Fees, Incentives, and Efficiency in Large Double Auctions

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Abstract

Fees are omnipresent in markets but, with few exceptions, are omitted in economic models—such as Double Auctions—of these markets. Allowing for general fee structures, we show that their impact on incentives and efficiency in large Double Auctions hinges on whether the fees are *homogeneous* (as, e.g., fixed fees and price fees) or *heterogeneous* (as, e.g., bid-ask spread fees). Double Auctions with homogeneous fees share the key advantages of Double Auctions without fees: markets with homogeneous fees are asymptotically strategyproof and efficient. We further show that these advantages are preserved even if traders have misspecified beliefs. In contrast, heterogeneous fees lead to complex strategic behavior (price guessing) and may result in severe market failures. Allowing for aggregate uncertainty, we extend these insights to market organizations other than the Double Auction.

Keywords: Double Auction, Fees, Transaction Costs, Incentives, Strategyproofness, Efficiency, Robustness.

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1 Introduction

Many markets, e.g., for commodities and stocks, are organized by intermediaries such as trading platforms, centralized clearing houses, designated liquidity providers, market makers, or brokers. Such intermediaries typically charge fees for their services; a fee being any difference between the prices that the buyers pay and the amounts that the sellers receive. Some markets operate with *price fees*, that is, fees that are a set percentage of the price; examples include price fees set by governments such as stamp duties and other transaction fees. Tobin taxes (implemented, e.g., in Sweden and Latin America) are also examples of price fees. Private intermediaries use price fees as well. For instance, Airbnb charges a service fee that is a percentage of the total price, and this percentage is listed explicitly on hosts' and guests' invoices.[1] Other private intermediaries use different fee models, among which one popular option can be described as *spread fees* in which the fee is a percentage of the difference between bid and ask. For instance, Robinhood, an online platform for stock trading, earns money on transactions, because market makers pay Robinhood for order flow. Because market makers in turn make money on the bid-ask spread, we can think of Robinhood's fee model as making money, indirectly, on the bid-ask spread too (how and what percentage exactly is unknown). Other platforms, e.g., Charles Schwab and E-Trade, charge market makers for order flow in the same way.[2]

What consequences do different fee schemes have on the strategies of market participants? Despite its importance for market design and regulation, the literature's focus on markets without fees left this question largely open.[3] This paper aims to fill this gap. In Double Auctions (DAs) with general fee structures, we investigate participants' strategic behavior and resulting market efficiency. We characterize optimal strategic behavior, and identify which classes of fees preserve—and which do not—the desirable properties of DAs, that is asymptotic truthfulness and efficiency, that are familiar from the analyses without fees (c.f., e.g., Rustichini et al. 1994). We also analyze the robustness of these properties to market participants having misspecified beliefs.

Our main insight is that these desirable properties of DA markets crucially hinge on whether the fees are *homogeneous* or *heterogeneous*. We say that a fee is homogeneous if, conditional on a market participant trading in the market, the participant's impact on the fee they pay vanishes as the market grows large; else we say that the fee is heterogeneous.[4] Price fees are examples of

[1]This fee is not only a transaction tax, as Airbnb's fee also covers additional services such as insurances. Our analysis applies equally to transaction taxes and fees covering additional services. Similar business models are pursued by other platforms such as Uber, Lyft, etc.

[2]Market makers provide a service by providing liquidity and carrying associated risks. Robinhood imposes no direct fees (commissions) on its users except for small transaction fees that it passes on from authorities such as from the Financial Industry Regulatory Authority (FINRA). Market makers paying platforms for order flow is not universal and it is actually illegal in some countries, including the United Kingdom.

[3]Below we discuss the notable exceptions: the analysis of efficiency under fixed fees in Tatur (2005), market entry in Marra (2019), and platform revenues in Chen and Zhang (2020).

[4]The two fee types are close to partitioning but do not completely partition the set of possible fees; see Section 3.2. We study both large finite and continuum models. In continuum models, the definition simplifies and, conditional on trade, the homogeneous fee paid by a participant is the same irrespective of the action of the participant, while the heterogeneous fees depend on participant's actions.

homogeneous fees as, in the limit, the market participants impact on the fees vanishes (and, relatedly, all participants who trade pay the same fee). Spread fees are examples of heterogeneous fees as, in the limit, the spread and hence the fee paid depends on the trading participant's action. Not surprisingly, under homogeneous fees, the traders behave similarly to traders in no-fee markets and they are approximately *price-taking* in large markets. In contrast, heterogeneous fees distort incentives fundamentally, and, asymptotically, lead to what we call *price-guessing* behavior whereby traders bid close to estimated market prices in order to try to minimize fee payments.

Homogeneous fees lead to some unavoidable welfare losses in finite markets that are due to strategic behavior and unprofitability of trades whose surplus is insufficient to cover the fee.[5] Because price-taking behavior emerges in the limit, in large markets the outcomes are not much affected when the fees are small; and the same obtains even when agents have misspecified beliefs.

In contrast, in large markets, heterogeneous fees lead to asymptotically full efficiency if the beliefs are correctly specified, but even slight belief misspecification often leads to substantive market failure. The risk of market failure occurs for all heterogeneous fees, and the degree of inefficiency does not vanish with decreasing fee size.

Allowing for aggregate uncertainty, we show that the aforementioned results qualitatively hold true for some market organizations other than the canonical DA. In particular, the insights continue to hold true in any market organization in which the participants believe that they have no impact on market prices, as in continuum markets and in Vickrey mechanisms.

Related literature

We know a lot about strategic behavior in DAs without fees as these mechanisms have been extensively studied.[6] Since the formal definition of the situation as one characterized by two-sided incomplete information (Chatterjee and Samuelson, 1983), the analysis of DAs focused on large markets because of the empirical relevance of this setting, and because in finite-size markets Myerson and Satterthwaite (1983) showed that there generally exists no budget-balanced, incentive-compatible, and individually rational mechanism that is Pareto efficient.[7]

In large DA markets, participants have incentives to be increasingly truthful, which results in asymptotic efficiency [Roberts and Postlewaite 1976, Rustichini et al. 1994, Cripps and Swinkels 2006, Reny and Perry 2006, Azevedo and Budish 2019]; any given participant's influence on the market price vanishes in larger markets, and market participants place increasing weight on maximizing their trading probability (as opposed to influencing the price), which they do by bidding close to truthfully.

[5]For simplicity, we evaluate efficiency while ignoring the add-on services (such as aforementioned insurances) provided by intermediaries. Note that even the second of the above mentioned types of inefficiency obtains when we take costs of services by the intermediaries into account except if fees are perfectly aligned with the individual costs of services provided by the intermediary.

[6]See Friedman and Rust (1993) for a survey of the DA as a market mechanism in history, theory and practice.

[7]The impossibility hinges on the quasilinearity of the preferences, which we also assume; see Garratt and Pycia (2016).

Rustichini et al. (1994) established this key insight for DAs with independent private values (c.f. Satterthwaite and Williams (1989b)). Their work assumes existence of symmetric equilibria, which was later established by Fudenberg et al. (2007) under correlated but conditionally independent private values.[8]

We know much less about DAs with fees, except for the case of fixed fees. Tatur (2005) analyzes incentives and efficiency in DAs but only with fixed fees; unlike us he does not require budget balance. Chen and Zhang (2020) study revenues in linear equilibria of DAs with fees; they allow fees to depend on the size of individual trade but not on price, bid-ask spread, nor other parameters of the market schemes. Marra (2019) studies market entry in DAs with fixed fees. Noussair et al. (1998) provides experimental evidence that fixed fees lead to efficiency loss. Fixed fees have also been the focus in the finance literature on limit order books [Colliard and Foucault 2012, Foucault et al. 2013, Malinova and Park 2015].[9] Where this literature focuses on specific fee structures (fixed fees), we look at fees more generally and our classification of fees has no counterpart in the literature. Our general incentive, efficiency, and robustness results are also new.

Our analysis also contributes to the burgeoning literature on market behavior in the presence of misspecified beliefs. The impact of misspecified beliefs on mechanism design has been analyzed by many authors, c.f., e.g., Ledyard (1978), Wilson (1987), Chung and Ely (2007), Bergemann and Morris (2005), Chassang (2013), Bergemann et al. (2015), Carroll (2015), Wolitzky (2016), Carroll (2017), Madarász and Prat (2017), Li (2017), Boergers and Li (2019), Pycia and Troyan (2019). The main thrust of this literature is that robustness to misspecification requires the mechanism to be simple. The impact of heterogeneous, misspecified, beliefs on Walrasian markets has been analyzed e.g., by Harrison and Kreps (1978) and Eyster and Piccione (2013).[10] We contribute to the studies of misspecified models by analyzing how misspecification impacts the efficiency of DAs with fees.

2 The model

2.1 The market

We consider a two-sided market populated by traders belonging to sets $\mathcal{B}, \mathcal{S} \subset \mathbb{R}$ of buyers ($b \in \mathcal{B}$) and sellers ($s \in \mathcal{S}$). Traders are interested in either buying or selling an indivisible good. We consider both the *finite* case, with m buyers $\mathcal{B} = \{1, 2, ..., m\}$ and n sellers $\mathcal{S} = \{1, 2, ..., n\}$, and the *infinite* case, with $\mathcal{B} \subset \mathbb{R}$ and $\mathcal{S} \subset \mathbb{R}$ being two closed intervals. Denote the *distributions* of buyers and

[8] They also generalized the convergence results of Rustichini et al. (1994). Earlier work on equilibrium existence in DAs includes Chatterjee and Samuelson (1983), Wilson (1985), Leininger et al. (1989), Satterthwaite and Williams (1989a), Williams (1991), and Cripps and Swinkels (2006). See also Jackson and Swinkels (2005) who studied equilibrium existence in a broad class of private value auctions that includes DAs.

[9] See also Shi et al. (2013) who study a numerical model of marketplace competition with fees.

[10] See also, e.g., (Heidhues et al., 2018) who study overconfidence in markets and (de Clippel and Rozen, 2018) who study the misperception of tastes.

sellers on $\mathcal{B}$ and $\mathcal{S}$ by μ_B and μ_S.[11] By $R = \frac{\mu_B(\mathcal{B})}{\mu_S(\mathcal{S})}$ we denote the ratio of buyers to sellers.

We are particularly interested in large markets. Say that a property $\mathcal{P}$ holds *in sufficiently large finite markets* (write $ISLFM$) if there exist $m, n \geq 1$ such that $\mathcal{P}$ holds in any finite market with at least m buyers and n sellers. If the property also holds in infinite markets, say that it holds *in sufficiently large markets* (write $ISLM$).

Every trader $i \in \mathcal{B} \cup \mathcal{S}$ has a *type* $t_i \in T = [\underline{t}, \overline{t}]$ giving valuation, reservation price or gross value. T is called the *type space*. Denote by $t_B : \mathcal{B} \to T$, $t_S : \mathcal{S} \to T$ measurable functions that assign a type to each trader. Let μ_B^t and μ_S^t be the push-forward measures of μ_B and μ_S with respect to t_B and t_S, i.e., $\mu_B^t(\cdot) = \mu_B(t_B^{-1}(\cdot))$ and $\mu_S^t(\cdot) = \mu_S(t_S^{-1}(\cdot))$. We call these the *type distributions*. They are σ-additive and finite measures on T, and specify the mass of traders with types inside any measurable subset of T.

Every trader i submits an *action* a_i representing a buyer's *bid* and a seller's *ask*. Denote by $a_B : \mathcal{B} \to A_B$ with $a_B(b) = a_b$ and by $a_S : \mathcal{S} \to A_S$ with $a_S(s) = a_s$ functions that assign an action for each trader. Let the *action distributions* μ_B^a and μ_S^a be two induced σ-additive and finite measures on $\mathbb{R}^{\geq 0}$ with support in the *action spaces* $A_B = [\underline{a}_B, \overline{a}_B]$ and $A_S = [\underline{a}_S, \overline{a}_S]$. That is, $\mu_B^a(\cdot) = \mu_B(a_B^{-1}(\cdot))$ and $\mu_S^a(\cdot) = \mu_S(a_S^{-1}(\cdot))$. Let a denote the joint distribution of bids and asks, specifying the mass of buyers and sellers with actions inside any measurable subset of A_B and A_S. We will often consider *strategies* $a_i : T \to A_i$, where $a_i(t_i)$ specifies the action given i's type.

In a finite market, a single trader influences the distribution of types and actions. Write, with some abuse of notation, $t = (t_i, t_{\text{-}i})$ and $a = (a_i, a_{\text{-}i})$, where $t_{\text{-}i}$ and $a_{\text{-}i}$ are the type and action distributions of all traders excluding trader i. In finite markets, t and a are obtained by adding a point mass at t_i and a_i to $t_{\text{-}i}$ and $a_{\text{-}i}$. Note that, in infinite markets, single traders do not change the distributions, and thus $t = t_{\text{-}i}$ $a = a_{\text{-}i}$.

2.2 The mechanism

Given action distributions a specifying bids and asks for buyers and sellers, the *generalized k-double auction with fees* selects a *market outcome* defined by an *allocation* identifying subsets of $\mathcal{B}^*(a) \subset \mathcal{B}$ and $\mathcal{S}^*(a) \subset \mathcal{S}$ who will be involved in trade together with a unique *market price* $\Pi(a)$ for all deals and *fees* $\Phi(a)$ for all active traders.[12] Denote all active traders by $A^*(a) = \mathcal{B}^* \cup \mathcal{S}^*$.

It will be useful to consider the set of traders whose actions are (strictly) above or below price P; for a relation $\mathcal{R} \in \{\geq, >, =, <, \leq\}$, we therefore introduce the shorthand notations $\mathcal{B}_{\mathcal{R}}(P) = \{b \in \mathcal{B} : a_b \mathcal{R} P\}$ and $\mathcal{S}_{\mathcal{R}}(P) = \{s \in \mathcal{S} : a_s \mathcal{R} P\}$.

[11] These are counting measures for finite, and Lebesgue-measures for infinite.

[12] Whenever the dependence on the action distribution is clear, we will simply write Π, $\mathcal{B}^*$ and $\mathcal{S}^*$. When focusing on a single trader with action a_i, we will write e.g. $\Pi(a_i, a_{\text{-}i})$.

The generalized k-DA with fees

Market price. For $k \in [0,1]$ set the market price as

$$\Pi(a) = k \cdot \min \mathcal{P}^{MC}(a) + (1-k) \cdot \max \mathcal{P}^{MC}(a),$$

where $\mathcal{P}^{MC}(a)$ is the set of *market clearing prices* that equilibrate revealed demand and supply.[a]

Allocation. Given $\Pi(a)$, the following allocations are carried out:

$$\mathcal{S}^*(a) = \mathcal{S}_<(\Pi(a)) \cup \tilde{\mathcal{S}}(a) \text{ and } \mathcal{B}^*(a) = \mathcal{B}_>(\Pi(a)) \cup \tilde{\mathcal{B}}(a),$$

where $\tilde{\mathcal{B}}(a) \subset \mathcal{B}_=(\Pi(a))$ (respectively $\tilde{\mathcal{S}}(a) \subset \mathcal{S}_=(\Pi(a))$) are uniformly random compact sets selecting players to balance trade in case there is market excess.[b]

Fees. Each trader i who is involved in trade has to pay a *fee* $\Phi_i(a) \geq 0$.

[a] A detailed account of demand, supply, and market-clearing prices is in Appendix A.1.

[b] See Appendix A.2 for details regarding the allocation and rationing.

We allow for general fees Φ_i. Commonly observed examples are *price*, *spread*, and *constant fees*: given a percentage $\phi_i \in [0,1]$ and constant $c_i \geq 0$, a fee Φ_i is a *price fee* if $\Phi_i(a) = \phi_i \Pi(a)$, a *spread fee* if $\Phi_i(a) = \phi_i |\Pi(a) - a_i|$, and a *constant fee* if $\Phi_i(a) = c_i$.[13]

2.3 Market performance

Here, we introduce various metrics that will be used to evaluate market outcomes (in Section 4).[14]

Demand and *supply* at a price P are defined as $D(P) = \mu_B(\mathcal{B}_\geq(P))$ and $S(P) = \mu_S(\mathcal{S}_\leq(P))$, that is, by the mass of all traders who weakly prefer trading over not trading at P.[15] The *trading volume* at P is $Q(P) = \min(D(P), S(P))$ and the *trading excess* is $Ex(P) = |D(P) - S(P)|$.

The individual *gains of trade* for a buyer b with gross value t_b are $t_b - \Pi$. Similarly, for a seller s with gross value t_s, the gains of trade are $\Pi - t_s$.[16] The *total gains of trade* GoT are $GoT = \mathbb{E}[\int_{\mathcal{B}^*} (t_b - \Pi)\, d\mu_B(b) + \int_{\mathcal{S}^*} (\Pi - t_s)\, d\mu_S(s)]$, where the expectation is taken with respect to the random allocation in case of excess. If agents report their gross values truthfully, the total gains of trade are maximized by market clearing at GoT_Φ. In the absence of fees this coincides with reporting their gross value, achieving the maximum total gains of trade, GoT_{id}. We refer to

[13] If $\phi_i = 0$ or $c_i = 0$, the setting simplifies to the classical feeless DA. Further, for spread fees, if $\phi_i = 1$ a trader has to pay their bid/ask. This setting resembles, for example, Priceline.com's *Name-Your-Own-Price* auction.

[14] Note that in these metrics we omit dependencies on types and action distributions, because those will not be varied when evaluated.

[15] Analytic properties of demand and supply are formulated in the Appendix A.1, and proven for the feeless generalized DA in Jantschgi et al. (2022).

[16] We focus on individually rational strategies $a_B(t_b) \leq t_b$ and $a_S(t_s) \geq t_s$, so that the individual gains of trade are non-negative.

$E_\Phi = {}^{GoT}/_{GoT_\Phi}$ as the *efficiency ratio*, which measures, how much of the achievable—subject to individual rationality given fee considerations—gains of trade are realized.

The *total fees collected* are $Fees = \int_{\mathcal{B}^*} \Phi_b d\mu_B(b) + \int_{\mathcal{S}^*} \Phi_s d\mu_S(s)$. Note that $Fees$ is deterministic, because the random allocation is only concerned with a set of traders with equal actions.

By the *surplus generated by the traders* we refer to the difference between the total gains of trade and the total fees generated: $Surplus = GoT - Fees$. Similarly, by *loss* we refer to $Loss = GoT_{id} - GoT$, which measures how much gains of trade are lost due to fee considerations and strategic behavior. GoT_{id} can therefore be decomposed into total fees, total surplus generated by the traders and the loss due to strategic behavior: $GoT_{id} = Surplus + Fees + Loss$.

2.4 Probabilistic types

We assume that traders' types are independent random variables on the type space T and that they are identically distributed for each of the two market sides. Let (F_B^t, F_S^t) be the pair of corresponding cumulative distribution functions, which are assumed to be differentiable with continuous derivative (i.e., C^1 functions). Let (f_B^t, f_S^t) be the corresponding probability density functions that have full support on the type space T. In a finite market, realizations of these random variables induce type distributions. Call the random empirical measures $\mu_B^t = \sum_{j=0}^m \delta_{t_b^j}$ and $\mu_S^t = \sum_{k=0}^n \delta_{t_s^k}$. Letting n and m tend to infinity, normalized versions of the random empirical measures converge uniformly to deterministic probability measures with densities f_B^t and f_S^t. In an infinite market, these measures are scaled to achieve the market ratio R. Strategies of traders induce random action distributions. If all traders use a symmetric strategy profile (a_B, a_S), where both strategies are strictly increasing C^1-functions, then actions are distributed according to $F_{B,i}^t(a_B^{-1}(\cdot))$ on A_B and $F_{S,i}^t(a_S^{-1}(\cdot))$ on A_S.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space describing the randomness of sampling type distributions and possible rationing.[17] Denote by $\mathbb{E}[\cdot]$ the *expectation* with respect to the probability measure $\mathbb{P}$.

2.5 Incentives and beliefs

Write $u_b(t_b, a_b, a_{\text{-}b}) = t_b - \Pi(a_b, a_{\text{-}b}) - \Phi_b(a_b, a_{\text{-}b})$ for the *utility* of a buyer b when trading with gross value t_b given own action a_b and all other actions $a_{\text{-}b}$. Analogously, for a seller s who trades, write $u_s(t_s, a_s, a_{\text{-}s}) = \Pi(a_s, a_{\text{-}s}) - t_s - \Phi_s(a_s, a_{\text{-}s})$.

In the feeless DA, bidding one's gross value, that is, $a_i(t_i) = t_i$, is the maximal bid for a buyer (minimal ask for a seller) that constitutes an undominated action.[18] The same is not necessarily the case if a fee is charged. Indeed, for some fees (in particular, price fees and constant fees) bidding one's gross value t_i is dominated. We therefore define the *net value*, t_i^Φ, as the largest (smallest) undominated action for a buyer (seller). Without fees and for spread fees the gross value equals the

[17] In finite markets, rationing is a probability zero event, because actions are assumed to have a continuous distribution, see Appendix A.2 and Appendix A.3.1.

[18] We say that an action a_i is undominated if it is not weakly dominated. That is, there exists no a_i' such that for all action distributions $a_{\text{-}i}$ $u_i(t_i, a_i, a_{\text{-}i}) \leq u_i(t_i, a_i', a_{\text{-}i})$.

net value. By contrast, for price fees the net value scales the gross value to account for the fee, that is, $t_b^\Phi = {}^{t_b}/_{1+\phi_b}$ and $t_s^\Phi = {}^{t_s}/_{1-\phi_s}$. Similarly, for constant fees, the net value shifts the gross value, that is, $t_b^\Phi = t_b - c_b$ and $t_s^\Phi = t_s + c_s$.[19] In the presence of fee considerations, it is natural for us to adapt the wording of *truthful* to mean that traders bid their net value. Without this scaling a trader might be involved in a trade that leads to a negative utility. To exclude pathological fee scenarios, and to allow for a meaningful analysis of market participation, we will assume that fee structures under considerations are such that the net value exists, is increasing in the gross value with $t_b^\Phi \leq t_b$ and $t_s^\Phi \geq t_s$, and that the expected utility when bidding the net value is non-negative. Price, spread and constant fees all satisfy these assumptions.

We assume traders know the market mechanism, but have incomplete information regarding the number of other traders, the distribution of gross values, market behavior of other traders and what fees are charged.[20] Traders may have heterogeneous and incorrect beliefs. A given trader i believes to be in *market environment* $\mathcal{M}_i$ with fees Φ_i and a ratio of buyers to sellers equal to R_i. We work with traders' beliefs that are specified directly over the distributions of actions.[21] Actions of other traders are assumed to be independent random variables, identically distributed for each of the two market sides. Let $(F_{B,i}, F_{S,i})$ be the pair of corresponding C^1 distribution functions, with densities $f_{B,i}$ and $f_{S,i}$ that have full support on action spaces $A_{B,i} = [\underline{a}_{B,i}, \overline{a}_{B,i}]$ and $A_{S,i} = [\underline{a}_{S,i}, \overline{a}_{B,i}]$.[22] Such beliefs induce random empirical measures describing the distributions of actions in both finite and infinite markets.[23]

In an infinite market, the market price is equal to the unique solution of the equation $F_{B,i}(\cdot) + R_i F_{S,i}(\cdot) = 1$. Call this solution the *critical value* Π_i^∞.[24] This threshold will be of central importance for the study of large markets, see Theorem 1.

Given the beliefs of trader i, let $(\Omega_{\text{-}i}, \mathcal{F}_{\text{-}i}, \mathbb{P}_{\text{-}i})$ be the probability space describing the randomness of action distributions and allocations in case of excess. Denote by $\mathbb{E}_{\text{-}i}[\cdot]$ the *expectation* with respect to the probability measure $\mathbb{P}_{\text{-}i}$. Furthermore, for an action a_i, let $\mathbb{P}_{\text{-}i}[i \in A^*(a_i, a_{\text{-}i})]$ denote the *probability of trading* for trader i. Let $A^*(i, a_i)$ denote the sub σ-algebra of $\mathcal{F}_{\text{-}i}$ generated by these events. Let $\mathbb{E}_{\text{-}i}[\cdot | A^*(i, a_i)]$ be the *expectation conditional on trading.* In infinite markets, the only random influence on the trading probability is the fair lottery that is used to deal with excess. If a trader is on the market side with no excess and their action is less aggressive then the critical value, then the trading probability is equal to 1. Otherwise there is *tie-breaking*, and for one market side,

[19]See Appendices B.1 and B.2 for details.

[20]We actually treat both cases, when traders know the exact fee and when they only know the fee type.

[21]This streamlined approach permits beliefs about distributions of gross values and strategies of other traders, but also more general beliefs.

[22]Assume that $\overline{a}_{S,i} \geq \overline{a}_{B,i} > t_i^\Phi > \underline{a}_{S,i} \geq \underline{a}_{B,i}$. That is, the action spaces intersect, which means that there are both buyers and sellers who are in and out of the market, so that a trader believes that being truthful ensures competing with both buyers and sellers.

[23]Note that infinite markets as a limit of finite markets have absolutely continuous action distributions. For some applications, we allow general action distributions in limit markets, see e.g. Theorem 2.

[24]Existence and uniqueness are proven in Appendix B.4.

the trading probability lies in $[0,1)$.[25] Assume that a trader in an infinite market has beliefs about the tie-breaking probability.

2.6 Solution Concept

Best responses maximize individual expected utility given beliefs. The two opposing forces are increasing the utility conditional on trading by being more aggressive and increasing the probability of trading by being less aggressive.[26] Aggressiveness refers to the amount of a bid's (or ask's) misrepresentation below a buyer's (above a seller's) gross value: A buyer's bid a_b^1 is *(strictly) more aggressive* than a_b^2, write $\underset{(\succ)}{\succcurlyeq}$, if $a_b^1 \underset{(>)}{\geq} a_b^2$ and similarly a seller's offer a_s^1 is *(strictly) more aggressive* than a_s^2, write $\underset{(\succ)}{\succcurlyeq}$, if a_s^1 is (strictly) less than a_s^2. Strategically optimal behavior finds the right amount of aggressiveness. Given trader i's market environment $\mathcal{M}_i$ and gross value t_i, an action a_i is an ϵ-*best response* if $\mathbb{E}_{\text{-}i}\left[u_i\left(t_i, a_i, a_{\text{-}i}\right)\right] \geq \sup_{a_i' \in \mathbb{R}} \mathbb{E}_{\text{-}i}\left[u_i\left(t_i, a_i', a_{\text{-}i}\right)\right] - \epsilon$. For $\epsilon = 0$ a_i is a best response.

The analysis of best responses includes the special case of *symmetric Bayesian Nash equilibria.* If all buyers use the same strictly increasing strategy a_B and all sellers use the same strictly increasing strategy a_S, call (a_B, a_S) a *symmetric strategy profile.* Given type distributions, the corresponding action distributions are given by $\mu_B^a(\cdot) = \mu_B\left(t_B^{-1}(a_B^{-1}(\cdot))\right)$ and $\mu_S^a(\cdot) = \mu_S\left(t_S^{-1}(a_S^{-1}(\cdot))\right)$. Assume that beliefs over action distributions originate from beliefs over gross value distributions and over the symmetric strategy profiles of the other traders (a_B, a_S). If, for every trader and every gross value, the action specified by these strategies are best responses, then the strategy profile constitutes a symmetric Bayesian Nash equilibrium.[27]

3 Large market asymptotics

3.1 Core properties of best responses

Underlying several of the key results that will follow in this section is the following observation: If indeed others' behaviors are consistent with a given trader's beliefs, then that trader can compute the market price with increasing accuracy as the market grows, and indeed precisely in the limit market. With more traders on both market sides, actions approximate a continuum, the variance of realized market prices decreases, and it becomes increasingly predictable who gets to trade. In the limit, the following proposition holds for a given trader's trading probability.

Proposition 1 (The trading probability converges to a step function at Π_i^∞)**.** *Consider a trader i with actions $a_i^1 \succ \Pi_i^\infty \succ a_i^2$. $\forall \epsilon > 0$ ISLM $\mathbb{P}_{\text{-}i}\left[i \in A^*(a_i^1, a_{\text{-}i})\right] \leq \epsilon$, $\mathbb{P}_{\text{-}i}\left[i \in A^*(a_i^2, a_{\text{-}i})\right] \geq 1 - \epsilon$.*

[25] See Appendix A.3 for more details on the trading probability in the limit market.

[26] A detailed analysis of this trade-off for price and spread fees in finite markets via first order conditions can be found in Appendix A.4.

[27] Therefore all of the results that we shall present in this paper about best responses directly apply to the study of symmetric Bayesian Nash equilibria.

Proof Outline. In infinite markets, the statement follows directly from the model. Growing market size in finite markets is formalized with respect to a single parameter. Consider a sequence of strictly increasing market sizes $(m(l), n(l))_{l\in\mathbb{N}}$ with $m(l), n(l) = \Theta(l)$ and $|R - \frac{n(l)}{m(l)}| = \mathcal{O}(l^{-1})$ for $R \in (0,\infty)$.[28] A buyer b is involved in trade, if their action a_b is greater (or equal, if they win tie-breaking) than at least $m(l)$ actions of other traders, that is $\mathbb{P}_{\text{-}b}\left[b \in A^*(a_b, a_{\text{-}b})\right] = \mathbb{P}_{\text{-}b}\left[a_b \geq a_{\text{-}b}^{m(l)}\right]$. The probability that the action of any other buyer and seller is below a_b is $p_{a_b} = F_{B,b}(a_b)$ and $q_{a_b} = F_{S,b}(a_b)$. If $X_i^{p_{a_b}}$ and $X_i^{q_{a_b}}$ are Bernoulli random variables with parameters p_{a_b} and q_{a_b}, then the total number of traders with actions below a_b has the same distribution as the sum $S_l^{a_b} = \sum_{i=1}^{m(l)-1} X_i^{p_{a_b}} + \sum_{i=1}^{n(l)} X_i^{q_{a_b}}$. It follows that $\mathbb{P}_{\text{-}b}\left[b \in A^*(a_b, a_{\text{-}b})\right] = \mathbb{P}[S_l^{a_b} \geq m(l)] = 1 - \mathbb{P}[S_l^{a_b} \leq m(l) - 1]$. By the Berry-Esseen Theorem (Tyurin, 2012) an appropriately normalized version of $S_l^{a_b}$ converges in distribution to a standard normal random variable with CDF Φ. We show that there exists a sequence $(A_{a_b}(l))_{l\in\mathbb{N}} = \Theta(\sqrt{l})$ with $|\mathbb{P}[S_l^{a_b} \leq m(l) - 1] - \Phi(A_{a_b}(l))| \in \mathcal{O}(l^{-\frac{1}{2}})$. For $a_b \prec \Pi_b^\infty$ we show for sufficiently large l that $A_{a_b}(l) < 0$, which yields that $A_{a_b}(l) \in \Theta(-\sqrt{l})$. Using a concentration inequality for a standard Gaussian random variable gives $\Phi(A_{a_b}(l)) \in \mathcal{O}(e^{-l})$. It therefore holds that $\mathbb{P}[S_l^{a_b} \leq m(l) - 1] = \mathcal{O}(l^{-\frac{1}{2}})$. The statement for $a_b \succ \Pi_b^\infty$ and for sellers can be derived analogously.[29] □

Note that, at the critical value, the trading probability in finite markets is determined by the action distributions and lies strictly between 0 and 1.[30] Next, we shall establish the existence of best responses under mild conditions on fees.

Proposition 2 (Existence of best responses)**.** *Provided that the expected fee payment conditional on trading,* $\mathbb{E}_{\text{-}i}\left[\Phi_i(\cdot, a_{\text{-}i}) | A^*(i, \cdot)\right]$*, is almost surely continuous, a best response exists in finite market environments and in infinite market environments without tie-breaking.*

Note that standard types of fees, such as constant, price, and spread fees, satisfy the continuity assumption of this proposition. For infinite markets, best responses might not exist for a player with $t_i \prec \Pi_i^\infty$. This is the case for certain fee types if there is rationing, e.g., spread fees. On the one hand, it is optimal for a trader to approximate Π_i^∞ in order to decrease the spread fee that is due, but, on the other hand, a trader will not want to be too aggressive in order to avoid the risk of not trading due to rationing.

We will sometimes focus on 'in-the-market' gross values t_i with $t_i^\Phi \prec \Pi_i^\infty$. Such gross values correspond to traders who are able to submit individually rational actions such that they are likely to be involved in trade in large markets. By contrast, for an 'out-of-the-market' trader with gross value $t_i^\Phi \succ \Pi_i^\infty$, the probability of trade (and therefore also the expected utility) goes to zero.

[28] If there exists a parameter l, such that for every $l' \geq l$ Proposition 1 holds in markets with $m(l')$ buyers and $n(l')$ sellers, then the statement also holds ISLFM.

[29] The proof is relegated to Appendix B.5.

[30] For uniform action distributions and equally many buyers and sellers, the trading probability is independent of the market size and equal to $\frac{1}{2}$ (see Theorem 8).

Proposition 3 (For 'out-of-the-market' gross values, truthfulness is close to optimal)**.** *Consider a trader i with $t_i^\Phi \succ \Pi_i^\infty$. $\forall \epsilon > 0$, truthfulness is an ϵ-best response ISLM.*

The proofs of Propositions 2 and 3 are relegated to Appendix B.6 and Appendix B.7.

3.2 Characterization of fees

We consider a general class of 'well-behaved' fees. What we require from a well-behaved fee is that it is *uniformly profit-permitting*. That is, if a trader is likely to trade by being truthful in ISLM, then this results in a strictly positive utility: For every gross value t_i with $t_i^\Phi \prec \Pi_i^\infty$, there exists $\epsilon > 0$ such that $\mathbb{E}_{\text{-}i}\left[u_i\left(t_i, t_i^\Phi, a_{\text{-}i}\right)\right] \geq \epsilon$ ISLM. As it turns out, optimal strategic behavior in large markets depends crucially on whether or not the associated fee asymptotically depends on one's own action or not.

Definition (Homogeneous vs. heterogeneous fees)**.** Two actions $a_i^1 \prec a_i^2 \prec \Pi_i^\infty$ lead to *asymptotically different fee payments*, if there exists $\epsilon > 0$ such that ISLM

$$\mathbb{E}_{\text{-}i}\left[\Phi_i(a_i^1, a_{\text{-}i}) | A^*\left(i, a_i^1\right)\right] - \mathbb{E}_{\text{-}i}\left[\Phi_i(a_i^2, a_{\text{-}i}) | A^*\left(i, a_i^2\right)\right] \geq \epsilon \tag{1}$$

almost surely. Otherwise, the two actions lead to *asymptotically equal fee payments*. Φ_i is *heterogeneous*, if every two actions $a_i^1 \prec a_i^2 \prec \Pi_i^\infty$ lead to asymptotically different fee payments. A fee Φ_i is called *homogeneous*, if $\forall \epsilon > 0$ ISLM almost surely

$$\sup_{a_i^1 \prec a_i^2 \prec \Pi_i^\infty} \mathbb{E}_{\text{-}i}\left[\Phi_i\left(a_i^1, a_{\text{-}i}\right) | A^*\left(i, a_i^1\right)\right] - \mathbb{E}_{\text{-}i}\left[\Phi_i\left(a_i^2, a_{\text{-}i}\right) | A^*\left(i, a_i^2\right)\right] \leq \epsilon. \tag{2}$$

In an infinite market, the definitions simplify: For heterogeneity, the conditional expected fee is strictly monotone for $a_i \prec \Pi_i^\infty$. For homogeneity, the conditional expected fee is constant for $a_i \prec \Pi_i^\infty$. Homogeneity and heterogeneity are not mutually exclusive, as one can construct fee schedules that are homogeneous in some price regions and heterogeneous at others. However, focusing on these two cases (rather than on hybrids) allows us to study the key strategic differences that in fact yield completely opposing behavior. In particular, the two canonical examples of fees, price and spread fees, fall under the two definitions: Price fees are homogeneous, and spread fees are heterogeneous. Fee structures may have significant strategic consequences.

Theorem 4 (Best responses $\Rightarrow$ asymptotically equal fee payments)**.** *Given two gross values t_i^1, t_i^2, the best responses $a_i^1(t_i^1)$ and $a_i^2(t_i^2)$ result in asymptotically equal fee payments.*

Proof Outline. Assume that two actions $a_i^1 \prec a_i^2 \prec \Pi_i^\infty$ lead to asymptotically different fee payments. We show that ISLM, a trader can increase their expected utility, when switching from action a_i^1 to a_i^2, proving that a_i^1 is not a best response. Formally, as $a_i^1 \prec a_i^2 \prec \Pi_i^\infty$, Theorem 1 yields $\forall \epsilon_1 > 0$ ISLM $\mathbb{P}_{\text{-}i}\left[i \in A^*(a_i^1, a_{\text{-}i})\right], \mathbb{P}_{\text{-}i}\left[i \in A^*(a_i^2, a_{\text{-}i})\right] \geq 1 - \epsilon_1$. The difference in trading probability between a_i^1

and a_i^2 is upper bounded by ϵ_1 ISLM. If ϵ_1 is sufficiently small, the loss in trading probability and possible influence on the market price is compensated by a decrease in expected fee payment by at least some $\epsilon_2 > 0$ because of asymptotically different fee payments. For sufficiently small ϵ_1, the difference in expected utility between actions a_i^1 and a_i^2 is negative ISLM, proving that a_i^1 is indeed not a best response.[31] □

Note that for homogeneous fees the condition holds by definition. For heterogeneous fees, the result is non-trivial and will be useful in later analyses (see Section 3.4).

3.3 Price-taking is approximately optimal with homogeneous fees

Strategic misrepresentation is driven by the incentive to influence market price and fee. Reporting truthfully maximizes one's trading probability. In large markets, the influence on the market price is vanishing 'faster' than the influence on one's trading probability, which is what drives the asymptotic truthfulness result in the literature. Therefore, if the influence on one's own fee payment is also vanishing 'fast' as the influence on the market price, then it is close to optimal to maximize one's trading probability by acting as a *price-taker*, that is, by reporting truthfully. Exactly that is the case for homogeneous fees, such as the price fee.

Theorem 5 (In large markets with homogeneous fees price-taking is an approximate best response)**.** *Suppose a homogeneous fee is charged. If trader i's best response is uniformly bounded away from their critical value, then* $\forall \epsilon > 0$ *truthfulness is an* ϵ*-best response ISLM.*

Proof Outline. Consider a best response a_i of trader i. If $a_i \prec t_i^\Phi$, then t_i^Φ is a best response by weak domination. Suppose now that $a_i \succ t_i^\Phi$. By assumption, there exists $\delta > 0$, such that ISLM, (i) $a_i \prec \Pi_i^\infty - \delta$ or (ii) $a_i \succ \Pi_i^\infty + \delta$ holds. If (i) holds, then Theorem 1 implies that $\mathbb{P}_{\text{-}i}\left[i \in A^*(a_i, a_{\text{-}i})\right]$ converges to zero as the market gets large. Therefore $\forall \epsilon > 0$ the expected utility of a_i is upper bounded by ϵ ISLM, which also proves that that the net value is an ϵ-best response, because it leads to a non-negative expected utility. If (ii) holds, consider $\mathbb{E}_{\text{-}i}[u_i(t_i, a_i, a_{\text{-}i})] - \mathbb{E}_{\text{-}i}[u_i(t_i, t_i^\Phi, a_{\text{-}i})]$. We split the difference into two components and show that for every $\forall \epsilon > 0$ both components are less or equal than $\frac{\epsilon}{2}$ ISLM: (a) Difference in expected fees and (b) Terms corresponding to a classical feeless DA. To bound (a), we can use Theorem 1 and homogeneity. For (b), we will use that for a feeless DA truthfulness is an ϵ-best response ISLM, see Theorem 6.2 with price fees equal to zero.[32] □

[31] The proof is relegated to Appendix B.8.
[32] The proof is relegated to Appendix B.9.

Price fees. Fixing a specific fee allows sharper results than Theorem 5. In particular, for a price fee, any best response can be explicitly shown to be close to truthful in large markets.

Theorem 6 (In large markets with price fees best responses are approximately truthful and truthfulness is an approximate best response)**.** *Suppose a price fee is charged. For every finite and infinite market, there exists a best response. Further,* $\forall \epsilon > 0$ *it holds that (1) all best responses are* ϵ*-truthful ISLFM and (2) truthfulness is an* ϵ*-best response ISLM.*

Proof Outline. Consider a buyer b. The expected fee is a percentage of the expected market price, which is shown to be continuous in a_i in the proof of Theorem 2. Therefore, the expected utility is continuous in a_i and the existence of a best response again follows from the Extreme Value theorem. For (1), a best response satisfies the first order condition $\frac{d\mathbb{E}_{\text{-}b}[u_b(t_b,a_b,a_{\text{-}b})]}{da_b} = 0$, see Appendix A.4. Explicit calculations yield that there exists a constant $\kappa > 0$, such that $t_b - (1+\phi_b)\, a_b \leq \kappa q(n,m)$, with $q(m,n) = \max\left\{\frac{1}{n}\left(1+\frac{m}{n}\right), \frac{1}{m}\left(1+\frac{n}{m}\right)\right\} = O(\max(m,n)^{-1})$, from which the statement follows.[33] For (2), we estimate $\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^{\Phi}, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right]$, where a_b denotes the best response. This difference is shown to be upper bounded by $-2k(1+\phi_b)|t_b^{\Phi} - a_b|$. It follows from (1) that $\forall \delta > 0$ it holds that $t_b^{\Phi} - a_b \leq \delta$ ISLFM. If for a given $\epsilon > 0$, $\delta > 0$ is chosen such that $\delta \leq \frac{\epsilon}{2k(1+\phi_b)}$, it holds ISLFM that t_b^{Φ} is ϵ-close to a best response a_b. In infinite markets, the expected utility is deterministic and truthfulness is a best response, as the only strategic incentive is to be involved in trade.[34] □

Example (Best responses and Bayesian Nash equilibria)**.** Set the price fee to $\phi_i = 0.1$ and consider a finite market with sizes (i) 2×2 (that is, two buyers and two sellers) and (ii) 5×5. Figure 1 shows best response strategies (for uniform beliefs over others' actions in $[1, 2]$) and a symmetric Bayesian Nash Equilibrium for the two market sides (for uniform beliefs over gross values in $[1, 2]$). In line with Theorem 6.1, optimal strategic behavior converges to truthfulness with growing market size. In a small market (2×2), traders have an incentive to be more aggressive and misrepresent their net value, as can be measured by the distance between their respective best response (dashed red/blue lines) and the net value (solid black lines). In contrast, and in line with Theorem 6.1, the best responses (dotted red/blue line) in the larger market (5×5) are approaching truth-telling.

[33] A similar proof technique has been used to show that Bayesian Nash equilibria are approximately truthful in DAs without fees, see Rustichini et al. (1994, Theorem 3.1).

[34] The proof is relegated to Appendix B.10.

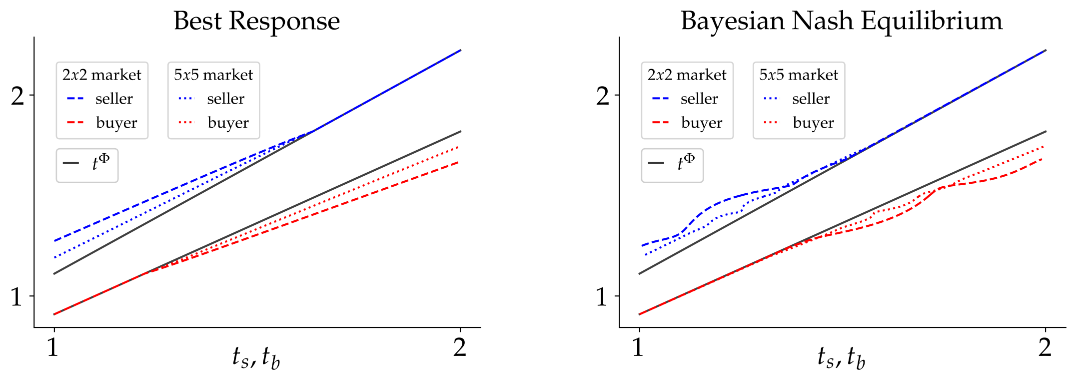

Figure 1: Best responses (left) and a symmetric Bayesian Nash equilibrium (right) for buyers (red) and sellers (blue) as functions of their gross value for 2×2 (dashed lines) and 5×5 (dotted lines) markets. k=0.5, price fee ϕ_i=0.1, and uniform beliefs over actions (left) and gross values (right).

3.4 Price-guessing is approximately optimal with heterogeneous fees

If a trader can influence their fee payment, then there remains a (non-vanishing) incentive to act strategically in large markets. Moreover, given a trader will almost certainly trade as long as their action meets the required threshold of the critical value, the incentive to influence their fee asymptotically outweighs the concern of loosing out on the deal. Therefore, it is optimal to bid close to the critical value that corresponds to the predicted price, which is why we shall call such behavior *Price-Guessing*.

Theorem 7 (In large markets with heterogeneous fees best responses are close to price guessing)**.** *Suppose a heterogeneous fee is charged to a trader i with $t_i^\Phi \prec \Pi_i^\infty$. $\forall \epsilon > 0$ all best responses are in an ϵ-neighbourhood of the critical value ISLM.*

Proof Outline. Consider a buyer with action $a_b > \Pi_b^\infty$. We show that if $a_b - \Pi_b^\infty \geq \epsilon$, then the difference in expected utility from playing a_b versus $\Pi_b^\infty + \frac{\epsilon}{2}$ is strictly negative ISLM, proving that a_b is not a best response ISLM. Similar to the proof of Theorem 4, we show that ISLM, the buyer will be involved in trade with high probability with both actions. Using that the fee is heterogeneous, the decrease of the fee when switching to the more aggressive action $\Pi_b^\infty + \frac{\epsilon}{2}$ outweighs the decrease in trading probability.[35] □

Spread fees. As a spread fee depends linearly on a trader's action, it is an example of a heterogeneous fee. A best response exists given the spread fee is continuous and must be close to the critical value. However, an analogous statement to Theorem 6.2, i.e. the utility at the critical value is close to optimal, is not true in general. We show that there exist markets, such that bidding the critical value is in general not ϵ-optimal in large markets.

[35] The proof is relegated to Appendix B.11.

Theorem 8 (In large markets with spread fees best responses are close, but not necessarily equal, to the critical value). *Suppose a positive spread fee is charged to a trader i with $t_i^\Phi \prec \Pi_i^\infty$. For a finite market and limit markets without rationing, a best response exists. In limit markets with rationing, there exists no best response. Further:*

1. *$\forall \epsilon > 0$ all best responses are in an ϵ-neighbourhood of the critical value ISLM.*

2. *For sufficiently small $\epsilon > 0$, there exist beliefs, such that the critical value is not an ϵ-best response ISLFM.*

Proof Outline. We show that the expected fee is and therefore the expected utility is continuous in a_i. The existence of a best response again follows as in Theorem 6. Consider a buyer b with $t_b^\Phi > \Pi_i^\infty$. (1) is proven in complete analogy to Theorem 5.1. For (2), consider beliefs such that the number of traders is equal to l for both market sides, where beliefs are uniformly distributed over $A_B = A_S = [0,1]$. It follows that $\Pi_b^\infty = \frac{1}{2}$. We prove that for every $l > 1$ it holds that $\mathbb{P}_{\text{-}b}[b \in \mathcal{B}^*(\Pi_b^\infty, a_{\text{-}b})] = \frac{1}{2}$. Therefore, for every bid $a_b > \Pi_b^\infty$ and for every $\epsilon > 0$, it follows from Theorem 1 that the buyer can increase their trading probability by $\frac{1}{2} - \epsilon$ when switching from Π_b^∞ to a_b. If a_b is chosen close to Π_b^∞, then this outweighs the increase in spread fee payment.[36] □

Example (Best responses and Bayesian Nash equilibria). Set the spread fee to $\phi_i = 1$ and consider finite markets with size (i) 2×2 and (ii) 5×5. Figure 2 shows best response strategies (for uniform beliefs over others' actions in $[1,2]$) and a symmetric Bayesian Nash equilibrium (for uniform beliefs over gross values in $[1,2]$). Note that in line with Theorem 8.1, best responses converge towards price-guessing with growing market size. In a small market with two buyers and two sellers traders have an incentive to be aggressive and misrepresent their true net value in order to influence the price and reduce their fee payment. In line with implications from Theorem 8, best responses in a larger market with five buyers and sellers (dotted line) do not approach truth-telling, if $t_i \prec \Pi_i^\infty$. Instead traders remain aggressive as they aim to reduce their fee payment. In contrast, their influence on the price diminishes which results in traders approximating the critical value Π_i^∞ provided it is individually rational.

4 Efficiency as a function of fees and beliefs

In this section, we evaluate efficiency of market outcomes under homogeneous and heterogeneous fees when traders adopt best responses as were characterized in the previous section. We show that homogeneous fees cause an inefficiency that scales with fee size and gets smaller in larger markets, while heterogeneous fees result in knife-edge results with either no or substantial efficiency loss that does not vanish asymptotically and does not scale in fee size.

[36]The proof is relegated to Appendix B.12.

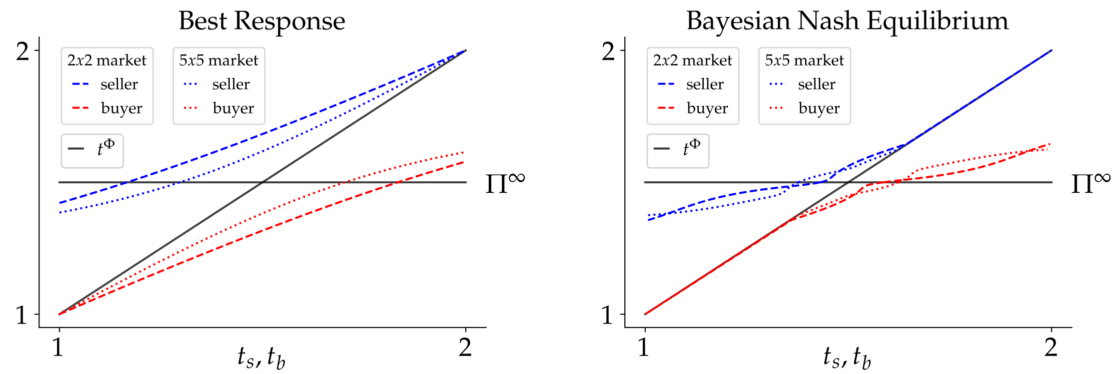

Figure 2: (left panel) and a symmetric Bayesian Nash equilibrium (right panel) for buyers (red) and sellers (blue) as functions of their gross value for 2×2 (dashed lines) and 5×5 (dotted lines) markets. $k = 0.5$, spread fee $\phi_i = 1$, and uniform beliefs over actions (left) and gross values (right).

In our analysis, we shall speak of traders having *belief systems* F about the market, allowing for heterogeneous beliefs in the population. F consists of two mappings, M_B, M_S, from type space T into the space of market environments, with $M_i(t_i)$ denoting what trader i with type t_i believes.

4.1 Homogeneous fees

Given beliefs F, for $\epsilon > 0$, define $\Upsilon^{\epsilon,opt}_{\Phi,F}$ as consisting of all strategies (a_B, a_S) that are strictly increasing and ϵ-truthful. Given homogeneous fees Φ, we know that ϵ-truthfulness emerges asymptotically (see Theorems 5 and 6).

Theorem 9 (In large markets with homogeneous fees, independent of the belief system, strategic behavior leads to almost full efficiency)**.** *Suppose a homogeneous fee Φ is charged. For all $\zeta > 0$, there exists a sequence of $\epsilon > 0$, such that for all strategies (a_B, a_S) in $\Upsilon^{\epsilon,opt}_{\Phi,F}$ it holds that $\mathbb{E}[E_\Phi] \geq 1 - \zeta$ ISLM.*

Proof Outline. We prove that $\frac{\mathbb{E}[GoT_\Phi - GoT]}{\mathbb{E}[GoT_\Phi]} \leq \zeta$. First, consider large finite markets.[37] We bound the numerator by showing that $\mathbb{E}[GoT_\Phi] \in \Theta\left(\min(m, n)\right)$. Next, we will bound the numerator $\mathbb{E}[GoT_\Phi - GoT]$. Denote by t^Φ a sample of $n+m$ net values. Denote by μ the distribution of the market price $\Pi(t^\Phi)$ and by $L(t^\Phi) = GoT_\Phi - GoT$ the total value of trades that inefficiently fail to occur given t^Φ and the strategies a_B, $a_S \in \Upsilon^{\epsilon,opt}_{\Phi,F}$. It holds that $\mathbb{E}\left[L(t^\Phi)\right] = \int_{-\infty}^{\infty} \mathbb{E}\left[L(t^\Phi) | \Pi(t^{\Phi,(m)})\right] d\mu(\Pi(t^{\Phi,(m)}))$. The latter can be bounded by $O(\min(m, n)^{\frac{1}{2}} + \min(m, n) \cdot \epsilon))$, thus yielding the result. In infinite markets, we prove that for continuous and increasing strategies GoT can be represented as a continuous and deterministic function $GoT(\cdot)$ evaluated at the trading volume Q. If strategies converge to truthfulness, then demand and supply converge uniformly to D_Φ and S_Φ. This implies

[37] The proof follows the methods from Rustichini et al. (1994, Theorem 3.2).

that also the market price and trading volume converge to Π_Φ and Q_Φ. As the efficiency ratio is equal to ${}^{GoT(Q)}/_{GoT(Q_\Phi)}$, the statement follows.[38] □

Example (Price fees in an infinite uniform market)**.** Consider an infinite market with type space $T = [1, 2]$ and μ_B^t and μ_S^t the Lebesgue-measures.[39] Assume that a symmetric price fee is charged, that is $\phi_b = \phi_s = \phi$, and, in line with the implications from Theorem 6, traders act as price-takers and truthfully report their net value. The table on the left-hand side of Fig. 3 gives different measures describing the outcome in a market with and without fees and the right-hand side shows the decomposition of the maximum gains of trade $GoT_{id} = {}^1/_4$ as a function of the fee. Note that, while the market price is independent of the fee and equal to ${}^3/_2$, the market volume is strictly decreasing in ϕ, equal to ${}^1/_2$, when $\phi = 0$, and complete market failure occurs at $\phi = {}^1/_3$. The gains of trade are also strictly decreasing in the fee. From a market maker's point of view, fee profits are maximized at $\phi = {}^1/_6$, where individuals' fee payments and market volume are balanced.

Buyer strategy $x_B(t_b)$	${}^{t_b}/_{(1+\phi)}$
Seller strategy $x_S(t_s)$	${}^{t_s}/_{(1-\phi)}$
Demand $D(P)$	2-(1+ϕ)P
Supply $\mathcal{S}(P)$	(1-ϕ)P-1
Market Price Π	${}^3/_2$
Market Volume Q^*	${}^{(1-3\phi)}/_2$
Market Excess Ex^*	0
Max. Gains of Trade GoT_{id}	${}^1/_4$
Gains of Trade GoT	${}^{\left(1-9\phi^2\right)}/_4$
Fees	${}^{\left(3\phi-9\phi^2\right)}/_2$
Surplus	${}^{\left(1-6\phi+9\phi^2\right)}/_4$
Loss	${}^{9\phi^2}/_4$

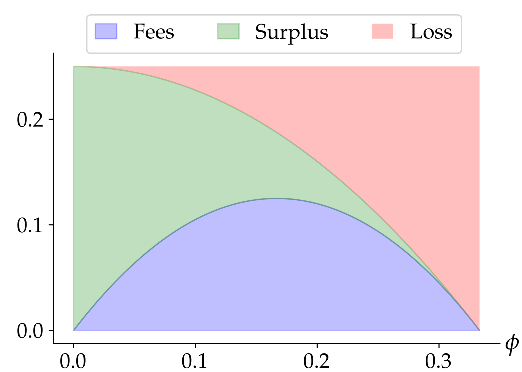

Figure 3: A symmetric infinite market with $T = [1, 2]$, $R = 1$, μ_B^t and μ_S^t the Lebesgue-measures, symmetric price fees ($\phi_b = \phi_s = \phi$) and the truthful strategy profiles t_b^Φ and t_s^Φ. *Left.* Market characteristics. *Right.* Decomposition of the maximum gains of trade into *Fees* (blue), *Surplus* (green), and *Loss* (red) as a function of the fee ϕ.

4.2 For heterogeneous fees, strategic behavior depending on beliefs can lead to any level of efficiency

Efficiency results change when heterogeneous fees Φ are charged. Given beliefs F, denote by $\Pi^\infty(t_i)$ the guess of the critical value of trader i with gross value t_i. Our characterizations of best responses under heterogeneous fees imply that price-guessing behavior approximates optimal strategic behavior for traders expecting to be in the market in large markets (see Theorems 7 and 8).[40] For $\epsilon \geq 0$,

[38] The proof is relegated to Appendix B.13.

[39] This is a setting with balanced market sides and uniformly distributed gross values.

[40] Recall best responses were such that traders chose actions equal to their belief of the critical value if this is individually rational, and are truthful otherwise.

define $\Psi^{\epsilon,opt}_{\Phi,F}$, which consists of all strategy pairs (a_B, a_S), that are ϵ-close to price-guessing, which we denote by (ρ_B, ρ_S). In contrast to price-taking which leads to full efficiency, price-guessing can lead to arbitrary efficiency outcomes.

Theorem 10 (In large markets, depending on the belief system, strategic behavior can lead to any level of efficiency with heterogeneous fees)**.** *Suppose a heterogeneous fee Φ is charged. $\forall \epsilon \geq 0$ and $\forall \zeta \in [0,1]$, there exist beliefs F and strategies in $\Psi^{\epsilon,opt}_{\Phi,F}$, such that the efficiency ratio is (1) equal to 0 ISLM and (2) equal to ζ in infinite markets.*

Proof Outline. For (1), suppose that all buyers and all sellers identify the same critical value, that is $\forall t_b \in T\ \Pi^{\infty}(t_b) = \Pi^{\infty}_B$ and $\forall t_s \in T\ \Pi^{\infty}(t_s) = \Pi^{\infty}_S$. Suppose that $\Pi^{\infty}_B < \Pi^{\infty}_S$ and traders act as price-guessers. For any realization of gross values, no profitable trade is possible and $GoT = 0$, which implies the result. For (2), we have that for continuous and strictly increasing strategies in an infinite market, GoT can be represented as a continuous function $G(\cdot)$ evaluated at Q with $G(Q_\Phi) = GoT_\Phi$ and $G(0) = 0$.[41] $E = {}^{G}/{}_{G_\Phi}$ can be represented as the continuous function $E(Q) = {}^{G(Q)}/{}_{G_\Phi}$ with $E(Q_\Phi) = 1$ and $E(0) = 0$. $\forall Q \in [0, Q_\Phi]$, we construct strategies in $\Psi^{\epsilon,opt}_{\Phi,F}$ with this trading volume. The result follows from the Intermediate Value Theorem.[42] □

Example (Spread fees in an infinite uniform market)**.** Consider an infinite market with type space $T = [1,2]$, μ^t_B and μ^t_S the Lebesgue-measures and a symmetric spread fee, that is, $\phi_b = \phi_s = \phi$. Best responses divide the population into price-guessers choosing actions at the critical value and price-takers. We suppose all buyers identify the critical value at $\beta \in [1,2]$, and all sellers at $\sigma \in [1,2]$.

Case (i) $\beta \geq {}^3/{}_2 \geq \sigma$**.** The market is fully efficient with $\Pi = {}^3/{}_2$, and $Q = {}^1/{}_2$. The fee is strictly increasing (decreasing) in $\rho_B(\rho_S)$. The surplus increases, if traders act more aggressively. Areas (i) in Fig. 4 illustrate these findings.

Case (ii) $\beta \geq \sigma > {}^3/{}_2$ **.** The market is partially efficient. $\Pi = \sigma$ and $Q = 2 - \sigma$, which are independent of β and strictly decreasing in σ. Because demand does not equal supply, tie-breaking selects sellers with Lebesgue-measure $2 - \sigma$ from all sellers asking for σ. The loss is increasing in σ, so more aggressive price-guessing by sellers leads to an efficiency loss. Part of the inefficiency is due to tie-breaking – see the dotted red lines in the figure. The generated fees depends on $\beta - \sigma$, but are generated entirely by buyers, as all sellers who trade offered the market price. Therefore, as in case (i), more aggressive behavior from both market sides leads to lower fees and a higher surplus. Areas (ii) in Fig. 4 illustrate these findings.

Case (iii) ${}^3/{}_2 > \beta \geq \sigma$**.** This case is analogue to case (ii).

Case (iv) $\beta < \sigma$**.** Market failure emerges as the highest bid of any buyer is below the lowest ask of any seller. Areas (iv) in Fig. 4 illustrate these findings.

[41] This was shown in the proof of Theorem 9.

[42] The proof is relegated to Appendix B.14.

	Case (i)	Case (ii)	Case (iii)	Case (iv)
Buyer strategy $x_B(t_b)$	β if $t_b \geq \beta$ and t_b if $t_b < \beta$			
Seller strategy $x_S(t_s)$	σ if $t_s \leq \sigma$ and t_s if $t_s > \sigma$			
Demand $D(P)$	$2-P$ if $P \leq \beta$ and 0 if $P > \beta$			
Supply $\mathcal{S}(P)$	0 if $P < \sigma$ and $P-1$ if $P \geq \sigma$			
Market Price Π	$3/2$	σ	β	$\in (\beta, \sigma)$
Market Volume Q^*	$1/2$	$2-\sigma$	$\beta-1$	0
Market Excess Ex^*	0	$2\sigma-3$	$3-2\beta$	0
Max. Gains of Trade GoT_{id}	$1/4$			
Gains of Trade GoT	$1/4$	$\frac{3\sigma-\sigma^2-2}{2(\sigma-1)}$	$\frac{3\beta-\beta^2-2}{2(2-\beta)}$	0
Fees	$\phi\Big((2\text{-}\beta)(\beta\text{-}{}^3\!/\!_2)+\frac{(\beta\text{-}3/2)^2}{2}$ $+(\sigma\text{-}1)({}^3\!/\!_2\text{-}\sigma)+\frac{(3/2\text{-}\sigma)^2}{2}\Big)$	$\phi\big((2\text{-}\beta)(\beta\text{-}\sigma)$ $+\frac{(\beta\text{-}\sigma)^2}{2}\big)$	$\phi\big((1\text{-}\sigma)(\beta\text{-}\sigma)$ $+\frac{(\beta\text{-}\sigma)^2}{2}\big)$	0
Surplus	$GoT - Fees$	$GoT - Fees$	$GoT - Fees$	0
Loss	0	$\frac{2\sigma^2\text{-}5\sigma+3}{4(\sigma\text{-}1)}$	$\frac{2\sigma^2\text{-}7\beta+6}{4(2\text{-}\sigma)}$	${}^1\!/\!_4$

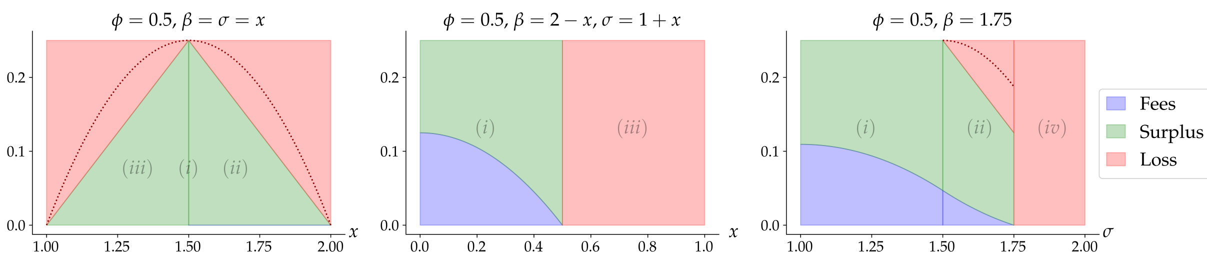

Figure 4: An infinite market with $T = [1, 2]$, $R = 1$, Lebesgue measures μ_B^t and μ_S^t, symmetric spread fees – $\phi_b = \phi_s = \phi$ – and the strategy profiles corresponding to symmetric price-guessing (β, σ). *Top.* Market characteristics. *Bottom.* Decomposition of the maximum gains of trade $GoT_{id} = {}^1\!/\!_4$ for $\phi = 0.5$ as functions of β and σ into *Surplus* (green), *Fees* (blue), and *Loss* (red). The dotted red line depicts, how much the green area could increase, if tie-breaking would be replaced by an instrument choosing the optimal allocation. *Left.* $\beta = \sigma$ varies in $[1, 2]$. *Middle.* $\beta = 2 - x$ and $\sigma = 1 + x$ as functions of a single parameter $x \in [0, 1]$ and therefore symmetrically varying both β and σ. *Right.* $\beta = 1.75$ and σ varies in $[1, 2]$.

5 Exogenous market price and aggregate uncertainty

Until now, our analysis was concerned with characterizing how individual behavior and best responses in large DAs are determined by beliefs about market price. We obtained these results in the DA context without uncertainty, but several of our core arguments remain valid for other non-DA mechanisms with aggregate uncertainty. The class of mechanisms that we consider has in common that individual buyers and sellers believe that what determines whether they are involved in trade is determined by whether their bids (asks) are above (below) an exogenous critical value Π that we call market price. When the trader bids or asks exactly the market price, they are involved in trade with tie-breaking probability $p \in [0, 1]$. Write $i \in A^*(a_i, \Pi)$, if trader i is involved in trade with action a_i, given the market price Π. Every trader i who is involved in trade has to pay a fee $\Phi_i(a_i, \Pi)$ that may depend on the market price and on their action, assuming continuity in a_i and Π.

In large DAs, the exogeneity of the market price emerged as individual traders had an asymptotically vanishing influence on market outcomes. In this section, we directly assume that individuals believe to have no direct influence on market prices, as would be the case in many continuum markets and Vickrey (VCG) mechanisms with rational Bayesian agents, as well as in markets where players believe to have no influence on the market for bounded rationality reasons.

The market price Π is not assumed to be deterministic and commonly known, but instead distributed according to a CDF F_Π on $[\underline{\Pi}, \overline{\Pi}] \subset \mathbb{R}^{\geq 0}$ with $\underline{\Pi} \leq \overline{\Pi}$ and corresponding probability measure $\mathbb{P}_\Pi$. Every individual trader i has incomplete information regarding the market price distribution, and believes that it is distributed according to some CDF $F_{\Pi i}$. We assume these distributions have convex support $[\underline{\Pi}_i, \overline{\Pi}_i]$ with either $\underline{\Pi}_i < \overline{\Pi}_i$ and continuous density function $f_{\Pi i} > 0$, or $\underline{\Pi}_i = \overline{\Pi}_i$ corresponding to deterministic beliefs. Additionally, traders also hold individual beliefs about the tie-breaking probability $p_i \in [0, 1]$ with corresponding probability measure denoted by $\mathbb{P}_i$. The individual beliefs may be different, wrong and misspecified. Moreover, market participants may be more or less certain about the market price, which, for some degree $\delta \geq 0$, we measure by δ-aggregate uncertainty as follows: given $\delta \geq 0$, there exists a price Π_i^*, such that $\mathbb{P}_i \left[\Pi \in [\Pi_i^* - \delta, \Pi_i^* + \delta\right] \geq 1 - \delta$.[43]

In terms of individual trader i's utility $u_i(t_i, a_i, \Pi)$ and net values, we follow the definitions from Section 2.5, again assuming that net values exist with $t_i^\Phi \succcurlyeq t_i$ and that t_i^Φ is in the true and believed support of the market price.

Homogeneous and heterogeneous fees are now defined as follows: A fee Φ_i is *homogeneous* if $\Phi_i(a_i, \Pi) \equiv \Phi_i(\Pi)$ is independent of a_i, and the functions $x \mapsto x + \Phi_b(x)$ and $x \mapsto x - \Phi_s(x)$ are increasing for buyers and sellers. Examples include price and constant fees. The net values are equal to the unique solutions of $t_b^\Phi + \Phi_b(t_b^\Phi) = t_b$ and $t_s^\Phi + \Phi_s(t_s^\Phi) = t_s$ for buyers and sellers respectively (see Appendix B.16). A fee Φ_i is *heterogeneous* if, given the market price Π, it holds that $\Phi_i(a_i, \Pi)$ is strictly increasing for buyers on $[\Pi, \infty)$, and strictly decreasing for sellers on $(-\infty, \Pi]$ as a function of the action a_i as well as the following condition: For two actions $a_i^1 \succ a_i^2$, there exists $\gamma > 0$, such that for all $\Pi \prec a_i^2$ it holds that $\Phi_i(a_i^1, \Pi) - \Phi(a_i^2, \Pi) \geq \gamma$. Spread fees are an example of a heterogeneous fee.

As in Section 2.5, we assume the fee is profit-permitting, that is $\forall t_i$ with $t_i^\Phi \prec \Pi$ $\exists \epsilon > 0$ such that $\mathbb{E}_i \left[u_i\left(t_i, t_i^\Phi, \Pi\right)\right] \geq \epsilon$.

We now analyze best responses (analogous definition to Section 2.6) and efficiency to extend our results from Section 3 and Section 4. As market prices are exogenous, there are now two opposing strategic incentives in this model: maximizing the trading probability and minimizing the fee payments.

For a trader i with gross value t_i and action a_i, we define—as in Section 2.3—the *efficiency ratio*

[43] $\delta = 0$ describes the case of deterministic beliefs, which corresponds to the limit case of our DA model.

as $E_\Phi = \frac{\mathbb{P}_\Pi[i \in A^*(a_i, \Pi)]}{\mathbb{P}_\Pi[i \in A^*(t_i^\Phi, \Pi)]}$. E_Φ measures the probability of a representative trader being involved in trade given their action compared to the maximal probability when being truthful.

For homogeneous fees, our results from Theorems 5 and 9 directly extend:

Theorem 11 (For an exogenous market price and homogeneous fees, truthfulness is a best response and fully efficient)**.** *Given δ-uncertainty, suppose a homogeneous fee is charged.*

1. *For $\delta > 0$, truthfulness is the unique best response, and ϵ-best responses approximate truthfulness. Therefore, all responses are fully efficient.*

2. *For $\delta = 0$, truthfulness is a best response and is fully efficient.*

The proof is relegated to Appendix B.17.

For heterogeneous fees, Theorems 7 and 10 also have their natural counterparts. In contrast to homogeneous fees, beliefs (in particular about tie-breaking) have a non-negligible impact on strategic incentives.

Theorem 12 (For an exogenous market price and heterogeneous fees best responses approximate price-guessing, which, dependent on beliefs, leads to any efficiency level)**.** *Given δ-uncertainty, suppose a heterogeneous fee is charged.*

1. *For $\delta > 0$, there exists a best response that depends on the trader's beliefs. If $t_i^\Phi \prec \Pi_i^*$ and $\delta > 0$ is sufficiently small, best responses approximate price-guessing and there exist beliefs such that the efficiency of best responses is zero.*

2. *For $\delta = 0$ and $t_i^\Phi \prec \Pi_i^*$, price-guessing is the unique best response for $p_i = 1$, and, for $p_i < 1$, there exists no best response and ϵ-best responses approximate price-guessing. If F_Π has a continuous density function $f_\Pi > 0$ on $[\underline{\Pi}, \overline{\Pi}]$, then $\forall \zeta \in [0, 1]$, there exist beliefs, such that the efficiency of best responses is equal to ζ,*

The proof is relegated to Appendix B.18.

6 Conclusion

Large markets, in particular large DAs, have been shown to be asymptotically efficient. However, much of the preexisting literature on the topic has abstracted away from fees. Our paper brings the importance of fees to the spotlight—they may fundamentally change incentives. In fact, fee considerations may become more important in larger markets, not less important, unlike strategic considerations related to prices. Different fee types—more so than their levels—have drastically different implications for incentives. In particular, spread fees, or heterogeneous fees more generally, even if small and charged implicitly, may alter bid/ask behavior and result in substantial market inefficiency.

Our results raise several natural empirical questions. What are the cost of strategic fee avoidance in markets, e.g., those we discussed in the Introduction? Are more experienced, more sophisticated, more informed traders better at avoiding fees? Charging the right kind and level of fee may have substantive efficiency and fairness consequences, and is an important question for regulators and intermediaries (e.g., platforms and brokers).

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A Additional results

A.1 Demand, supply, and market-clearing prices

In Appendix A.1 we clarify how the generalized k-DA chooses the market price. For a detailed treatment of the k-DA and the proofs of Lemmas 13, 14, and 15 see Jantschgi et al. (2022).

Recall the following notation: For a relation $\mathcal{R} \in \{\geq, >, =, <, \leq\}$, define $\mathcal{B}_{\mathcal{R}}(P) = \{b \in \mathcal{B} : t_b \mathcal{R} P\}$ and $\mathcal{S}_{\mathcal{R}}(P) = \{s \in \mathcal{S} : t_s \mathcal{R} P\}$.

Definition (Demand and supply functions)**.** The *demand* and *supply functions* at price P are defined as $D(P) = \mu_B(\mathcal{B}_{\geq}(P))$ and $S(P) = \mu_S(\mathcal{S}_{\leq}(P))$, that is, by the mass of all traders who weakly prefer trading over not trading at price P.

We define a special class of action distributions, which arise in infinite markets, e.g., if they are interpreted as the limit of finite markets where actions are modelled as independent random variables. Say that action distributions μ_B^a and μ_S^a are *continuous*, if they are equivalent to the Lebesgue-measure on A_B and A_S and moreover, their densities f_B and f_S are continuous, that is $\mu_B^a(\cdot) = \int_\cdot f_B(x)dx$ and $\mu_S^a(\cdot) = \int_\cdot f_S(x)dx$ for $A \subset \mathbb{R}$.

Lemma 13 (Analytic properties of demand and supply functions)**.** *The demand function is non-increasing, left-continuous with right limits. The supply function is non-decreasing, right-continuous with left limits. It holds that $D(P+) = \mu_B(\mathcal{B}_{>}(P))$ and $S(P-) = \mu_S(\mathcal{S}_{<}(P))$. If action distributions are continuous, then demand is continuous and decreasing on A_B and supply is continuous and increasing on A_S.*

The following concept corresponds to prices that equilibrate demand and supply.

Definition ((Strong) market clearing prices)**.** P is a market-clearing price if $D(P) \geq S(P)$ and $D(P+) \leq S(P)$ (*type I*) or $S(P) \geq D(P)$ and $S(P-) \leq D(P)$ (*type II*). P is a *strong market-clearing price* if $D(P) = S(P)$. Denote the set of all quasi-market-clearing prices by $\mathcal{P}^{MC}$ and the set of all strong market-clearing prices by $\mathcal{P}^{SMC}$.

Using the analytical properties of demand and supply, we can characterize the topology of the set of (strong) market clearing prices.

Lemma 14 (Topology of $\mathcal{P}^{SMC}$ and $\mathcal{P}^{MC}$)**.** *The set $\mathcal{P}^{SMC}$ is a convex subset of T. Every strong market-clearing price is a market-clearing price (of type I and II). The set of market-clearing prices is non-empty, convex and closed. The set $\mathcal{P}^{MC} \setminus \mathcal{P}^{SMC}$ has Lebesgue-measure zero. More precisely, if $\mathcal{P}^{SMC} \neq \emptyset$, then $\mathcal{P}^{MC} = \overline{\mathcal{P}^{SMC}}$, and if $\mathcal{P}^{SMC} = \emptyset$, then $\mathcal{P}^{MC}$ is a singleton.*
If action distributions are continuous, and $\overline{a}_S > \underline{a}_B$, then there exists a unique strong market clearing price with positive trading volume and it holds that $\mathcal{P}^{SMC} = \mathcal{P}^{MC}$.

Lastly, in finite markets, the generalized k-DA coincides with the classical DA (Rustichini et al., 1994), for which an explicit formula for the set of market-clearing prices is given. Let $a^{(m)}$ be the m'th smallest action in the set of all actions a.

Lemma 15. *In finite markets with m buyers and n sellers $\mathcal{P}^{MC} = [a^{(m)}, a^{(m+1)}]$. If $a^{(m)} \neq a^{(m+1)}$, then $P \in (a^{(m)}, a^{(m+1)}) \Rightarrow P \in \mathcal{P}^{SMC}$.*

A.2 Allocation and Tie-breaking

If the generalized k-DA results in a strong market-clearing price Π, that is $D(\Pi) = S(\Pi)$, then no fair lottery is needed. The allocation is set as $\mathcal{B}^* = \mathcal{B}_{\geq}(\Pi)$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(\Pi)$, which *balances trade*, that is $\mu_B(\mathcal{B}^*) = \mu_S(\mathcal{S}^*)$. Therefore, the allocation consists of all traders, who weakly prefer trading over not trading at Π.

Next, suppose that generalized k-DA results in a market clearing price of type I, which is not a strong market clearing price. Then, $D(\Pi) > S(\Pi)$ and $D(\Pi+) \leq S(\Pi)$. Set $\mathcal{S}^* = \mathcal{S}_{\leq}(\Pi)$, that is all sellers who, given their action, weakly prefer trading over not trading are involved in trade. Consider the set of all buyers who strictly prefer to trade at Π, that is $\mathcal{B}_{>}(\Pi)$. It follows from Theorem 13 that $D(\Pi+) = \mu_B(\mathcal{B}_{>}(\Pi))$. Let $x = S(\Pi) - \mu_B(\mathcal{B}_{>}(\Pi)) \geq 0$ and let $\tilde{\mathcal{B}}$ be a subset of $\mathcal{B}_{=}(\Pi)$ with μ_B-measure equal to x. Such a set exists because $D(\Pi) = \mu_B(\mathcal{B}_{\geq}(\Pi)) = \mu_B(\mathcal{B}_{>}(\Pi)) + \mu_B(\mathcal{B}_{=}(\Pi)) \geq S(\Pi)$ and $D(\Pi+) = \mu_B(\mathcal{B}_{>}(\Pi)) \leq S(\Pi)$. Set $\mathcal{B}^* = \mathcal{B}_{>}(\Pi) \cup \tilde{\mathcal{B}}$. That is, all buyers who strictly prefer to trade at Π are involved in trade, together with a subset of traders with bid equal to Π that are indifferent in order to balance trade.

Finally, if a market clearing price of type II is chosen, the allocation is set in analogy: $\mathcal{B}^* = \mathcal{B}_{\geq}(\Pi)$ and $\mathcal{S}^* = \mathcal{S}_{<}(\Pi) \cup \tilde{\mathcal{S}}$, where $\tilde{S}$ is a subset of $\mathcal{S}_{=}(\Pi)$ that balances trade.

In order to ensure fairness, suppose that $\tilde{B}$ (respectively $\tilde{S}$) are chosen uniformly at random. That is, they are random compact sets such that for all $b \in \mathcal{B}_{=}(\Pi)$ it holds that $\mathbb{P}[b \in \tilde{\mathcal{B}}] \equiv const$ (respectively for all $s \in \mathcal{S}_{=}(\Pi)$ it holds that $\mathbb{P}[s \in \tilde{\mathcal{S}}] \equiv const$). This constant is necessarily equal to $const = \mu_B(\tilde{\mathcal{B}})/\mu_B(\mathcal{B})$ (respectively $const = \mu_S(\tilde{\mathcal{S}})/\mu_S(\mathcal{S})$).

A.3 Explicit Formulas

In this section, we derive explicit formulas for some of the concepts introduced in the model in Section 2 that will be used in subsequent proofs. We will sometimes differentiate between finite markets with m buyers and n sellers and infinite markets.

Throughout this section, consider a buyer b with gross value t_b and bid a_b, and a seller s with gross value t_s and ask a_s. Let a denote an action distribution. Recall that in a finite market, $a^{(k)}$ denotes the k'th smallest element in the set of all taken actions.

A.3.1 Involvement in trade

Finite markets If $a_b < a^{(m)}_{\text{-}b}$, then it is strictly smaller than the $m+1$'st smallest element in the set of all actions a and buyer b is not involved in trade, because their bid is below the market price. If $a_b > a^{(m)}_{\text{-}b}$, then it is at least the $m+1$'st largest element and therefore sufficient to be involved in trade. If $a_b = a^{(m)}_{\text{-}b}$, then the buyer might be subject to tie-breaking.

If $a_s > a^{(m)}_{\text{-}s}$, then it is at least the $m+1$'st smallest element in the set of all actions and seller s is not involved in trade, because his ask was above the market price. If $a_s < a^{(m)}_{\text{-}s}$, then it is at most the m'th smallest action and therefore sufficient to be involved in trade. If $a_s = a^{(m)}_{\text{-}s}$s, then the seller might be subject to tie-breaking.

Infinite Markets If there exists no demand excess, then a buyer is involved in trade, if $a_b \geq \Pi(a)$. If $a_b < \Pi(a)$, then the buyer is not involved in trade. If there exists demand excess, it is generated by bids at $\Pi(a)$. If $a_b > \Pi(a)$, then the buyer is involved in trade. If $a_b = \Pi(a)$, then the buyer might be subject to tie-breaking.

If there exists no supply excess, then the seller is involved in trade, if $a_s \leq \Pi(a)$. If $a_s > \Pi(a)$, then the seller is not involved in trade. If there exists supply excess, it is generated by asks at $\Pi(a)$. If $a_s < \Pi(a)$, then the seller is involved in trade. If $a_s = \Pi(a)$, then the seller might be subject to tie-breaking.

Given these considerations, we can now express the probability of trade, given the beliefs of a trader.

A.3.2 Trading probabilities given beliefs

Finite Markets Given the belief that actions are random variables with continuous distribution, tie-breaking is a probability zero event in finite markets. It follows from Appendix A.3.1 that

$$\mathbb{P}_{a_{\text{-}b}}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] = \mathbb{P}_{a_{\text{-}b}}\left[a_b \geq a^{(m)}_{\text{-}b}\right] \quad \text{and} \quad \mathbb{P}_{a_{\text{-}s}}\left[s \in \mathcal{S}^*(a_s, a_{\text{-}s})\right] = \mathbb{P}_{a_{\text{-}s}}\left[a_s \leq a^{(m)}_{\text{-}s}\right]. \tag{3}$$

In section Appendix A.5, explicit formulas for such probabilities are derived in a more general context (see Equations (28) and (29).

Infinite Markets If there exists no excess demand at Π, then it holds that

$$\mathbb{P}_{a_{\text{-}b}}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] = \begin{cases} 1 & a_b \geq \Pi(a) \\ 0 & \text{else} \end{cases}. \tag{4}$$

Suppose that there is strictly positive excess demand. That is $\mu_B(\mathcal{B}_{\geq}(\Pi(a))) = Q(a) + x$ and $\mu_B(\mathcal{B}_{>}(\Pi(a))) = Q(a) - y$ for $x > 0$ and $y \geq 0$, see Appendix A.2. It holds that

$$\mathbb{P}_{a_{\text{-}b}}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] = \begin{cases} 1 & a_b > \Pi(a) \\ \frac{y}{x+y} & a_b = \Pi(a) \\ 0 & \text{else} \end{cases}. \tag{5}$$

If there exists no excess supply, then it holds that

$$\mathbb{P}_{a_{\text{-}s}}\left[s \in \mathcal{S}^*(a_s, a_{\text{-}s})\right] = \begin{cases} 1 & a_s \leq \Pi(a) \\ 0 & \text{else} \end{cases}. \tag{6}$$

Suppose that there is strictly positive excess supply. Then $\mu_S(\mathcal{S}_{\leq}(\Pi(a))) = Q(a)+x$ $\mu_S(\mathcal{S}_{<}(\Pi(a))) = Q(a) - y$ for $x > 0$ and $y \geq 0$. It holds that

$$\mathbb{P}_{a_{\text{-}s}}\left[s \in \mathcal{S}^*(a_s, a_{\text{-}s})\right] = \begin{cases} 1 & a_s < \Pi(a) \\ \frac{y}{x+y} & a_s = \Pi(a) \\ 0 & \text{else} \end{cases}. \tag{7}$$

Note that in the presence of strictly positive market excess, traders believe that if they are involved in tie-breaking in an infinite market, they have a fair chance of being involved in trade.

A.3.3 Market Price

Finite markets Recall that by Theorem 15, it holds that $\Pi(a) = ka^{(m)}+(1-k)a^{(m+1)}$. Interpreting the market price as a function of a single action yields that

$$\Pi\left(a_b, a_{\text{-}b}\right) = \begin{cases} (1\text{-}k)a_{\text{-}b}^{(m)}+ka_b & \text{if } a_{\text{-}b}^{(m)} \leq a_b \leq a_{\text{-}b}^{(m+1)} \\ (1\text{-}k)a_{\text{-}b}^{(m)}+ka_{\text{-}b}^{(m+1)} & \text{else} \end{cases} \tag{8}$$

$$\Pi\left(a_s, a_{\text{-}s}\right) = \begin{cases} (1\text{-}k)a_s+ka_{\text{-}s}^{(m)} & \text{if } a_{\text{-}s}^{(m\text{-}1)} \leq a_s \leq a_{\text{-}s}^{(m)} \\ (1\text{-}k)a_{\text{-}s}^{(m\text{-}1)}+ka_{\text{-}s}^{(m)} & \text{else} \end{cases}. \tag{9}$$

Note that $\Pi\left(a_b, a_{\text{-}b}\right)$ depends only on $a_{\text{-}b}^{(m)}$ and $a_{\text{-}b}^{(m+1)}$ and $\Pi\left(a_s, a_{\text{-}s}\right)$ depends only on $a_{\text{-}s}^{(m\text{-}1)}$ and $a_{\text{-}s}^{(m)}$. In some proofs, this dependence will be of importance and we will for example write $\Pi\left(a_b, a_{\text{-}b}^{(m)}, a_{\text{-}b}^{(m+1)}\right)$ instead of $\Pi\left(a_b, a_{\text{-}b}\right)$).

In addition, for a trader i, we will in some proofs consider $\tilde{\Pi}\left(a_i, a_{\text{-}i}\right)$, which is equal market price, if the trader is involved in trade, but zero otherwise.

Infinite Markets In an infinite market, a single trader cannot influence the market price. It therefore holds for a trader i and for all actions a_i and a_i' that $\Pi(a_i, a_{\text{-}i}) = \Pi(a_i', a_{\text{-}i})$. By abuse of notation, we will in some proofs write $\Pi(a_{\text{-}i})$.

A.3.4 Utility Functions

For a buyer the utility of being involved in trade is equal to the difference between their gross value and the market price minus the additional fee:

$$u_b\left(t_b, a_b, a_{\text{-}b})\right) = \begin{cases} t_b\text{-}\Pi\left(a_b, a_{\text{-}b}\right)\text{-}\Phi_b\left(a_b, a_{\text{-}b}\right) & b \in \mathcal{B}^* \\ 0 & \text{else} \end{cases} \tag{10}$$

For a seller the utility of being involved in trade is equal to the difference between the market price and their gross value minus the additional fee:

$$u_s\left(t_s, a_s, a_{\text{-}s})\right) = \begin{cases} \Pi\left(a_s, a_{\text{-}s}\right)\text{-}t_s\text{-}\Phi_s\left(a_s, a_{\text{-}s}\right) & s \in \mathcal{S}^* \\ 0 & \text{else} \end{cases} \tag{11}$$

A.3.5 Expected Utilities

Finite Markets Let $\mu_b\left(a_{\text{-}b}\right)$ denote the distribution of $a_{\text{-}b}$ according to the beliefs of trader b. It holds that

$$\begin{gathered} \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] = \\ \int\limits_{\{a_b \geq a_{\text{-}b}^{(m)}\}} \left(t_b\text{-}\Pi\left(a_b, a_{\text{-}b}\right)\text{-}\Phi_b\left(a_b, a_{\text{-}b}\right)\right) d\mu_b\left(a_{\text{-}b}\right) = \\ t_b \cdot \mathbb{P}_{\text{-}b}[b \in \mathcal{B}^*(a_b, a_{\text{-}b})] - \int\limits_{[\underline{a}_{S,b}, \overline{a}_{S,b}]^2} \tilde{\Pi}\left(a_b, a_{\text{-}b}^{(m)}, a_{\text{-}b}^{(m+1)}\right) d\mu_b(a_{\text{-}b}^{(m)}, a_{\text{-}b}^{(m+1)}) - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b})\right] \end{gathered} \tag{12}$$

Note that both $a_{\text{-}b}^{(m)}$ and have support in $[\underline{a}_{S,b}, \overline{a}_{S,b}]$. That is because $a_{\text{-}b}$ consists of $m-1$ bids and n asks. So there must be at least one ask below or equal to $a_{\text{-}b}^{(m)}$.

Let $\mu_s\left(a_{\text{-}s}\right)$ denote the distribution of $a_{\text{-}s}$ according to the beliefs of a seller s. It holds that

$$\begin{gathered} \mathbb{E}_{\text{-}s}\left[u_s(t_s, a_s, a_{\text{-}s})\right] = \\ \int\limits_{\{a_s \leq a_{\text{-}s}^{(m)}\}} \left(\Pi\left(a_s, a_{\text{-}s}\right)\text{-}t_s - \Phi_s\left(a_s, a_{\text{-}s}\right)\right) d\mu_s\left(a_{\text{-}s}\right) = \\ \int\limits_{[\underline{a}_{B,s}, \overline{a}_{B,s}]^2} \tilde{\Pi}\left(a_s, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right) d\mu_s(a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}) - t_s \cdot \mathbb{P}_{\text{-}s}[s \in \mathcal{S}^*(a_s, a_{\text{-}s})] - \mathbb{E}_{\text{-}s}\left[\Phi_s(a_s, a_{\text{-}s})\right]. \end{gathered} \tag{13}$$

Note that both $a_{\text{-}s}^{(m)}$ and have support in $[\underline{a}_{B,s}, \overline{a}_{B,s}]$.

Infinite Markets. The expectation is only concerned with uniform rationing, as both the market price and the fee are deterministic. Therefore,

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] = \left(t_b - \Pi(a_b, a_{\text{-}b}) - \Phi_b(a_b, a_{\text{-}b})\right)\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \tag{14}$$

and

$$\mathbb{E}_{\text{-}s}\left[u_s(t_s, a_s, a_{\text{-}s})\right] = \left(\Pi(a_s, a_{\text{-}s}) - t_s - \Phi_s(a_s, a_{\text{-}s})\right)\mathbb{P}_{\text{-}s}\left[s \in \mathcal{S}^*(a_s, a_{\text{-}s})\right]. \tag{15}$$

Difference in expected utility for two actions a_i^1 and a_i^2 in finite markets In multiple proofs, we will estimate the difference in expected utility in finite markets for two actions a_i^1 and a_i^2. The following Lemma yields and upper bound:

Lemma 16. *For two bids $a_b^1 > a_b^2$ and for two asks $a_s^1 < a_s^2$ it holds that*

$$\begin{aligned}&\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^2, a_{\text{-}b}\right)\right] \leq \\ &t_b\left(\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^1, a_{\text{-}b})\right] - \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^2, a_{\text{-}b})\right]\right) - \left(\mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b^2, a_{\text{-}b}\right)\right]\right).\end{aligned} \tag{16}$$

$$\begin{aligned}&\mathbb{E}_{\text{-}s}\left[u_s\left(t_s, a_s^1, a_{\text{-}s}\right)\right] - \mathbb{E}_{\text{-}s}\left[u_s\left(t_s, a_s^2, a_{\text{-}s}\right)\right] \\ &\leq 2\overline{a}_{B,s}\left(1 - \mathbb{P}_{\text{-}s}\left[s \in \mathcal{S}^*(a_s, a_{\text{-}s})\right]\right) - t_s\left(\mathbb{P}_{\text{-}s}\left[s \in \mathcal{S}^*(a_s^1, a_{\text{-}s})\right] - \mathbb{P}_{\text{-}s}\left[s \in \mathcal{S}^*(a_s^2, a_{\text{-}s})\right]\right) \\ &- \left(\mathbb{E}_{\text{-}s}\left[\Phi_s\left(a_s^1, a_{\text{-}s}\right)\right] - \mathbb{E}_{\text{-}s}\left[\Phi_s\left(a_s^2, a_{\text{-}s}\right)\right]\right).\end{aligned} \tag{17}$$

The proof of this Lemma is relegated to Appendix B.15.

A.4 Strategic incentives for price and spread fees

This section contains a detailed discussion of the opposing strategic incentives for the two main examples of price and spread fees in finite markets: (i) Utility when trading versus (ii) probability of trading.[44]

Recall that a trader i believes that actions are distributed in intervals $A_{B,i} = [\underline{a}_{B,i}, \overline{a}_{B,i}]$ and $A_{S,i} = [\underline{a}_{S,i}, \overline{a}_{B,i}]$ with the assumption that $\overline{a}_{S,i} \geq \overline{a}_{B,i} > t_i^{\Phi} > \underline{a}_{S,i} \geq \underline{a}_{B,i}$.

Consider a buyer b with action a_b. We can neglect the analysis of $a_b > \overline{a}_{B,b}$ and $a_b < \underline{a}_{S,b}$. For the first, such an action is by assumption not individually rational and strictly dominated by t_b^{Φ}. For the second, any action below $\underline{a}_{S,b}$ has probability of trade equal to 0, because no seller is believed to submit an action below it. Therefore, the expected utility at such a bid is equal to 0.

We therefore consider $a_b \in [\underline{a}_{S,b}, \overline{a}_{B,b}]$.

Recall that by Appendix A.3, the market price depends only on a_b, $a_{\text{-}b}^{(m)}$ and $a_{\text{-}b}^{(m+1)}$. For ease of notation, let $y = a_{\text{-}b}^{(m)}$ and $z = a_{\text{-}b}^{(m+1)}$ and denote by $e(y, z)$ the joint density of y and z given the beliefs of buyer b.

[44] The following section is closely related to methods used in Rustichini et al. (1994) to analyze strategic incentives in DAs without fees.

Price fees The expected utility of a buyer is of the form

$$\begin{aligned}\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] = &\int_{a_b}^{\overline{a}_{S,i}} \int_{\underline{a}_{S,b}}^{a_b} \left(t_b - (1+\phi_b)\left(ka_b + (1-k)\,y\right)\right) e(y,z) dy dz + \\ &\int_{\underline{a}_{S,b}}^{a_b} \int_{\underline{a}_{S,b}}^{z} \left(t_b - (1+\phi_b)\left(kz + (1-k)\,y\right)\right) e(y,z) dy dz.\end{aligned} \tag{18}$$

The expected utility is continuously differentiable as a function of a_b over the interval $[\underline{a}_{S,b}, \overline{a}_{S,b}]$. Straightforward computation using Leibniz's rule for differentiation under the integral sign yields

$$\frac{d\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right]}{da_b} = \left(t_b - (1+\phi_b)\,a_b\right) f_y(a_b) - (1+\phi_b)\,k\mathbb{P}_{\text{-}b}\left[y \leq a_b \leq z\right], \tag{19}$$

where $f_y(a_b)$ denotes the density function of y. If $a_b \in (\underline{a}_{S,b}, \overline{a}_{S,b})$ maximizes the expected utility, then the first order condition

$$\frac{d\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right]}{da_b} = 0, \tag{20}$$

holds. $f_y(a_b)$ is equal to $\frac{d\mathbb{P}_{\text{-}b}[y \leq a_b]}{da_b}$. A formula for $\mathbb{P}_{\text{-}b}[y \leq a_b]$ is stated in Appendix A.5. Therefore, we can explicitly state the first order condition in terms of distribution and density functions, see Equation (24) below.

The first order condition for a seller can be derived in analogy, see Equation (25) below.

Spread fees The expected utility of a buyer is of the form

$$\begin{aligned}\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b}\right] = &\int_{a_b}^{\overline{a}_{S,b}} \int_{\underline{a}_{S,b}}^{a_b} \left(t_b - \phi_b a_b - (1-\phi_b)\left(ka_b + (1-k)\,y\right)\right) e(y,z) dy dz + \\ &\int_{\underline{a}_{S,b}}^{a_b} \int_{\underline{a}_{S,b}}^{z} \left(t_b - \phi_b a_b - (1-\phi_b)\left(kz + (1-k)\,y\right)\right) e(y,z) dy dz.\end{aligned} \tag{21}$$

The expected utility is continuously differentiable as a function of a_b over the interval $[\underline{a}_{S,b}, \overline{a}_{S,b}]$. Straightforward computation using Leibniz's rule for differentiation under the integral sign yields

$$\frac{d\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b}\right]}{da_b} = \left(t_b - a_b\right) f_y(a_b) - \phi_b\mathbb{P}_{\text{-}b}\left[y \leq a_b\right] - (1-\phi_b)\,k\mathbb{P}_{\text{-}b}\left[y \leq a_b \leq x\right]. \tag{22}$$

where $f_y(a_b)$ denotes the density function of y. If $a_b \in (\underline{a}_{S,b}, \overline{a}_{S,b})$ maximizes the expected utility, then the first order condition

$$\frac{d\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right]}{da_b} = 0, \tag{23}$$

holds. $f_y(a_b)$ is equal to $\frac{d\mathbb{P}_{\text{-}b}[y\leq a_b]}{da_b}$. A formula for $\mathbb{P}_{\text{-}b}[y \leq a_b]$ is stated in Appendix A.5. Therefore, we can explicitly state the first order condition in terms of distribution and density functions, see Equation (24) below.

The first order condition for a seller can be derived in analogy, see Equation (25) below.

First Order Conditions To explicitly state the first order conditions, we introduce additional notation:

Define $a_{i,j}$ as an action distribution for i buyers and j sellers. In this notation, a as defined in Section 2 corresponds to $a_{m,n}$ and for any buyer b and seller s, $a_{\text{-}b}$ and $a_{\text{-}s}$ correspond to $a_{m-1,n}$ and $a_{m,n-1}$. Denote again by $a_{i,j}^{(l)}$ its l'th smallest element.

We say that an action a_b satisfies the *buyer's first order condition* for gross value t_b if

$$\left.\begin{matrix} (t_b\text{-}(1{+}\phi_b)\,a_b) \\ \\ (t_b\text{-}a_b) \end{matrix}\right\} \cdot \left(n\mathbb{P}_{\text{-}b}\left[a^{(m\text{-}1)}_{m\text{-}1,n\text{-}1} \leq a_b \leq a^{(m)}_{m\text{-}1,n\text{-}1}\right] f_{S,b}(a_b) + (m\text{-}1)\,\mathbb{P}_{\text{-}b}\left[a^{(m\text{-}1)}_{m\text{-}2,n} \leq a_b \leq a^{(m)}_{m\text{-}2,n}\right] f_{B,b}(a_b) \right) = \\ \begin{cases} (1+\phi_b)\,k\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m\text{-}1,n\text{-}1} \leq a_b \leq a^{(m+1)}_{m\text{-}1,n}\right] & \text{for price fees} \\ \phi_b\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m,n\text{-}1} \leq a_b\right] + (1\text{-}\phi_b)\,k\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m\text{-}1,n} \leq a_b \leq a^{(m+1)}_{m\text{-}1,n}\right] & \text{for spread fees} \end{cases}. \tag{24}$$

We say that an action a_s satisfies the *seller's first order condition* for gross value t_s if

$$\left.\begin{matrix} ((1\text{-}\phi_s)\,a_s\text{-}t_s) \\ \\ (a_s\text{-}t_s) \end{matrix}\right\} \cdot \left((n\text{-}1)\,\mathbb{P}_{\text{-}s}\left[a^{(m\text{-}1)}_{m,n\text{-}2} \leq a_s \leq a^{(m)}_{m,n\text{-}2}\right] f_{S,s}(a) + m\mathbb{P}_{\text{-}s}\left[a^{(m\text{-}1)}_{m\text{-}1,n\text{-}1} \leq a_s \leq a^{(m)}_{m\text{-}1,n\text{-}1}\right] f_{B,s}(a) \right) = \\ \begin{cases} (1\text{-}\phi_s)\,(1\text{-}k)\,\mathbb{P}_{\text{-}s}\left[a^{(m\text{-}1)}_{m,n\text{-}1} \leq a_s \leq a^{(m)}_{m,n\text{-}1}\right] & \text{for price fees} \\ \phi_s\mathbb{P}_{\text{-}s}\left[a^{(m)}_{m,n\text{-}1} \geq a_s\right] + (1\text{-}\phi_s)\,(1\text{-}k)\,\mathbb{P}_{\text{-}s}\left[a^{(m\text{-}1)}_{m,n\text{-}1} \leq a_s \leq a^{(m)}_{m,n\text{-}1}\right] & \text{for spread fees} \end{cases}. \tag{25}$$

Interpretation of a buyer's first order condition Despite the extensive and complex form of the condition, it has a natural interpretation: It balances between increasing the probability of trade versus increasing the utility when trading. In particular, an incremental increase Δa_b in a buyer's bid has two opposing impacts: If the bid a_b does not include the buyer amongst those who trade, then by increasing it to $a_b + \Delta a_b$, the buyer may surpass other bids and asks and be involved in trade. If the bid a_b is sufficient to include the buyer in trade, then increasing their bid by Δa_b leads to the following effects, depending on the fee structure: For a price fee it may increase the fee by $k(1 + \phi_b)\Delta a_b$ through a change in the market price. For a spread fee it may simply increase the part of the charged fee depending on the market price by $k(1 - \phi_b)\Delta a_b$ through the price setting rule and it directly increases the part of the charged fee depending on a_b by $\phi_b\Delta a_b$. In Equation (24) the sum in brackets times Δa_b is the probability that the buyer enters the set of buyers who trade as he incrementally raises his bid by Δa_b. The first term in the sum is the marginal probability of acquiring an item by passing a seller's offer and the second term is the marginal probability of acquiring an item by passing another buyer's bid. For a price fee the profit from such a trade is between $t_b - (1+\phi_b)a_b$ and $t_b - (1+\phi_b)a_b - (1+\phi_b)\Delta a_b$. Therefore the marginal expected profit for a buyer who raises their bid is $t_b - (1+\phi_b)a_b$ times the term in the brackets. In analogy, for a spread fee the marginal expected profit for a buyer who raises their bid is $t_b - \phi_b a_b$ times the term in the brackets. On the contrary, $\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m\text{-}1,n} \leq a_b \leq a^{(m+1)}_{m\text{-}1,n}\right]$ is the probability that a buyer who increases their bid by Δa_b increases the market price by $k(1+\phi_b)\Delta a_b$ for a price fee and by $k(1-\phi_b)\Delta a_b$ for a spread fee. Additionally, for a spread fee $\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m\text{-}1,n} \leq a_b\right]$ is the probability that a buyer who increases their bid by Δa_b increases the part of the charged fee depending on their bid by $\phi_b \Delta a_b$. Therefore the right hand side in both Equation (24) correspond to a buyer's marginal expected loss from increasing his bid above a_b.

A.5 Probabilities in the first order conditions

In this section we derive explicit formulas for the probabilities arising in the first order conditions in Equations (24) and (25), that are also used in the proof of Theorem 6 in Appendix B.10.

Instead of deriving expressions for all different probabilities, note that for general n, m, l all of them can be expressed as one of the following three probabilities for different l, m, n: (i) $\mathbb{P}_{\text{-}i}\left[a^{(l)}_{m,n} \leq a_i \leq a^{(l+1)}_{m,n}\right]$, (ii) $\mathbb{P}_{\text{-}i}\left[a^{(l)}_{m,n} \leq a_i\right]$ and (iii) $\mathbb{P}_{\text{-}i}\left[a^{(l)}_{m,n} \geq a_i\right]$.

For (i) that is the probability that action a_i lies between the l'th and $l+1$'st smallest element in a set of m bids and n asks. The probability that another buyer submits an action smaller or equal a_i is $F_{B,i}(a_i)$. The probability that a buyer submits an action greater or equal a_i is therefore $1 - F_{B,i}(a_i)$. Replace $F_{B,i}$ by $F_{S,i}$ for sellers. The event that exactly l bids and asks are below a_i can be split up in the following way: Suppose that i buyers and j sellers bid and offer less or equal than a_i. $i+j$ must be equal to l. Assuming that there are m buyers and n sellers in total, this means that exactly $m-i$ buyers and $n-j$ sellers bid and offer more than a_i. Selecting i buyers and j sellers, the probability that exactly $i+j = l$ bids and offers are below or equal to a_i is

$$F_{B,i}\left(a_i\right)^i F_{S,i}\left(a_i\right)^j \left(1\text{-}F_{B,i}\left(a_i\right)\right)^{m\text{-}i} \left(1\text{-}F_{S,i}\left(a_i\right)\right)^{n\text{-}j}, \tag{26}$$

because the actions of traders are assumed to be independent. There are $\binom{m}{i}$ possibilities to choose i buyers and $\binom{n}{j}$ possibilities to choose j sellers. Therefore, the total probability that exactly l traders submit below a_i is equal to

$$\mathbb{P}_{\text{-}i}\left[a^{(l)}_{m,n} \leq a_i \leq a^{(l+1)}_{m,n}\right] = \sum_{\substack{i+j=l \\ 0\leq i\leq m \\ 0\leq j\leq n}} \binom{m}{i}\binom{n}{j} F_{B,i}\left(a_i\right)^i F_{S,i}\left(a_i\right)^j \left(1\text{-}F_{B,i}\left(a_i\right)\right)^{m\text{-}i} \left(1\text{-}F_{S,i}\left(a_i\right)\right)^{n\text{-}j}. \tag{27}$$

For (ii), that is the probability that a_i is greater than the l'th action. That is, for some $k \in [l, m+n]$ the number of offers below a_i is exactly equal to k. Summing over k yields that

$$\mathbb{P}_{\text{-}i}\left[a_{m,n}^{(l)} \leq a_i\right] = \sum_{k=l}^{n+m} \sum_{\substack{i+j=k \\ 0\leq i\leq m \\ 0\leq j\leq n}} \binom{m}{i}\binom{n}{j} F_{B,i}\left(a_i\right)^i F_{S,i}\left(a_i\right)^j \left(1\text{-}F_{B,i}\left(a_i\right)\right)^{m\text{-}i} \left(1\text{-}F_{S,i}\left(a_i\right)\right)^{n\text{-}j}. \tag{28}$$

For (iii), note that because distributions are assumed to be atomless, $\mathbb{P}_{\text{-}i}\left[a_{m,n}^{(l)} = a_i\right] = 0$. It therefore holds that

$$\mathbb{P}_{\text{-}i}\left[a_{m,n}^{(l)} \geq a_i\right] = 1 - \mathbb{P}_{\text{-}i}\left[a_{m,n}^{(l)} \leq a_i\right], \tag{29}$$

which was computed above.

B Proofs.

B.1 Proof that for price fees the net values are $t_b^\Phi = \frac{t_b}{1+\phi_b}$ and $t_s^\Phi = \frac{t_s}{1-\phi_s}$.

Proof. Consider a buyer with gross value t_b. To show that $t_b^\Phi = \frac{t_b}{1+\phi_b}$, it suffices to prove two statements: (1) If a bid $a_b' > \frac{t_b}{1+\phi_b}$, then it is dominated by $\frac{t_b}{1+\phi_b}$ and (2) if $a_b' < \frac{t_b}{1+\phi_b}$, then there exists $a_{\text{-}b}$ such that $u_b\left(t_b, \frac{t_b}{1+\phi_b}, a_{\text{-}b}\right) > u_b\left(t_b, a_b', a_{\text{-}b}\right)$. For (1), if $a_{\text{-}b}$ is such that both a_b' and $\frac{t_b}{1+\phi_b}$ are not involved in trade, then both have utility equal to zero. If $a_{\text{-}b}$ is such that the buyer is involved in trade at a_b', but not at $\frac{t_b}{1+\phi_b}$, then the market price is greater or equal to $\frac{t_b}{1+\phi_b}$. It follows that $u_b\left(t_b, a_b', a_{\text{-}b}\right) = t_b - (1+\phi_b)\,\Pi\left(a_b, a_{\text{-}b}\right) \leq t_b - (1+\phi_b)\frac{t_b}{1+\phi_b} = 0$. If $a_{\text{-}b}$ is such that the buyer is involved in trade with both bids, then it follows that

$$\begin{aligned} u_b\left(t_b, a_b', a_{\text{-}b}\right) = t_b - (1+\phi_b)\,\Pi\left(a_b', a_{\text{-}b}\right) \leq \\ t_b - (1+\phi_b)\,\Pi\left(\frac{t_b}{1+\phi_b}, a_{\text{-}b}\right) = u_b\left(t_b, \frac{t_b}{1+\phi_b}, a_{\text{-}b}\right), \end{aligned} \tag{30}$$

because $\Pi\left(\cdot, a_{\text{-}b}\right)$ is non-decreasing, if a trader is involved in trade at the bid. For (2), consider $a_b' < \frac{t_b}{1+\phi_b}$. Consider $a_{\text{-}b}$, such that a buyer is involved in trade at bid $\frac{t_b}{1+\phi_b}$ but not with a_b' and it holds that $\Pi\left(\frac{t_b}{1+\phi_b}, a_{\text{-}b}\right) < \frac{t_b}{1+\phi_b}$. This yields

$$u_b\left(t_b, \frac{t_b}{1+\phi_b}, a_{\text{-}b}\right) = t_b - (1+\phi_b)\,\Pi\left(\frac{t_b}{1+\phi_b}, a_{\text{-}b}\right) > t_b - (1+\phi_b)\frac{t_b}{1+\phi_b} = 0. \tag{31}$$

The statement for sellers is proven in analogy. □

B.2 Proof that for spread fees the net values are $t_b^\Phi = t_b$ and $t_s^\Phi = t_s$.

Proof. Consider a buyer with gross value t_b. To show that $t_b^\Phi = t_b$, it suffices to prove two statements: (1) If a bid $a_b' > t_b$, then it is dominated by t_b and (2) if $a_b' < t_b$, there exists $a_{\text{-}b}$ such that $u_b\left(t_b, t_b, a_{\text{-}b}\right) > u_b\left(t_b, a_b', a_{\text{-}b}\right)$ holds. For (1), if $a_{\text{-}b}$ is such that at both bids a_b' and t_b the buyer is not involved in trade, then both have utility equal to zero. If $a_{\text{-}b}$ is such that the buyer is involved in trade at a_b', but not at t_b, then the market price is greater or equal to t_b. It follows that

$$u_b\left(t_b, a_b', a_{\text{-}b}\right) = t_b - \Pi\left(a_b', a_{\text{-}b}\right) - \phi_b|a_b' - \Pi\left(a_b', a_{\text{-}b}\right)| \leq t_b - t_b = 0. \tag{32}$$

If $a_{\text{-}b}$ is such that the buyer is involved in trade with both bids, then it follows that

$$\begin{aligned} u_b\left(t_b, a_b', a_{\text{-}b}\right) = t_b - \Pi\left(a_b', a_{\text{-}b}\right) - \phi_b|a_b' - \Pi\left(a_b', a_{\text{-}b}\right)| \leq \\ t_b - \Pi\left(t_b, a_{\text{-}b}\right) - \phi_b|a_b' - \Pi\left(t_b, a_{\text{-}b}\right)| u_b\left(t_b, t_b, a_{\text{-}b}\right), \end{aligned} \tag{33}$$

because $\Pi\left(\cdot, a_{\text{-}b}\right)$ is increasing, if a trader is involved in trade at the bid. For (2), consider $a_b' < t_b$. Consider $a_{\text{-}b}$, such that a buyer is involved in trade at bid t_b but not with a_b' and it holds that $\Pi\left(t_b, a_{\text{-}b}\right) < t_b$. This yields

$$u_b\left(t_b, t_b, a_{\text{-}b}\right) = t_b - \Pi\left(t_b, a_{\text{-}b}\right) - \phi_b|a_b' - \Pi\left(t_b, a_{\text{-}b}\right)| > t_b - t_b = 0. \tag{34}$$

The statement for sellers is proven in analogy. □

B.3 Proof that for constant fees the net values are $t_b^\Phi = t_b - c_b$ and $t_s^\Phi = t_s + c_s$.

Proof. Consider a buyer with gross value t_b. To show that $t_b^\Phi = t_b - c_b$, it suffices to prove two statements: (1) If a bid $a_b' > t_b - c_b$, then it is dominated by $t_b - c_b$ and (2) if $a_b' < t_b - c_b$, there exists $a_{\text{-}b}$ such that $u_b\left(t_b, t_b - c_b, a_{\text{-}b}\right) > u_b\left(t_b, a_b', a_{\text{-}b}\right)$ holds. For (1), if $a_{\text{-}b}$ is such that both a_b' and $t_b - c_b$ are not involved in trade, then both have utility equal to zero. If $a_{\text{-}b}$ is such that the buyer is involved in trade at a_b', but not at $t_b - c_b$, then the market price is greater or equal to $t_b - c_b$. It follows that $u_b\left(t_b, a_b', a_{\text{-}b}\right) = t_b - \Phi_b\left(a_b, a_{\text{-}b}\right) - c_b \leq t_b - \left(t_b - c_b\right) - c_b = 0$. If $a_{\text{-}b}$ is such that the buyer is involved in trade with both bids, then it follows that

$$\begin{aligned} u_b\left(t_b, a_b, a_{\text{-}b}\right) = t_b - \Phi_b\left(a_b, a_{\text{-}b}\right) - c_b \leq \\ t_b - \Phi_b\left(t_b - c_b, a_{\text{-}b}\right) - c_b = u_b\left(t_b, t_b - c_b, a_{\text{-}b}\right), \end{aligned} \tag{35}$$

because $\Phi_b\left(\cdot, a_{\text{-}b}\right)$ is non-decreasing, if a trader is involved in trade at the bid. For (2), consider $a_b' < t_b - c_b$. Consider $a_{\text{-}b}$, such that a buyer is involved in trade at bid $t_b - c_b$ but not with a_b' and it holds that $\Pi\left(t_b - c_b, a_{\text{-}b}\right) < t_b - c_b$. This yields

$$u_b\left(t_b, t_b - c_b, a_{\text{-}b}\right) = t_b - \Pi\left(t_b - c_b, a_{\text{-}b}\right) - c_b > t_b - \left(t_b - c_b\right) - c_b = 0. \tag{36}$$

The statement for sellers is proven in analogy. □

B.4 Proof that the critical value Π_i^∞ exists and is unique.

Proof. At the point $\underline{a}_{S,i}$, it holds that $F_{B,i}(\underline{a}_{S,i}) < 1$. That is because $F_{B,i}$ has a strictly positive density $f_{B,i}$ on $[\underline{a}_{B,i}, \overline{a}_{B,i}]$ and $\underline{a}_{S,i} < \overline{a}_{B,i}$ by assumption. Second, it holds that $F_{S,i}(\underline{a}_{S,i}) = 0$, because the corresponding density $f_{S,i}$ has support $[\underline{a}_{S,i}, \overline{a}_{B,i}]$. Therefore, at $\underline{a}_{S,i}$, it holds that

$$F_{B,i}(\underline{a}_{S,i}) + R_i F_{S,i}(\underline{a}_{S,i}) < 1. \tag{37}$$

A similar argument yields that at the point $\overline{a}_{B,i}$, it holds that $F_{B,i}(\overline{a}_{B,i}) = 1$ and $F_{S,i}(\overline{a}_{S,i}) > 0$. This implies that

$$F_{B,i}(\overline{a}_{B,i}) + R_i F_{S,i}(\overline{a}_{B,i}) > 1. \tag{38}$$

Because $F_{B,i}$ and $F_{S,i}$ are both continuous, it follows from the Intermediate Value theorem, that there exists $\Pi_i^\infty \in (\underline{a}_{S,i}, \overline{a}_{B,i})$ with

$$F_{B,i}(\Pi_i^\infty) + R_i F_{S,i}(\Pi_i^\infty) = 1. \tag{39}$$

Because both $F_{B,i}$ and $F_{S,i}$ are strictly monotone on $(\underline{a}_{S,i}, \overline{a}_{B,i})$, the uniqueness of Π_i^∞ follows. □

B.5 Proof of Proposition 1

Proof. For this proof, we will consider growing market size with respect to a single parameter. For trader i, consider a sequence of strictly increasing market sizes $(m(l), n(l))_{l\in\mathbb{N}}$ with $m(l), n(l) = \Theta(l)$ and $|R - \frac{n(l)}{m(l)}| = \mathcal{O}(l^{-1})$ for $R \in (0, \infty)$.[45]

Consider a buyer b. It follows from Appendix A.3 that $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] = \mathbb{P}_{\text{-}b}\left[a_b \geq a_{\text{-}b}^{m(l)}\right]$. This is equal to the probability that at least $m(l)$ actions are below a_b in a sample of actions from $m(l) - 1$ buyers and $n(l)$ sellers. Let $p_{a_b} = F_{B,b}(a_b) \in (0, 1)$ be the probability that another buyer's bid is below a_b. In analogy, define $q_{a_b} = F_{S,b}(a_b) \in (0, 1)$ for sellers. For $i > 0$ let $X_i^{p_{a_b}}$ denote an independent Bernoulli random variable with parameter p_{a_b} and for $j > 0$ let denote an independent Bernoulli random variable with parameter q_{a_b}. Define

$$S_l^{a_b} = \sum_{i=1}^{m(l)-1} X_i^{p_{a_b}} + \sum_{j=1}^{n(l)}. \tag{40}$$

$S_l^{a_b}$ has the same distribution as the number of traders in a sample of $m(l) - 1$ buyers and $n(l)$ sellers, whose actions are less or equal than a_b. It follows that

$$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] = \mathbb{P}[S_l^{a_b} \geq m(l)] = 1 - \mathbb{P}[S_l^{a_b} \leq m(l) - 1]. \tag{41}$$

[45] This means that both market sides are assumed to have linear growth with respect to a single parameter l, such that neither side of the market dominates the other asymptotically and the ratio of buyers to sellers converges and fluctuates only slightly in finite markets.

Next, we will show that a properly normalized version of $S_l^{a_b}$ converges in distribution to a standard normal random variable. This follows as an application of the following version of the Berry-Esseen theorem, see Tyurin (2012):

Theorem 17 (Berry-Esseen)**.** *Suppose $X_1, X_2, \ldots$ is a sequence of independent random variables with (i) $\mu_i = \mathbb{E}[X_i] < \infty$, (ii) $\sigma_i^2 = \mathbb{E}[(X_i - \mu_i)^2] < \infty$ and (iii) $\rho_i = \mathbb{E}[|X_i - \mu_i|^3] < \infty$. Set $r_n = \sum_{i=1}^n \rho_i$, $s_n^2 = \sum_{i=1}^n \sigma_i^2$, $F_n(x) = \mathbb{P}\left[\frac{\sum_{i=1}^n (X_i - \mu_i)}{\sqrt{s_n^2}} \leq x\right]$ and let $\Phi(x)$ be the distribution function of a standard random variable. There exists a constant $C = 0.5591$ such that for all $x \in \mathbb{R}$*

$$|F_n(x) - \Phi(x)| \leq \frac{Cr_n}{s_n^3} \tag{42}$$

In order to apply Theorem 17, we rewrite $S_l^{a_b}$ as a single sum of random variables and check all requirements. Define $Y_i^{p_{a_b}} = \sum_{j=0}^{m(i)-m(i-1)} X_{i,j}^{p_{a_b}}$ for $i \leq l-1$ and $Y_l^{p_{a_b}} = \sum_{j=1}^{m(l)-m(l-1)-1} X_{i,j}^{p_{a_b}}$ with $X_{i,j}^{p_{a_b}}$ independent Bernoulli random variables with parameter p_{a_b}. In analogy, define $Y_i^{q_{a_b}} = \sum_{j=1}^{n(i)-n(i-1)} X_{i,j}^{q_{a_b}}$ for $i \leq l$ independent Bernoulli random variables with parameter q_{a_b} and $Z_i^{a_b} = Y_i^{p_{a_b}} + Y_i^{q_{a_b}}$. This yields that in distribution

$$S_l^{a_b} \overset{d}{=} \sum_{i=1}^{l} Z_i^{a_b}. \tag{43}$$

Recall that a Bernoulli random variable with parameter p has expectation p and variance $p(1-p)$. Using linearity of expectation and, because the random variables are independent, linearity of variance, it holds for $i < l$, that the random variables satisfy (i) and (ii) in Theorem 17, i.e.

$$\begin{aligned} \mu_i &= (m(i) - m(i-1))\, p_{a_b} + (n(i) - n(i-1))\, q_{a_b} < \infty \\ \sigma_i^2 &= (m(i) - m(i-1))\, p_{a_b}(1 - p_{a_b}) + (n(i) - n(i-1))\, q_{a_b}(1 - q_{a_b}) < \infty. \end{aligned} \tag{44}$$

For $i = l$ it holds that

$$\begin{aligned} \mu_l &= (m(l) - m(l-1) - 1)\, p_{a_b} + (n(l) - n(l-1))\, q_{a_b} < \infty \\ \sigma_l^2 &= (m(l) - m(l-1) - 1)\, p_{a_b}(1 - p_{a_b}) + (n(l) - n(l-1))\, q_{a_b}(1 - q_{a_b}) < \infty. \end{aligned} \tag{45}$$

Furthermore, for $i < l$ it holds that

$$\begin{aligned} \rho_i &= \mathbb{E}\left[\left|\sum_{j=0}^{m(i)-m(i-1)} X_{i,j}^{p_{a_b}} + \sum_{j=0}^{n(i)-n(i-1)} X_{i,j}^{q_{a_b}} - (m(i) - m(i-1))\, p_{a_b} - (n(i) - n(i-1))\, q_{a_b}\right|^3\right] \\ &\leq \left((m(i) - m(i-1))(1 - p_{a_b}) + (n(i) - n(i-1))(1 - q_{a_b})\right)^3 \\ &\leq K < \infty. \end{aligned} \tag{46}$$

The first inequality in Equation (46) holds, because $X_{i,j}^{p_{a_b}} \leq 1$ and $X_{i,j}^{q_{a_b}} \leq 1$ almost surely. The second inequality follows for some finite $K > 0$ from the assumption $\sup_{i \geq 1} m(i) - m(i-1) < \infty$

and $\sup_{i\geq 1} n(i) - n(i-1) < \infty$. In analogy, for $i = l$ it holds that

$$\rho_l \leq K < \infty, \tag{47}$$

which proves that requirement (iii) is fulfilled. Finally, it holds that

$$s_l^2 = (m(l)-1)p_{a_b}(1-p_{a_b}) + n(l)q_{a_b}(1-q_{a_b}). \tag{48}$$

Next, define the sequence $(A_{a_b}(l))_{l\in\mathbb{N}}$ via

$$\begin{aligned} A_{a_b}(l) &= \frac{m\,(l)-1-((m\,(l)-1)\,p_{a_b}+n\,(l)\,q_{a_b})}{\sqrt{(m(l)-1)\,p_{a_b}\,(1-p_{a_b})+n(l)q_{a_b}\,(1-q_{a_b})}} \\ &= \sqrt{m(l)}\frac{\left(1-\frac{1}{m(l)}\right)-\left(\left(1-\frac{1}{m(l)}\right)p_{a_b}+\frac{n(l)}{m(l)}\right)q_{a_b})}{\sqrt{\left(1-\frac{1}{m(l)}\right)p_{a_b}\,(1-p_{a_b})+\frac{n(l)}{m(l)}q_{a_b}\,(1-q_{a_b})}}. \end{aligned} \tag{49}$$

Theorem 17 now implies that

$$|\mathbb{P}[\leq m(l)-1] - \Phi(A_{a_b}(l))| \leq \frac{Cr_l}{s_l^3} \leq \frac{CKl}{\left(s_l^2\right)^{3/2}} = \mathcal{O}(l^{-\frac{1}{2}}). \tag{50}$$

It follows from Equation (49) that $|A_{a_b}(l)| = \Theta(\sqrt{l})$. We now argue that for $a_b > \Pi_b^\infty$ and sufficiently large l $A_{a_b}(l) < 0$. This follows, if we show that for sufficiently large l

$$\left(1-\frac{1}{m(l)}\right) - \left(\left(1-\frac{1}{m(l)}\right)p_{a_b} + \frac{n(l)}{m(l)}q_{a_b}\right) < 0. \tag{51}$$

Given that a_b is strictly greater than the critical value Π_b^∞, there exists $\delta > 0$, such that $p_{a_b} + Rq_{a_b} = 1+\delta$. By adding and subtracting Rq_{a_b} it follows that Equation (51) is equivalent to

$$1 - \frac{1}{m(l)}(1-p_{a_b}) - (1+\delta) + (R - \frac{n(l)}{m(l)})q_{a_b} < 0 \tag{52}$$

and therefore to

$$R - \frac{n(l)}{m(l)} < \frac{1}{q_{a_b}}(\delta + \frac{(1-p_{a_b})}{m(l)}) \tag{53}$$

Because it is assumed that $|R - \frac{n(l)}{m(l)}| = \mathcal{O}(\frac{1}{l})$, Equation (51) holds for sufficiently large l. This implies that $A_{a_b}(l) = \Theta(-\sqrt{l})$. A standard concentration inequality for a standard Gaussian random variable Z and $x > 0$ using the Chernoff bound gives

$$\mathbb{P}\,|Z| \geq x] \leq 2\exp\left(\frac{-x^2}{2}\right) \tag{54}$$

It follows that

$$\Phi(A_{a_b}(l)) = \mathcal{O}(e^{-l}). \tag{55}$$

Equation (50) therefore implies that $\mathbb{P}[S_l^{a_b} \leq m(l) - 1] = \mathcal{O}(l^{-\frac{1}{2}})$. Recalling Equation (41) finishes the proof. The statements for $a_b < \Pi_b^\infty$ and for sellers can be proven analogous. □

B.6 Proof of Proposition 2

Proof. Consider a buyer b with private type t_b.

Finite Markets. As was shown in Equation (12) in Appendix A.3, the expected utility is of the form

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] = t_b \cdot \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Pi(a_b, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b})\right]. \tag{56}$$

First, we will show that the expected utility is continuous in a_b.[46] The first summand $t_b \cdot \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right]$ is continuous by Equation (3) in Appendix A.3 and Equation (28). To show that the expected market price is continuous, consider $\mathbb{E}_{\text{-}b}\left[\Pi(a_b'', a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Pi(a_b', a_{\text{-}b})\right]$ for two bids $a_b'' \geq a_b'$ as $a_b'' - a_b'$ approaches zero. The buyer increases the expected market price when raising their bid if (1) they are involved in trade at a_b'', but not at a_b' or (2) a_b' influences the market price. For (1), the market price is at most a_b'' and for (2) the change in market price is at most $a_b'' - a_b'$. This implies that

$$\begin{gathered}\mathbb{E}_{\text{-}b}\left[\Pi(a_b'', a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Pi(a_b', a_{\text{-}b})\right] \leq \\ a_b''\left(\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b'', a_{\text{-}b})\right] - \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b', a_{\text{-}b})\right]\right) + (a_b'' - a_b').\end{gathered} \tag{57}$$

The continuity of $\mathbb{E}_{\text{-}b}\left[\Pi(\cdot, a_{\text{-}b})\right]$ therefore follows from the continuity of $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\cdot, a_{\text{-}b})\right]$. To show that the expected fee is continuous, consider again $\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b'', a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b', a_{\text{-}b})\right]$ for two bids $a_b'' \geq a_b'$ as $a_b'' - a_b'$ approaches zero. The buyer increases their fee payment when raising their bid if (1) they are involved in trade at a_b'', but not at a_b' or (2) they are involved in trade for both bids. For (1), fee the payment is at most some finite number M. This implies that

$$\begin{gathered}\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b'', a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b', a_{\text{-}b})\right] \\ \leq M\left(\mathbb{P}_{\text{-}b}\left[(b, a_b'') \in \mathcal{B}^*\right] - \mathbb{P}_{\text{-}b}\left[(b, a_b') \in \mathcal{B}^*\right]\right) + \\ \left(\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b'', a_{\text{-}b}) | A^*(a_b'')\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b', a_{\text{-}b}) | A^*(a_b')\right]\right)\end{gathered} \tag{58}$$

The continuity of $\mathbb{E}_{\text{-}b}\left[\Phi_b(\cdot, a_{\text{-}b})\right]$ therefore follows from the continuity of $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\cdot, a_{\text{-}b})\right]$ that was proven above and $\mathbb{E}_{\text{-}b}\left[\Phi_b(\cdot, a_{\text{-}b}) | (\cdot)\right]$, which was an assumption. Therefore, the expected utility is indeed continuous.

[46] The same proof strategy for continuity is used in Williams (1991) for the expected utility in a buyer's bid DA without fees in the context of Bayesian Nash equilibria.

Every bid $a_b < \underline{a}_{S,b}$ results in zero utility, as the buyer is almost surely not involved in trade. For every bid $a_b > t_b^\Phi$, it follows from weak domination ex post that the expected utility for a_b is smaller or equal than for $t_b^\Phi \leq t_b$. If $t_b^\Phi \leq \underline{a}_{S,b}$, then t_b^Φ is a best response with expected utility equal to zero. Otherwise, in order to compute a best response, it is sufficient to consider the interval $\left[\underline{a}_{S,b}, t_b^\Phi\right]$. Because the expected utility is a continuous function on this compact set, it follows from the Extreme Value theorem that the expected utility attains a maximum. Therefore, a best response exists.

Infinite Markets. It was shown in Appendix A.3 that the expected utility is of the form

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] = (t_b - \Pi(a_b, a_{\text{-}b}) - \Phi_b(a_b, a_{\text{-}b})) \cdot \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right]. \tag{59}$$

In an infinite market, the market price $\Pi(a_b, a_{\text{-}b})$ and the fee $\Phi_b(a_b, a_{\text{-}b})$ are deterministic. The assumption that the conditional expected fee is continuous is therefore equivalent to the assumption that $\Phi_b(a_b, a_{\text{-}b})$ is continuous in the action a_b. By Appendix A.3 the assumption, that there is no tie-breaking implies that

$$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] = \begin{cases} 1 & a_b \geq \Pi(a) \\ 0 & \text{else} \end{cases}. \tag{60}$$

If $t_b^\Phi < \Pi(a)$, then buyer b has no undominated action with positive probability of trade. Therefore t_b^Φ is a best response with expected utility equal to zero. If $t_b^\Phi = \Pi(a)$, then the only undominated action with positive probability of trade is t_b^Φ. If this results in a strictly positive utility, then it is a best response. If not, then any bid below $\Pi(a)$ is a best response. Therefore, consider the case $t_b^\Phi > \Pi(a)$. If there is no tie-breaking, then the trading probability is constant and equal to 1 on the compact set $[\Pi(a), t_b^\Phi]$. Note that any bid above t_b^Φ is not a best response by weak domination. By similar arguments as before, the expected utility on this interval is equal to $(t_b - \Pi(a_b, a_{\text{-}b}) - \Phi_b(a_b, a_{\text{-}b}))$ and therefore a continuous function. The Extreme Value theorem implies again that the maximum is attained and a best response exists. The statement for sellers can be proven analogous. □

B.7 Proof of Proposition 3

Proof. Consider a buyer b with gross value t_b, such that $t_b^\Phi < \Pi_b^\infty$. A best response a_b with $a_b \leq t_b^\Phi$ must exist. That is because if there is a best response a_b with $a_b > t_b^\Phi$, then by weak domination of the net value, the expected utilities must be equal, proving that t_b^Φ is a best response as well. By the monotonicity of the trading probability, it then holds that

$$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \leq \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right] \tag{61}$$

For all $\gamma > 0$, it holds by Theorem 1 ISLMthat $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^{\Phi}, a_{\text{-}b})\right] \leq \gamma$. The expected utility is upper bounded by neglecting the payment of market price and fee, that is the gross value times the probability of trade:

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \leq t_b\gamma. \tag{62}$$

Choose $\gamma \leq \frac{\epsilon}{t_b}$. This implies that $ISLM$, the expected utility of a best response is upper bounded by ϵ. The expected utility of truthfulness is non-negative by assumption. This implies that truthfulness is an ϵ-best response. The statement for sellers can be proven analogous. □

B.8 Proof of Theorem 4

Proof. Consider a buyer b and two actions $a_b^1 > a_b^2 > \Pi_b^{\infty}$ that lead to asymptotically different fee payments. We will prove $ISLM$ that a buyer can improve his expected utility when switching from action a_b^1 to a_b^2. This in turn implies that best responses for two buyers with different gross values must lead to asymptotically equal fee payments. Otherwise, there is a buyer with a certain gross value, who has an incentive to switch $ISLM$ to increase their expected utility.

By assumption, there exists $\epsilon > 0$ such that $ISLM$ almost surely

$$\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b^1, a_{\text{-}b})|A^*\left(b, a_b^1\right)\right] - \mathbb{E}_{\text{-}i}\left[\Phi_b(a_b^2, a_{\text{-}b})|A^*\left(b, a_b^2\right)\right] \geq \epsilon. \tag{63}$$

We will show that $ISLM$ a_b^1 cannot be a best response. Assume that it was a best response for some gross value t_b. The expected utility $\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^1, a_{\text{-}b}\right)\right]$ is greater or equal than 0, otherwise it is trivially not a best response. We will prove that $ISLM$

$$\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^2, a_{\text{-}b}\right)\right] < 0, \tag{64}$$

which proves that a_b^1 is not a best response, because a_b^2 increases the expected utility.

Using the law of total expectation, the expected fee difference can be lower bounded by

$$\begin{aligned}
&\mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b^2, a_{\text{-}b}\right)\right] \\
&= \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b^1, a_{\text{-}b})|A^*\left(b, a_b^1\right)\right]\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^1, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b^2, a_{\text{-}b})|A^*\left(b, a_b^2\right)\right]\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^2, a_{\text{-}b})\right] \\
&\geq \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^2, a_{\text{-}b})\right]\left(\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b^1, a_{\text{-}b})|A^*\left(b, a_b^1\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b})|A^*\left(b, a_b^2\right)\right]\right)
\end{aligned} \tag{65}$$

The inequality from the last line follows by the monotonicity of the trading probability, which imples

$$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^1, a_{\text{-}b})\right] \geq \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^2, a_{\text{-}b})\right]. \tag{66}$$

It follows from Theorem 1 that for every γ it holds $ISLM$ that
$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^2, a_{\text{-}b})\right] \geq 1 - \gamma$ holds. Combining this with the assumption of asymptotically different fee payments yields that $ISLM$

$$\mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b^2, a_{\text{-}b}\right)\right] \geq (1 - \gamma)\epsilon. \tag{67}$$

Using Equation (16) in Theorem 16 it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^2, a_{\text{-}b}\right)\right] \leq t_b\gamma - (1-\gamma)\epsilon. \tag{68}$$

If we now choose $\gamma < \frac{\epsilon}{t_b+\epsilon}$, the difference in expected utility is strictly negative. The statement for sellers can be proven analogous. □

B.9 Proof of Theorem 5

Proof. Consider a buyer b with gross value t_b, such that the best response a_b is uniformly bounded away from the critical value. That is there exists $\delta > 0$, such that $ISLM$ either (i) $a_b \leq \Pi_b^\infty - \delta$ or (ii) $a_b \geq \Pi_b^\infty + \delta$. It suffices to prove that for every $\epsilon > 0$ it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \geq -\epsilon, \tag{69}$$

which implies that truthfulness is an ϵ-best response. If it holds that $t_b^\Phi \leq a_b$, it holds that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] = \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right], \tag{70}$$

because t_b^Φ weakly dominates every larger bid and since a_b is a best response, the expected utilities must be equal. Therefore, assume that $t_b^\Phi > a_b$.

If (i) holds, then Theorem 1 implies that for all $\gamma > 0$
$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \leq \gamma$ holds $ISLM$. If $\gamma < \frac{\epsilon}{t_b}$ it follows that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \leq t_b\gamma \leq \epsilon. \tag{71}$$

By assumption it also holds that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] \geq 0. \tag{72}$$

Combining Equation (71) and Equation (72) yields Equation (69).

If (ii) holds, then

$$\begin{aligned}\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \geq \\ t_b^\Phi\left(\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right]\right) - \left(\mathbb{E}_{\text{-}b}\left[\Pi(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Pi(a_b, a_{\text{-}b})\right]\right) \\ - \left(\mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b})\right]\right),\end{aligned} \tag{73}$$

because by assumption $t_b^\Phi \leq t_b$. It follows from Theorem 6 that for a DA without fees for every $\epsilon_1 > 0$ truthfulness is an ϵ_1-best response $ISLM$. Assume that a buyer has gross value equal to t_b^Φ.

It therefore holds $ISLM$ that for any other bid, i.e. also the best response a_b for gross value t_b

$$t_b^\Phi \left(\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right]\right) - \left(\mathbb{E}_{\text{-}b}\left[\Pi(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Pi(a_b, a_{\text{-}b})\right] \geq -\epsilon_1. \tag{74}$$

Using the law of total expectation, the expected fee difference in Equation (74) is equal to

$$\begin{aligned} &\mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b})\right] \\ &= \mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b}) | A^*\left(b, t_b^\Phi\right)\right] \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right] - \\ &\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b}) | A^*\left(b, a_b\right)\right] \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right]. \end{aligned} \tag{75}$$

Because both actions are by assumption greater or equal than $\Pi_b^\infty + \delta$, for every $\gamma > 0$ it holds $ISLM$ that $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right], \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \geq 1 - \gamma$. It therefore holds that

$$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(t_b^\Phi, a_{\text{-}b})\right] \leq \gamma. \tag{76}$$

This implies that $ISLM$

$$\begin{aligned} &\mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b})\right] \leq \\ &\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \left(\mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b}) | A^*\left(b, t_b^\Phi\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b}) | A^*\left(b, a_b\right)\right]\right) + \\ &\gamma \mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b}) | A^*\left(b, t_b^\Phi\right)\right]. \end{aligned} \tag{77}$$

Homogeneity implies that for every $\epsilon_2 > 0$ the first term in Equation (77) is less or equal than ϵ_2 $ISLM$ and for every $\epsilon_3 > 0$ the second term can be chosen to be less or equal than ϵ_3 $ISLM$ by choosing $\gamma \leq \frac{\epsilon_3}{\mathbb{E}_{\text{-}b}\left[\Phi_b(t_b^\Phi, a_{\text{-}b}) | A^*\left(b, t_b^\Phi\right)\right]}$. If ϵ_1, ϵ_2 and ϵ_3 are chosen such that their sum is less or equal than ϵ, plugging Equation (74) and Equation (77) in yields that $ISLM$

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \geq -(\epsilon_1 + \epsilon_2 + \epsilon_3) \geq -\epsilon, \tag{78}$$

which finishes the proof. The statement for sellers can be proven analogous. □

B.10 Proof of Theorem 6

Proof. Consider a buyer b with private type t_b.

Existence of a best response. The proof of the existence of a best response is closely related to the proof of Theorem 2. Because the fee is a percentage of the market price, the expected fee is a percentage of the expected market price, which is shown to be continuous in a_i in the proof of Theorem 2 in Appendix B.6. Therefore, the expected utility continuous in a_i and the existence of a best response again follows again from the Extreme Value theorem as in Appendix B.6.

Best responses are close to truthfulness We will show that there exists a constant $\kappa > 0$, such that

$$t_b - (1 + \phi_b)\, a_b \leq \kappa q(n, m), \tag{79}$$

with $q(m, n) = \max\left\{\frac{1}{n}\left(1 + \frac{m}{n}\right), \frac{1}{m}\left(1 + \frac{n}{m}\right)\right\} = O(\max(m, n)^{-1})$, from which the statement follows. It was proven in Appendix A.4, that a best response a_b necessarily satisfies the first order condition in Equation (24), which implies the following bound:

$$t_b - (1 + \phi_b)\, a_b \leq \frac{(1 + \phi_b)\, k\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m-1,n} \leq a_b \leq a^{(m+1)}_{m-1,n}\right]}{(m-1)\mathbb{P}_{\text{-}b}\left[a^{(m-1)}_{m-2,n} \leq a_b \leq a^{(m)}_{m-2,n}\right] f_{B,b}(a_b)}. \tag{80}$$

It can be proven analogous to Rustichini et al. (1994, Appendix) that

$$\frac{\mathbb{P}_{\text{-}b}\left[a^{(m)}_{m-1,n} \leq a_b \leq a^{(m+1)}_{m-1,n}\right]}{\mathbb{P}_{\text{-}b}\left[a^{(m-1)}_{m-2,n} \leq a_b \leq a^{(m)}_{m-2,n}\right]} \leq 2\left[F_{B,b}(a_b) + \frac{n}{m}\frac{(1 - F_{B,b}(a_b))\, F_{S,b}(a_b)}{1 - F_{S,b}(a_b)}\right]. \tag{81}$$

Defining

$$\tau_b \equiv 2 \max_{x \in [\underline{a}_{S,b}, \overline{a}_{B,b}]} \left\{\frac{F_{B,b}(x)}{f_{B,b}(x)}, \frac{(1 - F_{B,b}(x))\, F_{S,b}(x)}{f_{B,b}(x)\,(1 - F_{S,b}(x))}\right\} \tag{82}$$

yields that

$$t_b - (1 + \phi_b)\, a_b \leq \frac{\tau_b k\, (1 + \phi_b)}{m - 1}\left[1 + \frac{n}{m}\right]. \tag{83}$$

To obtain the bounds in the theorem, note that $\frac{n}{n-1}$ and $\frac{m}{m+1}$ are both less than 2. Setting $\kappa \equiv 2\tau_b k$ proves the statement for buyers. For a seller s with private type t_s an analogous argument yields

$$(1 - \phi_s)\, a_s - t_s \leq \frac{\tau_s(1\text{-}k)(1 - \phi_s))}{n - 1}\left[1 + \frac{m}{n}\right] \tag{84}$$

for τ_s with

$$\tau_s \equiv 2\max\left\{\frac{1 - F_{S,s}(x)}{f_{S,s}(x)}, \frac{(1 - F_{B,s}(x))\, F_{S,s}(x)}{f_{S,s}(x) F_{B,s}(x)}\right\}. \tag{85}$$

Truthfulness is an ϵ-best response We start by estimating the difference in utility when a buyer switches from a bid a^1_b to a smaller bid a^2_b, i.e. $\mathbb{E}_{\text{-}b}\left[u_b(t_b, a^1_b, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a^2_b, a_{\text{-}b})\right]$. The expected utility is not dependent on the entirety of $a_{\text{-}b}$, but only on $a^{(m)}_{\text{-}b}$ and $a^{(m+1)}_{\text{-}b}$. We consider all six possible cases for the realizations of $a^{(m)}_{\text{-}b}$ and $a^{(m+1)}_{\text{-}b}$ with respect to $a^1_b > a^2_b$.

		$u_b(t_b, a_b^1, a_{\text{-}b})$	$u_b(t_b, a_b^2, a_{\text{-}b})$
I	$a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)}$	$t_b\text{-}(1+\phi_b)\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	$t_b\text{-}(1+\phi_b)\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
II	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$t_b\text{-}(1+\phi_b)\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	$t_b\text{-}(1+\phi_b)\left(ka_b^2+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
III	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b\text{-}(1+\phi_b)\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	0
IV	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$t_b\text{-}(1+\phi_b)\left(ka_b^1+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	$t_b\text{-}(1+\phi_b)\left(ka_b^2+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
V	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b\text{-}(1+\phi_b)\left(ka_b^1+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	0
VI	$a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^1 \geq a_b^2$	0	0

Analogous, we consider the difference in utilities:

		$u_b(t_b, a_b^1, a_{\text{-}b}) - u_b(t_b, a_b^2, a_{\text{-}b})$
I	$a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)}$	0
II	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$\text{-}k(1+\phi_b)\left(a_{\text{-}b}^{(m+1)}\text{-}a_b^2\right)$
III	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b\text{-}(1+\phi_b)\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
IV	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$\text{-}k(1+\phi_b)\left(a_b^1\text{-}a_b^2\right)$
V	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b\text{-}(1+\phi_b)\left(ka_b^1+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
VI	$a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^1 \geq a_b^2$	0

We want to lower bound $\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b^1, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b^2, a_{\text{-}b})\right]$. It is therefore sufficient to lower bound the expression in **II** and **IV**, since they are negative and neglect the positive difference in the other cases. In order to prove truthfulness is close to optimal, consider $a_b^1 = t_b^\Phi$ and $a_b^2 = a_b$ a best response. We show that for any $\epsilon > 0$ it holds that $ISLFM$ the difference in expected utility is bounded from below by $-\epsilon$. Because best responses are ϵ-close to truthfulness $ISLFM$, it holds that for all $\delta > 0$ $t_b^\Phi - a_b \leq \delta$ $ISLFM$. Therefore the difference in **II** and **IV** is lower bounded by $-k(1+\phi_b)\delta$. It follows that

$$\begin{gathered}\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \leq \\ -k(1+\phi_b)\delta\left(\mathbb{P}\left[\mathbf{II}\right] + \mathbb{P}\left[\mathbf{IV}\right]\right) \leq -2k(1+\phi_b)\delta.\end{gathered} \tag{86}$$

If for a given $\epsilon > 0$, $\delta > 0$ is chosen such that $\delta \leq \frac{\epsilon}{2k(1+\phi_b)}$, it holds $ISLFM$ that t_b^Φ is ϵ-close to a best response a_b. In infinite markets, the expected utility is equal to

$$\mathbb{E}[u_b(t_b, a_b, a_{\text{-}b})] = \begin{cases} t_b - (1+\phi_b)\Pi & \text{if } a_b \geq \Pi \\ 0 & \text{if } a_b < \Pi, \end{cases} \tag{87}$$

If $t_b^\Phi \geq \Pi$, then the expected utility is equal to $t_b - (1+\phi_b)\Pi > 0$, and therefore a best response. If $t_b^\Phi \leq \Pi$, then the expected utility is equal to 0. Because every action $a_b > t_b^\Phi$ is dominated, t_b^Φ is again a best response. Therefore truthfully reporting t_b^Φ is a best response. The statement for sellers can be proven analogous. □

B.11 Proof of Theorem 7

Proof. Consider a buyer b with a gross value t_b and action a_b, such that $t_b^\Phi > \Pi_b^\infty$. First, assume that $a_b > \Pi_b^\infty$. That is, there exists $\epsilon > 0$ such that $a_b - \Pi_b^\infty \geq \epsilon$. We will prove that $ISLM$ it holds that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] < 0, \tag{88}$$

which proves that a_b is not a best response $ISLM$. Using the law of total expectation, the expected fee difference can be lower bounded by

$$\begin{gathered}\mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b\left(\Pi_b^\infty + \epsilon/2, a_{\text{-}b}\right)\right] = \\ \mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b}) | A^*\left(b, a_b\right)\right] \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] - \\ \mathbb{E}_{\text{-}b}\left[\Phi_b(\Pi_b^\infty + \epsilon/2, a_{\text{-}b}) | A^*\left(b, \Pi_b^\infty + \epsilon/2\right)\right] \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] \geq \\ \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] \left(\mathbb{E}_{\text{-}b}\left[\Phi_b(a_b, a_{\text{-}b}) | A^*\left(b, a_b\right)\right] - \right. \\ \left. \mathbb{E}_{\text{-}b}\left[\Phi_b(\Pi_b^\infty + \epsilon/2, a_{\text{-}b}) | A^*\left(b, \Pi_b^\infty + \epsilon/2\right)\right]\right)\end{gathered} \tag{89}$$

The inequality on the last line holds because the trading probability is monotone, which implies $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \geq \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right]$. It follows from Theorem 1 that for every γ it holds $ISLM$ that $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] \geq 1 - \gamma$. Combining this with the assumption of heterogeneity yields that there exists $\delta > 0$ such that it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[\Phi_b\left(a_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b\left(\Pi_b^\infty + \epsilon/2, a_{\text{-}b}\right)\right] \geq (1 - \gamma)\delta. \tag{90}$$

Using Equation (16) from Theorem 16, it therefore holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, \Pi_b^\infty + \epsilon/2, a_{\text{-}b}\right)\right] \leq t_b\gamma - (1 - \gamma)\delta. \tag{91}$$

If we now choose $\gamma < \delta/t_b + \delta$, the difference is strictly smaller than 0, which proves that a_b is not a best response $ISLM$.

Second, assume that $a_b < \Pi_b^\infty$. We will show that it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] > 0, \tag{92}$$

which again implies that a_b is not a best response. It follows from uniform profitability that there exists $\delta > 0$ such that it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, t_b^\Phi, a_{\text{-}b})\right] \geq \delta. \tag{93}$$

It therefore suffices to show that for $a_b < \Pi_b^\infty - \epsilon$ it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] < \delta. \tag{94}$$

We can upper bound the expected utility by neglecting the expected market price and the expected fee and get that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] \leq t_b \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right]. \tag{95}$$

Theorem 1 implies that for any $\gamma > 0$ it holds $ISLM$ that $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \leq \gamma$. If we choose $\gamma < \frac{\delta}{t_b}$, the statement follows. □

B.12 Proof of Theorem 8

Proof. For prove that best responses are in an ϵ-neighbourhood of the critical value $ISLM$, consider a buyer b with a gross value t_b and action a_b, such that $t_b^\Phi > \Pi_b^\infty$. First, assume that $a_b > \Pi_b^\infty$. That is, there exists $\epsilon > 0$ such that $a_b - \Pi_b^\infty \geq \epsilon$. We will prove that it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, a_b, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] < 0, \tag{96}$$

which proves that a_b is not a best response $ISLM$. For two bids $a_b^1 > a_b^2$ Theorem 16 implies in the presence of a spread fee that

$$\begin{aligned} &\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^1, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b^2, a_{\text{-}b}\right)\right] \\ \leq &\left(t_b - \phi_b a_b^1\right) \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^1, a_{\text{-}b})\right] - \left(t_b - \phi_b a_b^2\right) \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b^2, a_{\text{-}b})\right]. \end{aligned} \tag{97}$$

Now set $a_b^1 = a_b$ and $a_b^2 = \Pi_b^\infty + \epsilon/2$. It follows from Theorem 1 that for any $\gamma > 0$ it holds $ISLM$ that $\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right], \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] \geq 1 - \gamma$ and therefore also

$$\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a_b, a_{\text{-}b})\right] \leq \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(\Pi_b^\infty + \epsilon/2, a_{\text{-}b})\right] + \gamma. \tag{98}$$

Combining Equation (97) and Equation (98) implies that it holds $ISLM$ that

$$\begin{aligned} &\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, \Pi_b^\infty + \epsilon/2, a_{\text{-}b}\right)\right] \\ \leq &-\phi_b(1-\gamma)(a_b - (\Pi_b^\infty + \epsilon/2)) + \gamma\left(t_b - \phi_b a_b\right). \end{aligned} \tag{99}$$

By assumption, it holds that $a_b - (\Pi_b^\infty + \epsilon/2) \geq \epsilon/2$, which yields

$$\begin{aligned} &\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, \Pi_b^\infty + \epsilon/2, a_{\text{-}b}\right)\right] \\ \leq &-\phi_b(1-\gamma)\frac{\epsilon}{2} + \gamma\left(t_b - \phi_b a_b\right) \leq -\phi_b(1-\gamma)\frac{\epsilon}{2} + \gamma t_b. \end{aligned} \tag{100}$$

If γ is chosen such that $\gamma < \frac{\phi_b \epsilon}{2t_b + \phi_b \epsilon}$ holds, then ISLM

$$\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, \Pi_b^{\infty} + {}^{\epsilon}\!/2, a_{\text{-}b}\right)\right] < 0, \tag{101}$$

which implies that a_b is not a best response $ISLM$.

Next, we prove that for sufficiently small $\epsilon > 0$, there exist beliefs, such that the critical value is not an ϵ-ISLFM. Consider a buyer b with gross value $t_b^{\Phi} > \Pi_b^{\infty}$ in a sequence of market environment with $m(l) = l$, $n(l) = l$, $t_b = [0, 1]$ and uniformly distributed beliefs for both buyers and sellers. In this case, the critical value Π_b^{∞} is equal to $\frac{1}{2}$. By assumption, there exists $\epsilon > 0$, such that $t_b = \Pi_b^{\infty} + \epsilon$ for $\epsilon > 0$. We will show that it holds $ISLM$ that

$$\mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^{\infty} + {}^{\epsilon}\!/4, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^{\infty}, a_{\text{-}b})\right] > 0, \tag{102}$$

which proves that Π_b^{∞} is not a best response. In order to estimate the difference in expected utility for two bids $a_b^1 > a_b^2$, we use a table similar to the one in Appendix B.9 or Appendix B.10:

		$u_b(t_b, a_b^1, a_{\text{-}b})$	$u_b(t_b, a_b^2, a_{\text{-}b})$
I	$a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)}$	$t_b\text{-}\phi_b a_b^1\text{-}(1\text{-}\phi_b\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	$t_b\text{-}\phi_b a_b^2\text{-}(1\text{-}\phi_b\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
II	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$t_b\text{-}\phi_b a_b^1\text{-}(1\text{-}\phi_b\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	$t_b\text{-}\phi_b a_b^2\text{-}(1\text{-}\phi_b\left(ka_b^2+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
III	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b\text{-}\phi_b a_b^1\text{-}(1\text{-}\phi_b\left(ka_{\text{-}b}^{(m+1)}+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	0
IV	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$t_b\text{-}\phi_b a_b^1\text{-}(1\text{-}\phi_b\left(ka_b^1+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	$t_b\text{-}\phi_b a_b^2\text{-}(1\text{-}\phi_b\left(ka_b^2+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$
V	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b\text{-}\phi_b a_b^1\text{-}(1\text{-}\phi_b\left(ka_b^1+(1\text{-}k)a_{\text{-}b}^{(m)}\right)$	0
VI	$a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^1 \geq a_b^2$	0	0

Analogous, we consider the difference in utilities:

		$u_b(t_b, a_b^1, a_{\text{-}b}) - u_b(t_b, a_b^2, a_{\text{-}b})$
I	$a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)}$	$-\phi_b\left(a_b^1 - a_b^2\right)$
II	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$-\phi_b\left(a_b^1 - a_b^2\right) - k(1 - \phi_b\left(a_{\text{-}b}^{(m+1)} - a_b^2\right)$
III	$a_b^1 \geq a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b - \phi_b a_b^1 - (1 - \phi_b\left(ka_{\text{-}b}^{(m+1)} + (1\text{-}k)a_{\text{-}b}^{(m)}\right)$
IV	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_b^2 \geq a_{\text{-}b}^{(m)}$	$-\phi_b\left(a_b^1 - a_b^2\right) - k(1 - \phi_b\left(a_b^1 - a_b^2\right)$
V	$a_{\text{-}b}^{(m+1)} \geq a_b^1 \geq a_{\text{-}b}^{(m)} \geq a_b^2$	$t_b - \phi_b a_b^1 - (1 - \phi_b\left(ka_b^1 + (1\text{-}k)a_{\text{-}b}^{(m)}\right)$
VI	$a_{\text{-}b}^{(m+1)} \geq a_{\text{-}b}^{(m)} \geq a_b^1 \geq a_b^2$	0

In order to obtain a lower bound on the expected difference in utility, we bound all five non-zero terms from below. We set $a_b^1 = \Pi_b^{\infty} + {}^{\epsilon}\!/4$ and $a_b^2 = \Pi_b^{\infty}$, which implies that there difference is equal to ${}^{\epsilon}\!/4$. The expressions in **I**, **II** and **IV** are therefore greater or equal than $-{}^{\epsilon}\!/4$. For **III** and **V**, the lower bound $t_b - (\Pi_b^{\infty} + {}^{\epsilon}\!/4 = \frac{3\epsilon}{4}$ holds, because $t_b = \Pi_b^{\infty} + \epsilon$. Combining these bounds with the probabilities of each event, the following inequality holds:

$$
\begin{aligned}
&\mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^\infty + {\epsilon}/{4}, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^\infty, a_{\text{-}b})\right] \geq \\
&-\frac{\epsilon}{4}\mathbb{P}_{\text{-}b}\left[\Pi_b^\infty \geq a_{\text{-}b}^{(m)}\right] + \frac{3\epsilon}{4}\mathbb{P}_{\text{-}b}\left[\Pi_b^\infty + {\epsilon}/{4} \geq a_{\text{-}b}^{(m)} \geq \Pi_b^\infty\right] = \\
&-\frac{\epsilon}{2}\mathbb{P}_{\text{-}b}\left[\Pi_b^\infty \geq a_{\text{-}b}^{(m)}\right] + \frac{3\epsilon}{4}\left(\mathbb{P}_{\text{-}b}\left[a_{\text{-}b}^{(m)} \leq \Pi_b^\infty + {\epsilon}/{4}\right] - \mathbb{P}\left[a_{\text{-}b}^{(m)} \leq \Pi_b^\infty\right]\right)
\end{aligned} \tag{103}
$$

By definition $a_{\text{-}b}^{(m)}$ is the m'th smallest submission in a set of $m-1$ bids and n asks. Since buyer b assumes that those are uniformly distributed and that there are $m(l) = l$ and $n(l) = l$ many buyers and sellers, it follows from order statistics that $a_{\text{-}b}^{(m)} \sim Beta(l, l)$. This distribution is symmetric on $[0, 1]$ for every l and therefore at the critical value $\Pi_b^\infty = \frac{1}{2}$, it holds that $\mathbb{P}_{\text{-}b}\left[a_{\text{-}b}^{(m)} \leq \Pi_b^\infty\right] = \frac{1}{2}$. Furthermore, it follows from Theorem 1 that for any $\gamma > 0$ it holds in sufficiently large markets that $\mathbb{P}\left[a_{\text{-}b}^{(m)} \leq \Pi_b^\infty + {\epsilon}/{4}\right] \geq 1 - \gamma$. It follows that

$$
\begin{aligned}
&\mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^\infty + {\epsilon}/{4}, a_{\text{-}b})\right] - \mathbb{E}_{\text{-}b}\left[u_b(t_b, \Pi_b^\infty, a_{\text{-}b})\right] \geq \\
&-\frac{\epsilon}{8} + \frac{3\epsilon}{4}\left(\frac{1}{2} - \gamma\right),
\end{aligned} \tag{104}
$$

which is positive if γ is chosen to be smaller than $\frac{1}{3}$. □

B.13 Proof of Theorem 9

Proof. Recall that $E_\Phi = \frac{\mathbb{E}[GoT]}{\mathbb{E}[GoT_\Phi]}$. Because the allocation balances trade, that is $\mu_B(\mathcal{B}^*) = \mu_S(\mathcal{S}^*)$, it holds that

$$
\mathbb{E}[GoT] = \mathbb{E}\left[\int_{\mathcal{B}^*} (t_b - \Pi)\, d\mu_B(b) + \int_{\mathcal{S}^*} (\Pi - t_s)\, d\mu_S(s)\right] = \mathbb{E}\left[\int_{\mathcal{B}^*} t_b d\mu_B(b) - \int_{\mathcal{S}^*} t_s d\mu_S(s)\right]. \tag{105}
$$

Finite Markets. In finite markets, the integral representation of the gains of trade simplifies to the following sum:

$$
\mathbb{E}[GoT] = \mathbb{E}\left[\sum_{\mathcal{B}^*} t_b - \sum_{\mathcal{S}^*} t_s d\mu_S(s)\right] \tag{106}
$$

To show that $E_\Phi \geq 1 - \zeta$, it suffices to prove that $\frac{\mathbb{E}[GoT_\Phi - GoT]}{\mathbb{E}[GoT_\Phi]} \leq \zeta$.[47] We start by lower bounding the denominator. We pair off each of $\min(n, m)$ buyers and sellers. The expected gains of trade $\max(t_b - t_s, 0)$ of such a pair is equal to $\alpha > 0$. It therefore holds that $\mathbb{E}[GoT_\Phi] \geq \alpha \cdot \min(m, n)$. However the value of trade is bounded by $\beta = \overline{a}_B - \underline{a}_S$, proving that $\mathbb{E}[GoT_\Phi] \leq \beta \cdot \min(m, n)$ and therefore $\mathbb{E}[GoT_\Phi] \in \Theta(\min(m, n))$.

In a next step, we will bound the numerator $\mathbb{E}[GoT_\Phi - GoT]$. Let and be the distribution functions of net values on $A_B = [\underline{a}_B, \overline{a}_B] \subset \mathbb{R}^{\geq 0}$ and $A_S = [\underline{a}_S, \overline{a}_S] \subset \mathbb{R}^{\geq 0}$. Denote by t^Φ a sample

[47] The following proof is based on methods from Rustichini et al. (1994).

of $n+m$ net values. Denote by μ the distribution of the market price $\Pi(t^\Phi)$ and by $L(t^\Phi)$ the total value of trades that inefficiently fail to occur given t^Φ and the strategies a_B, $a_S \in \Upsilon^{\epsilon,opt}_{\Phi,F}$. It holds that

$$\mathbb{E}[GoT_\Phi - GoT] = \mathbb{E}\left[L(t^\Phi)\right] = \int_{-\infty}^{\infty} \mathbb{E}\left[L(t^\Phi)|\Pi(t^{\Phi,(m)})\right] d\mu(\Pi(t^{\Phi,(m)})). \tag{107}$$

We will bound the value of this integral over (i) $(-\infty, \underline{a}_S+\delta)$,(ii) $[\underline{a}_S+\delta, \overline{a}_B-\delta]$ and (iii) $[\overline{a}_B-\delta,\infty]$ for some $\delta>0$. δ is chosen small enough, such that $\underline{a}_S+\delta<\Pi^\infty$ and $\overline{a}_B-\delta>\Pi^\infty$, where Π^∞ denotes the critical value of and . The same proof-technique as in Theorem 1 shows that

$$\mathbb{P}[\Pi(t^{\Phi,(m)}) \leq \underline{a}_S+\delta], \mathbb{P}[\Pi(t^{\Phi,(m)}) \geq \overline{a}_B-\delta] \in O(\min(m,n)^{-\frac{1}{2}}). \tag{108}$$

Because it holds that $\mathbb{E}[L(t^\Phi)|\Pi(t^{\Phi,(m)})] \leq \beta \min(n,m)$, where $\beta = \overline{a}_B - \underline{a}_S$ we get that the integral in Equation (107) over (i) and (iii) is $O(\min(m,n)^{\frac{1}{2}})$. Next we bound the integral over (ii). Consider any symmetric strategy profile $a=(a_B,a_S)\in\Upsilon^{\epsilon,opt}_{\Phi,F}$ for some $\epsilon>0$. Given a realization of net values t^Φ, consider the set of values, if traders use a, and denote it by . If a is ϵ-close to truthfulness, it holds that

$$t^{\Phi,(m)} - \epsilon \leq t^{,(m)} \leq t^{\Phi,(m)} + \epsilon. \tag{109}$$

The value of a missed trade is at most some constant $\zeta>0$. A buyer with gross value t_b and a seller with gross value t_s fail to trade under a, but would trade when being truthful, if $t^\Phi_b \geq t^\Phi_s$, $a_B(t_b) \leq \Pi() \leq t^{,(m)}$ and $a_S(t_s) \geq \Pi() \geq t^{,(m)}$. We bound the expected number of missed trades conditional on $\Pi()$. It is bounded by the expected number of net values in the 2ϵ-neighbourhood of $\Pi()$. This is bounded by fixing $\Pi()$ and summing over the number i of buyers with net values above or equal to $\Pi()$. These i values are independently distributed according to $\frac{(\cdot)-(\Pi())}{1-(\Pi())}$ with density $\frac{f_B(\cdot)}{1-(\Pi())}$. Similarly, the remaining $n-i$ net values of sellers above or equal to $\Pi()$ are independently distributed according to $\frac{(\cdot)-(\Pi())}{1-(\Pi())}$ with density $\frac{f_S(\cdot)}{1-(\Pi())}$. Because $\Pi() \leq \overline{a}_B - \delta$ (case (ii)) and f_B and f_S are continuous, the densities are bounded from above by some number $\alpha(,,\delta)$ that is independent of m. Conditional upon $\Pi()$, the expected number of net values above and within 2ϵ of $\Pi()$ is thus no more than $n\cdot 2\epsilon\cdot\alpha(,,\delta)$. A similar argument shows that for some $\beta(,,\delta)$ the expected number of net values below and within 2ϵ of $t^{\Phi,(m)}$ is no more than $m\cdot 2\epsilon\cdot\beta(,,\delta)$. Thus the expected number of missed trades conditional on $t^{\Phi,(m)}$ is bounded by $\min(n,m)\cdot\epsilon\cdot\gamma(,,\delta)$. Therefore $\mathbb{E}[L(t^\Phi)|t^{\Phi,(m)}] \leq \min(m,n)\cdot\zeta\cdot\epsilon\cdot\gamma(,,\delta)$. Finally, we have that

$$\begin{gathered}\frac{\mathbb{E}[GoT_\Phi - GoT]}{\mathbb{E}[GoT_\Phi]} = \\ \frac{\int_{(i)+(iii)} \mathbb{E}\left[L(t^\Phi)|\Pi(t^{\Phi,(m)})\right] d\mu(\Pi(t^{\Phi,(m)}))}{\mathbb{E}[GoT_\Phi]} + \frac{\int_{(ii)} \mathbb{E}\left[L(t^\Phi)|\Pi(t^{\Phi,(m)})\right] d\mu(\Pi(t^{\Phi,(m)}))}{\mathbb{E}[GoT_\Phi]}.\end{gathered} \tag{110}$$

Recall that the denominator is of order $\Theta(\min(m,n))$. The numerator of the first summand is of order $O(\min(m,n)^{\frac{1}{2}})$. Therefore the whole summand is of order $O(\min(m,n)^{-\frac{1}{2}})$, so it goes to zero in sufficiently large market. The numerator of the second summand is of order $O(\min(m,n)\cdot\epsilon)$. Therefore the second summand is of order $O(\epsilon)$. Therefore, for any $\gamma>0$ and for any sequence of ϵ

that goes to zero, $\frac{\mathbb{E}[GoT_\Phi - GoT]}{\mathbb{E}[GoT_\Phi]} \leq \gamma \; ISLFM$.

Limit Markets. We consider symmetric strategy profiles (a_B, a_S) that are strictly increasing and continuous.

Observation. Demand and supply are continuous. Furthermore, demand is strictly decreasing on A_B and supply is strictly increasing on A_S.

Proof of Appendix B.13. It holds that

$$D(P) = \begin{cases} 0 & \text{if } P < \underline{a}_B \\ \mu_B^t([a_B^{-1}(P), \bar{t}]) & \text{if } P \in A_B \\ \mu_B^t(\Theta) & \text{if } P > \bar{a}_B \end{cases} \quad \text{and} \quad S(P) = \begin{cases} \mu_S^t(\Theta) & \text{if } P < \underline{a}_S \\ \mu_B^t([\underline{t}, a_B^{-1}(P)]) & \text{if } P \in A_S \\ 0 & \text{if } P > \bar{a}_S \end{cases}, \tag{111}$$

from which the observation directly follows. □

Observation. If it holds that $\underline{a}_S < \bar{a}_B$, then there exists a unique market price, which lies in the interval $[\underline{a}_S, \bar{a}_B]$ equating positive demand and supply. Otherwise, if $\underline{a}_S \geq \bar{a}_B$, then the trading volume is equal to zero. Note that in both cases, there is zero market excess, implying that the gains of trade GoT are deterministic.

Proof of Appendix B.13. This follows from Appendix B.13 and the Intermediate Value theorem. □

Observation. GoT can be represented as a continuous function $GoT(\cdot)$ evaluated at the point Q, if strategies are increasing and continuous.

Proof of Appendix B.13. Let $\mathcal{B}^*$ and $\mathcal{S}^*$ be the allocation and denote by $T_B^* = t_B(\mathcal{B}^*)$ and $T_S^* = t_S(\mathcal{S}^*)$ the set of gross values involved in trade. First, note that

$$GoT = \int_{T_B^*} x d\mu_B^t(x) - \int_{T_S^*} x d\mu_S^t(x). \tag{112}$$

Using that gross values are assumed to be continuously distributed, it holds that

$$GoT = \int_{\Theta_B^*} x f_B(x) dx - \int_{\Theta_S^*} x f_S(x) dx, \tag{113}$$

where f_B and f_S are the strictly positive and continuous Radon-Nikodym derivatives. Because of the strict monotonicity of strategies, the traders with the most profitable gross values are involved in trade. Therefore T_B^* is of the form $[a, \bar{t}]$ for some $a \in T$ and T_S^* is of the form $[\underline{t}, b]$ for some $b \in T$. If the trading volume $Q = 0$, then $a = \bar{t}$ and $b = \underline{t}$. If $Q > 0$, then $a < \bar{t}$ and $b > \underline{t}$. It therefore holds that

$$GoT = \int_a^{\bar{t}} x f_B(x) dx - \int_{\underline{t}}^b x f_S(x) dx. \tag{114}$$

Next, we prove that a and b can be expressed as continuous functions of the trading volume Q. Because the allocation balances trade, it holds that $\mu_B^t(T_B^*) = \mu_S^t(T_S^*) = Q$. Let $F_B(x) = \int_{\underline{t}}^{x} f_B(x)dx$ denote the anti-derivative of f_B, which is a continuous and increasing function. We can write $\mu_B^t(T_B^*) = \int_a^{\bar{t}} d\mu_B^t = \mu_B^t(T) - F_B(a)$ and $\mu_S^t(T_S^*) = \int_{\underline{t}}^{b} d\mu_S^t = F_S(b)$. This yields

$$a(Q) = \begin{cases} \bar{t} & \text{if} Q = 0 \\ F_B^{-1}(\mu_B^t(T) - Q) & \text{if } 0 < Q < \mu_B^t(T) \\ \underline{t} & \text{if} Q = \mu_B^t(\Theta) \end{cases} \quad \text{and} \quad b(Q) = \begin{cases} \underline{t} & \text{if} Q = 0 \\ F_S^{-1}(Q) & \text{if } 0 < Q < \mu_S^t(T) \\ \bar{t} & \text{if} Q = \mu_S^t(T) \end{cases} \tag{115}$$

$a(Q)$ is continuous on $(0, \mu_B^t(T))$, because F_B is continuous and strictly decreasing on T. Because $\lim_{x \uparrow \bar{t}} F_B(x) = 0$ and $\lim_{x \downarrow \underline{t}} F_B(x) = \mu_B^t(T)$, the continuity of $a(Q)$ extends to $Q = 0$ and $Q = \mu_B^t(T)$. Analogous reasoning yields that $b(Q)$ is continuous on $[0, \mu_S^t(t)]$. The gains of trade corresponding can therefore be represented as

$$GoT = \int_{a(Q)}^{\bar{t}} x f_B(x)dx - \int_{\underline{t}}^{b(Q)} x f_S(x)dx. \tag{116}$$

Because the integrands $xf_B(x)$ and $xf_S(x)$ are continuous in x, it follows that GoT is continuous in Q. □

Observation. Consider two symmetric, strictly increasing and continuous strategy profiles $a^1 = (a_B^1, a_S^1)$ and $a^2 = (a_B^2, a_S^2)$, such that for all $t \in T$ it holds that $a_{B1}(t) \succcurlyeq a_{B2}(t)$ and $a_{S1}(t) \succ a_{S2}(t)$. Then it holds that $GoT_{a_1} \geq GoT_{a_2}$.

Proof of Appendix B.13. By Appendix A.1 and Appendix B.13, for both strategy profiles the trading volume TV is equal to demand and supply at their unique crossing point. It follows from Equation (111) that $\forall P \; D_{a^1}(P) \geq D_{a^2}(P)$ and $S_{a^1}(P) \geq \mathcal{S}_{a^2}(P)$ holds, which implies that $Q_{a^1} \geq Q_{a^2}$. The observation now follows from Equation (116). □

Define the symmetric strategy profile a_n, which is equal to $t_b^\Phi - \frac{1}{n}$ and $t_s^\Phi + \frac{1}{n}$. Denote by the subscripts n and Φ market characteristics, when traders use a_n and truthfulness respectively.

Assume that the trading volume Q_Φ at the market price Π_Φ is strictly positive, that is $\underline{a}_{S\Phi} < \overline{a}_{B\Phi}$. Otherwise, it holds that $GoT_\Phi = 0$ and therefore also $GoT_n = 0$.

Observation. For sufficiently large n, there exists a unique market price Π_n with trading volume $Q_n > 0$.

Proof of Appendix B.13. According to Appendix B.13, demand $D_n(P)$ is continuous in P and strictly decreasing on an interval $A_{Bn} = [\underline{a}_{Bn}, \overline{a}_{Bn}]$. Supply $S_n(P)$ is continuous in P and strictly increasing on an interval $A_{Sn} = [\underline{a}_{Sn}, \overline{a}_{Sn}]$. $\underline{a}_{Bn}$ is for example the action of a buyer with gross value $\underline{t}$. Because $\lim_{n \to \infty} a_n(x) = x$, we can choose n large enough, such that also $\underline{a}_{Sn} < \overline{a}_{Bn}$. A unique market price $\Pi_n \in [\underline{a}_{Sn}, \overline{a}_{Bn}]$ with trading volume $Q_n > 0$ exists by Appendix B.13. □

Observation. It holds that

$$\lim_{n\to\infty} \sup_{P\in\Theta} |D_n(P) - D_\Phi(P)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{P\in\Theta} |S_n(P) - S_\Phi(P)| = 0. \tag{117}$$

Proof of Appendix B.13. Because larger n leads to a less aggressive strategy profile a_n, it follows that for fixed P $D_n(P) \leq D_{n+1}(P)$ and $S_n(P) \leq S_{n+1}(P)$. Furthermore, it holds that $\lim_{n\to\infty} D_n(P) = D_\Phi(P)$ and $\lim_{n\to\infty} S_n(P) = S_\Phi(P)$. Because D_Φ and S_Φ are continuous on Θ, the observation follows from Dini's theorem (Bartle and Sherbert, 2000, p.238). □

Observation. $\forall \delta_1 > 0$ and sufficiently large n, it holds that $|\Pi_\Phi - \Pi_n| \leq \delta_1$.

Proof of Appendix B.13. Π_Φ is unique and equates demand and supply, and it was proven above that Π_n has the same properties for sufficiently large n. Define the two continuous functions

$$F_\Phi(P) = D_\Phi(P) - S_\Phi(P) \quad \text{and} \quad F_n(P) = D_n(P) - S_n(P). \tag{118}$$

It holds that Π_Φ is the unique zero point of $F_\Phi(\cdot)$ and Π_n is the unique zero point of $F_n(\cdot)$. Because of the strict monotonicity of D_Φ and S_Φ, for every $\delta_1 > 0$ it holds that F_Φ is strictly negative at $\Pi_\Phi + \delta_1$ and strictly positive at $\Pi_\Phi + \delta_1$. Therefore, for small δ_1, there exists $\gamma_1 > 0$, such that

$$F_\Phi(\Pi_\Phi + \delta_1) \leq -\gamma_1 \quad \text{and} \quad F_\Phi(\Pi_\Phi - \delta_1) \geq \gamma_1. \tag{119}$$

We will now prove that for every $\gamma_2 > 0$ the distance between F_Φ and F_n at the two points $\Pi_\Phi + \delta_1$ and $\Pi_\Phi - \delta_1$ is smaller or equal than γ_2, if n is chosen sufficiently large. We have that

$$\begin{aligned} |F_\Phi(P) - F_n(P)| &= |D_\Phi(P) - S_\Phi(P) - D_n(P) + S_n(P)| \\ &\leq |D_\Phi(P) - D_n(P)| + |S_\Phi(P) - S_n(P)|. \end{aligned} \tag{120}$$

If δ_1 is chosen small enough, such that $\Pi_\Phi + \delta_1$ and $\Pi_\Phi - \delta_1$ are in Θ, then the uniform convergence observation from above implies that for every $\gamma_2 > 0$ and sufficiently large n

$$|F_\Phi(\Pi_\Phi + \delta_1) - F_n(\Pi_\Phi + \delta_1)| \leq \gamma_2 \quad \text{and} \quad |F_\Phi(\Pi_\Phi - \delta_1) - F_n(\Pi_\Phi - \delta_1)| \leq \gamma_2 \tag{121}$$

If γ_2 is chosen to be strictly less than γ_1, it follows that also

$$F_n(\Pi_\Phi + \delta_1) < 0 \quad \text{and} \quad F_n(\Pi_\Phi - \delta_1) > 0. \tag{122}$$

This then implies that Π_n, which is the unique zero of F_n, lies in the interval $(\Pi_\Phi - \delta_1, \Pi_\Phi + \delta_1)$, which proves the observation. □

Observation. $\forall \delta_2 > 0$ and sufficiently large n, it holds that $|Q_\Phi - Q_n| \leq \delta_2$.

Proof of Appendix B.13. Q_Φ is equal to $D_\Phi(\Pi_\Phi)$ and Q_n is equal to $D_n(\Pi_n)$. By adding and subtracting $D_n(\Pi_\Phi)$ and using the triangle-inequality, we get that

$$|Q_\Phi - Q_n| \leq |D_\Phi(\Pi_\Phi) - D_n(\Pi_\Phi)| + |D_n(\Pi_n) - D_n(\Pi_\Phi)|. \tag{123}$$

The first term on the right-hand side is less or equal than $\frac{\delta_2}{2}$ for sufficiently large n by Appendix B.13. For the second term, note that D_n is a continuous function. Appendix B.13 implies that for sufficiently large n, such that the distance between Π_Φ and Π_n gets small enough, the second term is also bounded from above by $\frac{\delta_2}{2}$, which proves the observation. □

Observation. For all $\delta_3 > 0$ and sufficiently large n, it holds that $|GoT_\Phi - GoT_n| \leq \delta_3$.

Proof of Appendix B.13. Because reporting the net-value is by assumption a continuous and increasing function, it was proven above that GoT_Φ and GoT_n can be represented as a continuous function $GoT(\cdot)$ evaluated at the two points Q_Φ and Q_n. If n is chosen sufficiently large, Appendix B.13 and the continuity of $GoT(\cdot)$ imply that the distance between Q_Φ and Q_n gets small enough to ensure that $G_n = G(Q_n)$ is close to $G_\Phi = G(Q_\Phi)$. □

Observation. For all $\zeta > 0$ and sufficiently large n, it holds that $E_n \geq 1 - \zeta$.

Proof of Appendix B.13. For the efficiency ratio E_n, it holds that

$$E_n = \frac{G_n}{G_\Phi} = 1 - \frac{G_\Phi - G_n}{G_\Phi}. \tag{124}$$

If n is now chosen large enough, such that by Appendix B.13 $|G_\Phi - G_n| \leq \zeta G_\Phi$, the statement follows. □

Observation. $\forall \zeta > 0$, there exists $\epsilon \in (0, 1]$, such that $\inf_{(a_B,a_S)\in\Upsilon^{\epsilon,opt}_{\Phi,F}} E_a \geq 1 - \zeta$.

Proof of Appendix B.13. Define $\epsilon_n = \frac{1}{n}$. By Appendix B.13, it holds for any strategy profile $(a_B, a_S) \in \Upsilon^{\epsilon,opt}_{\Phi,F}$ that $GoT_{\epsilon_n} \leq GoT_a$. Therefore, if n is sufficiently large, it holds that $E_a \geq E_\epsilon \geq 1 - \zeta$. □

Appendix B.13 finishes the proof for infinite markets. □

B.14 Proof of Theorem 10

Proof. For finite markets, we construct the following beliefs F. Assume that all buyers believe that they are facing the same market environment, independent of their gross value, which implies that they have the same belief about the critical value, that is $\forall t_b \in T$ it holds that $\Pi^\infty(t_b) = \Pi^\infty_B$. In analogy, assume that all sellers have the same beliefs, implying that $\forall t_s \in T$ it holds that $\Pi^\infty(t_s) = \Pi^\infty_S$. Suppose that $\Pi^\infty_B < \Pi^\infty_S$. For any $\epsilon \geq 0$, consider the strategy-profile corresponding to price-guessing $(\rho_B, \rho_S) \in \Psi^{\epsilon,opt}_{\Phi,F}$. Recall that for this strategy-profile, a buyers and sellers actions are equal to Π^∞_B and Π^∞_S respectively, if it is individually rational, and truthful otherwise. That is all buyers submit an action less or equal to Π^∞_B and all sellers submit an action greater or equal to Π^∞_S. Therefore, for any realization of gross values, no profitable trade is possible and the gains of trade are equal to zero almost surely. Therefore, the efficiency is equal to zero almost surely and therefore also in expectation.

For infinite markets, it was proven in Appendix B.13 in Appendix B.13 that for continuous and strictly increasing strategy profile in an infinite market, the gains of trade GoT can be represented as a continuous function $G(\cdot)$ evaluated at Q with $G(Q_\Phi) = GoT_\Phi$ and $G(0) = 0$. Therefore the efficiency ratio $E = \frac{G}{G_\Phi}$ can be represented as the continuous function $E(Q) = \frac{G(Q)}{G_\Phi}$. For $Q = Q_\Phi$ the efficiency ratio is equal to 1, for $Q = 0$, the efficiency ratio is equal to zero. If we show that for every $Q \in [0, Q_\Phi]$, it is possible to construct increasing strategies, such that the trading volume is equal to Q, the theorem follows from the Intermediate value theorem, because for every $\zeta \in [0,1]$, there exists $Q \in [0, Q_\Phi]$ with $E(Q) = \zeta$. One possible construction is as follows: For $a, b \geq 0$, consider beliefs F such that $\Pi^\infty(t_b) = t^\Phi_b - a$ and $\Pi^\infty(t_s) = t^\Phi_s + b$. For any $\epsilon \geq 0$, consider the strategy-profile $(\rho_B, \rho_S) \in \Psi^{\epsilon,opt}_{\Phi,F}$, which is continuous and strictly increasing. Note that for every trader, their belief about the critical value is individually rational. For any $Q \in [0, Q_\Phi]$, choose $a \geq 0$, such that $D(\Pi_\Phi) = D_\Phi(rho_B^{-1}(\Pi_\Phi)) = D_\Phi(\Pi_\Phi + a) = Q$. Such a constant exists in $[0, \bar{t} - \Pi_\Phi)]$ by the Intermediate value theorem, because D_Φ is continuous and decreasing on T with $D_\Phi(\Pi_\Phi) = Q_\Phi$ and $D_\Phi(\Pi_\Phi + (\bar{t} - \Pi_\Phi)) = Q_\Phi$. Next, choose $\tilde{P}$ as a price with $S_\Phi(\tilde{P}) = Q$. This price exists in $[\underline{t}, \Pi_\Phi]$ by the Intermediate Value theorem, because S_Φ is continuous and increasing on T with $S_\Phi(\underline{t}) = 0$ and $S_\Phi(\Pi_\Phi)) = Q_\Phi$. If we set $b = \Pi_\Phi - \tilde{P} \geq 0$, then $S(\Pi_\Phi) = S_\Phi(\tilde{P}) = Q$, which proves that the market price is equal to Π_Φ and the trading volume is equal to Q. This finishes the proof. □

B.15 Proof of Theorem 16

Recall that $\tilde{\Pi}$ denotes the market price, is a trader is involved in trade, and zero otherwise.

For a buyer b with private type t_b, Equation (12) yields that

$$\begin{aligned} &\mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a^1_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[u_b\left(t_b, a^2_b, a_{\text{-}b}\right)\right] = \\ &t_b\left(\mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a^1_b, a_{\text{-}b})\right] - \mathbb{P}_{\text{-}b}\left[b \in \mathcal{B}^*(a^2_b, a_{\text{-}b})\right]\right) - \\ \int_{[\underline{a}_{S,b}, \overline{a}_{S,b}]^2} &\left(\tilde{\Pi}\left(a^1_b, a^{(m)}_{\text{-}b}, a^{(m+1)}_{\text{-}b}\right) - \tilde{\Pi}\left(a^2_b, a^{(m)}_{\text{-}b}, a^{(m+1)}_{\text{-}b}\right)\right) d\mu(a^{(m)}_{\text{-}b}, a^{(m+1)}_{\text{-}b}) - \\ &\left(\mathbb{E}_{\text{-}b}\left[\Phi_b\left(a^1_b, a_{\text{-}b}\right)\right] - \mathbb{E}_{\text{-}b}\left[\Phi_b\left(a^2_b, a_{\text{-}b}\right)\right]\right). \end{aligned} \tag{125}$$

Note that the integral in the difference above is non-negative, because $\tilde{\Pi}(a_b, a^{(m)}_{\text{-}b}, a^{(m+1)}_{\text{-}b})$ is increasing in a_b for a fixed $a^{(m)}_{\text{-}b}$ and $a^{(m+1)}_{\text{-}b}$. Equation (16) follows by neglecting the term corresponding to the change in expected market price.

For a seller s with private type t_s, Equation (13) yields

$$\begin{gathered}\mathbb{E}_{\text{-}s}\left[u_s\left(t_s, a_s^1, a_{\text{-}s}\right)\right]-\mathbb{E}_{\text{-}s}\left[u_s\left(t_s, a_s^2, a_{\text{-}s}\right)\right]= \\ \int_{[\underline{a}_{B,s},\overline{a}_{B,s}]^2}\left(\tilde{MP}\left(a_s^1, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)-\tilde{\Pi}\left(a_s^2, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)\right) d\mu(a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)})- \\ t_s\left(\mathbb{P}_{\text{-}s}\left[s\in\mathcal{S}^*(a_s^1, a_{\text{-}s})\right]-\mathbb{P}_{\text{-}s}\left[s\in\mathcal{S}^*(a_s^2, a_{\text{-}s})\right]\right)-\left(\mathbb{E}_{\text{-}s}\left[\Phi_s\left(a_s^1, a_{\text{-}s}\right)\right]-\mathbb{E}_{\text{-}s}\left[\Phi_s\left(a_s^2, a_{\text{-}s}\right)\right]\right).\end{gathered} \tag{126}$$

$t_s\left(\mathbb{P}_{\text{-}s}\left[s\in\mathcal{S}^*(a_s^1, a_{\text{-}s})\right]-\mathbb{P}_{\text{-}s}\left[s\in\mathcal{S}^*(a_s^2, a_{\text{-}s})\right]\right)\geq 0$ holds, because the trading probability is decreasing for a seller in their ask. To see that the integral in Equation (126) is bounded from above by $2t_s\left(1-\mathbb{P}_{\text{-}s}\left[s\in\mathcal{S}^*(a_s^2, a_{\text{-}s})\right]\right)$, we split up the integral into all six possible cases for the realizations of and $a_{\text{-}s}^{(m\text{-}1)}$ with respect to $a_s^1 < a_s^2$. which is shown in the following table. [48]

		$\tilde{\Pi}\left(a_s^1, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)$	$\tilde{MP}\left(a_s^2, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)$
I	$a_{\text{-}s}^{(m)}\geq a_{\text{-}s}^{(m\text{-}1)}\geq a_s^2\geq a_s^1$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_{\text{-}s}^{(m\text{-}1)}$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_{\text{-}s}^{(m\text{-}1)}$
II	$a_{\text{-}s}^{(m)}\geq a_s^2\geq a_{\text{-}s}^{(m\text{-}1)}\geq a_s^1$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_{\text{-}s}^{(m\text{-}1)}$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_s^2$
III	$a_s^2\geq a_{\text{-}s}^{(m)}\geq a_{\text{-}s}^{(m\text{-}1)}\geq a_s^1$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_{\text{-}s}^{(m\text{-}1)}$	0
IV	$a_{\text{-}s}^{(m)}\geq a_s^2\geq a_s^1\geq a_{\text{-}s}^{(m\text{-}1)}$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_s^1$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_s^2$
V	$a_s^2\geq a_{\text{-}s}^{(m)}\geq a_s^1\geq a_{\text{-}s}^{(m\text{-}1)}$	$ka_{\text{-}s}^{(m)}+(1\text{-}k)a_s^1$	0
VI	$a_s^2\geq a_s^1\geq a_{\text{-}s}^{(m)}\geq a_{\text{-}s}^{(m\text{-}1)}$	0	0

For **I**, **II**, **IV** and **VI**, the difference between $\tilde{\Pi}\left(a_s^1, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)$ and $\tilde{\Pi}\left(a_s^2, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)$ is less or equal than 0. It follows that

$$\begin{aligned}\int_{[\underline{a}_{B,s},\overline{a}_{B,s}]^2}\left(\tilde{\Pi}\left(a_s^1, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)-\tilde{\Pi}\left(a_s^2, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)\right) d\mu(a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)})\leq \\ \int_{\mathbf{III}}(ka_{\text{-}s}^{(m)}+(1-k)a_{\text{-}s}^{(m\text{-}1)})d\mu_s^*(a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}) \\ +\int_{\mathbf{V}}(ka_{\text{-}s}^{(m)}+(1-k)a_s^1)d\mu(a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)})\end{aligned} \tag{127}$$

Because both integrands in Equation (127) are less or equal than $\overline{a}_{S,s}$, it follows that

$$\begin{aligned}\int_{[a_s,\underline{s}]^2}\left(\tilde{\Pi}\left(a_s^1, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)-\tilde{\Pi}\left(a_s^2, a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}\right)\right) d\mu(a_{\text{-}s}^{(m\text{-}1)}, a_{\text{-}s}^{(m)}) \\ \leq \overline{a}_{S,s}\mathbb{P}[\mathbf{III}]+\overline{a}_{S,s}\mathbb{P}[\mathbf{V}] \\ \leq 2\overline{a}_{S,s}\mathbb{P}[a_{\text{-}s}^{(m)}<a_s^2]=2\overline{a}_{S,s}\left(1-\mathbb{P}_{\text{-}s}\left[(s, a_s^2\in\mathcal{S}^*\right]\right),\end{aligned} \tag{128}$$

which finishes the proof.

[48]Different to $\tilde{\Pi}_b\left(a_b, y, z\right)$ it holds that $\tilde{\Pi}_s\left(a_s, y, z\right)$ is not increasing in a_s for fixed y and z.

B.16 Proof that for homogeneous fees in Section 5 the net values satisfy $t_b^\Phi + \Phi_b(t_b^\Phi) = t_b$ and $t_s^\Phi + \Phi_s(t_s^\Phi) = t_s$

Proof. Consider a buyer with gross value t_b. To show that the net value satisfies $t_b^\Phi + \Phi_b(t_b^\Phi) = t_b$, it suffices to prove two statements for the solution t_b^Φ of that equation: (1) If a bid $a_b' > t_b^\Phi$, then it is dominated by t_b^Φ and (2) if $a_b' < t_b^\Phi$, then there exists Π such that $u_b\left(t_b, t_b^\Phi, \Pi\right) > u_b\left(t_b, a_b', \Pi\right)$ holds. For (1), if Π is such that both a_b' and t_b^Φ are not involved in trade, then both have utility equal to zero. If Π is such that the buyer is involved in trade at a_b', but not at t_b^Φ, then the market price is greater or equal to t_b^Φ. Because $x \mapsto x + \Phi_b(x)$ is increasing, it follows that $u_b\left(t_b, a_b', \Pi\right) = t_b - \Pi - \Phi_b(\Pi) \leq t_b - t_b^\Phi - \Phi_b(t_b^\Phi) = 0$. If Π is such that the buyer is involved in trade with both bids, then it follows in analogy that

$$u_b\left(t_b, a_b', \Pi\right) = t_b - \Pi - \Phi_b(\Pi) \leq t_b - t_b^\Phi - \Phi_b(t_b^\Phi) = u_b\left(t_b, t_b^\Phi, \Pi\right). \tag{129}$$

For (2), consider $a_b' \leq t_b^\Phi$. Consider Π, such that a buyer is involved in trade at bid t_b^Φ but not with a_b' and it holds that $\Pi < t_b^\Phi$. This yields

$$u_b\left(t_b, t_b^\Phi, \Pi\right) = t_b - \Pi - \Phi_b(\Pi) > t_b - > t_b - t_b^\Phi - \Phi_b(t_b^\Phi) = 0. \tag{130}$$

The statement for sellers is proven in analogy. □

B.17 Proof of Theorem 11

Proof. Consider a buyer b with gross value t_b and action a_b. First, suppose that $\delta > 0$. Tie-breaking is a probability zero event. The expected utility is equal to

$$\mathbb{E}_b[u_b(t_b, a_b, \Pi)] = \int_{\underline{\Pi}_b}^{a_b} \left(t_b - x - \Phi_b(x)\right) f_\Pi(x) dx. \tag{131}$$

Recall, that it holds that $t_b - t_b^\Phi - \Phi_b(t_b^\Phi) = 0$, and $x \mapsto x + \Phi_b(x)$ is strictly increasing. Therefore, for $x \in [\underline{\Pi}_b, t_b^\Phi)$, the integrand is strictly greater than zero. For $x \in (t_b^\Phi, \overline{\Pi}_b]$, the integrand is strictly negative. Hence, the expected utility is maximized at the unique point t_b^Φ.[49] The function $a_b \mapsto \mathbb{E}_b[u_b(t_b, a_b, \Pi)]$ is continuous, increasing on $[\underline{\Pi}_b, t_b^\Phi]$ and decreasing on $[t_b^\Phi, \overline{\Pi}_b]$. ϵ-therefore approximate t_b^Φ. As truthfulness is the unique , it holds that $E_\Phi = \frac{\mathbb{P}_\Pi[b \in \mathcal{B}^*(,\Pi)]}{\mathbb{P}_\Pi[b \in \mathcal{B}^*(t_b^\Phi,\Pi)]} = \frac{\mathbb{P}_\Pi[b \in \mathcal{B}^*(t_b^\Phi,\Pi)]}{\mathbb{P}_\Pi[b \in \mathcal{B}^*(t_b^\Phi,\Pi)]} = 1.$

[49] Alternatively, this can be proven via the first order condition by differentiating the expected utility using Leibniz's rule and setting the derivative zero.

Second, suppose that $\delta = 0$. The expected utility is of the form

$$\mathbb{E}_b[u_b(t_b, a_b, \Pi)] = \begin{cases} t_b - \Pi - \Phi_b(\Pi) & \text{if } a_b > \Pi \\ p_b(t_b - \Pi - \Phi_b(\Pi)) & \text{if } a_b = \Pi \\ 0 & \text{if } a_b < \Pi, \end{cases} \tag{132}$$

where $p_b \in [0,1]$ depends on tie-breaking beliefs. If $t_b^\Phi > \Pi$, then the expected utility is equal to $t_b - \Pi - \Phi_b(\Pi) > t_b - t_b^\Phi - \Phi_b(t_b^\Phi) = 0$, and therefore a . If $t_b^\Phi \leq \Pi$, then the expected utility is equal to 0, regardless of tie-breaking assumptions. Because every action $a_b > t_b^\Phi$ is dominated, t_b^Φ is again a . Therefore truthfully reporting t_b^Φ is a for every gross value and as argued above, the efficiency ratio of truthfulness is equal to 1. The proof for sellers is analogous. □

B.18 Proof of Theorem 12

Proof. Consider a buyer b with gross value t_b and action a_b. First, consider $\delta > 0$. Tie-breaking is a probability zero event. The expected utility is equal to

$$\mathbb{E}_b[u_b(t_b, a_b, \Pi)] = \int_{\underline{\Pi}_b}^{a_b} (t_b - x - \Phi_b(a_b, x))\, f_\Pi(x) dx. \tag{133}$$

The expected utility is therefore continuous in a_b on $[\underline{\Pi}_b, \overline{\Pi}_b]$ and therefore attains a maximum by the Extreme Value theorem, which proves the existence of a . Suppose that $t_b^\Phi > \Pi_b^*$. First, consider an action a_b with $a_b - \Pi_b^* \geq \epsilon$ for some $\epsilon > 0$. We will show that if δ is chosen sufficiently small, than a_b is not a best response, proving that best responses must be ϵ-close to Π_b^*. More specifically, we prove that a buyer can increase their expected utility when switching to $\Pi_b^* + \epsilon/2$. For $\delta < \epsilon/2$ it holds that

$$\begin{gathered} \mathbb{E}_b[u_b(t_b, a_b, \Pi)] - \mathbb{E}_b[u_b(t_b, \Pi_b^* + \epsilon/2, \Pi)] = \\ \int_{\underline{\Pi}_b}^{a_b} (t_b - x - \Phi_b(a_b, x))\, d\mu_\Pi(x) - \int_{\underline{\Pi}_b}^{\Pi_b^* + \epsilon/2} (t_b - x - \Phi_b(\Pi_b^* + \epsilon/2, x))\, d\mu_\Pi(x) = \\ \int_{\Pi_b^* + \epsilon/2}^{a_b} (t_b - x)\, d\mu_\Pi(x) - \left(\int_{\underline{\Pi}_b}^{\Pi_b^* + \epsilon/2} (\Phi_b(a_b, x) - \Phi_b(\epsilon/2, x))\, d\mu_\Pi(x) + \int_{\Pi_b^* + \epsilon/2}^{a_b} \Phi_b(a_b, x) d\mu_\Pi(x) \right) \end{gathered} \tag{134}$$

It follows from heterogeneity, that there exists a constant $\gamma > 0$, such that $\forall P \in [\underline{\Pi}_b, \Pi_b^* + \epsilon/2]$ it holds that $(\Phi_b(a_b, x) - \Phi_b(\Pi_b^* + \epsilon/2, x)) \geq \gamma$. Together with δ-aggregate uncertainty, we get that

$$\int_{\underline{\Pi}_b}^{\Pi_b^* + \epsilon/2} (\Phi_b(a_b, x) - \Phi_b(\epsilon/2, x))\, d\mu_\Pi(x) \geq (1 - \delta)\gamma. \tag{135}$$

Moreover it holds that

$$\int_{\Pi_b^*+\epsilon/2}^{a_b} (t_b - x)\, d\mu_\Pi(x) \leq \delta t_b \quad \text{and} \quad \int_{\Pi_b^*+\epsilon/2}^{a_b} \Phi_b(a_b, x) d\mu_\Pi(x) \geq 0. \tag{136}$$

Combining Equation (134) with Equation (135) and Equation (136) yields

$$\mathbb{E}_b[u_b(t_b, a_b, \Pi)] - \mathbb{E}_b[u_b(t_b, \Pi_b^* + \epsilon/2, \Pi)] \leq t_b\delta - (1-\delta)\gamma. \tag{137}$$

If $\delta < \frac{\gamma}{t_b+\gamma}$, then the difference in expected utility is strictly negative, proving that a_b is not a best response. Second, assume that $a_b < \Pi_b^*$. We will show that it holds for sufficiently small δ that

$$\mathbb{E}_b\left[u_b(t_b, t_b^\Phi, \Pi)\right] - \mathbb{E}_b\left[u_b(t_b, a_b, \Pi)\right] > 0, \tag{138}$$

which again implies that a_b is not a best response. It follows from uniform profitability that there exists $\gamma > 0$ such that

$$\mathbb{E}_b\left[u_b(t_b, t_b^\Phi, \Pi)\right] \geq \gamma. \tag{139}$$

It therefore suffices to show that for $a_b < \Pi_b^* - \epsilon$ it holds for sufficiently small $\delta > 0$ that

$$\mathbb{E}_b\left[u_b(t_b, a_b, a_{\text{-}b})\right] < \gamma. \tag{140}$$

We can upper bound the expected utility by neglecting the expected market price and the expected fee and get that

$$\mathbb{E}_b\left[u_b(t_b, a_b, \Pi)\right] \leq t_b \mathbb{P}_b\left[b \in \mathcal{B}^*(a_b, \Pi)\right]. \tag{141}$$

δ-aggregate uncertainty implies that $\mathbb{P}_b\left[b \in \mathcal{B}^*(a_b, \Pi)\right] \leq \delta$. If we choose $\delta < \frac{\gamma}{t_b}$, a_b is not a .

Next, we construct beliefs, such that the efficiency of is zero. Suppose again that $t_b^\Phi > \Pi_b^*$. For sufficiently small δ, are ϵ-close to Π_b^*. It holds that $t_b^\Phi > \underline{\Pi}$ and suppose that beliefs are such that $\Pi_b^* < \underline{\Pi}$. That is, the buyer's prediction of the market price is not in the actual support of the market price, but their net value is. For small ϵ, $\Pi^* + \epsilon < \underline{\Pi}$. Therefore, the buyer is involved in trade with positive probability K when bidding truthful, but is almost surely not involved in trade with their , which is ϵ-close to Π^*. It follows that $E_\Phi = \frac{\mathbb{P}_\Pi[b \in \mathcal{B}^*(,\Pi)]}{\mathbb{P}_\Pi[b \in \mathcal{B}^*(t_b^\Phi,\Pi)]} = \frac{0}{K} = 0$.

Second, suppose that $\delta = 0$. The expected utility is of the form

$$\mathbb{E}_b[u_b(t_b, a_b, \Pi)] = \begin{cases} t_b - \Pi - \Phi_b(a_b, \Pi) & \text{if } a_b > \Pi \\ c_b(t_b - \Pi - \Phi_b(a_b, \Pi)) & \text{if } a_b = \Pi \\ 0 & \text{if } a_b < \Pi, \end{cases} \tag{142}$$

where $p_b \in [0,1]$ depends on tie-breaking assumptions. Consider a market without tie-breaking, that is $p_b = 1$. The minimum of $\Phi_b(\cdot, \Pi)$ on $[\Pi, \infty)$ is attained at Π. Therefore, the best response is equal to Π, if $t_b^\Phi \geq \Pi$. With tie-breaking, that is $p_b \in [0, 1)$, the fee payment $\Phi_b(\cdot, \Pi)$ decreases when a_b approximates Π. However, because $\Phi_b(a_b, \Pi)$ is continuous, there exists $\epsilon > 0$, such that

$$t_b - \Pi - \Phi_b(\Pi + \epsilon, \Pi) > p_b \left(t_b - \Pi - \Phi_b(\Pi, \Pi)\right). \tag{143}$$

Therefore it follows that Π is not a best response. Furthermore, because for any $a_b^1 > a_b^2 > \Pi$ it holds that $\Phi_b(a_b^1, \Pi) > \Phi_b(a_b^2, \Pi)$ and therefore also $\mathbb{E}_b[u_b(t_b, a_b^1, \Pi)] < \mathbb{E}_b[u_b(t_b, a_b^2, \Pi)]$, no best response exists, but ϵ-best responses approximate Π.

Finally, suppose that F_Π has a continuous density function $f_\Pi > 0$ on $[\underline{\Pi}, \overline{\Pi}]$. For all $\zeta \in [0, 1]$, we construct beliefs, such that the efficiency of is equal to ζ. First, $p_b = 1$, that is the buyer believes that there is no tie-breaking. Then the unique is equal to their deterministic belief Π_b^* of the market price. Therefore, for any value x, beliefs can be constructed, such that the is equal to x. The efficiency ratio is then equal to $E_\Phi = \frac{\mathbb{P}_\Pi[b \in \mathcal{B}^*(x, \Pi)]}{\mathbb{P}_\Pi[b \in \mathcal{B}^*(t_b^\Phi, \Pi)]} = \frac{1 - F_\Pi(x)}{1 - F_\Pi(t_b^\Phi)}$ with $1 - F_\Pi(t_b^\Phi)$ and therefore continuous for $x \in [\underline{\Pi}, \overline{\Pi}]$. If x is equal to $\underline{\Pi}$, the efficiency ratio is equal to 0, and if it is equal to t_b^Φ, the efficiency ratio is equal to 1. By the Intermediate value theorem, $\forall \zeta \in [0, 1]$ there exists $x \in [\underline{\Pi}, \overline{\Pi}]$, such that $E_\Phi = x$. The proof for sellers is analogous. □