Deliberate Choice under a Lack of Confidence*

Christian P. Traeger[1]

Department of Economics, University of Oslo

ifo Institute for Economic Research, Munich

Abstract: The paper introduces a degree of confidence into a recursive dynamic von Neumann-Morgenstern framework. The decision maker relies on probabilistic descriptions of the world, but she also assigns a degree of confidence to each probability measure. The model requires only minimal adaptations of von Neumann-Morgenstern's classical axioms. Yet, it results in a much richer decision framework where risk attitude depends on the degree of confidence. If the confidence labels can be ranked, the decision maker's attitude is a combination of risk aversion and aversion to the lack of confidence. A special case of aversion to the lack of confidence is recursive smooth ambiguity aversion.

JEL Codes: D81, Q54, D90, Q01

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[1]Email: traeger@uio.no

1 Introduction

Von Neumman & Morgenstern (1944) provided axiomatic guidance on merging probabilistic measures of uncertainty with the valuation of outcomes to rank scenarios and take decisions. Probabilistic descriptions of the world have also proven useful in structuring subjective decisions (Savage 1954, Anscombe & Aumann 1963). Yet, a large body of literature, including labels such as ambiguity, hard uncertainty, or Knightian uncertainty, keeps demonstrating that not all uncertainties are created equal.

Many economists still consider von Neumann & Morgenstern's (1944) axioms normatively attractive and a benchmark for "rational" decision making. Moreover, many applications of decision theory require the explicit merging of given beliefs and probabilities with an evaluation of outcomes. The present paper explores how far von Neumman & Morgenstern's (1944) axioms can take us if we explicitly acknowledge that not all uncertainties are created equal. Here, probabilistic descriptions of the world carry a label that can correspond to their source or the level of confidence. I show that the framework requires only minimal adjustments of the classical von Neumann & Morgenstern (1944) axioms and results in a much richer attitude towards uncertainty.

1.1 Motivation

The characterization of uncertainty in this paper distinguishes different types of probabilistic beliefs. For example, the model can distinguish objective probabilities from estimates based on a limited amount of data and from mere guesstimates. The Intergovernmental Panel on Climate Change adopted a similar threefold classification of probabilities to characterize different types of uncertainty (IPCC 2001, Box TS.1, p 22). Here, different classes of uncertainty can be ranked by a degree of confidence (or subjectivity). The main representation theorem of the paper will not assume an order of probability classes (for example in terms of confidence). This general representation only builds on the fact that we face different types of probabilistic beliefs. These can be characterized and distinguished by method of derivation, by level of confidence, by source of information, or a combination of these. The paper shows that different classes of probabilistic beliefs give rise to different degrees of risk aversion.

Outcomes can be subject to a combination of different types of uncertainty. For example, returns to investment in an industry or country might might be governed by a confidently known distribution, given a particular political regime. The stability of the political regime, however, is a less confidently known guess. The general characterization of uncertainty in this paper employs different layers of probabilistic beliefs that can differ in the class (or confidence) of probabilistic belief. The paper shows that the risk aversion used to assess various layers of uncertainty is only a function of the probability type (or confidence) and does not depend on the uncertainty layer per se. The layer structure relates the widespread framework of smooth ambiguity aversion by Klibanoff, Marinacci & Mukerji's (2009). Splitting the type of probability from the uncertainty layer allows me to extend an analogue of smooth ambiguity aversion to a general notion of aversion to the lack of confidence. If the decision maker's aversion to the lack of confidence is extreme, his assessment converges to an evaluation that only pays attention to the worst possible realizations. In the case of a single probability layer, an Arrow & Hurwicz (1972) type criterion for decision making under ignorance, and in the case of (extreme) aversion to the lack of confidence in second order probabilities, the decision criterion corresponds to Gilboa & Schmeidler's (1989) maximin expected utility. In particular, the framework permits a decision maker to employ a variety of decision criteria conditional on the type of uncertainty she faces.

The motivation of von Neumann & Morgenstern (1944) and of the present axiomatic system slightly differs from the current main stream of decision theoretic literature. This literature has a strong focus on sets of axioms that give rise to observed behavior, and revealed choice is usually translated into an "as if" separation of beliefs and their evaluation. The starting point of the current paper is different. I develop a decision support model for agents or an agencies that builds on a given set of beliefs, and derives the evaluation framework. Agencies or expert groups that assess complicated uncertain situations have no naturally given preference relation governing a rich choice set. In contrast, the agency obtains uncertainty assessments from a group of scientific experts or analysts whose assessment is independent of the valuation of the different scenarios. The current paper delivers a framework that links a comprehensive uncertainty characterization to a comprehensive evaluation approach. The interface between these two parts of the assessment are the confidence (or uncertainty class) labeled probabilistic beliefs.

The evaluation framework can help agencies or expert groups to evaluate and rank different uncertain scenarios deliberately, as in the case of the Intergovernmental Panel on Climate Change. It also forces agencies to apply evaluation consistently across different fields of application. In particular, it minimizes the ability to employ uncertainties or the lack of objective probabilities strategically to push or obstruct particular projects or pieces of legislation (as sometimes observed invoking the vaguely defined precautionary principle). It also helps individuals in their decision making by separating the reasoning into judgements of likelihoods, types of uncertainty, and rules for their evaluation. In application, the general framework's informational requirements on uncertainty attitude can be tamed by reducing it to a degree of risk aversion and a degree of aversion to the lack of confidence.[2]

The next section relates the paper to a selection of related decision theoretic literature. Section 2 introduces the setting of the paper, first graphically, then formally. Section 3 summarizes the axioms that underly the representation. Section 4 states the representation and demonstrates how to evaluate a simple example. Section 5 discusses the notions of smooth ambiguity aversion and aversion to the lack of confidence in beliefs as well as the disentanglement of the various dimensions of preference. Section 6 analyzes behavioral implications and sketches normative applications. Section 7 concludes. Proofs are gathered in the appendix.

1.2 Relating the Current Representation to the Literature

The idea of enriching probabilistic beliefs by a degree of confidence goes back to Ellsberg's (1961) suggestion for resolving the paradox today carrying his name. The paradox illustrates that people would prefer to bet on known as opposed to unknown probabilities. Over the last two decades, several strands of literature on decision making under uncertainty evolved around this paradox. One of these approaches abandons the concept of probabilities and replaces it with a non-additive set function called a capacity. In the resulting representations "expected values" are formed using the Choquet-integral, which resulted in the name Choquet expected utility (e.g. Schmeidler 1989, Chateauneuf, Grant & Eichberger 2007). Another approach assigns sets of probabilities to different scenarios and constructs decision criteria on these sets,

[2]It is not the degree of aversion to the lack of confidence by itself that matters, but how the aversion is projected onto an agreed scale for measuring confidence.

e.g., maximizing the minimal expected utility, (e.g. Gilboa & Schmeidler 1989, Ghirardato, Maccheroni & Marinacci 2004, Maccheroni, Marinacci & Rustichini 2006). The latter approach is often referred to as a "multiprior" approach. Various equivalence results between Choquet expected utility and the multiprior approach have been shown.

My paper relates most closely to a class of models that works with second order probabilities to capture non-risk uncertainty, including Segal (1990), Klibanoff, Marinacci & Mukerji (2005), Seo (2009), Ergin & Gul (2009), and Klibanoff et al. (2009). Similarly to the present study, these papers limit the reduction of compound lotteries. Halevy's (2007) experiment confirms that observed ambiguity aversion indeed seems tightly linked to the "failure" in reducing compound lotteries. The present paper connects this literature with the a literature on "source preference" developed by Tversky & Kahneman (1992), Tversky & Fox (1995), Fox & Weber (2002), Chew & Sagi (2008), and Abdellaoui, Baillon, Placido & Wakker (2011). Here, individuals judge uncertainty differently depending of the source of the uncertainty. They favor familiar over unfamiliar risk. In contrast to these literatures, the present paper develops a normatively motivated decision support system that sticks as close as possible to the classical framework of von Neumann & Morgenstern (1944). Apart from delivering a normatively differently motivated framework, the present paper formalized the idea of aversion to the lack of confidence.

For certain consumption paths my representation coincides with the intertemporally additive standard model. A utility function u_t evaluates outcomes in every period (and state of the world) and measures intertemporal substitutability. The aggregation over (various layers of) uncertainty is carried out by a generalized mean $f_t^{-1}\left[\mathrm{E} f_t(\cdot)\right]$ (Hardy, Littlewood & Polya 1964). A concave function f_t reduces the value of the generalized mean as opposed to taking mere expectations. The concavity of f_t is a measure for risk aversion. In general, uncertainty affects valuation through two channels. First, a stochastic process generates fluctuations. Agents with a preference for intertemporal consumption smoothing dislike these fluctuations. Second, agents can be intrinsically averse to risk. They dislike the mere existence of uncertainty. It is that second effect that is measure by the functions f_t. For the case of objective risk (Traeger 2007) gives a choice theoretic interpretation of this measure.[3]

[3]In a one commodity context, the model by Epstein & Zin (1989) and Weil (1990) disentangles intertemporal substitutability from Arrow Pratt risk aversion. In this framework, Arrow Pratt risk

In the smooth ambiguity model, Klibanoff et al. (2005) and Klibanoff et al. (2009) interpret an analog to the function f_t as a measure of ambiguity aversion. Here, the intrinsic risk aversion is with respect to second order subjective probability distributions. In the current paper, the functions f_t depend on the probability class (or the level of confidence).

Klibanoff et al. (2009) distinguish between objective versus subjective lotteries, which corresponds to a binary measure of confidence or subjectivity within my framework. In Klibanoff et al.'s (2009) model a subjective lottery is by definition a second stage lottery over first stage objective lotteries. In contrast, this paper makes the degree of subjectivity an explicit component of the uncertainty characterization and detaches it from a hierarchical structure of probabilities. Klibanoff et al. (2009) implicitly impose that objective lotteries are evaluated intertemporally risk neutral, which means that risk aversion to objective risk is only driven by aversion to intertemporal consumption fluctuations. There is no intrinsic aversion to risk. Formally, this assumption translates into the use of expected values rather than a generalized mean to aggregate over objective risk. In contrast, risk aversion to subjective lotteries incorporates intrinsic risk aversion and uses the generalized mean for evaluation. The authors identify the curvature of the corresponding weight-function with smooth ambiguity attitude. The generalized framework of this paper incorporates both, intrinsic risk aversion to objective as well as to subjective risk. Relating the two gives a better understanding and a more precise definition of the measure of smooth ambiguity aversion. Moreover, the current setting facilitates a three-fold disentanglement of dimensions of preference. One way to span these dimensions is in terms of intertemporal substitutability, aversion to objective risk, and ambiguity aversion. Alternative coordinates for these dimensions are offered. Finally, the present framework extends the concept of smooth ambiguity aversion to situations with an arbitrary number of subjectivity or confidence labels. Here, a generalized form of ambiguity aversion translates into an aversion to the degree of subjectivity of (or the lack of confidence in) probabilistic beliefs.

aversion measures both of the effects described here jointly. Therefore, the function f_t measures the difference between Arrow Pratt risk aversion in these models and the aversion to intertemporal substitution. Note that the Arrow Pratt measure of risk aversion is defined only in a one commodity setting. If it is extended to the multi-commodity setting as for example by Kihlstrom & Mirman (1974) the measure becomes good specific. In contrast, the measure of intertemporal risk aversion measure intrinsic aversion to uncertainty and is good-independent (Traeger 2007).

Hayashi & Miao (2011) extend Klibanoff et al.'s (2009) setting in a similar direction. The authors adopt a more technical setting using an Anscombe & Aumann (1963) version of Klibanoff et al. (2009) and an extension of Seo (2009). Similar to the present paper, the authors develop a framework that, at least in principle, permits to distinguish between intertemporal substitution, risk aversion, and ambiguity aversion. However, the function whose curvature the authors identify with risk aversion is only unique up to increasing transformations. The only function that is not subject to this indeterminacy is their extended definition of smooth ambiguity aversion. This measure coincides with my suggested measure of smooth ambiguity aversion, which I generalize to the notion of aversion to the subjectivity of belief. Hayashi & Miao (2011) stick to the more limiting hierarchical structure of subjective over objective lotteries discussed already for the setting of Klibanoff et al. (2009). Both, Klibanoff et al. (2009) and Hayashi & Miao (2011), go a step further than the present paper in discussing learning and in relating the paper to the standard Bayesian framework.

2 The Setting

I first provide a graphical illustration of the uncertainty structure underlying the model and explain the basic concepts necessary to understand the axioms and the representation. Then I formalize the general setting.

2.1 Graphical illustration

In every period uncertainty is described by an uncertainty tree that compasses an arbitrary number of individual lotteries. The left hand side of Figure 1 depicts such an uncertainty tree with three layers of uncertainty. Each node of the tree represents a (sub-) lottery. Each of these (sub-) lotteries is indexed with a label $s \in S$ representing the confidence in (or the subjectivity of) the corresponding lottery. Elements of S can specify verbal descriptions of relevant characteristics surrounding the derivation of the probabilities like "careful econometric analysis", "high frequency observation", "expert opinion", "causality poorly understood", "wild guess", "principle of insufficient reason", or "maximum entropy". Alternatively, the decision maker can employ labels such as "confident", "less confident", "not at all confident" or he can employ the labels "unpredictability", "structural uncertainty", and "value uncer-

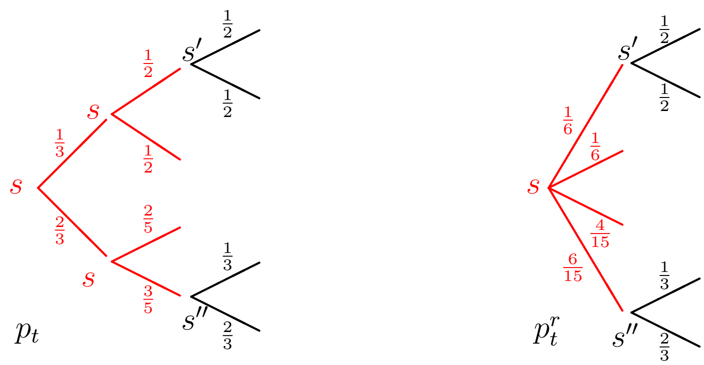

Figure 1: Example of two decision trees, $p_t \in Z^3(X^* \times P_{t+1})$ and $p_t^r \in Z^2(X^* \times P_{t+1})$, depicting uncertainty resolving in period t. Each uncertainty node is labeled with the degree of subjectivity of the corresponding lottery. The leaves of the trees are omitted and would consist of differing elements $(x_t, p_{t+1}) \in X^* \times P_{t+1}$. Lottery p_t^r is obtained from lottery p_t by collapsing the root lottery with the subsequent layer of uncertainty sharing the same degree of subjectivity. A decision maker satisfying axiom A1 is indifferent between the two depicted decision trees.

tainty" suggested by the International Panel on Climate Change (IPCC 2001, Box TS.1, p 22). The main representation theorem in section 4 does not assume that the set S is ordered. Only later in section 5 do I assume the existence of an order relation on S (such as "more confident than"). The branches of the uncertainty trees do not have to coincide in length. For example, a flip of a coin can decide whether an agent consumes a certain amount, or enters another lottery. Figure 1 omits the leaves of the uncertainty tree. The leaves specify the consumption payoff x_t of the decision maker in period t as well as the uncertainty he faces at the beginning of the next period p_{t+1}.

I define a function $\hat{s}(\cdot)$ that returns the subjectivity label of the root for every lottery p_t. In Figure 1 it is $\hat{s}(p_t) = \hat{s}(p_t^r) = s$. I refer to the degree of subjectivity $\hat{s}(p_t)$ of the root lottery as the degree of subjectivity of lottery p_t. Similarly, a function $\hat{n}(\cdot)$ returns the uncertainty layer of the root of a lottery p_t or the depth of the representing uncertainty tree in period t. The lotteries in Figure 1 yield $\hat{n}(p_t) = 3$ and $\hat{n}(p_t^r) = 2$. I refer to the number $\hat{n}(p_t)$ of a lottery p_t as its rank. Lottery p_t^r in Figure 1 is a rank 2 lottery over two lotteries of rank 1 and two certain outcomes of rank 0 (which take the form (x_t, p_{t+1})). In general, a lottery of rank n can be a lottery over a continuum of lotteries with rank smaller than n.

The first two uncertainty layers of lottery p_t on the left hand side of Figure 1 share the same degree of subjectivity. Given both uncertainty layers are of the same type, I define a reduction of these two uncertainty layers into a single layer by multiplying the

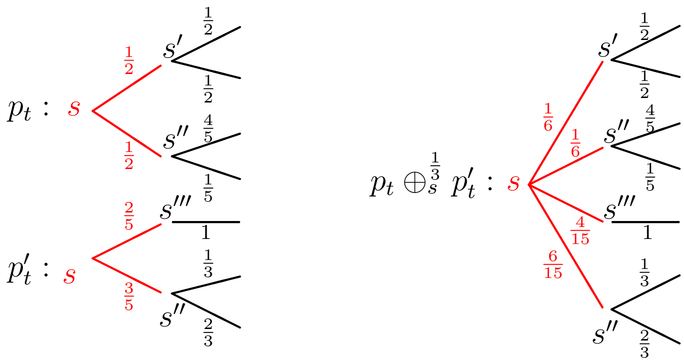

Figure 2 depicts the decision tree $p_t \oplus_s^{\frac{1}{3}} p'_t \in P_t^s$ that results from mixing the two simple lotteries $p_t, p'_t \in P_t^s$ with degree of subjectivity s.

corresponding probabilities. The resulting lottery p_t^r is depicted on the right hand side of Figure 1, where the superindex r denotes the reduction. Finally, Figure 2 shows a mixing of two lotteries p_t and p'_t. The mixing operator $\oplus_s^{\frac{1}{3}}$ mixes two lotteries with degree of subjectivity s assigning probability $\frac{1}{3}$ to the first lottery and probability $1 - \frac{1}{3}$ to the second lottery. Because the operator mixes both lotteries within the same uncertainty layer, both lotteries have to coincide in the degree of subjectivity (of their root lottery). In a remark at the end of the next section, I also introduce an alternative operator $\odot_s^\alpha$ that mixes two lotteries of arbitrary, and possibly differing, degree of subjectivity on the next higher uncertainty level. Here, α labels again the probability weight of the first lottery, while s labels the degree of subjectivity of the mixed lottery (whose rank is one more than that of the higher ranked lottery entering the mixture).

2.2 The formal setting

Time is discrete with a planning horizon $T \in \mathbb{N}$. In the usual abuse of notation T denotes at the same time the set $\{0, \ldots, T\}$. Elements x of a connected compact metric space X^* describe outcomes in any period $t \in T$. These elements represent consumption levels or a collection of general welfare relevant characteristics. To avoid repetition, I introduce several definitions using a generic compact metric space X instead of X^*. The Borel σ-algebra on X is denoted $\mathfrak{B}(X)$. Let S be a finite index set. The decision maker employs the index $s \in S$ to distinguish between lotteries (denoting general uncertain situations) that differ in terms of subjectivity of or confidence in the probabilistic belief. For every $s \in S$, I denote by $\Delta_s(X)$ a space of Borel probability measures on X that describe a lottery with degree of subjectivity s. Formally, these different lottery spaces are a family $\{(\Delta(X), s)\}_{s\in S}$. Each space $\Delta_s(X)$ is equipped with the Prohorov metric giving rise to the topology of weak convergence. I introduce an additional element $s^0 \notin S$ and define $\bar{S} = S \cup \mathrm{s}^0$. The element s^0 serves the purpose of defining under abuse of notation $\Delta_{\mathrm{s}^0}(X) = X$, making the space X part of the family $\{\Delta_{\mathrm{s}^0}(X)\}_{s\in\bar{S}}$. I introduce higher order lotteries inductively over the parameter $n \in N = \{0, 1, \dots, N\}$, which defines the maximal depth of the uncertainty tree within a period.[4] I start by setting $Z^0(X) = Y^0_{\mathrm{s}^0}(X) = X$. In the first induction step, I define for $n > 0$ the lottery spaces $Y^n_s(X) = \Delta_s(Z^{n-1}(X))$ for all $s \in \bar{S}$. These spaces describe the set of uncertainty trees of maximal depth n with a root lottery of subjectivity s. In the second induction step, I define the general choice space $Z^n(X) = \cup_{s\in\bar{S}} Y^n_s(X)$, which collects uncertainty trees with different degrees of subjectivity in the root. The inclusion of s^0 in the (disjoint) union allows the uncertainty tree to have branches of differing length. The spaces $Z^n(X)$ are equipped with the (disjoint) union topology and, thus, compact. In a static setting the decision maker's choice objects would be described as elements of $Z^N(X^*)$. These elements represent arbitrary concatenations of lotteries with differing degrees of subjectivity with a maximal uncertainty tree depth of N. Figure 1 depicts two examples of an uncertainty structure contained in $Z^3(\cdot)$.

I construct the general choice space in the intertemporal setting recursively. In the last period, choices are $p_T \in P_T = Z^N(X^*)$. Preceding choice spaces are defined by $P_{t-1} = Z^N(X^* \times P_t)$ for all $t \in \{1, \dots, T\}$, where $X^* \times P_t$ is equipped with the product topology. Thus, at the beginning of every period uncertainty is described as a composition of lotteries with differing degrees of subjectivity over current outcomes and over the uncertainty that describes the decision maker's future. I call the choice object $p_t \in P_t$ in period t a generalized temporal lottery. They generalize Kreps & Porteus's (1978) concept of a temporal lottery. I define the rank n of a lottery $p_t \in P_t$ by the function $\hat{n} : \cup_{t\in T} P_t \to N$ with

$$\hat{n}(p_t) = \min\left\{n \in N \mid \exists\, s \in \bar{S},\, t \in T \text{ s.th. } p_t \in Y^n_s(X^* \times P_{t+1})\right\}$$

[4]Decision nodes would be introduced at any point in the uncertainty trees the same way as done in Kreps & Porteus (1978), yielding a decision tree. Optimal choices in the framework always correspond to the best (sub-) tree and there is no explicit preference for flexibility as e.g. in Kreps (1979). Therefore, no additional insights derive from explicitly introducing decision nodes and the more complicated notation would be obstructive. The application of the stated evaluation functional in a dynamic programming framework with decision making in every period is immediate.

The rank captures the level of concatenation of a lottery, which corresponds to the depth of the representing uncertainty tree (within period t). I define the function $\hat{s} : \cup_{t\in T} P_t \to \bar{S}$ by

$$\hat{s}(p_t) = s \;\text{ iff }\; p_t \in Y_s^{\hat{n}(p_t)}(X^* \times P_{t+1})\,.$$

It maps a generalized temporal lottery into the degree of subjectivity of its root lottery and assigns s^0 if there is no uncertainty resolved in period t. The space $P_t^s = \{p_t \in P_t \,|\hat{s}(p_t) \in \{s, s^0\}\}$ denotes the space of all period t lotteries in which the root lottery has a degree of subjectivity s (as in Figure 1) and includes the certain outcomes.

I denote the sigma algebra of events evaluated by lotteries $p_t \in P_t$ of rank $\hat{n}(p_t) = n$ by $\mathfrak{B}_t^n = \mathfrak{B}\left(Z^{n-1}(X^* \times P_{t+1})\right)$, $0 < n \le N$. For a set B and $0 < n \le N$ I denote the set's restriction to events measurable by lotteries of rank n by $B_t^n = B \cap \mathfrak{B}_t^n$. If $p_t = (x_t, p_{t+1}) \in Z^0(X^* \times P_{t+1})$, i.e. no uncertainty resolves in period t, I introduce the notation

$$p_t(B_t^0) = (x_t, p_{t+1})(B_t^0) = \begin{cases} 1 \text{ if } (x_t, p_{t+1}) \in B \\ 0 \text{ if } (x_t, p_{t+1}) \notin B \end{cases}.$$

I use these restrictions B_t^n of the event set for composing lotteries of differing rank. The following composition of two lotteries lies at the core of the independence axiom. It composes two lotteries sharing the same degree of subjectivity. For any $s \in S$, $p_t, p'_t \in P_t^s$, $\alpha \in [0,1]$ and with $n^* = \max\{\hat{n}(p_t), \hat{n}(p'_t), 1\}$, I define a probability α mixture by the operation $\oplus_s^\alpha : P_t^s \times P_t^s \to P_t^s$ that maps $(p_t, p'_t) \mapsto p_t \oplus_s^\alpha p'_t \in Y_s^{n^*}$ defined by

$$p_t \oplus_s^\alpha p'_t\big(B\big) = \alpha\; p_t\Big(B_t^{\hat{n}(p_t)}\Big) + (1-\alpha)\; p'_t\Big(B_t^{\hat{n}(p'_t)}\Big)$$

for all $B \in \mathfrak{B}_t^{n^*}$. Note that the lottery resulting from this mixture lives in the same space as the higher ranked lottery of p_t and p'_t. An example of such a mixture is depicted in Figure 2.

Whenever the root lottery $p_t \in P_t$ shares the same degree of subjectivity with the subsequent layer of uncertainty (as in the left tree in Figure 1), I define a reduced lottery that collapses these layers sharing the same degree of subjectivity into a single layer. For any lottery $p_t \in \Delta_s\left(Y_s^n(X^* \times P_{t+1})\right)$ of rank $n+1$ I define the reduced lottery $p_t^r \in Y_s^n(X^* \times P_{t+1})$ of rank n by

$$p_t^r(B) = \int_{Y_s^n(X^* \times P_{t+1})} \tilde{p}_t(B) \, dp_t(\tilde{p}_t) \tag{1}$$

for all $B \in \mathfrak{B}_t^n$. An example is given in Figure 1. The lottery p_t^r collapses the root lottery and the subsequent layer of uncertainty in lottery p_t, both sharing the same degree of subjectivity, into a single layer of uncertainty.

The Cartesian product $\mathsf{X} = {X^*}^{T+1} \subset P_0$ characterizes the set of all certain consumption paths faced in the present. A consumption paths $\mathsf{x} \in \mathsf{X}$ is written $\mathsf{x} = (x_0, ..., x_T)$. Given $\mathsf{x} \in \mathsf{X}$, I define $(\mathsf{x}_{-i}, x) = (x_0, ..., x_{i-1}, x, x_{i+1}, ..., x_T) \in \mathsf{X}$ as the consumption path that coincides with x in all but the i^{th} period, in which it yields outcome x. I denote the set of certain consumption paths faced in period t by $\mathsf{X}^t = {X^*}^{T-t+1} \subset P_t$. In every period $t \in T$ the decision maker's preferences $\succeq_t$ are a binary relation on P_t.

Further Remarks: The operator $\oplus_s^\alpha$ mixes same degree of subjectivity lotteries within a given uncertainty layer (which is given by the lottery with the higher rank). An alternative composition mixes two arbitrary lotteries on an elevated level. For defining this alternative composition, I denote lotteries in P_t^s that are degenerate in the root by the indicator function $\mathbf{1}_{p_t}^s$, which is characterized by

$$\mathbf{1}_{p_t}^s(B) = \begin{cases} 1 \text{ if } p_t \in B \\ 0 \text{ if } p_t \notin B \end{cases}$$

for all $B \in \mathfrak{B}_t^N$. Note that, in principle, the lotteries $p_t, \mathbf{1}_{p_t}^s$, and $\mathbf{1}_{p_t}^{s'}$ are different for $s \neq s'$ (the axioms will imply that all three are evaluated the same). For any $s \in S$, $\alpha \in [0, 1]$, $p_t, p_t' \in P_t$, and with $n^* = \max\{\hat{n}(p_t), \hat{n}(p_t')\} + 1 \leq N$, I define an elevating probability α mixture by the operation $\odot_s^\alpha : P_t \times P_t \to P_t^s$ that maps $(p_t, p_t') \mapsto p_t \odot_s^\alpha p_t' \in Y_s^{n^*}$ defined by

$$p_t \odot_s^\alpha p_t'(B) = \alpha \mathbf{1}_{p_t}^s(B) + (1 - \alpha)\mathbf{1}_{p_t'}^s(B) \tag{2}$$

for all $B \in \mathfrak{B}_t^{n^*}$.

If both lotteries p_t and p_t' share the same degree of subjectivity, it stands to reason that a decision maker is indifferent whether probabilities are manipulated at the same lottery level or whether the manipulation takes place at an elevated level. Such an assumption corresponds to the statement

$$p_t \odot_s^\alpha p'_t \sim_t p_t \oplus_s^\alpha p'_t \qquad \text{for all} \quad p_t, p'_t \in P_t^s \text{ with } \hat{n}(p_t), \hat{n}(p'_t) < N \ . \tag{3}$$

Indifference in equation (is a special case of an axiom requiring indifference to the reduction of same degree of subjectivity lotteries introduced in the next section.

3 Axioms

The first axiom makes the decision maker indifferent to the reduction of same degree of subjectivity lotteries. Using the notation of a reduced lottery introduced in equation (the assumption is

A1 (indifference to reduction of lotteries with same degree of subjectivity)
For all $t \in T$, $s \in S$, $n < N$, $p_t \in \Delta_s\left(Y_s^n(X^* \times P_{t+1})\right)$: $\quad p_t \sim_t p_t^r$.

A decision maker who satisfies axiom A1 is indifferent between the two lotteries depicted in Figure 1. Note that the literature mentioned in the introduction that employs second order probabilities employs the uncertainty layer in order to distinguish between objective and subjective lotteries. In these papers, uncertainty attitude is tied to the layer and layers cannot be reduced. Instead, I tie the difference in uncertainty attitude directly to subjectivity and confidence as opposed to the level or order in which uncertainty strikes the agent. This way I can impose axiom A
(and satisfy equation

The following three axioms largely replicate the standard von Neumann & Morgenstern (1944) axioms for the compact metric space setting (e.g. Grandmont 1972).

A2 (weak order) For all $t \in T$ preferences $\succeq_t$ are transitive and complete, i.e.:
– transitive: For all $p_t, p'_t, p''_t \in P_t$: $p_t \succeq p'_t$ and $p'_t \succeq p''_t \Rightarrow p_t \succeq p''_t$
– complete: For all $p_t, p'_t \in P_t : p_t \succeq p'_t$ or $p'_t \succeq p_t$.

A3 (independence) For all $s \in S$, $\alpha \in [0,1]$, and $t \in T$:
For all $p_t, p'_t, p''_t \in P_t^s$: $\quad p_t \succeq_t p'_t \quad \Rightarrow \quad p_t \oplus_s^\alpha p''_t \succeq_t p'_t \oplus_s^\alpha p''_t$.

A4 (continuity) For all $t \in T$, for all $p_t \in P_t$:
$\{p'_t \in P_t : p'_t \succeq p_t\}$ and $\{p'_t \in P_t : p_t \succeq p'_t\}$ are closed in P_t .

The independence axiom is the only axiom that is slightly modified. I could call it "independence with respect to same degree of subjectivity mixing". It is mostly a technical assumption to require the same degree of subjectivity for the lotteries $p_t, p'_t, p''_t \in P^s_t$ and the $\oplus^\alpha_s$ operator. This assumption is necessary to permit a meaningful mixing at a given uncertainty layer. The fact that mixing takes place only for lotteries with coinciding degrees of subjectivity and within the uncertainty layer of the higher ranked lottery is further discussed in the remark at the end of this section. The remark also discuss an alternative independence axiom that mixes lotteries of differing degrees of subjectivity at a higher uncertainty level.

I add additive separability on certain consumption paths in order to replicate the predominant framework for certain intertemporal choice. I employ the axiomatization of Wakker (1988).[5]

A5 (certainty separability)
i) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}$, $x, x' \in X^*$ and $t \in T$:

$$(\mathsf{x}_{-t}, x) \succeq_1 (\mathsf{x}'_{-t}, x) \;\Leftrightarrow\; (\mathsf{x}_{-t}, x') \succeq_1 (\mathsf{x}'_{-t}, x')$$

ii) If $T = 1$ additionally: For all $x_t, x'_t, x''_t \in X^*$, $t \in \{0, 1\}$

$$(x_0, x_1) \sim_1 (x'_0, x''_1) \;\wedge\; (x'_0, x'_1) \sim_1 (x''_0, x_1) \;\Rightarrow\; (x_0, x'_1) \sim_1 (x''_0, x''_1) \; .$$

Wakker (1988) calls part *i*) of the axiom coordinate independence. It requires that the choice between two consumption paths does not depend on period t consumption, whenever the latter coincides for both paths. Part *ii*) is known as the Thomsen condition. It is required only if the model is limited to two periods.[6] Preferences in different periods are related by the following consistency assumption adapted from Kreps & Porteus (1978).

[5] Other axiomatizations of additive separability include Koopmans (1960), Krantz, Luce, Suppes & Tversky (1971), Jaffray (1974*a*), Jaffray (1974*b*), Radner (1982), and Fishburn (1992).

[6] In the case of two periods parts *i*) and *ii*) can also be replaced by the single requirement of triple cancellation (see Wakker 1988, 427).

A6 (time consistency) For all $t \in \{0, ..., T-1\}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \;\;\Leftrightarrow\;\; p_{t+1} \succeq_{t+1} p'_{t+1} \quad \forall\, x_t \in X^*,\; p_{t+1}, p'_{t+1} \in P_{t+1} \;.$$

The axiom is a requirement for choosing between two consumption plans in period t, both of which are degenerate and yield a coinciding outcome in the respective period. For these choice situations, axiom A6 demands that in period t, the decision maker prefers the plan that gives rise to the lottery that is preferred in period $t+1$.

Further Remarks: I pointed out that the operator $\oplus_s^\alpha$ and, thus, the independence axiom A3, mixes same degree of subjectivity lotteries within the root level of the higher ranked lottery. In the remark of the preceding section, I defined an alternative mixture composition $\odot_s^\alpha$ where the mixture of two lotteries happens at an elevated level, incrementing the rank. An alternative to axiom A3 is the following axiom

A3' (elevating independence) For all $s \in S$, $\alpha \in [0,1]$, $t \in T$, and $p_t, p'_t, p''_t \in P_t$ with $\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t) < N$: $\quad p_t \succeq_t p'_t \quad \Rightarrow \quad p_t \odot_s^\alpha p''_t \;\succeq_t\; p'_t \odot_s^\alpha p''_t \quad .$

The axiom differs from axiom A3 in two respects. First, it no longer requires that the lotteries p_t, p'_t, and p''_t share a common degree of subjectivity. Second, it creates the lottery mixture on a higher level than either of the individual lotteries, which is necessary to accommodate the differing degrees of subjectivity. The first change makes it stronger, however, the second change disconnects the levels of the primitive lotteries and the mixed lottery.

The final paragraph discusses the relation between axioms A3' and A3. It is easily verified that indifference between the $\oplus_s^\alpha$ and the $\odot_s^\alpha$ operations holds in the sense of equation (under the assumption of indifference to the reduction of same degree of subjectivity lotteries axiom A1.[7] Therefore, under assumption A1, axiom A3' implies axiom A3,[8] and axiom A3 implies axiom A3' restricted to same degree of subjectivity lotteries. It might be less obvious that already axiom A2 ensures that axiom A3 implies axiom A3' for same degree of subjectivity lotteries. The reason is that axiom A3 itself already contains a mild version of an assumption of indifference to the reduction of degenerate lotteries. See appendix A for details.

[7] Use the definition of $\odot_s^\alpha$ along with equation (and equation (.

[8] For lotteries satisfying $\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t) < N$. Otherwise the elevating independence axiom creates a mixture outside of the preference domain.

4 The Representation

This section gives a welfare representation for preferences satisfying the axioms introduced in the preceding section. A detailed discussion of the representation is delegated to section 5. I close the current section by illustrating how to apply the theorem to an evaluation of the uncertainty tree depicted in Figure 1.

4.1 The represenation theorem

The representation recursively constructs a welfare function $\hat{u}_t : X^* \times P_{t+1} \to \mathrm{I\!R}$ that evaluates degenerate outcomes in every period. Within a period, the representation recursively evaluates the different layers of uncertainty (subtrees of the uncertainty tree in Figure 1). The risk aversion in evaluating a lottery at a particular node is tied to the degree of subjectivity. This risk aversion can be captured by a set of continuous functions $\hat{f}_t = \{f_t^s\}_{s \in S}$, $f_t^s : \mathrm{I\!R} \to \mathrm{I\!R}$. I call these functions *uncertainty aggregation weights*. I define the *generalized uncertainty aggregator* $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} : P_t \to \mathrm{I\!R}$ for a given continuous bounded function $\hat{u}_t : X^* \times P_{t+1} \to \mathrm{I\!R}$ and a given set of uncertainty aggregation weights $\hat{f}_t = \{f_t^s\}_{s \in S}$ as follows. For degenerate lotteries $p_t = (x_t, p_{t+1}) \in P_t$ set $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t}(x_t, p_{t+1}) = \hat{u}_t(x_t, p_{t+1})$. Then inductively increase the domain to lotteries of rank $\hat{n}(p_t) = 1, 2, ..., N$ by defining

$$\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t = \left(f_t^{\hat{s}(p_t)} \right)^{-1} \circ \int_{\mathfrak{B}_t^{\hat{n}(p_t)}} f_t^{\hat{s}(p_t)} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \; p'_t \quad dp_t(p'_t) \; , \tag{4}$$

where the sign $\circ$ emphasizes functional composition as opposed to multiplication and the superindex -1 inverts the function in brackets. For any step in the recursion the expression $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \; p'_t$ captures certainty equivalent welfare for the lottery p'_t. The certainty equivalent welfare of each of these p'_t subtrees is transformed with the uncertainty aggregation weight $f_t^{\hat{s}(p_t)}$, corresponding to the degree of subjectivity of the lottery p_t. The integral sums over these probability weighted values and, finally, the inverse function $\left(f_t^{\hat{s}(p_t)}\right)^{-1}$ renormalizes the expression. The basic structure of the right hand side of equation (is that of a generalized mean of the form $f^{-1}\left[\mathrm{E}f(z)\right]$, where the variable z is the certainty equivalent welfare at a given layer of the uncertainty tree. A generalized mean of the form $f^{-1}\left[\mathrm{E}f(z)\right]$ results in a lower welfare equivalent than $\mathrm{E}z$ if the function f is increasing and concave. Therefore the concavity of f captures a form of risk aversion that will be discussed in detail in section 5. In equation (the function f, and thus risk aversion, generally depends on the subjectivity $\hat{s}(p_t)$ of the lottery over which expectations are taken.

Theorem 1: The sequence of preference relations $(\succeq_t)_{t\in T}$ satisfies axioms A1-A6 if, and only if, for all $t \in T$ there exist a set of strictly increasing and continuous functions $\hat{f}_t = \{f_t^s\}_{s\in S}$, $f_t^s : \mathbb{R} \to \mathbb{R}$, and a continuous and bounded function $u_t : X^* \to U \subset \mathbb{R}$ such that by defining recursively the functions $\hat{u}_T = u_T$ and $\hat{u}_{t-1} : X^* \times P_t \to \mathbb{R}$ by

$$\hat{u}_{t-1}(x_{t-1}, p_t) = u_{t-1}(x_{t-1}) \; + \; \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \, p_t \tag{5}$$

holds for all $t \in T$ and all $p_t, p'_t \in P_t$

$$p_t \;\succeq_t\; p'_t \quad \Leftrightarrow \quad \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \, p_t \;\geq\; \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \, p'_t \, . \tag{6}$$

Preferences $(\succeq_t)_{t\in T}$ over the space of generalized temporal lotteries can be represented by the sequence $(u_t, \hat{f}_t)_{t\in T}$. The functions u_t represent per period utility and inform the recursive construction of the intertemporal welfare function $\hat{u}_t$ (equation

Note that the representation in Theorem 1 is linear in every time step. In a setting where lotteries are not distinguished by their degree of subjectivity, the representation of this paper closely corresponds to Kreps & Porteus (1978). In their representation, Kreps & Porteus (1978) use a linear uncertainty aggregation at the expense of a non-linear time aggregation. Traeger (2007) shows how to shift this non-linearity between the time and the risk dimension. In the current setting, however, lotteries vary in their degree of subjectivity. Here, giving up linearity in the time step in equation (would only facilitate the linearization of f_t^s for one $s \in S$ and would not permit a linear aggregation over uncertainty in general. Thus, I consider the employed linearization over time as the preferred representation. Finally, note that affine transformation of the functions $\hat{f}_t^s$ leave the represented preferences unchanged. Affine transformation of the functions u_t have to share a common multiplicative constant (in the different periods) and have to be accompanied with a coinciding transformation of the functions $\left(\hat{f}_t^s\right)^{-1}$ for all $s \in S$.[9]

[9]This transformation is equivalent to composing the functions $\hat{f}_t^s$ with the inverse transformation from the right.

Further Remarks: The representation building on axioms A1 to A6 satisfies as well elevating independence, axiom A3', which mixes lotteries of differing degrees of subjectivity (the proof is appended to the proof of Theorem 1). Axiom A3' can be considered a normatively desirable property. Axiom A1 is responsible for connecting the uncertainty weights on the different layers. It implies the existence of a set $\hat{f}_t$ that is independent of the uncertainty layer.

4.2 Example

Assume that the decision maker faces a two period problem with a certain payoff in period 0 and an uncertain payoff in period 1 that is described by the lottery depicted on the left hand side of Figure 1. The payoffs at the leaves, omitted in the graph, are from top to bottom $\overline{x}$, $\underline{x}$, x^*, x^*, $\overline{x}$, $\underline{x}$. The payoff in the first period is x^*. Assume that the corresponding utility values are $u(\overline{x}) = 6$, $u(\underline{x}) = 0$, and $u(x^*) = 5$ and that second period utility is discounted by the factor $\beta = \frac{40}{41}$ implying a rate of pure time preference of approximately 2.5%. In a unidimensional setting these utilities can be generated by setting $\overline{x} = 20$, $\underline{x} = 0$, $x^* = 12$, and employing the utility functions $u_0(x_0) = \ln(1+x_0^2)$ and $u_1(x_1) = \beta \ln(1+x_1^2)$, rounding at the second decimal. Assume that the decision maker's risk aversion function is $f^s(z) = z$ for lotteries of confidence level s, $f^{s'}(z) = z^{\frac{1}{2}}$ for lotteries of confidence level s', and $f^{s''}(z) = z^{\frac{1}{3}}$ for lotteries of confidence level s'', where $z \in \mathbb{R}_+$. The scenario is evaluated recursively in time and, in every period, recursively in the uncertainty layer. First, the two lotteries of degree of subjectivity s' and s'' in the lowest uncertainty layer in period 1 are evaluated by calculating the certainty equivalent utilities

$$m_1 \equiv (f^{s'})^{-1} \left[\tfrac{1}{2} f^{s'}[u(\overline{x})] + \tfrac{1}{2} f^{s'}[u(\underline{x})]\right] = \left[\tfrac{1}{2}\, 6^{\frac{1}{2}} + \tfrac{1}{2}\, 0^{\frac{1}{2}}\right]^2 = \tfrac{3}{2}$$

$$m_2 \equiv (f^{s''})^{-1} \left[\tfrac{1}{2} f^{s''}[u(\overline{x})] + \tfrac{1}{2} f^{s''}[u(\underline{x})]\right] = \left[\tfrac{1}{2}\, 3^{\frac{1}{3}} + \tfrac{1}{2}\, 0^{\frac{1}{3}}\right]^3 = \tfrac{3}{4}\ .$$

The uncertainty tree comprises two more layers of uncertainty. By axiom A1 the decision maker could alternatively evaluate a single reduced layer of uncertainty, which is depicted in the tree on the right hand side of Figure 1. The following calculation illustrates this equivalence by tackling the two remaining uncertainty layers simultaneously, where curly brackets relate to the uncertainty aggregation in the root and square brackets correspond to uncertainty aggregation in the subsequent layer:

$$\begin{aligned} m_3 \;&\equiv (f^s)^{-1}\Big\{\; \tfrac{1}{3} f^s \left\{ (f^s)^{-1} \left[\tfrac{1}{2} f^s[m_1] + \tfrac{1}{2} f^s[u(x^*)] \right] \right\} \\ &\qquad\qquad + \tfrac{2}{3} f^s \left\{ (f^s)^{-1} \left[\tfrac{2}{5} f^s[u(x^*)] + \tfrac{3}{5} f^s[m_2] \right] \right\} \Big\} \\ &= (f^s)^{-1}\Big\{\; \tfrac{1}{6} f^s[m_1] + \tfrac{1}{6} f^s[u(x^*)] + \tfrac{4}{15} f^s[u(x^*)] + \tfrac{6}{15} f^s[m_2] \Big\} \\ &= \tfrac{1}{6}\tfrac{3}{2} + \tfrac{1}{6}\; 5 + \tfrac{4}{15}\; 5 + \tfrac{6}{15}\tfrac{3}{4} = \tfrac{41}{20} \end{aligned}$$

The intermediate step is equivalent to directly evaluating the reduced lottery. The certainty equivalent utility m_3 is discounted, resulting in a present value utility of $\beta\, m_3 = \frac{40}{41}\,\frac{41}{20} = 2$. Adding the utility $u(x^*) = 5$ that the decision maker obtains with certainty in period 0, he evaluates the scenario with an overall present value welfare of 7 units.

The above decision maker is risk averse with respect to lotteries of degree of subjectivity s' and s'' (in a way made precise in the next section). I compare his evaluation to that of a decision maker who is risk neutral with respect to all lotteries (and, thus, is described by the intertemporally additive standard model). Such an evaluation with $f^s(z) = f^{s'}(z) = f^{s''}(z) = z$ leads to an overall welfare of $5 + \frac{40}{41}\,\frac{52}{15} \approx$ 8.5 units. Using the utility functions $u_0(x_0) = \ln(1 + x_0^2)$ and $u_1(x_1) = \beta \ln(1 + x_1^2)$ I compare the risk neutral and the original evaluation in terms of certainty equivalent consumption. In the original evaluation the decision maker is indifferent to the lottery and to receiving 2.6 consumption units with certainty in the second period, while the risk neutral decision maker is indifferent between the lottery and receiving 6.8 consumption units with certainty. Observe that not only the risk averse, but also the risk neutral decision maker has a preference for smoothing consumption over time: He is willing to accept a reduction of 0.6 units in overall consumption in order to smooth his uneven certainty equivalent consumption path of 12 units in period 0 and 6.8 units in period 1 to a welfare equivalent consumption path where he consumes 9.1 units in both periods.

5 Discussion of the Representation

The discussion of the representation in Theorem 1 proceeds in two steps. First, I analyze a restricted version of the model limiting the space S to only two degrees of subjectivity. This restricted version of the model is a straight-forward generalization of Klibanoff et al.'s (2009) smooth ambiguity setting. I show that, in the generalized setting, Klibanoff et al.'s (2009) definition of smooth ambiguity aversion is "ambiguous" and I offer a more precise definition. Moreover, I disentangle intertemporal substitutability from risk aversion and ambiguity aversion. Then, I proceed to discuss the general setting with an arbitrary number of degrees of subjectivity in the lottery space. In particular, I generalize the definition of smooth ambiguity aversion to a notion of aversion to subjectivity or to the lack of confidence.

5.1 A binary classification of subjectivity or confidence

I start by interpreting a special case of the representation obtained from restricting the degrees of subjectivity to $\#S = 2$. I associate the two elements $s \in S = \{subj, obj\}$ with subjective and objective beliefs. Two further restrictions transform it into the smooth ambiguity model of Klibanoff et al. (2009) – translated into the von Neumann-Morgenstern setting. First, the evaluation of objective lotteries in Klibanoff et al.'s (2009) setting is (intertemporally) risk neutral in the sense that f_t^{obj} is absent from their representation. This latter point will be discussed in detail further below. Second, Klibanoff et al. (2009) restrict the number of uncertainty layers in every time period to $N = 2$ and impose a hierarchy of beliefs implying that decision makers can only face subjective lotteries over objective lotteries, but not vice versa. Uncertainty resolving in period t of the form depicted by lottery p_t^r on the right of Figure 1 would qualify for the restricted setting if $s = subj$ and $s' = s'' = obj$ (but not if $s = obj$ and $s' = s'' = subj$, e.g. representing a coin flip over whether to enter a situation of subjective risk). In contrast, the representation in Theorem 1 permits an arbitrary sequence of subjective and objective lotteries (within every period).

Maintaining these restrictions, the first interesting insight is that the representation in Theorem 1 is close to the standard von Neumann-Morgenstern setting. Lotteries simply have to be labeled by their degree of subjectivity and even the independence axiom is preserved. Thus, explicitly introducing the dimensions that Ellsberg (1961) already found missing in the Savage framework, i.e. a degree of confidence or subjectivity of belief, leads straight forwardly from von Neumann & Morgenstern (1944) to a model of smooth ambiguity aversion. The next insight concerns the interpretation of Klibanoff et al.'s (2009) concept of smooth ambiguity aversion. For this purpose, I briefly relate the representation in Theorem 1 to the generalized isoelastic model of Epstein & Zin (1989) and Weil (1990). A priori, a decision maker's propensity to smooth consumption over time is a different preference characteristic than his risk aversion. However, the intertemporally additive expected utility standard model implicitly assumes that these different dimensions of preference coincide. Epstein & Zin (1989) and Weil (1990) observed that in a one commodity version of Kreps & Porteus's (1978) recursive utility model of temporal lotteries disentangles these two dimensions of preference. Traeger (2007) shows in a setting corresponding to a $\#S = 1$ version of the current model, that the function f_t measures the difference between Arrow Pratt risk aversion and aversion to intertemporal substitution. As there is only one type of risk in the cited analysis, there is only one function f_t in every period used for uncertainty aggregation. He names f_t a measure of intertemporal risk aversion. It measures the part of risk aversion that is not simply a cause of a decision maker's propensity to smooth over time, but due an intrinsic aversion to risk. The concept of intertemporal risk aversion is not limited to the one-commodity setting of the Epstein & Zin (1989) framework, but generalizes to arbitrary dimensions and to settings without a naturally given measure scale of the good under observation. The following axiomatic characterization is put forth in Traeger (2007). For two given consumption paths $\mathrm{x}, \mathrm{x}' \in \mathrm{X}^t$, I define the 'best of combination' path $\mathrm{x}^{\text{high}}(\mathrm{x}, \mathrm{x}')$ by $(\mathrm{x}^{\text{high}}(\mathrm{x}, \mathrm{x}'))_\tau = \operatorname{argmax}_{x \in \{\mathrm{x}_\tau, \mathrm{x}'_\tau\}} u_\tau(x)$ and the 'worst off combination' path $\mathrm{x}^{\text{low}}(\mathrm{x}, \mathrm{x}')$ by $(\mathrm{x}^{\text{low}}(\mathrm{x}, \mathrm{x}'))_\tau = \operatorname{argmin}_{x \in \{\mathrm{x}_\tau, \mathrm{x}'_\tau\}} u_\tau(x)$ for all $\tau \in \{t, ..., T\}$.[10] In every period the consumption path $\mathrm{x}^{\text{high}}(\mathrm{x}, \mathrm{x}')$ picks out the better outcome of x and x', while $\mathrm{x}^{\text{low}}(\mathrm{x}, \mathrm{x}')$ collects the inferior outcomes. A decision maker is called (weakly)[11] *intertemporal risk averse* in period t if and only if for all consumption paths $\mathrm{x}, \mathrm{x}' \in \mathrm{X}^t$

$$\mathrm{x} \sim \mathrm{x}' \quad \Rightarrow \quad \mathrm{x} \ \succeq_t \ \tfrac{1}{2}\, \mathrm{x}^{\text{high}}(\mathrm{x}, \mathrm{x}') + \tfrac{1}{2}\, \mathrm{x}^{\text{low}}(\mathrm{x}, \mathrm{x}'), \tag{7}$$

where $\frac{1}{2}\, \mathrm{x}^{\text{high}}(\mathrm{x}, \mathrm{x}') + \frac{1}{2}\, \mathrm{x}^{\text{low}}(\mathrm{x}, \mathrm{x}')$ denotes a lottery of equal chance over the paths $\mathrm{x}^{\text{high}}(\mathrm{x}, \mathrm{x}')$ and $\mathrm{x}^{\text{low}}(\mathrm{x}, \mathrm{x}')$. The premise states that a decision maker is indifferent between the certain consumption paths x and x'. Then, an intertemporal risk averse decision maker prefers the consumption path x (or equivalently x') with certainty

[10] Traeger (2007) shows how these paths can be defined purely in terms of preferences.

[11] Analogously, a strict intertemporal risk averse decision maker can be defined by assuming in addition that there exists some period t^* such that $u(\mathrm{x}_{t^*}) \neq u(\mathrm{x}'_{t^*})$ and requiring a strict preference $\succ$ rather than the weak preference $\succeq$ in equation (.

over a lottery that yields with equal probability either a path combining all the best outcomes or a path combining all the worst outcomes. The cited paper shows that the function f_t in the representation is concave if and only if equation (holds. In a certainty additive representation, as employed in the current paper, intertemporal risk aversion can also be interpreted as risk aversion with respect to utility gains and losses.

The definition of intertemporal risk aversion extends straight forwardly to a setting with differing degrees of risk aversion to objective versus subjective lotteries. I characterize *intertemporal risk aversion to objective lotteries* by requiring for all $x, x' \in X^t$

$$x \sim x' \quad \Rightarrow \quad x \ \succeq_t \ x^{\text{high}}(x, x') \oplus^{\frac{1}{2}}_{obj} x^{\text{low}}(x, x') \tag{8}$$

implying concavity of f_t^{obj}, and similarly *intertemporal risk aversion to subjective lotteries* by requiring for all $x, x' \in X^t$

$$x \sim x' \quad \Rightarrow \quad x \ \succeq_t \ x^{\text{high}}(x, x') \oplus^{\frac{1}{2}}_{subj} x^{\text{low}}(x, x') \tag{9}$$

implying concavity of f_t^{subj}. Klibanoff et al. (2009) implicitly assume that $f^{obj} = \text{id}$, which corresponds to indifference in equation (. This assumption implies that uncertainty evaluation with respect to objective (or first order) lotteries is intertemporal risk neutral. Only when it comes to subjective lotteries, Klibanoff et al. (2009) introduce a non-trivial function f^{subj} and, thus, allow for intertemporal risk aversion. Klibanoff et al. (2009) define ambiguity aversion by the concavity of f_t^{subj} (in the setting assuming $f_t^{obj} = \text{id}$). This concept earned the name smooth ambiguity aversion in the decision theoretic literature. Relaxing the restriction $f_t^{obj} = \text{id}$ sheds more light onto this definition. In principle, there are two sensible ways of extending Klibanoff et al.'s (2009) representation to incorporate the missing non-linearity f_t^{obj}. The representation I have chosen in Theorem 1 introduces the function f_t^{obj} in such a way that it measures intertemporal risk aversion with respect to objective lotteries without changing the interpretation that f_t^{subj} measures intertemporal risk aversion with respect to subjective lotteries. Given the hierarchical order of subjective over objective lotteries in Klibanoff et al.'s (2009) setting, I can introduce an alternative function $f_t^{amb} \equiv f_t^{subj} \circ (f_t^{obj})^{-1}$ to eliminate f_t^{subj} from the representation. Observe the following transformation of the representing equation (where p_t and p'_t are different subjective lotteries over the set of objective lotteries, whose representatives are $\tilde{p}_t$

$$
\begin{aligned}
p_t \succeq_t p'_t \quad &\Leftrightarrow \quad \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\, p_t \;\geq\; \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\, p'_t \\
&\Leftrightarrow \left(f_t^{subj}\right)^{-1} \circ \int\limits_{Z^1(X^* \times P_{t+1})} f_t^{subj} \circ \left(f_t^{obj}\right)^{-1} \left[\int\limits_{X^* \times P_{t+1}} f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1})\, d\tilde{p}_t(x_t, p_{t+1}) \right] dp_t(\tilde{p}_t) \\
&\quad \geq \left(f_t^{subj}\right)^{-1} \circ \int\limits_{Z^1(X^* \times P_{t+1})} f_t^{subj} \circ \left(f_t^{obj}\right)^{-1} \left[\int\limits_{X^* \times P_{t+1}} f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1})\, d\tilde{p}_t(x_t, p_{t+1}) \right] dp'_t(\tilde{p}_t) \\
&\Leftrightarrow \left(f_t^{amb}\right)^{-1} \circ \int\limits_{Z^1(X^* \times P_{t+1})} f_t^{amb} \left[\int\limits_{X^* \times P_{t+1}} f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1})\, d\tilde{p}_t(x_t, p_{t+1}) \right] dp_t(\tilde{p}_t) \\
&\quad \geq \left(f_t^{amb}\right)^{-1} \circ \int\limits_{Z^1(X^* \times P_{t+1})} f_t^{amb} \left[\int\limits_{X^* \times P_{t+1}} f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1})\, d\tilde{p}_t(x_t, p_{t+1}) \right] dp'_t(\tilde{p}_t)
\end{aligned}
$$

This new function $f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ then measures the additional aversion to subjective risk as opposed to objective risk. For this interpretation, note that $f_t^{subj} \circ (f_t^{obj})^{-1}$ concave is a definition of f_t^{subj} being more concave than f_t^{obj} (Hardy et al. 1964).[12] Because Klibanoff et al.'s (2009) setting assumes $f_t^{obj} = \text{id}$, their definition of ambiguity aversion does not pin down whether smooth ambiguity aversion should be captured by intertemporal aversion to subjective risk, captured in f_t^{subj} and characterized by the lottery choice (, or whether it should be characterized by the functions f_t^{amb} measuring the additional risk aversion to subjective risk as opposed to objective risk. I suggest calling the latter a measure of smooth ambiguity aversion.

Definition 1: A decision maker exhibits (strict) smooth ambiguity aversion in period t if the function

$$f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$$

in the preference representation of Theorem 1 is (strictly) concave.

I follow Klibanoff et al. (2009) in defining the term by means of characteristics of the representation. However, (strict) concavity of the function f_t^{amb} is a characteristic of

[12]Hereto observe that f_t^{amb} concave and $f_t^{subj} = f_t^{amb} \circ (f_t^{obj})$ implies that f_t^{subj} is a concave transformation of f_t^{obj}.

preferences that is independent of a particular version of the representation. Employing equations (and (the condition $f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ concave translates smooth ambiguity aversion in period t into the requirement that for all $x, x', x'' \in X^t$

$$x \sim x' \succeq_t x^{\text{high}}(x, x') \oplus_{obj}^{\frac{1}{2}} x^{\text{low}}(x, x') \quad \Rightarrow \quad x \;\succeq_t\; x^{\text{high}}(x, x') \oplus_{subj}^{\frac{1}{2}} x^{\text{low}}(x, x') \; .$$

However, ambiguity aversion can be characterized more simply by recognizing that the intertemporal aspect of the risk comparison can be dropped.

Proposition 1: A decision maker exhibits (strict) smooth ambiguity aversion in the sense of Definition 1 if, and only if, for all $x, x' \in X^t$

$$x \oplus_{obj}^{\frac{1}{2}} x' \;\; \succeq_t \;\; (\succ_t) \;\; x \oplus_{subj}^{\frac{1}{2}} x' \; .$$

In a one-commodity setting,[13] the model gives rise to a three-fold disentanglement that can be expressed in terms of six different but related concepts (sharing three degrees of freedom):

- the functions u_t characterize aversion to intertemporal substitution,
- the functions f_t^{subj} characterize intertemporal risk aversion to objective risk,
- the functions f_t^{obj} characterize intertemporal risk aversion to subjective risk,
- the functions $f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ characterize smooth ambiguity aversion,
- the functions $g_t^{obj} \equiv f_t^{obj} \circ u_t$ measure Arrow Pratt risk aversion with respect to objective lotteries, and
- the functions $g_t^{subj} \equiv f_t^{subj} \circ u_t$ measure Arrow Pratt risk aversion with respect to subjective risk.

If follows immediately that, in the one-commodity setting, smooth ambiguity aversion can be expressed also as the difference in Arrow Pratt risk aversion with respect to subjective risk and Arrow Pratt risk aversion with respect to objective risk:

$$f_t^{amb} = g_t^{subj} \circ (g_t^{obj})^{-1} \; .$$

[13] Only in the one-commodity setting are the inverse of u, the Arrow Pratt measure of risk aversion, and the measure of intertemporal substitution unidimensional and well defined.

5.2 The general representation and aversion to the lack of confidence

A unique measure of ambiguity aversion is tied to the setting with $\#S = 2$. In general, a decision maker will not always be able to employ a binary classification scheme for the subjectivity of or confidence in lotteries. While objective probabilities are generally classified as those derived from symmetry reasoning or long-run, high frequency observations, subjective risk is basically any probabilistic belief not obtained in that way, which leaves a wide range of belief types for a single category. Examples include the odds based on a short time series or a slightly irregular dice, a horse race lottery, the odds of a 2°C global warming by 2050 due to climate change, or weather characteristics in Tomboctou on November 22nd 2012. In general, different decision makers are likely to classify different lotteries in different categories. Assume that a decision maker has a complete order over the elements in S. Let $s \triangleright s'$ denote that a lottery labeled s is more subjective than a lottery labeled s', or that s lacks more confidence than s'.

Definition 2: A decision maker is (strictly) averse to the lack of confidence if

$$s \triangleright s' \quad \Leftrightarrow \quad f_t^s \circ (f_t^{s'})^{-1} \text{ (strictly) concave} \quad \forall s, s' \in S \ .$$

For example, assume that the decision maker in the example of section 4.2 is most confident in the lotteries labeled s in Figure 1 and least confident when it comes to his probability estimates labeled s'', i.e., $s'' \triangleright s' \triangleright s$. It is easily verified that the decision maker exhibits aversion to the subjectivity of belief: $f_t^{s''} \circ (f_t^{s'})^{-1}(z) = z^{\frac{2}{3}}$ and $f_t^{s'} \circ (f_t^{s})^{-1}(z) = z^{\frac{1}{2}}$ are both concave (and the remaining case follow from transitivity).

Definition 1 of smooth ambiguity aversion is the special case of aversion to the lack of confidence in the case where $\#S = 2$. Its characterization in terms of preferences straight-forwardly carries over to the case of aversion to the lack of confidence.

Proposition 2: A decision maker exhibits (strict) aversion to the lack of confidence in the sense of Definition 2 if, and only if, for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^t$ and $s, s' \in S$ with $s \triangleright s'$

$$\mathsf{x} \oplus_{s'}^{\frac{1}{2}} \mathsf{x}' \ \succeq_t \ (\succ_t) \ \mathsf{x} \oplus_{s}^{\frac{1}{2}} \mathsf{x}' \ .$$

A decision maker with aversion to the lack of confidence would prefer a scenario with better known probabilities over one with more subjective probabilities. He would be willing to pay for reducing subjectivity and increasing confidence. The next section relates the analysis of this paper to the behaviorally motivated literature. In such a context one can employ proposition 2 in order to construct an order on S. As I will point out that for some behaviorally plausible situations such an order might not exist. Note that we can alternatively take the characterization in Proposition 2 as the definition of aversion to the lack of confidence and derive the characterization in Definition 2 as the result.

Finally, note that a decision maker who is averse to the lack of confidence might exhibit standard risk aversion (or even risk neutrality) with respect to objective risk, but shy away from situations where he feels that he cannot assess the risk involved. If he exhibits extreme aversion to the lack of confidence and feels that he completely lacks the ability to assess the involved probabilities his decision criteria gets arbitrarily close to the framework of decision making under ignorance suggested by Arrow & Hurwicz (1972). Here the decision maker simply maximizes the worst possible outcome. If this completely subjective lottery (or the complete lack of confidence) appears in the second uncertainty layer the decision maker behaves arbitrarily close to the decision maker in Gilboa & Schmeidler's (1989) wide-spread maximin expected utility model, where a decision maker maximizes the worst expected outcome (which here would be expectations over first order lotteries). While Arrow & Hurwicz's (1972) axioms yield maximin over deterministic outcomes and Gilboa & Schmeidler's (1989) axioms yield maximin over expectation, a general decision maker in the current framework can exhibit a decision criteria arbitrarily close to maximin on any layer, including maximin over evaluations employing the smooth ambiguity model, or risk (or ambiguity avers) expectations over maximin models. Thus, a single preference relation in the current framework can nest and interact the wide spread models of risk neutrality, Arrow Pratt risk aversion, maximin utility, maximin expected utility and smooth ambiguity aversion, depending on the situation the decision maker is facing.[14]

[14]As preferences are continuous in the set of probability distributions, the maximin decision criteria are only reached in the limit. However, the preferences permit an arbitrarily close approximation, e.g. by using $f^{ignorance}(z) = -z^{-L}$ with an arbitrarily large $L \in \mathbb{R}$.

6 Implications and applications

The section starts out relating the present framework to the Ellsberg paradox. I then discuss how the representation restricts behavior. Finally, I sketch two normative applications of the model.

6.1 Relation to behavioral analysis

The axioms underlying the representation are selected on a normative basis. Nevertheless, the framework incorporates observed behavior as in the Ellsberg (1961) paradox that cannot be captured within the economic standard model. In this section, I briefly discuss how the present framework relates to the behaviorally motivated ambiguity literature in accommodating Ellsberg type behavior. I then proceed to point out a type of behavior relating to subjectivity attitude that is ruled out by the rationality constraints of the current paper.

In the experiments underlying the Ellsberg (1961) paradox, a decision maker has to bet on the color of a ball that is drawn from an urn. The crucial feature of the various variants of the experiment can be reflected by the following simplified choice situation. In one urn, the decision maker knows that half of the balls are red. In another urn, the decision maker only knows that it contains nothing but red and blue balls. For the first urn, the draw can be characterized by an objective probability of $\frac{1}{2}$ for drawing a red ball. For the second urn, the principle of insufficient reason would give rise to a probability of $\frac{1}{2}$ as well. However, a good fraction of the individuals in comparable settings prefer betting on the first urn where they know the number of red balls.[15] The Choquet expected utility approach to explaining the seemingly paradoxical choice abandons the concept of a probability and replaces it with a non-additivity set function. The latter captures the decision maker's ambiguity about the red balls in the second urn. Choquet integrating over the capacities induces aversion to ambiguity. The multiple prior approach, instead, attaches a range of different probability distributions to drawing a red ball from the second urn and, e.g. in the simplest such approach formulated by Gilboa & Schmeidler (1989), evaluates the bet by the worst expected outcome possible within the range of priors. The Klibanoff

[15]The real versions of the Ellsberg (1961) paradox are set up slightly more sophisticatedly in order to assure that no possible probability assignment can explain the described choice within the standard expected utility setting.

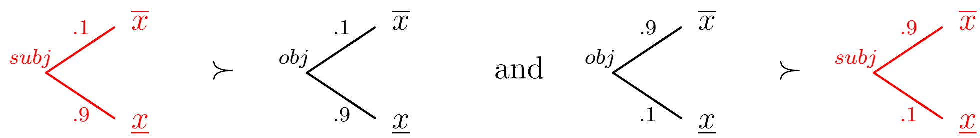

Figure 3 depicts a possible choice behavior corresponding to a non-global attitude with respect to subjectivity of belief.

et al. (2009) approach assigns two layers of probability distributions to the urn with the unknown number of balls. The lower level probability distributions are interpreted as the possible urn compositions. Each such urn composition is interpreted as giving rise to an objective lottery. The higher level distribution assigns a subjective probability weight to each of these possible urn compositions identified with the objective lotteries. Obviously, the representation in Theorem 1 can handle the Ellsberg paradox in the same way. However, there is an alternative way to describe the behavior by means of the representation in Theorem 1. The decision maker attaches a probability of one half to the event drawing a red ball for both urns. However, he labels the urn where he knows the number of balls to be an objective lottery and he labels the lottery where the probability of a half is only obtained from the principle of insufficient reason to be a subjective lottery. If the decision maker is averse to the subjectivity of probabilistic beliefs, he prefers to bet on the "objective urn". Note that, in general, some fraction of the participants of an Ellsberg type experiment do not show the "paradoxical behavior" discussed above. The current framework can explain their behavior in two different ways. Either, they are not averse to the lack of confidence (ambiguity averse), or they might simply label any fair urn setting as objective.

Behaviorally, aversion to subjectivity might not always be as convincing as in an Ellsberg (1961) type setting. Take a preference over lotteries as depicted on the left of Figure 3. Here, a decision maker faces a large probability of a terrible outcome $\underline{x}$ delivering welfare $\underline{u} = u(\underline{x})$ and a small probability of a great outcome $\overline{x}$ delivering welfare $\overline{u} = u(\overline{x})$.[16] Agents in such a choice situation prefer the subjective over the objective lottery if they prefer that the probability, stating a terrible event is likely,

[16]In keeping with the intertemporal nature of the general framework each of the explicitly depicted outcomes can be interpreted as a one period entry in a setting with a common future that is independent of the lottery realizations. Alternatively, the utility levels $\overline{u}$ and underline $\underline{u}$ can be interpreted as the welfare $\hat{u}(\cdot)$ of different futures.

is not objective or is of low confidence. Note that from a normative perspective, such a subjectivity loving behavior is probably judged undesirable as it implies that the decision maker would be willing to pay for reducing the quality of the probability assessment keeping expectations the same (a thought experiment only). Now assume that the same agent also exhibits the preference depicted on the right hand side of Figure 3. The choice situation yields the good outcome with a high probability and the terrible outcome with a low probability. I suggest that, in such a situation, the same agent might prefer the objective over the subjective lottery for a similar motive that implied the opposite attitude above: He prefers the objective lottery over the subjective lottery because it makes the small probability (or the smallness of the probability) of the terrible event objective.[17] In summary, the choice behavior in Figure 3 can be characterized as a preference for being less confident about distributions giving a bad outcome with high probability (or a bad expected outcome) as opposed to a preference for being confident about distributions that yield a good outcome with a high probability (or in expectation). As I show in Appendix A, such a preference is reflected by a convex-concave function $f^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ in the representation of Theorem 1. An example of such a convex-concave ambiguity aversion function is depicted on the left hand side of Figure 4.

The example above does not violate any of the axioms underlying the representation in Theorem 1, but goes against a global subjectivity attitude. However, a related behavior can violate the axioms of the representation themselves. Assume that a decision maker exhibits a behavior as depicted in Figure 3 for a sufficiently large set of lotteries, rather than just for a lottery over a worst outcome $\underline{x}$ and a best outcome $\overline{x}$. Given sufficient knowledge regarding the agent's choice on objective lotteries, I can select outcomes $x_1, x_2, x_3 \in X^*$ such that the points $y_i = f^{obj} \circ u(x_i)$ are spread approximately equidistantly with $\underline{y} = f^{obj}(\underline{u}) < y_1 < y_2 < y_3 < f^{obj}(\overline{u}) = \overline{y}$ as depicted on the horizontal axis of the right graph in Figure 4. Assume that the decision maker exhibits the subjectivity attitude reversal depicted in Figure 3 for a lottery over $\underline{x}$ and x_2 as well as for a lottery over x_2 and $\overline{x}$. Then these preferences can be represented by a function f^{amb} that is convex-concave on both, the interval $[\underline{y}, y_2]$ as also the interval $[y_2, \overline{y}]$. An example for an ambiguity attitude function satisfying these requirements is depicted by the solid line in the right graph of Figure 4. Now

[17] I would like to thank Steiner Holden and the participants of the departmental seminar at the University of Oslo for elaborating this example.

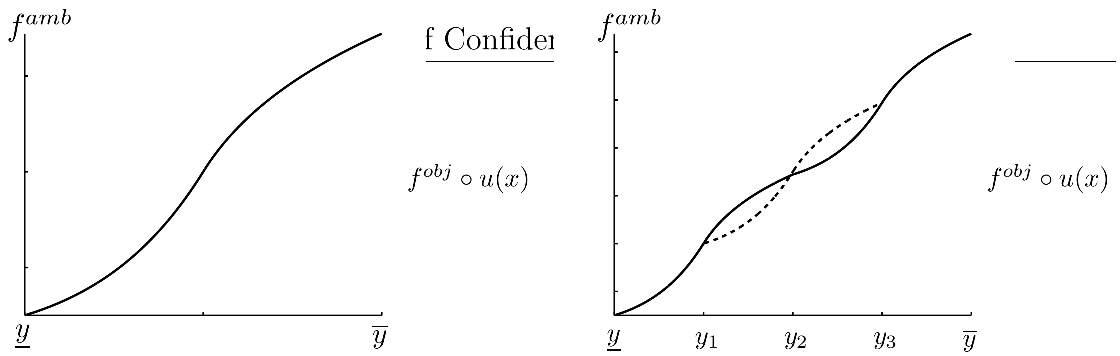

Figure 4 represents a function f^{amb} satisfying the necessary convex-concave characteristics for a global attitude reversal as in Figure 3 (left graph) and for multiple attitude reversals (right graph). The solid line in the right graph explains a choice according to Figure 3 for a lottery with outcomes $\underline{x}$ and x_2 as well as a lottery with outcomes x_2 and $\overline{x}$. The dashed line accommodates the convex-concave characteristics corresponding to a subjectivity attitude reversal in a choice as in Figure 3 for a lottery with outcomes x_1 and x_3, which is incompatible with the preference represented by the solid line. The points on the horizontal axis are $y_i = f^{obj} \circ u(x_i)$.

let the same type of subjectivity attitude reversal also hold for a lottery over x_1 and x_3. Then f^{amb} needs to exhibit convex-concave behavior also on the interval $[y_1, y_3]$ as represented by the dashed line in the right graph of Figure 4. However, the curvature of the dashed line is contradicting the curvature of f^{amb} implied by the earlier choices. The graph illustrates why a sufficiently rich set of subjectivity reversals as in Figure 3 with overlapping welfare implications cannot be represented by a single function f^{amb} and, therefore, violates the axiomatic framework underlying the representation in Theorem 1.

Finally, let me point out a slightly different interpretation of the model in a behavioral context. Individuals could identify the index s with a degree of familiarity with a particular risk. Even when they are rationally aware that, for example, the risk of a tragic plane accident is lower than the risk of dying in a car accident, their familiarity with exposure to ground traffic related risk could imply a relatively lower aversion, while they shy away relatively more from a means of transportation they use less frequently, even if they are aware of the information that the risk is lower.

6.2 Employing the model as a decision support framework - example and further thoughts

I briefly sketch two examples on how to apply the model as a decision support framework. The first is an open loop scenario assessment. The second relates to the question of learning. I draw both examples from the context of climate change economics, where the International Panel on Climate Change encourages a disentanglement of different types of uncertainties.

In the first example, an uncertainty tree for a given period in the future starts with the root lottery capturing uncertainty about the stock of greenhouse gases in the atmosphere. For a given pollution stock there is a subtree describing uncertainty about the temperature in the same period. For a given temperature there is uncertainty about precipitation. Given precipitation, there is uncertainty about agricultural yield. Given agricultural yield there is uncertainty about market prices and so on. Given such an uncertainty tree, the decision maker has to assign his degree of confidence or of subjectivity to each of these lotteries. For example, he assigns relatively more confidence to the subtrees determining a temperature and precipitation distribution if the parent corresponds to a low emission scenario resulting in a more familiar climate. In contrast, if the subtrees branch out from a very high realization of the greenhouse gas stock, the decision maker considers the probabilistic estimates of the temperature and the precipitation distribution less reliable, labeling the nodes with a lower confidence level. Assume that the decision maker is averse to lack of confidence as formalized in Definition 2. Then, he attaches a relatively lower value to the more subjective subtrees stemming from a higher perturbation of the climate system than a decision maker who does not distinguish lotteries by their confidence or subjectivity. Thus, a first conjecture would be that a decision maker with aversion to the lack of confidence would be willing to invest more into measures keeping him in a climate region that he can predict more confidently.

In the second example, a decision maker anticipates learning about the future as time goes by. The recursive structure of the welfare representation naturally invites a dynamic programming setup where the agent takes anticipated learning into account in his current decisions. Let me consider two layers of parametric uncertainty stacked over a layer of stochasticity that cannot be resolved. To make the example concrete, consider once more an agent deciding in the climate change context. A decision maker's payoffs are determined by local temperature and precipitation patterns, depending on the system's variables. In order to predict future temperatures and precipitation he employs a regional climate model that is coupled to a global model. However, there are unknowns θ_1 in the characterizations of the regional climate model and unknowns θ_2 characterizing the global model. Given both, θ_1 and θ_2, the weather characteristics w relevant to his payoffs are purely stochastic and given by the conditional distribution $\mu_0(w|\theta_1, \theta_2)$. Given θ_1 and θ_2 he trusts his model enough to label this lottery $\mu_0(w|\theta_1, \theta_2)$ objective. In contrast, he assigns a low confidence level to his prior $\mu_1(\theta_1|\theta_2)$ over the information state θ_1 characterizing the regional model (the prior might depend on the information underlying the global model θ_2). He assigns a higher confidence level to the prior $\mu_2(\theta_2)$, given that local climate models frequently face even harder challenges than global ones. The important difference to the first example is that the current decision maker spells out how the informational variables evolve over time in order to derive an optimal decision. A standard way to model this learning process would be Bayesian. The decision maker updates his priors μ_1 and μ_2 based on observing regional and global characteristics related to the informational states by means of a likelihood function. In general, the informational states will be informed by a variety of observations with a subset being the payoff relevant characteristics w governed by μ_0. However, in the climate context, the physical observations might even play a minor role as opposed to advancements in the models, driven by computer power and modeling techniques. These advances may be treated by considering the generated results as new observations. However, such a treatment would be somewhat arbitrary in deciding when the results of improved models should be considered a new observation. Moreover, it has to be decided whether old 'observations' based on outdated models should be eliminated from the observation set. With this example I want to point out that confidence in models that generate predictions is likely to change over time. Such a change is not easily captured by means of Bayesian learning. Bayesian learning in a multi-layer ambiguity model could only shrink priors within a given level. With sufficient information, in the long run, the priors could shrink to a singleton and the decision maker would be left with objective uncertainty or stochasticity. In contrast, it might be desirable to formulate a learning process that changes the confidence label of a lottery over time. The current framework permits an arbitrary number of confidence levels and changes over time. It thereby encourages the development of a richer framework for learning incorporating the confidence dimensions into the learning process. Finally, let me point out that the decision maker can calculate a reduced expected probability distribution over the weather characteristics by integrating over the priors: $p(w) = \int \int \mu_0(w|\theta_1, \theta_2) d\mu_1(\theta_1|\theta_2) d\mu_2(\theta_2)$. For inference purposes, or, for obtaining a 'best guess' of the final outcomes the decision maker can treat all probability distributions the same, at least in a straight forward probabilistic application. However, the different layers of uncertainty corresponding to different degrees of confidence or subjectivity have to be distinguished for the welfare evaluation. Here the layers have to be evaluated recursively, each with the corresponding degree of aversion.

7 Conclusions

The paper presents a model for evaluating scenarios based on probabilistic beliefs that differ in their type of degree of confidence. It respects the von Neumann & Morgenstern (1944) axioms and time consistency, which are often considered normatively attractive. The evaluation of scenarios employs only simple tools from risk analysis, where the risk measures become confidence dependent. The representation facilitates a unified framework for representing aversion to intertemporal substitution, aversion to objective risk, aversion to subjective risk, and smooth ambiguity aversion. Moreover, the representation facilitates a more general interpretation of smooth ambiguity aversion as the additional intertemporal risk aversion to subjective as opposed to objective lotteries. The previous literature formulates the concept of smooth ambiguity aversion in a hierarchical and binary context of purely subjective second order beliefs over purely objective first order beliefs. The present representation frees the subjectivity characterization form this straitjacket by incorporating the degree of subjectivity directly into the notion of a lottery.

I introduce the concept of aversion to the lack of confidence or, equivalently, aversion to the lack of confidence in beliefs. It generalizes the concept of smooth ambiguity aversion to settings with more than just two types of lotteries (a binary measurement of confidence). A given set of preferences can nest and interact behavior exhibiting risk neutrality, Arrow-Pratt risk aversion, and smooth ambiguity aversion as well as decision criteria arbitrarily close to Arrow & Hurwicz's (1972) maximin under ignorance and Gilboa & Schmeidler's (1989) maximin expected utility under ambiguity, depending on the situation the decision maker is facing, his confidence (or ignorance), and his aversion to the lack of confidence. I discussed behavioral implications and sketched two applications as a decision support framework in the context of climate change, an area where the International Panel on Climate Change has promoted an according qualitative distinction of probabilistic estimates.

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Appendix A

Axiom A2 implies $A3 \Rightarrow A3'$ for same degree of subjectivity lotteries:
In axiom A3 choose lotteries $p_t, p'_t, p''_t \in P^s_t$ satisfying $\hat{n}(p_t) = n < N$ and $\hat{n}(p'_t) = \hat{n}(p''_t) = n + 1$. Then, an $\alpha = 1$ mixture of the lotteries delivers

$$p_t \succeq_t p'_t \quad \Rightarrow \quad p_t \oplus^\alpha_s p''_t \;\succeq_t\; p'_t \oplus^\alpha_s p''_t \quad \Rightarrow \quad \mathbf{1}^s_{p_t} \;\succeq_t\; p'_t \;.$$

By completeness of preferences (axiom A2) I therefore obtain

$$\mathbf{1}^s_{p_t} \;\sim_t\; p_t \;. \tag{10}$$

Thus, for arbitrary lotteries $p_t, p'_t, p''_t \in P^s_t$ and $n^* = \max\{\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t)\} + 1 \leq N$:

$$\begin{aligned} p_t \succeq_t p'_t \quad &\Rightarrow \quad \mathbf{1}^s_{p_t} \;\succeq_t\; \mathbf{1}^s_{p'_t} \quad \Rightarrow \quad \mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p''_t} \;\succeq_t\; \mathbf{1}^s_{p'_t} \oplus^\alpha_s \mathbf{1}^s_{p''_t} \\ &\Rightarrow \quad p_t \odot^\alpha_s p''_t \;\succeq_t\; p'_t \odot^\alpha_s p''_t \end{aligned}$$

using first equation (and then axiom A3.

□

Choice in Figure 3 implies a convex-concave function f^{amb}:
Let $\overline{p} = 0.1$ and $\underline{p} = 0.9$ and let $\overline{u} = u(\overline{x})$ and $\underline{u} = u(\underline{x})$. Then, the choice on the left hand side of Figure 3 translates into the condition

$$\begin{aligned} & f^{subj^{-1}} \left[\overline{p} f^{subj}(\overline{u}) + \underline{p} f^{subj}(\underline{u}) \right] > f^{obj^{-1}} \left[\overline{p} f^{obj}(\overline{u}) + \underline{p} f^{obj}(\underline{u}) \right] \\ & \Leftrightarrow \overline{p} f^{subj} \circ f^{obj^{-1}}(\overline{y}) + \underline{p} f^{subj} \circ f^{obj^{-1}}(\underline{y}) > f^{subj} \circ f^{obj^{-1}} \left[\overline{p}\; \overline{y} + \underline{p}\; \underline{y} \right] \\ & \Leftrightarrow \overline{p} f^{amb}(\overline{y}) + \underline{p} f^{amb}(\underline{y}) > f^{amb} \left[\overline{p}\; \overline{y} + \underline{p}\; \underline{y} \right] \;, \end{aligned}$$

where $\overline{y} = f^{obj}(\overline{u})$ and $\underline{y} = f^{obj}(\underline{u})$. Thus, the function f^{amb} at the point $0.9\; \underline{y} + 0.1\; \overline{y}$ has to lie below the straight line connecting $f^{amb}(\underline{y})$ and $f^{amb}(\overline{y})$. Similarly, defining $\overline{p} = 0.9$ and $\underline{p} = 0.1$, the right hand side choice depicted in Figure 3 implies that f^{amb} at the point $0.1\; \underline{y} + 0.9\; \overline{y}$ has to lie above the straight line connecting $f^{amb}(\underline{y})$ and $f^{amb}(\overline{y})$. In consequence, the function f^{amb} has to be convex somewhere in the lower region $[\underline{y}, 0.9\; \overline{y} + 0.1\; \underline{y})$ and concave somewhere in the higher region $(0.1\; \overline{y} + 0.9\; \underline{y}, \overline{y}]$ (subsequently to being convex).

□

Appendix B

Proof of Theorem 1:

Part I develops the representation for a single layer of uncertainty in a given period. Part II builds the recursive evaluation of a general uncertainty tree within a given period. Part III constructs the intertemporal aggregation. Part IV shows that all axioms are satisfied by the representation.

Part I 1) Let $X_t = X^* \times P_{t+1} \, \forall \, t < T$ and $X_T = X^*$. By axioms A2 and A4 there exists an ordinal representation $V_t^0 : X_t \to \mathbb{R}$ of the preference relation $\succeq_t |_{X_t}$, which is restricted to degenerate period t outcomes. 2) For a given parameter s, axioms A2-A4 on $\Delta_s(X_t)$ are the standard von Neumann-Morgenstern axioms for a compact metric setting that permit an expected utility presentation on $\Delta_s(X_t)$. The only distinction of the current setting is that elements p_t are formally distinguished from the the degenerate lotteries $\mathbb{1}_{p_t}$. However, as I explain in the context of equation (in Appendix A, axioms A2 and A3 (or, alternatively, axiom A1) imply $p_t \sim \mathbb{1}^s_{p_t}$. Therefore, the standard mixture space arguments apply and the usual reasoning implies the existence of a particular V_t^{0*} that makes it possible to represent preferences over lotteries in the expected utility form. Instead of using this standard representation, I follow Traeger (2007) and build the representation on an arbitrary function $V_t^0 : X_t \to \mathbb{R}$ representing degenerate choices $\succeq_t |_{X_t}$. So far, V_t^0 can be the function singled out by von Neumann-Morgenstern as well as any strictly increasing and continuous transformation. For a given parameter s, Theorem 1 in Traeger (2007) translates into the following preference representation:

> Given is $V_t^0 : X_t \to \mathbb{R}$ with range$(V_t^0) = U$ representing preferences $\succeq_t |_{X_t}$. Then $\succeq_t |_{\Delta_s(X_t)}$ satisfies axioms A2-A4 if, and only if, there exists a strictly increasing and continuous function $f_t^s : U \to \mathbb{R}$ such that
>
> $$(f_t^s)^{-1} \circ \int_{X_t} f_t^s \circ V_t^0 \, dp$$
>
> represents $\succeq_t |_{\Delta_s(X_t)}$ for all $p \in \Delta_s(X_t)$. Moreover, f and f' both represent $\succeq$ in the above sense if, and only if, there exist $a, b \in \mathbb{R}, a > 0$ such that $f' = af + b$.

3) Carrying out step 2) for all $s \in S$ results in a set of increasing and continuous functions $\hat{f}_t = \{f_t^s\}_{s \in S}$, $f_t^s : \mathbb{R} \to \mathbb{R}$, as stated in the theorem, and a representation

of $\succeq_t |_{Z^1_t(X_t)}$ by

$$\bar{V}^1_t(p_t) = \begin{cases} V^0_t(p_t) & \text{if } \hat{n}(p_t) = 0 \\ V^1_t(p_t) = \left(f_t^{\hat{s}(p_t)}\right)^{-1} \int_{X_t} f_t^{\hat{s}(p_t)} \circ V^0_t \; dp_t & \text{if } \hat{n}(p_t) = 1 \; . \end{cases}$$

Part II constructs inductively a representation of $\succeq_t |_{Z^n_t(X_t)}$ for $n \in N$.
4) Let $\bar{V}^n_t : Z^n_t(X_t) \to \mathrm{I\!R}$ represent $\succeq_t |_{Z^n_t(X_t)}$. By equation (degenerate lotteries in $\Delta_s(Z^n_t(X_t))$ are assigned the same values as the corresponding elements in $Z^n_t(X_t)$. That identification makes $\bar{V}^n_t$ a representation for degenerate lotteries in $Z^{n+1}_t(X_t)$. Thus, for a given s, by axioms A2-A4 and Theorem 1 in Traeger (2007), cited in step 2, preference over lotteries in $\Delta_s(Z^n_t(X_t))$ can be represented by

$$V^{n+1}_s(p_t) = (\tilde{f}^s_t)^{-1} \int_{Z^n(X_t)} \tilde{f}^s_t \circ V_t^{\hat{n}(\tilde{p}_t)}(\tilde{p}_t) \; dp_t(\tilde{p}_t)$$

for some strictly increasing and continuous function $\tilde{f}^s_t$: range$(V^n_t) \to \mathrm{I\!R}$. Employing the representation theorem for each $s \in S$ delivers a representation over the union $Z^{n+1}(X_t) = \cup_{s \in \bar{S}} Y^{n+1}_s(X_t)$ (including $Y^{n+1}_{s_0}(X_t) = Z^n(X_t)$) that evaluates lotteries $p_t \in Z^{n+1}(X_t)$ by

$$\bar{V}^{n+1}_t(p_t) = \begin{cases} V^0_t(p_t) = \tilde{u}_t(p_t) & \text{if } \hat{n}(p_t) = 0 \\ V^1_t(p_t) = \left(f_t^{\hat{s}(p_t)}\right)^{-1} \circ \int\limits_{X_t} f_t^{\hat{s}(p_t)} \circ \tilde{u}_t \; dp_t & \text{if } \hat{n}(p_t) = 1 \\ \vdots \qquad\qquad \vdots & \qquad \vdots \\ V^{n+1}_t(p_t) = \left(\tilde{f}_t^{\hat{s}(p_t)}\right)^{-1} \circ \int\limits_{Z^n(X_t)} \tilde{f}_t^{\hat{s}(p_t)} \circ V_t^{\hat{n}(\tilde{p}_t)}(\tilde{p}_t) \; dp_t(\tilde{p}_t) & \text{if } \hat{n}(p_t) = n+1. \end{cases}$$

5) I show that the $\tilde{f}^s_t$ in V^{n+1}_t can be chosen to coincide with the f^s_t in V^n_t (and, thus, in all the $V^{i \leq n}_t$). Let $p_t, p'_t, p''_t \in Y^n_s \subset Z^n_t(X_t)$. Reduction of the lottery $\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}$ gives

$$\begin{aligned} \left[\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}\right]^r (B) &= \int\limits_{Y^n_s(X_t)} \tilde{p}_t(B) \; d\left(\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}\right)(\tilde{p}_t) \\ &= \alpha \int\limits_{Y^n_s(X_t)} \tilde{p}_t(B) \; d\left(\mathbf{1}^s_{p_t}\right)(\tilde{p}_t) + (1-\alpha) \int\limits_{Y^n_s(X_t)} \tilde{p}_t(B) \; d\left(\mathbf{1}^s_{p'_t}\right)(\tilde{p}_t) \\ &= \alpha \; p_t(B) + (1-\alpha) \; p'_t(B) \end{aligned}$$

for all $B \in \mathfrak{B}_t^n$ and, thus, $\left[\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}\right]^r = p_t \oplus^\alpha_s p'_t$. Then, by axiom A1

$$\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t} \;\sim_t\; \left[\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}\right]^r \;=\; p_t \oplus^\alpha_s p'_t$$

Evaluating the left hand side by means of the representation derived in step 4) I find:

$$\begin{aligned} V_s^{n+1}(\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}) = (\tilde{f}^s_t)^{-1}\Big[\, & \alpha \int\limits_{\mathfrak{B}_t^{n+1}} \tilde{f}^s_t \circ V_t^n(\tilde{p}_t)\; d\big(\mathbf{1}^s_{p_t}\big)(\tilde{p}_t) \\ & + (1-\alpha) \int\limits_{\mathfrak{B}_t^{n+1}} \tilde{f}^s_t \circ V_t^n(\tilde{p}_t)\; d\big(\mathbf{1}^s_{p'_t}\big)(\tilde{p}_t)\Big] \\ = (\tilde{f}^s_t)^{-1}\Big[\, & \alpha \tilde{f}^s_t \circ (f^s_t)^{-1} \int\limits_{\mathfrak{B}_t^n} f^s_t \circ V_t^{n-1}(\tilde{p}_t)\; dp_t(\tilde{p}_t) \\ & + (1-\alpha)\tilde{f}^s_t \circ (f^s_t)^{-1} \int\limits_{\mathfrak{B}_t^n} f^s_t \circ V_t^{n-1}(\tilde{p}_t)\; dp'_t(\tilde{p}_t)\Big] , \end{aligned}$$

which has to equal the evaluation of the right hand side:

$$\begin{aligned} V_s^n(p_t \oplus^\alpha_s p'_t) = (f^s_t)^{-1}\Big[\, & \alpha \int\limits_{\mathfrak{B}_t^n} f^s_t \circ V_t^{n-1}(\tilde{p}_t)\; dp_t(\tilde{p}_t) \\ & + (1-\alpha) \int\limits_{\mathfrak{B}_t^n} f^s_t \circ V_t^{n-1}(\tilde{p}_t)\; dp'_t(\tilde{p}_t)\Big] . \end{aligned}$$

Abbreviating $K(p) = \int_{\mathfrak{B}_t^n} f^s_t \circ V_t^{n-1}\; dp$, equivalence of the two expressions results in

$$\begin{aligned} & V_s^{n+1}(\mathbf{1}^s_{p_t} \oplus^\alpha_s \mathbf{1}^s_{p'_t}) = V_s^n(p_t \oplus^\alpha_s p'_t) \\ \Leftrightarrow\; & (\tilde{f}^s_t)^{-1}\left[\alpha \tilde{f}^s_t \circ (f^s_t)^{-1} \circ K(p_t) + (1-\alpha)\tilde{f}^s_t \circ (f^s_t)^{-1} \circ K(p'_t)\right] \\ & = (f^s_t)^{-1}\left[\alpha K(p_t) + (1-\alpha)K(p'_t)\right] \\ \Leftrightarrow\; & \alpha \tilde{f}^s_t \circ (f^s_t)^{-1} \circ K(p_t) + (1-\alpha)\tilde{f}^s_t \circ (f^s_t)^{-1} \circ K(p'_t) \\ & = \tilde{f}^s_t \circ (f^s_t)^{-1}\left[\alpha K(p_t) + (1-\alpha)K(p'_t)\right] . \end{aligned}$$

Because preferences are non-degenerate, $K(p)$ can be varied on a continuum and by Hardy et al. (1964, p 74) the continuous function $\tilde{f}^s_t \circ (f^s_t)^{-1}$ has to be linear implying $\tilde{f}^s_t = a f^s_t + b$ for some $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ (on the domain relevant to the representation). Affine transformations of the uncertainty aggregation weights do not change the representation (see step 2), thus, I can choose $\tilde{f}^s_t = f^s_t$.

6) Steps 4) and 5) can be applied inductively for $n \in \{1, \ldots, N-1\}$, yielding a

representation for $\succeq_t |_{Z_t^N(X_t)} = \succeq_t$. Given the uncertainty aggregation weights f_t^s coincide (step 5) for the different levels, I can construct the functions $\bar{V}_t^n$ as well inductively by defining $\bar{V}_t^0 = V_t^0$ and

$$\bar{V}_t^n(p_t) = (f_t^{\hat{s}(p_t)})^{-1} \circ \int_{\mathfrak{B}_t^{\hat{n}(p_t)}} f_t^{\hat{s}(p_t)} \circ \bar{V}_t^{\hat{n}(\tilde{p}_t)}(\tilde{p}_t)\, dp_t(\tilde{p}_t)$$

for $n \in N$ (noting that $\hat{n}(\tilde{p}_t) < n$). Then, for a given sequence of uncertainty weights $\hat{f}_t$ and a given function V_t^0 it is $\mathcal{M}_{V_t^0}^{\hat{f}_t}\, p_t = \bar{V}_t^N(p_t)$. I have established the existence of the sequences $\hat{f}_t$ as in the theorem and the existence of some V_t^0 such that the representation equation (in the theorem holds.

Part III shows that the sequence $\hat{u}_t, t \in T$ constructed in equation (indeed gives rise to a feasible set of Bernoulli utility functions $V_t^0, t \in T$.

7) Recall that the only requirement on the functions V_t^0 is that they have to be an ordinal representation of preferences on the space of degenerate outcomes in period t, i.e. for $\succeq_t |_{X_t}$. Axioms A2, A4, and A5 imply a certainty additive representation for preferences restricted to the subspace of certain consumption paths (Wakker 1988, theorem III.4.1).[18] I denote the corresponding continuous per period utility functions by $u_t : X^* \to \mathbb{R}$. They are unique up to affine transformations with a coinciding multiplicative constant (and heterogeneous additive constants).

8) For the last period I can choose $V_t^0 = \hat{u}_T = u_T$. I show recursively that $\hat{u}_{t-1}(x_{t-1}, p_t) = u_{t-1}(\mathrm{x}_{t-1}) + \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\, p_t$ is an (ordinal) representation of $\succeq_{t-1} |_{X_{t-1}}$ given that $\hat{u}_t$ is an (ordinal) representation of $\succeq_t |_{X_t}$. By construction of the uncertainty aggregator $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t}$, a certain consumption path $\mathrm{x}_t = (x_t, x_{t+1}, \ldots, x_T)$ is evaluated to $\hat{u}_t(\mathrm{x}^t) = \sum_{\tau=t}^T u_\tau(x_\tau)$. I define a certainty equivalent of a lottery $p_t \in P_t$ to be a lottery $(x_t^{p_t}, p_{t+1}^{p_t}) \in P_t$ that satisfies $(x_t^{p_t}, p_{t+1}^{p_t}) \sim_t p_t$. For any lottery there exists such a certainty equivalent and it does not matter which one is chosen.[19] By the representation already constructed, I know that $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\, p_t = \hat{u}_t(x_t^{p_t}, p_{t+1}^{p_t})$. Moreover, by inductively replacing $p_{t+1}^{p_t}$ with a

[18] A note on the details of the theorem's applicability: If the sets $\{p_0' \in P_0 : p_0' \succeq_0 \mathrm{x}\}$ and $\{p_0' \in P_0 : \mathrm{x} \succeq_0 p_0'\}$ are closed in P_0 for all $\mathrm{x} \in X^{T+1} \subset P_0$, then the sets $\{p_0' \in P_0 : p_0' \succeq_0 \mathrm{x}\} \cap X^{T+1} = \{\mathrm{x}' \in X^{T+1} : \mathrm{x}' \succeq_0 \mathrm{x}\}$ and $\{p_0' \in P_0 : \mathrm{x} \succeq_0 p_0'\} \cap X^{T+1} = \{\mathrm{x}' \in X^{T+1} : \mathrm{x} \succeq_0 \mathrm{x}'\}$ are closed in X^{T+1} endowed with the relative topology for all $\mathrm{x} \in X^{T+1}$. Moreover the relative topology on X^{T+1} is the product topology on X^{T+1}.

[19] The existence is most easily observed from the representation already constructed. The uncertainty aggregator is a generalized mean and, thus, the value of any lottery lies between the value of the worst and the best outcome. For more details see induction hypothesis H2 in proof of theorem 2 in Traeger (2007).

certainty equivalent, I obtain a certainty equivalent to the lottery p_t that is a certain consumption path, which I denote by $\mathrm{x}_t^{p_t}$.
9) By time consistency

$$\begin{aligned} p_t &\sim_t \mathrm{x}_t^{p_t} \\ \Leftrightarrow (x_{t-1}, p_t) &\sim_{t-1} (x_{t-1}, \mathrm{x}_t^{p_t}) \end{aligned}$$

and therefore

$$\begin{aligned} (x_{t-1}, p_t) &\succeq_{t-1} (x'_{t-1}, p'_t) \\ \Leftrightarrow (x_{t-1}, \mathrm{x}^{p_t}) &\succeq_{t-1} (x'_{t-1}, \mathrm{x}^{p'_t}) \\ \Leftrightarrow u_{t-1}(p_t) + \textstyle\sum_{\tau=t}^T u_\tau(x_\tau^{p_t}) &\geq u_{t-1}(p'_t) + \textstyle\sum_{\tau=t}^T u_\tau(x_\tau^{p'_t}) \\ \Leftrightarrow u_{t-1}(p_t) + \mathcal{M}_{p_t}^{\hat{f}_t}\, \hat{u}_t &\geq u_{t-1}(p'_t) + \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\, p_t \ . \end{aligned}$$

Hence $\hat{u}_{t-1} : X^* \times P_t \to \mathrm{I\!R}$ with $\hat{u}_{t-1}(x_{t-1}, p_t) = u_{t-1}(x_{t-1}) \ + \ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\, p_t$ is an (ordinal) representation of $\succeq_{t-1} |_{X^* \times P_t}$.
Part IV proofs necessity of the axioms. The lottery $p_t \in \Delta_s\left(Y_s^n(X^* \times P_{t+1})\right)$ on the left hand side of axiom A

evaluates as

$$\begin{aligned} &(f_t^s)^{-1} \circ \int\limits_{\mathfrak{B}_t^{n+1}} f_t^s \circ (f_t^s)^{-1} \circ \int\limits_{\mathfrak{B}_t^n} f_t^s \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ p'_t \quad d\tilde{p}_t(p'_t) \quad dp_t(\tilde{p}_t) \\ &= (f_t^s)^{-1} \circ \int\limits_{\mathfrak{B}_t^{n+1}} \int\limits_{\mathfrak{B}_t^n} f_t^s \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ p'_t \quad d\tilde{p}_t(p'_t) \quad dp_t(\tilde{p}_t) \\ &= (f_t^s)^{-1} \circ \int\limits_{\mathfrak{B}_t^n} f_t^s \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ p'_t \quad d\left[\int\limits_{\mathfrak{B}_t^{n+1}} \tilde{p}_t \ \ dp_t(\tilde{p}_t) \right](p'_t) \\ &= (f_t^s)^{-1} \circ \int\limits_{\mathfrak{B}_t^n} f_t^s \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ p'_t \quad dp_t^r(p'_t) \end{aligned}$$

and, thus, equivalently to the right hand side of axiom A

. Axiom A2 is obviously satisfied. Regarding axiom A3 observe that for all $t \in T$,

$p_t, p'_t, p''_t \in P^s_t$, and $\alpha \in [0,1]$:

$$\begin{aligned}
p_t \succeq_t p'_t &\Rightarrow \mathcal{M}^{\hat{f}_t}_{\tilde{u}_t}\ p_t \geq \mathcal{M}^{\hat{f}_t}_{\tilde{u}_t}\ p'_t \\
&\Rightarrow \left(f_t^{\hat{s}(p_t)}\right)^{-1} \circ \int_{\mathfrak{B}_t^{\hat{n}(p_t)}} f_t^{\hat{s}(p_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ dp_t(\tilde{p}_t) \\
&\quad \geq \left(f_t^{\hat{s}(p'_t)}\right)^{-1} \circ \int_{\mathfrak{B}_t^{\hat{n}(p'_t)}} f_t^{\hat{s}(p'_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ dp'_t(\tilde{p}_t) \\
&\Rightarrow \left(f_t^{\hat{s}(p_t)}\right)^{-1} \left[\int_{\mathfrak{B}_t^{\hat{n}(p_t)}} \alpha\ f_t^{\hat{s}(p_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ dp_t(\tilde{p}_t) + K \right] \\
&\quad \geq \left(f_t^{\hat{s}(p'_t)}\right)^{-1} \left[\int_{\mathfrak{B}_t^{\hat{n}(p'_t)}} \alpha\ f_t^{\hat{s}(p'_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ dp'_t(\tilde{p}_t) + K \right] \ ,
\end{aligned}$$

where $\hat{s}(p_t) = \hat{s}(p'_t)$. Setting

$$K = \int_{\mathfrak{B}_t^{\hat{n}(p''_t)}} (1-\alpha)\ f_t^{\hat{s}(p'_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ dp''_t(\tilde{p}_t)$$

it follows

$$\begin{aligned}
&\left(f_t^{\hat{s}(p_t)}\right)^{-1} \circ \int_{\mathfrak{B}_t^{n^*}} f_t^{\hat{s}(p_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ d(p_t \oplus^\alpha_s p''_t)(\tilde{p}_t) \\
&\quad \geq \left(f_t^{\hat{s}(p'_t)}\right)^{-1} \circ \int_{\mathfrak{B}_t^{n^*}} f_t^{\hat{s}(p'_t)} \circ \mathcal{M}^{\hat{f}_t}_{\hat{u}_t}\ \tilde{p}_t\ d(p'_t \oplus^\alpha_s p''_t)(\tilde{p}_t)
\end{aligned}$$

with $n^* = \max\{\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t)\}$ and, thus,

$$p_t \oplus^\alpha_s p''_t \ \succeq_t \ p'_t \oplus^\alpha_s p''_t \ .$$

To see that axiom A4 is satisfied note that in the union topology a set is closed if each preimage of the set under the injection maps[20] is closed. Thus, given that the functions $f^s_t \circ \hat{u}_t$ and V^n_t are continuous (in the topology of weak convergence) the sets in axiom A4 are closed. Axiom A5 is easily observed to be satisfied by recognizing that the evaluation on certain consumption paths reduces to the formula

[20] The s-th injection map inj_s assigns an element of $\Delta(\cdot)$ to the corresponding element in $(\Delta(\cdot), s) = \Delta_s(\cdot)$ (e.g. Cech 1966, 85).

$\hat{u}_t(\mathrm{x}^t) = \sum_{\tau=t}^{T} u_\tau(x_\tau)$. An inspecting of equation (shows that axiom A6 is satisfied. Finally, Axiom A3' is satisfied as well:

$$p_t \succeq_t p'_t \Leftrightarrow \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p_t \geq \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p'_t \Leftrightarrow (f_t^s)^{-1}\left[\alpha \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p_t\right] \geq (f_t^s)^{-1}\left[\alpha \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p'_t\right]$$

$$\Leftrightarrow f_t^s\left[\alpha \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p_t\right] + f_t^s\left[(1-\alpha)\mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p''_t\right] \geq f_t^s\left[\alpha \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p'_t\right] + f_t^s\left[(1-\alpha)\mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ p''_t\right]$$

$$\Leftrightarrow (f_t^s)^{-1}\left\{\int_{\mathfrak{B}_t^{\max\{\hat{n}(p_t),\hat{n}(p''_t)\}+1}} f_t^s \circ \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ \tilde{p}_t\ d\left[\alpha \mathbf{1}_{p_t}^s + (1-\alpha)\mathbf{1}_{p''_t}^s\right](\tilde{p}_t)\right\}$$

$$\geq (f_t^s)^{-1}\left\{\int_{\mathfrak{B}_t^{\max\{\hat{n}(p'_t),\hat{n}(p''_t)\}+1}} f_t^s \circ \mathcal{M}_{\tilde{u}_t}^{\hat{f}_t}\ \tilde{p}_t\ d\left[\alpha \mathbf{1}_{p'_t}^s + (1-\alpha)\mathbf{1}_{p''_t}^s\right](\tilde{p}_t)\right\}$$

$$\Leftrightarrow p_t \odot_s^\alpha p''_t \ \succeq_t\ p'_t \odot_s^\alpha p''_t\ .$$

□

Proof of Proposition 1:

For all $\mathrm{x}, \mathrm{x}' \in \mathrm{X}^t$ I have

$$\mathrm{x} \oplus_{obj}^{\frac{1}{2}} \mathrm{x}' \quad \succeq_t \quad \mathrm{x} \oplus_{subj}^{\frac{1}{2}} \mathrm{x}'$$

$$\Rightarrow \left(f_t^{obj}\right)^{-1}\left[\frac{1}{2} f_t^{obj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ \mathrm{x} + \frac{1}{2} f_t^{obj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ \mathrm{x}'\right]$$

$$\geq \left(f_t^{subj}\right)^{-1}\left[\frac{1}{2} f_t^{subj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ \mathrm{x} + \frac{1}{2} f_t^{subj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ \mathrm{x}'\right]\ .$$

Defining $K(\mathrm{x}) = f^{obj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t}\ \mathrm{x} = f^{obj} \circ \sum_{\tau=t}^{T} u_\tau(\mathrm{x}_\tau)$ I find

$$\Rightarrow f_t^{subj} \circ \left(f_t^{obj}\right)^{-1}\left[\frac{1}{2}\left[K(\mathrm{x})\right] + \frac{1}{2}\left[K(\mathrm{x})\right]\right]$$

$$\geq \frac{1}{2} f_t^{subj} \circ \left(f_t^{obj}\right)^{-1}\left[K(\mathrm{x})\right] + \frac{1}{2} f_t^{subj} \circ \left(f_t^{obj}\right)^{-1}\left[K(\mathrm{x}')\right]$$

and, thus, $f_t^{amb} = f_t^{subj} \circ \left(f_t^{obj}\right)^{-1}$ concave by Hardy et al. (1964, 75) on the range relevant for the representation. Analogously, I find strict concavity to hold by replacing $\succeq_t$ by $\succ_t$ and $\geq$ by $>$. □

Proof of Proposition 2:

For every pair $s, s' \in S$ with $s \triangleright s'$ the proof is a copy of the proof of Proposition 1.

$\square$