(Un-)Common Preferences, Ambiguity, and Coordination*

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Abstract

This paper studies the "common prior" assumption and its implications when agents have differential information and rational preferences beyond subjective expected utility (SEU). We characterize the class of consequentialist interim preferences that are dynamically consistent with respect to the same ex-ante preference in terms of common limits of higher-order expectations. We then relax common dynamic consistency by either allowing for non neutral attitudes towards the timing of resolution of uncertainty or by letting the agents only share benchmark beliefs with potentially heterogeneous preferences for uncertainty. Within this framework, we characterize the properties of equilibria of coordination games (e.g., financial beauty contests) in terms of the agents' private information, coordination motives, and attitudes toward uncertainty. When the agents share the same benchmark probabilistic model, high-coordination motives completely wash out their aversion to misspecification, producing outcomes that are indistinguishable from the ones obtained under SEU.

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1 Introduction

The common prior assumption is one of the most used and debated concepts in economic theory.[1] It captures the idea of mutual ex-ante agreement about a probabilistic model describing some aspect of the world. When paired with expected-utility maximization, it gives rise to correlated equilibrium, that is, arguably the plainest extension of the concept of Bayesian rationality of de Finetti [16] and Savage [61] from a single decision maker to a group of interacting agents (cf. Aumann [5] and Nau and McCardle [58]). However, systematic departures from subjective expected utility (SEU), such as the ones highlighted by the Ellsberg paradox, are at the same time normatively convincing and robust experimental findings. Notably, these departures are consistent with the rationality of decision makers that acknowledge their *ambiguity* (or "Knightian uncertainty", see Gilboa and Marinacci [31]) about an objective probabilistic model and have nonneutral attitudes toward it. Therefore, it is natural to wonder whether the ex-ante mutual agreement can be expressed independently of agents' attitudes toward ambiguity. More importantly, we wonder what the implications for the agents' interim preferences and behavior of this mutual agreement are.

This paper addresses these questions by first formalizing increasing degrees of mutual ex-ante agreement among agents with differential information and general rational preferences (cf. Cerreia-Vioglio et al. [11]) such as SEU, maxmin expected utility, Choquet expected utility (CEU), variational preferences, and, in general, uncertainty averse preferences. More concretely, we impose restrictions on the agents' interim preferences that guarantee the existence of a *single* ex-ante preference that is *jointly* "consistent" for all the agents.[2] Next, we show that, as for the baseline SEU case, all these restrictions can be fully characterized by properties of the *higher-order interim preferences* of the agents. In turn, this directly allows us to study the implications of ex-ante agreement for strategic reasoning and market behavior, which we address in the second and third leg of the paper, respectively.

Toward this goal, we next embed rational preferences in standard coordination games (e.g., beauty contests and price competition). We derive a complete characterization of equilibrium behavior in the high-coordination limit in terms of the agents' higher-order preferences without any ex-ante agreement restriction. However, when we also impose the forms of ex-ante agreement studied before, we find a striking result: the strong desire for coordination combined with ex-ante agreement considerably tame the nonneutral attitudes toward uncertainty and, in some critical cases, the limit equilibrium behavior is indistinguishable from the ones obtained under SEU. Finally, we provide necessary and sufficient conditions for ex-ante agreement in terms of no trade, highlighting a gap specific to non-SEU preferences.

[1] See, for example, Morris [56] and Bonanno and Nehring [8] for complete discussions on the foundation and the role of the common prior assumption in economics.

[2] The exact definition of "consistent" will specify the degree of the ex-ante agreement among agents.

Common ex-ante preferences and beyond First, we say that there exists a *common ex-ante preference* if the conditional preferences of all the agents are dynamically consistent with respect to the same unconditional preference.[3] In other words, we weaken the assumption of mutual agreement about an objective probabilistic model to that of *mutual dynamic consistency* with respect to a common ex-ante rational preference. The interpretation is that before observing their private information, the agents share the same perceived ambiguity about the probabilistic model and the same attitude toward it. Then, in the interim stage, the agents' preferences may differ, but only insofar the nature of their private information was different. Therefore, mutual dynamic consistency imposes restrictions between periods for each individual and restrictions across all individuals. As we show, this has non-trivial implications for the ambiguity that the agents perceive interim, their ambiguity attitudes, their betting/trading behavior in markets, or in general, their strategic behavior when interacting in games.

We provide a characterization of the existence of a common ex-ante preference that purely concerns the interim preferences of the agents. There is a common ex-ante preference if and only if all the interim higher-order (generalized) expectations of the agents converge to the same limit, which coincides with the common ex-ante preference. This result greatly generalizes the characterization of the common prior assumption in Samet [62], also highlighting that it is the *invariance property* of dynamic consistency that allows us to characterize mutual ex-ante agreement in terms of interim higher-order beliefs. Moreover, this characterization gives us a way to construct the implied common ex-ante preference of the agents.

Next, we altogether remove any pure dynamic-consistency restriction on the interim preferences. In this case, we can still define two extreme ex-ante preferences that are consistent if we allow for (minimal degrees) of aversion and attraction for later resolution of uncertainty. Similar to before, these ex-ante preferences are characterized via the extreme limits of higher-order expectations of the agents. This generalization is essential for applying our results to well-known updating rules that do not always induce dynamically consistent preferences (e.g., full Bayesian updating and proxy updating for maxmin preferences).

Finally, for the case of variational preferences, we consider an intermediate form of mutual consistency where the agents only share some ex-ante benchmark probabilistic models, but their interim preferences are otherwise unrelated. The existence of this mild form of ex-ante agreement naturally induces a common ex-ante preference that can be ranked with the unrestricted extreme ex-ante preferences defined above in terms of comparative uncertainty aversion. This further generalization allows us to consider coordination and market models where the agents share a common perception of the uncertain data-generating process(es) they face but have a heterogeneous level of confidence in it.

[3] Importantly, the information structures of the agents are fixed throughout the entire analysis. Therefore, the assumption of dynamic consistency concerning only a fixed information structure is weak enough to include a much richer set of preferences than SEU.

Coordination and ambiguity We next move to the implications of the assumptions on ex-ante agreement for variational preferences. We first consider an application of our result to network beauty contests in asset markets under incomplete information. Here, we show that, without further restrictions than a full-support assumption on interim preferences and connectedness of the network structure, the price dispersion in the unique equilibrium vanishes as coordination becomes more and more important. Notably, we provide bounds on the equilibrium price dispersion that only depends on the joint connectivity of the network and information structure.

Next, we analyze the unique equilibrium price in the limit for strong coordination motives. In general, this limit is characterized by a worst-case weighted average of the benchmark interim expectations of the agent. With this result, we can already see that a significant part of the ambiguity aversion of the agents disappears in the limit equilibrium. Moreover, we can provide meaningful bounds on the limit evaluation of the asset in terms of the ex-ante preferences studied before, thereby assessing the price effect of interim information.

Our theorem implies that whenever the agents share the same unique ex-ante benchmark probability model, the limit equilibrium price collapses to the expectation of the value of an asset for this unique benchmark. This establishes an important irrelevance result: as coordination motives prevail, the limit price is unaffected by ambiguity aversion, the information structure, or the information structure.[4]

In addition, if a common ex-ante preference exists, then the limit price can lie strictly above the ex-ante preference, pointing out a key difference with the limit result under SEU of Golub and Morris [35]. However, this wedge can only obtain if the agents are ambiguous with respect to each other information structure. Indeed, when agents are unambiguous about the aggregate information, the standard limit equivalence of the SEU case is restored. Notably, in this case, agents might still perceive ambiguity about the fundamental, and their full-coordination limit price decreases in their ambiguity aversion.

The previous results depend only on the best-response structure of the game. In particular, we can derive the same best-response functions from different games with strong coordination motives. An example is a coordination game where agents are firms that compete in producing partially differentiated goods under incomplete information about the demand function.

No-trade implications Finally, we study the relation between the existence of a common ex-ante preference and no-trade implications, which are usually used to characterize the common-prior assumption under SEU (cf. Morris [55], Samet [63], Feinberg [25], Gizatulina and Hellman [33]). There, we start with two interim expectations as primitive objects. We first show that if two agents with those interim expectations are willing to trade the same asset in any state,

[4]The irrelevance of the latter two aspects was already established by [35] for a very similar class of beauty contests.

they cannot be mutually dynamically consistent with respect to the same ex-ante preference. As already established, the exact converse of this statement does not hold in general (cf. Dow and Werlang [19]). However, the existence of a common prior is implied by the following stronger no-trade condition. If there is no endowment economy with two large populations of agents, each characterized by one of the primitive interim expectations, where trade can create a Pareto gain in every state, then these interim expectations are consistent with respect to the same ex-ante preference.

Related literature Our work lies at the intersection of several kinds of literature, including decision theory, game theory, and information economics. Our first theorem generalizes to rational preferences the common-prior characterization of Samet [62]. This has been previously extended to compact spaces of uncertainty in Hellman [42], and to more general payoff-relevant spaces in Golub and Morris [34]. More generally, both Samet's (for SEU) and our characterization (for rational preferences) can be used to study the implication of mutual ex-ante agreement of the agents. The support condition in Lipman [50] and the critical-path theorem in Kajii and Morris [43] are two standard examples of implications of the common prior assumption. Our work is a first step that provides the framework to obtain similar results in the more general case of rational preferences. More recently, the existence of a common ex-ante preference for non-ambiguity-neutral preferences but under both dynamic consistency and consequentialism has been studied by Ellis [20]. This paper shows that if the agents' information has a product structure in addition to the previous properties, then their interim preferences cannot exhibit violations of Savage's sure-thing principle for acts that are measurable with respect to the aggregate information.[5] The following facts nevertheless limit the implications of this critical result for our analysis: i) We also consider and characterize weaker versions of common dynamic consistency which allow for violations of Savage's sure-thing principle ii) We never impose a product structure for the information of the agents which in turn would rule out hard evidence about the interim types of the opponents (e.g., the E-mail game has such hard evidence) iii) For the class of games we consider in Section 4, even the residual ambiguity about the fundamental state is relevant for the equilibrium outcomes iv) As we discuss more in detail in Section 6, many of our results holds even without consequentialism.

Our applications generalize the standard beauty-contest settings in Shin and Williamson [65], Allen et al. [1], or Golub and Morris [35] by allowing for ambiguity aversion and obtaining notable equilibrium implications. More in general, our work proposes a viable theory for games under incomplete information without SEU. For example, Epstein and Wang [23] introduce a universal type space for a class of preferences very similar to the rational one analyzed in the

[5]More in detail, the agents' information has a product structure if each interim type of each player cannot rule out any interim type of the opponents. In [20], this property is implied by Assumption 3 (Full support) which, in general, is not implied by our full-support assumption.

current paper. We improve on this work by characterizing, within this universal type space, the collections of finite type spaces that admit some degree of ex ante mutual agreement. Relatedly, we improve on the analysis of incomplete-information games under uncertainty of Kajii and Ui [44] by considering variational preferences and deriving equilibrium properties for a specific class of coordination games. Moreover, we focus here on simultaneous-move games rather than analyzing the effect of ambiguity aversion in multistage-games such as Battigalli et al. [7], and Hanany et al. [39] which in turn provide a very different set of results.

Our results in the last part of the paper are related and complementary to the extended literature on no-trade results without SEU. On the one hand, Rigotti et al. [60] and Strzalecki and Werner [69] study efficient allocations under ambiguity with public information, as opposed to the private-information setting of the current paper. On the other hand, Kajii and Ui [45] and [46], and Martins-da-Rocha [54] provide no-trade characterizations of the existence of common ex-ante benchmark beliefs without analyzing the case of full mutual dynamic consistency as we do in Section 5.

Finally, our work is related to the extended literature on updating non-SEU preferences under (relaxations of) consequentialism and dynamic consistency as in Ghirardato [27], Pires [59], Epstein and Schneider [22], Ghirardato et al. [29], Maccheroni et al. [53], Gumen and Savochkin [37], Faro and Lefort [24], Bastianello et al. [6]. However, rather than deriving or studying a given updating rule as in the aforementioned works, we take an interim approach and derive the ex-ante preferences that are consistent with the given interim ones. In turn, this allows us to connect our results to existing updating rules by comparing the prescribed ex-ante preferences with the ones we obtain from the interim preferences, and obtain new insights for their implications in strategic interactions.

2 Nonlinear conditional expectations

In this section, we introduce nonlinear conditional expectations. We do so, as the examples at the end of this section will clarify, to move from a situation of risk, where probabilities are either known or trusted by agents, to a situation of uncertainty where agents might entertain several probability models (ambiguity) and/or might not trust them (misspecification aversion).[6]

We start by recalling the usual notion of (linear) conditional expectation. This will set the stage for discussing the generalization we consider in this paper and the formalization of our main theoretical question: when the expectations of different agents can be seen as generated by a common perception of uncertainty, but different sets of private information. As in Samet [62], we consider a finite state space Ω endowed with the power set $\mathcal{P}(\Omega)$.[7] We denote by Δ

[6] Appendix E contains a simple axiomatic preferential foundation for linear and nonlinear conditional expectations.

[7] Despite the finiteness of our setting, we maintain a more general notation. For instance, we keep the symbol

the set of all probabilities over Ω. We let Π denote a partition of Ω, and for every $\omega \in \Omega$, we let $\Pi(\omega)$ denote the unique element of Π that contains ω. Finally, we endow $\mathbb{R}^{\Omega}$ with the supnorm.

2.1 The linear case

Consider a probability $\mu \in \Delta$ and denote by $\mathbb{E}_{\mu} : \mathbb{R}^{\Omega} \to \mathbb{R}$ the functional

$$\mathbb{E}_{\mu}(f) = \int f d\mu \qquad \forall f \in \mathbb{R}^{\Omega}.$$

If Π is a partition of Ω, then a map $p_{\mu} : \Omega \times \mathcal{P}(\Omega) \to [0,1]$ is a regular conditional probability of μ given Π if and only if: (i) For each $\omega \in \Omega$ the function $p_{\mu}(\omega, \cdot) \in \Delta$; (ii) For each $F \in \mathcal{P}(\Omega)$ the function $p_{\mu}(\cdot, F) : \Omega \to [0,1]$ is a version of the conditional probability of F given Π.

Since Ω is finite, any probability μ on Ω admits a regular conditional probability p_{μ}. Moreover, the function $V_{\mu} : \Omega \times \mathbb{R}^{\Omega} \to \mathbb{R}$, defined by

$$V_{\mu}(\omega, f) = \int f dp_{\mu}(\omega, \cdot) \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^{\Omega},$$

is a regular conditional expectation and has the following properties:

a. For each $\omega \in \Omega$ the functional $V_{\mu}(\omega, \cdot) : \mathbb{R}^{\Omega} \to \mathbb{R}$ is normalized, monotone, and linear;[8]

b. For each $f \in \mathbb{R}^{\Omega}$ the function $V_{\mu}(\cdot, f) : \Omega \to \mathbb{R}$ is Π-measurable and satisfies

$$\mathbb{E}_{\mu}(f) = \mathbb{E}_{\mu}(V_{\mu}(\cdot, f)) \text{ and } V_{\mu}\left(\omega, f 1_{\Pi(\omega)} + h 1_{\Pi(\omega)^c}\right) = V_{\mu}(\omega, f) \quad \forall \omega \in \Omega, \forall h \in \mathbb{R}^{\Omega}. \tag{1}$$

In words, (1) contains two properties: the law of iterated expectations and the fact that the support of the update of μ must be contained in the cell of the partition which realized. Clearly, from a preferential viewpoint, the functionals $\mathbb{E}_{\mu}$ and V_{μ} can be axiomatized as the conditional representation of a subjective expected utility (SEU) decision maker who then satisfies dynamic consistency and consequentialism.

2.2 The nonlinear case

Mimicking what we discussed above, we consider two functions $\bar{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$ and $V : \Omega \times \mathbb{R}^{\Omega} \to \mathbb{R}$. In terms of interpretation, the functional $\bar{V}(f)$ is the unconditional expectation of f while $V(\cdot, f)$ describes its conditional expectation.

Definition 1. *Let $\bar{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$. We say that $\bar{V}$ is an* ex-ante (generalized) expectation *if and only if $\bar{V}$ is normalized and monotone.*

of integral in place of the one of sum.

[8] A functional $T : \mathbb{R}^{\Omega} \to \mathbb{R}$ is normalized if and only if $T(k1_{\Omega}) = k$ for all $k \in \mathbb{R}$.

This definition amounts to say that the preference $\succsim$ represented by an ex-ante expectation $\bar{V}$ is *rational* as in [11]. On the one hand, monotonicity is a conceptual (although mild) requirement implying that the agents prefer larger monetary outcomes. On the other hand, normalization requires that the representing $\bar{V}$ is the certainty equivalent for the preference. Moreover, under normalization, the comparative notion of ambiguity aversion of Ghirardato and Marinacci [30] is easily characterized: $\bar{V}$ is more ambiguity averse than $\bar{V}'$ if and only if $\bar{V}(f) \leq \bar{V}'(f)$ for all $f \in \mathbb{R}^{\Omega}$.

Definition 2. *Let Π be a partition of Ω and $V : \Omega \times \mathbb{R}^{\Omega} \to \mathbb{R}$. We say that (V, Π) is an* interim (generalized) expectation *if and only if for each $\omega \in \Omega$ the functional $V(\omega, \cdot) : \mathbb{R}^{\Omega} \to \mathbb{R}$ is normalized, monotone, and continuous and for each $f \in \mathbb{R}^{\Omega}$ the function $V(\cdot, f) : \Omega \to \mathbb{R}$ is Π-measurable and satisfies*

$$V\left(\omega, f 1_{\Pi(\omega)} + h 1_{\Pi(\omega)^c}\right) = V(\omega, f) \quad \forall \omega \in \Omega, \forall f, h \in \mathbb{R}^{\Omega}. \tag{2}$$

A generalized conditional expectation is a pair formed by an *ex-ante* (generalized) expectation (i.e., the functional $\bar{V}$) and an *interim* (generalized) expectation (i.e., the functional V paired with a partition Π) that are dynamically consistent.

Definition 3. *Let $\bar{V}$ be an ex-ante expectation and (V, Π) be an interim expectation. We say that $\left(\bar{V}, V, \Pi\right)$ is a* generalized conditional expectation *if and only if*

$$\bar{V}(f) = \bar{V}(V(\cdot, f)) \qquad \forall f \in \mathbb{R}^{\Omega}. \tag{3}$$

Compared to the standard notion of expectation, we only removed the assumption of linearity from both $\bar{V}$ and V. From a preferential viewpoint, this is tantamount to weaken the assumption of independence, but still retain consequentialism and dynamic consistency (see also Appendix E). Consequentialism takes care of (2), while dynamic consistency is the main axiom behind the law of iterated expectations in (3).

A natural question that emerges in this setting is whether the interim preferences of the agents are consistent with a common ex-ante expectation. More formally, we consider the following definition.

Definition 4. *We say that $\bar{V}$ is a* common ex-ante preference *for $\{(V_i, \Pi_i)\}_{i \in I}$ if and only if $\left(\bar{V}, V, \Pi\right)$ is a generalized conditional expectation for all $i \in I$.*

It is plain that in the case each $V_i(\omega, \cdot)$ is SEU our question amounts to study conditions which yield the existence of a *common prior*. Samet [62] addresses this special version of our question. As [62], we mostly focus on the case of full support which we next discuss.[9] Given a

[9] Theorem 1 does not rely on the full-support assumption per se but rather on a regularity condition of the sequences of higher-order beliefs (cf. Definition 5). Our full-support condition, paired with the absence of non-trivial public information, implies that the regularity condition holds. However, this can be verified directly and independently of the full-support assumption (cf. Example 2).

state $\bar{\omega} \in \Omega$, we say that $\bar{\omega}$ is $\bar{V}$-essential (resp., $V(\omega, \cdot)$-) if and only if there exists an $\varepsilon > 0$ such that for each $f \in \mathbb{R}^\Omega$ and for each $\delta \geq 0$

$$\bar{V}\left(f + \delta 1_{\{\bar{\omega}\}}\right) - \bar{V}(f) \geq \varepsilon\delta \text{ (resp., } V\left(\omega, f + \delta 1_{\{\bar{\omega}\}}\right) - V(\omega, f) \geq \varepsilon\delta). \tag{4}$$

In the linear case, we clearly have that $\bar{\omega}$ belongs to the support of μ (resp., $p_\mu(\omega, \cdot)$) if and only if $\bar{\omega}$ is $\bar{V}$-essential (resp., $V(\omega, \cdot)$-essential).[10] For the general case, we use this characterization to define the notion of support. In particular, we say that $\bar{V}$ (resp., $V(\omega, \cdot)$) has *full support* if and only if each $\bar{\omega} \in \Omega$ (resp., each $\bar{\omega} \in \Pi(\omega)$) is $\bar{V}$-essential (resp., $V(\omega, \cdot)$-essential). Moreover, we say that an interim expectation (V, Π) has full support if and only if $V(\omega, \cdot)$ has full support for all $\omega \in \Omega$.

Given a collection of partitions $\{\Pi_i\}_{i \in I}$ for the agents, that is, an *information structure*, we denote by $\Pi_{\sup}$ and $\Pi_{\inf}$ respectively the *meet* and the *join* of the partitions.[11] They respectively correspond to the public information among agents and the aggregate information collectively held by the agents. We conclude with few examples of generalized conditional expectations where we also illustrate the scope of our question.

Example 1 (Maxmin expectations)**.** Our first example considers maxmin expected utility functionals (see Gilboa and Schmeidler [32]) with full Bayesian updating. Consider a compact and convex set C of probabilities over Ω and a partition Π and set

$$\bar{V}_C(f) = \min_{\mu \in C} \int f d\mu \qquad \forall f \in \mathbb{R}^\Omega \tag{5}$$

and

$$V_C(\omega, f) = \min_{p \in C_\omega} \int f dp \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^\Omega, \tag{6}$$

where

$$C_\omega = \{p_\mu(\omega, \cdot) : \mu \in C\} \quad \forall \omega \in \Omega \tag{7}$$

and p_μ is the regular conditional probability of μ given Π. Note that in this case a state $\bar{\omega} \in \Omega$ is $\bar{V}$-essential if and only if $\mu(\bar{\omega}) > 0$ for all $\mu \in C$. A similar reasoning holds for $V_C(\omega, \cdot)$ and C_ω. It is well known that if C is rectangular and $\bar{V}$ has full support (see Epstein and Schneider [22]),[12] then $\left(\bar{V}_C, V_C, \Pi\right)$ is a generalized conditional expectation where both $\bar{V}$ and

[10] As usual, the support of a probability $p : \mathcal{P}(\Omega) \to [0, 1]$ is the set

$$\operatorname{supp} p = \{\omega \in \Omega : p(\{\omega\}) > 0\}.$$

[11] That is, $\Pi_{\sup}$ is the finest among all partitions that are coarser than each Π_i, and $\Pi_{\inf}$ is the coarsest among all partitions that are finer than each Π_i.

[12] C is rectangular if and only if

$$C = \left\{\sum_{l=1}^{L} p_{\mu_l}(E_l, \cdot)\, \mu(E_l) : \mu, \mu_1, ..., \mu_L \in C\right\}$$

(V_C, Π) have full support. Clearly, linear expectations are obtained when C is a singleton and rectangularity in that case is trivially satisfied. Next, consider a rectangular full-support set C as before and assume that each agent has an information partition Π_i which is coarser than Π and her conditional interim expectations (V_i, Π_i) depend on the collection of sets of probabilities $(C_{\omega,i})_{\omega\in\Omega, i\in I}$ that are computed according to (6) and (7). In this case, $\bar{V}$ is a common ex-ante preference for $\{(V_i, \Pi_i)\}_{i\in I}$, where $\bar{V}$ is defined as in (5). ▲

Example 2 (Multiplier expectations)**.** Our second example considers multiplier preferences functionals (see Hansen and Sargent [40]). Consider a probability with full support μ over Ω and a partition Π and set

$$\bar{V}_{\lambda,\mu}(f) = \min_{\nu\in\Delta}\left\{\int f d\nu + \lambda R(\nu||\mu)\right\} \qquad \forall f \in \mathbb{R}^{\Omega} \tag{8}$$

and

$$V_{\lambda,\mu}(\omega, f) = \min_{p\in\Delta: p(\Pi(\omega))=1}\left\{\int f dp + \lambda R(p||p_\mu(\omega,\cdot))\right\} \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^{\Omega} \tag{9}$$

where $\lambda > 0$ and $R(\cdot||\cdot)$ is the relative entropy. Compared to the previous example the agent has a probability model of reference μ, but she does not fully trust it. She is willing to consider other models ν, nevertheless the farther they are in terms of relative entropy from μ (resp., its update) the less plausible they are, and the smaller role they play in the minimization (8) (resp., (9)). In this perspective, λ is a parameter that captures the decision maker aversion to the potential misspecification of μ: the lower λ the more the decision maker considers other probability models p. It is well known that $(\bar{V}_{\lambda,\mu}, V_{\lambda,\mu}, \Pi)$ is a generalized conditional expectation (see Maccheroni, Marinacci, and Rustichini [53, Section 5.2]). One can show that linear expectations are obtained as the limit case when $\lambda \uparrow \infty$ (see Maccheroni, Marinacci, and Rustichini [51, Proposition 12]). Next, consider a full support probability μ and assume that each agent has an information partition Π_i and her conditional interim expectations (V_i, Π_i) are computed according to (9). In this case, $\bar{V}$ is a common ex-ante preference for $\{(V_i, \Pi_i)\}_{i\in I}$, where $\bar{V}$ is defined as in (5). In other words, in this case, a positive answer to our question amounts to find the existence of a common prior. ▲

Example 3 (Misspecification and ambiguity)**.** Our third example considers a particular case of variational preferences recently proposed by Cerreia-Vioglio et al. [12]. Consider a set $\Theta \subseteq \Delta(\Omega)$ of probabilities with full support over Ω and a partition Π. In particular, assume that $\mu_{|\Pi} = \mu'_{|\Pi}$ for all $\mu, \mu' \in \Theta$, that is, there is no model uncertainty with respect to the events

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where $\Pi = \{E_1, ..., E_L\}$. In this case, note that

$$p_\mu(\omega,\cdot) = p_\mu(\omega',\cdot) \quad \forall \omega, \omega' \in E_l, \forall l \in \{1, ..., L\}, \forall \mu \in C.$$

With a small abuse of notation, we can thus denote the update on the E_l cell by $p_\mu(E_l,\cdot)$.

that are Π-measurable. Next, set

$$\bar{V}_{\lambda,\Theta}(f) = \min_{\nu \in \Delta} \left\{ \int f d\nu + \lambda \min_{\mu \in \Theta} R(\nu || \mu) \right\} \qquad \forall f \in \mathbb{R}^{\Omega}.$$

Next, assume that each agent has an information partition Π_i and her conditional interim expectation (V_i, Π_i) is

$$V_{\lambda,\Theta}(\omega, f) = \min_{p \in \Delta : p(\Pi_i(\omega))=1} \left\{ \int f dp + \lambda \min_{\mu \in \Theta} R(p || p_{\mu,i}(\omega, \cdot)) \right\} \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^{\Omega}$$

where $\lambda > 0$ and $p_{\mu,i}(\omega, \cdot)$ is the conditional probability of μ given Π_i. For every $i \in I$, if Π_i is coarser than Π, then $(\bar{V}, V_i, \Pi_i)$ is a generalized conditional expectation. The interpretation is that the agents are uncertain about the probabilistic model beyond their aggregate information Π_{inf}. Moreover, the agents are averse to misspecification both about the (unique) model restricted on Π_{inf} as well as the set of models assigning likelihoods to events that are finer than Π_{inf}. ▲

3 Existence and implications of a common ex-ante expectation

3.1 Existence

We consider a finite set of agents $I = \{1, ..., n\}$. We assume that each agent is endowed with an interim expectation (V_i, Π_i). It might be convenient to view V_i as an operator from $\mathbb{R}^{\Omega}$ to $\mathbb{R}^{\Omega}$. In this case, the j-th component of this operator is $V_i(\omega_j, f)$ for all $f \in \mathbb{R}^{\Omega}$. With a small abuse of notation, we will still denote this operator by V_i. This rewriting turns out to be useful in order to formally discuss higher-order expectations. For instance, given two agents $i, j \in I$ and an act $f \in \mathbb{R}^{\Omega}$, the expectation of agent i at state ω about the evaluation of act f by agent j is $V_i(\omega, V_j(f))$. Moreover, if we do not fix a state $\omega \in \Omega$, we obtain the second-order evaluation (of i through j) $V_i \circ V_j : \mathbb{R}^{\Omega} \to \mathbb{R}^{\Omega}$. We next illustrate the relevance of this concept in a stylized asset-pricing model.

Example 4. ***[Forecasting the forecaster]*** Consider a state-contingent asset represented by an act $f \in \mathbb{R}^{\Omega}$ in a discrete-time economy with $t \in \mathbb{N}$ periods. Each index $i \in I$ represents a continuum of agents with interim expectations (V_i, Π_i). Let $(i_1, ..., i_t) \in I^t$, with $t \in \mathbb{N}$, be a finite sequence of agents' classes in I. At period 0, an external agent is endowed with the asset. At period 1 she has to sell the asset to one of the agents in class i_1. The price is determined by Bertrand competition among the potential buyers. At period 2, the agent of class i_1 holding the asset has to sell it to an agent in class i_2 according to the same procedure as above. This scheme proceeds until period t when the agent of class i_t holding the asset is paid its realized value.

We can easily solve for the unique equilibrium by backward induction. At period t, the willingness to pay for the asset of agent in class i_t, and therefore the (state-contingent) equilibrium price, is exactly $V_{i_t}(f)$. From the point of view of an agent in class i_{t-1}, the (state-contingent) value of the asset is then $V_{i_{t-1}} \circ V_{i_t}(f)$. Iterating this backward reasoning up to period 1, the initial (state-contingent) price of the asset is

$$V_{i_1} \circ V_{i_2} \circ \ldots \circ V_{i_{t-1}} \circ V_{i_t}(f) \in \mathbb{R}^{\Omega}.$$

Observe that the initial price is a random variable that is measurable with respect to the information of agent i_1.

This highlights the importance of the higher-order expectations in economic interactions. Of course, the model considered is stylized and simple. Most notably, toward pointing out the direct role of higher-order expectations, we assumed that agents know the class of the potential buyers (and hence their interim expectations). In Section 4, we characterize the equilibrium of the related beauty-contest game where the relevant class of buyers is uncertain. ▲

Following [62], we call a sequence $(i_t)_{t\in\mathbb{N}}$ in I an *I-sequence* if and only if for each individual $i \in I$, $i = i_t$ for infinitely many t indexes.

Definition 5. *We say that a collection* $\{(V_i, \Pi_i)\}_{i\in I}$ *of interim expectations exhibits* convergence to a deterministic limit *if and only if for all I-sequences $\iota = (i_t)_{t\in\mathbb{N}}$ and for all $f \in \mathbb{R}^{\Omega}$, there exists $k_{f,\iota} \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} V_{i_t} \circ V_{i_{t-1}} \circ \ldots \circ V_{i_2} \circ V_{i_1}(f) = k_{f,\iota} 1_{\Omega}.$$

In this case, for each I-sequence $\iota = (i_t)_{t\in\mathbb{N}} \in I^{\mathbb{N}}$ define $\bar{V}_\iota : \mathbb{R}^{\Omega} \to \mathbb{R}$ by $\bar{V}_\iota(f) = k_{f,\iota}$.

If there is convergence to a deterministic limit, then the sequences of higher-order expectations of the agents, capturing interactive higher-order reasoning, converge to a limit whose value is necessarily common knowledge.

Our first result shows that there is convergence to a deterministic limit, provided that all the interim expectations of the agents have full support and there is no non-trivial public event.[13] Moreover, the rate of convergence is *quasi-exponential*, that is, it is exponential in the number of times that all the agents have been repeated in the sequence.

[13] Note that the interim expectations in Examples 2 and 3 do not satisfy the full support assumption. However, in both cases, the interim expectation of each i can be written as

$$V_i(\omega, f) = \phi_\lambda^{-1}\left(\tilde{V}_i(\omega, \phi_\lambda(f))\right)$$

where $\phi_\lambda(z) = -\exp\left(-\frac{z}{\lambda}\right)$ and $\tilde{V}_i$ is an interim expectation with full support. Since, by taking iterated expectations, ϕ_λ and ϕ_λ^{-1} cancel out, the convergence to a deterministic limit of Proposition 1 still holds.

With non-trivial public information the results of this section apply in each cell of $\Pi_{\sup}$.

Proposition 1. *If $\{(V_i, \Pi_i)\}_{i \in I}$ is a collection of full support interim expectations such that $\Pi_{\text{sup}} = \{\Omega\}$, then $\{(V_i, \Pi_i)\}_{i \in I}$ exhibits convergence to a deterministic limit. Moreover, there exist $\varepsilon \in (0,1)$ and $C \in \mathbb{R}_+$ such that for each I-sequence $(i_m)_{m \in \mathbb{N}}$ and for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then*

$$\left|\left|\bar{V}_\iota (f) 1_\Omega - V_{i_t} \circ ... \circ V_{i_1} (f)\right|\right| \leq C\varepsilon^\tau \left|\left|f\right|\right|_\infty \qquad \forall f \in \mathbb{R}^\Omega.$$

Quasi-exponential convergence provides a bound on the approximation error for computing the limit higher-order expectation of f given ι using the t-th order expectation. In particular, the bound improves in t only if additional expectations of *all* the agents are involved.

We next illustrate the meaning of quasi-exponential convergence to a deterministic limit in the asset-pricing example.

(Forecasting the forecaster continued). Assume that the collection $\{(V_i, \Pi_i)\}_{i \in I}$ of interim expectations has full support and that $\Pi_{\text{sup}} = \{\Omega\}$. Then, rather than looking at a fixed-length sequence, we consider an infinite sequence of classes $(i_t)_{t \in \mathbb{N}}$. We can focus on I-sequences as, if the identity of classes are iid draws with full support on I, then with probability 1 an I-sequence is realized. With this, Proposition 1 guarantees that, for truncation $(i_1, ..., i_{\bar{t}})$ of $(i_t)_{t \in \mathbb{N}}$ such that each agent appears sufficiently many times, the dependence of the initial equilibrium price on the realized state of the world is arbitrarily (and exponentially) small. Intuitively, the willingness to pay of an agent in class i_1 does not significantly depend on the state as she knows that the selling value depends on a large number of subsequent transactions. More specifically, this and the assumption $\Pi_{\text{sup}} = \{\Omega\}$ imply that many of the subsequent buyers will care about the value of the asset also in states that are ruled out by the information of i_1. ▲

We are now ready for the main result of the paper. If there is convergence to a deterministic limit, then there exists a common ex-ante expectation if and only if the deterministic limit of all the I-sequences of higher-order expectations is the same.

Theorem 1. *Let $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of interim expectations that exhibits convergence to a deterministic limit. The following statements are equivalent:*

(i) There exists a common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$;

(ii) For each $f \in \mathbb{R}^\Omega$ there exists $k_f \in \mathbb{R}$ such that for each I-sequence $(i_t)_{t \in \mathbb{N}}$

$$\lim_{t \to \infty} V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1} (f) = k_f 1_\Omega.$$

In this case, for each $f \in \mathbb{R}^\Omega$, we have $\bar{V} (f) = k_f$.

Observe that as an immediate corollary of Proposition 1 and Theorem 1, we get that our characterization of common ex-ante preference holds provided that agents' interim preferences have full support and there is no public information. Next example first illustrates the (asset-pricing) equilibrium implications of the existence of a common ex-ante preference.

(Forecasting the forecaster continued). First, assume that the agents have a common ex-ante preference $\bar{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$. For a sufficiently long truncation $(i_1, ..., i_{\bar{t}})$ of $(i_t)_{t\in\mathbb{N}}$, the initial equilibrium price is approximately state-independent and equal to the common ex-ante evaluation $\bar{V}(f)$ of the asset. In words, under a common ex-ante preference, the particular order of trades does not affect the initial price. Conversely, for any two arbitrary I-sequences truncated at $\bar{t} \in \mathbb{N}$, we can falsify the existence of a common ex-ante preference by checking whether the corresponding equilibrium prices are sufficiently different.[14] ▲

3.2 Beyond dynamic consistency

In this section, we relax dynamic consistency between the ex-ante and the interim expectations. First, we observe that even if we restrict attention only to the subset of acts that are Π_{inf}-measurable, then the equivalence of Theorem 1 continues to hold. This class of acts is particularly relevant in strategic interactions where the payoff functions of the agents depend on their opponents' actions and on a payoff-relevant parameter. In this case, this weaker notion requires dynamic consistency with respect to acts that depend only on the agents' actions, since those are measurable with respect to the aggregate information Π_{inf}, while it does not impose it for acts that also depend on the payoff-relevant parameter.

Remark 1. *Consider the following weaker notion of common ex-ante expectation. As before, let $\{(V_i, \Pi_i)\}_{i\in I}$ be a profile of interim expectations. Fix any partition Π' that is finer that Π_{inf}. We say that the agents have a Π'-common ex-ante preference if there exists an ex-ante expectation $\bar{V}$ that satisfies*

$$\bar{V}(f) = \bar{V}(V_i(\cdot, f))$$

for all $i \in I$ and for all $f \in \mathbb{R}^{\Omega}$ that are Π'-measurable. By inspection of the proof of Theorem 1, it is easy to see that, if $\{(V_i, \Pi_i)\}_{i\in I}$ exhibits convergence to a deterministic limit, then the existence of this weaker form of common ex-ante expectation is equivalent to the following:

(ii') For each Π'-measurable $f \in \mathbb{R}^{\Omega}$ there exists $k_f \in \mathbb{R}$ such that for each I-sequence $(i_t)_{t\in\mathbb{N}}$

$$\lim_{t\to\infty} V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f) = k_f 1_{\Omega}.$$

Moreover, as in Theorem 1, this common limit coincides with the common ex-ante evaluation for every Π'-measurable act f, that is, $\bar{V}(f) = k_f$.

[14] Formally, consider two I-sequences with truncations $(i_1, ..., i_{\bar{t}})$ and $(\tilde{\imath}_1, ..., \tilde{\imath}_{\bar{t}})$ in which each $i \in I$ appears at least τ times. By inspection of the proof of Proposition 1, we have explicit expressions for the constant $C \in \mathbb{R}_+$ and $\varepsilon \in (0,1)$ of Definition 3. With this, we can say that a common rational preference does not exist if

$$||V_{\tilde{\imath}_{\bar{t}}} \circ ... \circ V_{\tilde{\imath}_1}(f) - V_{i_{\bar{t}}} \circ ... \circ V_{i_1}(f)|| > 2C\varepsilon^{\tau}.$$

Next, define the ex-ante preference $\bar{V}_i = \bar{V} \circ V_i$ for every $i \in I$.[15] *One can show that, for every $i \in I$, the functional $\bar{V}_i$ is the unique ex-ante preference that coincides with $\bar{V}$ on the Π'-measurable acts and is individually dynamically consistent in the sense that*

$$\bar{V}_i(g) = \bar{V}_i(V_i(\cdot, g)) \qquad \forall g \in \mathbb{R}^\Omega.$$

Moreover, for each I-sequence $(i_t)_{t\in\mathbb{N}}$,

$$\lim_{t\to\infty} V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g) = \bar{V}_{i_1}(g) 1_\Omega \qquad \forall g \in \mathbb{R}^\Omega,$$

that is, the ex-ante expectation of i_1 corresponds to the limit for the higher order expectations of every I-sequence where the first-order expectation is the one of i_1. ▲

3.2.1 Common preferences on resolution of uncertainty

We next consider a more conceptual relaxation of dynamic consistency. Indeed, it is well known that full-fledged dynamic consistency is restrictive outside the realm of subjective expected utility, especially with uncertainty averse preferences (see for example [2], [7], [21], [27], and [66]). Therefore, we consider the existence of a common ex-ante preference that is consistent with the interim expectations of all the agents yet possibly exhibiting a strict preference for earlier or later resolution of uncertainty.

Definition 6. *We say that an ex-ante expectation $V_\circ$ is a* ***lower common ex-ante expectation*** *for $\{(V_i, \Pi_i)\}_{i\in I}$ if and only if*

$$V_\circ(f) \le V_\circ(V_i(f)) \qquad \forall f \in \mathbb{R}^\Omega, \forall i \in I. \tag{10}$$

We say that an ex-ante expectation V° is a ***upper common ex-ante expectation*** *for $\{(V_i, \Pi_i)\}_{i\in I}$ if and only if*

$$V^\circ(f) \ge V^\circ(V_i(f)) \qquad \forall f \in \mathbb{R}^\Omega, \forall i \in I. \tag{11}$$

Let $\mathcal{V}_\circ$ and $\mathcal{V}^\circ$ denote respectively the collections of lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$.

Both relaxations have meaningful interpretations. Whenever the agents share a lower common ex-ante expectation, their interim preferences can be rationalized by the same ex-ante expectation provided that they exhibit preferences for earlier resolution of uncertainty (cf. Dillenberger [17] and Strzalecki [68]). Condition (10) is also equivalent to require that each agent $i \in I$ attaches a positive value to her information Π_i.[16] Moreover, such condition is satisfied by existing updating rules for preferences under uncertainty as we next show.

[15] Observe that each $\bar{V}_i$ is well defined since, for every $g \in \mathbb{R}^\Omega$, $V_i(g)$ is Π'-measurable, hence we can evaluate through $\bar{V}$.

[16] Formally, condition (10) is equivalent to assume that, for each finite set of acts $A \subseteq \mathbb{R}^\Omega$ and for each $i \in I$,

$$V_\circ\left(\max_{f\in A} V_i(\cdot, f)\right) \ge \max_{f\in A} V_\circ(f).$$

Example 5 (Choquet expected utility with proxy updating)**.** We analyze the class of preferences and updating rule recently proposed by Gul and Pesendorfer [36]. Formally, they consider a totally monotone capacity $\nu : 2^{\Omega} \to [0,1]$ and a collection of partitions $\{\Pi_i\}_{i\in I}$.[17] In the ex-ante stage, all the agents evaluate every act $f \in \mathbb{R}^{\Omega}$ with the Choquet integral of f with respect to ν, denoted as $V_{\circ}(f)$. Recall that the core of ν is defined as

$$core(\nu) = \left\{\mu \in \Delta(\Omega) : \forall E \in 2^{\Omega}, \mu(E) \geq \nu(E)\right\}$$

and that, in this case, $V_{\circ}(f) = \min_{\mu \in core(\nu)} \mathbb{E}_p[f]$. We let $\mu_\nu \in \Delta(\Omega)$ denote the Shapley value corresponding to ν. With this, the interim preferences of agent i at state ω are:

$$V_i(\omega, f) = \min_{\mu \in core_i(\nu)} \mathbb{E}_{p_{\mu,i}(\omega,\cdot)}[f] \qquad \forall f \in \mathbb{R}^{\Omega}$$

where $p_{\mu,i}(\omega,\cdot)$ is the conditional probability of μ given Π_i and

$$core_i(\nu) = \{\mu \in core(\nu) : \forall E \in \Pi_i, \mu(E) = \mu_\nu(E)\}.$$

In words, each agent updates her preferences with full Bayesian updating but starting from the restricted set $core_i(\nu)$. In this case, the results in [36, Axiom C.4 and Theorem 1] imply that $V_{\circ}$ is a lower common ex-ante expectation for $\{(V_i, \Pi_i)\}_{i\in I}$ but not a common ex-ante expectation in general. ▲

Instead, an upper common ex-ante expectation rationalizes the interim expectations of the agents provided that they exhibit preferences for later resolution of uncertainty. Notably, if the interim preferences $\{(V_i, \Pi_i)\}_{i\in I}$ are maxmin obtained by full Bayesian updating starting from the same maxmin ex-ante preference V°, then V° is a upper common ex-ante expectation for $\{(V_i, \Pi_i)\}_{i\in I}$.[18] This observation also holds for the class of divergence preferences introduced in [51] that generalizes Example 2 by allowing for other statistical distances beyond the relative entropy.

We next show that both $\mathcal{V}_{\circ}$ and $\mathcal{V}^{\circ}$ are nonempty and always admit respectively a maximal and a minimal element that we denote:

$$V_*(f) = \sup_{V_{\circ} \in \mathcal{V}_{\circ}} V_{\circ}(f) \quad \text{and} \quad V^*(f) = \inf_{V^{\circ} \in \mathcal{V}^{\circ}} V^{\circ}(f) \qquad \forall f \in \mathbb{R}^{\Omega}.$$

Lemma 1. *The sets $\mathcal{V}_{\circ}$ and $\mathcal{V}^{\circ}$ are nonempty and both V_* and V^* are well defined lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$.*

[17] A capacity ν is totally monotone if and only, for all $k \geq 2$ and all $E_1, ..., E_k \in 2^{\Omega}$,

$$\nu\left(\cup_{i=1}^{n} E_i\right) \geq \sum_{\{J:\emptyset \neq J \subseteq \{1,...,k\}\}} (-1)^{|J|+1} \nu\left(\cap_{j\in J} A_j\right).$$

[18] Recall from Example 1 that condition (11) is satisfied with equality if and only if each triple (V°, V_i, Π_i) satisfies rectangularity.

By construction, the lower (resp. upper) common ex-ante expectation V_* (resp. V^*) has the minimal attraction (resp. aversion) for earlier resolution of uncertainty among the elements in $\mathcal{V}_\circ$ (resp. $\mathcal{V}^\circ$). In Online Appendix G, we provide an algorithm to compute V_* and V^* starting from the interim preferences of the agents, which are, in principle, observable.

We now provide a characterization of the extreme common ex-ante expectations in terms of the higher-order expectations of the agents. Notably, such characterization holds regardless of the existence of a common ex-ante preference.

Proposition 2. *If $\{(V_i, \Pi_i)\}_{i\in I}$ is a collection of full support interim expectations such that $\Pi_{\sup} = \{\Omega\}$, then, for every $f \in \mathbb{R}^\Omega$,*

$$V_*(f) = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f) \quad \text{and} \quad V^*(f) = \sup_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f).$$

The interpretation is that by looking at the lowest (resp. highest) limit of the iterated expectations, we exactly identify the minimal attraction (resp. aversion) to earlier resolution of uncertainty needed to jointly rationalize the interim preferences of the agents. Moreover, observe that the previous proposition implies that $V_*(f) \leq V^*(f)$ for all $f \in \mathbb{R}^\Omega$, that is, the ex-ante preferences V_* and V^* are ranked in terms of their uncertainty aversion.

(Forecasting the forecaster continued). Consider our running example under all the previous assumptions with the exception of the existence of a common ex-ante preference. In particular, fix an I-sequence $\iota = (i_n)_{n\in\mathbb{N}}$ and recall that the equilibrium initial price of asset f, for the game with length t, is equal to the random variable

$$V_{i_t} \circ V_{i_{t-1}} \circ \ldots \circ V_{i_2} \circ V_{i_1}(f).$$

In this case, by Proposition 1, as we let t go to infinity, the limit price is deterministic and equal to $\bar{V}_\iota(f)$. Moreover, by Lemma 1 and Proposition 2, the limit initial price satisfies

$$V_\circ(f) \leq \bar{V}_\iota(f) \leq V^\circ(f) \tag{12}$$

for all upper and lower common ex-ante expectations $V_\circ \in V_\circ$ and $V^\circ \in V^\circ$, and, more accurately,

$$\bar{V}_\iota(f) \in [V_*(f), V^*(f)].$$

Equation (12) has direct implications for the equilibrium price with preferences that do not satisfy dynamic consistency. For example, if the traders are maxmin agents and share the same set of ex-ante probabilistic models $C \subseteq \Delta(\Omega)$, then, under full Bayesian updating, the limit initial price with private information $\bar{V}_\iota(f)$ is smaller than the common ex-ante evaluation $V^\circ(f) = \min_{p\in C}\int f dp$. Indeed, the initial equilibrium price is the result of a compounded pessimistic evaluation due to full Bayesian updating and iterated minimization across all the updated probabilistic models. ▲

Proposition 2 has also important implications for the characterization of the existence of a common ex-ante preference, even in the SEU case.

Corollary 1. *Let $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of full support interim expectations such that $\Pi_{\sup} = \{\Omega\}$. The following statements are equivalent:*

(i) There exists a common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$;

(ii) For each $f \in \mathbb{R}^\Omega$, we have $V_*(f) = V^*(f)$.

Moreover, $\bar{V}(f) = V_*(f) = V^*(f) = \bar{V}_\iota(f)$ for all $f \in \mathbb{R}^\Omega$ and all I-sequences $\iota \in I^{\mathbb{N}}$.

The previous corollary provides an alternative characterization of the existence of a common ex-ante preference in terms of the weakenings of the common ex-ante expectations that we have proposed.

Corollary 2. *Let $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of full support interim expectations such that $\Pi_{\sup} = \{\Omega\}$ and such that V_i is SEU for all $i \in I$. The following statements are equivalent:*

(i) There exists a common prior $p \in \Delta(\Omega)$ for $\{(V_i, \Pi_i)\}_{i \in I}$;

(ii) Both V_* and V^* are SEU.

Moreover, $\mathbb{E}_p(f) = V_*(f) = V^*(f)$ for all $f \in \mathbb{R}^\Omega$.

This second corollary provides a new characterization of the common prior assumption in the setting of Samet [62]. In particular, there exists a common prior if and only if both the extreme ex-ante preferences of the agents are neutral with respect to the timing of resolution of uncertainty.

We close this section with a result bounding the difference between the iterated expectations along two different I-sequences without assuming the existence of a common ex-ante preference. This bound is the sum of the wedge between the two extreme ex-ante evaluations V_* and V^* and a quasi-exponentially vanishing term due to Proposition 1.

Corollary 3. *Let $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of full support interim expectations such that $\Pi_{\sup} = \{\Omega\}$. There exist $\varepsilon \in (0,1)$ and $C \in \mathbb{R}_+$ such that for every pair of I-sequence $\iota = (i_m)_{m \in \mathbb{N}}$ and $\iota' = (i'_m)_{m \in \mathbb{N}}$, and for each $\tau, t \in \mathbb{N}$, if every $i \in I$ appears at least τ times in both $(i_1, ..., i_t)$ and $(i'_1, ..., i'_t)$, then*

$$\left\| V_\iota^t(f) - V_{\iota'}^t(f) \right\|_\infty \leq \| V_*(f) - V^*(f) \|_\infty + C\varepsilon^\tau \|f\|_\infty \qquad \forall f \in \mathbb{R}^\Omega.$$

Observe that, in the two-agent case, when there exists a common ex-ante preference, the previous result gives a bound on the higher-order disagreement between agents, by getting rid of the first term on the right-hand side.

3.2.2 Dynamic consistency of local subjective beliefs

In this section, we consider a minimal notion of mutual dynamic consistency that only involves the most trusted probabilistic models. We show below that it strictly linked to the concept of *subjective beliefs at an act* introduced by Rigotti et al. [60, Definition 1 and Proposition 3] to study Pareto optimal allocations under ambiguity. To formalize this concept we need to restrict ourselves to the class of variational preferences (cf. [51]).

Definition 7. *A collection of interim expectations $\{(V_i, \Pi_i)\}_{i\in I}$ is variational if and only if and for every $i \in I$ and $\omega \in \Omega$, there exists a lower semicontinuous, grounded, and convex cost function $c_{i,\omega} : \Delta(\Omega) \to [0, \infty]$ such that*

$$V_i(\omega, f) = \min_{p\in\Delta(\Omega)} \{\mathbb{E}_p[f] + c_{i,\omega}(p)\} \tag{13}$$

for all $f \in \mathbb{R}^{\Omega}$.[19]

Variational interim expectations exhibit violations of subjective expected utility due to aversion to ambiguity, a widely documented trait. The interpretation is that each agent considers the evaluation of the act under many probabilistic models and $c_{i,\omega}$ penalizes more the models (subjectively) deemed less plausible. In particular, the probabilistic models p for which $c_{i,\omega}(p) = 0$ represent the ones that i trusts the most in state ω. All the examples of preferences we have introduced are variational.[20] For instance, in the case of maxmin preferences, $c_{i,\omega}$ is the support function of the set of probabilistic models $C_{i,\omega}$.

Define the following set which captures a minimal extent of mutual dynamic consistency among the agents:

$$\Theta = \bigcap_{i\in I} co\{p \in \Delta(\Omega) : \exists\omega \in \Omega, c_{i,\omega}(p) = 0\}.$$

In words, Θ contains all the ex-ante probabilistic models that, when updated, are among the most trusted by every agent in every state, that is, those that minimize the interim cost function. Following Ghirardato and Marinacci [30], we call these probability measures as *benchmark models*.[21] Incidentally, Θ also coincides with the set of ex-ante probabilistic models that are consistent with a selection from the subjective beliefs at any constant act (cf. [60, Definition 1 and Proposition 3], [46], [54]) of the interim preferences of the agents.

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[19] A cost function c is grounded if and only $\min_{p\in\Delta(\Omega)} c(p) = 0$.

[20] Imposing the representation in equation (13) is equivalent to assume that each $V_i(\omega, \cdot)$ is concave and translation invariant, that is,

$$V_i(\omega, f + ke) = V_i(\omega, f) + k$$

for all $f \in \mathbb{R}^{\Omega}$ and $k \in \mathbb{R}$. From a preferential viewpoint, these functional properties are equivalent to two axioms: uncertainty aversion and weak c-independence (cf. [51]).

[21] These probability measures correspond to SEU preferences that are less ambiguity averse than the interim preference of the agent as formally showed in [51].

Definition 8. *We say that a variational collection of interim expectations* $\{(V_i, \Pi_i)\}_{i\in I}$ *has a common local subjective belief if and only if* $\Theta \neq \emptyset$. *In this case, we define* $V^{\Theta} : \mathbb{R}^{\Omega} \to \mathbb{R}$ *as*

$$V^{\Theta}(f) = \min_{\mu \in \Theta} \mathbb{E}_{\mu}[f].$$

In words, V^{Θ} is a caution evaluation of acts that only relies on the benchmark ex-ante probabilistic models. In the next result, we relate these intermediate notions of common preferences with the ones we have already studied.

Proposition 3. *Let* $\{(V_i, \Pi_i)\}_{i\in I}$ *be a variational collection of interim expectations. The following facts are true:*

1. *If* $\{(V_i, \Pi_i)\}_{i\in I}$ *has a common local subjective belief, then* V^{Θ} *is a upper common ex-ante expectation for* $\{(V_i, \Pi_i)\}_{i\in I}$, *hence* $V^{\Theta} \geq V^*$.

2. *If there exists a common ex-ante preference* $\bar{V}$ *for* $\{(V_i, \Pi_i)\}_{i\in I}$, *then* $\{(V_i, \Pi_i)\}_{i\in I}$ *has a common local subjective belief, hence* $V^{\Theta} \geq \bar{V}$.

Not surprisingly, the new notion of ex-ante expectation introduced V^{Θ} is less ambiguity averse than the previous ones. The reason is that each $\mu \in \Theta$ is obtained by mixing the interim beliefs of the agents that correspond to SEU preferences that are less ambiguity averse.

4 Equilibrium and (un-)common ex-ante preferences

In this section, we consider the equilibrium implications of our previous analysis. To do so, we maintain the assumption that the interim preferences have full support and belong to the class of variational preferences (cf. Definition 7):

Assumption 1 The collection of interim expectations $\{(V_i, \Pi_i)\}_{i\in I}$ has full support, is such that $\Pi_{\text{sup}} = \{\Omega\}$, and is variational.

Under this assumption, Proposition 1 guarantees that $\{(V_i, \Pi_i)\}_{i\in I}$ exhibits quasi-exponential convergence to a deterministic limit.

In each of the following applications, the equilibrium outcomes $\sigma^{\beta} = \left(\sigma_i^{\beta}\right)_{i\in I} \in \left(\mathbb{R}^{\Omega}\right)^n$ of the agents will always be described by the following fixed-point condition:

$$\sigma_i^{\beta}(\omega) = V_i\left(\omega, (1-\beta)\hat{f} + \beta \sum_{j\in I} w_{ij}\sigma_j^{\beta}\right) \qquad \forall \omega \in \Omega, \forall i \in I. \tag{14}$$

Here, $\hat{f} \in \mathbb{R}^{\Omega}$ is a payoff-relevant fundamental, $\beta \in (0,1)$ parametrizes the relative importance of *coordination* with other agents over *adaptation* to the fundamental, and $W = \{w_{ij}\}_{i,j\in I} \in \mathbb{R}^{n\times n}$ is a stochastic matrix where each w_{ij} captures the relative importance of agent j for i.[22]

[22] A matrix $W = (w_{ij})_{i,j\in I} \in \mathbb{R}^{I\times I}$ is stochastic if and only if $w_{ij} \geq 0$ for all $i, j \in I$ and $\sum_{j\in I} w_{ij} = 1$ for all $i \in I$.

The interpretation is that the equilibrium outcome for agent i coincides with her (generalized) expectation of a combination of the fundamental and the equilibrium outcomes of the other players. These kind of fixed-point conditions are ubiquitous in models of asset pricing with beauty-contests (cf. Morris and Shin [57]), networks of financial institutions (cf. Jackson and Pernoud [49]), and price competition (cf. Angeletos and Pavan [3]) as we show below. In particular, in all these cases, the high-coordination limit ($\beta \to 1$) of the equilibrium outcomes is used to select an equilibrium of the pure-coordination games (cf. Shin and Williamson [65] and Golub and Morris [35]). Therefore, this will be the main focus of our analysis.

4.1 Beauty contests: coordination and equilibrium

As a leading application, we consider a beauty-contest model with random matching and private information (as in [35]) that generalizes the leading example of Section 3.

Each $i \in I$ represents a continuum of agents sharing the same information partition Π_i. Time is discrete $t \in \{1, ..., T, ...\}$ and there is a random variable $\hat{f} \in \mathbb{R}^\Omega$ denoting the only asset in this economy which is sequentially traded with random matching. Let $\beta \in (0, 1)$. At every period t, if an agent in class i holds the asset, with probability $(1 - \beta)$ she has to liquidate the asset and obtain its fundamental (uncertain) value $\hat{f}$. With complementary probability β, she privately has to sell the asset to an agent from a randomly selected class and then leaves the game. The matching probabilities, conditional on not liquidating the asset, are described by a stochastic and strongly connected matrix W, where w_{ij} is the probability with which an agent in class i is matched to class j. In particular, the random matching is independent of the state $\omega \in \Omega$ and plays the role of objective lotteries a la Anscombe and Aumann in our setting.[23]

After the realization of the matched class j, the agents in class j compete a la Bertrand offering a price to asset holder in i who decides to whom to sell the asset. This mechanism implies that in equilibrium the offered price is equal to the (common) willingness to pay for the asset of the agents in class j. If an agent in class j acquires the asset, then the game continues to period $t + 1$. Observe that there is no relevant learning over time since the past owners of the asset have left the game. Moreover, conditional on non liquidation, even if the asset holder would learn something about the state $\omega \in \Omega$ from the offers of the agents in j, accepting the highest offer is still a dominant strategy given the absence of outside options.

A *strategy* for an agent in class $i \in I$ is a random variable $\sigma_i \in \mathbb{R}^\Omega$ that is measurable with respect to the information structure Π_i.[24] In particular, from the point of view of agents in i, the strategies $\sigma_j \in \mathbb{R}^\Omega$ of agents in any class j are state-dependent offers that can be evaluated through their interim preferences V_i as standard acts. Let Σ_i and Σ denote respectively the set

[23] In other words, the matching probabilities are used to take convex linear combinations of acts of the form $h \in \mathbb{R}^\Omega$.

[24] For this application, we are implicitly restricting our attention to Markov strategies where all the agents condition their actions only on their private information.

of strategies for agents in class i and the set of profiles of strategies for n agents, one for each class. With this, for every $\beta \in (0,1)$, if we fix a profile $\sigma = (\sigma_j)_{j \in I} \in \Sigma$ of strategies for all the agents in all classes j, the corresponding (state-dependent) willingness to pay for asset $\hat{f}$ of any agent in class $i \in I$ is:

$$S_{\beta,i,\omega}(\sigma) = V_i\left(\omega, (1-\beta)\hat{f} + \beta \sum_{j \in I} w_{ij}\sigma_j\right) \qquad \forall \omega \in \Omega.$$

The *equilibria* of this game correspond to the fixed points of the map $S_\beta(\cdot) : \Sigma \to \Sigma$, that is, $\sigma^\beta \in \Sigma$ is a equilibrium if and only if it satisfies equation (14).

Proposition 4. *There exists $C \in \mathbb{R}_+$ such that, for every $\beta \in (0,1)$, the operator $S_\beta : \Sigma \to \Sigma$ is a contraction with respect to the supnorm and it admits a unique equilibrium $\sigma^\beta \in \Sigma$ that satisfies*

$$\max_{i,j \in I, \omega, \omega' \in \Omega} \left|\sigma_i^\beta(\omega) - \sigma_j^\beta(\omega')\right| \leq (1-\beta)\, C \max_{\omega,\omega' \in \Omega} \left|\hat{f}(\omega) - \hat{f}(\omega')\right|. \tag{15}$$

The inequality in equation (15) gives a bound on the maximum level of disagreement among the equilibrium asset evaluations. First, we observe that the right hand side is monotonically decreasing in β and vanishes as we let coordination become more important, that is $\beta \to 1$. This implies that the price of the asset becomes constant across states and agents in the limit. Second, the speed of this convergence is disciplined by C which can be linked back to the preferences, information, and network primitives, as we explain in the next remark.

Remark 2. *In [10], we further elaborate on the estimate on the range of the fixed points of equations like (14) and find an explicit expression for the estimate in Proposition 4 in terms of the properties of S_β. In the current setting, this translates in the following way. Define the adjacency matrix $A \in \{0,1\}^{(I\times\Omega)\times(I\times\Omega)}$ over $(I \times \Omega)$ by letting, for all $i, j \in I$ and $\omega', \omega \in \Omega$, $a_{(i,\omega')(j,\omega)} = 1$ if and only if $w_{ij} > 0$ and $\omega \in \Pi_i(\omega')$. Also, for all $i \in I$, $\omega' \in \Omega$, and $\omega \in \Pi_i(\omega')$, let $\varepsilon_{i,\omega,\omega'} > 0$ denote the ε satisfying the full-support equation (4) for agent i at state ω' with respect the essential state ω. Next, let*

$$\underline{\varepsilon} = \min_{i,j \in I, \omega, \omega' \in \Omega : a_{(i,\omega')(j,\omega)} = 1} \varepsilon_{i,\omega,\omega'} w_{ij}$$

and with this define the bound

$$C = \sum_{\tau=0}^{d-1} \left(1 + \frac{1}{\underline{\varepsilon}}\right)^{2d-\tau}$$

where d is the diameter of the graph corresponding to A. The number of connections in A depends on both the number of connections in the network among agents W as well as on the dependence of their information structures. In turn, increasing the number of connections in A has two contrasting effects: first it reduces the diameter of the graph, making C smaller, second it reduces the maximal possible magnitude of $\underline{\varepsilon}$, making C larger. For example, the diameter *is low when all the agents are connected and the information structure has a product form, i.e., whenever $\Pi_i(\omega) \cap \Pi_j(\omega') \neq \emptyset$ for all $i, j \in I$ and $\omega', \omega \in \Omega$, and is high under a circular information structure, i.e., whenever for every i and $\omega \in \Omega$, $\Pi_i(\omega)$ has nonempty intersection only with two partition cells of the coplayers.* ▲

4.2 Beauty contests: high coordination and misspecification neutrality

In this section, we characterize the unique equilibrium σ^β as coordination becomes more and more important, i.e. $\beta \to 1$. To do so, we need to introduce some ancillary objects. First, for all $i \in I$ and $\omega \in \Omega$, let $\partial V_i(\omega, 0)$ denote the superdifferential of $V_i(\omega, \cdot)$ at 0 which is nonempty since the latter is concave. It is easy to see that our assumption guarantees that

$$\partial V_i(\omega, 0) = \{p \in \Delta(\Omega) : c_{i,\omega}(p) = 0\}, \tag{16}$$

that is, each $\partial V_i(\omega, 0)$ consists of the benchmark probability models by agent i at state ω (cf. [51, Theorem 18]). With this, define the set of interim benchmark beliefs

$$\partial V(0) = \left\{q \in \Delta(\Omega)^{I\times\Omega} : \forall (i,\omega) \in I \times \Omega, q_{i,\omega} \in \partial V_i(\omega, 0)\right\}.$$

With a slight abuse of notation, for every $q \in \partial V(0)$, we let $\mathbb{E}_q\left[\hat{f}\right] \in \mathbb{R}^{I\times\Omega}$ denote the vector $\left(\mathbb{E}_{q_{i,\omega}}\left[\hat{f}\right]\right)_{(i,\omega)\in I\times\Omega}$.

Each $q \in \partial V(0)$ can be combined with the network structure W to obtain an *interaction structure* $W^q \in \mathbb{R}_+^{(I\times\Omega)\times(I\times\Omega)}$ among agent-state pairs capturing both the interim beliefs of the agents as well as the strength of their links. Formally, we let

$$w^q_{(i,\omega)(j,\omega')} = w_{ij} q_{i,\omega}(\omega') \qquad \forall i, j \in I, \forall \omega, \omega' \in \Omega. \tag{17}$$

Under SEU interim preferences, there is a unique interaction structure pinned down by the network W and the posterior beliefs of the agents. In the SEU case, the interaction structure was introduced by Golub and Morris [35] and used to characterize the limit equilibrium of a similar coordination game. In the present setting, model uncertainty translates into a multiplicity of interim relevant beliefs, hence into a multiplicity of interaction structures. However, this multiplicity is disciplined by both the information and the interim preferences of the agents. For example, if $\omega' \notin \Pi_i(\omega)$, then we immediately have that $w^q_{(i,\omega)(j,\omega')} = 0$ for all $q \in \partial V(0)$.

Lemma 2. *For each $q \in \partial V(0)$, there exists a unique probability vector $\gamma^q \in \Delta(I \times \Omega)$ such that $\gamma^q = \gamma^q W^q$.*

This is a consequence of the connectedness properties of each W^q implied by $\Pi_{\inf} = \{\Omega\}$, full support of $\{V_i, \Pi_i\}_{i\in I}$, and that W is strongly connected. We are now ready to state the main result of this section.

Theorem 2. *For all* $i \in I$ *and* $\omega \in \Omega$,

$$V_*\left(\hat{f}\right) \le \lim_{\beta \to 1} \sigma_i^\beta(\omega) = \min_{q \in \partial V(0)} \sum_{(i,\omega) \in I \times \Omega} \gamma_{i,\omega}^q \mathbb{E}_{q_{i,\omega}}\left[\hat{f}\right] \le \inf_{\mu \in \Theta} \mathbb{E}_\mu\left[\hat{f}\right]. \tag{18}$$

Moreover, if there exists a common ex-ante preference $\bar{V}$ *for* $\{(V_i, \Pi_i)\}_{i \in I}$*, then, for all* $i \in I$ *and* $\omega \in \Omega$,

$$\lim_{\beta \to 1} \sigma_i^\beta(\omega) \in \left[\bar{V}\left(\hat{f}\right), V^\Theta\left(\hat{f}\right)\right].$$

First, we observe that, in the limit where the coordination motive prevails, the equilibrium price is independent on the realized state and the identity of the agent. In particular, the limit selects an equilibrium of the pure coordination game where the asset is payoff irrelevant. This generalizes a well-known fact for subjective expected utility (cf. [35] and [65]).

Second, the constant limit price is equal to the most cautious average of the benchmark evaluations of $\hat{f}$ that are consistent with the network structure. Notably, the cautious selection of the benchmark models q from $\partial V(0)$ induced by the market interaction has two roles. While selecting beliefs that evaluate the asset in a cautious way (i.e., to keep the first-order evaluations $\mathbb{E}_{q_{i,\omega}}\left[\hat{f}\right]$ low), it also determines how the heterogeneous evaluations are aggregated through the eigenvector centrality γ^q of the interactions structure.

Third, our formula for the limit equilibrium points out that the strong coordination motives in the market attenuates the ambiguity concern exhibited by the equilibrium evaluation. Intuitively, the asymmetric information of the traders combined with their coordination motive imply that the equilibrium prices are less variable across states than the fundamental itself. Therefore, the uncertainty averse traders evaluate the asset more favorably than the fundamental. More formally, we have

$$\lim_{\beta \to 1} \sigma_i^\beta(\omega) \ge V_i\left(\omega, \hat{f}\right) \qquad \forall i \in I, \forall \omega \in \Omega,$$

since each collection of beliefs $q \in \partial V(0)$ satisfy $c_{i,\omega}(q_{i,\omega}) = 0$ for all $i \in I$ and $\omega \in \Omega$. In turn, this immediately yields the lower bound in equation (18) and, when there exists a common ex-ante evaluation, we actually have $V_*\left(\hat{f}\right) = \bar{V}\left(\hat{f}\right)$ (cf. Corollary 1), implying that the equilibrium price is higher than the *shared* ex-ante evaluation. This is a sharp difference with respect to the case of SEU interim preferences where, under a common prior, the equilibrium price coincides with the prior expectation.

Fourth, the equilibrium price cannot be higher than the evaluation of the fundamental under any *ex-ante* probabilistic model that is consistently trusted by all the agents in all the states. Importantly, while the specific value of the limit equilibrium price depends on the network structure, the two bounds we have just described are robust in the sense that they hold across all the strongly connected network structures. Moreover, as we next show, the upper bound is actually attained in several important cases.

Corollary 4. *Assume that, for all $i \in I$ and $\omega \in \Omega$, it holds* $\arg\min_{p\in\Delta(\Omega)} c_{i,\omega} = \left\{q^*_{i,\omega}\right\}$. *For all $i \in I$ and $\omega \in \Omega$,*

$$\lim_{\beta\to 1} \sigma_i^\beta(\omega) = \sum_{(i,\omega)\in I\times\Omega} \gamma_{i,\omega}^{q^*} \mathbb{E}_{q^*_{i,\omega}}\left[\hat{f}\right].$$

Moreover, if $\{(V_i, \Pi_i)\}_{i\in I}$ has a common local subjective belief, then $\Theta = \{\mu^*\}$ and

$$\lim_{\beta\to 1} \sigma_i^\beta(\omega) = \mathbb{E}_{\mu^*}\left[\hat{f}\right].$$

This results characterizes an extreme form of ambiguity reduction. Indeed, whenever each interim preference has a unique benchmark model (e.g., all the agents have divergence preferences in the interim), the equilibrium price is equal to a SEU evaluation of the asset, implying that only the interim benchmark models matter as the importance of coordination grows. This reduction assumes a particularly stark form whenever the agents share a common local subjective belief μ^*. In this case, the ex-ante evaluation of the asset according to this probabilistic model is the limit price equilibrium and this limit is the same regardless of the ambiguity attitudes and the network structure. In the next example, we illustrate this phenomenon within the class of multiplier preferences with Bayesian updating from a common ex-ante probabilistic model.

Example 6. Suppose that, in the ex-ante stage, the agents share the same unique benchmark model $\mu^* \in \Delta(\Omega)$ but they are adverse to misspecification with possibly different attitudes: each $i \in I$ evaluates $\hat{f}$ as

$$\min_{p\in\Delta}\left\{\mathbb{E}_p\left[\hat{f}\right] + \lambda_i R(p||\mu^*)\right\}$$

where $(\lambda_i)_{i\in I} \in \mathbb{R}^I_{++}$ is a profile of misspecification fear indexes. After having observed their own private information, the agents update the benchmark model to $p_{\mu^*,i}(\omega,\cdot)$. Therefore, the interim evaluation of i at ω is

$$V_i(\omega, f) = \min_{p\in\Delta}\{\mathbb{E}_p[f] + \lambda_i R(p||p_{\mu^*,i}(\omega,\cdot))\} \qquad \forall f \in \mathbb{R}^\Omega.$$

In this case, Corollary 4 implies that

$$\lim_{\beta\to 1} \sigma_i^\beta(\omega) = \mathbb{E}_{\mu^*}\left[\hat{f}\right] \qquad \forall i \in I, \forall \omega \in \Omega.$$

That is, the ambiguity is completely washed out in the limit and the price converges to the expected evaluation of the asset, independently of the attitudes towards misspecification. If these attitudes are homogeneous, i.e., $\lambda_i = \lambda$ for all $i \in I$, there exists a common ex-ante expectation

$$\bar{V}(f) = \min_{p\in\Delta}\{\mathbb{E}_p[f] + \lambda R(p||\mu^*)\} \qquad \forall f \in \mathbb{R}^\Omega$$

and a wedge between $\bar{V}\left(\hat{f}\right)$ and $\lim_{\beta\to 1}\sigma_i^\beta(\omega)$ arises whenever the asset pays a different amount in each state. More generally, this wedge remains present between V_* and $\lim_{\beta\to 1}\sigma_i^\beta(\omega)$ even when the misspecification attitudes are heterogeneous. ▲

The simple Example 9 in the Online Appendix illustrate how the ambiguity-attenuating effect of the market interaction becomes relevant already at intermediate levels of coordination, i.e., for β far from 1.

Even beyond the case of interim preferences with single benchmark models, Theorem 2 has important implications for games with incomplete information with existing updating rules. For example, it implies that if all the trades share the same set $C \subseteq \Delta(\Omega)$ of ex-ante benchmark probability models, are maxmin, and update with full Bayesian updating, then equation (18) tells us that the equilibrium price is lower than $\min_{p \in C} \mathbb{E}_p \left[\hat{f} \right]$, which is the common ex-ante willingness to pay for the asset, hence the price of the asset in absence of information. When the upper bound is actually attained, this effect can be interpreted as *contagion of ambiguity* as the next example illustrates.

Example 7 (Contagion of ambiguity)**.** Consider two traders $I = \{1, 2\}$ that are uncertain about an asset $\hat{f} \in \mathbb{R}^{\Omega}$ with $\Omega = \{l, m, h\}$ and $\hat{f}(l) < \hat{f}(m) < \hat{f}(h)$. The agents are endowed with the following information structures

$$\Pi_1 = \{\{l\}, \{m, h\}\} \qquad \text{and} \qquad \Pi_2 = \{\{l, m\}, \{h\}\}.$$

Fix $\gamma \in (0, 1)$ and $\varepsilon \in (0, 1/2)$, and assume that the agents have a common set of ex-ante probabilistic models

$$C = \{\alpha\delta_l + (1-\alpha)(\gamma\delta_m + (1-\gamma)\delta_h) : \alpha \in [\varepsilon, 1-\varepsilon]\}.$$

In the interim stage, conditional on each ω, each agent i has maxmin preferences with respect to $C_{i,\omega}$ obtained via full Bayesian updating. In particular, we have

$$C_{1,l} = \{\delta_l\} \qquad \text{and} \qquad C_{1,m} = C_{1,h} = \{\gamma\delta_m + (1-\gamma)\delta_h\},$$

and

$$C_{2,l} = C_{2,m} = \left\{\alpha\delta_l + (1-\alpha)\delta_m : \alpha \in \left[\frac{\varepsilon}{\varepsilon + \gamma(1-\varepsilon)}, \frac{(1-\varepsilon)}{(1-\varepsilon) + \gamma\varepsilon}\right]\right\} \qquad \text{and} \qquad C_{2,h} = \{\delta_h\}.$$

In particular, in the interim stage, only agent 2 conditional on $\omega \in \{l, m\}$ perceives ambiguity. For every β, it is easy to guess and verify that the equilibrium strategy satisfies

$$\sigma_1^{\beta}(l) \leq \sigma_1^{\beta}(m) = \sigma_1^{\beta}(h).$$

Therefore, conditional on $\omega \in \{l, m\}$, agent 2 behaves as if her probabilistic model assigns the highest possible probability to l, that is, $\alpha = \frac{(1-\varepsilon)}{(1-\varepsilon)+\gamma\varepsilon}$. With some tedious algebra, this observation allows us to compute the equilibrium in closed form and obtain the limit equilibrium

$$\lim_{\beta \to 1} \sigma_i^{\beta}(\omega) = \min_{p \in C} \mathbb{E}_p \left[\hat{f} \right] = (1-\varepsilon)\hat{f}(l) + \varepsilon\gamma\hat{f}(m) + \varepsilon(1-\gamma)\hat{f}(h).$$

In words, the ambiguity aversion of agent 2 conditional on $\omega \in \{l, m\}$ is strong enough to infect both types of agent 1 as well as her own type when she observes h. This effect leads to full coordination on the ex-ante ambiguity averse evaluation. This is particularly sharp as we increase the ex-ante ambiguity of the players by letting $\varepsilon \to 0$. In this case, in the high-coordination limit, the unique price will converge to the lowest evaluation possible $\hat{f}(l)$ at every state of the world. ▲

The previous two examples may suggest that, whenever $\lim_{\beta\to 1} \sigma_i^{\beta}(\omega)$ is well defined and Θ is nonempty, we have $\lim_{\beta\to 1} \sigma_i^{\beta}(\omega) = \min_{\mu\in\Theta} \mathbb{E}_{\mu}\left[\hat{f}\right] = V^{\Theta}$, that is, the upper bound in Theorem 2 is always achieved even beyond the scope of Corollary 4. However, the next simple example shows that this is not always the case under these assumptions.

Example 8. Let $I = \{1, 2\}$, $\hat{f} \in \mathbb{R}^{\Omega}$, and endow the two traders with no information, that is, $\Pi_1 = \Pi_2 = \{\Omega\}$. In the ex-ante stage, both the agents have maxmin preferences with corresponding sets of probabilistic models $C_1, C_2 \subseteq \Delta(\Omega)$ such that $C_1 \neq C_2$ and $C_1 \cap C_2 \neq \emptyset$. In this case, we have $\Theta = C_1 \cap C_2$ given that both agents have no information. Moreover, for every $\beta \in (0, 1)$, the unique equilibrium σ^{β} is given by

$$\sigma_i^{\beta} = \frac{\min_{p\in C_i}\left\{\mathbb{E}_p\left[\hat{f}\right]\right\} + \beta \min_{p\in C_{-i}}\left\{\mathbb{E}_p\left[\hat{f}\right]\right\}}{1+\beta} \qquad \forall i \in I.$$

With this, the high-coordination limit price is given by

$$\lim_{\beta\to 1} \sigma_i^{\beta}(\omega) = \frac{\min_{p\in C_1}\left\{\mathbb{E}_p\left[\hat{f}\right]\right\} + \min_{p\in C_2}\left\{\mathbb{E}_p\left[\hat{f}\right]\right\}}{2} \leq \min_{p\in C_1\cap C_2} \mathbb{E}_p\left[\hat{f}\right],$$

and, in general, the previous inequality may be strict.[25] ▲

The previous example with maxmin preferences and full Bayesian updating crucially relies on the non existence of a common ex-ante expectation $\bar{V}$. Indeed, the next corollary of Theorem 2 shows that, in this setting, if $\bar{V}$ exists, then the lower and upper bound collapses and are equal to the limit price, regardless of the network structure.

Corollary 5. *Let $\{(V_i, \Pi_i)\}_{i\in I}$ be a collection of maxmin (cf. Example 1) interim preferences. If there exists a common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i\in I}$, then, $\bar{V}$ is a maxmin ex-ante expectation and, for all $i \in I$ and $\omega \in \Omega$,*

$$\bar{V}\left(\hat{f}\right) = \lim_{\beta\to 1} \sigma_i^{\beta}(\omega) = V^{\Theta}\left(\hat{f}\right).$$

[25] To see this concretely, let $\Omega = \{L, H\}$, $C_1 = \{p \in \Delta(\Omega) : p(H) \in [1/4, 1/2]\}$, $C_2 = \{p \in \Delta(\Omega) : p(H) \in [1/3, 1/2]\}$, and $\hat{f}(L) = 1 - \hat{f}(H) = 0$.

In stark contrast with Corollary 4 and Example 6, under maxmin preferences the perception and aversion of ambiguity is still present in the high-coordination limit. Moreover, the higher the ex-ante ambiguity about the underlying fundamental (i.e., a lower $\bar{V}$) the lower the equilibrium price. However, the fact that all the ex-ante ambiguity is preserved in the limit is driven by the fact that full Bayesian updating is an overly cautious updating rule under maxmin preferences.[26] We next illustrate our results in the less cautious proxy updating of [36] already introduced in Example 5. In particular, recall that a *lower common ex-ante expectation* $V_\circ = \min_{\mu \in core(\nu)} \mathbb{E}_p$ describes the ex-ante preferences of the agents. Moreover, by Theorem 2, we have, for every network structure, the equilibrium price in the high-coordination limit belongs to

$$\left[\min_{\mu \in core(\nu)} \mathbb{E}_p\left[\hat{f}\right], \min_{\mu \in \cap_{i \in I} core_i(\nu)} \mathbb{E}_\mu\left[\hat{f}\right]\right]$$

as $\cap_{i \in I} core_i(\nu)$ is included in Θ and this intersection is always nonempty since it contains the Shapley value μ_ν. Importantly, whenever the probabilities in $core(\nu)$ agree on the events that are $\Pi_{\inf}$-measurable, the two bounds collapse as, in this case, each $core_i(\nu) = core(\nu)$. In the next section, we generalize this result to the whole class of variational models.

4.3 Beauty contests: unambiguous information structure

Here, we consider an important particular case: the agents are unambiguous with respect to the information structure while still possibly perceiving ambiguity about the fundamental $\hat{f}$, i.e., there is no *strategic ambiguity*. In this case, the first-order expectations of the agents exhibits perceived ambiguity and ambiguity aversion whereas the higher-order expectations do not, that is, they are SEU. Formally, we say that the *information structure is unambiguous* if and only if for every $i \in I$, V_i is $\Pi_{\inf}$-affine, that is

$$V_i(\omega, (1-\alpha) h + \alpha g) = (1-\alpha) V_i(\omega, h) + \alpha V_i(\omega, g)$$

for all $\alpha \in (0,1)$, for all $\omega \in \Omega$, and for all $g, h \in \mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable. This implies that V_i is linear over the vector space of elements $g \in \mathbb{R}^\Omega$ that are $\Pi_{\inf}$-measurable. This restriction is reasonable, for instance, in games where the agents repeatedly interact and have the ability to observe the actions of the coplayers after each interaction. In this case, if the agents are correctly specified, then their beliefs will converge to the true distribution on $\Pi_{\inf}$.

Proposition 5. *For all $i \in I$ and $\omega \in \Omega$,*

$$\lim_{\beta \to 1} \sigma_i^\beta(\omega) \in \left[V_*\left(\hat{f}\right), V^*\left(\hat{f}\right)\right].$$

[26] The existence of a common ex-ante expectation implies that each V_i is obtained via full Bayesian updating from $\bar{V}$ (cf. [22]).

Moreover, if there exists a common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$, then, for all $i \in I$ and $\omega \in \Omega$,

$$\lim_{\beta \to 1} \sigma_i^\beta (\omega) = \bar{V} \left(\hat{f} \right).$$

Whenever the traders are not ambiguous regarding events in their information structure, the extreme ex-ante preferences give both an upper and lower bound for any possible equilibrium selection. In particular, observe that the upper bound given by the previous proposition improves on the one of Theorem 2 since V^Θ is a common upper ex-ante preference. Next, observe that, whenever a common ex-ante preference exists, the identity $\bar{V} = V_* = V^*$ implies that the limit equilibrium $\lim_{\beta \to 1} \sigma^\beta$ is well defined and equal to the ex-ante evaluation. This is an implication of the common prior assumption under SEU (cf. [35]) that we extend to the unambiguous-partition case. Finally, comparing the second parts of Theorem 2 and of Proposition 5, we observe that the only ambiguity that can be tamed by the market interaction is the one about the information structures of the agents.

4.4 Additional application: price competition

Next, we consider an alternative foundation for the equilibrium equation (14) that is the starting point of the equilibrium characterization given in all the results in this section. Concretely, there are n firms competing on prices. We fix a random variable $\hat{f} \in \mathbb{R}^\Omega$ representing the state of the economy and we let $y \in \mathbb{R}$ denote its realization. The interpretation is that there is aggregate uncertainty about the state y. Each firm i chooses the price $x_i \in \mathbb{R}$ for its good, has 0 production costs, and its payoff function $u_i : \mathbb{R}^I \times \mathbb{R} \to \mathbb{R}$ depends on the state y as well as the entire profile of prices $x \in \mathbb{R}^I$:

$$u_i (x, y) = D_i (x, y) x_i$$

where $D_i : \mathbb{R}^I \times \mathbb{R} \to \mathbb{R}$ is the demand function faced by firm i and is defined as

$$D_i (x, y) = \beta \sum_{j \in I} w_{ij} x_j + (1 - \beta) y - \frac{x_i}{2}$$

for some $\beta \in (0, 1)$ and a stochastic and strongly connected matrix W with $w_{jj} = 0$ for all $j \in I$. The demand faced by firm i negatively depends on its own price and positively depends on the state of the economy and on the prices of the other firms respectively with coefficients $(1 - \beta)$ and β. As usual, the interpretation is that the firms compete on the same market with partially differentiated products and w_{ij} captures the similarity of products i and j. For the rest of this section we strengthen Assumption 1 by letting $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of maxmin (cf. 1) interim preferences. In particular, let $C_{i,\omega} \subseteq \Delta (\Omega)$ denote the set of interim probabilistic models of agent i at state ω.

As before, a strategy $\sigma_i \in \Sigma_i$ of agent i is measurable with respect to Π_i. Given a strategy profile $\sigma_{-i} \in \prod_{j \neq i} \Sigma_j$ for the co-players of i, the problem faced by i given state $\omega \in \Omega$ is

$$\max_{x_i \in \mathbb{R}} \min_{p \in C_{i,\omega}} \mathbb{E}_p \left[\left((1-\beta) \hat{f} + \beta \sum_{j \in I} w_{ij} \sigma_j \right) x_i - \frac{x_i^2}{2} \right].$$

With this, the first-order condition characterizing the equilibrium σ^β for every $\beta \in (0,1)$ is

$$\sigma_i^\beta (\omega) = \min_{p \in C_{i,\omega}} \mathbb{E}_p \left[(1-\beta) \hat{f} + \beta \sum_{j \in I} w_{ij} \sigma_j^\beta \right] \qquad \forall \omega \in \Omega, \forall i \in I, \tag{19}$$

which is just a particular case of equation (14).

5 No trade and betting implications

In this section, we give both necessary and sufficient conditions, in terms of interim trade and betting behavior, for the existence of a common ex-ante expectation.[27] For simplicity, we let $I = \{1, 2\}$ and we suppose that the only feasible acts are $f \in F = [-k, k]^\Omega$, $k \in \mathbb{R}_+$. The additional restriction we impose with respect to Section 3 is translation invariance of the interim preferences. The class of variational preferences, considered in Section 4 and in all the examples of the paper, satisfies this property.

First, we show that if there exist an asset $f \in F$ and a price $r \in \mathbb{R}$ such that in each state $\omega \in \Omega$, if endowed with the asset player 2 would like to sell it, while player 1 would like to buy it, then there is no common ex-ante preference. Formally, we say that there exists an *interim Pareto improving transaction* if there exists $f \in F$ and $r \in \mathbb{R}$ such that, for all $\omega \in \Omega$, we have $V_1 (\omega, f) > r > V_2 (\omega, f)$.

Proposition 6. *Let* $\{(V_i, \Pi_i)\}_{i \in \{1,2\}}$ *be a collection of full support and translation invariant interim expectations such that* $\Pi_{\sup} = \{\Omega\}$. *If there is an interim Pareto improving transaction, then there is no common ex-ante expectation* $\bar{V}$ *for* $\{(V_i, \Pi_i)\}_{i \in \{1,2\}}$.

This result clarifies that common dynamic consistency, even without purely probabilistic beliefs, already implies the absence of trade between the agents. This does not come as a surprise since, as showed by Kajii and Ui [46] for maxmin preferences and by Martins-da-Rocha [54] for more general preferences, the absence of interim trade is equivalent to the existence of a common local subjective belief $\Theta \neq \emptyset$, which is always implied by the existence of a common ex-ante expectation $\bar{V}$ (cf. Proposition 3). However, the latter property is in general much

[27] Here, we do not consider the interim no-trade characterizations of the existence of the extreme common ex-ante expectations V_* and V^*, as well as the existence of a common local subjective belief, i.e., $\Theta \neq \emptyset$. Indeed, the former always exist as shown in Proposition 2, whereas the no-trade implications of the latter have been extensively studied in [46] and [54].

stronger than the former, thereby establishing an important difference with the SEU case, where common dynamic consistency is equivalent to the absence of interim Pareto improving transaction (cf. [63]).

The sufficient conditions for the existence of an ex-ante expectation can instead be expressed in terms of the existence of an interim Pareto gain in a large economy with a unit mass of agents endowed with the interim expectation (V_1, Π_1) and a unit mass of agents endowed with the interim expectation (V_2, Π_2). Formally, a *two-population endowment economy* $\{(\chi_i, V_i, \Pi_i)\}_{i\in\{1,2\}}$ is composed by a pair of functions with finite range $(\chi_1, \chi_2) \in F^{[0,1]} \times F^{[0,1]}$ and a pair of interim expectations $\{(V_i, \Pi_i)\}_{i\in\{1,2\}}$. Here $\chi_i(x)$ is the initial asset position of agent x of population i. We say that an endowment economy is interim Pareto improvable if there exists $(\chi_1', \chi_2') \in F^{[0,1]} \times F^{[0,1]}$ such that

1. **Market clearing:**
$$\int_{[0,1]} \chi_1(x)(\omega)\,dx + \int_{[0,1]} \chi_2(x)(\omega)\,dx = \int_{[0,1]} \chi_1'(x)(\omega)\,dx + \int_{[0,1]} \chi_2'(x)(\omega)\,dx \qquad \forall \omega \in \Omega;$$

2. **Interim Pareto improvement:**
$$V_i(\omega, \chi_i(x)) < V_i(\omega, \chi_i'(x)) \qquad \forall i \in \{1,2\}, \forall x \in [0,1], \forall \omega \in \Omega.$$

Theorem 3. *Let $\{(V_i, \Pi_i)\}_{i\in\{1,2\}}$ be a set of full support translation invariant interim expectations such that $\Pi_{\sup} = \{\Omega\}$. If there is no two-population endowment economy $\{(\chi_i, V_i, \Pi_i)\}_{i\in\{1,2\}}$ that is interim Pareto improvable, then there exists a translation invariant common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i\in\{1,2\}}$.*

There are two reasons behind the gap between the necessary and sufficient conditions for the existence of a common ex-ante preference. First, for non SEU agent the value of shortening a position, $V_i(\omega, -f)$, is in general different from the negative of the value of the position, $-V_i(\omega, -f)$. Therefore, to guarantee the existence of a common prior the absence of profitable trade must be verified at every initial asset position, and it is not enough to look at neutral initial positions. Moreover, the non additivity of V_i over the different assets implies that ruling out bilateral improvements is not enough, and instead joint transfers between multiple agents must be considered.

6 Discussion and conclusion

The results of this paper can be also used as a stepping stone for further analysis of games with non subjective expected utility. Here we highlight some open questions and future research avenues.

First, as already stressed, despite our analysis follows an interim approach, our results can be used in games of incomplete information with general preferences under uncertainty and a *given* set of updating rules. Indeed, the disagreement bound in Proposition 4 and the limit characterization in Theorem 2 did not put any intertemporal restriction on the agents' preferences. So, for example, if all the agents are maxmin, share the same ex ante set of probability models, and update their beliefs with full Bayesian updating, then our results give tools to study how the equilibrium outcomes changes with respect to the agents' private information. Therefore, our results can be seen as a stepping stone toward a model of information design in beauty contests under non SEU preferences.

Second, our framework enables us to revisit some classical results for SEU agents on incomplete information games to understand whether they carry on with more general preferences. An example is the result established in [26] that if a stochastically monotone function (often interpreted as the price of an asset) of the beliefs is common knowledge across the players, their beliefs actually coincide. The result extends if the information structure is unambiguous, but may fail more generally.

Finally, our framework and results can be used to obtain sharper equilibrium refinements in complete information games. Indeed, in the SEU world, Monderer and Samet [52] and Kajii and Morris [43] pioneered a robust approach that selects only the subset of equilibria that are limit points of *every* sequence of incomplete information games that is approximating the original complete information game. An even sharper refinement would only select equilibria that are limit points including elaborations under incomplete information and non-SEU preferences.

A Appendix: Mathematical preliminaries

Recall that we have a finite state space Ω and a finite set of individuals $I = \{1, ..., n\}$. Since Ω is finite, we can enumerate its elements $\Omega = \{\omega_1, ..., \omega_{\bar{n}}\}$ with $\bar{n} \in \mathbb{N}$. With a small abuse of notation, we equivalently view the state space as either the set $\Omega = \{\omega_1, ..., \omega_{\bar{n}}\}$ or as the set $J = \{1, ..., \bar{n}\}$. In this way, $\mathbb{R}^\Omega$ is isomorphic to the set of vectors $\mathbb{R}^{\bar{n}}$, where both are endowed with the supnorm. For this reason, we still denote the elements of $\mathbb{R}^{\bar{n}}$ by f. We also denote the elements of the canonical basis of $\mathbb{R}^{\bar{n}}$ by e^j for all $j \in J$. Finally, we denote the vector whose components are all 1s by e: it corresponds to the function 1_Ω in $\mathbb{R}^\Omega$.

In this section, we focus our attention on operators $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$. In what follows any such operator will be assumed to be normalized, monotone, and continuous with the exception of Definition 9 and Lemma 3.[28] Clearly, a normalized, monotone, and continuous operator

[28] That is, T is normalized if and only if $T(ke) = ke$ for all $k \in \mathbb{R}$. Obviously, T is monotone if and only if for each $f, g \in \mathbb{R}^{\bar{n}}$

$$f \geq g \implies T(f) \geq T(g).$$

$T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ is linear if and only if there exists a stochastic $\bar{n} \times \bar{n}$ matrix M such that $T(f) = Mf$ for all $f \in \mathbb{R}^{\bar{n}}$.[29] The composition of normalized, monotone, and continuous operators is an operator which shares the same properties. In this work, all products of $\bar{n} \times \bar{n}$ matrices are to be intended backward/left, that is, $\Pi_{l=1}^{k+1} M_l = M_{k+1} \Pi_{l=1}^{k} M_l = M_{k+1} ... M_1$ for all $k \in \mathbb{N}$. Define $I_{\bar{n}}$ to be the $\bar{n} \times \bar{n}$ identity matrix. Given $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$, we define by T_j the j-th component of T, that is, $T_j(f)$ is the j-th component of the vector $T(f)$ for all $f \in \mathbb{R}^{\bar{n}}$. Given $j, j' \in J$ we say that j *is strongly monotone with respect to* j' (under T) if and only if there exists $\varepsilon_{jj'} \in (0,1)$ such that for each $f \in \mathbb{R}^{\Omega}$ and for each $\delta \geq 0$

$$T_j\left(f + \delta e^{j'}\right) - T_j(f) \geq \varepsilon_{jj'} \delta. \tag{20}$$

We also say that j *is constant with respect to* j' if and only if for each $f \in \mathbb{R}^{\Omega}$ and for each $\delta \geq 0$

$$T_j\left(f + \delta e^{j'}\right) - T_j(f) = 0. \tag{21}$$

Given T and $j, j' \in J$, observe that it might be the case that neither j is strongly monotone with respect to j' nor j is constant with respect to j'. In light of this, we say that T is *dichotomic* if and only if for each $j, j' \in J$, j is either strongly monotone with respect to j' or constant. Our operators T are typically nondifferentiable, when they are though, condition (20) (resp., (21)) amounts to require that the partial derivative of T_j with respect to j' is uniformly bounded away from zero (resp., is zero) at each f.

We next define the notion of indicator matrix for an operator T.

Definition 9. *Let T be a monotone operator. We say that $A(T)$ is the* indicator matrix *of T if and only if its jj'-th entry is such that*

$$a_{jj'} = \begin{cases} 1 & j \text{ is strongly monotone wrt } j' \\ 0 & \text{otherwise} \end{cases} \qquad \forall j, j' \in J.$$

The above notion of indicator matrix generalizes the notion of indicator matrix for positive matrices. In fact, the indicator matrix $A(M)$ of an $\bar{n} \times \bar{n}$ nonnegative matrix M is defined to be such that $a_{jj'} = 1$ if and only if $m_{jj'} > 0$ and $a_{jj'} = 0$ if and only if $m_{jj'} = 0$.[30] We say that $A(T)$ is *nontrivial* if and only if for each $j \in J$ there exists $j' \in J$ such that $a_{jj'} = 1$. The indicator matrix $A(T)$ of a monotone operator T induces a natural partition of J, associated to T. Recall that given a nonnegative $\bar{n} \times \bar{n}$ matrix A with *nonnull rows*, we can partition the set $J = \{1, ..., \bar{n}\}$ with the partition $\{J_l(A)\}_{l=1}^{m_A+1}$ of essential and inessential indexes of A. The first m_A sets consist of the essential classes while $J_{m_A+1}(A)$ consists of all inessential indexes and it might be empty. This is the case if A is symmetric, that is, $a_{jj'} = a_{j'j}$ for all $j, j' \in J$.

[29] As usual, a stochastic matrix is a square matrix whose entries are nonnegative and the entries of each row sum up to 1.

[30] To see this, define $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ by $T(f) = Mf$ for all $f \in \mathbb{R}^{\bar{n}}$. It is then immediate to see that $A(T) = A(M)$.

Instead, there always exists at least a nonempty class of essential indexes $J_1(A)$.[31] We call $\Pi(A) = \{J_l(A)\}_{l=1}^{m_A+1}$ the partition of A. When $A = A(T)$ where T is normalized, monotone, and continuous and $A(T)$ is nontrivial, we denote by $\Pi(T)$ the partition $\Pi(A(T))$.

Lemma 3. *Let $\{B_k\}_{k\in\{1,...,K\}}$ be a finite collection of $\bar{n}\times\bar{n}$ nonnegative matrices such that $b_{k,jj} > 0$ for all $k \in \{1,...,K\}$ and for all $j \in J$. If $A(B_k)$ is symmetric for all $k \in \{1,...,K\}$, then $A(B_K...B_1) \geq A(B_k)$ for all $k \in \{1,...,K\}$ and $\Pi(A(B_K...B_1))$ is coarser than $\Pi(B_k)$ for all $k \in \{1,...,K\}$.*

We already observed that a normalized, monotone, and continuous operator T is linear if and only if $T(f) = Mf$ for all $f \in \mathbb{R}^{\bar{n}}$ where M is an $\bar{n}\times\bar{n}$ stochastic matrix. Intuitively, the next two results show that dropping the linearity assumption allows M to depend on f. The first result will not impose much discipline on the replicating matrices $M(f)$ while the second one will connect the indicator matrix of $M(f)$ to the one of T. As usual, we denote by $\Delta_{\bar{n}}$ the collection of all vectors in $\mathbb{R}_+^{\bar{n}}$ whose entries sum up to 1.

Lemma 4. *If $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ is normalized, monotone, and continuous, then there exists a compact and convex set $\mathcal{M}(T)$ of $\bar{n}\times\bar{n}$ stochastic matrices such that for each $f \in \mathbb{R}^{\bar{n}}$ there exists $M(f) \in \mathcal{M}(T)$ such that*

$$T(f) = M(f)f.$$

Moreover, if j is constant with respect to j', then $m_{jj'} = 0$ for all $M \in \mathcal{M}(T)$.

The next result builds on [9, Proposition 8].

Proposition 7. *If $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ is normalized, monotone, continuous, and such that $A(T)$ is nontrivial, then there exists a compact and convex set $\mathcal{M}(T)$ of $\bar{n}\times\bar{n}$ stochastic matrices such that $A(M) \geq A(T)$ for all $M \in \mathcal{M}(T)$ and for each $f \in \mathbb{R}^{\bar{n}}$ there exists $M(f) \in \mathcal{M}(T)$ such that*

$$T(f) = M(f)f.$$

Moreover, if T is dichotomic, then $\mathcal{M}(T)$ can be chosen to be such that $A(M) = A(T)$ for all $M \in \mathcal{M}(T)$.

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[31] We follow Seneta [64]. Denote by $a_{jj'}^{(t)}$ the jj'-th entry of A^t. We write $j \overset{A}{\to} j'$ if and only if $a_{jj'}^{(t)} > 0$ for some $t \in \mathbb{N}$. It is immediate to see that if $j \overset{A}{\to} j'$ and $j' \overset{A}{\to} j''$, then $j \overset{A}{\to} j''$. We also write $j \overset{A}{\longleftrightarrow} j'$ if and only if $j \overset{A}{\to} j'$ and $j' \overset{A}{\to} j$. In this case, clearly, we have that $j \overset{A}{\to} j$, that is, $a_{jj}^{(t)} > 0$ for some $t \in \mathbb{N}$. Next, we classify each index $j \in J$ as essential or inessential. An index $j \in J$ is *essential* if and only if for each $j' \in J$

$$j \overset{A}{\to} j' \implies j \overset{A}{\longleftrightarrow} j'.$$

If instead there exists $j' \in J$ such that $j \overset{A}{\to} j'$, but $j' \overset{A}{\not\to} j$, we say that j is *inessential*. In other words, j is inessential if and only if it is not essential. Note that there always exists at least one essential index (see Seneta [64, Lemma 1.1]). For each essential $j \in J$, define $[j] = \left\{j' \in J : j \overset{A}{\longleftrightarrow} j'\right\}$. Note that given two essential indexes j and j' in J we have that either $[j] = [j']$ or $[j] \cap [j'] = \emptyset$. In particular, $j'' \in [j]$ if and only if j'' is essential and $j \overset{A}{\longleftrightarrow} j''$. Moreover, given $j, j' \in J$ such that $j \overset{A}{\longleftrightarrow} j'$, j is essential if and only if j' is.

Proof. For each $j, j' \in J$ if j is strongly monotone with respect to j', consider $\varepsilon_{jj'} \in (0,1)$ as in (20) otherwise let $\varepsilon_{jj'} = 1/2$. Define $\tilde{M}$ to be such that $\tilde{m}_{jj'} = a_{jj'}\varepsilon_{jj'}$ for all $j, j' \in J$ where $a_{jj'}$ is the jj'-th entry of $A(T)$. Since each row of $A(T)$ is not null, for each $j \in J$ there exists $j' \in J$ such that $a_{jj'} = 1$ and, in particular, $\tilde{m}_{jj'} > 0$. This implies that $\sum_{l=1}^{\bar{n}} \tilde{m}_{jl} > 0$ for all $j \in J$. Define also $\varepsilon = \min\left\{\min_{j\in J} \sum_{l=1}^{\bar{n}} \tilde{m}_{jl}, 1/2\right\} \in (0,1)$. Define the stochastic matrix $\bar{M}$ to be such that $\bar{m}_{jj'} = \tilde{m}_{jj'}/\sum_{l=1}^{\bar{n}} \tilde{m}_{jl}$ for all $j, j' \in J$. Clearly, we have that for each $j, j' \in J$

$$\bar{m}_{jj'} > 0 \iff \tilde{m}_{jj'} > 0 \iff a_{jj'} = 1.$$

This yields that $A\left(\bar{M}\right) = A(T)$. Next, consider $f, g \in \mathbb{R}^{\bar{n}}$ such that $f \geq g$. Define $g^0 = g$. For each $j' \in \{1, ..., \bar{n}-1\}$ define $g^{j'} \in \mathbb{R}^{\bar{n}}$ to be such that $g_j^{j'} = f_j$ for all $j \leq j'$ and $g_j^{j'} = g_j$ for all $j \geq j'+1$. Define $g^{\bar{n}} = f$. Note that $f = g^{\bar{n}} \geq ... \geq g^1 \geq g^0 = g$. It follows that

$$\begin{aligned}
T_j(f) - T_j(g) &= \sum_{j'=1}^{\bar{n}} \left[T_j\left(g^{j'}\right) - T_j\left(g^{j'-1}\right)\right] \geq \sum_{j'=1}^{\bar{n}} a_{jj'}\varepsilon_{jj'}\left(g_{j'}^{j'} - g_{j'}^{j'-1}\right) \\
&= \sum_{j'=1}^{\bar{n}} \tilde{m}_{jj'}\left(f_{j'} - g_{j'}\right) = \left(\sum_{l=1}^{\bar{n}} \tilde{m}_{jl}\right)\left(\sum_{j'=1}^{\bar{n}} \frac{\tilde{m}_{jj'}}{\sum_{l=1}^{\bar{n}} \tilde{m}_{jl}}\left(f_{j'} - g_{j'}\right)\right) \\
&= \left(\sum_{l=1}^{\bar{n}} \tilde{m}_{jl}\right)\left(\sum_{j'=1}^{\bar{n}} \bar{m}_{jj'}\left(f_{j'} - g_{j'}\right)\right) \\
&\geq \varepsilon \sum_{j'=1}^{\bar{n}} \bar{m}_{jj'}\left(f_{j'} - g_{j'}\right) \qquad \forall j \in J.
\end{aligned}$$

This implies that

$$f \geq g \implies T(f) - T(g) \geq \varepsilon\bar{M}(f-g) = \varepsilon\left(\bar{M}f - \bar{M}g\right). \tag{22}$$

Define $S : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ by

$$S(f) = \frac{T(f) - \varepsilon\bar{M}f}{1-\varepsilon} \qquad \forall f \in \mathbb{R}^{\bar{n}}.$$

By definition of S and (22) and since $\bar{M}$ is a stochastic matrix and T is normalized, monotone, and continuous, it is immediate to see that S is normalized, monotone, and continuous. We can rewrite T to be such that

$$T(f) = \varepsilon\bar{M}f + (1-\varepsilon)S(f) \qquad \forall f \in \mathbb{R}^{\bar{n}}. \tag{23}$$

Consider the set $\mathcal{M}(S)$ of Lemma 4. Define $\mathcal{M}(T) = \varepsilon\bar{M} + (1-\varepsilon)\mathcal{M}(S)$. Since $\mathcal{M}(S)$ is compact and convex, $A(T) = A\left(\bar{M}\right)$, and $\varepsilon \in (0,1)$, it follows that $\mathcal{M}(T)$ is compact and convex and $A(M) \geq A\left(\bar{M}\right) = A(T)$ for all $M \in \mathcal{M}(T)$. By (23) and since for each $f \in \mathbb{R}^{\bar{n}}$ there exists $\hat{M}(f) \in \mathcal{M}(S)$ such that $S(f) = \hat{M}(f)f$, for each $f \in \mathbb{R}^{\bar{n}}$ we have that $T(f) = M(f)f$ where $M(f) = \varepsilon\bar{M} + (1-\varepsilon)\hat{M}(f) \in \mathcal{M}(T)$.

Finally, consider $j, j' \in J$. Since $A(M) \geq A(T)$, if the jj'-entry of $A(T)$ is 1 so is the one of $A(M)$ for all $M \in \mathcal{M}(T)$. Assume that the jj'-entry of $A(T)$ is 0. Since $A(T) = A(\bar{M})$, the jj'-entry of $A(\bar{M})$ is 0 too. Since T is dichotomic, it follows that for each $f \in \mathbb{R}^{\bar{n}}$ and for each $\delta \geq 0$

$$\begin{aligned}\varepsilon \sum_{l=1}^{\bar{n}} \bar{m}_{jl} f_l + (1-\varepsilon) S_j\left(f + \delta e^{j'}\right) &= \varepsilon \sum_{l=1}^{\bar{n}} \bar{m}_{jl}\left(f_l + \delta e_l^{j'}\right) + (1-\varepsilon) S_j\left(f + \delta e^{j'}\right) \\ &= T_j\left(f + \delta e^{j'}\right) = T_j(f) = \varepsilon \sum_{l=1}^{\bar{n}} \bar{m}_{jl} f_l + (1-\varepsilon) S_j(f).\end{aligned}$$

Since $\varepsilon \in (0,1)$, we can conclude that $S_j\left(f + \delta e^{j'}\right) = S_j(f)$ for all $f \in \mathbb{R}^{\bar{n}}$ and for all $\delta \geq 0$, that is, j is constant with respect to j' under S. By Lemma 4, we have that $m_{jj'} = 0$ for all $M \in \mathcal{M}(S)$. Since $\mathcal{M}(T) = \varepsilon \bar{M} + (1-\varepsilon)\mathcal{M}(S)$ and $\bar{m}_{jj'} = 0$, we can conclude that the jj'-entry of $A(M)$ is 0 for all $M \in \mathcal{M}(T)$. Since j and j' were arbitrarily chosen, we can conclude that $A(M) = A(T)$ for all $M \in \mathcal{M}(T)$. ■

The next lemma is an extension to our framework of Lemma 2 of Samet [62]. In order to discuss it, we need to introduce some notation. Given a stochastic matrix M, we denote by $\delta(M) = \min_{j,j' \in J: m_{jj'} > 0} m_{jj'}$ and $d(M) = \min_{j \in J} m_{jj}$.

Lemma 5. *Let M and $\bar{M}$ be two $\bar{n} \times \bar{n}$ stochastic matrices. If $A(\bar{M})$ is symmetric and $0 < d(\bar{M})$, then we have that $A(\bar{M}M) \geq A(M)$ and*

1. *$\delta(\bar{M}M) \geq \delta(M)$, provided $A(\bar{M}M) = A(M)$.*
2. *$\delta(\bar{M}M) \geq \delta(M)\delta(\bar{M})$, provided $A(\bar{M}M) > A(M)$.*

Moreover, if $\{M_k\}_{k=1}^{\infty}$ is a sequence of $\bar{n} \times \bar{n}$ stochastic matrices such that $A(M_k)$ is symmetric, $\delta(M_k) \geq \delta > 0$, and $d(M_k) > 0$ for all $k \in \mathbb{N}$, then

$$\delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta^{\bar{n}^2} \qquad \forall m \in \mathbb{N}. \tag{24}$$

Define

Theorem 4. *Let $\{T_i\}_{i \in I}$ be a finite collection of normalized, monotone, and continuous dichotomic operators. If*

1. *$A(T_i)$ is symmetric for all $i \in I$,*
2. *$a_{i,jj} = 1$ for all $i \in I$ and for all $j \in J$,*
3. *the meet of the partitions $\{\Pi(T_i)\}_{i \in I}$ is $\{\Omega\}$,*

then for each I-sequence $(i_m)_{m\in\mathbb{N}}$ and for each $f \in \mathbb{R}^{\bar{n}}$ we have that

$$\lim_{m\to\infty} T_{i_m} \circ ... \circ T_{i_1}(f)$$

exists and is a constant vector. Moreover, for each I-sequence $(i_m)_{m\in\mathbb{N}}$ and for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then

$$\left|\left|\lim_{m\to\infty} T_{i_m} \circ ... \circ T_{i_1}(f) - T_{i_t} \circ ... \circ T_{i_1}(f)\right|\right|_\infty \leq \left(1 - \delta^{2^{\bar{n}^2}\bar{n}^2}\right)^{\tau 2^{-\bar{n}^2}-1} ||f||_\infty,$$

where $\delta = \inf_{i\in I, M\in\mathcal{M}(T_i)} \delta(M) > 0$.

Proof. Define $\hat{t} = 2^{\bar{n}^2}$. By Proposition 7, we have that $I_{\bar{n}} \leq A(T_i) = A(M)$ for all $M \in \mathcal{M}(T_i)$ and for all $i \in I$. Since $\mathcal{M}(T_i)$ is compact for all $i \in I$ and I is finite, this implies that $\delta = \inf_{i\in I, M\in\mathcal{M}(T_i)} \delta(M) > 0$. Define $\hat{\delta} = \delta^{\hat{t}\bar{n}^2} > 0$. Consider $f \in \mathbb{R}^{\bar{n}}$ and an I-sequence $(i_t)_{t\in\mathbb{N}}$. Define $f_t = T_{i_t} \circ ... \circ T_{i_1}(f) \in \mathbb{R}^{\bar{n}}$ for all $t \in \mathbb{N}$ and set $f_0 = f$. By Proposition 7, there exists a sequence $\{M_t\}_{t\in\mathbb{N}}$ of $\bar{n} \times \bar{n}$ stochastic matrices such that $M_t \in \mathcal{M}(T_{i_t})$ and $T_{i_t}(f_{t-1}) = M_t f_{t-1}$ for all $t \in \mathbb{N}$. Set $t_0 = 0$. Define recursively the following subsequence

$$t_{h+1} = \min\{m > t_h : \{i_{t_h+1}, ..., i_m\} \supseteq I\} \quad \forall h \geq 0.$$

We next proceed by steps.

Step 1: $A\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right) \geq I_{\bar{n}}$ and $\Pi\left(A\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right)\right) = \{\Omega\}$ for all $h \in \mathbb{N}_0$.

Proof of the Step. Fix $h \in \mathbb{N}_0$. Since $I_{\bar{n}} \leq A(T_{i_t}) = A(M_t)$ for all $t \in \{t_h + 1, ..., t_{h+1}\}$, we have that $A(M_t)$ has a strictly positive diagonal and it is symmetric for all $t \in \{t_h + 1, ..., t_{h+1}\}$. By Lemma 3 and since $\{t_h + 1, ..., t_{h+1}\} \supseteq I$ and the meet of the partitions $\{\Pi(T_i)\}_{i\in I}$ is $\{\Omega\}$, so is the meet of the partitions $\{\Pi(M_t)\}_{t=t_h+1}^{t_h+1}$, yielding that $\Pi\left(A\left(M_{t_{h+1}}...M_{t_h+1}\right)\right) = \{\Omega\}$. By Lemma 3, we also have that $A\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right) \geq A(M_t) \geq I_{\bar{n}}$ for all $t \in \{t_h + 1, ..., t_{h+1}\}$. □

Step 2: $\delta\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right) \geq \delta^{\bar{n}^2}$ for all $h \in \mathbb{N}_0$.

Proof of the Step. Fix $h \in \mathbb{N}_0$. By Lemma 5 and since $A(M_t) = A(T_{i_t})$ is symmetric, $\delta(M_t) \geq \delta > 0$, and $d(M_t) > 0$ for all $t \in \mathbb{N}$, the statement follows. □

Define $\bar{M}_h = \Pi_{t=t_h+1}^{t_{h+1}} M_t$ for all $h \in \mathbb{N}_0$. By Steps 1 and 2 and [64, Lemma 4.8 and Theorem 4.19], we have that $\Pi_{h=0}^{m} \bar{M}_h$ converges to a stochastic matrix M whose rows coincide to each other and, in particular, that

$$\left|\left|M - \Pi_{h=0}^{\tau-1} \bar{M}_h\right|\right|_\infty \leq \left(1 - \hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1} \qquad \forall \tau \in \mathbb{N}.$$

This implies that $\Pi_{l=1}^{m} M_l \to M$ and, in particular, that for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then

$$\left|\left|M - \Pi_{l=1}^{t} M_t\right|\right|_\infty \leq \left|\left|M - \Pi_{h=0}^{\tau-1} \bar{M}_h\right|\right|_\infty \leq \left(1 - \hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1}.$$

Finally, it follows that

$$\lim_{m\to\infty} T_{i_m} \circ ... \circ T_{i_1}(f) = \lim_{m\to\infty} \Pi_{l=1}^m M_l f = Mf,$$

and, in particular, that for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then

$$\left|\left|\lim_{m\to\infty} T_{i_m} \circ ... \circ T_{i_1}(f) - T_{i_t} \circ ... \circ T_{i_1}(f)\right|\right|_\infty = \left|\left|Mf - \left(\Pi_{l=1}^t M_t\right) f\right|\right| \leq \left(1 - \hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1} ||f||_\infty$$
$$= \left(1 - \delta^{2^{\bar{n}^2}\bar{n}^2}\right)^{\tau 2^{-\bar{n}^2}-1} ||f||_\infty$$

proving the statement. ■

B Appendix: Existence and implications

In this section, we use the results previously discussed. For such a reason, we equivalently refer to $\mathbb{R}^\Omega$ and $\mathbb{R}^{\bar{n}}$, since they are isomorphic.

Lemma 6. *If $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ is an ex-ante expectation, then it is continuous at constant functions.*

Lemma 7. *Let (V, Π) be an interim expectation with full support. The following statements are equivalent:*

(i) $a_{jj'} = 1$;

(ii) $\Pi(\omega_j) = \Pi(\omega_{j'})$.

In particular, $A(V)$ is symmetric, $a_{jj} = 1$ for all $j \in J$, $\Pi(V) = \Pi$, and V is dichotomic.

Proof of Proposition 1. By Lemma 7 and since $\{(V_i, \Pi_i)\}_{i\in I}$ is a finite set of full support interim expectations, we have that $A(V_i)$ is symmetric, $\Pi(V_i) = \Pi_i$, and V_i is dichotomic for all $i \in I$. Moreover, we have that $a_{i,jj} = 1$ for all $j \in J$ and for all $i \in I$. By Theorem 4 and since the meet of $\{\Pi(V_i)\}_{i\in I}$ is $\{\Omega\}$, we can conclude that for each I-sequence $\iota = (i_t)_{t\in\mathbb{N}}$ and for each $f \in \mathbb{R}^\Omega$ we have that $\lim_{m\to\infty} V_{i_m} \circ ... \circ V_{i_1}(f) = k_{\iota,f} 1_\Omega$ for some $k_{\iota,f} \in \mathbb{R}$. Moreover, there exist $\hat{\delta} = \left(\inf_{i\in I, M\in\mathcal{M}(T_i)} \delta(M)\right)^{2^{\bar{n}^2}\bar{n}^2} \in (0,1)$ and $\hat{t} = 2^{\bar{n}^2} \in \mathbb{N}$ such that for each I-sequence $(i_m)_{m\in\mathbb{N}}$ and for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then

$$||k_{f,\iota} 1_\Omega - V_{i_t} \circ ... \circ V_{i_1}(f)|| \leq \left(1 - \hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1} ||f||.$$

Finally, the last part of the statement follows from the previous claim by setting $C = \frac{1}{1-\hat{\delta}}$ and $\varepsilon = \left(1 - \hat{\delta}\right)^{\frac{1}{\hat{t}}}$. ■

Denote by P the set of permutations of agents, that is, bijections $\rho : \{1, ..., n\} \to \{1, ..., n\}$. Given $\rho \in P$, we denote by $V_\rho : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ the operator defined by

$$V_\rho = V_{\rho(1)} \circ V_{\rho(2)} \circ ... \circ V_{\rho(n)}. \tag{25}$$

As usual, we also denote by V_ρ^t the composition $\underbrace{V_\rho \circ ... \circ V_\rho}_{t\text{-times}}$ for all $t \in \mathbb{N}$ and for all $\rho \in P$.

Proof of Theorem 1. (i) implies (ii). By assumption, for each I-sequence $\iota = (i_t)_{t\in\mathbb{N}}$ and for each $f \in \mathbb{R}^\Omega$ we have that $\lim_{m\to\infty} V_{i_m} \circ ... \circ V_{i_1}(f) = k_{\iota,f} 1_\Omega$ for some $k_{\iota,f} \in \mathbb{R}$. By Lemma 6 and since $\bar{V}$ is an ex-ante expectation and $(\bar{V}, V_i, \Pi_i)$ is a generalized conditional expectation, we have that

$$\begin{aligned} k_{\iota,f} &= \bar{V}(k_{\iota,f} 1_\Omega) = \bar{V}\left(\lim_{m\to\infty} V_{i_m} \circ ... \circ V_{i_1}(f)\right) = \lim_{m\to\infty} \bar{V}(V_{i_m} \circ ... \circ V_{i_1}(f)) \\ &= \lim_{m\to\infty} \bar{V}\left(V_{i_{m-1}} \circ ... \circ V_{i_1}(f)\right) = ... = \lim_{m\to\infty} \bar{V}(V_{i_1}(f)) = \bar{V}(f), \end{aligned}$$

proving the implication.

(ii) implies (i). Fix a permutation $\bar{\rho} \in P$. Define the I-sequence $(i_k)_{k\in\mathbb{N}}$ by $i_k = \bar{\rho}(k \bmod n)$ for all $k \in \mathbb{N}$ such that $k \bmod n \neq 0$ and $i_k = \bar{\rho}(n)$ for all $k \in \mathbb{N}$ such that $k \bmod n = 0$.[32] Define $\hat{V} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ by $\hat{V}(f) = \lim_{\tau\to\infty} V_{\bar{\rho}}^\tau(f)$ for all $f \in \mathbb{R}^\Omega$. By assumption, we have that $\hat{V}$ is well defined and $\hat{V}(f)$ is a constant function for all $f \in \mathbb{R}^\Omega$. Since $V_{\bar{\rho}}$ is the composition of normalized, monotone, and continuous operators, so is $V_{\bar{\rho}}^\tau$ for all $\tau \in \mathbb{N}$ and, by passing to the limit, $\hat{V}$ is normalized and monotone. By assumption, we also have that

$$\hat{V}(f) = \lim_{\tau\to\infty} V_\rho^\tau(f) \qquad \forall f \in \mathbb{R}^\Omega, \forall \rho \in P.$$

Since $\hat{V}$ is normalized and monotone and $\hat{V}(f)$ is a constant function for all $f \in \mathbb{R}^\Omega$, we also have that $\hat{V}\left(\hat{V}(f)\right) = \hat{V}(f)$ for all $f \in \mathbb{R}^\Omega$, that is, $\hat{V} \circ \hat{V} = \hat{V}$. Define also $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ by $\bar{V}(f) = \hat{V}_1(f)$ for all $f \in \mathbb{R}^\Omega$. Since $\hat{V} \circ \hat{V} = \hat{V}$, it is immediate to see that $\bar{V}$ is an ex-ante expectation such that $\bar{V} \circ \hat{V} = \bar{V}$. This implies that for each $f \in \mathbb{R}^\Omega$ and for each $\rho \in P$

$$\bar{V}(V_\rho(f)) = \bar{V}\left(\hat{V}(V_\rho(f))\right) = \bar{V}\left(\lim_{\tau\to\infty} V_\rho^\tau(V_\rho(f))\right) = \bar{V}\left(\lim_{\tau\to\infty} V_\rho^{\tau+1}(f)\right) = \bar{V}\left(\hat{V}(f)\right) = \bar{V}(f). \tag{26}$$

Consider $i \in I$. Consider any permutation such that $\tilde{\rho}(1) = i$. By (26), we have that $\bar{V} \circ V_{\tilde{\rho}} \circ V_i = \bar{V} \circ V_i$. Consider the permutation $\hat{\rho}$ such that $\hat{\rho}(i') = \tilde{\rho}(i'+1)$ for all $i' \in \{1, ..., n-1\}$ and $\hat{\rho}(n) = i$. Define also $\tilde{V} = \bar{V} \circ V_i$. It follows that $\tilde{V}$ is an ex-ante expectation. Since $\bar{V} \circ V_{\tilde{\rho}} \circ V_i = \bar{V} \circ V_i$, we can conclude that

$$\tilde{V} \circ V_{\hat{\rho}} = \bar{V} \circ V_i \circ V_{\hat{\rho}} = \bar{V} \circ V_{\tilde{\rho}} \circ V_i = \bar{V} \circ V_i = \tilde{V}.$$

[32] This is the sequence

$$\bar{\rho}(1)\, \bar{\rho}(2) ... \bar{\rho}(n)\, \bar{\rho}(1)\, \bar{\rho}(2) ... \bar{\rho}(n)\, \bar{\rho}(1)\, \bar{\rho}(2) ... \bar{\rho}(n) ...$$

By induction, this implies that $\tilde{V} \circ V_{\hat{\rho}}^{\tau} = \bar{V} \circ V_i = \tilde{V}$ for all $\tau \in \mathbb{N}$. By (26) and Lemma 6 and since $\tilde{V}$ is an ex-ante expectation, $\bar{V} \circ \hat{V} = \bar{V}$, and $\tilde{V} \circ V_{\hat{\rho}}^{\tau} = \bar{V} \circ V_i = \tilde{V}$ for all $\tau \in \mathbb{N}$, we can conclude that

$$\begin{aligned}\bar{V}(f) &= \bar{V}\left(\hat{V}(f)\right) = \bar{V}\left(V_i\left(\hat{V}(f)\right)\right) = \tilde{V}\left(\hat{V}(f)\right) = \tilde{V}\left(\lim_{\tau\to\infty} V_{\hat{\rho}}^{\tau}(f)\right) \\ &= \lim_{\tau\to\infty} \tilde{V}\left(V_{\hat{\rho}}^{\tau}(f)\right) = \bar{V}(V_i(f)) \quad \forall f \in \mathbb{R}^{\Omega},\end{aligned}$$

yielding that $\bar{V} \circ V_i = \bar{V}$. Since i was arbitrarily chosen, the statement follows. ■

Proof of Lemma 1. Define

$$V_{\circ}(f) = \min_{\omega\in\Omega} f(\omega) \quad \text{and} \quad V^{\circ}(f) = \max_{\omega\in\Omega} f(\omega) \qquad \forall f \in \mathbb{R}^{\Omega}.$$

It is immediate to see that both $V_{\circ}$ and V° are ex-ante expectations. Next, fix $f \in \mathbb{R}^{\Omega}$, and observe that given

$$V_i(\omega, f) \in \left[\min_{\omega'\in\Omega} f(\omega'), \max_{\omega'\in\Omega} f(\omega')\right] \qquad \forall \omega \in \Omega, \forall i \in I,$$

we have that

$$V_{\circ}(V_i(f)) = \min_{\omega\in\Omega} V_i(\omega, f) \geq \min_{\omega'\in\Omega} f(\omega') = V_{\circ}(f) \qquad \forall i \in I$$

and

$$V^{\circ}(V_i(f)) = \max_{\omega\in\Omega} V_i(\omega, f) \leq \max_{\omega'\in\Omega} f(\omega') = V^{\circ}(f) \qquad \forall i \in I.$$

This proves that $V_{\circ}$ and V° are respectively lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$, hence that $\mathcal{V}_{\circ}$ and $\mathcal{V}^{\circ}$ are nonempty. We next show that V_* and V^* are well defined lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$. First, observe that

$$V_*(ke) = \sup_{V_{\circ}\in\mathcal{V}_{\circ}} V_{\circ}(ke) = \sup_{V_{\circ}\in\mathcal{V}_{\circ}} k = k \qquad \forall k \in \mathbb{R}$$

and that, for all $f, g \in \mathbb{R}^{\Omega}$ with $f \geq g$, we have

$$V_*(f) = \sup_{V_{\circ}\in\mathcal{V}_{\circ}} V_{\circ}(f) \geq \sup_{V_{\circ}\in\mathcal{V}_{\circ}} V_{\circ}(g) = V_*(g),$$

where the inequality follows from monotonicity of each $V_{\circ} \in \mathcal{V}_{\circ}$. With this, V_* is an ex-ante expectation. With exactly the same steps we can show that V^* is also an ex-ante expectation. Next, fix $f \in \mathbb{R}^{\Omega}$ and $V_{\circ} \in \mathcal{V}_{\circ}$. For each $i \in I$, we have

$$V_{\circ}(f) \leq V_{\circ}(V_i(f)) \leq \sup_{V'_{\circ}\in\mathcal{V}_{\circ}} V'_{\circ}(V_i(f)) = V_*(V_i(f)).$$

Given that $V_{\circ} \in \mathcal{V}_{\circ}$ was arbitrarily chosen, it follows that

$$V_*(f) = \sup_{V_{\circ}\in\mathcal{V}_{\circ}} V_{\circ}(f) \leq V_*(V_i(f))$$

proving that V_* is a lower common ex-ante expectation. With exactly the same steps we can show that V^* is also an upper common ex-ante expectation. ■

Before proving Proposition 2, we define $V_\star : \mathbb{R}^\Omega \to \mathbb{R}$ and $V^\star : \mathbb{R}^\Omega \to \mathbb{R}$ by

$$V_\star(f) = \inf_{\iota \in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f) \text{ and } V^\star(f) = \sup_{\iota \in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f) \quad \forall f \in \mathbb{R}^\Omega.$$

Clearly, we have that $V_\star \leq V^\star$.

Proof of Proposition 2. Since $V_\star$ (resp. $V^\star$) is a pointwise infimum (resp. supremum) of normalized and monotone functionals, so is $V_\star$ (resp. $V^\star$). Fix $f \in \mathbb{R}^\Omega$ and $i \in I$. Consider also an I-sequence ι'. Note that, by Proposition 1, we have

$$\begin{aligned} k_{V_i(f),\iota'} 1_\Omega &= \lim_{t\to\infty} V_{i'_t} \circ V_{i'_{t-1}} \circ ... \circ V_{i'_2} \circ V_{i'_1}(V_i(f)) \\ &= \lim_{t\to\infty} V_{i''_t} \circ V_{i''_{t-1}} \circ ... \circ V_{i''_2} \circ V_{i''_1}(f) = k_{f,\iota''} 1_\Omega \end{aligned}$$

where ι'' is the I-sequence such that $\iota''_1 = i$ and $\iota''_t = \iota'_{t-1}$ for all $t \in \mathbb{N}$. This implies that

$$k_{V_i(f),\iota'} = k_{f,\iota''} \geq \inf_{\iota \in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} k_{f,\iota} = \inf_{\iota \in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f) = V_\star(f).$$

Since ι' was arbitrarily chosen, this implies that

$$V_\star(V_i(f)) = \inf_{\iota \in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(V_i(f)) = \inf_{\iota \in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} k_{V_i(f),\iota} \geq V_\star(f),$$

proving that $V_\star \in \mathcal{V}_\circ$. Next, consider $V' \in \mathcal{V}_\circ$ and suppose by contradiction that $V'(g) > V_\star(g)$ for some $g \in \mathbb{R}^\Omega$. Since $V'(g) > V_\star(g)$, there exists an I sequence ι' such that

$$V'(g) 1_\Omega > \lim_{t\to\infty} V_{i'_t} \circ V_{i'_{t-1}} \circ ... \circ V_{i'_2} \circ V_{i'_1}(g) = k_{g,\iota} 1_\Omega.$$

Since V' is normalized and continuous at $k_{g,\iota} 1_\Omega$ by Lemma 6,

$$\begin{aligned} V'(g) &= V'(V'(g) 1_\Omega) \\ &> V'(k_{g,\iota} 1_\Omega) = V'\left(\lim_{t\to\infty} V_{i'_t} \circ V_{i'_{t-1}} \circ ... \circ V_{i'_2} \circ V_{i'_1}(g)\right) \\ &= \lim_{t\to\infty} V' \circ V_{i'_t} \circ V_{i'_{t-1}} \circ ... \circ V_{i'_2} \circ V_{i'_1}(g) \\ &\geq V'(g) \end{aligned}$$

obtaining a contradiction. This proves that $V_\star = V_*$. A completely symmetric argument shows that $V^\star = V^*$. ■

Proof of Corollary 2. (i) $\Longrightarrow$ (ii) By Proposition 1, Theorem 1, and Corollary 1, we have that $\mathbb{E}_p(f) = V_\star(f) = V^\star(f)$ for all $f \in \mathbb{R}^\Omega$. This immediately implies (ii) via Proposition

2. (ii) $\implies$ (i) Given that both V_* and V^* are SEU, there exist $p_*, p^* \in \Delta(\Omega)$ such that $V_*(f) = \mathbb{E}_{p_*}(f)$ and $V^*(f) = \mathbb{E}_{p^*}(f)$ for all $f \in \mathbb{R}^\Omega$. By Proposition 2, it follows that

$$\mathbb{E}_{p_*}(f) = V_*(f) \leq V^*(f) = \mathbb{E}_{p^*}(f) \qquad \forall f \in \mathbb{R}^\Omega.$$

Therefore, we have that $p_* = p^*$, implying that $V_* = V^*$. By Corollary 1, it follows that there exists an ex-ante expectation $\bar{V}$ such that $(\bar{V}, V_i, \Pi_i)$ is a generalized conditional expectation for all $i \in I$ and such that $\bar{V} = \mathbb{E}_{p_*} = \mathbb{E}_{p^*}$, proving (i). ■

Proof of Proposition 3. 1. It is immediate to see that V^Θ is monotone and normalized provided that $\Theta \neq \emptyset$. Fix $i \in I$, $f \in \mathbb{R}^\Omega$, and $\mu \in \Theta$. For every $\omega \in \Omega$, we have

$$\mathbb{E}_\mu[f] \geq \min_{p \in \Delta} \{\mathbb{E}_p[f] + c_{i,\omega}(p)\} = V_i(\omega, f).$$

In particular, we have $\mathbb{E}_\mu[f] \geq \mathbb{E}_\mu[V_i(f)]$. Given that μ was arbitrarily chosen, it follows that

$$V^\Theta(f) = \min_{\mu \in \Theta} \mathbb{E}_\mu[f] \geq \min_{\mu \in \Theta} \mathbb{E}_\mu[V_i(f)] = V^\Theta(V_i(f)).$$

Given that i and f were arbitrarily chosen, it follows that $V^\Theta \in \mathcal{V}^\circ$.

2. We first prove an ancillary claim.

Claim 1. *If there exists a common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$, there is no $(k_i)_{i \in I} \in \mathbb{R}^I_{++}$ and $(f_i)_{i \in I} \in (\mathbb{R}^\Omega)^I$ such that*

$$\min_{i \in I, \omega \in \Omega} \{k_i + f_i(\omega)\} \geq 0,$$
$$\sum_{i \in I} f_i(\omega) = 0 \qquad \forall \omega \in \Omega,$$
$$\min_{i \in I, \omega \in \Omega} \{V_i(\omega, k_i e + f_i) - V_i(\omega, k_i e)\} > 0.$$

Proof. Suppose by contradiction that there exist $(k_i)_{i \in I} \in \mathbb{R}^I_{++}$ and $(f_i)_{i \in I} \in (\mathbb{R}^\Omega)^I$ as in the statement. Observe that

$$\begin{aligned}
\frac{1}{n}\sum_{i \in I} k_i = \bar{V}\left(\frac{1}{n}\sum_{i \in I} k_i e + \frac{1}{n}\sum_{i \in I} f_i\right) &\geq \frac{1}{n}\sum_{i \in I} \bar{V}(k_i e + f_i) \\
&= \frac{1}{n}\sum_{i \in I} \bar{V}(V_i(k_i e + f_i)) \\
&\geq \frac{1}{n}\sum_{i \in I} \bar{V}\left(V_i(k_i e) + \left(\min_{j \in I, \omega \in \Omega}\{V_j(\omega, k_j e + f_j) - V_j(\omega, k_j e)\}\right) e\right) \\
&= \frac{1}{n}\left[\sum_{i \in I} \bar{V}(V_i(k_i e))\right] + \min_{j \in I, \omega \in \Omega}\{V_j(\omega, k_j e + f_j) - V_j(\omega, k_j e)\} \\
&= \frac{1}{n}\sum_{i \in I} k_i + \min_{j \in I, \omega \in \Omega}\{V_j(\omega, k_j e + f_j) - V_j(\omega, k_j e)\} > \frac{1}{n}\sum_{i \in I} k_i
\end{aligned}$$

yielding a contradiction. □

We are now ready to prove the statement. By the previous claim, [54, Theorem 6.2], and [15, Corollary 5] if there exists a common ex-ante preference $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$, then

$$\bigcap_{i \in I} co \{p \in \Delta(\Omega) : \exists \omega \in \Omega, p \in \partial V_i(\omega, e)\} \neq \emptyset.$$

Finally, by [51, Lemma 32], we have that $\Theta \neq \emptyset$. ■

C Appendix: Equilibrium

We consider the vector space $(\mathbb{R}^\Omega)^n$. The elements of $(\mathbb{R}^\Omega)^n$ are vectors of n components, $\mathbf{f}$, where each component i, f_i, is an element of $\mathbb{R}^\Omega$. We endow $(\mathbb{R}^\Omega)^n$ with the norm $\| \ \|_* : (\mathbb{R}^\Omega)^n \to [0, \infty)$ defined by

$$\|\mathbf{f}\|_* = \sup_{i \in I} \sup_{\omega \in \Omega} |f_i(\omega)| = \sup_{i \in I} \|f_i\|_\infty \qquad \forall \mathbf{f} \in (\mathbb{R}^\Omega)^n.$$

We say that an interim expectation V_i is *nonexpansive* if and only if

$$\|V_i(f) - V_i(g)\|_\infty \leq \|f - g\|_\infty$$

for all $f, g \in \mathbb{R}^\Omega$. Recall that variational preferences $V_i(\omega, \cdot)$ as in equation (13) are concave and translation invariant, hence $V_i(\omega, \cdot)$ is concave and translation invariant. Therefore, by [14, p. 346], this assumption implies that each $V_i(\omega, \cdot)$ is nonexpansive.

Fix $\hat{f} \in \mathbb{R}^\Omega$ and let W be an $n \times n$ stochastic matrix and assume that $|I| \geq 2$. Recall that, for each $\beta \in (0, 1]$, we define $S_\beta : (\mathbb{R}^\Omega)^n \to (\mathbb{R}^\Omega)^n$ by

$$S_{\beta,i}(\mathbf{f}) = V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^{n} w_{il} f_l\right) \qquad \forall \mathbf{f} \in (\mathbb{R}^\Omega)^n, \forall i \in I, \tag{27}$$

where $S_{\beta,i}(\mathbf{f})$ is the i-th component of $S_\beta(\mathbf{f})$ for all $\mathbf{f} \in (\mathbb{R}^\Omega)^n$. Also, define $\hat{\mathbf{f}} \in (\mathbb{R}^\Omega)^n$ as $\hat{f}_i = \hat{f}$ for all $i \in I$. In addition, observe that S_1 is normalized, monotone, translation invariant, and concave.

Lemma 8. *If $\beta \in (0, 1]$ and V_i is nonexpansive for all $i \in I$, then S_β is a β-contraction. In particular, for each $\beta \in (0, 1)$, there exists a unique $\sigma^\beta \in (\mathbb{R}^\Omega)^n$ such that*

$$S_\beta^\tau\left(\hat{\mathbf{f}}\right) \overset{\| \|_*}{\to} \sigma^\beta, \ S_\beta(\sigma^\beta) = \sigma^\beta, \text{ and } \|\sigma^\beta\|_* \leq \left\|\hat{f}\right\|_\infty.$$

Lemma 9. *Let W be strongly connected, $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of full support interim expectations such that $\Pi_{\sup} = \{\Omega\}$, and $\mathbf{f} \in (\mathbb{R}^\Omega)^n$. The following statements are equivalent:*

(i) $S_1(\mathbf{f}) = \mathbf{f}$;

(ii) There exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m1_\Omega$ for all $i, i' \in I$.

Recall that $\bar{n} = |\Omega|$. For every monotone operator $R : \left(\mathbb{R}^\Omega\right)^n \to \left(\mathbb{R}^\Omega\right)^n$ define the adjacency matrices $\underline{A}(R), \bar{A}(R) \in \{0,1\}^{(n\times\bar{n})\times(n\times\bar{n})}$ as follows. For every $i, j \in I$ and $\omega, \omega' \in \Omega$, we set $\underline{a}_{(i,\omega)(j,\omega')}(R) = 1$ if and only if there exists $\varepsilon_{(i,\omega)(j,\omega')} > 0$ such that for each $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$,

$$R_{i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - R_{i,\omega}(\mathbf{f}) \geq \varepsilon_{(i,\omega)(j,\omega')}\delta,$$

and we set $\bar{a}_{(i,\omega)(j,\omega')}(R) = 1$ if and only if there exist $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$ such that

$$R_{i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - R_{i,\omega}(\mathbf{f}) > 0.$$

Moreover, we say that a class of indices Z, $\emptyset \neq Z \subseteq I \times \Omega$, is closed and strongly connected with respect to an adjacency matrix $A \in \{0,1\}^{(n\times\bar{n})\times(n\times\bar{n})}$ if and only if (i) for each $z, z' \in Z$ there exists a path $\{z_l\}_{l=1}^K \subseteq Z$ such that $a_{z_l z_{l+1}} = 1$ for all $l \in \{1, ..., K-1\}$, $z_1 = z$ and $z_K = z'$; (ii) for each $z \in Z$, $a_{zz'} = 1$ implies $z' \in Z$.

Lemma 10. *There exists a unique class of indices Z, $\emptyset \neq Z \subseteq I \times \Omega$, that is closed and strongly connected with respect to $\underline{A}(S_1)$ and, in addition, every row of $\underline{A}(S_1)$ is not null.*

Proof of Proposition 4. By Lemma 8, it follows that, for every $\beta \in (0,1)$, S_β is a contraction with respect to the supnorm and it admits a unique fixed point $\sigma^\beta \in \Sigma$. With this, the result follows by Lemma 10 and applying [10, Proposition ?] with $T = S_1$. ■

Next, let $\mathcal{W} \subseteq \mathbb{R}_+^{(n\times\bar{n})\times(n\times\bar{n})}$ denote the set of stochastic matrices over $I \times \Omega$ and define

$$\partial S_1(0) = \left\{\hat{W} \in \mathcal{W} : \forall (i,\omega) \in I \times \Omega, w_{i,\omega} \in \partial S_{1,i,\omega}(0)\right\},$$

where $\partial S_{1,i,\omega}(0) \subseteq \Delta(I \times \Omega)$ is the superdifferential of the concave functional $S_{1,i,\omega}$ at 0. In particular, the fact that $\partial S_1(0) \subseteq \mathcal{W}$ easily follows from the fact that S_1 is normalized, monotone, and translation invariant.

Lemma 11. *We have*

$$\partial S_1(0) = \{W^q \in \mathcal{W} : q \in \partial V(0)\}.$$

Lemma 12. *Let $\{(V_i, \Pi_i)\}_{i\in I}$ be a collection of full support interim expectations such that $\Pi_{\sup} = \{\Omega\}$. The following facts are true*

1. *If V_i is concave for all $i \in I$, then V_* is concave. If in addition V_i is positive homogeneous for all $i \in I$, then V_* is positive homogeneous.*

2. *If V_i is $\Pi_{\inf}$-affine for all $i \in I$, then*

$$V_*((1-\lambda)h + \lambda g) \geq (1-\lambda)V_*(h) + \lambda V_*(g)$$

and

$$V^*\left(\left(1-\lambda\right)h+\lambda g\right)\le\left(1-\lambda\right)V^*\left(h\right)+\lambda V^*\left(g\right)$$

for all $\lambda\in(0,1)$ *and for all* $g,h\in\mathbb{R}^\Omega$ *where* g *is* $\Pi_{\inf}$*-measurable.*

Lemma 13. *Let* $\{(V_i,\Pi_i)\}_{i\in I}$ *be a collection of full support interim expectations such that* $\Pi_{\sup}=\{\Omega\}$*. If* V_i *is concave for all* $i\in I$*, then for each* $\beta\in(0,1)$

$$V_*\left(S^\tau_{\beta,i}\left(\hat{\mathbf{f}}\right)\right)\ge V_*\left(\hat{f}\right)\qquad\forall i\in I,\forall\tau\in\mathbb{N},$$

where $\hat{\mathbf{f}}\in\left(\mathbb{R}^\Omega\right)^n$ *is such that* $\hat{f}_i=\hat{f}$ *for all* $i\in I$*. Moreover, if* V_i *is nonexpansive for all* $i\in I$*, then* $V_*\left(\sigma_i^\beta\right)\ge V_*\left(\hat{f}\right)$ *for all* $i\in I$ *and for all* $\beta\in(0,1)$*.*

Lemma 14. *If the collection of interim expectations* $\{(V_i,\Pi_i)\}_{i\in I}$ *has full support, is such that* $\Pi_{\sup}=\{\Omega\}$*, is variational and* $\Theta\neq\emptyset$*, then* $\Theta\subseteq$ *int* $(\Delta(\Omega))$*.*

Proof of Lemma 2. Fix $q\in\partial V(0)$ and observe that, by Lemma 11, we have $W^q\in\partial S_1(0)$. By Lemma 10, there exists a unique class of indices Z, $\emptyset\neq Z\subseteq I\times\Omega$, that is closed and strongly connected with respect to $\underline{A}(S_1)$ and, in addition, every row of $\underline{A}(S_1)$ is not null. Given that S_1 is concave, it follows easily from the definition of $\partial S_1(0)$ that, for each $\hat{W}\in\partial S_1(0)$, Z is the unique closed and strongly connected with respect to $\underline{A}\left(\hat{W}\right)$.[33] In particular, Z is the unique closed and strongly connected with respect to $\underline{A}(W^q)$. Next, observe that, for each $\gamma\in\Delta(I\times\Omega)$, we have $\gamma=\gamma\left(\frac{I+W^q}{2}\right)$ if and only if $\gamma=\gamma W^q$. In addition, given that $\underline{A}\left(\frac{I+W^q}{2}\right)\ge\underline{A}(W^q)$, it follows by [48, Corollaries 8.1 and 8.2] and [67, Theorem 2.2.5] that there exists a unique $\gamma^q\in\Delta(I\times\Omega)$ such that $\gamma^q=\gamma^q\left(\frac{I+W^q}{2}\right)$. By the previous claim, γ^q is also the unique probability vector such that $\gamma^q=\gamma^qW^q$. Given that $q\in\partial V(0)$ was arbitrarily chosen, the statement follows. ■

Let $s\in$ *int* $(\Delta(I))$ denote the unique probability vector that satisfies $s=sW$, where uniqueness and strict positivity follow from the fact that W is strongly connected.

Proof of Theorem 2. First, recall that S_1 is normalized, monotone, translation invariant, concave and, by Lemma 9, $S_1(\mathbf{f})=\mathbf{f}$ if and only if there exists $m\in\mathbb{R}$ such that $f_i=f_{i'}=m1_\Omega$ for all $i,i'\in I$. With this, by [10, Corollary ?], we have that

$$\lim_{\beta\to1}\sigma_i^\beta(\omega)=\min_{\left\{\eta\in\Delta(I\times\Omega):\exists\hat{W}\in\partial S_1(0),\eta=\eta\hat{W}\right\}}\sum_{(i,\omega)\in I\times\Omega}\eta_{i,\omega}\hat{f}(\omega),$$

for all $(i,\omega)\in I\times\Omega$. By Lemmas 11 and 2, it follows that

$$\lim_{\beta\to1}\sigma_i^\beta(\omega)=\min_{q\in\partial V(0)}\sum_{(i,\omega)\in I\times\Omega}\gamma^q_{i,\omega}\hat{f}(\omega).$$

[33] Here, with an abuse of notation we identify the linear operator induced by the matrix $\hat{W}$ with $\hat{W}$ itself.

Next, fix $q \in \partial V(0)$ and observe that

$$\sum_{(i,\omega)\in I\times\Omega} \gamma^q_{i,\omega} \mathbb{E}_{q_{i,\omega}}\left[\hat{f}\right] = \sum_{(i,\omega)\in I\times\Omega} \gamma^q_{i,\omega} \mathbb{E}_{q_{i,\omega}}\left[\sum_{j=1}^{n} w_{ij}\hat{f}\right] = \sum_{(i,\omega)\in I\times\Omega} \gamma^q_{i,\omega}\left[\sum_{(j,\omega')\in I\times\Omega} q_{i,\omega}(\omega') w_{ij}\hat{f}(\omega)\right]$$

$$= \sum_{(i,\omega)\in I\times\Omega} \gamma^q_{i,\omega}\left[\sum_{(j,\omega')\in I\times\Omega} w^q_{(i,\omega)(j,\omega')}\hat{f}(\omega)\right] = \sum_{(i,\omega)\in I\times\Omega} \gamma^q_{i,\omega}\hat{f}(\omega),$$

where the third equality follows from the definition of W^q and the last equality follows from the fact that $\gamma^q = \gamma^q W^q$. This proves the equality in (18).

We now prove the left inequality in (18). Fix $\bar{\imath} \in I$. By the previous part, we know that there exists $m \in \mathbb{R}$ such that $\lim_{\beta\to 1}\sigma^\beta_i(\omega) = m$ for all $(i,\omega) \in I\times\Omega$. By contradiction, assume that $V_*\left(\hat{f}\right) > m$. By Lemma 13 and since V_* is a nonexpansive ex-ante expectation, we can conclude that

$$m = V_*(m1_\Omega) = \lim_{\beta\to 1} V_*\left(\sigma^\beta_i\right) \geq V_*\left(\hat{f}\right) > m$$

yielding a contradiction.

Next, we prove the right inequality in (18). First, observe that, if $\Theta = \emptyset$, then $\inf_{\mu\in\Theta}\mathbb{E}_\mu\left[\hat{f}\right] = \infty$ and the right inequality in (18) trivially holds. Next, assume that $\Theta \neq \emptyset$ and fix $\mu \in \Theta$. In particular, we have $\mu \in int(\Delta(\Omega))$ by Lemma 14, hence $p_{\mu,i}(\omega,\cdot)$ is uniquely defined for all $(i,\omega) \in I\times\Omega$. By definition of Θ and equation (16), we have that $p_{\mu,i}(\omega,\cdot) \in \partial V_i(\omega, 0)$ for all $(i,\omega) \in I\times\Omega$. With this, define $q^\mu \in \Delta(\Omega)^{I\times\Omega}$ as $q^\mu_{i,\omega} = p_{\mu,i}(\omega,\cdot)$ for all $(i,\omega) \in I\times\Omega$ and observe that $q^\mu \in \partial V(0)$ by construction. With this, by Lemma 2 there exists a unique probability vector $\gamma^{q^\mu} \in \Delta(I\times\Omega)$ such that $\gamma^{q^\mu} = \gamma^{q^\mu}W^{q^\mu}$. Now, define $\gamma^\mu \in \Delta(I\times\Omega)$ as $\gamma^\mu_{i,\omega} = s_i\mu(\omega)$ for all $(i,\omega) \in I\times\Omega$. Observe that, for all $(i,\omega) \in I\times\Omega$, we have

$$\sum_{(j,\omega')\in I\times\Omega} \gamma^\mu_{j,\omega'} w^{q^\mu}_{(j,\omega')(i,\omega)} = \sum_{(j,\omega')\in I\times\Omega} s_j\mu(\omega') w_{ji} q^\mu_{j,\omega'}(\omega) = \sum_{j\in I} s_j w_{ji} \sum_{\omega'\in\Omega} \mu(\omega') p_{\mu,j}(\omega',\omega)$$

$$= \mu(\omega)\sum_{j\in I} s_j w_{ji} = \mu(\omega) s_i = \gamma^\mu_{i,\omega}.$$

This show that $\gamma^\mu = \gamma^\mu W^{q^\mu}$, proving that $\gamma^\mu = \gamma^{q^\mu}$. This in turn yields the right inequality in (18).

The second part of the statement directly follows by the first part and by Corollary 1 (left inequality) and Proposition 3 (right inequality). ■

Proof of Corollary 4. The first part of the statement follows from Theorem 2 and from the fact that, by assumption, $\partial V(0) = \{q^*\}$. Next, assume that $\Theta \neq \emptyset$. Observe that, for each $\mu \in \Theta$, we have that, by Lemma 14, $p_{\mu,i}(\omega,\cdot) = q^*_{i,\omega}$ is uniquely defined for all $(i,\omega) \in I\times\Omega$. Assume by contradiction that there exist $\mu, \mu' \in \Theta$ with $\mu \neq \mu'$ and consider the collection $\left\{\left(\mathbb{E}_{q^*_i}, \Pi_i\right)\right\}_{i\in I}$ of interim expectations. This collection has full support by Lemma

14. Therefore, $\{(\mathbb{E}_{q_i^*}, \Pi_i)\}_{i\in I}$ exhibits convergence to a deterministic limit by Proposition 1. In particular, both $\mathbb{E}_\mu$ and $\mathbb{E}_{\mu'}$ are common ex-ante expectations for $\{(\mathbb{E}_{q_i^*}, \Pi_i)\}_{i\in I}$ by construction, yielding a contradiction with Theorem 1. Therefore, we obtain $\Theta = \{\mu^*\}$ for some $\mu^* \in \Delta(\Omega)$. Moreover, by Lemma 2, there exists a unique probability vector $\gamma^{q^*} \in \Delta(I \times \Omega)$ such that $\gamma^{q^*} = \gamma^{q^*} W^{q^*}$. Now, for each $(i, \omega) \in I \times \Omega$, define $\gamma^{\mu^*} \in \Delta(I \times \Omega)$ as $\gamma^{\mu^*}_{i,\omega} = s_i \mu^*(\omega)$ and observe that

$$\sum_{(j,\omega')\in I\times\Omega} \gamma^{\mu^*}_{i,\omega} w^{q^*}_{(j,\omega')(i,\omega)} = \sum_{(j,\omega')\in I\times\Omega} s_j \mu^*(\omega') w_{ji} q^*_{j,\omega'}(\omega) = \sum_{j\in I} s_j w_{ji} \sum_{\omega'\in\Omega} \mu^*(\omega') p_{\mu^*,j}(\omega', \omega)$$
$$= \mu^*(\omega) \sum_{j\in I} s_j w_{ji} = \mu^*(\omega) s_i = \gamma^{\mu^*}_{i,\omega}.$$

This show that $\gamma^{\mu^*} = \gamma^{\mu^*} W^{q^*}$, proving that $\gamma^{q^*} = \gamma^{\mu^*}$. Finally, we have

$$\sum_{(i,\omega)\in I\times\Omega} \gamma^{q^*}_{i,\omega} \mathbb{E}_{q^*_{i,\omega}}\left[\hat{f}\right] = \sum_{(i,\omega)\in I\times\Omega} \gamma^{\mu^*}_{i,\omega} \mathbb{E}_{q^*_{i,\omega}}\left[\hat{f}\right] = \sum_{(i,\omega)\in I\times\Omega} s_i \mu^*(\omega) \mathbb{E}_{p_{\mu^*,i}(\omega,\cdot)}\left[\hat{f}\right] = \mathbb{E}_{\mu^*}\left[\hat{f}\right],$$

proving the second part of the statement. ■

Proof of Corollary 5. By Lemma 12, we have that V_* is a maxmin ex-ante expectation. By Corollary 1, it follows that $\bar{V}$ is a maxmin ex-ante expectation as well. Let $\bar{C} \subseteq \Delta(\Omega)$ denote the set of probabilities such that $\bar{V}(f) = \min_{\mu\in\bar{C}} \mathbb{E}_\mu[f]$ for all $f \in \mathbb{R}^\Omega$. Fix $\mu \in \bar{C}$, $i \in I$, and $\omega \in \Omega$. Since $\bar{V}$ and V_i satisfy dynamic consistency, it follows by [22] that $\bar{C}$ is Π_i-rectangular. With this, we have that $p_{\mu,i}(\omega, \cdot) \in C_{i,\omega}$ where $C_{i,\omega}$ is the set of probabilities such that $V_i(\omega, f) = \min_{p\in C_{i,\omega}} \mathbb{E}_p[f]$ for all $f \in \mathbb{R}^\Omega$. With this, it follows that $c_{i,\omega}(p_{\mu,i}(\omega, \cdot)) = 0$. Given that $i \in I$ and $\omega \in \Omega$ were arbitrarily chosen, it follows that $\mu \in \Theta$. This in turn proves that $\bar{C} \subseteq \Theta$, hence that $\bar{V}\left(\hat{f}\right) \leq V^\Theta\left(\hat{f}\right)$. Finally, the result follows by the second part of Theorem 2. ■

Proof of Proposition 5. Fix $\beta \in (0, 1)$. By Lemma 8, we have that $\sigma^\beta_i = S_{\beta,i}(\sigma^\beta) = V_i\left((1-\beta)\hat{f} + \beta\sum_{l=1}^n w_{il}\sigma^\beta_l\right)$ for all $i \in I$. This implies that σ^β_i is Π_i-measurable and, in particular, $\Pi_{\inf}$-measurable for all $i \in I$. Since V_i is $\Pi_{\inf}$-affine, this implies that

$$\sigma^\beta_i = V_i\left((1-\beta)\hat{f} + \beta\sum_{l=1}^n w_{il}\sigma^\beta_l\right) = (1-\beta)V_i\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_i\left(\sigma^\beta_l\right) \quad \forall i \in I. \tag{28}$$

By Lemma 12, since V_i is $\Pi_{\inf}$-affine for every $i \in I$, we have that V_* is such that

$$V_*((1-\alpha)h + \alpha g) \geq (1-\alpha)V_*(h) + \alpha V_*(g) \tag{29}$$

and V^* is such that

$$V^*((1-\alpha)h + \alpha g) \leq (1-\alpha)V^*(h) + \alpha V^*(g) \tag{30}$$

for all $\alpha \in (0,1)$ and for all $g, h \in \mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable. By (28), (29), (30) and since each $V_i\left(\hat{f}\right)$ is Π_i-measurable, hence $\Pi_{\inf}$-measurable, we have that, for each $i \in I$,

$$\begin{aligned} V_*\left(\sigma_i^\beta\right) &= V_*\left((1-\beta)V_i\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_i\left(\sigma_l^\beta\right)\right) \\ &\geq (1-\beta)V_*\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_*\left(\sigma_l^\beta\right), \end{aligned}$$

and

$$\begin{aligned} V^*\left(\sigma_i^\beta\right) &= V^*\left((1-\beta)V_i\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_i\left(\sigma_l^\beta\right)\right) \\ &\leq (1-\beta)V^*\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V^*\left(\sigma_l^\beta\right). \end{aligned}$$

Define $x_* \in \mathbb{R}^n$ to be such that $x_{*i} = V_*\left(\sigma_i^\beta\right) - V_*\left(\hat{f}\right)$ for all $i \in I$. We can conclude that $x_* \geq \beta W x_*$. Assume by contradiction that $x_{*i'} = \min_{i\in I} x_{*i} < 0$. Since W is a stochastic matrix, we have $x_{*i'} \leq (Wx_*)_{i'}$. Since $\beta \in (0,1)$ was arbitrarily chosen, it follows that $x_{*i'} < \beta (Wx_*)_{i'}$, yielding the contradiction

$$x_{*i'} < \beta (Wx_*)_{i'} \leq x_{*i'}.$$

Therefore, we must have $V_*\left(\sigma_i^\beta\right) \geq V_*\left(\hat{f}\right)$ for all $i \in I$ and for all $\beta \in (0,1)$. By taking the limit for $\beta \to 1$ in the previous inequality and by Lemma 6 and Theorem 2, we get $\lim_{\beta\to 1}\sigma_i^\beta(\omega) \geq V_*\left(\hat{f}\right)$ for all $\omega \in \Omega$ and for all $i \in I$. Analogous steps yield that $\lim_{\beta\to 1}\sigma_i^\beta(\omega) \leq V^*\left(\hat{f}\right)$ for all $\omega \in \Omega$ and for all $i \in I$. The second part of the statement follows from the first part and Corollary 1. ■

C.1 Table of bounds

Preferences	Lower bound	Upper bound
SEU CP μ	$\mathbb{E}_\mu\left[\hat{f}\right]$	$\mathbb{E}_\mu\left[\hat{f}\right]$
MEU, FBU, RECT C	$\min_{\mu\in C}\mathbb{E}_\mu\left[\hat{f}\right]$	$\min_{\mu\in C}\mathbb{E}_\mu\left[\hat{f}\right]$
MEU, FBU, $(C_i)_{i\in I}$	$V_*\left(\hat{f}\right)$	$\inf_{\mu\in\cap_{i\in I}C_i}\mathbb{E}_\mu\left[\hat{f}\right]$
MEU, PROXY, ν	$\min_{\mu\in core(\nu)}\mathbb{E}_\mu\left[\hat{f}\right]$	$\min_{\mu\in\cap_{i\in I}core_i(\nu)}\mathbb{E}_\mu\left[\hat{f}\right]$
HS, CP μ, $(\lambda_i)_{i\in I}$	$\min_{p\in\Delta(\Omega)}\left\{\mathbb{E}_p\left[\hat{f}\right] + \min_{i\in I}\lambda_i R(p\|\|\mu)\right\}$	$\mathbb{E}_\mu\left[\hat{f}\right]$
CHMM, FBU, RECT, C, λ	$\min_{p\in\Delta(\Omega)}\left\{\mathbb{E}_p\left[\hat{f}\right] + \lambda\min_{\mu\in C}R(p\|\|\mu)\right\}$	$\min_{\mu\in C}\mathbb{E}_\mu\left[\hat{f}\right]$

D Appendix: No trade

Let $\mathcal{M}(F)$ denote the set of countably additive measures over F, and as $\mathcal{M}_0(F)$ the subset of measures in $\mathcal{M}(F)$ with finite support, and let $\Delta_0(F) \subseteq \mathcal{M}(F)$ be the set of finite support probability measures. Define

$$\mathcal{V} = \left\{ \bar{V} \in C(F) : \bar{V} \text{ is normalized, monotone, translation invariant} \right\}.$$

Given that F is compact and each $\bar{V} \in \mathcal{V}$ is 1-Lipschitz continuous, (see, [14, p. 346]) it follows by Arzelà-Ascoli Theorem that $\mathcal{V}$ is compact in the topology of uniform convergence.

Consider the interim expectations $\{(V_i, \Pi_i)\}_{i \in I}$ and, for every $i \in I$, define

$$\mathcal{V}_i = \left\{ \bar{V} \in \mathcal{V} : \forall f \in F, \bar{V}(f) = \bar{V}(V_i(f)) \right\}.$$

Proof of Proposition 6. Suppose there exists $\bar{V} \in \mathcal{V}_1 \cap \mathcal{V}_2$. But then

$$\bar{V}(f) = \bar{V}(V_1(f)) > r > \bar{V}(V_2(f)) = \bar{V}(f)$$

yields a contradiction. ■

Lemma 15. *Let $\{(V_i, \Pi_i)\}_{i \in \{1,2\}}$ be a set of full support translation invariant interim expectations such that $\Pi_{\sup} = \{\Omega\}$. If there is no $\nu \in \mathcal{M}_0(F)$ such that*

$$\int_F V_1(\omega, f)\, d\nu(f) > 0 > \int_F V_2(\omega, f)\, d\nu(f) \qquad \forall \omega \in \Omega,$$

then there exists a translation invariant ex-ante expectation $\bar{V}$ such that $\left(\bar{V}, V_i, \Pi_i\right)$ is a generalized conditional expectation for all $i \in \{1, 2\}$.

Proof of Theorem 3. We show that if there is $\nu \in \mathcal{M}_0(F)$ such that

$$\int_F V_1(\omega, f)\, d\nu(f) > 0 > \int_F V_2(\omega, f)\, d\nu(f) \qquad \forall \omega \in \Omega, \tag{31}$$

then there exists a two populations endowment economy $\{(\chi_i, V_i, \Pi_i)\}_{i \in \{1,2\}}$ that is interim Pareto improvable. By Lemma 15, this proves the statement. We define the endowment economy in the following way. Let ν^+ and ν^- be the positive and negative components of ν, and enumerate their respective finite supports as $S^+ = (f_1, ..., f_k)$ and $S^- = (g_1, ..., g_l)$. Observe that at least one between $\nu^+(F)$ and $\nu^-(F)$ is strictly larger than 0. We prove the case in which $\nu^-(F) > 0$, the proof of the other case being analogous. We construct the endowment economy in the following way. There are two cases $\frac{\nu^+(F)}{\nu^-(F)} \leq 1$ and $\frac{\nu^+(F)}{\nu^-(F)} > 1$. In the first case, define $(c_1, ..., c_{l-1})$ as

$$c_1 = \frac{\nu^-(g_1)}{\nu^-(F)} \text{ and } c_{i+1} = c_i + \frac{\nu^-(g_{i+1})}{\nu^-(F)} \text{ for all } i \in \{2, ..., l-1\}$$

and $(d_1, ..., d_{k-1})$ as

$$d_1 = \frac{\nu^+(g_1)\,\nu^+(F)}{\nu^-(F)} \text{ and } d_{i+1} = d_i + \frac{\nu^+(g_{i+1})\,\nu^+(F)}{\nu^-(F)} \text{ for all } i \in \{2, ..., k-1\}.$$

Let

$$\chi_1(x) = g_i \qquad x \in [c_{i-1}, c_i), i \in \{1, ..., l\}$$

and

$$\chi_2(x) = \begin{cases} f_i & x \in [d_{i-1}, d_i), i \in \{1, ..., k\} \\ 0 & x \in [d_k, 1) \end{cases}.$$

We know show that the two populations endowment economy $\{(\chi_i, V_i, \Pi_i)\}_{i\in\{1,2\}}$ is interim Pareto improvable. Indeed, if let $(\chi_1', \chi_2') \in F^{[0,1]} \times F^{[0,1]}$ be given by

$$\chi_1'(x) = \chi_2(x) + V_1(\cdot, \chi_1(x)) - V_1(\cdot, \chi_2(x)) + \hat{f}(\cdot)$$

where

$$\hat{f}(\omega) = \int_F V_1(\omega, f)\, d\frac{\nu(f)}{\nu^-(F)} - \int_F V_2(\omega, f)\, \frac{\nu(f)}{\nu^-(F)}.$$

Notice that by construction (χ_1', χ_2') satisfies market clearing, and it is an interim Pareto improvement if $\hat{f}(\omega) > 0$ for all $\omega \in \Omega$. But by equation (31), this is indeed the case. ∎

E Appendix: An axiomatic foundation

In this section, we consider a single decision maker with preferences over monetary acts or utility profiles, that is, $\mathbb{R}^\Omega$. We model the decision maker preferences via a binary relation $\succsim$ on $\mathbb{R}^\Omega$. We next list four important properties:

A 1 (Weak order)**.** *The binary relation $\succsim$ is complete and transitive.*

A 2 (Certainty equivalent)**.** *For each $f \in \mathbb{R}^\Omega$ there exists $k \in \mathbb{R}$ such that $f \sim k1_\Omega$.*

A 3 (Continuity)**.** *For each $f, g, h \in \mathbb{R}^\Omega$ the sets*

$$\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\} \text{ and } \{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$$

are closed.

A 4 (Monotonicity)**.** *For each $f, g \in \mathbb{R}^\Omega$ and for each $h, k \in \mathbb{R}$*

$$f \geq g \implies f \succsim g$$

and

$$h > k \implies h1_\Omega \succ k1_\Omega.$$

On the one hand, transitivity and monotonicity are common assumptions of rationality while completeness reflects the burden of choice the decision maker faces. On the other hand, continuity is a technical assumption which will allow us to represent preferences through a continuous utility function. The assumption of certainty equivalent shares both features. It allows us to show that preferences admit a utility function, possibly not continuous, yet it takes a clear behavioral interpretation: the decision maker for each random variable admits an equivalent amount which received with certainty makes her indifferent to the random prospect. The above axioms define the following two nested class of preferences.

Definition 10. *Let $\succsim$ be a binary relation on $\mathbb{R}^\Omega$. We say that $\succsim$ is a* rational preference *if and only if it satisfies weak order, certainty equivalent, and monotonicity. We say that $\succsim$ is a* continuous rational preference *if and only if it satisfies weak order, continuity, and monotonicity.*

It is easy to show that continuous rational preferences are rational preferences. Continuous rational preferences were studied by Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [11]. The next result is a version of their Proposition 1.

Proposition 8. *Let $\succsim$ be a binary relation on $\mathbb{R}^\Omega$. The following statements are equivalent:*

(i) $\succsim$ is a rational preference;

(ii) There exists a normalized and monotone functional $\tilde{V} : \mathbb{R}^\Omega \to \mathbb{R}$ such that

$$f \succsim g \iff \tilde{V}(f) \geq \tilde{V}(g). \tag{32}$$

Moreover, we have that:

1. *The functional $\tilde{V}$ is continuous if and only if $\succsim$ is a continuous rational preference.*

2. *The functional $\tilde{V}$ is the unique normalized functional satisfying (32).*

Proof. (ii) implies (i). It is routine.

(i) implies (ii). Since $\succsim$ satisfies certainty equivalent, for each $f \in \mathbb{R}^\Omega$ define k_f to be such that $k_f 1_\Omega \sim f$. Since $\succsim$ satisfies weak order and monotonicity, we have that k_f is unique. Define $\tilde{V} : \mathbb{R}^\Omega \to \mathbb{R}$ by $\tilde{V}(f) = k_f$ for all $f \in \mathbb{R}^\Omega$. Since $\succsim$ satisfies weak order and monotonicity, we have that

$$f \succsim g \iff k_f 1_\Omega \succsim k_g 1_\Omega \iff k_f \geq k_g \iff \tilde{V}(f) \geq \tilde{V}(g),$$

proving (32). Clearly, if $f = k1_\Omega$ for some $k \in \mathbb{R}$, we have that $\tilde{V}(k1_\Omega) = \tilde{V}(f) = k_f = k$, proving that $\tilde{V}$ is normalized. Finally, since $\succsim$ satisfies monotonicity, if $f \geq g$, then $f \succsim g$ and $\tilde{V}(f) \geq \tilde{V}(g)$, proving that $\tilde{V}$ is monotone.

1. The "Only if" is routine. "If". Since $\succsim$ satisfies weak order, continuity, and monotonicity, we have that $\succsim$ satisfies certainty equivalent. It follows that $\tilde{V}$ as defined above represents $\succsim$. Since $\succsim$ satisfies continuity, it follows that for each $f, g \in \mathbb{R}^\Omega$ and for each $c \in \mathbb{R}$

$$\left\{\lambda \in [0,1] : \tilde{V}(\lambda f + (1-\lambda) g) \leq c\right\} = \left\{\lambda \in [0,1] : \tilde{V}(\lambda f + (1-\lambda) g) \leq \tilde{V}(c1_\Omega)\right\}$$
$$= \{\lambda \in [0,1] : c1_\Omega \succsim \lambda f + (1-\lambda) g\}$$

where the latter set is closed. By [13, Lemma 42], we have that $\tilde{V}$ is lower semicontinuous. By [13, Appendix A.3], upper semicontinuity follows similarly.

2. Assume that $\hat{V}$ is normalized and satisfies (32). We have that for each $f \in \mathbb{R}^\Omega$

$$\hat{V}(f) = \hat{V}\left(\hat{V}(f) 1_\Omega\right) \implies f \sim \hat{V}(f) 1_\Omega \implies \tilde{V}(f) = \tilde{V}\left(\hat{V}(f) 1_\Omega\right) = \hat{V}(f),$$

proving that $\hat{V} = \tilde{V}$. ■

We can now discuss conditional preferences. We assume that there are two periods 0 and 1. At 0, the decision maker has no information and has also preferences over $\mathbb{R}^\Omega$. At time 1, the decision maker observes an event E from a partition Π of Ω and updates her preferences. We model this by a pair $\left(\succsim, \{\succsim_\omega\}_{\omega \in \Omega}\right)$. Given $\omega \in \Omega$, as before, we denote by $\Pi(\omega)$ the only element of Π which contains ω. We consider the following assumptions.

A 5 (Rationality)**.** *The binary relation $\succsim$ is a rational preference and $\succsim_\omega$ is a continuous rational preference for all $\omega \in \Omega$.*

A 6 (Conditional preferences)**.** *For each $\omega, \omega' \in \Omega$*

$$\Pi(\omega) = \Pi(\omega') \implies \quad \succsim_\omega = \succsim_{\omega'}.$$

We thus assume that original and updated preferences are rational, where the latter are also assumed to be continuous. At the same time, we assume that if two states belong to the same event, then the corresponding updated preferences must be the same, incorporating exactly nothing more than the information embedded in Π.

A 7 (Consequentialism)**.** *For each $f \in \mathbb{R}^\Omega$ and for each $\omega \in \Omega$*

$$f 1_{\Pi(\omega)} + h 1_{\Pi(\omega)^c} \sim_\omega f \qquad \forall h \in \mathbb{R}^\Omega.$$

A 8 (Dynamic consistency)**.** *For each $f, g \in \mathbb{R}^\Omega$*

$$f \succsim_\omega g \quad \forall \omega \in \Omega \implies f \succsim g.$$

On the one hand, consequentialism imposes that updated preferences over are only influenced by the states that are still relevant/possible. On the other hand, dynamic consistency is a form of monotonicity and it states that if interim f is weakly better than g, no matter which event realized in Π, then f is weakly better than g also at time 0.

Definition 11. *Let* $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ *be a collection of binary relations on* $\mathbb{R}^\Omega$*. We say that* $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ *is a* dynamic rational preference *if and only if it satisfies the properties of rationality, conditional preferences, consequentialism, and dynamic consistency.*

The next result provides a behavioral foundation for generalized conditional expectations.

Proposition 9. *Let* $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ *be a collection of binary relations on* $\mathbb{R}^\Omega$*. The following statements are equivalent:*

(i) $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ *is a dynamic rational preference;*

(ii) There exists two functions $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ *and* $V : \Omega \times \mathbb{R}^\Omega \to \mathbb{R}$ *such that* $\left(\bar{V}, V, \Pi\right)$ *is a generalized conditional expectation and for each* $\omega \in \Omega$

$$f \succsim_\omega g \iff V(\omega, f) \geq V(\omega, g) \text{ and } f \succsim g \iff \bar{V}(f) \geq \bar{V}(g).$$

Proof. (ii) implies (i). It is routine.

(i) implies (ii). By Proposition 8 and since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies rationality, we have that there exists a normalized and monotone function $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ and a collection of normalized, monotone, and continuous functions $\{V_\omega\}_{\omega\in\Omega}$ from $\mathbb{R}^\Omega$ to $\mathbb{R}$ such that $\bar{V}$ represents $\succsim$ and V_ω represents $\succsim_\omega$ for all $\omega \in \Omega$. Define $V : \Omega \times \mathbb{R}^\Omega \to \mathbb{R}$ by $V(\omega, f) = V_\omega(f)$ for all $(\omega, f) \in \Omega \times \mathbb{R}^\Omega$. It follows that $\bar{V}$ and V satisfy the first two properties of generalized conditional expectation. By point 2 of Proposition 8 and since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies conditional preferences, we have that for each $\omega, \omega' \in \Omega$

$$\Pi(\omega) = \Pi(\omega') \implies \succsim_\omega = \succsim_{\omega'} \implies V(\omega, \cdot) = V(\omega', \cdot),$$

proving that $V(\cdot, f)$ is Π-measurable for all $f \in \mathbb{R}^\Omega$. Since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies consequentialism, we have that for each $\omega \in \Omega$ and for each $f, h \in \mathbb{R}^\Omega$

$$f1_{\Pi(\omega)} + h1_{\Pi(\omega)^c} \sim_\omega f \implies V\left(\omega, f1_{\Pi(\omega)} + h1_{\Pi(\omega)^c}\right) = V(\omega, f).$$

Finally, for each $f \in \mathbb{R}^\Omega$ define $\bar{f} \in \mathbb{R}^\Omega$ by $\bar{f}(\omega) = V(\omega, f)$ for all $\omega \in \Omega$. It follows that $\bar{f} \sim_\omega \bar{f}1_{\Pi(\omega)} \sim_\omega f$ for all $\omega \in \Omega$ and for all $f \in \mathbb{R}^\Omega$. Since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies dynamic consistency, we can conclude that $\bar{f} \sim f$ and, in particular, $\bar{V}(f) = \bar{V}\left(\bar{f}\right) = \bar{V}(V(\cdot, f))$ for all $f \in \mathbb{R}^\Omega$. ■

Clearly, in Proposition 9, linear conditional expectations are obtained by requiring in (i) $\succsim$ and each $\succsim_\omega$ to satisfy the axiom of independence. Similarly, maxmin conditional expectations, as in Example 1, are obtained by imposing c-independence.

References

[1] F. Allen, S. Morris, and H.S. Shin, Beauty contests and iterated expectations in asset markets, The Review of Financial Studies, 19, 719-752, 2006.

[2] N.I. Al-Najjar and J. Weinstein, The ambiguity aversion literature: a critical assessment, *Economics & Philosophy*, 25, 249-284, 2009.

[3] G.M. Angeletos, and A. Pavan, Efficient use of information and social value of information. *Econometrica*, 75, 1103-1142, 2007.

[4] R.J. Aumann, Agreeing to disagree, *The annals of statistics*, 4, 1236-1239, 1976.

[5] R.J. Aumann, Correlated equilibrium as an expression of Bayesian rationality, *Econometrica*, 55, 1-18, 1987.

[6] L. Bastianello, J.H. Faro, and A. Santos, Dynamically consistent objective and subjective rationality, arXiv preprint arXiv:2004.12347, 2020.

[7] P. Battigalli, E. Catonini, G. Lanzani, and M. Marinacci, "Ambiguity attitudes and self-confirming equilibrium in sequential games", *Games and Economic Behavior,* 115, 1-29, 2019.

[8] G. Bonanno and K. Nehring, How to make sense of the common prior assumption under incomplete information, *International Journal of Game Theory*, 28, 409-434, 1999.

[9] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Dynamic Opinion Aggregation: Long-run Stability and Disagreement, mimeo, 2021.

[10] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Nonlinear fixed points and Ergodicity: Economic Applications, mimeo, 2022.

[11] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 341–375, 2011.

[12] S. Cerreia-Vioglio, L.P. Hansen, F. Maccheroni, and M. Marinacci, Making decisions under model misspecification, *University of Chicago, Becker Friedman Institute for Economics Working Paper*, (2020-103), 2020.

[13] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, *Journal of Economic Theory*, 146, 1275–1330, 2011.

[14] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and A. Rustichini, Niveloids and their extensions: Risk measures on small domains, *Journal of Mathematical Analysis and Applications*, 413, 343–360, 2014.

[15] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and A. Rustichini, The structure of variational preferences, *Journal of Mathematical Economics*, 57, 12-19, 2015.

[16] B. De Finetti, *Theory of probability: A critical introductory treatment,* Wiley, 1974.

[17] D. Dillenberger, Preferences for one-shot resolution of uncertainty and Allais-type behavior, *Econometrica*, 78, 1973-2004, 2010.

[18] A. Di Tillio, E. Faingold, and D. Samet, What is common to common knowledge and common priors?, mimeo, 2021.

[19] J. Dow, and S.R. da Costa Werlang, Uncertainty aversion, risk aversion, and the optimal choice of portfolio, *Econometrica*, 60, 197-204, 1992.

[20] A. Ellis, On dynamic consistency in ambiguous games, *Games and Economic Behavior,* 111, 241-249, 2018.

[21] L. G. Epstein and M. Le Breton, Dynamically consistent beliefs must be Bayesian, *Journal of Economic Theory*, 61, 1-22, 1993.

[22] L. G. Epstein and M. Schneider, Recursive multiple-priors, *Journal of Economic Theory*, 113, 1–31, 2003.

[23] L.G. Epstein and T. Wang, "Beliefs about beliefs" without probabilities, *Econometrica,* 64, 1343-1373, 1996.

[24] J.H. Faro, and J.P. Lefort, 2019, Dynamic objective and subjective rationality, *Theoretical Economics,* 14, 1-14.

[25] Y. Feinberg, Characterizing common priors in the form of posteriors, *Journal of Economic Theory*, 91, 127-179, 2000.

[26] L.T. Nielsen, A. Brandenburger, J. Geanakoplos, R. McKelvey, and T. Page, Common knowledge of an aggregate of expectations, *Econometrica,* 58, 1235-1239, 1990.

[27] P. Ghirardato, Revisiting Savage in a conditional world, *Economic Theory,* 20, 83-92, 2002.

[28] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.

[29] P. Ghirardato, F. Maccheroni, and M. Marinacci, Revealed ambiguity and its consequences: updating. In Advances in decision making Under Risk and Uncertainty, Springer, Berlin, Heidelberg, 2008.

[30] P. Ghirardato and M. Marinacci, Ambiguity made precise: A comparative foundation, *Journal of Economic Theory*, 102, 251-289, 2002.

[31] I. Gilboa and M. Marinacci, Ambiguity and the Bayesian paradigm, in *Readings in formal epistemology*, 385-439,. Springer, 2016.

[32] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 141–153, 1989.

[33] A. Gizatulina and Z. Hellman, No trade and yes trade theorems for heterogeneous priors, *Journal of Economic Theory*, 182, 161-184, 2019.

[34] B. Golub and S. Morris, Higher-Order Expectations, *SSRN* 2979089, 2017.

[35] B. Golub and S. Morris, Expectations, Networks, and Conventions, mimeo, 2018.

[36] F. Gul and W. Pesendorfer, Evaluating ambiguous random variables from Choquet to maxmin expected utility, *Journal of Economic Theory*, 192, 105129, 2021.

[37] A. Gumen, and A. Savochkin, Dynamically stable preferences, *Journal of Economic Theory,* 148, 1487-1508, 2013.

[38] E. Hanany, and P. Klibanoff, Updating Ambiguity Averse Preferences, *The B.E. Journal of Theoretical Economics,* 9, 2009.

[39] E. Hanany, P. Klibanoff, and S. Mukerji, Incomplete information games with ambiguity averse players, *American Economic Journal: Microeconomics,* 12, 135-87, 2020.

[40] L. P. Hansen and T. J. Sargent, Robust control and model uncertainty, *American Economic Review*, 91, 60–66, 2001.

[41] A. Heifetz, The positive foundation of the common prior assumption, *Games and Economic Behavior*, 56, 105-120, 2006.

[42] Z. Hellman, Iterated expectations, compact spaces, and common priors, *Games and Economic Behavior*, 72, 163-171, 2011.

[43] A. Kajii and S. Morris, The robustness of equilibria to incomplete information, *Econometrica*, 65, 1283-1309, 1997.

[44] A. Kajii and T. Ui, Incomplete information games with multiple priors, *The Japanese Economic Review,* 56, 332-351, 2005.

[45] A. Kajii and T. Ui, Agreeable bets with multiple priors, *Journal of Economic Theory,* 128, 299-305, 2006.

[46] A. Kajii and T. Ui, Interim efficient allocations under uncertainty, *Journal of Economic Theory,*144, 337-353, 2009.

[47] P. Klibanoff, M. Marinacci, and S. Mukerji, A smooth model of decision making under ambiguity, *Econometrica,* 73, 1849-1892, 2005.

[48] M. O. Jackson, *Social and Economic Networks*, Princeton University Press, Princeton, 2008.

[49] M.O. Jackson, and A. Pernoud, Systemic risk in financial networks: A survey, *Annual Review of Economics*, 13, 171-202, 2021.

[50] B.L. Lipman, Finite order implications of common priors, *Econometrica*, 71, 1255-1267, 2003.

[51] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.

[52] D. Monderer, and D. Samet, Approximating common knowledge with common beliefs, *Games and Economic Behavior,* 1, 170-190, 1989.

[53] F. Maccheroni, M. Marinacci, and A. Rustichini, Dynamic variational preferences, *Journal of Economic Theory*, 128, 4–44, 2006.

[54] V.F. Martins-da-Rocha, Interim efficiency with MEU-preferences, *Journal of Economic Theory,* 145, 1987-2017, 2010.

[55] S. Morris, Trade with heterogeneous prior beliefs and asymmetric information, *Econometrica*, 62, 1327-1347, 1994.

[56] S. Morris, The common prior assumption in economic theory, *Economics & Philosophy*, 11, 227-253, 1995.

[57] S. Morris, and H.S. Shin, Social value of public information, *American Economic Review*, 92, 1521-1534, 2002.

[58] R.F. Nau and K.F. McCardle, Coherent behavior in noncooperative games, *Journal of Economic Theory*, 50, 424-444, 1990.

[59] C.P. Pires, A rule for updating ambiguous beliefs, *Theory and Decision,* 53, 137-152, 2002.

[60] L. Rigotti, C. Shannon, and T. Strzalecki, Subjective beliefs and ex ante trade, *Econometrica,* 76, 1167-1190, 2008.

[61] L.J. Savage, *The foundations of statistics*, Courier Corporation, 1972.

[62] D. Samet, Iterated expectations and common priors, *Games and Economic Behavior*, 24, 131–141, 1998.

[63] D. Samet, Common priors and separation of convex sets, *Games and Economic Behavior*, 24, 172-174, 1998.

[64] E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd ed., Springer, New York, 1981.

[65] H.S. Shin, T. Williamson, How much common belief is necessary for a convention?, *Games and Economic Behavior*, 13, 252-268, 1996.

[66] M. Siniscalchi, Dynamic choice under ambiguity, *Theoretical Economics*, 6, 379-421, 2011.

[67] D. W. Stroock, *An Introduction to Markov Processes*, 2nd ed., Springer Verlag, Berlin, 2014.

[68] T. Strzalecki, Temporal resolution of uncertainty and recursive models of ambiguity aversion, *Econometrica*, 81, 1039-1074, 2013.

[69] T. Strzalecki, and J. Werner, Efficient allocations under ambiguity, *Journal of Economic Theory,* 146, 1173-1194, 2011.

F Online appendix: Omitted proofs

Here, we collect the proofs omitted from the main appendix. We start with an ancillary result that elaborates on how the indicator matrix of the composition of a finite collection of operators $\{T_h\}_{h\in\{1,...,H\}}$ is related to the product of their indicator matrices.

Lemma 16. *Let $S, T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ be monotone and define $\hat{A} = A(T \circ S)$, $\tilde{A} = A(S)$, and $A = A(T)$. If there exists $k \in J$ such that $a_{jk} > 0$ and $\tilde{a}_{kj'} > 0$, then $\hat{a}_{jj'} > 0$. In particular, we have that:*

1. *If $\{T_h\}_{h\in\{1,...,H\}}$ is a collection of H monotone operators from $\mathbb{R}^{\bar{n}}$ to $\mathbb{R}^{\bar{n}}$ and the jj'-th entry of $\Pi_{h=1}^{H} A(T_h)$ is strictly positive, then the jj'-th of $A(T_H \circ ... \circ T_1)$ is strictly positive.*

2. *If $t \in \mathbb{N}$ and the jj'-th entry of $A(T)^t$ is strictly positive, then the jj'-th of $A(T^t)$ is strictly positive.*

Proof. By assumption, there exists $k \in \{1, ..., \bar{n}\}$ such that $a_{jk}, \tilde{a}_{kj'} > 0$, that is, there exist $\varepsilon_{jk}, \varepsilon_{kj'} \in (0, 1)$ such that for each $f \in \mathbb{R}^{\bar{n}}$ and for each $\delta \geq 0$

$$S_k\left(f + \delta e^{j'}\right) - S_k(f) \geq \varepsilon_{kj'}\delta \text{ and } T_j\left(f + \delta e^k\right) - T_j(f) \geq \varepsilon_{jk}\delta.$$

Since S is monotone, this implies that $S\left(f + \delta e^{j'}\right) \geq S(f) + \varepsilon_{kj'}\delta e^k$ for all $f \in \mathbb{R}^{\bar{n}}$ and for all $\delta \geq 0$. Since T is monotone, this yields that for each $f \in \mathbb{R}^{\bar{n}}$ and for each $\delta \geq 0$

$$T_j\left(S\left(f + \delta e^{j'}\right)\right) \geq T_j\left(S(f) + \varepsilon_{kj'}\delta e^k\right) \geq T_j(S(f)) + \varepsilon_{jk}\varepsilon_{kj'}\delta.$$

Since $\varepsilon_{jk}\varepsilon_{kj'} \in (0, 1)$, this proves that, under $T \circ S$, j is strongly monotone with respect to j', proving that $\hat{a}_{jj'} > 0$ and the main part of the statement.

1. Consider a collection of H monotone operators from $\mathbb{R}^{\bar{n}}$ to $\mathbb{R}^{\bar{n}}$: $\{T_h\}_{h\in\{1,...,H\}}$. We prove by finite induction the statement that, for each $l \in \{1, ..., H\}$, if the jj'-th entry of $\Pi_{h=1}^{l} A(T_h)$ is strictly positive, then the jj'-th of $A(T_l \circ ... \circ T_1)$ is strictly positive.

Initial step. Assume $l = 1$. In this case, we have that $A(T_1) = \Pi_{h=1}^{l} A(T_h)$. This proves that if the jj'-th entry of $\Pi_{h=1}^{l} A(T_h)$ is strictly positive, so is the jj'-th entry of the indicator matrix of the composition.

Inductive step. Assume the statement is true for l. We prove it is true for $l+1$. Define $S = T_l \circ ... \circ T_1$ and $T = T_{l+1}$. As before, set $\tilde{A} = A(S)$, $A = A(T)$, and $\hat{A} = A(T \circ S) = A(T_{l+1} \circ ... \circ T_1)$. Finally, define by $a_{jj'}^{(l)}$ (resp., $a_{jj'}^{(1)}$ and $a_{jj'}^{(l+1)}$) the generic jj'-th entry of $\Pi_{h=1}^{l} A(T_h)$ (resp., $A(T_{l+1})$ and $\Pi_{h=1}^{l+1} A(T_h)$). Observe that

$$a_{jj'}^{(l+1)} = \sum_{k=1}^{\bar{n}} a_{jk}^{(1)} a_{kj'}^{(l)}.$$

If the jj'-th entry of $\Pi_{h=1}^{l+1} A(T_h)$ is strictly positive, then $a_{jj'}^{(l+1)} > 0$, yielding that $a_{jk}^{(1)} a_{kj'}^{(l)} > 0$ for some $k \in J$, that is, $a_{jk}^{(1)}, a_{kj'}^{(l)} > 0$ for some $k \in J$. By inductive hypothesis, we have that $a_{kj'}^{(l)} > 0$ implies that $\tilde{a}_{kj'} > 0$ as well as $a_{jk} > 0$. By the main part of the statement, we can conclude that $\hat{a}_{jj'} > 0$, proving the inductive step.

The statement follows by finite induction.

2. By point 1, the statement trivially follows by considering the collection $\{T_h\}_{h=1}^{H}$ where $H = t$ and $T_h = T$ for all $h \in \{1, ..., H\}$. ■

Proof of Lemma 3. Define $B = \Pi_{k=1}^{K} B_k$. By induction, we prove that $A(\Pi_{k=1}^{m} B_k) \geq A(B_k) \geq I_{\bar{n}}$ for all $k \in \{1, ..., m\}$ and for all $m \in \{1, ..., K\}$. By definition and since $b_{1,jj} > 0$ for all $j \in J$, if $m = 1$, then $A(\Pi_{k=1}^{1} B_k) = A(B_1) \geq I_{\bar{n}}$. By point 1 of Lemma 16 and inductive hypothesis and since $b_{k,jj} > 0$ for all $k \in \{1, ..., K\}$ and for all $j \in J$, if $m, m+1 \in \{1, ..., K\}$, then $A(B_{m+1}) A(\Pi_{k=1}^{m} B_k) \geq I_{\bar{n}} A(B_k)$ and $A\left(\Pi_{k=1}^{m+1} B_k\right) = A(B_{m+1} \Pi_{k=1}^{m} B_k) \geq A(A(B_{m+1}) A(\Pi_{k=1}^{m} B_k)) \geq A(I_{\bar{n}} A(B_k)) = A(B_k) \geq I_{\bar{n}}$ for all $k \in \{1, ..., m\}$. By point 1 of Lemma 16 and inductive hypothesis, we also have that $A(B_{m+1}) A(\Pi_{k=1}^{m} B_k) \geq A(B_{m+1}) I_{\bar{n}}$ and $A\left(\Pi_{k=1}^{m+1} B_k\right) = A(B_{m+1} \Pi_{k=1}^{m} B_k) \geq A(A(B_{m+1}) A(\Pi_{k=1}^{m} B_k)) \geq A(A(B_{m+1}) I_{\bar{n}}) = A(B_{m+1}) \geq I_{\bar{n}}$. The statement follows by finite induction. In particular, this yields that

$$A(B_K ... B_1) \geq A(B_k) \geq I_{\bar{n}} \qquad \forall k \in \{1, ..., K\}.$$

Consider $k \in \{1, ..., K\}$. Since $A(B_k)$ is symmetric, any index $j \in J$ is essential under B_k. Let $l \in \{1, ..., m_{B_k}\}$ and $j \in J_l(B_k)$. We have two cases:

1. $j \in J_{l'}(A(B))$ for some $l' \in \left\{1, ..., m_{A(B)}\right\}$. Consider $j' \in J_l(B_k)$. It follows that $j \overset{B_k}{\longleftrightarrow} j'$. Since $A(B) \geq A(B_k)$, we have that $j \overset{A(B)}{\longleftrightarrow} j'$, yielding that $j' \in J_{l'}(A(B))$. This implies that $J_l(B_k) \subseteq J_{l'}(A(B))$.

2. $j \in J_{m_B+1}(A(B))$. Consider $j' \in J_l(B_k)$. It follows that $j \overset{B_k}{\longleftrightarrow} j'$. Since $A(B) \geq A(B_k)$, we have that $j \overset{A(B)}{\longleftrightarrow} j'$, yielding that $j' \in J_{m_{B+1}}(A(B))$. Otherwise, since $j \overset{A(B)}{\longleftrightarrow} j'$, if $j' \notin J_{m_{B+1}}(B)$, then j' would be essential under $A(B)$ and so would be j, a contradiction. This implies that $J_l(B_k) \subseteq J_{m_B+1}(A(B))$. ■

Proof of Lemma 4. Before starting, we denote by $\langle \, , \, \rangle$ the inner product of $\mathbb{R}^{\bar{n}}$. Let $j \in J$. Define the binary relation $\succsim_j^*$ on $\mathbb{R}^{\Omega}$ by

$$f \succsim_j^* g \iff T_j(\lambda f + (1-\lambda) h) \geq T_j(\lambda g + (1-\lambda) h) \quad \forall \lambda \in (0,1], \forall h \in \mathbb{R}^{\bar{n}}.$$

By [3] and since T_j is normalized, monotone, and continuous, we have that there exists a compact and convex set C_j of $\Delta_{\bar{n}}$ such that

$$f \succsim_j^* g \iff \langle f, p \rangle \geq \langle g, p \rangle \qquad \forall p \in C_j \tag{33}$$

and

$$T_j(f) = \alpha_j(f) \min_{p \in C_j} \langle f, p \rangle + (1 - \alpha_j(f)) \max_{p \in C_j} \langle f, p \rangle \quad \forall f \in \mathbb{R}^{\bar{n}} \tag{34}$$

where $\alpha_j : \mathbb{R}^{\bar{n}} \to [0,1]$. Observe also that if j is constant with respect to j', then $e^{j'} \sim_j^* 0$. By (33), it follows that

$$p_{j'} = 0 \qquad \forall p \in C_j. \tag{35}$$

Since C_j is compact, for each $f \in \mathbb{R}^{\bar{n}}$ define $p_{\min,f}, p_{\max,f} \in C_j$ such that $\langle f, p_{\min,f} \rangle = \min_{p \in C_j} \langle f, p \rangle$ and $\langle f, p_{\max,f} \rangle = \max_{p \in C_j} \langle f, p \rangle$. By (34) and since C_j is convex, it follows that $p_{j,f} = \alpha_j(f) p_{\min,f} + (1 - \alpha_j(f)) p_{\max,f} \in C_j$ such that $T_j(f) = \langle f, p_{j,f} \rangle$ for all $f \in \mathbb{R}^{\bar{n}}$. Fix $f \in \mathbb{R}^{\bar{n}}$. Since j was arbitrarily chosen, define $M(f)$ to be the matrix whose j-th row entries correspond to the entries of $p_{j,f}$. It follows that $T(f) = M(f) f$. Moreover, $M(f)$ belongs to the set $\mathcal{M}(T)$ of matrices M whose j-th row belongs to C_j. Since each of these sets is compact and convex, so is $\mathcal{M}(T)$. Since f was arbitrarily chosen, the statement follows. By construction of $\mathcal{M}(T)$ and (35), it follows that if j is constant with respect to j', then $m_{jj'} = 0$ for all $M \in \mathcal{M}(T)$. ■

Proof of Lemma 5. Since $d(\bar{M}) > 0$, it follows that $\bar{m}_{jj} > 0$ for all $j \in J$. This implies that the jj-th entry of $A(\bar{M})$ is 1 for all $j \in J$, and, in particular, if the jj'-th entry of $A(M)$ is strictly positive, so is the one of $A(\bar{M}) A(M)$. By point 1 of Lemma 16, we can conclude that $A(\bar{M}M) \geq A(M)$. We have two cases:

1. $A(\bar{M}M) = A(M)$. Set $\hat{M} = \bar{M}M$ and consider $\hat{m}_{jj'} > 0$. We next prove that for each $l \in \{1, ..., \bar{n}\}$
$$m_{lj'} = 0 \implies \bar{m}_{jl} = 0. \tag{36}$$
By contradiction, assume that there exists $\bar{l} \in \{1, ..., \bar{n}\}$ such that $m_{\bar{l}j'} = 0$ and $\bar{m}_{j\bar{l}} > 0$. Since $A(\hat{M}) = A(\bar{M}M) = A(M)$ and $\hat{m}_{jj'} > 0$ and $m_{\bar{l}j'} = 0$, we would have that $m_{jj'} > 0$ and $\hat{m}_{\bar{l}j'} = 0$. Since $A(\bar{M})$ is symmetric, we would also have that $\bar{m}_{\bar{l}j} > 0$, yielding that $\hat{m}_{\bar{l}j'} \geq \bar{m}_{\bar{l}j} m_{jj'} > 0$, a contradiction with $\hat{m}_{\bar{l}j'} = 0$. By (36), we can conclude that $\hat{m}_{jj'} = \sum_{l=1}^{\bar{n}} \bar{m}_{jl} m_{lj'} \geq \sum_{l=1}^{\bar{n}} \bar{m}_{jl} \delta(M) = \delta(M)$, proving the statement.

2. $A(\bar{M}M) > A(M)$. Set $\hat{M} = \bar{M}M$. In this case, if $\hat{m}_{jj'} > 0$, then $\bar{m}_{j\bar{l}} m_{\bar{l}j'} > 0$ for some $\bar{l} \in \{1, ..., \bar{n}\}$ and, in particular, $\bar{m}_{j\bar{l}}, m_{\bar{l}j'} > 0$. It follows that $\hat{m}_{jj'} = \sum_{l=1}^{\bar{n}} \bar{m}_{jl} m_{lj'} \geq \bar{m}_{j\bar{l}} m_{\bar{l}j'} \geq \delta(\bar{M}) \delta(M)$, proving the statement.

Consider a sequence $\{M_k\}_{k=1}^{\infty}$ of $\bar{n} \times \bar{n}$ stochastic matrices such that $A(M_k)$ is symmetric, $\delta(M_k) \geq \delta > 0$, and $d(M_k) > 0$ for all $k \in \mathbb{N}$. By induction and the previous part, we have that $A\left(\prod_{k=1}^{m+1} M_k\right) = A\left(M_{m+1} \prod_{k=1}^{m} M_k\right) \geq A\left(\prod_{k=1}^{m} M_k\right)$ for all $m \in \mathbb{N}$. Define $f : \mathbb{N} \to \{0,1\}$ by

$f(1) = 1$ and

$$f(m+1) = \begin{cases} 1 & \text{if } A\left(\prod_{k=1}^{m+1} M_k\right) > A\left(\prod_{k=1}^{m} M_k\right) \\ 0 & \text{if } A\left(\prod_{k=1}^{m+1} M_k\right) = A\left(\prod_{k=1}^{m} M_k\right) \end{cases} \qquad \forall m \in \mathbb{N}.$$

By induction, we prove that

$$\delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta^{\sum_{k=1}^{m} f(k)} \qquad \forall m \in \mathbb{N}. \tag{37}$$

Initial step. Assume $m = 1$. Since $f(1) = 1$, $\delta\left(\prod_{k=1}^{m} M_k\right) = \delta(M_1) \geq \delta = \delta^{\sum_{k=1}^{m} f(k)}$.

Inductive step. Assume the statement is true for $m \in \mathbb{N}$. We prove it is true for $m+1$. Since $A\left(\prod_{k=1}^{m+1} M_k\right) \geq A\left(\prod_{k=1}^{m} M_k\right)$, we have two cases:

1. $A\left(\prod_{k=1}^{m+1} M_k\right) > A\left(\prod_{k=1}^{m} M_k\right)$. In this case, we have that $f(m+1) = 1$. By the first part of the statement and inductive hypothesis, we have that

$$\delta\left(\prod_{k=1}^{m+1} M_k\right) = \delta\left(M_{m+1}\prod_{k=1}^{m} M_k\right) \geq \delta(M_{m+1})\,\delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta\delta^{\sum_{k=1}^{m} f(k)} = \delta^{\sum_{k=1}^{m+1} f(k)}.$$

2. $A\left(\prod_{k=1}^{m+1} M_k\right) = A\left(\prod_{k=1}^{m} M_k\right)$. In this case, we have that $f(m+1) = 0$. By the first part of the statement and inductive hypothesis, we have that

$$\delta\left(\prod_{k=1}^{m+1} M_k\right) = \delta\left(M_{m+1}\prod_{k=1}^{m} M_k\right) \geq \delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta^{\sum_{k=1}^{m} f(k)} = \delta^{\sum_{k=1}^{m+1} f(k)}.$$

Thus, (37) follows by induction. Since $\left\{A\left(\prod_{k=1}^{m} M_k\right)\right\}_{m\in\mathbb{N}}$ is an increasing sequence with upper bound the $\bar{n} \times \bar{n}$ square matrix whose entries are all 1s, we observe that $f(k) = 1$ for at most $\bar{n}^2$ indices, yielding that $\sum_{k=1}^{m} f(k) \leq \bar{n}^2$ for all $m \in \mathbb{N}$, proving (24) . ■

Proof of Lemma 6. Consider $k \in \mathbb{R}$ and a sequence of functions $\{f_m\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^\Omega$ such that $f_m \to k1_\Omega$. Since $f_m \to k1_\Omega$ and Ω is finite, we have that $\lim_{m\to\infty} \min_{\omega\in\Omega} f_m(\omega) = k = \lim_{m\to\infty} \max_{\omega\in\Omega} f_m(\omega)$. Since $\bar{V}$ is normalized and monotone, we also have that $\min_{\omega\in\Omega} f_m(\omega) \leq \bar{V}(f_m) \leq \max_{\omega\in\Omega} f_m(\omega)$ for all $m \in \mathbb{N}$. By passing to the limit and since $\bar{V}$ is normalized, we have that

$$\lim_{m\to\infty} \bar{V}(f_m) = k = \bar{V}(k1_\Omega),$$

proving continuity at $k1_\Omega$. ■

Proof of Lemma 7. (i) implies (ii). Let $j, j' \in J$. Since $a_{jj'} = 1$, we have that j is strongly monotone with respect to j'. By contradiction, assume that $\Pi(\omega_j) \neq \Pi(\omega_{j'})$. Since Π is a partition, it follows that $\Pi(\omega_j) \cap \Pi(\omega_{j'}) = \emptyset$. Since (V, Π) is an interim expectation and j is strongly monotone with respect to j', we thus have that there exists $\varepsilon_{jj'} \in (0,1)$ such that

$$\begin{aligned} 0 &= V\left(\omega_j, 01_{\Pi(\omega_j)} + 1_{\{\omega_{j'}\}} 1_{\Pi(\omega_j)^c}\right) - V(\omega_j, 0) \\ &= V\left(\omega_j, 1_{\{\omega_{j'}\}} 1_{\Pi(\omega_j)} + 1_{\{\omega_{j'}\}} 1_{\Pi(\omega_j)^c}\right) - V(\omega_j, 0) \\ &= V\left(\omega_j, 1_{\{\omega_{j'}\}}\right) - V(\omega_j, 0) \geq \varepsilon_{jj'} > 0, \end{aligned}$$

a contradiction.

(ii) implies (i). Note that $\Pi(\omega_j) = \Pi(\omega_{j'})$ only if $\omega_{j'} \in \Pi(\omega_j)$. Since (V, Π) is an interim expectation with full support, we have that each $\bar{\omega} \in \Pi(\omega_j)$ is $V(\omega_j, \cdot)$-essential and, in particular, so is $\omega_{j'}$, yielding that $a_{jj'} = 1$.

By the previous part of the proof and since $\Pi(\omega_j) = \Pi(\omega_j)$ for all $j \in J$ and $A(V)$ is $\{0,1\}$-valued, we thus have that

$$a_{jj'} = 1 \iff \Pi(\omega_j) = \Pi(\omega_{j'}) \iff \Pi(\omega_{j'}) = \Pi(\omega_j) \iff a_{j'j} = 1,$$

proving that $A(V)$ is symmetric, $a_{jj} = 1$ for all $j \in J$, and $\Pi(V) = \Pi$. Finally, for all $j, j' \in J$, if j is not strongly monotone with respect to j', we can conclude that $a_{jj'} = 0$ and $\omega_{j'} \notin \Pi(\omega_j)$. Since $V\left(\omega, f1_{\Pi(\omega)} + h1_{\Pi(\omega)^c}\right) = V(\omega, f)$ for all $\omega \in \Omega$ and for all $f, h \in \mathbb{R}^\Omega$, this implies that

$$\begin{aligned} V\left(\omega_j, f + \delta 1_{\{\omega_{j'}\}}\right) &= V\left(\omega_j, f1_{\Pi(\omega_j)} + \delta 1_{\{\omega_{j'}\}} 1_{\Pi(\omega_j)} + 01_{\Pi(\omega_j)^c}\right) \\ &= V\left(\omega_j, f1_{\Pi(\omega_j)} + 01_{\Pi(\omega_j)^c}\right) = V(\omega_j, f) \end{aligned}$$

for all $f \in \mathbb{R}^\Omega$ and for all $\delta \geq 0$, yielding that j is constant with respect to j'. This implies that V is dichotomic. ■

Proof of Lemma 8. Since each V_i is nonexpansive, we have that

$$\begin{aligned} \|S_{\beta,i}(\mathbf{f}) - S_{\beta,i}(\mathbf{g})\|_\infty &= \left\| V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} f_l\right) - V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} g_l\right) \right\|_\infty \\ &\leq \left\| (1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} f_l - (1-\beta)\hat{f} - \beta \sum_{l=1}^n w_{il} g_l \right\|_\infty \\ &= \left\| \beta \sum_{l=1}^n w_{il}(f_l - g_l) \right\|_\infty \leq \beta \sum_{l=1}^n w_{il} \|f_l - g_l\|_\infty \\ &\leq \beta \sum_{l=1}^n w_{il} \|\mathbf{f} - \mathbf{g}\|_* \leq \beta \|\mathbf{f} - \mathbf{g}\|_* \quad \forall i \in I, \forall \mathbf{f}, \mathbf{g} \in \left(\mathbb{R}^\Omega\right)^n, \end{aligned}$$

proving that $\|S_\beta(\mathbf{f}) - S_\beta(\mathbf{g})\|_* = \sup_{i \in I} \|S_{\beta,i}(\mathbf{f}) - S_{\beta,i}(\mathbf{g})\|_\infty \leq \beta \|\mathbf{f} - \mathbf{g}\|_*$ for all $\mathbf{f}, \mathbf{g} \in \left(\mathbb{R}^\Omega\right)^n$. By the Banach contraction principle, for each $\beta \in (0,1)$ we have that $S_\beta^\tau\left(\hat{\mathbf{f}}\right) \overset{\|\ \|_*}{\to} \sigma^\beta$ as well as $S_\beta\left(\sigma^\beta\right) = \sigma^\beta$ where σ^β is the unique fixed point of S_β for all $\beta \in (0,1)$. Finally, since V_i is nonexpansive and normalized, observe that

$$\begin{aligned}\left\|S_{\beta,i}(\mathbf{f})\right\|_\infty &= \left\|V_i\left((1-\beta)\hat{f} + \beta\sum_{l=1}^n w_{il}f_l\right)\right\|_\infty \le \left\|(1-\beta)\hat{f} + \beta\sum_{l=1}^n w_{il}f_l\right\|_\infty \\ &\le (1-\beta)\left\|\hat{f}\right\|_\infty + \beta\sum_{l=1}^n w_{il}\left\|f_l\right\|_\infty \quad \forall i \in I, \forall \mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n.\end{aligned}$$

By induction, this implies that $\left\|S_\beta^\tau\left(\hat{\mathbf{f}}\right)\right\|_* \le \left\|\hat{f}\right\|_\infty$ for all $\tau \in \mathbb{N}$. By passing to the limit, the statement follows. ∎

Proof of Lemma 9. (i) implies (ii). Before starting, since Ω is finite, we enumerate its elements $\Omega = \{\omega_1, ..., \omega_{\bar{n}}\}$ and set as before $J = \{1, ..., \bar{n}\}$. By assumption, we have that $f_i = V_i\left(\sum_{l=1}^n w_{il}f_l\right)$ for all $i \in I$. By Proposition 7 and Lemma 7, for each $i \in I$ there exists an $\bar{n} \times \bar{n}$ stochastic matrix M_i whose diagonal is strictly positive and it is such that: 1) $A(V_i) = A(M_i)$ is symmetric, 2) $\Pi(M_i) = \Pi_i$, and 3) $V_i\left(\sum_{l=1}^n w_{il}f_l\right) = M_i\left(\sum_{l=1}^n w_{il}f_l\right) = \sum_{l=1}^n w_{il}M_if_l$.[34] It follows that $\mathbf{f}$ is also a fixed point of the operator $\tilde{S} : \left(\mathbb{R}^\Omega\right)^n \to \left(\mathbb{R}^\Omega\right)^n$ where

$$\tilde{S}_i(\mathbf{g}) = \sum_{l=1}^n w_{il}M_ig_l \qquad \forall i \in I.$$

We next show that $\tilde{S}(\mathbf{f}) = \mathbf{f}$ only if there exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m1_\Omega$ for all $i, i' \in I$. By contradiction, assume that there exists $\bar{\imath}, \bar{\imath}' \in I$ and $\omega_{\bar{j}}, \omega_{\bar{j}'} \in \Omega$ such that $f_{\bar{\imath}}\left(\omega_{\bar{j}}\right) = \max_{i\in I}\max_{j\in J} f_i(\omega_j) > \min_{i\in I}\min_{j\in J} f_i(\omega) = f_{\bar{\imath}'}\left(\omega_{\bar{j}'}\right)$. We begin with an observation. For each $t \in \mathbb{N}$ denote by I^t the set of (finite) sequences in I with t elements, that is, $\mathbf{i} \in I^t$ if and only if $\mathbf{i} = (i_1, ..., i_t)$ with $i_l \in I$ for all $l \in \{1, ..., t\}$. By induction, note that for each $t \in \mathbb{N}$

$$\tilde{S}_i^t(\mathbf{g}) = \sum_{\mathbf{i}\in I^{t+1}:i_1=i} w_{i_1i_2}...w_{i_ti_{t+1}}M_{i_1}...M_{i_t}g_{i_{t+1}} \qquad \forall \mathbf{g} \in \left(\mathbb{R}^\Omega\right)^n$$

and

$$w_{i_1i_2}...w_{i_ti_{t+1}} \ge 0 \text{ for all } \mathbf{i} \in I^{t+1} \text{ such that } i_1 = i \text{ and } \sum_{\mathbf{i}\in I^{t+1}:i_1=i} w_{i_1i_2}...w_{i_ti_{t+1}} = 1.$$

Since W is strongly connected, there exists a sequence of agents $(\bar{\imath}_1, ..., \bar{\imath}_{\bar{t}+1})$ such that $\bar{t} \in \mathbb{N}$, $\{\bar{\imath}_1, ..., \bar{\imath}_{\bar{t}}\} \supseteq I$, and $\bar{\imath}_1 = \bar{\imath}_{\bar{t}+1} = \bar{\imath}$ with $w_{\bar{\imath}_l\bar{\imath}_{l+1}} > 0$ for all $l \in \{1, ..., \bar{t}\}$. By Lemma 3 and since $\{\bar{\imath}_1, ..., \bar{\imath}_{\bar{t}}\} \supseteq I$, we have that $\Pi\left(A\left(M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}\right)\right)$ is coarser than $\Pi(M_i) = \Pi_i$ for all $i \in I$. Since $\Pi_{\sup} = \{\Omega\}$, we can conclude that $\Pi\left(A\left(M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}\right)\right) = \{\Omega\}$, yielding that $M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}$ is strongly connected. By Lemma 3 and since the diagonal of each $M_{\bar{\imath}_l}$ is strictly positive, we also have that $M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}$ has a strictly positive diagonal. This implies that $M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}$ is primitive, that is, there exists $\tau \in \mathbb{N}$ such that each entry of $\left(M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}\right)^\tau$ is strictly positive. Since W is strongly connected there exists a sequence of agents $\left(\hat{\imath}_1, ..., \hat{\imath}_{\hat{t}+1}\right)$ such that $\hat{t} \in \mathbb{N}$, $\hat{\imath}_1 = \bar{\imath}$, and

[34] Given an $\bar{n} \times \bar{n}$ stochastic matrix M and $h, h' \in \mathbb{R}^\Omega$, we write $h' = Mh$ when $h'(\omega_j) = \sum_{j'=1}^{\bar{n}} m_{jj'}h(\omega_{j'})$ for all $j \in J$.

$\hat{\imath}_{\hat{t}+1} = \bar{\imath}'$ with $w_{\hat{\imath}_l \hat{\imath}_{l+1}} > 0$ for all $l \in \{1, ..., \hat{t}\}$. Next, recall that by Euclid's algorithm for each $l \in \{1, ..., \tau\bar{t} + 1\}$ there exists unique $q_l \in \mathbb{N}_0$ and $r'_l \in \{0, ..., \bar{t} - 1\}$ such that

$$l = q_l \bar{t} + r'_l.$$

We define $r_l = r'_l$ if $r'_l \in \{1, ..., \bar{t} - 1\}$ and $r_l = \bar{t}$ if $r'_l = 0$. Finally, consider the sequence of agents $(\tilde{\imath}_1, ..., \tilde{\imath}_{\tau\bar{t}+\hat{t}+1})$ where $\tilde{\imath}_l = \bar{\imath}_{r_l}$ for all $l \in \{1, ..., \tau\bar{t} + 1\}$ and $\tilde{\imath}_l = \hat{\imath}_{l-\tau\bar{t}}$ for all $l \in \{\tau\bar{t} + 1, ..., \tau\bar{t} + 1 + \hat{t}\}$. By construction, we have that $w_{\tilde{\imath}_l \tilde{\imath}_{l+1}} > 0$ for all $l \in \{1, ..., \tau\bar{t} + 1 + \hat{t}\}$. Since $\mathbf{f}$ is a fixed point of $\tilde{S}$, note that $\tilde{S}^{\tau}(\mathbf{f}) = \mathbf{f}$ for all $\tau \in \mathbb{N}$ and

$$f_{\bar{\imath}} = \tilde{S}^{\tau\bar{t}+\hat{t}}_{\bar{\imath}}(\mathbf{f}) = \sum_{\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1} : i_1 = \bar{\imath}} w_{i_1 i_2} ... w_{i_{\tau\bar{t}+\hat{t}} i_{\tau\bar{t}+\hat{t}+1}} M_{i_1} ... M_{i_{\tau\bar{t}+\hat{t}}} f_{i_{\tau\bar{t}+\hat{t}+1}}.$$

Define $f^{\mathbf{i}} = M_{i_1} ... M_{i_{\tau\bar{t}+\hat{t}}} f_{i_{\tau\bar{t}+\hat{t}+1}}$ for all $\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1}$ such that $i_1 = \bar{\imath}$. We have that

$$f_{\bar{\imath}} = \sum_{\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1} : i_1 = \bar{\imath}} w_{i_1 i_2} ... w_{i_{\tau\bar{t}+\hat{t}} i_{\tau\bar{t}+\hat{t}+1}} f^{\mathbf{i}}. \tag{38}$$

Since each M_i is an $\bar{n} \times \bar{n}$ stochastic matrix and $\max_{j \in J} f_i(\omega_j) \leq f_{\bar{\imath}}(\omega_{\bar{j}})$ for all $i \in I$, we have that $\max_{j \in J} f^{\mathbf{i}}(\omega_j) \leq f_{\bar{\imath}}(\omega_{\bar{j}})$ for all $\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1}$ such that $i_1 = \bar{\imath}$. We focus on the summand

$$w_{\tilde{\imath}_1 \tilde{\imath}_2} ... w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}} \tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} M_{\tilde{\imath}_1} ... M_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}} f_{\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} = w_{\tilde{\imath}_1 \tilde{\imath}_2} ... w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}} \tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} f^{\tilde{\imath}}.$$

By construction, we have that $w_{\tilde{\imath}_1 \tilde{\imath}_2} ... w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}} \tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} > 0$ and

$$M_{\tilde{\imath}_1} ... M_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}} f_{\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} = (M_{\bar{\imath}_1} ... M_{\bar{\imath}_{\bar{t}}})^{\tau} M_{\hat{\imath}_1} ... M_{\hat{\imath}_{\hat{t}}} f_{\hat{\imath}_{\hat{t}+1}}.$$

Set $g = M_{\hat{\imath}_1} ... M_{\hat{\imath}_{\hat{t}}} f_{\hat{\imath}_{\hat{t}+1}} = M_{\hat{\imath}_1} ... M_{\hat{\imath}_{\hat{t}}} f_{\bar{\imath}'}$. Since each $M_{\hat{\imath}_l}$ is an $\bar{n} \times \bar{n}$ stochastic matrix with strictly positive diagonal, so is $M_{\hat{\imath}_1} ... M_{\hat{\imath}_{\hat{t}}}$. Since $\max_{j \in J} f_{\bar{\imath}'}(\omega_j) \leq f_{\bar{\imath}}(\omega_{\bar{j}})$ and $f_{\bar{\imath}'}(\omega_{\bar{j}'}) < f_{\bar{\imath}}(\omega_{\bar{j}})$, this implies that $\min_{j \in J} g(\omega) \leq g(\omega_{\bar{j}'}) < f_{\bar{\imath}}(\omega_{\bar{j}})$ and $\max_{\omega \in \Omega} g(\omega) \leq f_{\bar{\imath}}(\omega_{\bar{j}})$. Since each entry of $(M_{\bar{\imath}_1} ... M_{\bar{\imath}_{\bar{t}}})^{\tau}$ is strictly positive and $f^{\tilde{\imath}} = (M_{\bar{\imath}_1} ... M_{\bar{\imath}_{\bar{t}}})^{\tau} g$, we can conclude that $f^{\tilde{\imath}}(\omega) < f_{\bar{\imath}}(\omega_{\bar{j}})$ for all $\omega \in \Omega$. By (38) and since $w_{\tilde{\imath}_1 \tilde{\imath}_2} ... w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}} \tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} > 0$ and $\max_{j \in J} f^{\mathbf{i}}(\omega_j) \leq f_{\bar{\imath}}(\omega_{\bar{j}})$ for all $\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1}$, this implies that

$$\begin{aligned} 0 &= \sum_{\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1} : i_1 = \bar{\imath}} w_{i_1 i_2} ... w_{i_{\tau\bar{t}+\hat{t}} i_{\tau\bar{t}+\hat{t}+1}} \left[f^{\mathbf{i}}(\omega_{\bar{j}}) - f_{\bar{\imath}}(\omega_{\bar{j}}) \right] \\ &\leq w_{\tilde{\imath}_1 \tilde{\imath}_2} ... w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}} \tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} \left[f^{\tilde{\imath}}(\omega_{\bar{j}}) - f_{\bar{\imath}}(\omega_{\bar{j}}) \right] < 0, \end{aligned}$$

a contradiction.

(ii) implies (i). Since each V_i is normalized and W is a stochastic matrix, the statement is trivial. ■

Lemma 17. *Fix $i, j \in I$ and $\omega, \omega' \in \Omega$. The following are equivalent:*

(i) $w_{ij} > 0$ *and* $\omega' \in \Pi_i(\omega)$*;*

(ii) $\underline{a}_{(i,\omega)(j,\omega')}(S_1) = 1$*;*

(iii) $\bar{a}_{(i,\omega)(j,\omega')}(S_1) = 1.$

Proof. (i) implies (ii). By Lemma 7, there exists $\varepsilon > 0$ such that for each $f \in \mathbb{R}^\Omega$ and for each $\delta \geq 0$

$$V_i\left(\omega, f + \delta e^{\omega'}\right) - V_i(\omega, f) \geq \varepsilon\delta.$$

Next, fix $\mathbf{f} = (f_l)_{l=1}^n \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$, and observe that

$$S_{1,i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - S_{1,i,\omega}(\mathbf{f}) = V_i\left(\omega, \sum_{l=1}^n w_{il} f_l + w_{ij}\delta e^{\omega'}\right) - V_i\left(\omega, \sum_{l=1}^n w_{il} f_l\right) \geq \varepsilon w_{ij}\delta$$

proving the statement by setting $\varepsilon_{(i,\omega)(j,\omega')} = \varepsilon w_{ij}$.

(ii) implies (iii). Immediate.

(iii) implies (i). We prove the statement by contradiction. Fix $\mathbf{f} = (f_l)_{l=1}^n \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$ and observe that

$$S_{1,i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - S_{1,i,\omega}(\mathbf{f}) = V_i\left(\omega, \sum_{l=1}^n w_{il} f_l + w_{ij}\delta e^{\omega'}\right) - V_i\left(\omega, \sum_{l=1}^n w_{il} f_l\right).$$

Therefore, if either $w_{ij} = 0$ or $\omega' \in \Pi_i(\omega)$, then $S_{1,i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) = S_{1,i,\omega}(\mathbf{f})$. Given that $\mathbf{f}$ and δ were arbitrarily chosen, we obtain a contradiction. ■

Proof of Lemma 10. We have that $S_\beta(\mathbf{f}) = S_1\left((1-\beta)\hat{\mathbf{f}} + \beta\mathbf{f}\right)$ for all $\beta \in (0,1)$ and recall that S_1 is normalized, monotone, and translation invariant. Fix $\lambda \in (0,1)$ and define $S_1^\lambda = \lambda I + (1-\lambda) S_1$. Clearly, we have that, for each $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$,

$$S_1^\lambda(\mathbf{f}) = \mathbf{f} \iff S_1(\mathbf{f}) = \mathbf{f}.$$

Therefore, by Lemma 9, $S_1^\lambda(\mathbf{f}) = \mathbf{f}$ if and only if there exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m1_\Omega$ for all $i, i' \in I$. By [2, Corollary 1 and part 2 of Proposition 2], it follows that there exists a unique class of indices Z', $\emptyset \neq Z' \subseteq I \times \Omega$, that is closed and strongly connected with respect to $\bar{A}\left(S_1^\lambda\right)$. It is easy to see that every row of $\bar{A}(S_1)$ is not null and that Z' is also closed and strongly connected with respect to $\bar{A}(S_1)$. In addition, by Lemma 17, every row of $\underline{A}(S_1)$ is not null and Z' is closed and strongly connected with respect to $\underline{A}(S_1)$. Finally, the statement follows by setting $Z = Z'$. ■

Proof of Lemma 11. For every $(i,\omega) \in I \times \Omega$, by Theorem [4, Theorem 2.3.9], we have that

$$\partial S_{1,i,\omega}(0) = \left\{w_i \tilde{q}_{i,\omega} \in \Delta(I \times \Omega) : \tilde{q}_{i,\omega} \in \partial V_i(\omega, 0)\right\}.$$

With this, the statement follows by the definitions of $\partial S_1(0)$ and of each W^q in equation (17). ■

Proof of Lemma 12. 1. Consider an I-sequence $\iota = (i_k)_{k\in\mathbb{N}} \in I^{\mathbb{N}}$. Consider $f, g \in \mathbb{R}^\Omega$ and $\lambda \in (0,1)$. Since each V_i is concave, we have that

$$V_{i_1}(\lambda f + (1-\lambda) g) \geq \lambda V_{i_1}(f) + (1-\lambda) V_{i_1}(g).$$

By induction, assume that

$$V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)\geq\lambda V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right).$$

Since $V_{i_{k+1}}$ is a concave interim expectation, we have that

$$\begin{aligned}&V_{i_{k+1}}\circ V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)=V_{i_{k+1}}\left(V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)\right)\\&\geq V_{i_{k+1}}\left(\lambda V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right)\right)\\&\geq\lambda V_{i_{k+1}}\circ V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_{k+1}}\circ V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right).\end{aligned}$$

By passing to the limit, we obtain that

$$\bar{V}_\iota\left(\lambda f+\left(1-\lambda\right)g\right)1_\Omega\geq\lambda\bar{V}_\iota\left(f\right)1_\Omega+\left(1-\lambda\right)\bar{V}_\iota\left(g\right)1_\Omega,$$

proving that $\bar{V}_\iota$ is concave. Since ι was arbitrarily chosen, we have that $\bar{V}_\iota$ is concave for every I-sequence ι. Finally, given that, by Proposition 2, we have

$$V_*\left(f\right)=\inf_{\iota\in I^{\mathbb{N}}:\iota\text{ is an }I\text{-sequence}}\bar{V}_\iota\left(f\right)\qquad\forall f\in\mathbb{R}^\Omega,$$

it follows that V_* is concave. With similar steps we can prove the second part of the first item.

2. Consider an I-sequence $\iota=\left(i_k\right)_{k\in\mathbb{N}}\in I^{\mathbb{N}}$. Consider $f,g\in\mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable, and $\lambda\in(0,1)$. Since each V_i is $\Pi_{\inf}$-affine, we have that

$$V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)=\lambda V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_1}\left(g\right).$$

By induction, assume that

$$V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)=\lambda V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right).$$

Since $V_{i_{k+1}}$ is $\Pi_{\inf}$-affine and $V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right)$ is $\Pi_{\inf}$-measurable, we have that

$$\begin{aligned}&V_{i_{k+1}}\circ V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)=V_{i_{k+1}}\left(V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(\lambda f+\left(1-\lambda\right)g\right)\right)\\&=V_{i_{k+1}}\left(\lambda V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right)\right)\\&=\lambda V_{i_{k+1}}\circ V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(f\right)+\left(1-\lambda\right)V_{i_{k+1}}\circ V_{i_k}\circ V_{i_{k-1}}\circ\ldots\circ V_{i_2}\circ V_{i_1}\left(g\right).\end{aligned}$$

By passing to the limit, we obtain that

$$\bar{V}_\iota\left(\lambda f+\left(1-\lambda\right)g\right)1_\Omega=\lambda\bar{V}_\iota\left(f\right)1_\Omega+\left(1-\lambda\right)\bar{V}_\iota\left(g\right)1_\Omega,$$

proving that $\bar{V}_\iota$ is $\Pi_{\inf}$-affine. Since ι was arbitrarily chosen, we have that $\bar{V}_\iota$ is $\Pi_{\inf}$-affine for every I-sequence ι. Finally, given that, by Proposition 2, we have

$$V_*\left(f\right)=\inf_{\iota\in I^{\mathbb{N}}:\iota\text{ is an }I\text{-sequence}}\bar{V}_\iota\left(f\right)\qquad\forall f\in\mathbb{R}^\Omega,$$

it follows that

$$\begin{aligned}V_*\left(\left(1-\lambda\right)h+\lambda g\right)&=\inf_{\iota\in I^{\mathbb{N}}:\iota\text{ is an }I\text{-sequence}}\bar{V}_\iota\left(\left(1-\lambda\right)h+\lambda g\right)\\&=\inf_{\iota\in I^{\mathbb{N}}:\iota\text{ is an }I\text{-sequence}}\left\{\lambda\bar{V}_\iota\left(f\right)+\left(1-\lambda\right)\bar{V}_\iota\left(g\right)\right\}\\&\geq\lambda\inf_{\iota\in I^{\mathbb{N}}:\iota\text{ is an }I\text{-sequence}}\bar{V}_\iota\left(f\right)+\left(1-\lambda\right)\inf_{\iota\in I^{\mathbb{N}}:\iota\text{ is an }I\text{-sequence}}\bar{V}_\iota\left(g\right)\\&=\left(1-\lambda\right)V_*\left(h\right)+\lambda V_*\left(g\right)\end{aligned}$$

for all $\lambda \in (0,1)$ and for all $g, h \in \mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable. The statement for V^* follows from completely symmetric steps. ■

Proof of Lemma 13. Fix $\beta \in (0,1)$. By Lemma 12, V_* is concave. This implies that

$$V_* \left(S_{\beta,i}(\mathbf{f})\right) = V_* \left(V_i \left(\left(1-\beta\right) \hat{f} + \beta \sum_{l=1}^{n} w_{il} f_l\right)\right) \geq V_* \left(\left(1-\beta\right) \hat{f} + \beta \sum_{l=1}^{n} w_{il} f_l\right)$$
$$\geq \left(1-\beta\right) V_* \left(\hat{f}\right) + \beta \sum_{l=1}^{n} w_{il} V_* \left(f_l\right) \quad \forall i \in I, \forall \mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n.$$

We now prove the statement for $\tau = 1$. We have that

$$V_* \left(S^1_{\beta,i}\left(\hat{\mathbf{f}}\right)\right) = V_* \left(S_{\beta,i}\left(\hat{\mathbf{f}}\right)\right) \geq \left(1-\beta\right) V_* \left(\hat{f}\right) + \beta \sum_{l=1}^{n} w_{il} V_* \left(\hat{f}_l\right) = V_* \left(\hat{f}\right) \quad \forall i \in I.$$

Assume that the statement is true for $\tau \in \mathbb{N}$. Observe that for each $i \in I$

$$V_* \left(S^{\tau+1}_{\beta,i}\left(\hat{\mathbf{f}}\right)\right) = V_* \left(S_{\beta,i}\left(S^{\tau}_{\beta}\left(\hat{\mathbf{f}}\right)\right)\right) \geq \left(1-\beta\right) V_* \left(\hat{f}\right) + \beta \sum_{l=1}^{n} w_{il} V_* \left(S^{\tau}_{\beta,l}\left(\hat{\mathbf{f}}\right)\right) \geq V_* \left(\hat{f}\right).$$

The statement follows by induction. Next, assume that V_i is also nonexpansive for all $i \in I$. By Propositions 1 and 2 and since each V_i is nonexpansive, we have that V_* is nonexpansive.[35] By Lemma 8 and the previous part of the proof and since V_* is a continuous ex-ante expectation, we have that

$$V_* \left(\sigma^{\beta}_i\right) = V_* \left(\lim_{\tau} S^{\tau}_{\beta,i}\left(\hat{\mathbf{f}}\right)\right) = V_* \left(S^{\tau}_{\beta,i}\left(\hat{\mathbf{f}}\right)\right) \geq V_* \left(\hat{f}\right) \qquad \forall i \in I, \forall \beta \in (0,1),$$

proving the statement. ■

Proof of Lemma 14. Assume that $\Theta \neq \emptyset$ and fix $\mu \in \Theta$. We next show that $\mu \in int\left(\Delta\left(\Omega\right)\right)$. First, observe that the full-support assumption on $\left\{\left(V_i, \Pi_i\right)\right\}_{i \in I}$ implies that, for all $i \in I$, $\omega' \in \Omega$, $\omega \in \Pi_i\left(\omega'\right)$, and $p \in \arg\min_{\tilde{p} \in \Delta(\Omega)} c_{i,\omega'}\left(\tilde{p}\right)$, we have $p\left(\omega\right) > 0$.[36] Second, let supp$\mu = E$ and assume by contradiction that $E \neq \Omega$. Since $\Pi_{\sup} = \{\Omega\}$, we have that there exists $\omega \in \Omega \backslash E$, $i \in I$, and $\omega' \in E$ such that $\omega \in \Pi_i\left(\omega'\right)$. Given that $\mu\left(\omega\right) = 0$ and $\mu\left(\omega'\right) > 0$, we obtain

$$p_{\mu,i}\left(\omega', \omega\right) = 0,$$

yielding a contradiction with the fact that $p_{\mu,i}\left(\omega', \cdot\right) \in \arg\min_{\tilde{p} \in \Delta(\Omega)} c_{i,\omega'}\left(\tilde{p}\right)$. ■

Proof of Lemma 15. We first prove an ancillary claim.

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[35] Recall that for any collection of functionals that are nonexpansive, their pointwise infimum is also nonexpansive.

[36] Indeed, for every $i \in I$, the operator $V_i : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is monotone and such that its indicator matrix $A\left(V_i\right)$ (cf. Definition 9) satisfies

$$\omega \in \Pi_i\left(\omega'\right) \implies a_{\omega'\omega} = 1 \qquad \forall \omega, \omega' \in \Omega.$$

In particular, this implies that

$$\arg \min_{\tilde{p} \in \Delta(\Omega)} c_{i,\omega'}\left(\tilde{p}\right) = \partial V_i\left(\omega, 0\right) \subseteq int\left(\Delta\left(\Pi_i\left(\omega'\right)\right)\right),$$

where the first equality follows from [5, Lemma 32]. The inclusion follows from concavity and the definition of the superdifferential.

Claim 2. *For every $i \in N$, $\mathcal{V}_i \subseteq C(F)$ is convex and compact in the topology of uniform convergence.*

Proof. Fix $i \in N$ and consider $\bar{V}, \bar{V}' \in \mathcal{V}_i$ as well as $\lambda \in [0,1]$. Fix $f \in F$ and note that

$$\begin{aligned}\left(\lambda\bar{V} + (1-\lambda)\bar{V}'\right)(f) &= \lambda\bar{V}(f) + (1-\lambda)\bar{V}'(f) \\ &= \lambda\bar{V}(V_i(f)) + (1-\lambda)\bar{V}'(V_i(f)) = \left(\lambda\bar{V} + (1-\lambda)\bar{V}'\right)V_i(f),\end{aligned}$$

showing that $\lambda\bar{V} + (1-\lambda)\bar{V}' \in \mathcal{V}_i$. Next, consider a sequence $\{\bar{V}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{V}_i$ such that $\bar{V}_n \to \bar{V}$. Given that uniform convergence implies pointwise convergence, it is standard to show that $\bar{V}$ is normalized, monotone, translation invariant and such that, for every $f \in F$, $\bar{V}(f) = \bar{V}(V_i(f))$. Therefore, $\mathcal{V}_i$ is closed, hence compact, in the topology of uniform convergence. □

Suppose that there exists no ex-ante expectation $\bar{V}$ such that $\left(\bar{V}, V_i, \Pi_i\right)$ is a generalized conditional expectation for all $i \in \{1,2\}$, that is,

$$\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset.$$

By the Hahn-Banach separation theorem, there exists a linear continuous functional $L : C(F) \to \mathbb{R}$ and $c \in \mathbb{R}$, such that

$$L\left(\bar{V}_1\right) > c > L\left(\bar{V}_2\right)$$

for all $\bar{V}_1 \in \mathcal{V}_1$ and $\bar{V}_2 \in \mathcal{V}_2$. By the Riesz representation theorem, there exists $\nu \in \mathcal{M}(F)$ such that

$$L(V) = \int_F V(f)\,d\nu(f) \qquad \forall f \in F.$$

Therefore,

$$\int_F \bar{V}_1(f)\,d\nu(f) > c > \int_F \bar{V}_2(f)\,d\nu(f)$$

for all $\bar{V}_1 \in \mathcal{V}_1$ and $\bar{V}_2 \in \mathcal{V}_2$. Fix $a \in [-k,k]$ such that $a \neq 0$ and define the measure $\nu_c \in \mathcal{M}(F)$ as

$$\nu_c = \nu - \frac{c}{a}\delta_{\{ae\}}$$

where $ae \in F$ is the constant act assigning a to all states and $\delta_{\{ae\}}$ is the Dirac measure on ae. For every $\bar{V}_1 \in \mathcal{V}_1$ and $\bar{V}_2 \in \mathcal{V}_2$, it follows that

$$\int_F \bar{V}_1(f)\,d\nu_c(f) = \int_F \bar{V}_1(f)\,d\nu(f) - \frac{c}{a}\bar{V}_1(ae) = \int_F \bar{V}_1(f)\,d\nu(f) - c > 0.$$

Symmetrically, we also have

$$\int_F \bar{V}_2(f)\,d\nu_c(f) < 0.$$

Therefore,

$$\int_F \bar{V}_1(f)\,d\nu_c(f) > 0 > \int_F \bar{V}_2(f)\,d\nu_c(f),$$

for all $\bar{V}_1 \in \mathcal{V}_1$ and $\bar{V}_2 \in \mathcal{V}_2$. Given that for every $i \in I$ and $\omega \in \Omega$

$$V_i(\omega, \cdot) \in \mathcal{V}_i,$$

we obtain

$$\int_F V_1(\omega, f)\, d\nu_c(f) > 0 > \int_F V_2(\omega, f)\, d\nu_c(f) \qquad \forall \omega \in \Omega,$$

as desired. Moreover, by [1, Corollary 5.108] $\mathcal{M}_0(F)$ is dense in $\mathcal{M}(F)$ endowed with the weak*-topology, and since Ω is finite and $V_i(\omega, \cdot)$ is continuous for all $i \in \{1, 2\}$ and for all $\omega \in \Omega$, the implication follows. ■

G Online appendix: algorithm to construct extreme ex-ante expectations

In this section, we propose an algorithm to compute V_*. Consider any ex-ante expectation $\hat{V} : \mathbb{R}^\Omega \to \mathbb{R}$ such that $\hat{V} \geq V_*$. For example, one can choose

$$\hat{V}(f) = \max_{\omega \in \Omega} f(\omega) \qquad \forall f \in \mathbb{R}^\Omega.$$

Define recursively the sequence $\left\{\hat{V}^\tau\right\}_{\tau \in \mathbb{N}}$ of real-valued functionals over $\mathbb{R}^\Omega$ by $\hat{V}^1 = \hat{V}$ and

$$\hat{V}^{\tau+1}(f) = \min_{i \in I} \hat{V}^\tau(V_i(f)) \qquad \forall f \in \mathbb{R}^\Omega, \forall \tau \in \mathbb{N}.$$

By induction, we have that each $\hat{V}^\tau$ is an ex-ante expectation. Fix $f \in \mathbb{R}^\Omega$. Since each V_i is an interim expectation, if $\tau \geq 2$, then we have that

$$\begin{aligned}\hat{V}^{\tau+1}(f) &= \min_{i \in I} \hat{V}^\tau(V_i(f)) = \min_{i \in I} \min_{i' \in I} \hat{V}^{\tau-1}(V_{i'}(V_i(f))) \leq \min_{i \in I} \hat{V}^{\tau-1}(V_i(V_i(f))) \\ &= \min_{i \in I} \hat{V}^{\tau-1}(V_i(f)) = \hat{V}^\tau(f).\end{aligned}$$

Since f was arbitrarily chosen, this implies that $\hat{V}^{\tau+1} \leq \hat{V}^\tau$ for all $\tau \in \mathbb{N} \setminus \{1\}$. Define $\hat{V}^\infty : \mathbb{R}^\Omega \to \mathbb{R}$ by $\hat{V}(f) = \lim_\tau \hat{V}^\tau(f)$ for all $f \in \mathbb{R}^\Omega$.

Proposition 10. *For every ex-ante expectation $\hat{V} : \mathbb{R}^\Omega \to \mathbb{R}$ such that $\hat{V} \geq V_*$, we have $\hat{V}^\infty = V_*$.*

Proof of Proposition 10. Since $\left\{\hat{V}^\tau(f)\right\}_{\tau \in \mathbb{N}}$ is an eventually decreasing sequence bounded from below by $\min_{\omega \in \Omega} f(\omega)$, $\hat{V}^\infty$ is a well defined ex-ante expectation. By construction, we have that

$$\hat{V}^{\tau+1}(f) \leq \hat{V}^\tau(V_i(f)) \qquad \forall f \in \mathbb{R}^\Omega, \forall i \in I.$$

By passing to the limit, we obtain that $\hat{V}^\infty(f) \leq \hat{V}^\infty(V_i(f))$ for all $f \in \mathbb{R}^\Omega$ and for all $i \in I$, which in turn yields that $\hat{V}^\infty \leq V_*$ by definition of V_*. Conversely, note that

1. Since $\hat{V}^1 = \hat{V} \geq V_*$, if $\tau = 1$, then $\hat{V}^{\tau+1}(f) = \min_{i \in I} \hat{V}^\tau(V_i(f)) \geq \min_{i \in I} V_*(V_i(f)) \geq V_*(f)$ for all $f \in \mathbb{R}^\Omega$.

2. By induction assume that $\hat{V}^\tau \geq V_*$. It follows that

$$\hat{V}^{\tau+1}(f) = \min_{i \in I} \hat{V}^\tau(V_i(f)) \geq \min_{i \in I} V_*(V_i(f)) \geq V_*(f) \quad \forall f \in \mathbb{R}^\Omega,$$

proving the inductive step.

By induction, we conclude that $\hat{V}^\tau \geq V_*$ for all $\tau \in \mathbb{N}$, yielding that $\hat{V}^\infty \geq V_*$ and, in particular, $\hat{V}^\infty = V_*$. ■

H Online appendix: Omitted examples

Example 9 (Extreme information asymmetry)**.** Consider two traders $I = \{1, 2\}$ that are uncertain about an asset $\hat{f} \in \mathbb{R}^{\Omega}$ and are endowed respectively with full-information $\Pi_1 = 2^{\Omega}$ and no-information $\Pi_2 = \{\Omega\}$. In this case, the interim expectation of an act f by agent 1 in each state $\omega \in \Omega$ must coincide with $f(\omega)$. As Agent 2 does not receive any information, both her ex-ante and interim expectations are variational and given by

$$V_2(f) = \min_{p \in \Delta(\Omega)} \{\mathbb{E}_p[f] + c(p)\}.$$

With this, the interim preferences of the agents admit a common ex-ante expectation which must coincide with the preference of agent 2, that is $\bar{V} = V_2$.[37] Next, for all $\beta \in (0, 1)$, the equilibrium strategy of player 2 does not depend on the realized state

$$\sigma_2^{\beta} = \min_{p \in \Delta(\Omega)} \left\{\mathbb{E}_p\left[(1 - \beta)\hat{f} + \beta\sigma_1^{\beta}\right] + c(p)\right\},$$

while the equilibrium strategy of player 1 is adapted to the realized state

$$\sigma_1^{\beta}(\omega) = (1 - \beta)\hat{f}(\omega) + \beta\sigma_2^{\beta} \qquad \forall\omega \in \Omega.$$

By simple substitution, we get

$$\sigma_2^{\beta} = \min_{p \in \Delta(\Omega)} \left\{\mathbb{E}_p\left[\hat{f}\right] + \frac{1}{(1 - \beta^2)}c(p)\right\} \geq \bar{V}\left(\hat{f}\right),$$

that is, the equilibrium willingness to pay of player 2 coincides with a less ambiguity-averse version of the ex-ante common expectation. In the high-coordination limit, the ambiguity of the agents is restricted only among the least penalized probabilistic models:

$$\lim_{\beta \to 1} \sigma_2^{\beta} = \lim_{\beta \to 1} \sigma_1^{\beta}(\omega) = \min_{\mu \in \Theta} \mathbb{E}_{\mu}\left[\hat{f}\right] \geq \bar{V}\left(\hat{f}\right) \qquad \forall\omega \in \Omega,$$

where $\Theta = \arg\min_{p \in \Delta(\Omega)} c(p)$.[38] In words, the equilibrium price is converging to a cautious evaluation consistent with the most trusted probabilistic models, i.e., $p \in \Delta(\Omega)$ such that $c(p) = 0$. In the maxmin model, where c is the (convex-analysis) indicator function of a set $C \subseteq \Delta(\Omega)$, this cautious evaluation coincides with the common ex-ante expectation, so that $\lim_{\beta \to 1} \sigma_i^{\beta}(\omega) = \bar{V}\left(\hat{f}\right)$. ▲

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, 3rd ed., Springer-Verlag, Berlin, 2006.

[2] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Dynamic Opinion Aggregation: Long-run Stability and Disagreement, mimeo, 2021.

[37] Observe that, given the extreme nature of the information structures considered, there is no need to specify an updating rule for the preferences of the agents.

[38] This last step follows by [51, Proposition 12].

[3] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 341–375, 2011.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, 1990.

[5] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.