A solution to the global identification problem in DSGE models

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Abstract

We develop an analytical framework to study global identification in structural models with forward-looking expectations. Our identification condition combines the similarity transformation linking the observationally equivalent state space systems with the constraints imposed on them by the model parameters. The key step of solving the identification problem then reduces to finding all roots of a system of polynomial equations. We show how it can be done using the concept of a Gröbner basis and recently developed algorithms to compute it analytically. In contrast to papers relying on numerical search, our approach can prove whether a model is identified or not at a given parameter point, explicitly delivering the complete set of observationally equivalent parameter vectors. We present the solution to the global identification problem for several popular DSGE models. Our findings indicate that observational equivalence in medium-sized models might be not as widespread as suggested by earlier, small model-based evidence.

JEL: C10, C51, C65, E32
Keywords: global identification, state space systems, DSGE models, Gröbner basis

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1 Introduction

Parameter identification is one of the primary concerns of structural macroeconomic modeling. In the context of traditional simultaneous equations systems, the essence of the problem and its treatment has already been formalized in the 1940s, mainly by various authors connected to the Cowles Commission for Research in Economics (see e.g. Koopmans, 1949). In recent decades, this class of purely backward-looking models has been gradually replaced, in both academic circles and policy making institutions, by the so-called dynamic stochastic general equilibrium (DSGE) models. In these mathematical constructs, the dynamics is driven by unobserved stochastic processes and crucially depends on agents' expectations, typically assumed rational. The key difficulty with this class of models in the context of identification is that while their solution has a state-space representation, for which the global identification problem is fairly well understood, the coefficients defining this solution are only implicit rather than analytical functions of the original model parameters. As a result, a new approach to identification became necessary.

Early contributions highlighting the identification problem in simple DSGE models include Beyer and Farmer (2007), Fukac et al. (2007), Canova and Sala (2009) and Cochrane (2011). A more formal analysis soon followed, focusing first on local identification issues, and resulting in the rank conditions on an appropriately defined Jacobian matrix (Iskrev, 2010; Komunjer and Ng, 2011) or spectral density matrix (Qu and Tkachenko, 2012). Important progress has also been made towards resolving the problem of global identification. Qu and Tkachenko (2017) present a numerical routine that searches for observationally equivalent parameters by minimizing the Kullback-Leibler distance in the frequency domain. Kociecki and Kolasa (2018) develop an alternative algorithm that relies on the conditions linking observationally equivalent state space representations from Komunjer and Ng (2011), thus avoiding the need to solve the model for each candidate parameter.[1]

However, while these two existing approaches to global identification problem are very useful tools for detecting possible identification failure, they have one important limitation in that they cannot strictly prove that a given model is globally identified. If a numerical search routine fails to find an observationally equivalent vector of parameters to the one at which one checks identification, this does not necessarily mean that such a vector does not exist. It might be that the algorithm simply neglected some support of the (multi-dimensional) parameter space, where observationally equivalent points are situated. Therefore, the lack of a solution to the problem of global identification in DSGE models should be considered a serious methodological gap.

Against this backdrop, this paper develops an analytical framework to study global identification in dynamic linear systems with rational expectations. The framework is comprehensive in that it encompasses both determinate models, in which the rational expectations solution is unique, as well as indeterminate ones, where the dynamics may be additionally driven by sunspot shocks. The essence of our approach consists of two insights. The first one establishes a formal identification condition that reduces checking identification of the model's parameters (or their appropriately

[1] While all of this literature deals with linearized DSGE models, there have been some attempts to study local identification of their higher-order approximations, see e.g. Mutschler (2015).

defined analytical functions) to finding all roots of a system of polynomial equations. This condition is derived by linking the observationally equivalent state space systems with the inherent constraints imposed on them by the deep parameters of the underlying structural model. The second insight relies on applying the concept of the Gröbner basis to analytically solve this system of polynomial equations. In short, and postponing the details for later, this last step boils down to transforming the original system of polynomials into an equivalent triangular system, which is done in a way resembling Gaussian elimination in linear algebra.

The key advantage of our framework is that it explicitly derives and directly checks the global identification conditions at a given point in the parameter space. This gives the formal proof of global identification or lack thereof at this point as the calculation of the Gröbner basis is exact in principle. Our explicit approach is hence a major advantage over the two existing methods to check global identification in DSGE models (Kociecki and Kolasa, 2018; Qu and Tkachenko, 2017), both of which rely on searching numerically over the parameter space, and hence cannot formally prove that the model is identified. Another useful feature of our framework is that it generalizes and unifies the conditions linking observationally equivalent state space representations, which in the previous identification literature were derived separately for singular and non-singular cases.

While designed to solve the global identification problem, our framework offers also some additional insights over the existing and well-established approaches to handle its local variant (Iskrev, 2010; Komunjer and Ng, 2011). Most importantly, a Gröbner basis applied to our identification condition explicitly produces the complete set of parameter vectors that are observationally equivalent to the one at which one checks identification. For example, it may reveal that some parameters "live" on the intersection of hyperplanes. Knowing that is very useful as it explicitly tells which of them need to be fixed to attain global identification

Gröbner basis methods are a fast developing field in computational algebraic geometry. Despite their great potential, they are still very rarely used in economics, with only few exceptions. Kubler and Schmedders (2010a) and Kubler and Schmedders (2010b) successfully apply these methods to determine the exact number of equilibria in several economic models and to calculate them analytically. Datta (2010) exploits this concept to find Nash equilibria in games. Foerster et al. (2016) apply Gröbner bases to obtain higher-order approximations to the solutions of Markov-switching DSGE models.

Despite the existence of analytical algorithms that are proved to succeed after a finite number of iterations, computing a Gröbner basis for large systems of equations can be quite time and memory consuming in practice. However, there are several features of our application that help alleviate this curse of dimensionality. The key one is that, for a typical DSGE model, the system of polynomials generated by our identification condition is of limited degree and very sparse.

In fact, we show that our identification analysis can be applied not only to small-scale DSGE models, but also to their richer versions represented e.g. by the Smets and Wouters (2007) setup. To our knowledge, we are actually the first to solve the global identification problem in this popular model, showing that observational equivalence can be ruled out by fixing only two structural

parameters, which is less than suggested by the previous literature. Unlike other related papers, we also apply our framework to study identification in several variants of open economy DSGE models, including those featuring a line of promising extensions suggested by the recent literature. Strikingly, we can prove that all of them are globally identified, at least for a standard selection of observable variables. Together with the conclusions obtained for the Smets-Wouters model, our findings indicate that observational equivalence in medium-sized DSGE models might be actually not as widespread as some earlier small model-based evidence suggested.

The rest of this paper proceeds as follows. Section 2 presents the setup and establishes notation for a typical dynamic linear system with rational expectations and its state-space representation. Section 3 derives the conditions linking observationally equivalent state-space representations. Section 4 combines these links with the original (structural) form of the model to establish the formal global identification conditions. Section 5 offers a brief introduction to the concept of Gröbner basis and describes its application to checking the identification condition. Several illustrative examples, including popular DSGE models from the literature and their extensions, are presented in Section 6. Section 7 concludes and discusses some possible further research directions. All proofs and more involved analytical details are relegated to the Appendix.

2 Structural model

Let us cast a linearized DSGE model in the following general form

$$\Gamma_0(\theta)\begin{bmatrix} s_t \\ p_t \end{bmatrix} = \Gamma_1(\theta)\mathbb{E}_t\begin{bmatrix} s_{t+1} \\ p_{t+1} \end{bmatrix} + \Gamma_2(\theta)s_{t-1} + \Gamma_3(\theta)\varepsilon_t \tag{1}$$

where s_t is an $n \times 1$ vector of states, p_t is a $q \times 1$ vector of other endogenous (policy) variables, and $\varepsilon_t \sim i.i.d.\, N(0, \Sigma(\theta))$ is a $k \times 1$ vector of shocks, which can include both innovations to structural (fundamental) disturbances, sunspot shocks and measurement errors, the latter entering with zero loadings. Matrices $\Gamma_0(\theta)$, $\Gamma_1(\theta)$, $\Gamma_2(\theta)$, $\Gamma_3(\theta)$ and symmetric positive definite $k \times k$ matrix $\Sigma(\theta)$ are explicit functions of deep model parameters collected in an $m \times 1$ vector $\theta \in \Theta \subseteq \mathbb{R}^m$.

A stable solution to (1) can be written as

$$s_t = A(\theta)s_{t-1} + B(\theta)\varepsilon_t \tag{2}$$

$$p_t = F(\theta)s_{t-1} + G(\theta)\varepsilon_t \tag{3}$$

where $A(\theta)$ is an $n \times n$ matrix, $B(\theta)$ is an $n \times k$ matrix, $F(\theta)$ is a $q \times n$ matrix and $G(\theta)$ is a $q \times k$ matrix, all of which implicitly depend on deep model parameters θ. This is always the case if the non-explosive equilibrium is unique. Under indeterminacy, the solution has still the form given by equations (2)-(3) as long as one allows for a sufficient number of sunspot shocks in ε_t, see Lubik and Schorfheide (2003). This becomes even more straightforward if, in the case of indeterminacy, one equivalently transforms the model as suggested by Farmer et al. (2015), i.e. redefines a sufficient

number of errors in expectations as fundamentals.[2]

Suppose the measurement equations relates the model variables to the data as follows

$$y_t = H(\theta) \begin{bmatrix} s_t \\ p_t \end{bmatrix} + J(\theta)\varepsilon_t \tag{4}$$

where y_t is an $r \times 1$ vector of observable variables, $H(\theta)$ is an $r \times (n+q)$ matrix and $J(\theta)$ is an $r \times k$ matrix, both of which explicitly depend on θ.

Decomposing $H(\theta)$ into blocks corresponding to the state and policy variables $H(\theta) = [\ H^s(\theta) \quad H^p(\theta)\ ]$ and using equations (2) and (3) allows us to rewrite measurement equation (4) as

$$y_t = C(\theta)s_{t-1} + D(\theta)\varepsilon_t \tag{5}$$

where an $r \times n$ matrix $C(\theta)$ and an $r \times k$ matrix $D(\theta)$ are defined as

$$C(\theta) = H^s(\theta)A(\theta) + H^p(\theta)F(\theta) \tag{6}$$

$$D(\theta) = H^s(\theta)B(\theta) + H^p(\theta)G(\theta) + J(\theta) \tag{7}$$

Consequently, the law of motion for observable variables y_t has a state space form, given by transition equation (2) and measurement equation (5). For future reference, such a representation will be called the ABCD-representation.

3 Observational equivalence of state-space representations

One of the key insights from Komunjer and Ng (2011) is that the ABCD-representation of a DSGE model is not identified, and hence its elements cannot be treated as reduced-form parameters. In this section we generalize their results by developing a set of conditions linking the observationally equivalent ABCD-representations that encompass both singular and non-singular cases.[3] From now on, to save on notation, let us denote any matrix $X(\theta)$ that depends on θ simply as X. Similarly, when referring to this matrix evaluated at an alternative parameter vector $\bar{\theta}$, we will write in short $\bar{X}$.

3.1 Theoretical setup

To proceed, we need two assumptions to get our most general identification result for the ABCD-representation of a DSGE model. The first one concerns stability of the model solution.

[2] As our identification analysis requires fixed structure of the model solution, we need to rule out indeterminacy while checking identification at θ that implies uniqueness and vice versa. This can be easily done by imposing appropriate restrictions on Θ.

[3] Non-singular models are the cases in which there are more shocks than observables ($k > r$) or when the system is square ($k = r$) but non-invertible. See the table on page 2010 in Komunjer and Ng (2011).

Assumption 1. *(Stability) For every $\theta \in \Theta$ and for any $z \in \mathbb{C}$ (a set of complex numbers) $det(z\mathrm{I}_n - A) = 0$ implies $|z| < 1$.*

The purpose of Assumption 1 is to restrict the analysis to stationary models. As a consequence, we can define the steady-state value $P = E(s_t s_t')$, which is a unique solution to the Lyapunov equation $P = APA' + B\Sigma B'$ implied by equation (2). Bearing in mind measurement equation (5), the autocovariance sequence $\Lambda_l = E(y_t y_{t-l}')$ is readily seen as $\Lambda_0 = CPC' + D\Sigma D'$ and $\Lambda_l = CA^{l-1}N$, for $l > 0$, where $N = APC' + B\Sigma D'$. Needless to say, we have $\Lambda_{-l} = \Lambda_l'$.

To state the second assumption, let us define $\mathcal{O} = [C' \vdots A'C' \vdots A'^2C' \vdots \ldots \vdots A'^{n-1}C']'$ and $\mathcal{C} = [N \vdots AN \vdots A^2N \vdots \ldots \vdots A^{n-1}N]$.

Assumption 2. *(Stochastic minimality) For every $\theta \in \Theta$, matrices $\mathcal{O}$ and $\mathcal{C}$ have, respectively, full column and full row rank, i.e. $rank(\mathcal{O}) = rank(\mathcal{C}) = n$.*

Assumption 2 (under the name that we use) is well known in the linear system literature, see e.g. Lindquist and Picci (1996). It is exactly the same as in e.g. Komunjer and Zhu (2020), who term it as autocovariance minimality. Its main purpose is to confine the analysis only to those ABCD-representations (consistent with given autocovariance sequences) in which the dimension of the state vector is as small as possible. To this end, Assumption 2 ensures that the underlying infinite block Hankel matrix has the same rank n (see Appendix A.2).

Assumption 2 differs from the assumptions made by Komunjer and Ng (2011) in how matrix $\mathcal{C}$ is defined. In their framework, N is replaced either by B (Assumption 5-S, applicable to the singular case) or the steady-state Kalman gain associated with the innovations representation of the original state-space system (Assumption 5-NS, for the non-singular case). Moreover, Komunjer and Ng (2011) additionally impose left-invertibility of the transfer function (Assumption 4-S for the singular case)[4] or full row rank of matrix D (Assumption 4-NS for the non-singular case). In our most general form of the identification condition, we do not need any of these additional assumptions. We also do not have to distinguish between singular and non-singular models, which spares us reformulation of the original problem into its innovations representation in the latter case. In this sense, our framework can be seen both as unification and some generalization (as it relies on weaker conditions) of that developed by Komunjer and Ng (2011).

Even though well established in the linear system literature, one may question the practical aspect of Assumption 2 since it is impossible to check its validity for all $\theta \in \Theta$. However, in Appendix A.1 we show that if Assumption 2 is valid for some θ at which we check identification, then in fact it holds for almost all $\theta \in \Theta$. This allows us to safely proceed with our analysis, with the understanding that the underlying deep parameter space Θ excludes those θ's that violate the assumption, which however form the nowhere dense subset of measure zero. In fact, as we discuss at the end of this section using a simple example, the excluded parameter values can just correspond to

[4] Under left-invertibility, finding all observationally equivalent ABCD-representations closely resembles the so-called deterministic realization problem, hence standard assumptions concerning observability and controllability are sufficient.

particular degenerate cases, and hence are not the relevant candidates for observational equivalence to those θ's for which Assumption 2 holds.

As implied by our model formulation, we deal with a stationary Gaussian environment and ignore intercept in the measurement equation so that the unconditional mean of all observables is zero.[5] This allows us to define observational equivalence by using only second moments. More formally, let us define the spectral density of the ABCD-representation as $\Phi(z) = H(z)\Sigma H'(z^{-1})$, where $H(z) = D + C(z\mathrm{I}_n - A)^{-1}B$ is the transfer function and z^{-1} corresponds to its backward shift. Then we have the following definition

Definition 1. θ and $\bar{\theta}$ are observationally equivalent (written as $\theta \sim \bar{\theta}$) if $\Phi(z) = \bar{\Phi}(z)$ for all $z \in \mathbb{C}$.

What Definition 1 conveys is that two deep parameters sets are observationally equivalent if they result in the same autocovariance sequence, so that we cannot distinguish between them using second moments of the observable variables. We are now ready to state the key theorem.

Theorem 1. *Let Assumptions 1 and 2 hold. Then $\theta \sim \bar{\theta}$ if and only if 1) $\bar{A} = TAT^{-1}$, 2) $\bar{C} = CT^{-1}$, 3) $AQA' - Q = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$, 4) $CQC' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D'$, 5) $AQC' = T^{-1}\bar{B}\bar{\Sigma}\bar{D}' - B\Sigma D'$, for some nonsingular matrix T and symmetric matrix Q. In addition, if $\theta \sim \bar{\theta}$ then both T and Q are unique.*

This theorem is an adapted version of Corollary 4.5 in Glover (1973), which probably belongs to "folk wisdom" among specialists in linear system theory, but, to our knowledge, it has not been yet used to study economic systems.[6] From the perspective of identification in DSGE models, Theorem 1 generalizes and unifies the key propositions 1-S and 1-NS in Komunjer and Ng (2011), who consider separately the singular and non-singular cases, for which they need to assume left-invertibility of the transfer function in the former case and the full row rank of D in the latter case. Most importantly, the general form of the theorem allows us to treat the case $r < k$, which arises naturally under indeterminacy as full characterization of the model solutions requires adding sunspot shocks (Lubik and Schorfheide, 2003).

It may be useful to know under what further conditions our Theorem 1 nests the conclusions of Propositions 1-S and 1-NS in Komunjer and Ng (2011) for the singular and non-singular case, respectively. Starting with the latter, let us define the Riccati equation (in symmetric matrix Π)

$$\Pi = A\Pi A' + B\Sigma B' - K\Sigma_a K' \tag{8}$$

where $\Sigma_a = C\Pi C' + D\Sigma D'$ and $K = (A\Pi C' + B\Sigma D')\Sigma_a^{-1}$ (in what follows we assume that Σ_a is positive definite). Let us also formulate the following assumption.

[5] If the measurement equation includes an intercept that depends analytically on θ, additional identification conditions can be obtained from the first moments of observable variables. Adding this information to our identification analysis is straightforward, so we do not deal with this possibility here.

[6] For this reason, and also because Glover (1973) contains only the proof of a continuous time version of Corollary 4.5, we prove Theorem 1 in Appendix A.2.

Assumption 3. *For every $\theta \in \Theta$, the Riccati equation (8) possesses a unique, positive semidefinite solution.*

Clearly, Assumption 3 is a high level assumption. However, as we show in Appendix A.3, checking whether it holds is quite easy. Then we have the following proposition.

Proposition 1. *Let Assumptions 1, 2 and 3 hold. Then $\theta \sim \bar{\theta}$ if and only if 1) $\bar{A} = TAT^{-1}$, 2) $\bar{C} = CT^{-1}$, 3) $\bar{K} = TK$ and 4) $\bar{\Sigma}_a = \Sigma_a$ for some nonsingular matrix T. In addition, if $\theta \sim \bar{\theta}$ then T is unique.*

The conclusions of this proposition, which we prove in Appendix A.4, are exactly as in Proposition 1-NS in Komunjer and Ng (2011). Obviously, from an operational point of view, they should be read together with the definition of Riccati equation (8), which links K and Σ_a ($\bar{K}$ and $\bar{\Sigma}_a$) to the ABCD-representation via matrix Π ($\bar{\Pi}$).

Let us now move to the case, in which the number of observable variables is equal to the number of shocks, i.e. $r = k$. This is by far the most relevant case in the DSGE literature, which uses likelihood-based methods to estimate the model parameters. To nest the square case in our framework, we need the following assumption.

Assumption 4. *For every $\theta \in \Theta$, D is nonsingular.*

Needless to say, as in the case of Assumption 2, if Assumption 4 holds for one $\theta \in \Theta$, then it applies for almost all $\theta's$. Then we have the next proposition.

Proposition 2. *Let Assumptions 1-4 hold. Then $\theta \sim \bar{\theta}$ if and only if 1) $\bar{A} = TAT^{-1}$, 2) $\bar{B} = TBU$, 3) $\bar{C} = CT^{-1}$, 4) $\bar{D} = DU$, 5) $\bar{\Sigma} = U^{-1}\Sigma U'^{-1}$, for some nonsingular matrix T and nonsingular matrix U. In addition, if $\theta \sim \bar{\theta}$ then both T and U are unique.*

The conclusions of this proposition are the same as in Proposition 1-S in Komunjer and Ng (2011). From the perspective of deep parameter identification that we describe in the following section, the conditions in Proposition 2 are a bit more convenient to handle than those stated in Theorem 1, so we recommend using the former whenever Assumptions 3 and 4 are satisfied.

3.2 Simple example

To illustrate the meaning and interaction between our all assumptions, theorem and propositions, consider a simple $MA(1)$ model, i.e. $y_t = \phi\varepsilon_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$. This model is nested in the ABCD-representation by putting $A = 0$, $B = \phi$, $C = 1$, $D = 1$, $\Sigma = \sigma^2$. Since $A = 0$, our model is stable (i.e. Assumption 1 holds). Moreover, $rank(\mathcal{O}) = rank([1, 0, \ldots, 0]') = 1$ for all θ, but $rank(\mathcal{C}) = rank([\phi\sigma^2, 0, \ldots, 0]) = 1$ only for $\phi \neq 0$. This shows why if the latter rank assumption is satisfied for some ϕ at which we check identification, then it holds for almost all ϕ. Evidently, Assumption 2 simply excludes the white noise model (i.e. $\phi = 0$) from the considerations, i.e. a point at which $\mathcal{C}$ drops rank. Note that this exclusion is not restrictive as white noise cannot be observationally equivalent to any non-degenerate $MA(1)$ model.

In the case $\phi \neq 0$, we can safely apply Theorem 1. Since C is restricted to 1, we immediately have $T = 1$. Then, from 3), 4) and 5) in this theorem, we get $Q = \phi^2\sigma^2 - \bar{\phi}^2\bar{\sigma}^2$, $Q = \bar{\sigma}^2 - \sigma^2$ and $\phi\sigma^2 = \bar{\phi}\bar{\sigma}^2$, respectively. Solving these three equations in three unknowns $Q, \bar{\phi}, \bar{\sigma}^2$ gives us exactly two solutions $(Q, \bar{\phi}, \bar{\sigma}^2) = (0, \phi, \sigma^2)$ and $(Q, \bar{\phi}, \bar{\sigma}^2) = (\sigma^2(\phi^2 - 1), \frac{1}{\phi}, \phi^2\sigma^2)$ for $\phi \neq \pm 1$, and one solution $(Q, \bar{\phi}, \bar{\sigma}^2) = (0, \phi, \sigma^2)$ for $\phi = \pm 1$. Hence, the model is not globally identified at $\phi \neq \pm 1$.

Let us now demonstrate how our analysis works in more specialized cases. Note that, since D is restricted to 1, Assumption 4 holds automatically. Hence, provided that Assumption 3 is fullfiled, we can apply both Proposition 1 and 2. In our case, the Riccati equation (8) possesses two solutions $\Pi = 0$ and $\Pi = \sigma^2(\phi^2 - 1)$. For Assumption 3 to hold, we then need to restrict $|\phi| < 1$ as in this case the only positive semidefinite solution to the Ricatti equation is $\Pi = 0$. As a matter of fact, in Appendix A.3 we show that Assumption 3 holds if and only if $\Psi = A - BD^{-1}C = -\phi$ is stable, i.e. $|\phi| < 1$. With this restriction, Proposition 1 gives us $\bar{\sigma}^2 = \sigma^2$, $T = 1$ and $\bar{\phi} = \phi$, while from Proposition 2 we have $T = 1$ and $U = 1$. Hence, the model is globally identified when $|\phi| < 1$, which is consistent with what Theorem 1 gave us as the alternative solution $\frac{1}{\phi}$ is precluded from the space of allowable parameters when Assumption 3 is imposed.

To sum up, our general Theorem 1 comprises both invertible and noninvertible $MA(1)$ models, proving their global identification failure unless $\phi = \pm 1$. The less general and obtained under stronger assumptions Propositions 1 and 2 allow only for invertible $MA(1)$ processes, in which case their global identification holds.

4 Global identification condition for structural parameters

The ABCD-representation is defined by matrices that, except for some very special cases like the $MA(1)$ model elaborated above, are only implicit functions of θ. Therefore, to check identification of this vector of deep parameters, we typically cannot apply Theorem 1 directly, but additionally need to impose restrictions on the observationally equivalent ABCD-representations defined in this theorem that would guarantee consistence with the underlying DSGE model structure. As in Kociecki and Kolasa (2018), these can be readily obtained by substituting the model solution (2)-(3) into model formulation (1), which we rewrite for convenience in a block form

$$\begin{bmatrix} \Gamma_0^s & \Gamma_0^p \end{bmatrix} \begin{bmatrix} s_t \\ p_t \end{bmatrix} = \begin{bmatrix} \Gamma_1^s & \Gamma_1^p \end{bmatrix} \mathbb{E}_t \begin{bmatrix} s_{t+1} \\ p_{t+1} \end{bmatrix} + \Gamma_2 s_{t-1} + \Gamma_3 \varepsilon_t \tag{9}$$

Using $\mathbb{E}_t \varepsilon_{t+1} = 0$ results in the following two matrix equation restrictions

$$\Gamma_0^s A + \Gamma_0^p F - \Gamma_1^s A^2 - \Gamma_1^p FA = \Gamma_2 \tag{10}$$

$$\Gamma_1^s AB + \Gamma_1^p FB - \Gamma_0^s B + \Gamma_3 = \Gamma_0^p G \tag{11}$$

A similar operation using the original measurement equation (4) results in two other matrix restrictions, that are already available as equations (6) and (7).

We hence arrive at the following final set of conditions that have to be met by any parameter vector $\bar{\theta}$ that is observationally equivalent to some θ

$$\bar{\Gamma}_0^s \bar{A} + \bar{\Gamma}_0^p \bar{F} - \bar{\Gamma}_1^s (\bar{A})^2 - \bar{\Gamma}_1^p \bar{F} \bar{A} = \bar{\Gamma}_2 \tag{12}$$

$$\bar{\Gamma}_1^s \bar{A} \bar{B} + \bar{\Gamma}_1^p \bar{F} \bar{B} - \bar{\Gamma}_0^s \bar{B} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \tag{13}$$

$$\bar{C} = \bar{H}^s \bar{A} + \bar{H}^p \bar{F} \tag{14}$$

$$\bar{D} = \bar{H}^s \bar{B} + \bar{H}^p \bar{G} + \bar{J} \tag{15}$$

$$\bar{A} = T A T^{-1} \tag{16}$$

$$\bar{C} = C T^{-1} \tag{17}$$

$$A Q A' - Q = -B \Sigma B' + T^{-1} \bar{B} \bar{\Sigma} \bar{B}' (T^{-1})' \tag{18}$$

$$A Q C' = T^{-1} \bar{B} \bar{\Sigma} \bar{D}' - B \Sigma D' \tag{19}$$

$$C Q C' = \bar{D} \bar{\Sigma} \bar{D}' - D \Sigma D' \tag{20}$$

$$Q = Q' \tag{21}$$

In this system of equations, the unknowns are $\bar{\theta}$ (on which the following depend explicitly: $\bar{\Gamma}_0^s$, $\bar{\Gamma}_0^p$, $\bar{\Gamma}_1^s$, $\bar{\Gamma}_1^p$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, $\bar{\Sigma}$, $\bar{H}^s$, $\bar{H}^p$, $\bar{J}$), as well as $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{F}$, $\bar{G}$, T, Q. All remaining matrices are functions of θ, and hence known while checking identification at this point in the parameter space. Therefore, our final global identification condition for the structural (deep) model parameters can be stated as follows

Definition 2. *The model given by equations (1) and (4) is globally identified if and only if all admissible solutions to system (12)-(21) are such that $\bar{\theta} = \theta$.*

Note that being able to solve the system of equations above analytically, i.e. giving the full set of $\bar{\theta} \in \Theta$ that satisfy it, essentially resolves the problem of identification in a given DSGE model. However, this is not easy as equations (12)-(21) are non-linear and their number is fairly large even for small-scale models. Naturally, one can try to solve this system numerically, as it is done in a less general framework by Kociecki and Kolasa (2018), but numerical methods can give only one solution at a time rather than their full set. Solving the identification problem hence requires analytical methods, and to this end we will use some concepts developed in computational algebraic geometry.

To apply these methods, we first need to write our model such that the coefficients on the model variables that show up in the equations of the original model formulation (1) and (4) form polynomials. In many cases this is straightforward and can be achieved by basic algebraic operations on the model equations. For example, if some coefficients in a model equation form a fraction, we can simply multiply all terms in this equation by the denominator of this fraction. Whenever this is not possible, e.g. when one parameter enters as an exponent of another, we need to define auxiliary

parameters that add to the original ones, possibly replacing some of them.[7] We will denote the thus obtained modified parameter vector as α, and will refer to its elements as semi-structural parameters, as opposed to deep parameters collected in θ. Naturally, since the deep and semi-structural parameters are linked analytically via time-invariant restrictions, it is straightforward to use the solution of the global identification problem defined for α to make inference about identification of θ.

To see the nature of the underlying problem a bit more clearly, let us eliminate some terms in the system of equations (12)-(21) and reorganize to get

$$\bar{\Gamma}_0^s TA + \bar{\Gamma}_0^p \bar{\tilde{F}} - \bar{\Gamma}_1^s TA^2 - \bar{\Gamma}_1^p \bar{\tilde{F}} A = \bar{\Gamma}_2 T \tag{22}$$

$$\bar{\Gamma}_1^s TA\bar{\tilde{B}} + \bar{\Gamma}_1^p \bar{\tilde{F}}\bar{\tilde{B}} - \bar{\Gamma}_0^s T\bar{\tilde{B}} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \tag{23}$$

$$C = \bar{H}^s TA + \bar{H}^p \bar{\tilde{F}} \tag{24}$$

$$\bar{D} = \bar{H}^s T\bar{\tilde{B}} + \bar{H}^p \bar{G} + \bar{J} \tag{25}$$

$$AQA' - Q = -B\Sigma B' + \bar{\tilde{B}}\bar{\Sigma}\bar{\tilde{B}}' \tag{26}$$

$$AQC' = \bar{\tilde{B}}\bar{\Sigma}\bar{D}' - B\Sigma D' \tag{27}$$

$$CQC' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D' \tag{28}$$

$$Q = Q' \tag{29}$$

where $\bar{\tilde{F}} = \bar{F}T$ and $\bar{\tilde{B}} = T^{-1}\bar{B}$.[8] We have thus turned our identification conditions into a system of polynomial equations. In this alternative formulation, the unknowns are: $\bar{\alpha}$ (on which the following depend analytically: $\bar{\Gamma}_0^s$, $\bar{\Gamma}_0^p$, $\bar{\Gamma}_1^s$, $\bar{\Gamma}_1^p$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, $\bar{\Sigma}$, $\bar{H}^s$, $\bar{H}^p$, $\bar{J}$), as well as matrices $\bar{\tilde{B}}$, $\bar{D}$, $\bar{\tilde{F}}$, $\bar{G}$, T and Q.

It is straightforward to derive a similar set of identification conditions for the square case, when we can use the similarity transformation defined by Proposition 2. These are

$$\bar{\Gamma}_0^s TA + \bar{\Gamma}_0^p \bar{F}T - \bar{\Gamma}_1^s TA^2 - \bar{\Gamma}_1^p \bar{F}TA = \bar{\Gamma}_2 T \tag{30}$$

$$\bar{\Gamma}_1^s TABU + \bar{\Gamma}_1^p \bar{F}TBU - \bar{\Gamma}_0^s TBU + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \tag{31}$$

$$C = \bar{H}^s TA + \bar{H}^p \bar{F}T \tag{32}$$

[7] Rewriting the model equations using auxiliary parameters may be useful even if it is not necessary to obtain a polynomial structure. Take for example the New Keynesian Phillips curve $\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} x_t$, which can be easily cast in the form required by our analysis by multiplying it by ξ. Obviously, for any value of the Calvo probability ξ there exists an observationally equivalent alternative number, which lies outside the unit interval and hence should not be taken into account. However, while solving the identification problem mathematically, this alternative parametrization will be found, only to be discarded after applying economic restrictions. If we instead replace ξ in the vector of model parameters with a semi-structural parameter $\kappa = \frac{(1-\xi)(1-\beta\xi)}{\xi}$, this validation step can be avoided.

[8] Note that, since T is nonsingular, there is a one-to-one relationship between these two newly defined matrices and their parents $\bar{F}$ and $\bar{B}$.

$$DU = \bar{H}^s TBU + \bar{H}^p \bar{G} + \bar{J} \tag{33}$$

$$U\bar{\Sigma}U' = \Sigma \tag{34}$$

forming a system of polynomial equations in $\bar{\alpha}$, $\bar{F}$, $\bar{G}$, T and U.

5 Implementation

5.1 Gröbner basis

As we have demonstrated in the previous section, the key step in solving the identification problem in a DSGE model boils down to solving a system of polynomial equations. In our implementation we draw on the concept of a Gröbner basis. Intuitively, calculating it is analogous to Gaussian elimination in systems of linear equations, and it is entirely analytical. There exist many algorithms that produce a Gröbner basis in finitely many steps and, since the first algorithm proposed by Buchberger in the 1960s, enormous progress in computational efficiency has been made. Below we offer a brief introduction to the key concepts. For details, we refer the interested readers to widely suggested introductive textbooks on computational algebraic geometry (and Gröbner basis in particular) by Cox et al. (1997) and Cox et al. (2005). An excellent introduction in the context of finding all equilibria in economic models can be found in Kubler et al. (2014).

Let $\mathbb{K}$ denote any field. For us, the most important field will be $\mathbb{Q}$, i.e. that of rational numbers, however the field of real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ will also be in use. In addition, let us denote as $\bar{\mathbb{K}}$ an algebraically closed field containing $\mathbb{K}$. Without going into details, one can think of $\bar{\mathbb{K}}$ as $\mathbb{C}$. The set of polynomials in variables $x_1, \ldots, x_l$ with coefficients in $\mathbb{K}$ will be denoted $\mathbb{K}(x_1, \ldots, x_l)$. Each polynomial equation is a finite sum of terms $cx_1^{d_1}x_2^{d_2}\cdots x_l^{d_l}$, where c is a coefficient (in $\mathbb{K}$) and $x_1^{d_1}x_2^{d_2}\cdots x_l^{d_l}$ is called a monomial, where each d_i is a non-negative integer. The degree of a monomial is $d_1 + \cdots + d_l$, and the degree of a polynomial equation is the maximum of the degrees of its all monomials.

Suppose we have a set of s polynomials $f_1, f_2, \ldots, f_s \in \mathbb{K}(x_1, \ldots, x_l)$. Then, a variety V is defined to be a set of all solutions to $f_1 = 0, f_2 = 0, \ldots, f_s = 0$, i.e. $V(f_1, \ldots, f_s) = \{(a_1, \ldots, a_l) \in \bar{\mathbb{K}}^l | f_1 = 0, \ldots, f_s = 0\}$. Of course, the initial polynomials $f_1, \ldots, f_s$ only represent the variety. There are many other alternative sets of polynomials, some of which could do a better job in the sense of the ease with which the underlying solutions can be read. In particular, this opens a way to a Gröbner basis. To this end, define an ideal generated by $f_1, \ldots, f_s$ as $I = \langle f_1, \ldots, f_s \rangle = \{u_1 f_1 + \cdots + u_s f_s | u_i \in \mathbb{K}(x_1, \ldots, x_l), i = 1, \ldots, s\}$. The ideal is just a weighted sum of all initial polynomials (called its generators), in which the coefficients (weights) are polynomials themselves. What makes the ideal useful is that $V(f_1, \ldots, f_s) = V(I)$, i.e. the solution set of the initial finite system of polynomials and that of an ideal generated by these polynomials (i.e. an infinite system) are the same. Evidently, any ideal can have different generators. As a matter of fact, if $\langle f_1, \ldots, f_s \rangle = \langle f'_1, \ldots, f'_{s'} \rangle$, then $V(f_1, \ldots, f_s) = V(I) = V(f'_1, \ldots, f'_{s'})$. Hence, the solutions to $f_1 = 0, \ldots, f_s = 0$ and to $f'_1 = 0, \ldots, f'_{s'} = 0$ are the same. In the essence, what defines the solution set is the ideal and

not the initial polynomials. The main idea of a Gröbner basis is to find alternative generators that represent the ideal in a "better" way. For example, in the case of linear polynomials (equations), this "better" way is to find their row echelon form. Importantly, by the Hilbert basis theorem, each ideal must be generated by finite number of polynomials, hence the algorithmic methods to find the "better" representation of the variety may be safely applied.

Before we can define (and obtain) a Gröbner basis, we have to take a stand on the ordering of monomials since every algorithm to compute the basis must involve polynomial divisions. An ordering is just a rule that allows for a unique placement of terms in a polynomial. It turns out that the chosen ordering greatly influences the ultimate Gröbner basis, and some orderings are particularly useful. For our purposes, the most important ordering is the so-called lexicographic ordering, i.e. monomial $x_1^{d_1}x_2^{d_2}\cdots x_l^{d_l} \succ$ (is greater than) $x_1^{e_1}x_2^{e_2}\cdots x_l^{e_l}$ if $d_1 = e_1, \ldots, d_m = e_m$ and $d_{m+1} > e_{m+1}$ (where possibly $m = 0$). For example, $x_1x_2^2x_3 \succ x_1x_2^2x_4^3$, since $d_3 = 1 > e_3 = 0$. Let us define $x^d := x_1^{d_1}x_2^{d_2}\cdots x_l^{d_l}$. If we choose a monomial ordering, each polynomial may be written uniquely as $f = c_dx^d + \cdots$. Then x^d is called the leading monomial and c_dx^d is the leading term. We say that polynomials $g_1, \ldots, g_t \in I$ constitute the Gröbner basis for ideal I if the leading term of any (nonzero) polynomial in I is divisible by the leading term of one of $g_1, \ldots, g_t$.[9] Needless to say, $g_1, \ldots, g_t$ are generators for I, every (nonzero) I possesses a Gröbner basis, and solutions to $g_1 = 0, \ldots, g_t = 0$ and to the initial polynomials $f_1 = 0, \ldots, f_s = 0$ are the same. When there is only a finite number of solutions, the underlying ideal is called zero-dimensional.

Using the lexicographic ordering, the resulting Gröbner basis represents an ideal particularly well since the polynomial system becomes "triangularized". The Gröbner basis contains a lot of information about the solutions set of the initial polynomial system. For example, the system does not have any solution if and only if the Gröbner basis contains only 1. Further, the fact whether an ideal is zero-dimensional or not is explicitly "coded" in the Gröbner basis and may be easily read off. The initial system of polynomials possesses a finite number of solutions (i.e. I is zero-dimensional) if and only if for every variable x_i there exists a polynomial in the Gröbner basis such that its leading monomial is equal to x_i^m, for some $m > 0$. Importantly, calculation of a Gröbner basis is analytical, i.e. numerical approximations are not involved.

5.2 Computation

Going back to our problem of identification in DSGE models, we showed that the key step in solving it amounts to finding all roots to a polynomial system of equations. In a typical case, the number of equations s exceeds the number of variables l, which is the so-called overdetermined case. For generic overdetermined polynomial systems, the solution set is empty, but of course in our formulation we do know that at least one solution exists. In order to deal effectively with overdetermined systems, we exploit the approaches presented in Lazard (1992) or Moller (1993). The idea is to decompose the original ideal so that the solution set of the initial polynomials will be the disjoint (finite) union of solutions to some smaller systems of l equations in l variables. This leads to the so-called triangular

[9] A nonzero term c_dx^d is divisible by a nonzero term c_ex^e if $d_i \geq e_i$ for all i.

decomposition. See Kubler et al. (2014), section 2.2.3, for some intuition.

For all our calculations that follow, we use SINGULAR, a free and open source computer algebra system specialized in polynomial calculations, see at `www.singular.uni-kl.de`. It has implemented many routines for calculation of the Gröbner basis, which is useful as there is no single algorithm that beats in terms of computational efficiency all alternatives for all possible cases. Importantly, SINGULAR can be considered a repository of most state-of-the-art algorithms, with active community of users sharing their experience in approaching various problems. In the next section, we demonstrate that applying the concept of a Gröbner basis is feasible for widely used DSGE models.

5.3 Checking global identification

By calculating the Gröbner basis for a given DSGE model, we obtain a formally proved verdict about global identification of its semi-structural parameters. To complete the identification analysis about the deep model parameters, we need to use the restrictions mapping $\bar{\theta}$ into $\bar{\alpha}$. In particular, this step also allows us to rule out those observationally equivalent semi-structural parameter sets that violate the restrictions imposed on them by the deep parameters.

These restrictions can be of two types. One concerns possible remaining dependencies between the semi-structural parameters, which are imposed by the deep parameters and which were ignored while defining the former. The second type of restrictions is related to the range of admissible values of $\bar{\theta}$ summarized by their space Θ, which may also impose restrictions on $\bar{\alpha}$. Accommodating such restrictions in the existing algorithms used to calculate the Gröbner basis is not easy, and hence they have to be verified ex post. As we have argued before, and as we demonstrate in our examples, since the links between α and θ are analytical, this step of the identification analysis is straightforward.

To summarize, a complete identification analysis in our framework can proceed as follows.

1. Calculate the Gröbner basis associated with identification conditions (22)-(29), or (30)-(34) if Assumptions 3 and 4 hold.

2. If the obtained Gröbner basis suggests multiple solutions, use the mapping between $\bar{\alpha}$ and $\bar{\theta}$ to rule out those resulting in $\bar{\theta} \notin \Theta$ (and in particular those for which $\bar{\theta} \in \emptyset$).

3. If at least one of the alternative solutions remains, the model is not globally identified.

4. If instead the Gröbner basis implies only one admissible solution, then the model is globally identified if and only if the mapping between α and θ is unique (for $\theta \in \Theta$).

This analysis can be further refined to distinguish between global and local identification failure. Again, doing it at the level of mapping between α and θ is straightforward and boils down to checking if a possible non-uniqueness of the mapping imply a finite number or infinitely many admissible θ's consistent with a given α. In the former case, the model is globally unidentified, but local identification holds. Similar information at the level of semi-structural parameters is coded in the Gröbner basis. If it implies multiple admissible solutions, but is zero-dimensional, the model is locally identified.

6 Examples

We demonstrate the working of our identification framework with several examples. The first one is based on Cochrane (2011), and its simplicity allows us to show in detail the key steps of our analysis, including the use of the Gröbner basis. We next exploit a small-scale DSGE model by An and Schorfheide (2007), AS henceforth, modified to allow for correlation between government spending and productivity as in Herbst and Schorfheide (2016). This is a very instructive example as it allows to nest various non-trivial types of identification issues, including the case when the model is only locally (but not globally) identified. Our next example is the widely cited medium-sized DSGE model of Smets and Wouters (2007), and the goal here is to show that our approach can also handle models of this size. Finally, we study global identification in an open economy framework, which has not been done in the literature before, starting from the model developed by Justiniano and Preston (2010), and extending it in several directions.

6.1 Cochrane model

6.1.1 Model summary and its analytical solution

Consider a very simple model

$$i_t = \mathbb{E}_t \pi_{t+1} \tag{35}$$

$$i_t = \phi \pi_t + x_t \tag{36}$$

where i_t is the nominal interest rate, π_t denotes inflation and x_t is a monetary policy shock that follows a stationary AR(1) process. The first equation can be interpreted as the log-linearized Fisher relationship, while the second as a simple monetary policy feedback rule. We restrict here our attention to the case of determinacy so that $\phi > 1$.

Substituting out i_t and writing the process driving x_t explicitly leads to the following system

$$x_t = \rho x_{t-1} + \varepsilon_t \tag{37}$$

$$\phi \pi_t + x_t = \mathbb{E}_t \pi_{t+1} \tag{38}$$

where $\mid \rho \mid < 1$ and $\varepsilon_t \sim N(0, v)$, with $v > 0$ denoting the variance. This system can be easily cast into form (1), with $s_t = x_t$, $p_t = \pi_t$, $\theta = [\ \rho \quad \phi \quad v\ ]'$ and

$$\Gamma_0 = \begin{bmatrix} 1 & 0 \\ 1 & \phi \end{bmatrix};\ \Gamma_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\ \Gamma_2 = \begin{bmatrix} \rho \\ 0 \end{bmatrix};\ \Gamma_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \Sigma = v$$

Note that all deep parameters collected in θ enter the model equations linearly so that we do not need to rewrite them using semi-structural parameters, which we can formally write as $\alpha = \theta$. If the only observable variable is inflation, i.e. $y_t = \pi_t$, and there is no measurement error, we have

$$H = \begin{bmatrix} 0 & 1 \end{bmatrix}; \qquad J = 0$$

The model is simple enough to have an analytical solution, which, given the restriction on ϕ, is uniquely given by formulas (2)-(3) with the following coefficients

$$A = \rho; \;\; B = 1; \;\; F = -\frac{\rho}{\phi - \rho}; \;\; G = -\frac{1}{\phi - \rho}$$

We also obviously have $C = F$ and $D = G$. This solution implies that the observable variable can be written as an AR(1) process

$$y_t = \rho y_{t-1} - \frac{1}{\phi - \rho}\varepsilon_t \tag{39}$$

Having such an analytical solution, the identification analysis is straightforward and we can immediately conclude that, of the three model parameters, only ρ is globally identified while ϕ and v cannot be separately identified.

6.1.2 Calculating the Gröbner basis

To demonstrate the working of our framework, suppose now that, as it is typically the case, we do not know the analytical solution of the model, so that A, B, C and D are just numbers. Let us apply our framework at a generic point $\theta = [\ 0.8 \quad 1.8 \quad 1\ ]'$ so that $A = 0.8$, $B = 1$, $C = -0.8$ and $D = -1$. The identification conditions (22)-(29) can then be written as follows

$$\begin{aligned}
0.8 + \bar{\bar{F}} &= 0 \\
\bar{D} - \bar{G} &= 0 \\
0.8T - \bar{\rho}T &= 0 \\
0.8T + \bar{\phi}\bar{\bar{F}} - 0.8\bar{\bar{F}} &= 0 \\
-T\bar{\bar{B}} + 1 &= 0 \\
\bar{\bar{F}}\bar{\bar{B}} - T\bar{\bar{B}} - \bar{\phi}\bar{G} &= 0 \\
-0.36Q + 1 - (\bar{\bar{B}})^2\bar{v} &= 0 \\
0.64Q + \bar{\bar{B}}\bar{v}\bar{D} + 1 &= 0 \\
0.64Q - (\bar{D})^2\bar{v} + 1 &= 0
\end{aligned} \tag{40}$$

where the last one was omitted as it becomes an identity when the dimension of Q is one. The unknown variables are: $\bar{\rho}$, $\bar{\phi}$, $\bar{v}$, as well as $\bar{\bar{B}}$, $\bar{D}$, $\bar{\bar{F}}$, $\bar{G}$, T and Q, all of which are one-dimensional objects. The identification conditions are hence given by a system of polynomial equations of degree three. Finding its all solutions is not straightforward even in this simple case. We will show now how this goal can be achieved by calculating the Gröbner basis of the ideal generated by these polynomials.

As we mentioned in the previous section, defining a Gröbner basis involves ordering of monomials, which allows for a unique placement of terms in each polynomial. In applications like ours, the most convenient one is the so-called lexicographic ordering, applied to variables arranged such that the objects of interest, which are the model parameters $\bar{\rho}$, $\bar{\phi}$ and $\bar{v}$, come first. The sequence in which we

listed the unknown variables in the previous paragraph meets this criterion, so we use it here. After applying the lexicographic ordering to our polynomials and running the Buchberger algorithm, we obtain the following Gröbner basis

$$\begin{aligned}
0.64Q^2 + Q &= 0 \\
TQ + 0.8Q &= 0 \\
\bar{G}Q &= 0 \\
\bar{G}T + 0.64Q + 1 &= 0 \\
\bar{\tilde{F}} + 0.8 &= 0 \\
\bar{D} - \bar{G} &= 0 \\
\bar{\tilde{B}} + \bar{G} - 0.8Q &= 0 \\
\bar{v} - T^2 + 0.2304Q &= 0 \\
\bar{\phi} - T - 0.8 &= 0 \\
\bar{\rho} - 0.8 &= 0
\end{aligned} \tag{41}$$

where all numbers showing up in the equations above are exact rational numbers (even though we present them using a decimal notation) as they are derived by analytical algebraic operations that do not involve any numerical approximation.

6.1.3 Identification analysis

One important thing to note is a triangular structure of the obtained Gröbner basis. The first of polynomials includes only Q, the second adds to it T, the third may additionally contain $\bar{G}$, and so on until the model parameters are finally added. This is exactly what makes finding all solutions of the system of polynomial equations (41) easy, in contrast to the original set of identification conditions (40), and we know that the solutions are exactly the same. Naturally, the particular sequence in which the unknown variables add to this triangle is no coincidence, but simply reflects the ordering that we have chosen. As we will show now, this often allows for straightforward conclusions on identification of the model parameters even without having to solve for all other objects.

In our particular example, we immediately see that all possible solutions must be such that $\bar{\rho} = 0.8 = \rho$, hence this parameter is globally identified at the θ we consider. As regards the other two parameters, they depend on Q and T, which are fully determined by the first two equations in (41). From the first one we obtain that $Q = -1.5625$ or $Q = 0$. The first case leads to $T = -0.8$ and further to $\bar{\phi} = 0$, which violates the restriction imposed on this parameter, and hence can be ruled out. If instead $Q = 0$, the second equation does not put any restriction on T, and hence $\bar{v}$ and $\bar{\phi}$ are not identified. For $T = 1$ we obtain $\bar{\theta} = \theta$, but any deviation of T from unity results in an alternative $\bar{\theta}$ that is observationally equivalent to θ. This deviation can be arbitrarily small, which means that the identification failure is local.

A useful feature of our approach is that having the Gröbner basis also allows to establish the

explicit relationship between the unidentified parameters, which (by eliminating T from the penultimate two equations) is $\bar{v} - (\bar{\phi} - 0.8)^2 = 0$. Any pair of $\bar{v}$ and $\bar{\phi}$ meeting this restriction and consistent with the underlying support for deep parameters Θ is observationally equivalent to $v = 1$ and $\phi = 1.8$. Naturally, this conclusion perfectly matches that following from the analytical solution given by equation (39), but we arrived at it as if we did not know the latter. It also immediately follows that fixing either $\bar{v}$ or $\bar{\phi}$ renders the model globally identified at the considered θ.

6.2 An-Schorfheide model

6.2.1 Model summary

When written in a log-linearized form, the model is given by the following equations

$$x_t = \mathbb{E}_t x_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau}(R_t - \mathbb{E}_t \pi_{t+1} - \mathbb{E}_t z_{t+1}) \tag{42}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa(x_t - g_t) \tag{43}$$

$$R_t = \rho_m R_{t-1} + (1 - \rho_m)[\psi_1 \pi_t + \psi_2 (x_t - g_t)] + \varepsilon_{m,t} \tag{44}$$

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t} \tag{45}$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t} \tag{46}$$

There are three endogenous variables in the model: detrended output x_t, inflation π_t and the interest rate R_t. They are driven by two exogenous AR(1) processes for productivity growth z_t and government spending g_t, with innovations $\varepsilon_{z,t}$ and $\varepsilon_{g,t}$, respectively, and by an i.i.d. monetary policy shock $\varepsilon_{m,t}$. All of the i.i.d. innovations are assumed to be mutually uncorrelated and their variances are v_z, v_g and v_m, respectively. The 13-dimensional vector of deep parameters is hence $\theta = [\ \tau \quad \beta \quad \kappa \quad \psi_1 \quad \psi_2 \quad \rho_z \quad \rho_{zg} \quad \rho_g \quad \rho_{gz} \quad \rho_m \quad v_z \quad v_g \quad v_m\ ]'$.

The model can be cast in form (1), with states $s_t = [\ z_t \quad g_t \quad R_t\ ]'$, policy variables $p_t = [\ x_t \quad \pi_t\ ]'$, shocks $\varepsilon_t = [\ \varepsilon_{z,t} \quad \varepsilon_{g,t} \quad \varepsilon_{m,t}\ ]'$ and matrices Γ_0, Γ_1, Γ_2, Γ_3 and Σ given by

$$\Gamma_0 = \begin{bmatrix} 0 & -\tau & 1 & \tau & 0 \\ 0 & \kappa & 0 & -\kappa & 1 \\ 0 & (1-\rho_m)\psi_2 & 1 & -(1-\rho_m)\psi_2 & -(1-\rho_m)\psi_1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \qquad \Gamma_1 = \begin{bmatrix} 1 & -\tau & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_m \\ \rho_z & \rho_{zg} & 0 \\ \rho_{gz} & \rho_g & 0 \end{bmatrix}; \qquad \Gamma_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \qquad \Sigma = \begin{bmatrix} v_z & 0 & 0 \\ 0 & v_g & 0 \\ 0 & 0 & v_m \end{bmatrix}$$

where we have multiplied equation (42) by τ so that all coefficients in the model equilibrium conditions are polynomials. As a result, we do not need to define any auxiliary parameters and can implement our identification analysis directly on the deep model parameters, i.e. $\alpha = \theta$.[10] The vector of observable variables is $y_t = [\ R_t \quad x_t \quad \pi_t\ ]'$ and there are no measurement errors, which means that $H = [\ 0_{3\times 2} \quad \mathrm{I}_3\ ]$ and $J = [\ 0_{3\times 3}\ ]$.

6.2.2 Global identification failure in a locally identified model

Let us start with the following benchmark parametrization: $\tau = 2$, $\beta = 0.9975$, $\kappa = 0.33$, $\psi_1 = 1.5$, $\psi_2 = 0.125$, $\rho_z = 0.9$, $\rho_g = 0.95$, $\rho_{zg} = 0.1$, $\rho_{gz} = -0.075$, $\rho_m = 0.75$, $v_z = 0.09$, $v_g = 0.36$, $v_m = 0.04$. These values are the same as in An and Schorfheide (2007), except for ρ_{zg} and ρ_{gz}, which are taken from Kociecki and Kolasa (2018). Calculating the Gröbner basis results in the following set of solutions for $\bar{\theta}$:

$$
\begin{aligned}
0 &= u^2 - 1.8697u + 0.8697 \\
&\ldots \\
\bar{\rho}_z &= 0.9155 - 0.0155u \\
\bar{\rho}_{zg} &= 0.2415 - 0.1415u \\
\bar{\rho}_g &= 0.9345 + 0.0155u \\
\bar{\rho}_{gz} &= 0.0209 - 0.0959u \\
\bar{\tau} &= 2 \\
\bar{\beta} &= 0.5351 + 0.4624u \\
\bar{\kappa} &= 0.4912 - 0.1612u \\
\bar{\rho}_m &= 0.75 \\
\bar{\psi}_1 &= 1.3131 + 0.1869u \\
\bar{\psi}_2 &= 0.2516 - 0.1266u \\
\bar{v}_z &= 0.1279 - 0.0379u \\
\bar{v}_g &= 0.3128 + 0.6728u \\
\bar{v}_m &= 0.04
\end{aligned}
\tag{47}
$$

where u is the second element of the second row in matrix T. To save space, we skip above the equations determining the solutions for other "unknowns" in the system of identification conditions (22)-(29) as they are not needed to arrive at identification conclusions for θ. We also show the

[10] More precisely, in the original An-Schorfheide model κ is actually a semi-structural parameter, linked to the deep model parameters via $\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}$. Since ν, π and ϕ do not show up anywhere else in the model equations, including them separately in θ instead of combining into κ trivially leads to (local) identification failure.

numbers rounded to four decimal digits, even though they are in fact arbitrarily accurate numbers.[11]

As we can see, three structural parameters, namely $\bar{\tau}$, $\bar{\rho}_m$ and $\bar{v}_m$, are equal to their respective elements of θ, at which we check identification. The remaining elements of $\bar{\theta}$ are parametrized by u, which needs to be consistent with the quadratic restriction in the first equation, implying that $u = 1$ or $u = 0.8697$. It is easy to verify that, in the former case, we get our benchmark parameter vector θ, while the latter case results in an observationally equivalent model parametrization that is exactly the same as that obtained by Kociecki and Kolasa (2018) with their numerical algorithm. We have thus a formal and constructive proof that the AS model is locally but not globally identified at θ, and that the identification failure concerns all deep parameters but τ, ρ_m and v_m. Moreover, looking at the Gröbner basis (47) immediately reveals that fixing any of the unidentified parameters renders the model globally identified.

6.2.3 Local identification failure

Let us now consider the same benchmark parameter vector θ, except that we rule out any spillovers between productivity and government spending shocks, i.e. fix $\rho_{zg} = \rho_{gz} = 0$. Calculating the Gröbner basis yields:

$$
\begin{aligned}
0 &= u^2 - 49.3312u \\
0 &= uw \\
&\ldots \\
\bar{\rho}_z &= 0.9 - 0.0078u \\
\bar{\rho}_g &= 0.95 \\
\bar{\tau} &= 2 - 0.9618u \\
\bar{\beta} &= 0.9975 \\
\bar{\kappa} &= 0.33 \\
\bar{\rho}_m &= 0.75w \\
\bar{\psi}_1(w - 1.3333) &= 3.7211w - 4.2211 \\
0 &= (\bar{\psi}_1 - 3.1658)u \\
\bar{\psi}_2 &= 2.7302 - 1.7368\bar{\psi}_1 \\
\bar{v}_z &= 0.09 + 8.1277u \\
\bar{v}_g &= 0.36 \\
\bar{v}_m &= 0.04w^2
\end{aligned}
\tag{48}
$$

[11] To get arbitrary precision, it is important that the matrices describing the model solution 2-3 for the parameter vector at which we check identification are also sufficiently precise. Therefore, we recommend to do all computation (including obtaining matrices A, B, F and G) in SINGULAR, which works at arbitrary precision.

where, as before, we save space by skipping those elements of the basis that are not necessary for our identification analysis.

Of the two roots of the first equation, only $u = 0$ does not violate the restrictions on the deep model parameters. In particular, the other root implies $\tau < 0$, so we can rule it out. If $u = 0$, the second equation does not put any restrictions on w. Setting $w = 1$ results in $\bar{\theta} = \theta$, any other value of w meeting the restrictions on the deep parameters gives an alternative parameter vector $\bar{\theta}$ that is observationally equivalent to θ. The identification failure concerns exclusively ρ_m, ψ_1, ψ_2 and v_m, which is now proved in a constructive way.

One can think of this failure as local since it applies to any vicinity of $w = 1$. This conclusion is consistent with previous papers dealing with this version of the AS model. Importantly, however, and in contrast to any of the existing approaches to analyze local identification (also Iskrev, 2010; Komunjer and Ng, 2011), our framework analytically produces the whole set of parameter vectors that are observationally equivalent to the one at which we check identification.[12] In this example, the set is one-dimensional and can be written, after some rearrangement, as follows

$$\begin{aligned}
\bar{\rho}_m &= 0.75w \\
\bar{\psi}_1 &= \frac{3.1658 - 2.7908w}{1 - 0.75w} \\
\bar{\psi}_2 &= \frac{-2.7682 + 2.7994w}{1 - 0.75w} \\
\bar{v}_m &= 0.04w^2
\end{aligned} \tag{49}$$

where w is any real number that keeps the alternative model parametrization $\bar{\theta}$ in the determinacy (and stability) region. Having such an explicitly defined set can be useful. For example, one can see from it that our baseline parametrization, which features a positive response of the interest rate to both inflation and output, can be observationally equivalent to one which implies that the central bank's reaction to output is negative (e.g. for $w = 0.8$).

More generally, our method gives a new insight into the concept of local identification, that seems to be new in the literature. In fact, what we demonstrated is that all observationally equivalent parameters in this example live on the intersection of some hyperplanes, whose dimension is 1. It is not difficult to imagine other cases that possibly could emerge in other models, e.g. an intersection of hyperplanes of higher dimension or some polynomials in several variables (e.g. 2 polynomial equations of second degree in 3 variables).

6.2.4 Handling indeterminacy

In the previous two parametrizations of the AS model, we have considered the parameter vectors that imply a unique stable solution. However, our framework can also handle indeterminate cases. To demonstrate it, let us consider the same benchmark θ as before, except that now $\psi_1 = 0.75$, i.e.

[12] Though we are the first to offer analytical insight into the local identification problem in this model, a similar (but numerical) concept called nonidentification curves was proposed earlier by Qu and Tkachenko (2012).

half the previously assumed value. It can be easily verified, e.g. by checking the Blanchard-Kahn conditions, that there are infinitely many stable equilibria under such parametrization. As shown by Lubik and Schorfheide (2003), the full set of these equilibria are still given by equations (2) and (3), except that the vector of shocks ε_t must include a sufficient number of sunspots. Moreover, expectations of forward-looking variables become new states, and hence need to be included in vector s_t. As demonstrated by Farmer et al. (2015), an equivalent characterization of indeterminate equilibria is to redefine a subset of expectational errors as new fundamentals. This is what we do in this example.

The order of indeterminacy in the considered model is one, so we need to pick one expectational error. Without loss of generality, let us pick the one associated with the output gap x_t. Then, the AS model can be written as

$$x_t = \tilde{x}_t + g_t - \mathbb{E}_t g_{t+1} - \frac{1}{\tau}(R_t - \mathbb{E}_t \pi_{t+1} - \mathbb{E}_t z_{t+1}) \tag{50}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa(x_t - g_t) \tag{51}$$

$$R_t = \rho_m R_{t-1} + (1-\rho_m)[\psi_1 \pi_t + \psi_2(x_t - g_t)] + \varepsilon_{m,t} \tag{52}$$

$$x_t - \tilde{x}_{t-1} = \rho_{sz}\varepsilon_{z,t} + \rho_{sg}\varepsilon_{g,t} + \rho_{sm}\varepsilon_{m,t} + \varepsilon_{s,t} \tag{53}$$

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t} \tag{54}$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t} \tag{55}$$

where $\tilde{x}_t = \mathbb{E}_t x_{t+1}$, $\varepsilon_{s,t}$ is an i.i.d. sunspot shock with variance v_s and, as evident from equation (53), we allow for possible correlation between expectational errors and other structural shocks.

As an illustration, we check identification of this model at $\rho_{sz} = \rho_{sg} = \rho_{sm} = 0.1$, $v_s = 0.01$. Calculating the Gröbner basis implies a unique solution to our identification restrictions, i.e. $\bar{\theta} = \theta$. We have hence proved that the AS model is globally identified at this indeterminate parametrization θ. We arrive at the same conclusion also if we fix $\rho_{zg} = \rho_{gz} = 0$, thus confirming the outcome obtained by Qu and Tkachenko (2017) with a numerical algorithm that searches over the parameter space.

6.3 Smets-Wouters model

We next apply our identification framework to a variant of the widely cited medium-sized DSGE model of Smets and Wouters (2007). The only deviation from the original model is that we define the output gap in the monetary policy rule as the deviation of output from its deterministic trend rather than from its potential level. This allows us to leave out the flexible price block and reduce the time to calculate the Gröbner basis to just around 10 seconds using a computing unit with CPU speed 2.90 GHz and 16 GB of RAM memory. Since the model is quite large and its full version well documented in the literature, we describe its structure and all steps in our identification analysis in Appendix A.6, and here we only discuss the conclusions.

By applying our identification framework, complemented with restrictions imposed by the first moments of the observable variables, we can prove the following results. First, if none of the 41 deep parameters in the model are fixed, the model is not locally (and hence also not globally) identified at the posterior mean reported by Smets and Wouters (2007). Second, after fixing two appropriately selected parameters, which are either the curvature of the Kimball goods market aggregator or the Calvo probability for prices, and either the curvature of the Kimball labor market aggregator or the Calvo probability for wages, the model is globally (and hence also locally) identified.

The part of these findings that concerns local identification hence confirms the results obtained by Iskrev (2010) and Komunjer and Ng (2011). The results on global identification are new. The only paper that has studied the Smets-Wouters model from a global identification perspective is Qu and Tkachenko (2017). By applying a numerical routine that searches for observationally equivalent parameters, they conclude that the model is globally identified after fixing the five parameters that were originally calibrated in the original paper.[13] The novel finding obtained using our framework is that only two parameters need to be fixed to obtain global identification.

Summing up, by formally solving the global identification problem in the Smets-Wouters model, we can conclude that identification issues in this framework are less severe than one could suspect based on the previous studies.

6.4 Open economy models

We finally use our framework to study global identification in several variants of open economy models, which has not been done before. Our departure point is the setup developed by Justiniano and Preston (2010), which can be considered a more empirically-oriented version of the small open economy setup by Gali and Monacelli (2005). Similarly to medium-sized closed economy DSGE models in the spirit of Smets and Wouters (2007), the model features imperfect competition, price rigidities, indexation and habits. It also includes two important open economy frictions, namely incomplete international financial markets and local currency pricing in imports. The model is estimated using eight time series, which are home and foreign output, inflation and the short-term interest rates, as well as the terms of trade and real exchange rate.

We additionally consider several extensions to this baseline setup that have been recently emphasized as key to resolving several important puzzles in the open economy literature, see Itskhoki and Mukhin (2017) and Gopinath et al. (2020). These include local currency pricing in exports, strategic complementarities in pricing, and use of imported intermediate inputs in production sold abroad. In Appendix A.7, we present the log-linearized equilibrium conditions of the richest version of the open economy model, and explain how it can be reduced all the way back to the original Justiniano-Preston setup by putting appropriate restrictions on selected parameters. We check global identification at the parameter values calibrated and estimated for Canada (with foreign economy represented by the United States), while the extended versions are parametrized using

[13] The additional three parameters are the depreciation rate, steady-state wage markup and the steady-state share of government purchases in output.

typical values from the literature.

The striking finding of our identification analysis is that, at least when we use the above-mentioned eight time series as observables, all the model variants prove to be globally identified. This is despite we do not fix any of the structural parameters that are usually calibrated rather than estimated when such models are taken to the data. The same conclusion holds if we treat the three variables describing the foreign economy as unobservable, bringing the model close to that considered by Lubik and Schorfheide (2007).

All of this suggests that, as long as one uses the standard set of observables, observational equivalence is not the key source of problems encountered while estimating open economy DSGE models.

7 Conclusions

In this paper we have developed a comprehensive framework to analyze local and global identification in DSGE models or, more generally, dynamic linear systems with rational or model-consistent expectations. Its main advantage is an analytical flavor, which effectively allows to prove identification or lack thereof. The essence of our approach is application of a Gröbner basis to solve analytically for all roots of a system of polynomial equations, which make up a formal identification condition that we derive.

Calculation of the Gröbner bases is known to be computationally involved for large systems, but we have shown that it can be still successfully applied to small and even some medium-sized DSGE models. One of the conclusions that emerge from the set of studied examples is that observational equivalence might be actually not as widespread in this class of models, and especially in their richer versions, as some earlier small model-based evidence suggested. Instead, problems with estimating these models using maximum likelihood, commonly resolved by resorting to Bayesian methods, are much more likely to stem from misspecification or weak identification issues associated with short data series or irregularities in the likelihood function (Al-Sadoon, 2021).

Finally, it is worth stressing that using the concept of a Gröbner basis is not the only possible way to make use of our formal identification condition. One potentially attractive avenue to explore is application of all-solution homotopy methods, recently brought to the attention of economists by Kubler et al. (2014). While numerical in its nature, it may be a useful complement to the Gröbner basis due to its computational advantage, arising from the use of parallelizability.

Acknowledgements

We thank Frank Schorfheide for very useful guidelines and comments to an early concept of this paper, and three anonymous referees for helpful suggestions on its first draft. This paper also benefited from discussions with Majid Al-Sadoon and participants to the IAAE conference in Cyprus, virtual QRFE workshop on identification of DSGE models at Durham University and virtual CEF conference in Tokyo. The project was financed by the Polish National Science Center research grant No. 2019/35/B/HS4/00135. The views expressed in this paper are solely those of the authors.

Appendix

A.1 Discussion of Assumption 2

Let us denote $s = n^2 + nk + rn + rk + \frac{1}{2}k(k+1)$ and $\mathcal{S} = \{A, B, C, D, \Sigma \in \mathbb{R}^s | rank(\mathcal{O}) = rank(\mathcal{C}) = n\} = \mathcal{S}_1 \cap \mathcal{S}_2$, where $\mathcal{S}_1 = \{A, B, C, D, \Sigma \in \mathbb{R}^s |\ rank(\mathcal{O}) = n\} = \{A, B, C, D, \Sigma \in \mathbb{R}^s | det(\mathcal{O}'\mathcal{O}) \neq 0\}$, $\mathcal{S}_2 = \{A, B, C, D, \Sigma \in \mathbb{R}^s | rank(\mathcal{C}) = n\} = \{A, B, C, D, \Sigma \in \mathbb{R}^s |\ det(\mathcal{C}\mathcal{C}') \neq 0\}$. Evidently, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are open subsets of $\mathbb{R}^s$ (being the inverse image of the open set $\mathbb{R} \setminus \{0\}$). Since a finite intersection of open subsets is open, we conclude that $\mathcal{S}$ is open. Further, since the determinant is a polynomial that is an analytic function of its elements, it implies that $\mathcal{S}$ is dense in $\mathbb{R}^s$. This is because an analytic function such as the determinant cannot be equal to 0 on an open subset of $\mathbb{R}^s$ unless it is identically equal to zero. Since $\mathcal{S}$ is an open and dense subset of $\mathbb{R}^s$, we conclude that if Assumption 2 is valid for one $\theta \in \Theta$, all $\theta \in \Theta$ such that Assumption 2 is violated form a nowhere dense subset of $\mathbb{R}^s$ of measure zero.

A.2 Proof of Theorem 1

Recalling the notation introduced in the main text, let us define the infinite block Hankel matrix as

$$\mathcal{H} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots \\ \Lambda_2 & \Lambda_3 & \Lambda_4 & \cdots \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{bmatrix} \begin{bmatrix} N & AN & A^2N & A^3N & \cdots \end{bmatrix} \tag{A.1}$$

Assuming stochastic minimality (Assumption 2) and using Sylvester's rank inequality, it may be easily shown that $rank(\mathcal{H}) = n$. Suppose that two sets of deep parameters $\bar{\theta} \neq \theta$ generate the same autocovariances. Looking at the Hankel matrix, this implies $\mathcal{O}A\mathcal{C} = \bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}$. By Assumption 2, it follows that $\bar{A} = (\bar{\mathcal{O}}'\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}'\mathcal{O}A\mathcal{C}\bar{\mathcal{C}}'(\bar{\mathcal{C}}\bar{\mathcal{C}}')^{-1}$. Let us denote $T = (\bar{\mathcal{O}}'\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}'\mathcal{O}$, which is nonsingular also by Assumption 2. Since we additionally have $\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$, we get $T^{-1} = \mathcal{C}\bar{\mathcal{C}}'(\bar{\mathcal{C}}\bar{\mathcal{C}}')^{-1}$, hence $\bar{A} = TAT^{-1}$. Looking at the first block row of the Hankel matrix, we have $C\mathcal{C} = \bar{C}\bar{\mathcal{C}} \Rightarrow \bar{C} = CT^{-1}$. We have hence arrived at the first two conclusions of Theorem 1. Uniqueness of T follows from equality $\bar{\mathcal{O}} = \mathcal{O}T^{-1}$ and full column rank of $\mathcal{O}$.

Now suppose that $\bar{\theta} \neq \theta$ results in the same spectral density $\Phi(z) = \bar{\Phi}(z)$ for all $z \in \mathbb{C}$ (i.e. $\bar{\theta} \sim \theta$), that is

$$[C(z\mathrm{I}_n - A)^{-1} \vdots \mathrm{I}_r] \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1} \vdots \mathrm{I}_r]' =$$

$$[\bar{C}(z\mathrm{I}_n - \bar{A})^{-1} \vdots \mathrm{I}_r] \begin{bmatrix} \bar{B}\bar{\Sigma}\bar{B}' & \bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}' & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} [\bar{C}(z^{-1}\mathrm{I}_n - \bar{A})^{-1} \vdots \mathrm{I}_r]' \tag{A.2}$$

Using $\bar{A} = TAT^{-1}$ and $\bar{C} = CT^{-1}$, we obtain

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]\begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix}[C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]'$$

$$= [C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]\begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix}[C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' \tag{A.3}$$

Define the Lyapunov equation evaluated at θ as $P = APA' + B\Sigma B'$, and that evaluated at $\bar{\theta}$ as $\bar{P} = \bar{A}\bar{P}\bar{A}' + \bar{B}\bar{\Sigma}\bar{B}'$. Using $\bar{A} = TAT^{-1}$, the latter may be written as $T^{-1}\bar{P}T'^{-1} = AT^{-1}\bar{P}T'^{-1}A' + T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1}$. Since A is stable , $\tilde{P} = T^{-1}\bar{P}T'^{-1}$ is unique.

To proceed further, we need to use a well known lemma, see e.g. Lindquist and Picci (2015), p. 199. Let X be any symmetric $n \times n$ matrix X, then

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]\begin{bmatrix} X - AXA' & -AXC' \\ -CXA' & -CXC' \end{bmatrix}[C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' = 0 \tag{A.4}$$

Let us use this lemma and subtract (A.4) evaluated at $X = P$ and at $X = \tilde{P}$ from, respectively, the left and right-hand side of equation (A.3). Keeping in mind the Lyapunov equations, we get

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]\begin{bmatrix} 0 & S \\ S' & R \end{bmatrix}[C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' = 0 \tag{A.5}$$

where $S = B\Sigma D' - T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A(P - \tilde{P})C'$ and $R = D\Sigma D' - \bar{D}\bar{\Sigma}\bar{D}' + C(P - \tilde{P})C'$.

Since $(z\mathrm{I}_n - A)^{-1} = z^{-1}\mathrm{I}_n + z^{-2}A + z^{-3}A^2 + \cdots$, multiplying all terms yields

$$S'(z\mathrm{I}_n + z^2A' + z^3A'^2 + \cdots)C' + C(z^{-1}\mathrm{I}_n + z^{-2}A + z^{-3}A^2 + \cdots)S + R = 0 \tag{A.6}$$

Any polynomial is identically (i.e. for all $z \in \mathbb{C}$) equal to zero iff its all coefficients are zeros. Using this fact, we get $R = 0$ and, by stacking (part of) the remaining restrictions on the polynomial coefficients together we have $[C'\vdots A'C'\vdots A'^2C'\vdots\ldots\vdots A'^{n-1}C']'S = 0$. By Assumption 2, the latter yields $S = B\Sigma D' - T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A(P - \tilde{P})C' = 0$. Lastly, combining the two Lyapunov equations into one equation we have $A(P - \tilde{P})A' - (P - \tilde{P}) = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$. By setting $Q = P - \tilde{P}$, we arrive at conclusions 3)-5) of Theorem 1. Symmetry and uniqueness of Q follows from symmetry and uniqueness of P and $\tilde{P}$.

The implication in the other direction, which amounts to checking if the spectral density remains the same if conclusions 1)-5) in Theorem 1 hold, is easy to demonstrate.[14]

A.3 Discussion of Assumption 3

We first show that, if $D\Sigma D'$ is non-singular,[15] then a sufficient condition to make Assumption 3 hold is that matrix $\Psi = A - B\Sigma D'(D\Sigma D')^{-1}C$ is stable, i.e. its all eigenvalues are strictly less than

[14] A similar note also applies to the proofs of Propositions 1 and 2 to be presented below.

[15] Non-singularity of $D\Sigma D'$ is also imposed by Komunjer and Ng (2011) as Assumption 4-NS.

1 in modulus. To see it, let us additionally define $M = B\Sigma B' - B\Sigma D'(D\Sigma D')^{-1}D\Sigma B'$. Then, by Theorem 5.4. in Katayama (2005), Assumption 3 holds if and only if, for any $z \in \mathbb{C}$ with $|z| \geq 1$, both $rank[\Psi - z\mathrm{I}_n \vdots M^{\frac{1}{2}}] = n$ and $rank \begin{bmatrix} \Psi - z\mathrm{I}_n \\ C \end{bmatrix} = n$. When all eigenvalues of Ψ are such that $|z| < 1$, then for all $z \in \mathbb{C}$ with $|z| \geq 1$, $\Psi - z\mathrm{I}_n$ is nonsingular, i.e. $rank(\Psi - z\mathrm{I}_n) = n$.

Moreover, when D is nonsingular, so that $\Psi = A - BD^{-1}C$, then stability of Ψ is necessary and sufficient for Assumption 3. To see it, note that, when D is nonsingular, then $M = 0$. Suppose that Assumption 3 holds, but Ψ is not stable, which means that there is at least one $|z| \geq 1$ such that $\Psi - z\mathrm{I}_n$ is singular, i.e. $rank(\Psi - z\mathrm{I}_n) < n$. In such a case $rank[\Psi - z\mathrm{I}_n \vdots M^{\frac{1}{2}}] = rank[\Psi - z\mathrm{I}_n \vdots 0] = rank(\Psi - z\mathrm{I}_n) = n$, for all $|z| \geq 1$, cannot hold. Hence we arrive at a contradiction, which implies that Ψ must be stable.

As a matter of fact, if D is nonsingular, then stability of Ψ is equivalent to the "poor man's" invertibility condition in Fernández-Villaverde et al. (2007) and is almost identical to Assumption 4-S in Komunjer and Ng (2011), i.e. left-invertibility of the transfer function.[16] Moreover, to the extent that Assumption 4-S generalizes the "poor man's" invertibility condition for the case $r > k$, the condition concerning stability of Ψ can be thought of as generalizing the latter for the case $r < k$.

A.4 Proof of Proposition 1

By definition $\Lambda_0 - CPC' = D\Sigma D'$, $N - APC' = B\Sigma D'$, $P - APA' = B\Sigma B'$. Let us write those equations in the form of the following symmetric matrix

$$\begin{bmatrix} P - APA' & N - APC' \\ N' - CPA' & \Lambda_0 - CPC' \end{bmatrix} = \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} \tag{A.7}$$

On the other hand, by Assumption 2 we have $\bar{A} = TAT^{-1}$ and $\bar{C} = CT^{-1}$. This implies $\Lambda_l = CA^{l-1}N = \bar{C}\bar{A}^{l-1}\bar{N} = CA^{l-1}T^{-1}\bar{N}$, for $l > 0$. Using assumption $rank(\mathcal{O}) = n$, we conclude that $\bar{N} = TN$. Hence, for all $\bar{\theta}$, we can write an analogous symmetric matrix

$$\begin{bmatrix} \tilde{P} - A\tilde{P}A' & N - A\tilde{P}C' \\ N' - C\tilde{P}A' & \Lambda_0 - C\tilde{P}C' \end{bmatrix} = \begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \tag{A.8}$$

where $\tilde{P} = T^{-1}\bar{P}T'^{-1}$. To proceed, we will need the following lemma

Lemma 1. *Let Assumptions 1 and 3 hold. Then $W = AWA' + K\Sigma_a K'$ has a unique solution with respect to W. In addition $P = W + \Pi$, where $P = E(s_t s_t')$ and Π is the solution to the Riccati equation (8).*

Proof. Since $\Pi = A\Pi A' + B\Sigma B' - K\Sigma_a K'$ and $P = APA' + B\Sigma B'$, we have $P - \Pi = A(P - \Pi)A' + K\Sigma_a K'$. Since A is stable by Assumption 1 and $K\Sigma_a K'$ is unique (because Π is unique), $W = P - \Pi$

[16] Assumption 4-S in the case of nonsingular D is equivalent to the statement that all eigenvalues of $A - BD^{-1}C$ are less than or equal to 1 in modulus.

is also unique. □

Using Lemma 1, we can rewrite (A.7) and (A.8) as

$$\begin{bmatrix} W - AWA' & N - AWC' \\ N' - CWA' & \Lambda_0 - CWC' \end{bmatrix} = \begin{bmatrix} A\Pi A' - \Pi + B\Sigma B' & A\Pi C' + B\Sigma D' \\ C\Pi A' + D\Sigma B' & C\Pi C' + D\Sigma D' \end{bmatrix} \tag{A.9}$$

and

$$\begin{bmatrix} \tilde{W} - A\tilde{W}A' & N - A\tilde{W}C' \\ N' - C\tilde{W}A' & \Lambda_0 - C\tilde{W}C' \end{bmatrix} = \begin{bmatrix} A\tilde{\Pi}A' - \tilde{\Pi} + T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & A\tilde{\Pi}C' + T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ C\tilde{\Pi}A' + \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & C\tilde{\Pi}C' + \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \tag{A.10}$$

where $\tilde{W} = T^{-1}\bar{W}T'^{-1}$ and $\tilde{\Pi} = T^{-1}\bar{\Pi}T'^{-1}$. Using (A.9), we can equivalently write the equation for W as $W = AWA' + (N - AWC')(\Lambda_0 - CWC')^{-1}(N - AWC')'$. Evaluating the latter at $\bar{\theta}$, after some simple algebra, we can show that $\tilde{W} = A\tilde{W}A' + (N - A\tilde{W}C')(\Lambda_0 - C\tilde{W}C')^{-1}(N - A\tilde{W}C')'$. By Lemma 1, we conclude that $\tilde{W} = W$.

Hence, the right hand sides of (A.9) and (A.10) are equal, and $\bar{K} = TK$, $\bar{\Sigma}_a = \Sigma_a$. Moreover, $AQ^*A' - Q^* = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$, where $Q^* = \Pi - \tilde{\Pi}$. However, since $P = W + \Pi$, $\tilde{P} = \tilde{W} + \tilde{\Pi}$ and $\tilde{W} = W$, we have $Q^* = \Pi - \tilde{\Pi} = P - \tilde{P} = Q$. Hence the additional conclusion 3) from Theorem 1 reappears automatically.

A.5 Proof of Proposition 2

Let us consider the following matrix

$$\begin{bmatrix} \mathrm{I}_n & -V \\ 0 & \mathrm{I}_r \end{bmatrix} \begin{bmatrix} P - APA' & N - APC' \\ N' - CPA' & \Lambda_0 - CPC' \end{bmatrix} \begin{bmatrix} \mathrm{I}_n & 0 \\ -V' & \mathrm{I}_r \end{bmatrix} \tag{A.11}$$

where $V = (N - APC')(\Lambda_0 - CPC')^{-1}$. Clearly, using expression (A.7) and the fact that, by Assumption 4, $rank \begin{bmatrix} B \\ D \end{bmatrix} = r$, the inner matrix in (A.11) has rank r, and the whole matrix (A.11) has also rank r. Multiplying all terms in (A.11) we have

$$\begin{bmatrix} Z & 0 \\ 0 & \Lambda_0 - CPC' \end{bmatrix} \tag{A.12}$$

where $Z = P - APA' - (N - APC')(\Lambda_0 - CPC')^{-1}(N - APC')'$. Using Assumption 4, we have $rank(\Lambda_0 - CPC') = rank(D\Sigma D') = r$ and

$$r = rank \begin{bmatrix} P - APA' & N - APC' \\ N' - CPA' & \Lambda_0 - CPC' \end{bmatrix} = r + rank(Z) \tag{A.13}$$

It follows $Z = 0$ i.e. P solves the equation for W. By Lemma 1, we conclude that $P = W$, hence $\Pi = 0$. Proceeding similarly we can get an analogous result for any other $\bar{\theta}$, which leads us to the

finding that $\tilde{P} = W$. Since $W = \tilde{W}$ (see the proof of Proposition 1), we get $\tilde{\Pi} = 0$.

We conclude that not only the right hand sides of (A.9) and (A.10) are equal, but we can also put $\Pi = \tilde{\Pi} = 0$. As a result, we arrive at

$$\begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} = \begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \tag{A.14}$$

From (A.14), it follows quite easily that $\bar{B} = TBU$, $\bar{D} = DU$ and $\bar{\Sigma} = U^{-1}\Sigma U'^{-1}$, for some (unique) nonsingular $k \times k$ matrix U.

A.6 Identification of the Smets-Wouters model

The considered model is made of the following 24 equations:

$$y_t = \alpha_1 i_t + (1 - \alpha_1 - g_y)c_t + \varpi z_t + \varepsilon_t^g \tag{A.15}$$

$$c_t = \alpha_2 c_{t-1} + (1 - \alpha_2)\mathbb{E}_t c_{t+1} + \alpha_3(l_t - \mathbb{E}_t l_{t+1}) - \alpha_4(r_t - \mathbb{E}_t \pi_{t+1} + \varepsilon_t^b) \tag{A.16}$$

$$i_t = \alpha_5 i_{t-1} + (1 - \alpha_5)\mathbb{E}_t i_{t+1} + \alpha_6 q_t + \varepsilon_t^i \tag{A.17}$$

$$q_t = \alpha_7 \mathbb{E}_t q_{t+1} + (1 - \alpha_7)\mathbb{E}_t r_{t+1}^k - (r_t - \mathbb{E}_t \pi_{t+1} + \varepsilon_t^b) \tag{A.18}$$

$$y_t = \phi_p \left[\varpi k_t^s + (1 - \varpi)l_t + \varepsilon_t^a\right] \tag{A.19}$$

$$k_t^s = k_{t-1} + z_t \tag{A.20}$$

$$\psi z_t = (1 - \psi)r_t^k \tag{A.21}$$

$$r_t^k = -(k_t^s - l_t) + w_t \tag{A.22}$$

$$k_t = \alpha_8 k_{t-1} + (1 - \alpha_8)i_t + \alpha_9 \varepsilon_t^i \tag{A.23}$$

$$\mu_t^p = \varpi(k_t^s - l_t) + \varepsilon_t^a - w_t \tag{A.24}$$

$$\pi_t = \alpha_{10}\pi_{t-1} + \alpha_{11}\mathbb{E}_t \pi_{t+1} - \alpha_{12}\mu_t^p + \varepsilon_t^p \tag{A.25}$$

$$\mu_t^w = w_t - \sigma_l l_t - \alpha_{13} c_t + (\alpha_{13} - 1)c_{t-1} \tag{A.26}$$

$$w_t = \alpha_5 w_{t-1} + \alpha_{14}\pi_{t-1} + (1 - \alpha_7)\mathbb{E}_t(w_{t+1} + \pi_{t+1}) - \alpha_{15}\pi_t - \alpha_{16}\mu_t^w + \varepsilon_t^w \tag{A.27}$$

$$\tilde{r}_t = \rho r_t - r_{\Delta y} y_t \tag{A.28}$$

$$r_t = \tilde{r}_{t-1} + (1 - \rho)r_\pi \pi_t + [(1 - \rho)r_y + r_{\Delta y}]y_t + \varepsilon_t^r \tag{A.29}$$

$$\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \eta_t^a \tag{A.30}$$

$$\varepsilon_t^b = \rho_b \varepsilon_{t-1}^b + \eta_t^b \tag{A.31}$$

$$\varepsilon_t^g = \rho_g \varepsilon_{t-1}^g + \rho_{ga}\eta_t^a + \eta_t^g \tag{A.32}$$

$$\varepsilon_t^i = \rho_i \varepsilon_{t-1}^i + \eta_t^i \tag{A.33}$$

$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + \eta_t^r \tag{A.34}$$

$$\varepsilon_t^p = \tilde{\varepsilon}_{t-1}^p + \eta_t^p \tag{A.35}$$

$$\tilde{\varepsilon}_t^p = \rho_p \tilde{\varepsilon}_{t-1}^p + (\rho_p - \mu_p)\eta_t^p \tag{A.36}$$

$$\varepsilon_t^w = \tilde{\varepsilon}_{t-1}^w + \eta_t^w \tag{A.37}$$

$$\tilde{\varepsilon}_t^w = \rho_w \tilde{\varepsilon}_{t-1}^w + (\rho_w - \mu_w)\eta_t^w \tag{A.38}$$

and the covariance matrix of shocks, denoted by η_t with appropriate superscripts, is $\Sigma = \text{diag}([v_a\ v_b\ v_g\ v_i\ v_r\ v_p\ v_w])$.

Compared to the original paper by Smets and Wouters (2007), we define the output gap in the monetary policy feedback rule as the deviation of output from its deterministic trend rather than from its hypothetical level in the absence of nominal rigidities and markup shocks. This follows the practice in many policy making institutions and allows us to simplify the model by leaving out the flexible price block, which anyway includes only parameters that already show up in the main part of the model.[17] Additionally, to fit the model to our identification framework, and in particular to meet Assumption 2, we use a state-space representation of the ARMA processes for markup shocks and rewrite the monetary policy rule using an appropriately defined auxiliary variable. See Kociecki and Kolasa (2018) for more details.

There are 41 deep parameters in the model, i.e. $\theta = [\gamma\ \pi\ l\ \beta\ \delta\ g_y\ \sigma_c\ \lambda\ \varphi\ \phi_p\ \varpi\ \psi\ \iota_p\ \xi_p\ \varepsilon_p\ \sigma_l\ \phi_w\ \iota_w\ \xi_w\ \varepsilon_w\ \rho\ r_\pi\ r_y\ r_{\Delta y}\ \rho_a\ \rho_b\ \rho_g\ \rho_r\ \rho_r\ \rho_p\ \rho_w\ \rho_{ga}\ \mu_p\ \mu_w\ v_a\ v_b\ v_g\ v_r\ v_r\ v_p\ v_w]'$.[18] While writing the model equations above using the semi-structural parameters, we have defined the following objects: $\alpha_1 = \frac{(\gamma-1+\delta)\varpi}{\beta^{-1}\gamma^{\sigma_c}-1+\delta}$, $\alpha_2 = \frac{\lambda\gamma^{-1}}{1+\lambda\gamma^{-1}}$, $\alpha_3 = \frac{(1-\varpi)(\sigma_c-1)}{\phi_w\sigma_c(1+\lambda\gamma^{-1})(1-\alpha_1-g_y)}$, $\alpha_4 = \frac{1-\lambda\gamma^{-1}}{(1+\lambda\gamma^{-1})\sigma_c}$, $\alpha_5 = \frac{1}{1+\beta\gamma^{1-\sigma_c}}$, $\alpha_6 = \frac{1}{(1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2}$, $\alpha_7 = \beta\gamma^{-\sigma_c}(1-\delta)$, $\alpha_8 = (1-\delta)\gamma^{-1}$, $\alpha_9 = (1-\alpha_8)(1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2$, $\alpha_{10} = \frac{\iota_p}{1+\beta\gamma^{1-\sigma_c}\iota_p}$, $\alpha_{11} = \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p}$, $\alpha_{12} = \frac{(1-\beta\gamma^{1-\sigma_c}\xi_p)(1-\xi_p)}{(1+\beta\gamma^{1-\sigma_c}\iota_p)\xi_p[(\phi_p-1)\varepsilon_p+1]}$, $\alpha_{13} = \frac{1}{1-\lambda\gamma^{-1}}$, $\alpha_{14} = \frac{\iota_w}{1+\beta\gamma^{1-\sigma_c}}$, $\alpha_{15} = \frac{1+\beta\gamma^{1-\sigma_c}\iota_w}{1+\beta\gamma^{1-\sigma_c}}$, $\alpha_{16} = \frac{(1-\beta\gamma^{1-\sigma_c}\xi_w)(1-\xi_w)}{(1+\beta\gamma^{1-\sigma_c})\xi_w[(\phi_w-1)\varepsilon_w+1]}$. Applying them substitutes the following elements of θ: γ, β, δ, σ_c, λ, φ, ι_p, ξ_p, ε_p, ϕ_w, ι_w, ξ_w, ε_w. The vector of semi-structural parameters is then $\alpha = [\alpha_1\ ...\ \alpha_{16}\ \pi\ l\ g_y\ \phi_p\ \varpi\ \psi\ \sigma_l\ \rho\ r_\pi\ r_y\ r_{\Delta y}\ \rho_a\ \rho_b\ \rho_g\ \rho_r\ \rho_r\ \rho_p\ \rho_w\ \rho_{ga}\ \mu_p\ \mu_w\ v_a\ v_b\ v_g\ v_r\ v_r\ v_p\ v_w]'$, and hence has three elements more than θ. This is because we do not take into account all cross-equation restrictions implied by the model's deep parameters while defining α. This means that, if our identification conditions generate any $\bar{\alpha} \neq \alpha$, we will need to check if it is consistent with some $\bar{\theta} \in \Theta$.

Calculating the Gröbner basis associated with this identification problem at the posterior mean reported in Smets and Wouters (2007), and for the seven observable variables that they use, reveals

[17] Without this simplification, memory requirements become prohibitively expensive and the Gröbner basis cannot be calculated using a computing unit equipped with 16 GB of RAM.

[18] The notation follows exactly Smets and Wouters (2007), except that (i) we replace α with ϖ as the former is already reserved in our paper to denote the vector of semi-structural parameters, (ii) we denote the steady state levels of inflation and labor simply as π and l, i.e. without bars, as these we use to indicate the observationally equivalent alternative parameter values, and (iii) we use variance v rather than standard deviation σ to measure shock volatility as the latter are obviously identified only up to a sign.

that the only solution to the system (22)-(29) is such that $\bar{\alpha} = \alpha$, which formally proves that all semi-structural parameters that show up in equations (A.15)-(A.38) and in the shock covariance matrix Σ are globally identified. To formulate the conclusions about the deep parameters, one needs to examine the analytical links between α and θ listed above. This is relatively straightforward, especially if we take into account that γ, π and l can be identified from the first moments, which leaves us with 38 deep parameters. In particular, for given γ, α_2 uniquely determines λ, then α_4 pins down σ_c, α_7 determines δ, while α_5 pins down β . This further allows to obtain uniquely φ from α_6, ϕ_w from α_3, ι_p from α_{11}, and ι_w from α_{14}. Out of the remaining four deep parameters, ξ_p and ε_p are linked only to α_{12} while ξ_w and ε_w show up only in the definition of α_{16}, and hence they are not identified. To achieve identification, one needs to fix one parameter in each of these two pairs.

A.7 Identification of open economy models

The richest version of the considered open economy model is given by the following 22 equations:

$$(1+h)\alpha_1 c_t = \alpha_1 E_t c_{t+1} + h\alpha_1 c_{t-1} + (g_t - E_t g_{t+1}) - (i_t - E_t \pi_{t+1}) \tag{A.39}$$

$$(1-\varpi)\left[(1-\omega)c_t + \omega x_t\right] = y_t - (1-\varpi)\eta\varpi s_t + \varpi\lambda p^*_{H,t} - \varpi y^*_t \tag{A.40}$$

$$\beta a_t = a_{t-1} + \beta[y_t - (1-\omega)c_t - \omega x_t + \varpi p^*_{H,t}] \tag{A.41}$$

$$(1+\varphi)(y_t - z_t) + (1-\omega)\alpha_1(c_t - hc_{t-1}) = \omega\varphi x_t \tag{A.42}$$

$$mc_t = \varphi y_t - (1+\varphi)z_t - \varphi\omega x_t + \varpi s_t + (1-\omega)\alpha_1(c_t - hc_{t-1}) \tag{A.43}$$

$$s_t - s_{t-1} = \pi_{F,t} - \pi_{H,t} \tag{A.44}$$

$$q_t = \psi_{F,t} + (1-\varpi)s_t \tag{A.45}$$

$$\pi^*_{H,t} = p^*_{H,t} - p^*_{H,t-1} + \pi^*_t \tag{A.46}$$

$$\pi_{H,t} - \delta_H \pi_{H,t-1} = \beta(E_t\pi_{H,t+1} - \delta_H \pi_{H,t}) + \alpha_2\left[(1-\gamma_H)mc_t + \gamma_H \varpi s_t\right] \tag{A.47}$$

$$\pi^*_{H,t} - \delta^*_H \pi_{H,t-1} = \beta(E_t\pi^*_{H,t+1} - \delta^*_H \pi_{H,t}) + \alpha_3\left[(1-\gamma^*_H)(mc_t - q_t - \varpi s_t) - p^*_{H,t}\right] \tag{A.48}$$

$$\pi_{F,t} - \delta_F \pi_{F,t-1} = \beta(E_t\pi_{F,t+1} - \delta_F \pi_{F,t}) + \alpha_4\left[(1-\gamma_F)\psi_{F,t} - \gamma_F(1-\varpi)s_t\right] + cp_t \tag{A.49}$$

$$\pi_t = \pi_{H,t} + \varpi(s_t - s_{t-1}) \tag{A.50}$$

$$(i_t - E_t\pi_{t+1}) - (i^*_t - E_t\pi^*_{t+1}) = E_t q_{t+1} - q_t - \chi a_t - \phi_t \tag{A.51}$$

$$\tilde{i}_t = \psi_i i_t - (1-\psi_i)\psi_{\Delta y} y_t - (1-\psi_i)\psi_e q_t \tag{A.52}$$

$$i_t = \tilde{i}_{t-1} + (1-\psi_i)\psi_\pi \pi_t + (1-\psi_i)(\psi_y + \psi_{\Delta y})y_t + (1-\psi_i)\psi_e(q_t - \pi^*_t + \pi_t) + \eta^m_t \tag{A.53}$$

$$z_t = \rho_z z_{t-1} + \eta^z_t \tag{A.54}$$

$$g_t = \rho_g g_{t-1} + \eta_t^g \tag{A.55}$$

$$cp_t = \rho_{cp} cp_{t-1} + \eta_t^{cp} \tag{A.56}$$

$$\phi_t = \rho_\phi \phi_{t-1} + \eta_t^\phi \tag{A.57}$$

$$\begin{bmatrix} \pi_t^* \\ y_t^* \\ i_t^* \end{bmatrix} = A_V \begin{bmatrix} \pi_{t-1}^* \\ y_{t-1}^* \\ i_{t-1}^* \end{bmatrix} + \begin{bmatrix} \eta_t^{\pi^*} \\ \eta_t^{y^*} \\ \eta_t^{i^*} \end{bmatrix} \tag{A.58}$$

and the covariance matrix of shocks denoted by η_t with appropriate superscripts is

$$\Sigma = \begin{bmatrix} \Sigma_M & 0 \\ 0 & \Sigma_V \end{bmatrix} \quad \Sigma_M = \text{diag}([\ v_m \quad v_z \quad v_g \quad v_{cp} \quad v_\phi\ ]) \quad \Sigma_V = \begin{bmatrix} v_{\pi^*} & v_{\pi^* y^*} & v_{\pi^* i^*} \\ v_{\pi^* y^*} & v_{y^*} & v_{y^* i^*} \\ v_{\pi^* i^*} & v_{y^* i^*} & v_{i^*} \end{bmatrix} \tag{A.59}$$

where A_V and Σ_V are 3×3 matrices with, respectively, foreign VAR coefficients and the covariance structure of VAR innovations. Note that A_V contains 9 independent elements, while the number of independent elements in Σ_V is 6. As in the case of the Smets-Wouters model described in section A.6, the monetary policy rule is written using an appropriately defined auxiliary variable to meet Assumption 2.

There are 47 deep parameters in the model, i.e. $\theta = [h\ \sigma\ \varpi\ \omega\ \eta\ \lambda\ \varphi\ \beta\ \delta_H\ \xi_H\ \gamma_H\ \delta_H^*\ \xi_H^*\ \gamma_H^*\ \delta_F\ \xi_F\ \gamma_F\ \chi\ \psi_i\ \psi_\pi\ \psi_y\ \psi_{\Delta y}\ \psi_e\ \rho_z\ \rho_g\ \rho_{cp}\ \rho_\phi\ v_m\ v_z\ v_g\ v_{cp}\ v_\phi\ v_{\pi^*}\ v_{y^*}\ v_{i^*}\ v_{\pi^* y^*}\ v_{\pi^* i^*}\ v_{y^* i^*}\ \text{vec}(A_V)']'$. While writing the model equations we define $\alpha_1 = \frac{\sigma}{1-h}$, $\alpha_2 = \frac{(1-\xi_H)(1-\beta\xi_H)}{\xi_H}$, $\alpha_3 = \frac{(1-\xi_H^*)(1-\beta\xi_H^*)}{\xi_H^*}$ and $\alpha_4 = \frac{(1-\xi_F)(1-\beta\xi_F)}{\xi_F}$. Using these auxiliary definitions eliminates σ, ξ_H, ξ_H^* and ξ_F, respectively, so that the vector of semi-structural parameters is $\alpha = [h\ \varpi\ \omega\ \eta\ \lambda\ \varphi\ \beta\ \delta_H\ \gamma_H\ \delta_H^*\ \gamma_H^*\ \delta_F\ \gamma_F\ \chi\ \psi_i\ \psi_\pi\ \psi_y\ \psi_{\Delta y}\ \psi_e\ \rho_z\ \rho_g\ \rho_{cp}\ \rho_\phi\ v_m\ v_z\ v_g\ v_{cp}\ v_\phi\ v_{\pi^*}\ v_{y^*}\ v_{i^*}\ v_{\pi^* y^*}\ v_{\pi^* i^*}\ v_{y^* i^*}\ \text{vec}(A_V)'\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]'$, which also has 47 elements. Note that the mapping from α to θ is straightforward, so by solving the identification problem for the former we immediately obtain the outcome for the latter.

Compared to the original Justiniano-Preston setup, the equilibrium conditions (A.39)-(A.58) feature several additional structural parameters.[19] These are: ω, which is the share of intermediate inputs in output, γ_H,γ_H^*, γ_F, which controls the degree of strategic complementarity in domestic, export and import pricing, and ξ_H^*, which measures the degree of price stickiness in export sales.

We first analyze global identification for the model described in Justiniano and Preston (2010), which obtains by setting $\omega = \gamma_H = \gamma_H^* = \gamma_F = 0$ and replacing the Phillips curve for exports (A.48) with the law of one price for domestic production $-\varpi s_t = p_{H,t}^* + q_t$. We check identification at the point corresponding to the posterior median estimated by these authors for Canada, see Table I in their paper. We next move to extensions, adding successively the new features until we reach

[19] Otherwise, the notation here follows exactly Justiniano and Preston (2010), except that (i) we replace α with ϖ as the former is already reserved in our paper to denote the vector of semi-structural parameters, (ii) we replace the Calvo probabilities θ with ξ as the former we use to denote the vector of deep parameters, (iii) we use variance v rather than standard deviation sd to measure shock volatility, thus avoiding the need to impose an additional sign restriction.

the full version described above. For all of these models, we find that the only solution to our identification conditions (22)-(29) is such that $\bar{\alpha} = \alpha$, and hence (after restricting each ξ_H, ξ_H^* and ξ_F to lie in the unit interval) $\bar{\theta} = \theta$.

We additionally consider a simpler case resembling the setup of Lubik and Schorfheide (2007). Compared to the baseline Justinano-Preston model, it drops cost-push shocks cp_t, risk premium shocks ϕ_t and foreign interest rate i_t^*, and treats the foreign VAR as unobservable. Also for this variant, our identification analysis proves that the observational equivalence set is a singleton that contains only the benchmark parameter vector.

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