, \\
0, & \text { otherwise. }\end{cases} \tag{4}
\end{gather*}
$$

Observe that $A$ voters maximise their utility when candidate $a$ is elected regardless of what state is drawn. Similarly, $B$ voters maximise their utility when candidate $b$ is elected. Truth-seeking $I$ and $U$ voters have common values, that is, their utility is maximised when the best candidate is elected, i.e. their utility does not depend only on the identity of the candidate but also on the drawn state of nature.

Interim probability that alternative $a$ wins when types are publicly observed: Given a publicly observed realisation of types, an interim strategy profile $P$ and a state $s_{c}$, for $c \in\{a, b\}$, define the probability that alternative $a$ wins by

$$
\begin{equation*}
W\left(P, s_{c}\right) \tag{5}
\end{equation*}
$$

Ex-ante probability that alternative $a$ wins in state $s_{c}$ : Given an ex-ante strategy profile $Q$ and a state $s_{c}$, define the probability that alternative $a$ wins by

$$
\begin{equation*}
w\left(Q, s_{c}\right) . \tag{6}
\end{equation*}
$$

Moreover, given an ex-ante strategy profile $Q$, a state $s_{c}$, and a type $t_{i} \in T$, define the probability that the alternative $a$ wins conditional on player $i$ being assigned type $t_{i} \in T$ by

$$
\begin{equation*}
w_{i}\left(Q, s_{c}, t_{i}\right) . \tag{7}
\end{equation*}
$$

Definition 1 (Interim expected utilities when types are publicly observed). Given a strategy profile $P$, we define the expected utility of a voter $i \in N$ as,

$$
E U_{i}(P)= \begin{cases}p \cdot W\left(P, s_{a}\right) \cdot 1+(1-p) \cdot W\left(P, s_{b}\right) \cdot 1, & \text { if } t_{i}=A  \tag{8}\\ p \cdot\left(1-W\left(P, s_{a}\right)\right) \cdot 1+(1-p) \cdot\left(1-W\left(P, s_{b}\right)\right) \cdot 1, & \text { if } t_{i}=B \\ p \cdot W\left(P, s_{a}\right) \cdot 1+(1-p) \cdot\left(1-W\left(P, s_{b}\right)\right) \cdot 1, & \text { if } t_{i}=I \text { or } t_{i}=U\end{cases}
$$

Definition 2 (Ex-ante expected utilities:). Given an ex-ante strategy profile $Q$, we define the ex-ante expected utility of a voter $i \in N$ as,

$$
\begin{align*}
E U_{i}(Q)= & p_{A}^{i} \cdot\left(p \cdot w_{i}\left(Q, s_{a}, t_{A}\right)+(1-p) \cdot w_{i}\left(Q, s_{b}, t_{A}\right)\right) \\
& +p_{B}^{i} \cdot\left(p \cdot\left(1-w_{i}\left(Q, s_{a}, t_{B}\right)\right)+(1-p) \cdot\left(1-w_{i}\left(Q, s_{b}, t_{B}\right)\right)\right) \\
& +p_{I}^{i} \cdot\left(p \cdot w_{i}\left(Q, s_{a}, t_{I}\right)+(1-p) \cdot\left(1-w_{i}\left(Q, s_{b}, t_{I}\right)\right)\right) \\
& +p_{U}^{i} \cdot\left(p \cdot w_{i}\left(Q, s_{a}, t_{U}\right)+(1-p) \cdot\left(1-w_{i}\left(Q, s_{b}, t_{U}\right)\right)\right) . \tag{9}
\end{align*}
$$

Welfare benchmarks: Our welfare benchmark is the choice of a utilitarian informed planner, in the sense that the "efficient" outcome is the one that maximises the sum of (ex-post) individual utilities.

An efficient equilibrium is such that it yields the efficient outcome. When an equilibrium of a given mechanism leads to the efficient outcome with a higher probability than another equilibrium of the same or a different mechanism, we say that the former is better than the latter. We refer to the alternative that matches the state of the world as the "correct" one. Notice that the correct alternative does not always coincide with the efficient one due to the presence of partisan voters.

Definition 3 (Weakly dominated interim strategy when types are publicly observed.). When the realisation of types is publicly observed, an interim strategy $P_{i}$ is weakly dominated for voter $i$ if there exists a strategy $\hat{P}_{i}$ such that for all $P_{-i}$, we have

$$
\begin{equation*}
E U_{i}\left(\hat{P}_{i}, P_{-i}\right) \geq E U_{i}\left(P_{i}, P_{-i}\right) \tag{10}
\end{equation*}
$$

and, for at least one choice of $P_{-i}$, we have

$$
\begin{equation*}
E U_{i}\left(\hat{P}_{i}, P_{-i}\right)>E U_{i}\left(P_{i}, P_{-i}\right) . \tag{11}
\end{equation*}
$$

In this case, we say that $\hat{P}_{i}$ weakly dominates $P_{i}$.
Definition 4 (Dominance solvable (DS) game when types are publicly observed.). When types are publicly observable, the game is dominance solvable if for every type profile $t$, the following is true: For every pair of strategy interim profiles $P, P^{\prime}$ that survive the iterated elimination of weakly dominated strategies (IEWDS), it holds either that

$$
\begin{equation*}
W\left(P, s_{c}\right)=W\left(P^{\prime}, s_{c}\right), \text { for any } c \in\{a, b\} \tag{12}
\end{equation*}
$$

or IEWDS results in a unique strategy profile.

## Timing

1. Nature draws a state.
2. The types of voters are drawn. We investigate both of the cases that types are either: (i) public information, or (ii) private information.
3. Voters simultaneously decide on their strategies
4. Uncertainty is resolved and the voters' payoffs are realised.

## 4 Types are public information

In this section, we analyse the first interesting case: when the state of the world is unobservable, but voters have complete information on the types of other voters.

As discussed in the introduction, voting games admit Nash equilibria which are based on unreasonable beliefs on the part of voters: there could be Nash equilibria in this game where all voters vote for candidate $a$, regardless of the state, even though $a$ is the worst outcome for $B$ voters, and for $I$ and $U$ voters in state $s_{b}$. Multiple equilibria exist -some of which lead to inefficient outcomes- both when delegation is allowed and when it is not. ${ }^{1}$ We show (in Proposition 1) that when delegation of votes is permitted, the voting game is DS , and all $U$ voters adopt a similar behavior. Moreover, the DS outcome is also the efficient outcome. It is intuitive that partisan and $I$ voters would prefer to vote sincerely for their preferred alternatives- this is obvious when the strategy set does not contain delegation. When it does, intuitively partisans gain nothing by delegating and $I$ voters could only do worse by delegation. Given this, $U$ voters lose nothing by delegating to an $I$ voter. This is what the DS outcome captures. Note too that in the reduced game the strategy of delegating to $I$ voters is a natural focal point - however, it is important that voters understand that $I$ voters will play their undominated strategy. Moreover, it is a "symmetric" (upto the identity of the delegatee) strategy, which could be argued to have lower cognitive costs. In contrast, the game without delegation is not DS (Proposition 2). As a result, reaching the efficient pure strategy Nash equilibria (PSNE) is not a straightforward task: it requires a lot of coordination among the $U$ voters as efficient equilibria are in type-asymmetric strategies. Moreover, other types of equilibria that require less coordination (type-symmetric PSNE or mixed strategy Nash equilibria) exist, but they frequently lead to inefficient outcomes.

We first present a simple example to build intuition for the proof.
Consider an election where there are three $A$ voters, one $B$ voter, one $I$ voter and five $U$ voters. Assume first that delegation of votes is allowed. We show that the game is DS by IEWDS. It is easy to see that $I$ voters have a weakly dominant strategy $(\mathbf{a}, \mathbf{b})$ : such a strategy increases the probability that the winning alternative is the correct one for all profiles. Consider the profile where $A$ voters choose a, $B$ voters choose $\mathbf{b}$, two $U$ voters choose $\mathbf{b}$ and three choose $\mathbf{x}$. Then, the $I$ voter is pivotal and clearly $(\mathbf{a}, \mathbf{b})$ ensures that the correct candidate wins in both states. The strategy ( $\mathbf{a}, \mathbf{a}$ ) implies that $a$ wins in both states while ( $\mathbf{b}, \mathbf{b}$ ) leads to candidate $b$ winning in both states. Any strategy involving abstention in any state will lead to a tie in that state. Finally, if the informed voter delegates his vote to another voter, then the wrong candidate wins in at least one state. For this profile the strategy ( $\mathbf{a}, \mathbf{b}$ ) is a Unique Best Response (UBR), hence it is a dominant strategy.

We continue by showing that the $A$ and $B$ voters have weakly dominant strategies a and $\mathbf{b}$ respectively. Consider an $A$ voter. By choosing a, an $A$ voter always weakly increases the probability of obtaining $a$. Moreover, suppose two of the $U$ voters choose $\mathbf{b}$, one chooses a and the other two abstain, $I$ voters choose their dominant strategy, the other $2 A$ voters choose a while the $B$ voter chooses $\mathbf{b}$. Then, choosing a creates a tie between $a$ and $b$ in state $s_{b}$ and in state $s_{a}$ the outcome is $a$-this yields a strictly higher payoff than any of the other strategies, including delegation to $B$ and $I$ voters.

Consider now a $B$ voter. Let three $U$ voters choose $\mathbf{b}$, two $U$ voters abstain and all $A$ voters choose a, while $I$ voters play their dominant strategy. Then in state $s_{a}$ the $B$ voter can create a tie by choosing strategy $\mathbf{b}$, while in state $s_{b}$ the outcome is $b$. Strategy $\mathbf{b}$ is strictly better than other

[^7]strategies including delegation to $A$ or $I$ voters in this profile. Hence, we can eliminate all strategies except a for $A$ voters, all strategies except $\mathbf{b}$ for all $B$ voters, all strategies except ( $\mathbf{a}, \mathbf{b}$ ) for $I$ voters. Now we turn to the $U$ voters.

In the reduced game, consider an individual $U$ voter. Suppose out of the other $U$ voters two voters choose $\mathbf{b}$ and two choose $\mathbf{x}$, then the $I$ voter is always decisive, without $i$, so $a$ wins in state $s_{a}$ and $b$ wins in state $s_{b}$. This remains the same if $i$ chooses to delegate to the $I$ voter, or to abstain. Now if $i$ chooses a then a tie is created in state $s_{b}$ and if $i$ chooses $\mathbf{b}$ then a tie is created in state $s_{a}$. Both choices are strictly worse than choosing abstention or delegation. If three other $U$ voters choose $\mathbf{b}$ and one abstains then in state $s_{a}$ there is a tie with four votes each for $a$ and $b$. In state $s_{b}, b$ gets a minimum of five votes while $a$ gets a maximum of three, excluding voter $i$. If $i$ chooses $\mathbf{x}$ then in state $s_{a}$, there is a tie while if $i$ chooses delegation, $a$ wins, and in state $s_{b}, b$ always wins. Therefore, delegation is a strictly better response than abstention. In all profiles, delegation is at least as good as the other three strategies. This establishes that the game is DS and all $U$ voters prefer to delegate their votes to $I$ voters.

Now consider the variation of the game in which delegation is not allowed. The majority of voters are truth-seeking voters. Hence, the efficient PSNE is obtained when the correct candidate is chosen. Using IEWDS we can arrive at the reduced game where $i$ voters for $i=\{A, B\}$ choose $\mathbf{a}, \mathbf{b}$ respectively, $I$ voters choose $(\mathbf{a}, \mathbf{b})$ (the same argument applies since we did not use profiles where $U$ voters choose $\mathbf{d}$ to construct the strict best response profiles for $A, B, I$ voters). Consider the profile where two of the $U$ voters choose $\mathbf{b}$ and the other three $U$ voters abstain, while the $I$ voter chooses ( $\mathbf{a}, \mathbf{b}$ ). The outcome is $a$ in state $s_{a}$ and $b$ in state $s_{b}$. Note that each of the $U$ voters has a unique best response, of playing $\mathbf{a}, \mathbf{b}, \mathbf{x}$ respectively. Therefore the game is not DS.

The correct outcome is therefore reached in the reduced game when the $U$ voters exactly neutralise the partisan voters and the rest abstain -this allows informed voters to decide the election. This is the unique equilibrium in which the correct outcome is reached. However, this equilibrium can only occur with extreme coordination among the $U$ voters, and it is easy to make mistakes.

Let us turn to symmetric PSNE. Assume that all five $U$ voters vote for the same candidate. If they vote for candidate $a$, regardless of the state of nature, the election outcome is always candidate $a$ there are eight out of ten votes for candidate $a$ - hence no single voter is pivotal in either state, so this is a PSNE. If all $U$ voters vote for $b$ then there are six out of ten votes for $b$. In state $s_{b}$ there are seven votes for $b$ so no single voter is pivotal. In state $s_{a}$ there four votes for $a$ and six votes for $b$. No single voter is pivotal so this is an equilibrium. Finally, if all $U$ voters abstain then the outcome is $a$ in both states. Each of these is inefficient. If mixed symmetric equilibria exist, we have a positive probability that the $U$ voters vote for either $a$ or $b$. Let $q_{q}, q_{b} \in(0,1)$ be the probability of playing $\mathbf{a}, \mathbf{b}$ in the mixed strategy equilibrium then with a probability $q_{a}^{5}>0$ the outcome is always $a$ regardless of state, with a probability $q_{b}^{5}>0$ the outcome is always $b$ regardless of the state.

There are two reasons why we might want the game to be DS. First, because, as we see in the example, this property rules out many other equilibria which are not efficient and plausible in the game with delegation.Second, without this property, reaching the efficient equilibria requires a fine balancing
act, where $U$ voters must use type-asymmetric strategies, when delegation is not in the strategy set. The chances of coordination are low and it is highly likely that mistakes are made when delegation is not allowed. Next, we extend the intuitions presented in this example to more general contexts.

### 4.1 With delegation

Proposition 1 below shows that the election game with delegation is dominance solvable, and characterizes the DS outcome.

Proposition 1. Assume that $n_{I} \geq 1, \max \left(n_{A}+n_{I}, n_{B}+n_{I}\right)<\left\lceil\frac{n}{2}\right\rceil, n_{U} \geq n_{I}+\left|n_{A}-n_{B}\right|+1$, then the election game with delegation is dominance solvable. The DS outcome is the efficient outcome.

The proof of this proposition is in the Appendix.
The dominant strategies for $A, B$ and $I$ voters are $\mathbf{a}, \mathbf{b},(\mathbf{a}, \mathbf{b})$ respectively, while the strategy for voter $i$ of type $U$ in the DS equilibrium is $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$. Given the conditions on numbers of each type, this profile ensures that candidate $a$ wins in state $s_{a}$ and candidate $b$ wins in state $s_{b}$. If these conditions are not satisfied, e.g if $\max \left(n_{A}+n_{I}, n_{B}+n_{I}\right)>\left\lceil\frac{n}{2}\right\rceil$, then there does not exist profiles where a $U$ voter is pivotal in both states. Suppose e.g. that $\max \left(n_{A}+n_{I}, n_{B}+n_{I}\right)=n_{A}+n_{I} \geq\left\lceil\frac{n}{2}\right\rceil$, then there does not exist a profile where a $U$ voter is pivotal in state $s_{a}$. In this case, the strategy $\mathbf{b}$ is equivalent to delegating to an informed voter in terms of maximising the chance of obtaining the correct outcome. The two conditions together ensure that there are profiles where a $U$ voter is pivotal on every pair of strategies involving delegation to an informed voter -e.g. a $U$ voter must prefer delegation to an informed voter, $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, than any other strategy $\mathbf{a}, \mathbf{b}, \mathbf{x}$ and $P_{i}=\mathbf{d}_{j: P_{j}=b}, P_{i}=\mathbf{d}_{j: P_{j}=a}$.

Next, we move to the game without delegation and show that it is not DS. We will consider equilibria in undominated strategies.

### 4.2 Without delegation

Proposition 2. Assume that $n_{I} \geq 1, \max \left(n_{A}+n_{I}, n_{B}+n_{I}\right)<\left\lceil\frac{n}{2}\right\rceil, n_{U} \geq n_{I}+\left|n_{A}-n_{B}\right|+1$ and that delegation is not allowed. (A) The game is not dominance solvable. (B) There exist multiple efficient equilibria in asymmetric strategies for $U$ voters where the outcome is a in state $s_{a}$ and $b$ in state $s_{b}$. Assume $n_{U} \geq n_{I}+\left|n_{A}-n_{B}\right|+2$. If $n_{I}>\left|n_{A}-n_{B}\right|$, there exists an efficient equilibrium where all $U$ voters abstain. If $n_{I}<\left|n_{A}-n_{B}\right|$ then there exists an inefficient equilibrium where all $U$ voters abstain and the correct outcome is not chosen in at least one state. There also exist inefficient symmetric PSNE where the outcome is $i \in\{a, b\}$ regardless of the state.

The proof of this proposition can also be found in the Appendix.
Overall, we observe that the efficient outcome is more "likely" (in the sense of DS of the game) to be reached in the game with delegation than in the game without delegation. The reason is that it is much easier for voters to coordinate their strategies in the game with delegation. The strategy is symmetric and requires all $U$ voters to choose delegation to an informed voter: In contrast, if the game is not DS,
the efficient equilibrium requires high level of coordination as there are multiple such equilibria (any permutation of an equilibrium strategy profile across $U$ voters is an equilibrium). Moreover there are also multiple inefficient equilibria even if we restrict ourselves to PSNE -all outcomes are possible in this game.

Unlike much of the literature on information aggregation (e.g. Feddersen and Pesendorfer, 1996), we do not need to assume large elections for this result. With delegation, when types are known as e.g. in committee elections, delegation ensures that the efficient outcome is reached. When delegation of votes is not allowed it is "likely" that inefficient outcomes prevail.

## 5 Types are Private Information

In this section, we investigate the case where each voter's type is her private information. Similar to Section 4 above, we show that there exists a better equilibrium -i.e. an equilibrium in which the probability of electing the efficient alternative is higher- when the voters have the option to delegate their voting rights, compared to any equilibrium of the case in which there is no option of delegation.

The road-map for the proof follows. First, we construct a new auxiliary game with the following two features: $(i)$ all voters are of type $U$ and (ii) after voters choose their actions, nature changes them to other actions with some positive probability. Then, we show that, for given mappings between the parameters of the original model and the new one, the sets of undominated equilibria of the two games are equivalent in terms of probability of implementation of the correct alternative. Finally, we prove that the best equilibrium of the new game with delegation -in terms of likelihood of implementation of the efficient alternative- is always at least as good -and sometimes strictly better- compared to the best equilibrium of the new game without delegation, and hence delegation is welfare improving in the original game too.

### 5.1 Auxiliary game

Consider the original game that we study, with the following modification. All the voters $N=\{1, \ldots, n\}$ are of the same type and this is common knowledge. More specifically, they are all $U$ voters, i.e. all of them wish that the correct alternative is chosen, but they do not observe the state of nature. Each voter $i \in N$ selects a strategy $\bar{P}_{i} \in \mathcal{P}$ if there is no delegation, and a strategy $\bar{P}_{i} \in \mathcal{P}_{\mathbf{d}}$ if there is delegation. After strategies are chosen, they may change with some exogenously given probabilities. We consider exogenous events which change voter $i$ 's strategy to one of the following: to vote for alternative $a$, to vote for alternative $b$, or, to vote for alternative $a$ in state $s_{a}$ and for alternative $b$ in state $s_{b}$.

Definition 5 (Exogenous shock). There exists a probability $q_{i}$ that the strategy of $\bar{P}_{i}$ of voter $i$ does not change at all, a probability $q_{\boldsymbol{a}}^{i}$ that $\bar{P}_{i}$ changes to $\boldsymbol{a}$, a probability $q_{\boldsymbol{b}}^{i}$ that $\bar{P}_{i}$ changes to $\boldsymbol{b}$, and a probability $q_{(a, b)}^{i}$ that $\bar{P}_{i}$ changes to ( $\left.\boldsymbol{a}, \boldsymbol{b}\right)$ (i.e. to $\boldsymbol{a}$ when the state is $s_{a}$ and to $\boldsymbol{b}$ when the state is $s_{b}$ ), with

$$
\begin{equation*}
q^{i}+q_{a}^{i}+q_{b}^{i}+q_{(\boldsymbol{a}, \boldsymbol{b})}^{i}=1 . \tag{13}
\end{equation*}
$$

Definition 6. For every strategy profile $\bar{P}$ of the new game define the ex-ante strategy profile $Q^{\bar{P}}$ of the original game which prescribes to every player the following behavior: use action $\boldsymbol{a}(\boldsymbol{b})$ when assigned type $t_{A}\left(t_{B}\right)$, action $(\boldsymbol{a}, \boldsymbol{b})$ when assigned type $t_{I}$, and action $\bar{P}_{i}$ when assigned type $t_{U}$.

We consider both pure and mixed strategies, For simplicity -with slight abuse in notation- we use the same symbols for both kinds of strategy profiles.

Definition 7. Given a strategy profile $\bar{P}$ of the new game and a state $s_{c}$, define the probability that alternative a wins by

$$
\begin{equation*}
\bar{W}\left(\bar{P}, s_{c}\right) \tag{14}
\end{equation*}
$$

and the probability that alternative a wins conditional on player i's action not having changed by nature, by

$$
\begin{equation*}
\bar{W}_{i}\left(\bar{P}, s_{c}\right) . \tag{15}
\end{equation*}
$$

Lemma 1. Assume that

$$
\begin{aligned}
q_{a}^{i} & =p_{A}^{i}, \\
q_{b}^{i} & =p_{B}^{i}, \\
q_{(a, b)}^{i} & =p_{I}^{i}, \\
q^{i} & =p_{U}^{i},
\end{aligned}
$$

for every $i \in N$. Then, $\bar{P}^{*}$ is an equilibrium of the new game, if and only if $Q^{\bar{P}^{*}}$ is an equilibrium of the original game. Moreover,

$$
\begin{equation*}
\bar{W}\left(\bar{P}^{*}, s_{c}\right)=w\left(Q^{\bar{P}^{*}}, s_{c}\right) \tag{16}
\end{equation*}
$$

for every $c \in(a, b)$.
Proof. Due to the finiteness of the number of players, and of the action and the type space, an equilibrium (possibly in mixed strategies) is guaranteed to exist both in the new game and in the original one. Consider an equilibrium $\bar{P}^{*}$ of the new game and observe that the expected utility of voter $i$ is given by

$$
\begin{equation*}
q^{i} \cdot\left(p \cdot \bar{W}_{i}\left(\bar{P}^{*}, s_{a}\right)+(1-p) \cdot\left(1-\bar{W}_{i}\left(\bar{P}^{*}, s_{b}\right)\right)\right)+\left(1-q^{i}\right) \cdot h\left(\bar{P}_{-i}^{*}\right), \tag{17}
\end{equation*}
$$

where $h\left(\bar{P}_{-i}^{*}\right)$ is the expected utility of $i$ conditional on her action having changed exogenously. Since $\bar{P}^{*}$ is an equilibrium, $\bar{P}_{i}^{*}$ is the strategy that maximises

$$
\begin{equation*}
p \cdot \bar{W}_{i}\left(\bar{P}^{*}, s_{a}\right)+(1-p) \cdot\left(1-\bar{W}_{i}\left(\bar{P}^{*}, s_{b}\right)\right) . \tag{18}
\end{equation*}
$$

Now take the ex-ante strategy profile $Q^{\bar{P}^{*}}$ of the original game and notice that the expected utility of player $i$, conditional on $i$ being assigned type $U$, is given by

$$
\begin{equation*}
p \cdot w_{i}\left(Q^{\bar{P}^{*}}, s_{a}, t_{U}\right)+(1-p) \cdot\left(1-w_{i}\left(Q^{\bar{P}^{*}}, s_{b}, t_{U}\right)\right) . \tag{19}
\end{equation*}
$$

However, notice that, given our parametric assumptions, it is the case that

$$
\begin{equation*}
\bar{W}_{i}\left(\bar{P}^{*}, s_{c}\right)=w_{i}\left(Q^{\bar{P}^{*}}, s_{c}, t_{U}\right) \tag{20}
\end{equation*}
$$

for every $i \in N$ and $c \in\{a, b\}$. Therefore, if $\bar{P}_{i}^{*}$ is maximising (17), then $Q^{\bar{P}_{i}^{*}}$ maximises (19) and vice versa. Since $Q^{\bar{P}_{i}^{*}}$ maximises (17), and also prescribes dominant actions for the cases in which the voter is assigned a type different than type $U$, it follows that $Q^{\bar{P}^{*}}$ is an equilibrium of the original game.

Proposition 3. The best undominated equilibrium of the original game with delegation is at least as good as the best undominated equilibrium of the original game without delegation.

Proof. Let $\hat{\bar{P}}=\left[\bar{P}_{i}\right]_{i \in N}$ and $\hat{\bar{P}}^{\mathbf{d}}=\left[\bar{P}_{i}^{\mathbf{d}}\right]_{i \in N}$ be the strategy profile that maximises voters' ex-ante expected utilities in our new game (without and with the strategy option of delegation, respectively) such that, for every voter $i \in N$, we have $\bar{P}_{i} \in \mathcal{P}^{3} \times \mathcal{P}_{I}$ and $\bar{P}_{i}^{\mathbf{d}} \in \mathcal{P}_{\mathbf{d}}^{3} \times \mathcal{P}_{I, \mathbf{d}}$. Note that, since the strategy set $\mathcal{P}$ is a subset of $\mathcal{P}_{\mathbf{d}}$, every strategy profile that is feasible in the new game without delegation is also feasible in the new game with delegation.

Then, observe that the strategy set $\mathcal{P}_{\mathbf{d}}$ (allowing the additional strategy of delegation), the ex-ante expected utility maximiser $\hat{\bar{P}}^{\text {d }}$ offers a weakly higher expected utility to each voter $i \in N$ than the ex-ante expected utility maximiser $\hat{\bar{P}}$,

$$
\begin{equation*}
E U_{i}\left(\hat{\bar{P}}^{\mathbf{d}}\right) \geq E U_{i}(\hat{\bar{P}}) \tag{21}
\end{equation*}
$$

By McLennan (1998), and since our new game is a common value game, we get that each one of the strategy profiles $\hat{\bar{P}}$ and $\hat{\bar{P}}^{\mathrm{d}}$ is also an ex-ante equilibrium of the new game for the case of without delegation and for the case of with delegation, denoting this profile update as $\hat{\bar{P}}^{*}$ and $\hat{\bar{P}}^{\mathbf{d} *}$, respectively. Thus, we can say that the undominated equilibrium with delegation $\hat{\bar{P}}^{\mathrm{d} *}$ offers at least the same ex-ante expected utility to each voter as the ex-ante expected utility achieved in the undominated equilibrium without delegation $\hat{\bar{P}}^{*}$, i.e,

$$
\begin{equation*}
E U_{i}\left(\hat{\bar{P}}^{\mathbf{d} *}\right) \geq E U_{i}\left(\hat{\bar{P}}^{*}\right) \tag{22}
\end{equation*}
$$

For each one of the strategy profiles $\hat{\bar{P}}^{*}$ and $\hat{\bar{P}}{ }^{\mathbf{d} *}$ of the new game, define an ex-ante strategy profile $Q^{\hat{P}^{*}}$ and $Q^{\hat{P}^{\mathrm{d}} *}$ in the original game, respectively, as described in Definition 6. Then observe that, for every voter $i \in N, p_{A}^{i}=q_{\mathbf{a}}^{i}, p_{B}^{i}=q_{\mathbf{b}}^{i}, p_{I}^{i}=q_{(\mathbf{a}, \mathbf{b})}^{i}$ and $p_{U}^{i}=q^{i}$. Using Lemma 1 and inequality (23), we get

$$
\begin{equation*}
E U_{i}\left(Q^{\hat{P^{\mathrm{d}}}}\right) \geq E U_{i}\left(Q^{\hat{P}^{*}}\right) \tag{23}
\end{equation*}
$$

Thus, it follows that the undominated equilibrium of the original game with delegation that maximises the probability of electing the correct alternative is weakly better than the undominated equilibrium of the original game without delegation that maximises the probability of electing the correct alternative.

Finally, notice that if an equilibrium of a mechanism implements the correct alternative with higher probability than another equilibrium of the same or of a different mechanism, then it also implements the efficient alternative with higher probability. This is so because for all type draws such that the truth-seeking voters are not pivotal as a group (i.e. their choice cannot affect the outcome when players use undominated strategies), then both equilibria lead to the efficient outcome with certainty and to the correct outcome with the same probability. Observe that in the remaining type draws the correct alternative coincides with the efficient alternative. Therefore, since one equilibrium implements the correct alternative with higher probability than the other unconditionally, it must be the case that it implements the correct alternative -and hence also the efficient alternative- with higher probability conditional on the truth-seeking types being decisive. Since the efficient alternative is implemented with higher probability under the former equilibrium than the latter conditional on the group of truth-seeking voters being decisive, and with equal probability conditional on truth-seeking voters not being decisive, it leads to the implementation of the efficient alternative with higher probability unconditionally.

The intuition behind Proposition 3 is the following: consider an incomplete information game where the types of voters are private information. In any undominated equilibrium of the interim game, the $A, B, I$ voters must be playing their strategies $\mathbf{a}, \mathbf{b},(\mathbf{a}, \mathbf{b})$ respectively. Assume that in the interim game without delegation there exists an equilibrium where $U$ voters manage to coordinate in expectation, so that the probability that voters with the highest probability $p_{I}^{i}$ decide the election is positive in this equilibrium. Then adding a strategy of delegation can only make the $U$ voters better off as they can delegate to the set of voters who have the highest $p_{I}^{i}$, across all $i \in N$.

Notice that the proof strategy that we have employed here is not only valid for the case in which truth-seeking voters are either perfectly informed or fully uninformed. The result also generalizes to the case in which truth-seeking voters are partially and heterogeneously informed (i.e. the quality of information held by each truth-seeking voter is allowed to differ). As explained in the introduction, in such a setting delegation introduces an additional trade-off. Less informed voters by delegating their votes to more informed agents, enhance the electoral impact of more informed voters (which is desirable), but also remove valuable pieces of information from the aggregation procedure (which is undesirable). This observation is, arguably, of independent interest, since the literature has paid special attention to this trade-off of delegation and tried to assess its sign under alternative behavioral assumptions (see, Kahng et al., 2018). Our work demonstrates rational individuals who choose behaviors endogenously to maximise their individual welfare, can gain from being allowed to delegate votes, despite the partially adverse consequences of vote concentration. ${ }^{1}$

[^8]So far we showed that delegation is weakly welfare improving when welfare is measured by the probability of implementing the efficient outcome. When there is full information on who the informed voters are, the election game is dominance solvable and the outcome is efficient as compared to the game without delegation where efficient equilibria are fragile in the sense that they require a lot of coordination between uninformed voters. When there is incomplete information on the types of voters, we showed that the best (highest welfare) equilibrium of the game with delegation is weakly better than the best equilibrium in the game without delegation. In the next section, we show that we can do even better with delegation - we present a mechanism by which we can ensure that, when the number of voters is sufficiently large, there exists an equilibrium in which all uninformed voters delegate their votes to the informed truth-seeking voters. This maximises the probability of reaching the efficient outcome even ex-post, relative to the game without delegation, just as in the complete information case.

## 6 A mechanism for secure transfers of votes

Can we guarantee that delegation aggregates information better even ex-post, than direct voting? We show that the answer is yes, with a mechanism that induces indirect delegation in a restricted model for sufficiently large $n$.

In this section, we propose a mechanism that will guarantee that any delegated votes go only to informed voters, even when the types of voters are not complete information. We show the existence of an equilibrium where all $U$ voters drop their votes into a "box" from which only informed voters pick up an extra vote. The mechanism we propose should not be taken too literally - we only show that it is possible to construct a mechanism that can do the job but it is not necessarily the most practical one. A general description of the mechanism we propose is the following:

## The Mechanism.

- Voters who wish to delegate leave their vote in a box.
- Voters who would like additional votes then pick up one and only one vote from the box.
- Each voter who picks a vote agrees to pay a fixed penalty $\rho>0$ in case she votes for the losing candidate (we assume that votes are observed by the penalty-issuing entity).

Admissible parameterisations. Consider the original game that we study, with the following modifications: the electorate is $N$, the number of the three sub-electorates (informed, uninformed and partisans) is fixed and known. While there is private information about types, the size of each sub-electorate is known. There are $n_{U}$ uninformed voters, $n_{I}$ informed voters and $n_{P}$ partisan voters with $n_{I}<n_{U}$. W.l.o.g. assume that $n_{P}$ is an even number, with $n_{P}=h_{I} \times n_{I}$, where $h_{I}>1$. Each partisan voter supports either alternative with equal probability. We also assume that the two states of nature are equally likely. ${ }^{1}$

[^9]Equilibrium analysis. We will show that there is a Bayesian equilibrium in this game where only uninformed voters leave votes in the box and only informed voters pick up votes. We investigate the conditions under which equilibria of this form exist.

It is easy to see that informed and partisan voters never have incentives to place their votes in the box, and uninformed voters never have incentives to pick up votes given the beliefs that other uninformed voters are expected to leave votes in the box and only informed ones pick them up. We therefore need to ensure that there exist values of the penalty $\rho$ that lead to the desirable redistribution of votes from uninformed to informed voters, without the risk of partisan voters gaining extra power. If it is incentive compatible for informed voters to pick up additional votes and for partisans not to, then uninformed voters prefer to delegate their votes than keep them. Below we show that we can choose the penalty $\rho$ in such a way that partisans are disincentivized to pick up any additional votes while informed voters are encouraged to do so.

To conduct our equilibrium analysis, we see the game from the perspective of a $t$ voter who believes that the remaining players behave as follows: all informed voters vote for the correct alternative and pick up an additional vote from the box, all uninformed put their votes in a box and all partisans vote for their preferred alternative. We define as $P_{\text {tie }}^{(C, t)}$ the probability that an extra vote of our $t$ voter for the correct alternative creates or breaks a tie, and as $P_{t i e}^{(W, t)}$ the probability that an extra vote of our $t$ voter for the wrong alternative creates or breaks a tie. Similarly, we define as $P_{\text {lose }}^{(C, t)}$, the probability of the correct alternative losing, conditional on being a type- $t$ voter, picking up an additional vote and voting for the correct alternative; and as $P_{\text {lose }}^{(W, t)}$ the probability of the wrong alternative losing, conditional on being a type- $t$ voter, picking up an additional vote and voting for the wrong alternative.

Denote the expected utility of a voter $i$ when picking up an extra ballot and casting it for her preferred alternative by $U^{+}(i)$, and the expected utility from not picking up an extra ballot and simply voting for her preferred alternative by $U(i)$. Then, their difference is given by

$$
\begin{array}{rll}
\Delta U(i) & =U^{+}(i)-U(i) \\
& = \begin{cases}\frac{1}{2} \cdot P_{\text {tie }}^{\left(C, t_{i}\right)}-\rho \cdot P_{\text {lose }}^{\left(C, t_{i}\right)}, & \text { if } t_{i}=I, \\
\frac{1}{2} \cdot\left(\frac{1}{2} \cdot P_{\text {tie }}^{\left(C, t_{i}\right)}-\rho \cdot P_{\text {lose }}^{\left(C, t_{i}\right)}\right)+\frac{1}{2} \cdot\left(\frac{1}{2} \cdot P_{\text {tie }}^{\left(W, t_{i}\right)}-\rho \cdot P_{\text {lose }}^{\left(W, t_{i}\right)}\right), & \text { if } t_{i}=A \text { or } B .\end{cases} \tag{24}
\end{array}
$$

The first expression is the expected extra utility of the informed truth-seeking voter. When she creates a tie, she gets a utility $\frac{1}{2} 1+\frac{1}{2} 0=\frac{1}{2}$ and when she breaks a tie she gets a utility $1-\left(\frac{1}{2} 1+\frac{1}{2} 0\right)=\frac{1}{2}$. When the correct alternative loses, she pays the price $\rho$. The second expression is the utility of a partisan. Given that the probability of each state is $1 / 2$, a partisan voter is voting for the correct alternative half the time and the wrong alternative half the time. In the equilibrium an informed voter is supposed to pick up one delegated vote from the box.

Thus we need to set the first part of (24) to be at least equal to 0 . Then, we get

$$
\begin{equation*}
\frac{1}{2} \cdot P_{\text {tie }}^{(C, I)}-\rho \cdot P_{\text {lose }}^{(C, I)} \geq 0 \Rightarrow \rho \leq \frac{P_{\text {tie }}^{(C, I)}}{2 \cdot P_{\text {lose }}^{(C, I)}} \tag{25}
\end{equation*}
$$

In equilibrium, type- $A$ partisans (and, similarly, type- $B$ partisans) should not want to pick up a vote from the box, i.e. the second part of (24) must be less than 0 , that is

$$
\begin{equation*}
\frac{1}{2} \cdot\left(\frac{1}{2} \cdot P_{\text {tie }}^{(C, A)}-\rho \cdot P_{l o s e}^{(C, A)}\right)+\frac{1}{2} \cdot\left(\frac{1}{2} \cdot P_{\text {tie }}^{(W, A)}-\rho \cdot P_{\text {lose }}^{(W, A)}\right)<0 \tag{26}
\end{equation*}
$$

Whenever $\rho$ satisfies inequality (25) then the first part of inequality (26) is also positive, which implies $\rho$ can be increased until (25) is satisfied with equality. If we can find a $\rho$ that satisfies (25) with equality and also satisfies (26) then an equilibrium such that only uninformed voters leave their votes in the box and only informed ones pick them up exists. The maximum price that $\rho$ can take so an informed voter picks up a vote should be the one that makes her indifferent between picking a vote or not, which is $\rho=\frac{P_{t i e}^{(C, I)}}{2 \cdot P_{\text {lose }}^{(C, I)}}$. By substituting this value of $\rho$ into (26), we get that (26) is satisfied as long as the following holds,

$$
\begin{equation*}
\frac{P_{t i e}^{(C, A)}+P_{t i e}^{(W, A)}}{P_{t i e}^{(C, I)}}<\frac{P_{\text {lose }}^{(C, A)}+P_{l o s e}^{(W, A)}}{P_{\text {lose }}^{(C, I)}} \tag{27}
\end{equation*}
$$

Proposition 4. For any admissible $\left(h_{I}, h_{U}\right)$, there exists $\tilde{n}>0$ such that the described mechanism admits an equilibrium where uninformed voters transfer their votes only to informed voters for all $n \geq \tilde{n}$.

The idea of this proof is as follows. Given our parametric assumptions, the right hand side trivially diverges to infinity as $n \rightarrow \infty$, by a standard Central Limit Theorem argument. ${ }^{1}$ To show that inequality (27) holds, we then turn to the second ratio and show that

$$
\begin{equation*}
\frac{P_{t i e}^{(C, A)}+P_{t i e}^{(W, A)}}{P_{t i e}^{(C, I)}}=2 \tag{28}
\end{equation*}
$$

To do this, it is enough to investigate only the cases where either a partisan is pivotal (case 1) or an informed voter is pivotal (case 2). You can find the analysis of both these cases in page 33 of the Appendix.

The intuition behind Proposition 4 is as follows. Note that in any state the expected number of partisans voting with the informed voters is half of the total partisans. Now suppose that a partisan voter deviates to pick up a vote. His incentive to do so is based on the chance that he tilts the outcome towards the preferred candidate. As the society grows large, the probability

[^10]that he is pivotal becomes very small. But, because of the presence of informed voters, the overall probability of his preferred candidate losing is always sizable (in fact, why? it goes to $1 / 2$ as the size of the electorate increases). Therefore, a partisan voter has little to gain from picking up a vote, and much to lose. On the contrary, informed truth-seeking voters have a tiny gain from picking up a vote (since their pivotal probability also converges to zero as society grows), but also a vanishing cost. Indeed, since the probability that the correct outcome prevails converges to one, it follows that the probability that they receive a penalty goes to zero. Hence, for a properly configured penalty size, only such voters have incentives to pick up votes, and their overall electoral impact doubles compared to simple voting. Note that the size of the penalty to ensure that this happens goes to 0 as $n \rightarrow \infty$.

## 7 Robustness

We proved Proposition 4 for the special case of the two states having equal probability of occurring and for the probability of each partisan type being equally probable. It is easy to show that the same result holds for any probability $p$ of state $s_{a}$. It is more difficult to show analytically that the result holds for probabilities on partisan voter types $A, B$ that are different from $\frac{1}{2}$ - however Section 7.1 below shows that this is true, using numerical simulations around $p=\frac{1}{2}$.

## 7.1 $A$ and $B$ voters occur with different probability

In the previous section, we proved the desired inequality (27) assuming that a partisan could be either an $A$ or a $B$ voter with an equal probability of $\frac{1}{2}$. In this section, we relax this symmetry assumption and numerically show that we still get that the desired inequality (27) holds for sufficiently large societies. To do this, we assume without loss of generality that $a$ is the correct alternative and we define as $p$ the probability that a random partisan voter is a supporter of $a$. Then, with respect to this probability $p$, we calculate $P_{\text {tie }}^{(C, A)}, P_{\text {tie }}^{(W, A)}, P_{\text {tie }}^{(C, I)}, P_{\text {lose }}^{(C, A)}, P_{\text {lose }}^{(W, A)}$, and $P_{\text {lose }}^{(C, I)}$, and use these formulas to numerically show the desired inequality.

To do this, we first define the following ratio

$$
\mathcal{R}=\frac{\frac{P_{\text {tie }}^{(C, A)}+P_{\text {tie }}^{(W, A)}}{P_{\text {tie }}^{(C, I)}}}{\frac{P_{\text {lose }}^{(C, A)}+P_{\text {lose }}^{(W, A)}}{P_{\text {lose }}^{(C l l}}}
$$

and then we compute it as a function of the cardinality of the set of partisans, considering alternative parametrizations to illustrate that $\mathcal{R}<1$ even for values of $p$ other than $\frac{1}{2}$. To simplify the input in ratio $\mathcal{R}$, we set $n_{U}>n_{I}=h \cdot n_{P}$, where $h<1$. Our numerical results are displayed in Figure 2 and the computation details of the probabilities' formulas can be found in the Appendix.


Figure 2: Ratio $\mathcal{R}$ converges to 0 for large number of partisans: $(i)$ even when the number of informed is half of the number of partisans and the probability $p$ (of being an $A$ voter) is relaxed to values around 0.5 ; (ii) even when the number of informed is only the $1 / 5$ of the number of partisans and the probability $p$ (of being an $A$ voter when partisan) is relaxed to values around 0.5 .

### 7.2 Partially and Heterogenously Informed Voters

We have mentioned earlier that although our benchmark model is one with either fully informed or fully uninformed independent voters, the results do apply to a more general model with partially and heterogenously informed voters, both for the complete and the incomplete information game. In this section, we present a simple example to illustrate the intuition for complete information games.

Assume $p=\frac{1}{2}$. Consider a set of 9 independent voters (we assume partisans play their dominant strategies so that the reduced game is one with common values): Voter 1 who is best informed with precision (probability of inferring the correct state of nature) $P_{1}$, voters $2,3,4,5$ who are all moderately
informed with precision $P=0.73$ and uninformed voters who have precision 0.5 . By Nitzan and Paroush (1982) the weighting votes that maximize information aggregation are (1) When $P_{1} \leq 0.88$, Voters 1-5 have equal voting weights: i.e. each has one vote while the voters $6-9$ abstain. (2) When $P_{1} \geq 0.98$, then all voters except voter 1 should abstain. (3) However when $p_{1} \in[0.88,0.98]$ there are no voting weights in direct democracy that will lead to full information aggregation. However if delegation is allowed, then not only are cases (1) and (2) possible to implement but (3) is also possible by assigning votes $5>v_{1}>1$. Since this is a common values game, by McLennan (1998) these weights are also Nash equilibria. While we do not discuss the case with incomplete information, we claim that the result holds even for that case.

## 8 Conclusion

In this paper we formalised the notion of liquid democracy and argued that delegation of votes improves information aggregation relative to direct democracy in finite elections of any size. Our results are relevant both when the types of voters are known, and when they are not, and show that delegation can improve outcomes even in the presence of partisan voters.

The reason for studying liquid democracy is not just a theoretical curiosity: there is an increasing interest in large corporations and organizations to fully exploit advances in digital and encryption technology to improve collective decision making (e.g. "Google Votes" allows delegation of votes to decide on the menu of the day). ${ }^{1}$ Since such new ways of voting are consistently gaining ground, we need to understand the positive and normative properties of these rules. Our paper is a step in that direction.

Clearly more work is needed to have a more complete understanding of the prospects and caveats of delegating votes. Experimental testing of the welfare properties of liquid democracy in the laboratory and in the field seems a natural next step. While such additional methodological approaches are clearly beyond the scope of the current analysis, we are hopeful that our work provides useful insights to properly inform such analyses in the near future.

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## 9 Appendix

## Proof of Proposition 1

Proof. We first focus on $A, B$ and $I$ voters showing that there is a dominant strategy for each of these categories (strategies $\mathbf{a} \in \mathcal{P}_{\mathbf{d}}, \mathbf{b} \in \mathcal{P}_{\mathbf{d}}$ and $(a, b) \in \mathcal{P}_{I, \mathbf{d}}$ ). In the reduced game where $A, B$ and $I$ voters play only their dominant strategies, we show that delegation is a dominant strategy for the $U$ voters.
$A$ voters: Consider an $A$ voter and let $P$ be a strategy profile such that $P \in \mathcal{P}_{\mathbf{d}}$ and $P_{i} \neq \mathbf{a}$. Observe that if $i$ deviates from $P_{i}$ to a $\hat{P}_{i}=\mathbf{a}$, then

$$
\begin{equation*}
W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{a}\right) \geq W\left(P, s_{a}\right) \quad \text { and } \quad W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{b}\right) \geq W\left(P, s_{b}\right) \tag{29}
\end{equation*}
$$

which yields the following

$$
\begin{equation*}
p \cdot W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{a}\right)+(1-p) \cdot W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{b}\right) \geq p \cdot W\left(P, s_{a}\right)+(1-p) \cdot W\left(P, s_{b}\right) . \tag{30}
\end{equation*}
$$

To complete this part of the proof, we next present a profile which satisfies (11). Let $n=|N|$ be an odd number and $P$ be a strategy profile in which $\frac{n-1}{2}$ voters vote for alternative $a$ and another $\frac{n-1}{2}$ vote for alternative $b$ in both cases (independently of the state of nature) such that there is a tie (without considering voter $i$ ). Then observe that any strategy $P_{i}$ of $i$ among $\{\mathbf{b}, \mathbf{x}\}$ either keeps alternative $b$ as the winner, or keeps a tie. However, strategy $\hat{P}_{i}=\mathbf{a}$ would result in a win for alternative $a$. Thus any $P_{i} \in\{\mathbf{b}, \mathbf{x}\}$ satisfies (11). Moreover, suppose $i$ delegates to some $j \neq i$, i.e. $P_{i}=\mathbf{d}_{j: j \in N-\{i\}}$, then there exists a profile $P_{j}$ for every such $j$ such that

$$
P_{j}=\left\{\begin{array}{lll}
\mathbf{b}, & \text { for } & t_{j} \in\{A, B, U\},  \tag{31}\\
(\mathbf{b}, \mathbf{b}), & \text { for } & t_{j}=I
\end{array}\right.
$$

That is to say that $j$ is one of the $\frac{n-1}{2}$ voters who vote for $b$. Then (11) is also satisfied, since voting for $a$ would lead to a win for $a$ while delegation would lead to a win for $b$. Since such a strategy profile exists for any $j \neq i$, then $P_{i}=\mathbf{d}_{j: j \in N-\{i\}}$ is dominated by strategy a for every $j \neq i$.

Similarly, for an even number $n=|N|$, let $P$ be a strategy profile in which $\frac{n-1}{2}$ voters vote for alternative $a$ and another $\frac{n+1}{2}$ vote for alternative $b$, so that alternative $b$ wins (without considering voter $i$ ). Then observe that any strategy $P_{i}$ of $i$ among $\{\mathbf{b}, \mathbf{x}\}$ keeps alternative $b$ as the winner. However, strategy $\hat{P}_{i}=\mathbf{a}$ would result in a tie. Thus any $P_{i} \in\{\mathbf{b}, \mathbf{x}\}$ satisfies (11). Moreover, suppose $i$ delegates to some $j \neq i$, there exists a profile $P_{j}$ for every $j$ such that:

$$
P_{j}= \begin{cases}\mathbf{b}, & \text { for } \quad t_{j} \in\{A, B, U\}  \tag{32}\\ (\mathbf{b}, \mathbf{b}), & \text { for } \quad t_{j}=I\end{cases}
$$

and if $P_{i}=\mathbf{d}_{j: j \in N-\{i\}}$, then (11) is also satisfied since $i$ would prefer to vote for alternative $a$ to create a tie. Such a strategy profile exists for any $j \neq i$, therefore $P_{i}=\mathbf{d}_{j: j \in N-\{i\}}$ is dominated by strategy a for every $j \neq i$. This completes the proof that $A$ voters have a weakly dominant strategy (strategy a).
$\underline{B}$ voters: Using $\hat{P}_{i}=b$ and following exactly the same technique as for $A$ voters, we can show that $\hat{P}_{i}=\mathbf{b}$ is a dominant strategy for any $B$ voter.
$I$ voters: Consider a voter with type $t_{i}=I$ and let $P$ be a strategy profile such that $P \in \mathcal{P}_{\mathbf{d}}$ and $P_{i} \neq(\mathbf{a}, \mathbf{b})$. Observe that if $i$ deviates from $P_{i}$ to a $\hat{P}_{i}=(\mathbf{a}, \mathbf{b})$, then

$$
\begin{equation*}
W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{a}\right) \geq W\left(P, s_{a}\right) \quad \text { and } \quad W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{b}\right) \leq W\left(P, s_{b}\right) . \tag{33}
\end{equation*}
$$

By (33) and Definition 2 of the expected utility of $i$ for $t_{i}=I$, we get that inequality (10) is satisfied, since

$$
\begin{equation*}
E U_{i}\left((\mathbf{a}, \mathbf{b}), P_{-i}\right) \geq E U_{i}\left(P_{i}, P_{-i}\right) \tag{34}
\end{equation*}
$$

for any $P_{i} \neq(\mathbf{a}, \mathbf{b})$.

Next we present a profile for which (11) is also satisfied. Let $n=|N|$ be an odd number and $P$ be a strategy profile in which $\frac{n-1}{2}$ voters vote for alternative $a$ and another $\frac{n-1}{2}$ vote for $b$ in both states, so that there is a tie (without considering voter $i$ ). Let $P_{i} \neq \hat{P}_{i}$ be any strategy of $i$ which does not include delegation, that is

$$
\begin{equation*}
P_{i} \in\{\{\mathbf{a}, \mathbf{b}, \mathbf{x}\} \times\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}\}-(\mathbf{a}, \mathbf{b}) . \tag{35}
\end{equation*}
$$

Observe that any such strategy $P_{i}$ gives a strictly lower expected utility for $i$ compared to the utility gained when ( $\mathbf{a}, \mathbf{b}$ ) is chosen (which equals to 1 ). Therefore any $P_{i}$ satisfying (35) is weakly dominated by $(\mathbf{a}, \mathbf{b})$. We now focus on $I$ voters' strategies that include only the option of delegation in one state. W.l.o.g. we assume that

$$
\begin{equation*}
\left(\mathbf{d}_{j: j \in N-\{i\}}, \xi\right), \quad \text { where } \xi \in\left\{\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{d}_{j: j \in N-\{i\}}\right\}, \tag{36}
\end{equation*}
$$

and show that strategies following the (36) structure are weakly dominated by strategy ( $\mathbf{a}, \mathbf{b}$ ). To do this, suppose voter $i$ delegates to some $j \neq i$. By the same argument as we used for the $A$ and $B$ voters, note that there is always a profile in which voter $i$ delegates to a $j$ who chooses strategy $b$, i.e. strategy $P_{i}=\mathbf{d}_{j: P_{j}=b}$, yielding a vote cast from $i$ to candidate $b$ (through $j$ ). Observe that, if the state of nature is $s_{a}$, the $I$ voter $i$ prefers to deviate to strategy $(\mathbf{a}, \xi)$ as her expected utility would be higher.

Together with (34), we can write that

$$
\begin{equation*}
\left.E U_{i}\left((\mathbf{a}, \mathbf{b}), P_{-i}\right) \geq E U_{i}\left((\mathbf{a}, \xi), P_{-i}\right)>E U_{i}\left(\mathbf{d}_{j: j \in N-\{i\}}, \xi\right\}, P_{-i}\right) \tag{37}
\end{equation*}
$$

for any $\xi \in\left\{\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{d}_{j: j \in N-\{i\}}\right\}$. Following a similar argument, we end up to the following,

$$
\begin{equation*}
E U_{i}\left((\mathbf{a}, \mathbf{b}), P_{-i}\right) \geq E U_{i}\left((\xi, \mathbf{b}), P_{-i}\right)>E U_{i}\left(\left(\xi, \mathbf{d}_{j: j \in N-\{i\}}\right\}, P_{-i}\right) \tag{38}
\end{equation*}
$$

for any $\xi \in\left\{\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{d}_{j: j \in N-\{i\}}\right\}$.

From (34), (37) and (38), we get that ( $\mathbf{a}, \mathbf{b}$ ) is a dominant strategy for any $I$ voter.
$\underline{U}$ voters: Since we have shown that $A, B$ and $I$ voters have dominant strategies $\mathbf{a}, \mathbf{b}$ and $(\mathbf{a}, \mathbf{b})$, respectively, the rest of this proof we will consider the reduced game where $A, B$ and $I$ voters can only choose their dominant strategies. Recall the definition of dominance solvability adapted to the rediced game below:

1. for every pair of strategy profiles $P, P^{\prime}: W\left(P, s_{c}\right)=W\left(P^{\prime}, s_{c}\right) \quad \forall c \in\{a, b\}$,
2. if there exist strategy profiles $P, P^{\prime}$ such that $W\left(P, s_{c}\right) \neq W\left(P^{\prime}, s_{c}\right)$ for at least one of the states of nature, then strategy $\mathbf{d}_{j: t_{j}=I}$ is weakly dominant for any $U$ voter, where $j \in N$ is an $I$ voter.

In case 1, the strategies of the uninformed voters have no impact on the election result. An example of that is when the number of partisans of one voter is so large that they can decide the result regardless of the actions of other voters. In such a situation there is no difference in the reduced game with delegation and without delegation. Therefore we focus on Case 2 below.

Lemma 2. Assume that $n_{I} \geq 1, \max \left(n_{A}+n_{I}, n_{B}+n_{I}\right)<\left\lceil\frac{n}{2}\right\rceil, n_{U} \geq n_{I}+\left|n_{A}-n_{B}\right|+1$. Then the election game is dominance solvable, i.e. strategy $\boldsymbol{d}_{j: t_{j}=I}$ is weakly dominant for any $U$ voter, where $j \in N$ is an I voter.

Proof. Our conditions above ensure that $U$ voters can affect the election results. Consider a $U$ voter and let $P$ be a strategy profile such that $P_{i} \neq \mathbf{d}_{j: t_{j}=I}$. Observe that if $i$ deviates from $P_{i}$ to $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, then

$$
\begin{equation*}
W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{a}\right) \geq W\left(P, s_{a}\right) \quad \text { and } \quad W\left(\left(\hat{P}_{i}, P_{-i}\right), s_{b}\right) \geq W\left(P, s_{b}\right), \tag{39}
\end{equation*}
$$

which implies (10).
To complete the proof, we present a profile which satisfies (11). Consider an election game where $n_{A}, n_{B}, n_{I}$ and $n_{U}$ are the number of $A, B, I$ and $U$ voters, respectively, and $n=n_{A}+n_{I}+n_{B}+n_{U}$. W.l.o.g, let $n_{A}-n_{B} \geq 0$. Then, we get $\left(n_{A}+n_{I}\right)-n_{B}=z \geq 0$.
W.l.o.g, let $n_{A} \geq n_{B}$ and let $z=n_{I}+n_{A}-n_{B}$.
(1) Consider that $n_{I}=1$.

If $z=1$, and $n_{U}$ is even, let $\frac{n_{U}}{2}$ voters vote for $a$ and $\frac{n_{U}}{2}$ voters vote for $b$. If $n_{U}$ is odd then let one of the voters abstain and others divide votes equally between $a$ and $b$. This ensures that $I$ voters decide the outcome. Consider $i$ who is voting for $a$ in this profile. If $P_{i}=\mathbf{b}$ or $P_{i}=\mathbf{x}$ or $P_{i}=\mathbf{d}_{j: P_{j}=b}$, the outcome is a tie in state $s_{b}$. Consider a voter $i$ who is voting for $b$ in this profile. If instead $P_{i}=\mathbf{a}$ or $P_{i}=\mathbf{x}$ or $P_{i}=\mathbf{d}_{j: P_{j}=a}$, the outcome is a tie in state $s_{a}$. If voter $i$ deviates to strategy $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, then the outcome is $a$ in state $s_{a}$ and $b$ in state $s_{b}$. This shows that $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, is strictly better response than $\mathbf{a}, \mathbf{b}, \mathbf{x}$ and $P_{i}=\mathbf{d}_{j: P_{j}=b}, P_{i}=\mathbf{d}_{j: P_{j}=a}$.
Thus, for the voters who vote for $a$, delegation to the informed voter gives the same payoff while voting for $b$, delegating to a voter who votes $b$ or abstaining will change the outcome to a tie in state $s_{a}$. Similarly for those voters who are voting for $b$, delegation gives the same payoff as $\mathbf{b}$ but strictly higher payoff than $\mathbf{a}, \mathbf{x}$ and delegation to a voter who votes for $a$ Alternately let all $U$ voters abstain, then abstention is a strict best response along with delegation to the informed voter.
If $z>1$ then let $z-n_{I}$ voters vote for $b$ and the rest abstain. In this case, $a$ wins in stats $s_{a}$ and $b$ wins by one vote in state $b$. This profile requires sufficiently many U voters such that $n_{U} \geq z-n_{I}=n_{A}-n_{B}$. First consider a $U$ voter $i$ who is voting for $b$.If $i$ delegates to an informed voter then the outcome does not change. If voter $i$ switches to $\mathbf{a}$ or $\mathbf{x}$ then the outcome is a tie in state $b$ as also when $i$ switches to $P_{i}=\mathbf{d}_{j: P_{j}=a}$. Therefore $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, is strictly better response than $\mathbf{a}, \mathbf{x}$ and $P_{i}=\mathbf{d}_{j: P_{j}=a}$.
Second, consider a voter $i$ (if any) who is abstaining. If she switches to $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, the outcome is $a$ in state $s_{a}$ and $b$ in state $s_{b}$. If she switches to a or $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=a}$ then there is a tie in state $s_{b}$ while $a$ wins in state $s_{a}$. If she switches to $\hat{P}_{i}=\mathbf{b}$ or $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=b}$ then there is a tie in state $s_{b}$ while $a$ wins in state $s_{a}$. Therefore $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$, is strictly better than both $\mathbf{a}$ and $\mathbf{b}$ or delegating to non informed voters.
We have found profiles where $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$ is strictly better than each of the other strategies.
(2) Consider the case where $n_{I}>1$.

Consider the profile where one of the U voters votes for $a$ and exactly $z$ of the $U$ voters vote for $b$ and the rest abstain. This requires $n_{U} \geq z+1$. Such a profile ensures that in state $s_{a}$ the preferred candidate wins by only one vote. In state $s_{b}$ however the total votes for $b$ would be $z+n_{I}$ compared to votes for $a$ which are $n_{A}+1$, so none of the $U$ voters are pivotal. First, consider a voter $i$ who is voting for $a$ or delegating to an informed voter, in this profile. If she deviates to abstention, then $a$ ties with $b$ and if she deviates to $b$ then $b$ wins in state $s_{a}$, and $b$ wins in state $b$ . So in this profile, delegating to an informed voter is strictly better than $\mathbf{b}$ and $\mathbf{x}$ or $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=b}$. In the same way consider the profile where exactly $z$ of the $U$ voters vote for $a$, one $U$ voter votes for $b$ and the rest abstain. This requires $n_{U} \geq z+1$. Such a profile ensures that in state $s_{b}$ the preferred candidate wins by only one vote. In state $s_{a}$ however the total votes for $a$ would be $z+n_{I}$ compared to votes for $b$ which are $n_{B}+1$, so that $a$ wins by more than 1 vote and no $U$ voter is pivotal. Consider a voter $i$ who is voting for $b$ or delegating to an informed voter in this profile. If she deviates to abstention, then $a$ ties with $b$ and if she deviates to $a$ or delegating to a
voter who votes $a$ regardless of the state, then $a$ wins in state $s_{b}$. So in this profile, delegating to an informed voter is strictly better than $\mathbf{a}$ and $\mathbf{x}$ or $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=a}$.
Therefore we have shown that both equations (10) and (11) are satisfied for $\hat{P}_{i}=\mathbf{d}_{j: t_{j}=I}$.
The dominant strategy for $U$ voters is to delegate to informed voters. Therefore the DS outcome is $a$ in state $s_{a}$ and $b$ in state $s_{B}$.

## Proof of Proposition 2

Proof. (A) In Proposition 1 we showed that voters A, B, I have dominant strategies among all strategies in $\mathcal{P}_{\mathbf{d}}$,. Since we now restrict the strategy space by removing the option of delegation, voters of type A, B, I, still have the same dominant strategy. However, in the reduced game where A, B, I voters play their dominant strategies, we show that $U$ voters have no dominant strategy so the game is not dominance solvable.
W.l.o.g let $n_{A} \geq n_{B}$ and let $z=n_{I}+n_{A}-n_{B}$.
(1) Consider that $n_{I}=1$.
(a) Suppose $z=1$. If $n_{U}$ is even, let $\frac{n_{U}}{2}$ voters vote for $a$ and $\frac{n_{U}}{2}$ voters vote for $b$. If $n_{U}$ is odd then let one of the voters abstain and others divide votes equally between $a$ and $b$. This ensures that $I$ voters decide the outcome. Thus, voting for $a$ and $b$ are strictly preferred to abstention, so abstention is not a dominant strategy. Alternately let all $U$ voters abstain, then abstention is a strict best response.
(b) If $z>1$ then let $z-n_{I}$ voters vote for $b$ and the rest abstain, then both abstention and voting for $b$ are strict best responses. This requires sufficiently many U voters such that $n_{U} \geq z-n_{I}=n_{A}-n_{B}$ (c) Alternately let $z-n_{I}+1$ vote for $b$ and $1 U$ voter vote for $a$, while the rest abstain then a is a also strict best response. Clearly if $n_{B}>n_{A}$ the same logic applies. This requires sufficiently many U voters such that $n_{U} \geq z-n_{I}+2=n_{A}-n_{B}+2$. Therefore we can find profiles such that voting for $a, b$ and abstention are all strict best responses.
(2) Consider the case $n_{I}>1$.
(d) We show that a is a unique best response (UBR) to the profile where one of the U voters votes for $a$ and exactly $z$ of the $U$ voters vote for $b$ and the rest abstain. This requires $n_{U} \geq z+1$. Such a profile ensures that in state $s_{a}$ the preferred candidate wins by only one vote. In state $s_{b}$ however the total votes for $b$ would be $z+n_{I}$ compared to votes for $a$ which are $n_{A}+1$, so none of the $U$ voters are pivotal. Therefore voting for $a$ is a UBR. Consider a voter $i$ who is voting for $a$ in this profile. If she deviates to abstention, then $a$ ties with $b$ and if she deviates to $b$ then $b$ wins. So in this profile, a is a UBR and $\mathbf{x}$ is strictly better than $\mathbf{b}$.
(e) In the same way $\mathbf{b}$ is a UBR to the profile where exactly $z$ of the $U$ voters vote for $a$, one $U$ voter votes for $b$ and the rest abstain. This requires $n_{U} \geq z+1$. Such a profile ensures that in state $s_{b}$ the preferred candidate wins by only one vote. In state $s_{a}$ however the total votes for $a$ would be $z+n_{I}$ compared to votes for $b$ which are $n_{B}+1$, so that $a$ wins by more than 1 vote and no $U$ voter is pivotal. Consider a voter $i$ who is voting for $b$ in this profile. If she deviates to abstention, then $a$ ties with $b$ and if she deviates to $a$ then $a$ wins. So in this profile, $\mathbf{b}$ is a UBR and $\mathbf{x}$ is strictly better than $\mathbf{a}$.

Thus we have shown that $\mathbf{a}(\mathbf{b})$ is not dominated by either $\mathbf{b}(\mathbf{a})$ or $\mathbf{x}$. We also showed that there are profiles where $\mathbf{x}$ is strictly better than $\mathbf{a}$ and profiles where $\mathbf{x}$ is strictly better than $\mathbf{b}$. Therefore none of the strategies can be eliminated by weak dominance and the game is not dominance solvable.
(B). Here we show that for each of these cases we have multiple PSNE. It is obvious that if $n_{I}>$ $\left|n_{A}-n_{B}\right|$, there exists an efficient equilibrium where all $U$ voters abstain. Since in this case $I$ voters decide the outcome, this is a best response for $U$ voters. If $n_{I}<\left|n_{A}-n_{B}\right|$ and $n_{U}>2$ then there exists an inefficient equilibrium where all $U$ voters abstain and $a$ is the outcome. Now we show that there exists asymmetric PSNE where the outcome is $i$ regardless of the state. The profile described in (1a) above is an efficient equilibrium in asymmetric strategies. If $n_{U}=2$, these are the only two PSNE. If $n_{U}>2$, then in addition to these two PSNE we have other inefficient PSNE. If all $U$ voters choose a then, given that $n_{U}>2$, the outcome is $a$ regardless of the state. If all $U$ voters choose $\mathbf{b}$ then the outcome is $b$. Both these are equilibria since no single $U$ voter is pivotal.

Profiles described in 1(d) and 1(e) above are both Nash equilibria which achieve the efficient outcome. There are other PSNE however: let all $U$ voters choose a then if $n_{A}-n_{B} \geq 1$ then given $n_{U} \geq z+1$, no single voter is pivotal, and $a$ is the outcome regardless of state. To obtain $b$ as an outcome, it is sufficient that, excluding the $I$ voters, the difference in votes between $b$ and $a$ is greater than 1 in both states, so that no single $U$ voter is pivotal. Let all $U$ voters choose $\mathbf{b}$. This equilibrium exists when $n_{U} \geq z+2$.

## Proof of Proposition 4

## Proof. Case 1: A partisan is pivotal

We separate this case into two categories where the pivotal partisan supports the correct alternative and the case where the pivotal partisan supports the wrong alternative.

## (i) The pivotal partisan supports the correct alternative

Consider a partisan $i$ who supports the correct alternative. In the putative equilibrium profile, all informed voters pick up an extra vote, and none of the partisans do. Suppose that a partisan deviates by picking up an extra vote. Then, the probability that she becomes pivotal by picking an additional vote is the probability where, without this additional vote, the election outcome would have been a tie. This is because we assume that $n_{P}$ is even and $2 n_{I}$ is always even- therefore the probability of being pivotal is the same as the probability of breaking a tie. In turn, a tie only arises when the number of voters voting for the correct (or wrong) alternative is exactly $1 / 2$. For this to happen we need exactly $k^{*}$ partisans voting for the correct alternative, where $k^{*}$ satisfies

$$
\begin{equation*}
k^{*}+2 \cdot n_{I}=\frac{2 \cdot n_{I}+n_{P}}{2} \tag{40}
\end{equation*}
$$

By excluding partisan $i$ from the $k^{*}$ partisans and defining $k=k^{*}-1$, we get that (40) translates to

$$
\begin{align*}
& k+1+2 \cdot n_{I}=\frac{2 \cdot n_{I}+n_{P}}{2}  \tag{41}\\
& \Rightarrow k=\frac{1}{2} \cdot\left(-2 \cdot n_{I}+n_{P}\right)-1 . \tag{42}
\end{align*}
$$

Given an equal probability of $\frac{1}{2}$ for an $A$ or $B$ voter to occur, our desired probability can be computed as a $k$-combination of the partisan electorate $N_{P}-\{i\}$, with $\left|N_{P}\right|=n_{P}$.

$$
\begin{equation*}
\frac{\left(n_{P}-1\right)!}{k!\cdot\left(\left(n_{P}-1\right)-k\right)!} \cdot\left(\frac{1}{2}\right)^{n_{P}-1}=\frac{2^{1-n_{P}} \cdot\left(n_{P}-1\right)!}{\left(\frac{1}{2} \cdot\left(2 \cdot n_{I}-n_{P}\right)+n_{P}\right)!\cdot\left(-1+\frac{1}{2} \cdot\left(-2 \cdot n_{I}+n_{P}\right)\right)!} . \tag{43}
\end{equation*}
$$

## (ii) The pivotal partisan supports the wrong alternative

Consider a partisan $i$ supporting the wrong alternative. The case where an additional vote for $i$ would be pivotal is the case where if $i$ does not pick up an additional vote, then the election outcome would be a tie, i.e. $\frac{2 n_{I}+n_{P}}{2}=k^{*}$, where $k^{*}$ is the number of partisans (voting for the wrong alternative) needed for a tie to occur.

Let now $k$ be the number of partisans needed to support the wrong alternative so that there is a tie without $i$. Then observe that

$$
\begin{align*}
& k=k^{*}-1=\frac{2 \cdot n_{I}+n_{P}}{2}-1  \tag{44}\\
& \Rightarrow k=\frac{2 \cdot n_{I}+n_{P}}{2}-1 .
\end{align*}
$$

Similarly, the probability in this case is given by

$$
\begin{equation*}
\frac{\left(n_{P}-1\right)!}{k!\cdot\left(\left(n_{P}-1\right)-k\right)!} \cdot\left(\frac{1}{2}\right)^{n_{P}-1}=\frac{2^{1-n_{P}} \cdot\left(-1+n_{P}\right)!}{\left(\frac{1}{2} \cdot\left(-2 \cdot n_{I}-n_{P}\right)+n_{P}\right)!\cdot\left(-1+\frac{1}{2} \cdot\left(2 \cdot n_{I}+n_{P}\right)\right)!} \tag{45}
\end{equation*}
$$

given that, an $A$ or $B$ partisan have an equal probability of $\frac{1}{2}$ to occur.

## Case 2. An informed voter is pivotal

In our equilibrium profile all informed voters have two votes and all partisans have one vote. Suppose a single informed voter deviates to a single vote then the probability that he is pivotal is the probability that without the extra vote, the correct alternative is one behind the wrong alternative (again because $2 n_{I}-1+n_{P}$ is odd). I.e. exactly $k^{*}$ partisans vote for the correct alternative where $k^{*}$ satisfies:

$$
\begin{equation*}
2 \cdot\left(n_{I}-1\right)+1+k^{*}=\frac{2 \cdot\left(n_{I}-1\right)+1+n_{P}-1}{2} \tag{46}
\end{equation*}
$$

Since the pivotal voter $i$ is an informed voter, note that $k^{*}=k$ and that the " -1 " on the right-hand side of (46) is added due to the fact that the numerator is an odd number. By solving for $k^{*}$, we get $k^{*}=k=\frac{1}{2} \cdot\left(-2 \cdot n_{I}+n_{P}\right)$. Therefore the probability that $i$ picking an additional vote is pivotal equals to the probability that each of the $k$ partisans supports $i$ (each of the $k$ partisans votes for the correct alternative). This probability can be computed as a $k$-combination of the partisan electorate $N_{P}$, with $\left|N_{P}\right|=n_{P}$, where an $A$ or $B$ partisan have an equal probability of $\frac{1}{2}$ to occur,

$$
\begin{equation*}
\frac{\left(n_{P}\right)!}{k!\cdot\left(n_{P}-k\right)!} \cdot\left(\frac{1}{2}\right)^{n_{P}}=\frac{2^{-n_{P}} \cdot n_{P}}{\frac{1}{2} \cdot\left(-2 \cdot n_{I}+n_{P}\right)!\cdot\left(\frac{1}{2} \cdot\left(2 \cdot n_{I}-n_{P}\right)+n_{P}\right)!} . \tag{47}
\end{equation*}
$$

## Final computation

After computing the probabilities of the pivotal cases, observe that the relevant ratio is given by the following linear combination of equations (45), (43) and (47),

$$
\begin{equation*}
\frac{(45)+(43)}{(47)} . \tag{48}
\end{equation*}
$$

We then substitute accordingly in (48) and by performing algebraic calculations, as follows, we get that ratio (48) equals to 2 . Intuitively, this following result shows that regardless of the voter type (partisan or informed), the probability of being pivotal by picking up an additional vote is the same. We get

$$
\begin{gathered}
\frac{2^{1-n_{P}} \cdot\left(-1+n_{P}\right)!}{\left(\frac{1}{2} \cdot\left(-2 \cdot n_{I}-n_{P}\right)+n_{P}\right)!\cdot\left(-1+\frac{1}{2} \cdot\left(2 \cdot n_{I}+n_{P}\right)\right)!}+\frac{2^{1-n_{P} \cdot\left(n_{P}-1\right)!}}{\frac{2^{-n_{P}} P \cdot n_{P}}{\left(\frac{1}{2} \cdot\left(2 \cdot n_{I}-n_{P}\right)+n_{P}\right)!\cdot\left(-1+\frac{1}{2} \cdot\left(-2 \cdot n_{I}+n_{P}\right)\right)!}} \\
=\frac{\frac{1}{2} \cdot\left(-2 \cdot n_{I}+n_{P}\right)!\cdot\left(\frac{1}{2} \cdot\left(2 \cdot n_{I}-n_{P}\right)+n_{P}\right)!}{\left(\frac{2^{1-n_{P}} \cdot\left(-1+n_{P}\right)!}{2}\right)!\left(-1+\frac{n_{P}}{2}\right)!}+\frac{2^{1-n_{P}} \frac{n_{P} \cdot\left(-1+n_{P}\right)!}{\left(\frac{n_{P}}{2}\right)!\left(-1+\frac{n_{P}}{2}\right)!}}{\frac{2^{-n_{P} \cdot n_{P}!}}{\left(\frac{1}{2} \cdot\left(-2 n_{I}+n_{P}\right)\right)!\cdot\left(\frac{1}{2} \cdot\left(2 n_{I}-n_{P}\right)+n_{P}\right)!}}=\frac{\frac{2^{1-n_{P} \cdot\left(-1+n_{P}\right)!}}{\left(-1+n_{P}\right)!}+\frac{2^{1-n_{P} \cdot\left(-1+n_{P}\right)!}}{\frac{2}{2-n_{P} \cdot n_{P}!}}}{\frac{\left.1+n_{P}\right)!}{\left(\frac{1}{2}\left(-2 n_{I}+n_{P}\right)\right)!\left(\frac{1}{2} \cdot\left(2 n_{I}-n_{P}\right)+n_{P}\right)!}} \\
=\frac{\frac{2^{-n} P \cdot\left(-1+n_{P}\right)!}{\left(-1+n_{P}\right)!}+\frac{2^{-n_{P}} \cdot\left(-1+n_{P}\right)!}{\left(-1+n_{P}\right)!}}{\frac{2^{-n_{P} \cdot\left(n_{P}\right)!}}{n_{P}!}}=\frac{\frac{2^{-n} P \cdot\left(-1+n_{P}\right)!+2^{-n_{P} \cdot\left(-1+n_{P}\right)!}}{\left(-1+n_{P}\right)!}}{2^{-n_{P}}}=2,
\end{gathered}
$$

which proves inequality (27) and thus completes the proof of Theorem 4.

## Computation of probabilities of section 7.1

First, we focus on the computation of $P_{t i e}^{(C, A)}$, which is given by a $k$-combination of the partisan electorate $N_{P}-\{i\}$, where $\left|N_{P}\right|=n_{P}$, that is,

$$
\begin{equation*}
\frac{\left(n_{P}-1\right)!}{k!\cdot\left(\left(n_{P}-1\right)-k\right)!} \cdot(p)^{k} \cdot(1-p)^{n_{P}-1-k} . \tag{49}
\end{equation*}
$$

Similarly, for $P_{t i e}^{(W, A)}$ and $P_{t i e}^{(C, I)}$ we get that

$$
\begin{equation*}
P_{t i e}^{(W, A)}=\frac{\left(n_{P}-1\right)!}{k!\cdot\left(\left(n_{P}-1\right)-k\right)!} \cdot(p)^{k} \cdot(1-p)^{n_{P}-1-k}, \tag{50}
\end{equation*}
$$

where $k=\frac{2 \cdot n_{I}+n_{P}}{2}-1$, and

$$
\begin{equation*}
P_{t i e}^{(C, I)}=\frac{\left(n_{P}\right)!}{k!\cdot\left(n_{P}-k\right)!} \cdot(p)^{k} \cdot(1-p)^{n_{P}-k}, \tag{51}
\end{equation*}
$$

where $k=\frac{-2 \cdot n_{I}+n_{P}}{2}$.

Secondly, we define each one of the probabilities in the ratio

$$
\frac{P_{\text {lose }}^{(C, A)}+P_{\text {lose }}^{(W, A)}}{P_{\text {lose }}^{(C, I)}} .
$$

Recall that $P_{\text {lose }}^{(C, A)}$ is the probability of losing when an $A$ voter is picking up an extra vote in favour of the correct alternative, $P_{\text {lose }}^{(W, A)}$ is the probability of losing when an $A$ voter picks up an extra vote in favour of the wrong alternative, and $P_{\text {lose }}^{(C, I)}$ is the probability of losing when an $I$ voter picks up an extra vote in favour of the correct alternative.

For the computation of $P_{\text {lose }}^{(C, A)}$, consider a partisan $i$ who supports the correct alternative $A$. In the putative equilibrium profile, all informed voters pick up an extra vote, and none of the partisans do. Suppose that a partisan deviates by picking up an extra vote. In this case the total votes number equals to $2 n_{I}+2+n_{P}-1$, which is an odd number. Thus, the only possibility of losing is to have the votes in favour of the correct alternative strictly smaller than votes for the wrong alternative. Then, the probability that she loses when voting for the correct alternative with the extra vote is the probability that

$$
\begin{aligned}
2 n_{I}+k+2<n_{P}-1-k & \Rightarrow k<\frac{n_{P}-1-2 n_{I}-2}{2} \\
& \Rightarrow k \leq 1+\frac{n_{P}-2 n_{I}-3}{2}
\end{aligned}
$$

Given that an $A$ voters occurs with probability $p$, we get that our desired probability is given by

$$
\begin{equation*}
\sum_{0}^{k^{*}} \frac{\left(n_{P}-1\right)!}{k!\cdot\left(\left(n_{P}-1\right)-k\right)!} \cdot(p)^{k} \cdot(1-p)^{n_{P}-1-k} \tag{52}
\end{equation*}
$$

where $k^{*}=1+\frac{n_{P}-2 n_{I}-3}{2}$.

Similarly, for the computation of $P_{\text {lose }}^{(W, A)}$, consider a partisan $i$ who supports the wrong alternative $B$. In the putative equilibrium profile, all informed voters pick up an extra vote, and none of the partisans do. Suppose that a partisan deviates by picking up an extra vote. In this case the total votes number equals to $2 n_{I}+2+n_{P}-1$, which is an odd number. Thus the only possibility of losing is to have the votes in favour of the wrong alternative strictly smaller than votes for the correct alternative. Then the probability that she loses even with the extra vote when voting for the wrong alternative is the probability that

$$
k+2<2 n_{I}+n_{P}-1-k \Rightarrow k \leq \frac{2 n_{I}+n_{P}-3}{2}+1 .
$$

Given that a $B$ partisan occurs with probability $1-p$, we get that our desired probability is given by

$$
\begin{equation*}
\sum_{0}^{k^{*}} \frac{\left(n_{P}-1\right)!}{k!\cdot\left(\left(n_{P}-1\right)-k\right)!} \cdot(1-p)^{k} \cdot(p)^{n_{P}-1-k} \tag{53}
\end{equation*}
$$

where $k^{*}=\frac{2 n_{I}+n_{P}-3}{2}+1$.

For the computation of $P_{\text {lose }}^{(C, I)}$, we do the following analysis. In our equilibrium profile all informed voters have two votes and all partisans have one vote. Suppose that a single informed voter $i$ deviates to a single vote. Note that the total votes number is $2 n_{I}+n_{P}$, which is even. Therefore it is possible that there is a tie result in which the correct alternative loses with probability $\frac{1}{2}$. We assume that in this case there is no penalty. However, the probability that she loses for sure when picking up an extra vote is given by

$$
2 n_{I}+k<n_{P}-k \Rightarrow k \leq \frac{n_{P}-2 n_{I}}{2}+1 \equiv k^{*}
$$

Given that an $A$ partisan occurs with probability $p$, we get that our desired probability is given by

$$
\begin{equation*}
\sum_{0}^{k^{*}} \frac{\left(n_{P}\right)!}{k!\cdot\left(\left(n_{P}\right)-k\right)!} \cdot(p)^{k} \cdot(1-p)^{n_{P}-k} \tag{54}
\end{equation*}
$$

where $k^{*}=\frac{n_{P}-2 n_{I}}{2}+1$.

From the previous calculations, it follows that the inequality that guarantees the existence of the described equilibrium -and which we study numerically- is given by

$$
\frac{(49)+(50)}{(51)}<\frac{(52)+(53)}{(54)} \Leftrightarrow \frac{\frac{(49)+(50)}{(51)}}{\frac{(52)+(53)}{(54)}}<1 .
$$


[^0]:    *King's College, London
    ${ }^{\dagger}$ London School of Economics
    ${ }^{\ddagger}$ Humboldt University
    ${ }^{\text {§ }}$ University of Cyprus

[^1]:    ${ }^{1}$ Brill (2018) discusses some research directions on liquid democracy in the context of the need of new interactive collective decision-making processes.

[^2]:    ${ }^{1}$ The concept of IEWDS has the drawback that the order of elimination may matter in reaching the dominance solvable outcome. However, when voters have strict preferences over alternatives, as in our setting, Marx and Swinkels (2000) show that the order of elimination does not affect the outcome. Finally, a common knowledge justification for IEWDS was provided by Rajan (1998).
    ${ }^{2}$ McLennan (1998) says " When the environment is symmetric, in that the agents are interchangeable, an equilibrium assigning different strategies to identical agents embodies a degree of coordination that may seem implausible when the population is large.Myerson (1998) has argued that population uncertainty, in that the number of players of each type is random, is best modeled by a framework in which asymmetric behavior is not merely implausible but in fact very difficult to describe at all.

[^3]:    ${ }^{1}$ Although this looks a lot like vote trading, anonymity is an important part of the Liquid Democracy concept- who delegates the vote and to whom is private information- therefore it differs from the usual vote trading concept.

[^4]:    ${ }^{1}$ As shown by Shotts (2006) and Meirowitz and Shotts (2009) moderate voters need not only abstain to avoid diluting the informativeness of the election, but also to signal their private preferences to the politicians in settings of repeated elections.

[^5]:    ${ }^{1} \mathrm{~A}$ Turing-complete system of data-manipulation rules (e.g. a programming language) is a system in which any algorithm's logic can be designed and then computed.

[^6]:    ${ }^{1}$ We make this assumption to simplify our model and avoid cycles. The results do not change even if we allow further delegations.

[^7]:    ${ }^{1}$ Section 7 shows that when voters are informed with different levels of precision, the game with delegation aggregates information better than the game without delegation.

[^8]:    ${ }^{1}$ The formal construction of the appropriate auxiliary game when truth-seeking voters are partially and heterogeneously informed, is similar to Tsakas and Xefteris (2021).

[^9]:    ${ }^{1}$ We partially relax these symmetry assumptions at the end of this section.

[^10]:    ${ }^{1}$ This is equivalent to saying that the probability that the correct alternative loses goes to 0 as $n \rightarrow \infty$ - this is because in any state, for large $n$ by the CLT, at least half the partisans and all informed voters are voting for the correct alternative vs half the partisans voting for the wrong alternative.

[^11]:    ${ }^{1}$ See Hardt and Lopes (2015) for details.

