The drifting natural rate of interest and optimal inflation*

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Abstract

Empirical analyses starting from Laubach and Williams (2003) find that the natural rate of interest is best described by an integrated stochastic process. Estimates suggest that it has followed a downward trend over the past decades, reaching levels around zero in the 2010s. This paper models an integrated natural rate of interest in a simple new Keynesian framework and studies its implications for monetary policy. If one abstracts from the effective lower bound, permanent shocks to the natural rate do not prevent an optimizing central bank from achieving perfect inflation and output gap stabilization. Taking the lower bound into account, systematic increases in the optimal rate of inflation become warranted in response to downward shocks to the long-run natural rate, once this drifts below 1%. Nevertheless a targeting rule of the form put forward in Eggertsson and Woodford (2003) continues providing a good approximation to optimal commitment. Our results underpin the need for periodic revisions of central banks' inflation targets, if the long-run natural rate of interest is subject to permanent shocks, but they suggest that the good performance of catch-up strategies based on a constant price level target is not impaired.

Keywords: nonlinear optimal policy, zero lower bound, commitment, liquidity trap, New Keynesian, natural rate of interest.

JEL Codes: C63, E31, E52.

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1 Introduction

One of the key stylised facts highlighted in Kaldor (1957) is that the real interest rate (or the return on capital) is roughly constant over long periods of time. Recent empirical research, however, has found that the "equilibrium" real, or *natural*, interest rate – the real interest rate consistent with output equaling its natural rate and stable inflation – has a time-varying low-frequency component best described by an integrated stochastic process (Laubach and Williams, 2003). A related finding is that the low frequency component, or long-run level, of the natural rate has declined over the past few decades (e.g. Laubach and Williams, 2016) and may currently hover around zero in the U.S. and around negative values in the euro area (Holston, Laubach and Williams, 2017, Fiorentini et al., 2018).

These findings raise questions for the conduct of monetary policy. Due to the effective lower bound (ELB) constraint on nominal interest rates, a permanently low level of the long-run natural interest rate would reduce the room for manoeuvre of monetary policy.[1] Various authors – e.g. Blanchard, Dell'Ariccia and Mauro (2010), Ball (2013), Krugman (2014) and Andrade et al. (2018) – have called for an increase in the inflation target in this case. If, in addition, there is a risk that the long-run natural rate falls to even lower levels in the future, these arguments are strenghtened. A central bank may want to act preemptively and provide economic stimulus even before the new permanent shocks materialise; and it may want to adjust the optimal rate of inflation on a recurrent basis.

It is also conceivable that optimal policy would be harder to approximate through simple rules when the natural rate is subject to permanent shocks. More specifically the constant price level targeting rule of the form put forward in Eggertsson and Woodford (2003) has been shown to work well in the context of a model with a stationary natural rate. The rule is also appealing for its simplicity, which makes it easy to communicate. However, the rule may be unsuitable for a world in which recurrent adjustments in the optimal rate of inflation are necessary. Would a constant price level targeting rule work poorly when the natural rate of interest is subjecto to permanent shocks?

[1] The recent experience has shown that the lower bound on nominal interest rates is not zero, as previously assumed, but negative due to cash storage costs. In our theoretical model, cash storage costs are ignored, so the lower bound is equal to zero. Nevertheless we refer to is as the effective lower bound for the sake of generality.

This paper provides an answer to the aforementioned two questions in the context of a version of the new Keynesian model modified along two dimensions to be consistent with the aforementioned empirical findings.

Our first modification, consistent with the assumptions in Laubach and Williams (2003, 2016), is to allow for a stochastic trend not only in the level of productivity (as is common, for example, in Altig et al., 2011, Christiano, Motto and Rostagno, 2014, Christoffel, Coenen and Warne, 2008), but also in its rate of growth. In terms of the time-series literature, productivity will be integrated of order 2. Intuitively, this implies that, at any point in time, the current, low-frequency component of productivity growth is expected to remain unchanged over the future, but it is not constant. As a result, the natural rate of interest – in the model: the real interest rate which would prevail in the absence of nominal rigidities (Woodford, 2003) – will no longer be constant in the long-run. Consistently with the empirical results, permanent declines (or increases) in the rate of growth of productivity will induce low-frequency declines (increases) in the natural rate of interest.

Our second moditication of the new Keynesian model capturs the fact that other determinants are likely to have contributed to the secular decline in the natural rate. They may include the demographic transition which is ongoing in many Western economies (Carvalho, Ferrero and Nechio, 2016, Gagnon, Johannsen and López-Salido, 2016) and an increase in the required premium for safety and liquidity (e.g. Caballero and Farhi, 2017, Krishnamurthy and Vissing-Jorgensen, 2012). Del Negro et al. (2017) make a particularly compelling case about the role of the liquidity premium. We therefore follow Michaillat and Saez (2018) in postulating that people derive utility from their holdings of real bonds relative to everyone else. As a result, government bonds will incorporate a (negative) convenience yield. The convenience yield allows us to rationalise a negative value of the long-run natural rate without assuming a negative rate of productivity growth in the long-run.

Our results can be summarised as follows. We first demonstrate analytically that, absent the effective lower bound (ELB) on nominal interest rates, optimal inflation stabilization remains possible even if the natural rate has a stochastic trend. As in the standard, stationary model, a central bank optimising under commitment will want to adjust its policy rate so as to "track" the natural rate of interest. This will be possible also in the face of permanent shocks to the natural rate, provided actual nominal interest rates inherit the stochastic trend of the natural rate.

We then switch to numerical methods to account for the ELB as an occasionally binding constraint and study optimal monetary policy when the natural interest rate hovers around very low levels. We calibrate the process followed by the natural rate of interest based on the results in Fiorentini et al. (2018) and Del Negro et al. (2017). We solve the model using projection methods to account for the expectational effects induced by the ELB nonlinearity.

We show that the optimal rate of inflation should be higher, the closer to zero the steady state level of the natural rate of interest. This the case in spite of our assumption that the central bank is able to influence outcomes through a credible commitment to future actions, a type of "forward guidance" which may be unrealistically powerful–see Del Negro, Giannoni and Patterson (2015). More specifically, compared to a version of the model with low, but stationary, natural rate, we show that optimal inflation increases more, because of the risk of new, permanent downward shocks to the natural rate. As the long-run natural interest rate approaches zero from above – a value close to the current estimates of the long-run natural rate in Holston, Laubach and Williams (2017) and Fiorentini et al. (2018) – the central bank will tolerate an inflation rate twice as high as in the case with stationary shocks.

These results suggest that the optimal rate of inflation needs to be recurrently adjusted when the natural rate is subject to permanent shocks. The adjustment is negligible if the initial level of the long-run natural rate is relatively high, e.g. at 3.5%; it becomes larger, the closer to zero the initial level of the long-run natural rate. In practice, our results underpin the need for periodic revisions in the central bank inflation target. They also highlight that such revisions are really necessary once the long-run natural rate of interest is estimated to be below 1%.

We finally compare welfare implications of optimal policy to those of a constant price level targeting rule of the sort put forward by Eggertsson and Woodford (2003). Clearly, a constant price level target cannot fully replicate the properties of optimal policy, if the latter requires a positive inflation rate when the long-run natural rate falls below 1%. Intuitively, one would expect the price level to have to incorporate a drift, and that the drift should be contingent on the current value of the long-run natural rate of interest. A rule of this sort would loose the property of simplicity which constitutes its main strength for practical implementation and for public communication purposes. However, our results show that a price level target adjusted with a constant drift continues to provide a reasonably good approximation to optimal commitment, as long as the long-run natural rate remains in positive territory. We therefore conclude that, for practical purposes, catch-up policies in a liquidity trap situation preserve their desirable features even if the natural rate of interest follows an integrated time series process.

Our paper is obviously related to the literature on the optimal rate of inflation reviewed in Schmitt-Grohé and Uribe (2011), where the optimal rate of inflation is found to range from negative values to numbers insignificantly above zero. Zero inflation is found to be optimal when prices are sticky (Woodford 2003). Taking the ELB into account does not alter this conclusion in Eggertsson and Woodford (2003), Adam and Billi (2006), Nakov (2008) and Levin et al. (2010). These papers, however, focus on calibrations consistent with a relatively high level of the nominal interest rate (3.5% or higher). Our results replicate those in Adam and Billi (2006) when the natural rate of interest follows a stationary process around 3.5% steady state level. Optimal inflation increases above zero only when the long-run natural rate of interest falls below 1%. Intuitively, the benefits of a permanently higher inflation rate in terms of reducing the incidence of the ELB must be weighed against its economic costs in terms of inefficient price dispersion. The optimal monetary literature has concluded that the costs tend to outweigh the benefits (Schmitt-Grohé and Uribe, 2011), although possibly due to an implausibly high cost of price dispersion in standard models (Nakamura et al., 2018).

Our paper is most closely related to Andrade et al. (2018), which first studied the monetary policy implications of different levels of the natural rate of interest. A key distinguishing feature of our paper is to allow for an integrated natural rate. Moreover, Andrade et al. (2018) adopts a richer and more realistic model specification and analyses the optimal inflation target which should be assigned to a central bank following a Taylor rule. By contrast, we adopt a simpler model specification and solve for fully optimal monetary policy taking into account the stochastic effects induced by the ELB constraint.

The paper is organised as follows. Section 2 presents the model and derives optimal policy. The numerical approach that we use to solve the model and key features of the calibration are described in Section 3. Consistently with the empirical literature, permanent shocks to the natural rate of interest are calibrated to have very low variance. Section 4 builds an intuition for our main results. It studies a few simulation moments and impulses to temporary shocks in our full model and in a version of the model without permanent shocks. We show that differences between the two versions of the model start appearing when the long-run natural rate is as low as 1%. We also demonstrate that the model with integrated natural rate shocks can be seen as the limiting case of a model with extremely persistent, stationary shocks. We present our main results on optimal inflation in section 4, which starts with some analytical results on optimal policy in a static version of the model. This special case serves as a useful benchmark to interpret the numerical results on optimal inflation as a function of the long-run natural rate. This section also presents impulse responses to permanent natural rate shocks. Finally, section 6 studies the performance of price level targeting rules. The final section 7 offers some concluding remarks.

2 The model

Households consume a composite good C_t which is the aggregate of intermediate goods $C_{k,t}$ according to the aggregator

$$C_t = \left\{ \int_0^1 C_{k,t}^{\frac{\theta-1}{\theta}} \mathrm{d}k \right\}^{\frac{\theta}{\theta-1}}$$

The composite good aggregating sector takes output and input prices as given. Profit maximization implies that the demand for $C_{k,t}$ is

$$P_t = \left\{ \int_0^1 P_{k,t}^{1-\theta} \mathrm{d}k \right\}^{\frac{1}{1-\theta}}$$

The representative household j demands an amount $C_{j,t}$ of the good to maximise utility

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U_{j,t} \left(C_{j,t}, H_{j,k,t}, \frac{M_{j,t}}{P_t} \right)$$

where, as in Woodford (2003), $H_{j,k,t}$ are hours worked in all firms $k \in [0,1]$ and $M_{j,t}$ are nominal non-state contingent bonds issued by the government. The assumption of bonds-in-the-utility is as in Del Negro et al. (2017), Fisher (2015) and Krishnamurthy and Vissing-Jorgensen (2012). We will specifically follow Michaillat and Saez (2018) and postulate that households derive utility from their relative real bond holdings, so that temporary utility is

$$U_{j,t} = \bar{C}_t \left(\log C_{j,t} + S_t \upsilon \left(\frac{M_{j,t}}{P_t} - \frac{M_t}{P_t} \right) - \frac{\gamma}{1+\upsilon} \bar{H}_t^{-\upsilon} \int_0^1 H_{j,k,t}^{1+\upsilon} \mathrm{d}k \right)$$

where the function $\upsilon(\cdot)$ is increasing and concave and S_t is a liquidity/safety shock.

The Michaillat and Saez (2018) specification is consistent with the more common one in Fisher (2015), which goes back to Sidrauski (1967). Beyond producing a convenience yield on bonds, it has the potential benefit of leading to a form of discounting in the linearised Euler equation of the model–see equation (3) below. Michaillat and Saez (2018) demonstrate that this assumption can help mitigate the so-called forward guidance puzzle. In our analysis, however, the effects of the convenience yield on the power of forward guidance are quantitatively modest.

To maximise utility, household j chooses a path of consumption, hours worked, government bond and state contingent assets subject to the period budget constraints

$$\mathrm{E}_t Q_{t,t+1} B_{j,t+1} + M_{j,t} \leq I_{t-1}^m M_{j,t-1} + B_{j,t} + \int_0^1 W_{k,t} H_{j,k,t} \mathrm{d}k + \Pi_{j,t} + T_t - P_t C_{j,t}$$

where T_t are lump-sum taxes/transfers, $\Pi_{j,t}$ are firms' distributed profits, $W_{k,t}$ is the wage rate in sector k, I_t^m is the non-state-contingent gross return on government bonds, $B_{j,t}$ is a portfolio of state contingent assets and $Q_{t,t+1}$ is a stochastic discount factor which is unique, under the assumption of complete markets. Assuming that $\lim_{n\to\infty} \mathrm{E}_t Q_{t,t+n} \left(M_{j,t+n} + B_{j,t+n}\right) = 0$, the above sequence of period budget constraints can be rewritten as a single intertemporal budget constraints.

We assume that $\bar{C}_t = \Delta_t \bar{C}_{t-1}$, for $t > 1$ and $\bar{C}_0 = 1$. We also assume that economy-wide productivity evolves around a stochastic trend $\overline{A}_t$. Defining detrended consumption and detrended real government bond holdings as $\tilde{C}_t = C_t/\overline{A}_t$ and $\tilde{m}_t = \left(M_t/P_t\right)/\overline{A}_t$, respectively, assuming that $\upsilon'(\cdot)$ is homogeneous of degree -1, so that $\upsilon'\left(\tilde{m}_{j,t}\bar{A}_t - \tilde{m}_t\bar{A}_t\right) = \upsilon'\left(\tilde{m}_{j,t} - \tilde{m}_t\right)\bar{A}_t^{-1}$, and assuming further that everyone has same preferences and the same initial wealth, so that $\frac{M_{j,t}}{P_t} - \frac{M_t}{P_t} = 0$, the first order conditions of the household will include

$$\frac{1}{I_t} = \beta \mathrm{E}_t \Delta_{t+1} \frac{\tilde{C}_t}{\tilde{C}_{t+1}} \frac{\bar{A}_t}{\bar{A}_{t+1}} \frac{1}{\Pi_{t+1}} \tag{1}$$

$$\Delta_t^m = \tilde{C}_t S_t \upsilon'(0) \tag{2}$$

where $\Delta_t^m \equiv \left(I_t - I_t^m\right)/I_t$ and where we used $I_t^{-1} = \mathrm{E}_t Q_{t,t+1}$.

Note that equation (1) is the usual Euler equation in a growth model, suggesting that the gross nominal interest rate I_t is related to the rate of growth of productivity $\bar{A}_{t+1}/\bar{A}_t$. If $\bar{A}_{t+1}/\bar{A}_t$ were stationary, this equation could be linearised as usual. Equation (2) describes the spread between the return on government bonds and the safe portfolio of state-contingent assets. It shows that the return on nominal bonds, I_t^m, includes a convenience yield related to their marginal utility benefit $S_t v'(0)$. Assuming that $v'(0) > 0$, the convenience yield is negative and household will be happy to hold bonds at a discount compared to the safe portfolio of state-contingent assets.

We will assume that, within large but finite boundaries ξ^H and ξ^L, the rate of growth of productivity is integrated, i.e. that $\bar{A}_{t+1}/\bar{A}_t = \Xi_{t+1}$ and $\xi_t = \log \Xi_t$ follows a random walk

$$\xi_t = \xi_{t-1} + \psi_t$$

where ψ_t is a stationary process defined below.

Note that under this assumption all nominal returns will inherit the stochastic trend Ξ_{t+1}. If we detrend them as $\breve{I}_t^m = I_t^m/\Xi_t$ and $\breve{I}_t = I_t/\Xi_t$, conditions (1) and (2) can be linearised as

$$\tilde{c}_t = \left(1 - \breve{\Delta}^m\right)\left(\mathrm{E}_t\tilde{c}_{t+1} - (\breve{\imath}_t^m - \mathrm{E}_t\pi_{t+1}) + \mathrm{E}_t\hat{\psi}_{t+1} - \mathrm{E}_t\delta_{t+1}\right) - \breve{\Delta}^m s_t$$

where $\breve{\Delta}_t^m \equiv \left(\breve{I}_t - \breve{I}_t^m\right)/\breve{I}_t$ and $\breve{\Delta}^m = v'(0)\,\tilde{C}$ is its steady state value. In the absence of nominal rigidities the above equation would be similar apart from the fact that the real rate $\breve{\imath}_t^m - \mathrm{E}_t\pi_{t+1}$ would be independent of monetary policy and equal to the natural rate $\breve{r}_t^n$. The above equation could thus be rewritten as

$$x_t = \left(1 - \breve{\Delta}^m\right)\left[\mathrm{E}_t x_{t+1} - (\breve{\imath}_t^m - \mathrm{E}_t\pi_{t+1} - \breve{r}_t^n)\right] \quad (3)$$

where $x_t \equiv \tilde{c}_t - \tilde{c}_t^n$ and $\breve{r}_t^n = \mathrm{E}_t\tilde{c}_{t+1}^n - \frac{1}{1-\breve{\Delta}^m}\tilde{c}_t^n + \mathrm{E}_t\hat{\psi}_{t+1} - \mathrm{E}_t\delta_{t+1} - \frac{\breve{\Delta}^m}{1-\breve{\Delta}^m}s_t$.

Note that equation (3) is very similar to the standard linearised Euler equation of the new Keynesian model apart from the factor $1 - \breve{\Delta}^m$. Recall that $\breve{\Delta}^m$ is a positive spread, so that $1 - \breve{\Delta}^m$ is a coefficient smaller than 1. Hence, $1 - \breve{\Delta}^m$ acts as a discount factor in the Euler equation and, as such, it will tend to mitigate the forward guidance puzzle for the reasons described in Michaillat and Saez (2018).

There is a countinuum of firms indexed by $k \in [0, 1]$ owned by the households and producing intermediate goods under monopolistic competition and sticky prices. The intermediate goods are bundled into a final good by a competitive industry. The intermediate goods are produced according to the production function $Y_{k,t} = A_t (H_{k,t})^{\frac{1}{\phi}}$. In each period, the profit of firm k is $\Pi_{k,t} = (1 - \tau_t)\, p_{k,t} Y_{k,t} - W_{k,t} H_{k,t}$, where τ_t is a sales tax/subsidy. Given the assumed

Dixit-Stiglitz specification of preferences, the demand for intermediate good $Y_{k,t}$ will be $Y_{k,t} = \left(\frac{p_{k,t}}{P_t}\right)^{-\theta} Y_t$. Firms will choose prices to maximise profits subject to the demand schedule and to a Calvo lottery implying that with probability α they will not be able to adjust prices again in the future. Assuming zero inflation in the non-stochastic steady state, the firms' first order conditions can be linearised to yield

$$\pi_t = \kappa x_t + \beta \mathrm{E}_t \pi_{t+1}$$

for $\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \frac{1+\omega}{1+\omega\theta}$. This is the standard new-Keynesian Phillips curve, provided that the output gap x_t is defined in terms of detrended output in deviation from detrended natural output.

Under flexible prices, the same conditions yield $\tilde{y}_t^n = 0$ and the detrended natural rate can be solved out explicitly as

$$\breve{r}_t^n = -\rho_\delta \delta_t - \frac{\breve{\Delta}^m}{1 - \breve{\Delta}^m} s_t$$

where we used the assumptions

$$\psi_{t+1} = \psi + \sigma_\psi \varepsilon_{t+1}^\psi$$
$$\delta_{t+1} = \rho_\delta \delta_t + \sigma_\delta \varepsilon_{\delta,t+1}$$
$$s_{t+1} = \rho_s s_t + \sigma_s \varepsilon_{s,t+1}.$$

for $\psi_t \equiv \log \Psi_t$, $\delta_t \equiv \log \Delta_t$ and $s_t \equiv \log S_t$.

Since, by definition, $\breve{R}_t^n = \frac{R_t^n}{\Xi_t}$ we can also write the log level of the natural rate as

$$r_t^n = \psi - \log \beta + \ln\left(1 - \breve{\Delta}^m\right) + \xi_t - \rho_\delta \delta_t - \frac{\breve{\Delta}^m}{1 - \breve{\Delta}^m} s_t \tag{4}$$

Since ξ_t is a random walk, the natural rate will be I(1). Around the stochastic trend produced by ξ_t, the natural rate will also be subject to stationary fluctuations induced by both demand shocks δ_t and liquidity/safety shocks s_t. Note that, as in Laubach and Williams (2003), natural output $\tilde{y}_t^n$ and the natural equilibrium $\breve{r}_t^n$ are both trending variables, but they have different stochastic trends, i.e. $\log \bar{A}_t$ and ξ_t, respectively.

2.1 Optimal policy

The appendix shows that, up to a second order approximation, household temporary utility can be written as in the stationary case as

$$U_t^{CB} = -\pi_t^2 - \lambda x_t^2$$

for $\lambda = \kappa/\theta$, where the only difference is that x_t is detrended output in deviation from detrended natural output.

We will solve for optimal policy under commitment taking into account the ELB constraint $I_t^m \geq 1$. Optimal policy under commitment requires

$$2\lambda x_t = -\lambda_{x,t} + \beta^{-1}\left(1 - \breve{\Delta}^m\right)\lambda_{x,t-1} + \kappa\lambda_{p,t}$$
$$2\pi_t = \beta^{-1}\left(1 - \breve{\Delta}^m\right)\lambda_{x,t-1} - \lambda_{p,t} + \lambda_{p,t-1}$$

plus $\lambda_{x,t} = 0$ when $\breve{i}_t^m > -\xi_t - \ln\left(1 - \breve{\Delta}^m\right) - \psi + \ln\beta$ and $\lambda_{x,t} > 0$ when the nominal rate is at the ELB and $\breve{i}_t^m = -\xi_t - \ln\left(1 - \breve{\Delta}^m\right) - \psi + \ln\beta$. In the above equations $\lambda_{x,t}$ is the lagrange multiplier associated to equation (3) and $\lambda_{p,t}$ the multiplier associated to the Phillips curve.

Note that if the ELB were not binding at any point in time, $\lambda_{x,t} = 0$ for any t and optimal policy would boil down to the conditions that hold in the standard new Keynesian model with constant, long-run natural rate and without convenience yield on government bonds:

$$\pi_t = -\frac{\lambda}{\kappa}\Delta x_t$$

where $\Delta x_t = x_t - x_{t-1}$. Together with the Phillips curve, this implies that, absent cost-push shocks, price stability can be maintained at all times in spite of the integrated productivity shocks. As a result the nominal interest rate will be integrated of order 1.

3 Numerical methods

3.1 Solution algorithm

This section offers a brief overview of the numerical method used to solve the different variants of the model considered in the next sections. More details are provided in appendix, that also demonstrates that the solution is quite accurate.

We account for the occasionally binding zero bound constraint on the nominal interest rate by solving the model with policy function iteration. The rate of inflation, π, and the output gap x, are approximated with piecewise linear splines. These functions are known to yield a reasonably good approximations of irregular functions because they have narrow supports, provided the state space is dense. To ensure that this is the case in an efficient fashion, in the spirit of Maliar & Maliar 2015, we use a grid with relatively more points in the region where the cloud of simulated points is relatively more dense.

We assume that the rate of productivity growth follows a bounded random walk such that $\xi_t \in [\xi_L, \xi_H]\ \forall t$. We wish to ensure that ξ_L and ξ_H identify a plausible set of possible values for ξ_t, but we also assume that the two boundaries have never been reached in reality (because empirical studies have not detected evidence of reflective behaviour). To calibrate their values we rely on the long time-series for utilization-adjusted TFP growth constructed in Fernald (2014). We take 20-year moving averages to capture the low-frequency component of TFP growth. Starting from levels around 2% in the early 1970s, the 20-year moving average of utilization-adjusted TFP growth undergoes a slow, but persistent decline all the way to almost 0.5% in the mid-1990s, before increasing again at the end of the 1990s and in the early 2000s. This suggests that plausible values of the boundaries should be below 0.5% and above 2%. We therefore set $\xi_L = 0\%$ and $\xi_H = 4\%$.

This reflects the range of values that are obtained by simulating 10000 random walks for 20 years, each starting from ξ_0 equal to 2%, which is the value that we obtain for the US in the 1970s by using Fernald (2014) data. The distance between the boundaries and the fact that they are centered around the value observed in the 1970s is consistent with Holston et al. (2017). They find that the US productivity trend growth rate was about 3% in the 70s and fluctuates between 1% and 5% over the period 1960-2015.

Under the bounded random walk assumption, the unconditional distribution of the rate of productivity growth becomes near-uniform, and the rate of productivity growth averages 2% in annualised values.

3.2 Calibration

Table 1 summarises our parameter calibration.

Table 1: Calibration of key parameters (quarterly)

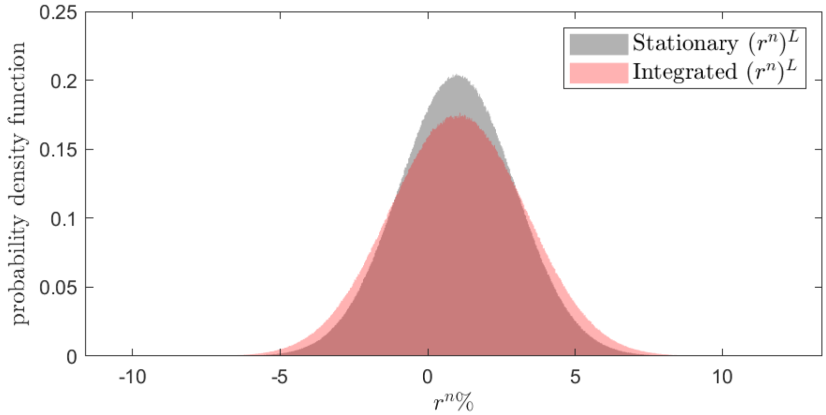
	σ_δ	σ_ψ	ψ	β	$\breve{\Delta}^m$	ξ	$(r^n)^L$
Historical	2.9e-3	2.5e-4	0	0.9945	1.8e-3	5.0e-3	$\frac{3.5}{400}$
Recent	2.9e-3	2.5e-4	0	0.9945	8.0e-3	5.0e-3	$\frac{1}{400}$

This table presents the calibration of key parameters. ξ and $(r^n)^L$ correspond to the unconditional means of the rate of productivity growth and of the natural rate of interest respectively.

To illustrate the key features of our model, we compare two calibrations for $\breve{\Delta}^m$. The first one is taken from Krishnamurthy and Vissing-Jorgensen (2012), which estimates the long-run value of the convenience yield at 72 basis points (in annualised values), corresponding to $\breve{\Delta}^m = 0.0018$. We interpret this as the value of $\breve{\Delta}^m$ prevailing in the past and refer to it as the "historical calibration" in Table 1. For more recent years, we follow Del Negro et al. (2017), which finds that the convenience yield increased markedly and persistently, especially since the early 2000s. More specifically, Del Negro et al. (2017) estimates the persistent conponent of the convenience yield to have increased by up to 3.0% (in annualised terms), considering the top of a 90% confidence interval. This implies approximately $\breve{\Delta}^m = 0.008$, which we take as the value of the convenience yield prevailing in recent years–or "recent calibration" in Table 1.

We assume no drift in productivity growth, so we set $\psi = 0$, which is consistent with the assumption that productivity growth is expected to remain unchanged over the future.

Figure 1: Distribution of the natural rate when $E[r^n]$=1%



This figure presents the distributions of the stationary and of the integrated natural rate. These distributions are computed from a simulation of 1e4 economies over 1000 periods.

We finally set $\beta = 0.9945$. Assuming an average productivity growth rate, ξ_t, equal to 2%, this yields a calibration for the average long-run natural rate equal to 3.5% when $\breve{\Delta}^m = 0.0018$. This is roughly comparable to the steady state level of the natural rate typically used in the new Keynesian literature.

The alternative calibration $\breve{\Delta}^m = 0.008$ would yield an average long-run natural rate of 1%, and lower values of the long-run natural rate when ξ_t falls towards the ξ_L boundary. This is consistent with recent empirical estimates. For example, Holston, Laubach and Williams (2017) finds that, in 2016, the long-run natural rate was between 0 and 1% in the United States and possibly slightly negative in the euro area. Using an alternative approach, Fiorentini et al. (2018) estimates that the long-run level of the natural rate in 2016 was slightly above 1% in the U.S. and as low as -1% in the euro area.

Compared to standard new Keynesian model, we also need to set the variance for the productivity growth shock

$$\xi_t = \xi_{t-1} + \sigma_\psi \varepsilon_{\psi,t}$$

We calibrate σ_ψ based on the results in Fiorentini et al. (2018). More specifically, we used their estimates based on historical data at annual frequency over the period 1891-2016 for a set of 17 advanced economies. The conditional standard deviation σ_ψ takes an annualised value of 0.1%, which is about 10 times lower than the conditional standard deviation that we use for the stationary component $\breve{r}_t^n$, i.e. 1.18% in annualised terms.

It is important to highlight that permanent shocks alter the shape of the unconditional distribution of the natural rate. However, since permanent shocks have very small variance, the unconditional distribution of the natural rate in the integrated case does not depart substantially from its counterpart in the stationary case. Figure 1 illustrates this by showing the two empirical distribution based on 1e4 simulations of shocks for 1000 periods. Another implication of low volatility is that the boundaries should affect the natural rate only when ξ_t is in a very small neighborood above (below) the lower (upper) boundary. Over the rest of its support, ξ_t displays a conditional random walk behaviour, i.e. its expected value is equal to the current value.

All remaining parameters are standard in the literature (with the exception of the intertemporal elasticity of substitution, which is 1 in this paper). More specifically, they are as in Adam & Billi (2006), which in turn draws on Woodford (2003). Note that we do not directly calibrate ϕ and v but, again following Woodford (2003), we only calibrate $\omega \equiv \phi(1+v) - 1$.

4 Drifting vs. stationary natural rate and the ELB

This section highlights the impact of permanent shocks to the natural rate on the equilibrium of the model under optimal monetary policy. it does so though a comparison of simulation moments and impulses to temporary shocks in our full model and in a version of the model without permanent shocks. We make this comparison for each of the two calibrated values of the natural rate of interest reported in Table 1.

4.1 Simulation moments

We start analysing simulation moments of key variables of interest when the unconditional mean of the natural rate is equal to 3.5%. Table 2 considers three model variants. The first variant is based on the Adam and Billi (2007) calibration, which has a steady state natural rate of 3.5%. It is obtained in our model if $\beta = 0.9913$, $\xi_t = 0$ and $\breve{\Delta}^m = 0$. The second variant is our historical calibration, but in the absence of permanent shocks to ξ_t. In this case we assume that productivity grows at a constant rate ψ whose value is set to 2% (in annualised terms) to produce the same steady state natural rate as in Adam and Billi (2007). The third variant is the historical calibration of our full model, which includes permanent shocks to the natural rate.

Table 2: Sample moments when $E[r^n]$=3.5%

	Stationary $(r^n)^L$		Integrated $(r^n)^L$
	Adam & Billi 2007	Baseline	Baseline
β	0.9913	0.9945	0.9945
ψ or $E[\xi_t]$	0	$\frac{2}{400}$	$\frac{2}{400}$
$\breve{\Delta}^m$	0	$\frac{0.72}{400}$	$\frac{0.72}{400}$
r^n	3,494	3,498	3,519
x	0	0	-0,001
π	0,001	0,001	0,003
i	3,494	3,498	3,519
RR spread	-0,001	-0,001	-0,003
ELB frequency (x100)	5,541	5,515	9,577
ELB duration (quarters)	2,643	2,635	3,768

This table reports the simulation moments obtained by simulating 10000 economies, each for 1000 quarters long. It contains the annualized means of the natural rate of interest, the output gap, the inflation rate, the nominal interest rate, the real rate spread.

Focusing on the two stationary versions of the model, columns 1 and 2 of Table 2 show that the unconditional means of the endogenous variables are not sensitive to permutations in the calibration of β, $\breve{\Delta}^m$ and ψ which deliver the same steady state natural rate value of 3.5%. Both in columns 1 and 2, the average output gap and the average inflation rate are positive but very closed to zero. The unconditional spread between the real rate and the natural rate is only slightly negative. In both cases, the frequency of ELB episodes is so low that forward guidance can be effectively deployed to prevent the ELB constraint from generating a skew in unconditional means.

Table 2 also shows that adding integrated natural rate shocks to the model makes only a tiny difference, if the unconditional mean of the natural rate is 3.5%.

Table 3 performs a similar exercise for a higher calibration of the convenience yield.

Table 3: Sample moments when E$[r^n]$=1%

	Stationary $(r^n)^L$	Integrated $(r^n)^L$	Persistent $(r^n)^L$
$\breve{\Delta}^m$	$\frac{3.2}{400}$	$\frac{3.2}{400}$	$\frac{3.2}{400}$
r^n	0,998	1,019	1,009
x	0	-0,017	-0,003
π	0,058	0,18	0,115
i	1,044	1,174	1,092
RR spread	-0,013	-0,028	-0,033
ELB frequency (x100)	54,463	55,811	55,354
ELB duration (quarters)	8,953	12,92	10,799
Stdev(r^n)	1,96	2,26	2,104

See table 2 for details. The fourth column corresponds to a case in which the low frequency component of the natural rate follows a highly persistent AR(1) process instead of an integrated process: $\Xi_t = \Psi \cdot \Psi_t$ where $\psi = \log(\Psi) = 2\%$ annually, $\mathrm{E}_t[\log \Psi_{t+1}] = \bar{\psi}_t = \rho_\psi \bar{\psi}_{t-1} + \sigma_\psi \varepsilon_t$, and $\rho_\psi = 0.99$.

The first two columns in the table compare economies with and without permanent shocks (and temporary shocks characterised by an autocorrelation coefficient of 0.8). We observe that the average output gap and the average spread between the real rate and the natural rate are very similar in the two cases. The main differences charaterise nominal variables. On average, inflation and the nominal rate are somewhat higher in the case with permanent shocks.

These outcomes arise because commitment plays a quantitatively larger role in the model with permanent shocks. Due to the higher likelihood that the natural rate will spend a long time around very low levels, the central bank will more frequently operate through promises of higher future inflation. Such promises will affect private expectations biasing them upwards. Due to the Euler equation, if average inflation expectations increase, so does the average nominal interest rate.

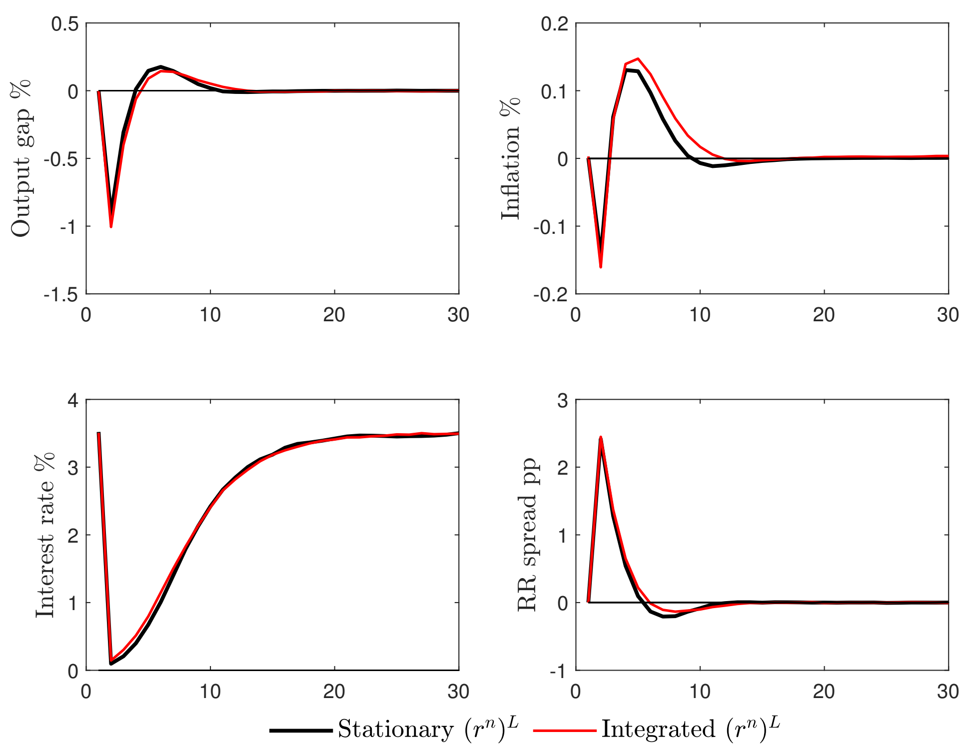
The third column in the Table 3 highlights that numerical results similar to the case with permanent shocks would be obtained if shocks to the rate of growth of productivity ξ_t were stationary, but highly persistent (i.e. characterised by an autocorrelation coefficient of 0.99). The model with integrated shocks can thus be seen as the limiting case of a model with extremely persistent, stationary shocks.

4.2 Impulse responses to temporary shocks

Figure 2 compares the dynamic properties of the models with stationary or integrated natural rate under optimal policy. The unconditional mean of the natural rate is 3.5%. The figure shows impulse responses to a negative shock to the temporary component of the natural rate of interest. The shock is large enough to warrant a reduction of the policy interest rate all the way to the ELB.

In both cases, the shape of the impulse responses is consistent with previous results in the literature. Due to the ELB constraint, the economy cannot be fully stabilised, so inflation falls on impact by about 15 basis points. Thereafter, the central bank implements forward guidance. It keeps the interest rate low for longer, ie. it raises the policy rate more slowly than the natural rate (the spread between the actual and natural real interest rates turns negative) in order to produce a temporary overshooting of inflation.

Figure 2: Average IRFs to a temporary real rate shock when $\mathrm{E}[r^n] = 3.5\%$



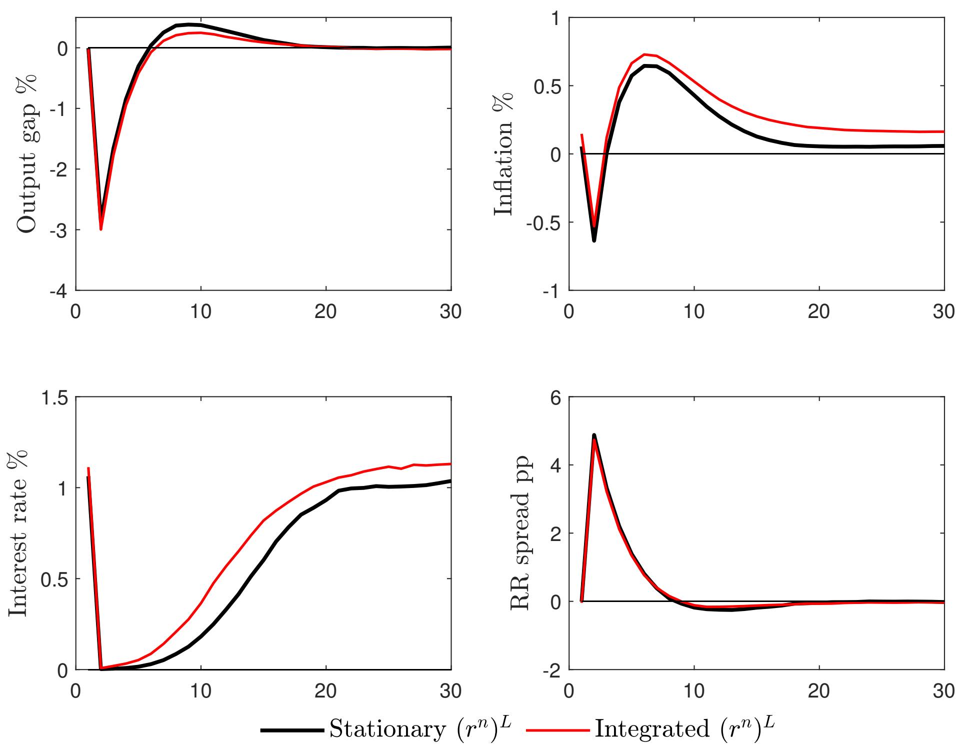
This figure reports average responses to a large negative shock of 3 times the unconditional standard deviation of temporary real rate shocks $\bar{\delta}_t$ (about 5.88 pp annually). They are computed from a simulation of 10000 economies, all starting from a random draw in the unconditional distribution of the state of the economy.

All in all, when the unconditional mean of the natural rate is 3.5%, the model with integrated natural interest rate is not strikingly different from a stationary model.

Figure 3 repeats the experiment with an unconditional mean of the natural rate of 1%.

Average inflation and the average nominal interest rate are higher in the economy with permanent natural rate shocks. This creates more space to cut policy rates, when necessary. Since inflation expectations will return to a higher average level, policy is more expansionary and there is less of a need to rely on forward guidance on future policy rates. The output gap is stabilised equally well, independently of permanent shocks.

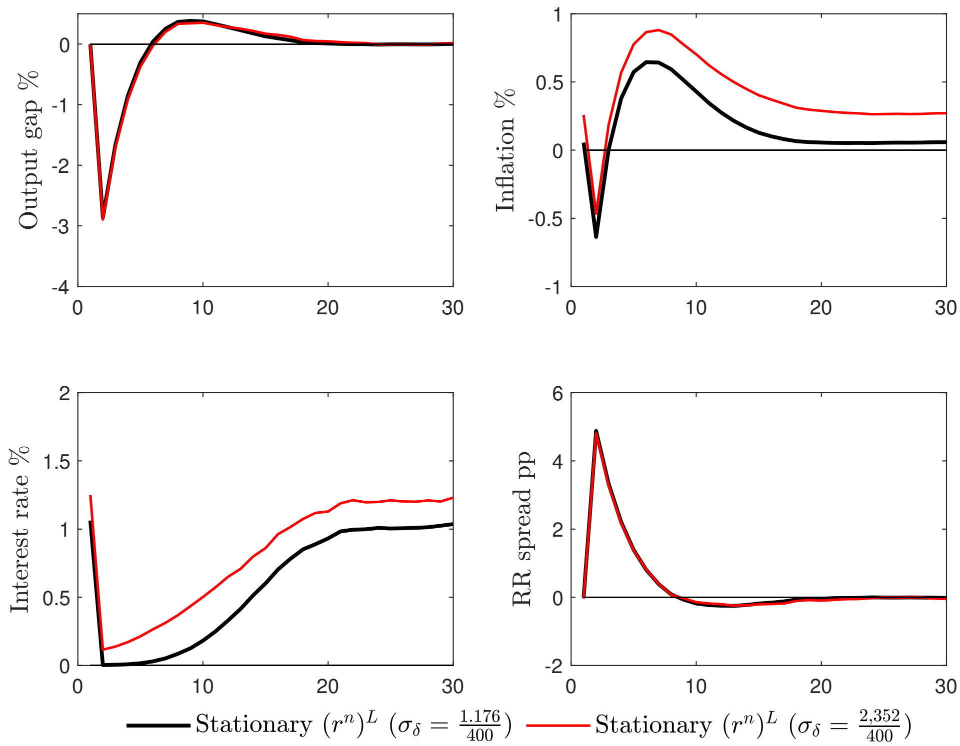
Figure 3: Average IRFs to a temporary real rate shock when $\mathrm{E}[r^n] = 1\%$



See figure 2 for details.

Finally, Figure 4 demonstrates that effects on expectations and averages would also be produced in the model without permanent shocks, if stationary shocks had larger volatility. In both cases, what matters is the probability that the natural rate may fall, either persistently or permanently, to very low levels. For illustrative purposes, figure 4 considers the case in which transitory shocks have a volatility twice as large as in the baseline calibration. The qualitative impact of this exercise on expectations and impulse responses is comparable to that observed in figure 3. Permanent shocks have the impact of boosting this expectational mechanism without producing at the same time an unrealistically high conditional volatility of the natural rate.

Figure 4: Optimal policy implications of more volatility when $E[r^n] = 1\%$



See figure 2 for details.

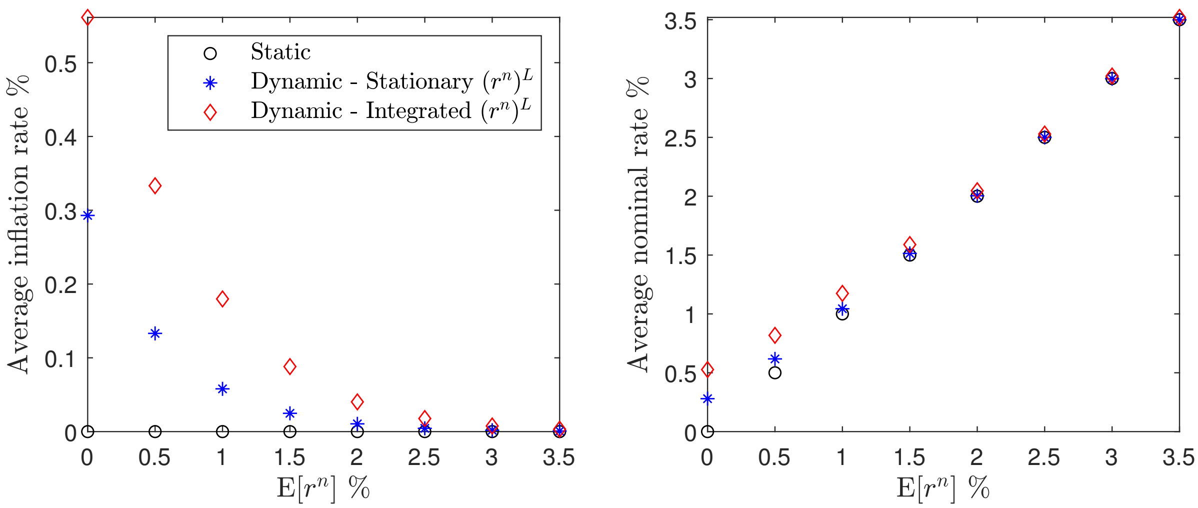
5 The drifting natural rate and optimal inflation

5.1 Optimal inflation

Table 2 suggests that the drifting natural rate has no substantive implications on optimal inflation, when the unconditional mean of the natural rate is 3.5%. The lower bound constraint does induce stochastic effects on inflation, but they are tiny. This is consistent with previous results in Adam and Billi (2006), Nakov (2008). Those papers have shown that forward guidance on the path of future policy rates is quite effective in stabilising the macroeconomy in the face of a temporary fall in the natural rate which forces the policy rate against the ELB. As a result, it is not desirable for the central bank to engineer a positive average inflation rate in order to reduce the likelihood of hitting the ELB. Positive average inflation would not bring significant benefits in terms of macroeconomic stabilization, but it would produce sizable costs in terms of price dispersion. The central bank prefers to tolerate occasionally large fluctuations in output and inflation when the ELB becomes binding.

Table 3 showed that optimal inflation starts increasing somewhat above zero when the unconditional mean of the natural rate is 1%. In this section, we compute optimal inflation for lower and lower average values of the natural rate. Once again we consider the versions of the model with both stationary and integrated natural rate. More specifically, starting from the calibrations in columns 2 and 3 of Table 2, we progressively lower $\breve{\Delta}^m$ towards value which push the long-run natural rate towards zero. Figure 5 illustrates the results.

Figure 5: Optimal inflation and interest rate



This figure reports the average inflation rate and the average nominal interest rate as a function of the average natural rate for both, an economy in which the natural rate is stationary, and an economy in which it is integrated.

To interpretate our numerical results, we also use as a benchmak a static steady state optimization problem, i.e. the maximisation of the nonlinear, steady state welfare by choice of an appropriate inflation rate. For this special case, we assume away the I(1) component of productivity growth. The appendix shows that the solution is a step function such that $\pi = 0$, if $r^n \geq 0$, and $\pi = -r^n$, if $r^n < 0$. Intuitively, the optimal inflation rate is zero, because this is the value which minimises price distortions in the economy. If, however, the steady state natural rate is negative, then zero inflation is infeasible, because it would require a negative steady state policy rate, which would violate the ELB constraint. Hence, when the steady state natural rate is negative, the constrained optimal inflation rate is equal to $-r^n$, that is, the smallest inflation rate consistent with the ELB constraint. This benchmark is represented by the solid line in figure Figure 5.

In the stochastic model, the central bank reacts to the risk of finding itself in a state of the world in which the natural rate may fall below zero. When shocks are transitory, but persistent, the risk is sufficiently high to warrant a precautionary increase in optimal inflation already when the natural rate falls below 1%. This is because a negative natural rate would significantly weaken the central bank's ability to stimulate the economy. Forward guidance would remain effective, but it would need to stretch forward to the point when the natural rate returns to above-zero values. This risk is obviously increasing in the variance of natural rate shocks. For our standard calibration, optimal inflation increases to about 0.3% when the natural rate is close to zero.

In the non-stationary version of the model, optimal inflation rises almost twice as fast, moving towards 0.6% when the natural rate is close to zero. On the one hand, this additional increase is due to the somewhat higher conditional variance of the natural rate. On the other hand, if shocks are permanent, a possible dip of the long run natural rate into negative territory would also be permanent (in expectations). In this situation the central bank would become completely powerless. Forward guidance is (in expectations) permanently ineffective if the natural rate moves below zero. All the central bank can do is hope for a lucky combination of future positive shocks.

All in all, the lower the long-run value of the natural rate, the higher the optimal level of inflation in the economy. This result echoes that in Andrade et al. (2018), but it is at the same time more powerful and less pronounced. On the one hand, it is more powerful because it is obtained in a set up where the central bank can exploit the full benefits of forward guidance. Even under a mildly positive natural interest rate, e.g. at just 10 basis points above zero, the central bank could promise to keep the policy rate at zero for a very long future, so as to create a very persistent, negative interest rate gap of -10 basis points. For given natural rate level, such a promise would stimulate the economy and stabilise inflation. However, the natural rate is not constant over time. The central bank, therefore, acts against the risk that the natural rate will fall further and even the 10 basis points room for manouvre will disappear. On the other hand, our results are quantitatively less pronounced than in Andrade et al. (2018), because the full power of forward guidance does play an important role in stabilising inflation. By contrast, Andrade et al. (2018) assume that the central bank follows a Taylor rule, hence it is more severely hampered by low levels of the natural rate. When unconditional mean of the natural rate of interest falls towards 0, the optimal inflation target in Andrade et al. (2018) must increase by a larger amount.

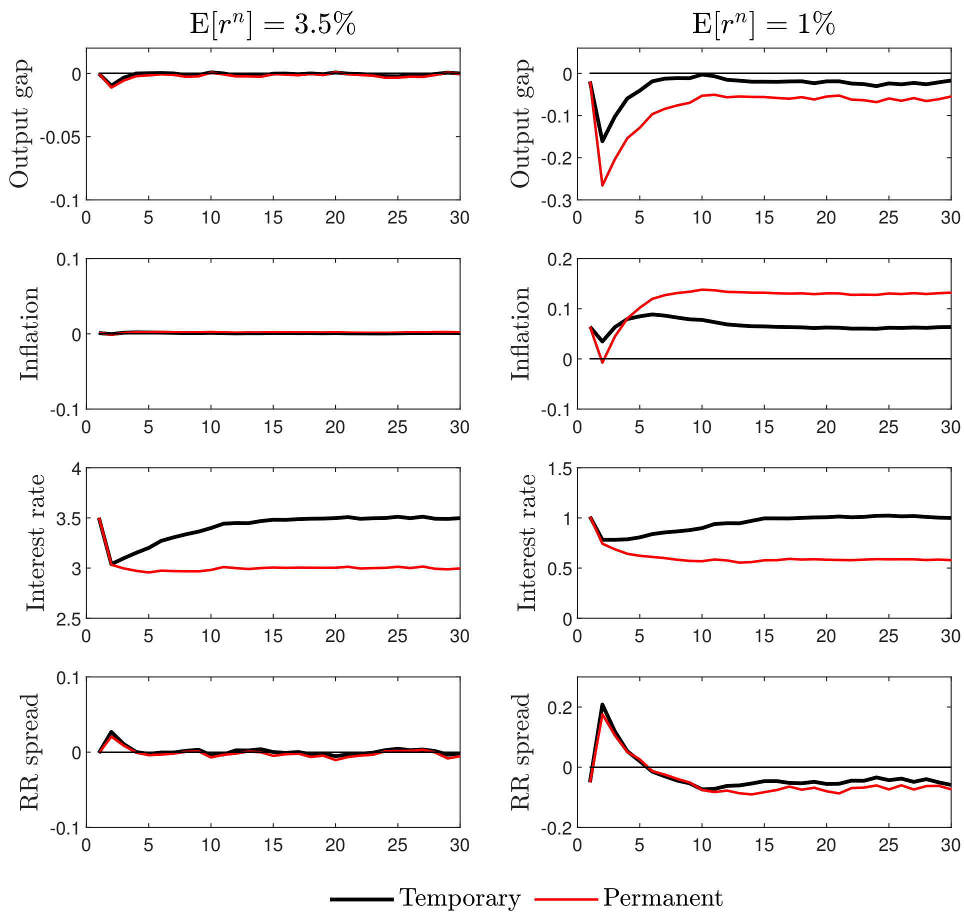
5.2 Permanent shocks

Figure 6 shows impulse responses to a permanent, 0.5 percentage points shock to the natural rate. On the left panel of figure 6, the unconditional mean of the natural rate is 3.5%; on the right panel it is equal to 1.0%. As a benchmark, the figure also shows the responses of a temporary shock of the same size.

The left panel demonstrates that inflation and the output gap can be almost perfectly stabilised after a permanent shock which reduces the long-run natural rate to 3%. In response to the fall in the long-run natural rate, the policy interest rate falls one-to-one with the long-run natural rate. The policy interest rate must therefore inherit the stochastic trend in the natural rate. Optimal inflation increases slightly, due to the somewhat higher likelihood of hitting the ELB, but the increase is tiny.

The right panel shows reponses to the same shock starting from a long-run natural rate level of 1.0%. Such level is sufficiently low to warrant a positive optimal level of inflation of approximately 0.1% before the shock occurs. The economy can no longer be stabilised after the negative, permanent shock. Inflation falls on impact by about 0.1% and the output gap by 0.3%. To achieve this outcome, the central bank engineers a combination of two measures. On the one hand, it cuts the policy interest rate as in the case of 3.5%, but less than one-to-one with the long-run natural rate. As a result, the spread between the real interest rate and the natural rate increases on impact, imparting some downward pressure on inflation. On the other hand, the central bank commits to a permanent upward revision of optimal inflation, by about 10 basis points. As a result, the real interest rate is somewhat lower in the new long-run equilibrium.

Figure 6: Average IRFs to temporary and permanent real rate shocks



This figure reports average responses to a temporary and to a permanent negative real rate shock of 0.5 pp (about 0.4 (4.63) time the conditional standard deviation of $\bar{\delta}_t$ (ξ_t)). They are computed from a simulation of 10000 economies, all starting from a random draw in the unconditional distribution of the state of the economy, except the rate of productivity growth which falls from 2% to 1.5%.

Figure 6 suggests that low values of the long-run natural rate of interest have an impact on the course of monetary policy. The lower the long-run natural rate, the smaller the response in policy rates and the larger the adjustment in the inflation objective.

More specifically, the inflation objective must be ratcheted up every time there is a negative shock of the long-run natural rate. (Conversely, it must be adjusted downwards if the long-run natural rate increases.) The adjustment is nonlinear. For given shock size, the lower the initial long-run natural rate, the bigger the required increase in the inflation objective.

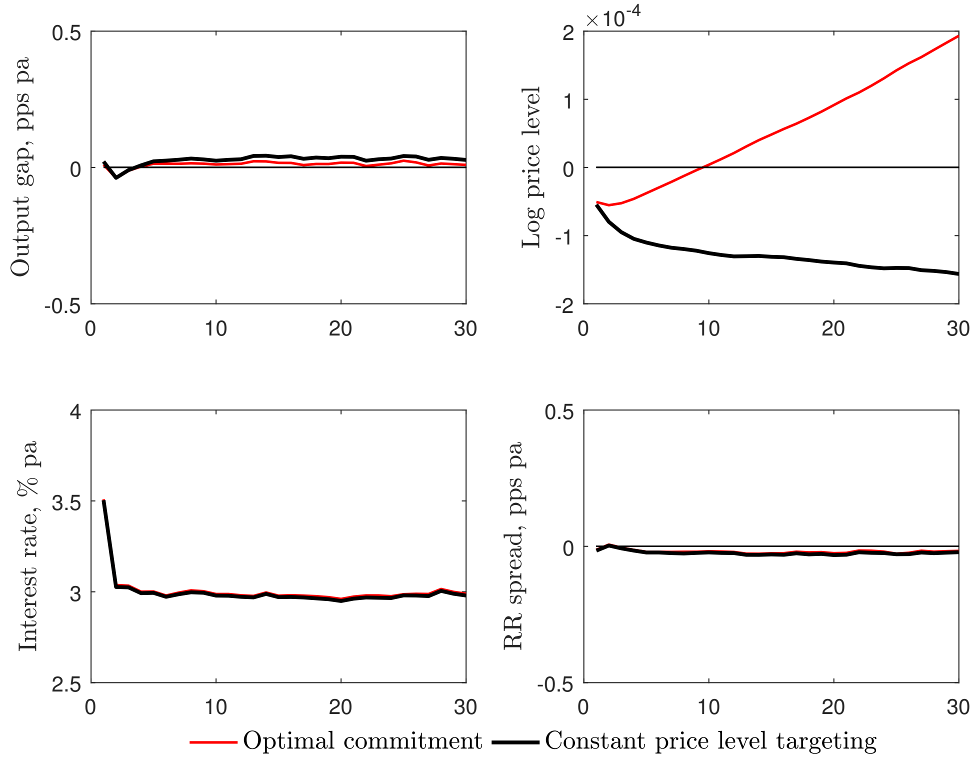
The above results suggest that the integrated nature of the natural rate of interest underpins the need for periodic revisions in the inflation target. This is consistent with the existing practice of some central banks, such as the Bank of Canada, which every five years reviews its periodic agreement with the Government of Canada. Such revisions are increasingly desirable, the lower the estimated value of the long-run natural rate.

6 Price level targeting and permanent natural rate shocks

Eggertsson and Woodford (2003) demonstrate that the type of commtiment required by optimal policy can be equivalently expressed in terms of a path for an "output gap-adjusted" price level, which is given by the log of the price index plus a positive multiple of the output gap. The target path for the gap-adjusted price level is not constant, but should be adjusted upwards any time the ELB prevents the central bank from achieving it. The policy rate should then remain at the ELB as long as the gap-adjusted price level remains below the target path.

A similar target path for the gap-adjusted price level can be defined in our model, provided one interprets the output gap as the detrended output gap–see the demonstration in the appendix. Note, however, that the inflation objective is positive under optimal policy, if the long-run natural rate of interest is sufficiently low. Hence, the price level is no longer constant when shocks die out, but it has a positive trend. Moreover, after a permanent, negative shock that induces a target path shortfall because the policy interest rate reaches the ELB, we have seen that the inflation target must be ratcheted up. Hence not only must the gap-adjusted price level be adjusted upwards after such shock, but its trend growth must also become steeper, consistently with the higher inflation objective.

Figure 7: IRFs to a permanent shock under price level targeting when $(r_0^n)^L = 3.5\%$



This figure provides generalized impulse response functions to a negative permanent shock of 0.5% (about 4.5 times the conditional standard deviation of the permanent component). They are computed from a simulation of $1e^4$ economies, starting from the unconditional distribution of state variables around $(r_0^n)^L$ equal to 3.5%.

In a stationary model, Eggertsson and Woodford (2003) demonstrates that a simpler criterion, such that the gap-adjusted price level is kept at a constant value, achieves nearly as good stabilization outcomes as optimal policy. The reason is that, while not permitting an upward revision in the price level after adverse shocks, a constant gap-adjusted price level has at least the benefit not to let the target path shift downwards, as would be the case under a purely forward-looking inflation objective. In this section, we therefore analyse the properties of a simple rule of this kind, i.e.

$$p_t + \frac{\lambda}{\kappa} x_t = P^* \tag{5}$$

where P^* is the target gap-adjusted (log-)price level (a given constant).

Figure 7 shows impulse responses to a permanent shock starting from a long-run natural rate of 3.5%. The figure compares outcomes under optimal policy and under the price level target in rule (5). Recall that, due to the high long-run natural rate of interest, optimal inflation is essentially zero. A constant price level target is therefore a reasonably good approximation to optimal policy.

Nevertheless, after the shock the long-run natural rate falls permanently to 3%, so optimal inflation must increase slightly. This implies that the price level path induced by optimal policy will permanently incorporate a positive trend–see the top, right panel in figure 7. The price level targeting rule (5) does not capture this trend. As a consequence, an ever increasing gap arises between the price level outcomes under optimal policy and under the price level target. Expectations react to the worse performance of the price level targeting rule and induce a downward bias in the actual price level, which remains permanently below its target (the initial level in the figure).

Rule (5) will work less and less well for lower initial values of the natural rate, because it does not allow for positive inflation, which is a feature of optimal policy under commitment. An obvious refinement of rule (5) is therefore to allow for a trend in the price level target, such that

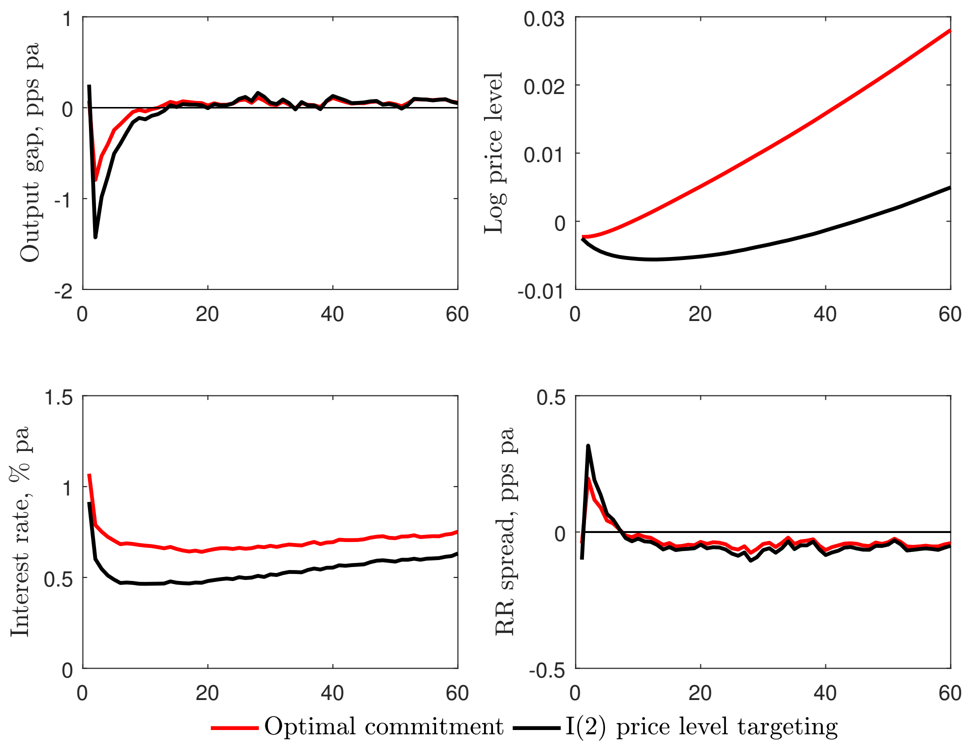
$$p_t + \frac{\lambda}{\kappa} x_t = P_t^* \tag{6}$$

where P_t^* now follows a deterministic trend given by

$$P_t^* = P_{t-1}^* + \pi^*$$

The adjusted rule (6) requires a sufficiently high π^*. When the unconditional mean of the natural rate is 1%, we are only able to solve the model if the trend π^* is above 1%, which implies a much higher rate of increase in prices than under optimal commitment. Intuitively, the constant price trend must be sufficiently high to ensure the existence of an equilibrium even after a possible sequence of negtive, permanent shocks to the natural rate of interest.

Figure 8: IRFs to a permanent shock under price level targeting when $(r_0^n)^L = 1\%$



This figure provides generalized impulse response functions to a negative permanent shock of 0.5% (about 4.5 times the conditional standard deviation of the permanent component). They are computed from a simulation of $1e^4$ economies, starting from the unconditional distribution of state variables around $(r_0^n)^L$ equal to 1%.

Figure 8 shows generalised impule responses to a permanent natural rate shock under this type of price level targeting and under optimal commitment. The shock occurs when the initial long-run natural rate is equal to 1%, it is negative and equal to 50 basis points.

Recall that a positive inflation target is desirable when the steady state natural rate is at 1%. This is illustrated by the growing (log-)price level path under optimal commitment. More specifically, optimal commitment does not allow for any drop in the price level after the shock. The price level only suffers a temporary slow down of its growth rate. By contrast, the price level initially falls on impact under the trend-adjusted price level targeting rule, so that a gap opens with respect to the price level under optimal commitment. As a result, the real interest rate tends to increase more under the price level targeting rule. (The real interest rate falls less than the natural rate because of the occasionally binding ELB constraint). Consequently, stabilization outcomes are worse under price level targeting than under optimal commitment:

the peak fall in the output gap is twice as large under the price level targeting rule. After about 4 years, the price levels return to their trends. The trend is higher under optimal commitment, corresponding to a higher average inflation rate, so the nominal interest rate is also somewhat higher in this case. This increases the room for manouvre available for policy interest rates in the face of additional negative shocks.

Overall, we conclude that an inflation-adjusted price level target continues to provide a reasonably good approximation to optimal commitment, as long as the long-run natural rate remains in positive territory. For practical purposes, simple catch-up policies in a liquidity trap situation preserve their desirable features even if the natural rate of interest follows an integrated time series process.

7 Conclusions

Motivated by recent estimates suggesting that the natural rate of interest is best described by an integrated stochastic process, we have analysed the optimal policy implications of a non-stationary natural rate in a new Keynesian model that accounts for the effective lower bound on nominal interest rates.

We have shown that, in spite of the strong power of forward guidance in the model, permanent, negative shocks to the long-run natural rate eventually warrant a non-negligible increase in the optimal rate of inflation. Once the long-run natural rate becomes close to zero, as in recent empirical estimates, the optimal inflation rate in the model increases to positive values. This is the case in spite of the strong power of forward guidance in the model. Taking into account the practical limitations of forwad guidance, the optimal inflation rate when the long-run natural rate is around zero would likely be much higher.

We have also shown that price level targeting rules have a harder time to mimic optimal policy, if the natural rate is subject to permanent shocks. This is because, in response to such shocks, the optimal inflation rate must be adjusted by variable amounts, depending on initial conditions. Nevertheless, an inflation-adjusted price level target continues to provide a reasonably good approximation to optimal commitment, as long as the long-run natural rate remains in positive territory.

Throughout the paper, we have assumed that the effective lower bound is actually zero. However, the experience of the ECB and of other central banks has demonstrated that policy rates can go slightly negative. Our results would be somewhat sensitive to a reduction of the ELB, but our overall conclusion that optimal inflation will rise is robust. Indeed our premise that the long-run natural rate of interest is close to zero is conservative. For example, some cross-country estimates in Fiorentini et al. (2018) place the long-run natural rate in 2016 as low as -1% in the euro area and -2% in the U.S..

In the paper, we have assumed that both the central bank and the private sector can observe the long-run natural rate of interest. However, the uncertainty surrounding empirical estimates of the long-run natural rate of interest suggests that accounting for filtering uncertainty in optimal monetary policy analyses would be important. Unfortunately, the existing methods, e.g. Svensson and Woodford (2003), are not immediately applicable in a nonlinear context which takes the ELB into account. We leave this extension to future research.

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8 Appendix

8.1 Households' and firms' optimization problems

The problem of household j is to

$$\max_{C_{j,t},H_{j,k,t},M_{j,t},B_{j,t+1}} \mathrm{E}_0 \sum_{t=0}^{\infty} U_{j,t}\left(C_{j,t}, H_{j,k,t}, \frac{M_{j,t}}{P_t}\right)$$

where $M_{j,t}$ are government bonds,

$$U_{j,t} = \bar{C}_t \left(\log C_{j,t} + s_t \upsilon \left(\frac{M_{j,t}}{P_t} - \frac{M_t}{P_t}\right) - \frac{\gamma}{1+v}\bar{H}_t^{-v}\int_0^1 H_{j,k,t}^{1+v}\mathrm{d}k\right)$$

and where the function $\upsilon(\cdot)$ is increasing and concave, s_t is a shock, and T_t are taxes/trasnfers.

Utility maximisation is subject to the budget constraint

$$\mathrm{E}_t Q_{t,t+1} B_{j,t+1} + M_{j,t} \leq I_{t-1}^m M_{j,t-1} + B_{j,t} + \int_0^1 W_{k,t} H_{j,k,t}\mathrm{d}k + \Pi_{j,t} + T_t - P_t C_{j,t}$$

where we assume complete markets and $B_{j,t+1}$ is a portfolio of state contingent assets.

The first order conditions include

$$\bar{C}_t C_{j,t}^{-1} = \lambda_t P_t$$
$$\frac{W_{k,t}}{P_t} = \gamma \mu_t^W \bar{H}_t^{-v} H_{j,k,t}^{v} C_{j,t}$$

and state by state

$$Q_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}.$$

Note that the short term interest rate is

$$\frac{1}{I_t} = \mathrm{E}_t Q_{t,t+1},$$

which can be used to rewrite the first order condition with respect to $M_{j,t}$ as

$$1 - \frac{I_t^m}{I_t} = s_t C_{j,t} \upsilon' \left(\frac{M_{j,t}}{P_t} - \frac{M_t}{P_t}\right)$$

We assume $\bar{C}_t = \Delta_t \bar{C}_{t-1}$, for $t > 1$ and $\bar{C}_0 = 1$. We will also assume that productivity includes a stochastic trend $\bar{A}_t$, such that $A_t = \bar{A}_t Z_t$. Before linearising, we detrended consumption as $\tilde{C}_t = C_t/\bar{A}_t$ and detrended real money as $\widetilde{m}_t = (M_t/P_t)/\bar{A}_t$. We also assume that $\upsilon'(\cdot)$ is homogeneous of degree -1, so that $\upsilon'\left([\widetilde{m}_{j,t} - \widetilde{m}_t]\bar{A}_t\right) = \upsilon'(\widetilde{m}_{j,t} - \widetilde{m}_t)\bar{A}_t^{-1}$. This implies

$$\begin{aligned}
\frac{W_{k,t}}{P_t} &= \gamma\mu_t^W \bar{H}_t^{-v} H_{j,k,t}^v \tilde{C}_{j,t}\bar{A}_t \\
1-\frac{I_t^m}{I_t} &= s_t \tilde{C}_{j,t} \upsilon'\left(\widetilde{m}_{j,t}-\widetilde{m}_t\right) \\
1 &= \beta \mathrm{E}_t \Delta_{t+1} \frac{\tilde{C}_{j,t}}{\tilde{C}_{j,t+1}} \frac{\bar{A}_t}{\bar{A}_{t+1}} \frac{1}{\Pi_{t+1}} I_t
\end{aligned}$$

Since everyone has same preferences and same initial wealth, we can drop the $j's$ and note that $\widetilde{m}_{j,t} = \widetilde{m}_t$ to obtain

$$\begin{aligned}
\frac{W_{k,t}}{P_t} &= \gamma\mu_t^W \bar{H}_t^{-v} H_{k,t}^v \tilde{C}_t\bar{A}_t \\
1-\frac{I_t^m}{I_t} &= s_t \tilde{C}_{j,t} \upsilon'(0) \\
1 &= \beta \mathrm{E}_t \Delta_{t+1} \frac{\tilde{C}_t}{\tilde{C}_{t+1}} \frac{\bar{A}_t}{\bar{A}_{t+1}} \frac{1}{\Pi_{t+1}} I_t
\end{aligned}$$

where we assume that $\upsilon'(0) > 0$.

We will assume that $\delta_t = \ln \Delta_t$ and $s_t = \ln S_t$ follow AR(1) processes

$$\begin{aligned}
\delta_{t+1} &= \rho_\delta \delta_t + \sigma_\delta \varepsilon_{\delta,t+1}, \\
s_{t+1} &= \rho_s s_t + \sigma_s \varepsilon_{s,t+1},
\end{aligned}$$

and that productivity growth, $\Xi_{t+1} = \frac{\bar{A}_{t+1}}{\bar{A}_t}$ is itself integrated, i.e. that its $\xi_t = \log \Xi_t$ follows

$$\begin{aligned}
\xi_t &= \xi_{t-1} + \psi_t \\
\psi_t &= (1-\rho_\psi)\psi + \rho_\psi \psi_{t-1} + \sigma_\psi \varepsilon_t^\psi
\end{aligned}$$

where ψ_t is the (rate of) change in the rate of productivity growth.

Note that in steady state, for $\breve{\Delta}_t^m \equiv \frac{\breve{I}_t - \breve{I}_t^m}{\breve{I}_t}$, $\breve{\Delta}^m = \tilde{C}\upsilon'(0)$ and $\breve{I} = \frac{\Psi}{\beta}$. To first order

$$\frac{\breve{I}^m}{\breve{I}-\breve{I}^m}\left(\breve{\imath}_t - \breve{\imath}_t^m\right) = \tilde{c}_t + s_t$$

and

$$\tilde{c}_t = \mathrm{E}_t\tilde{c}_{t+1} - \left(\breve{\imath}_t - \mathrm{E}_t\pi_{t+1}\right) + \mathrm{E}_t\widehat{\psi}_{t+1} - \mathrm{E}_t\delta_{t+1}$$

These two equations can be combined to obtain

$$\tilde{c}_t = \left(1-\breve{\Delta}^m\right)\left[\mathrm{E}_t\tilde{c}_{t+1} - \left(\breve{\imath}_t^m - \mathrm{E}_t\pi_{t+1}\right) + \mathrm{E}_t\widehat{\psi}_{t+1} - \mathrm{E}_t\delta_{t+1}\right] - \breve{\Delta}^m s_t$$

In a natural equilibrium where prices are flexible $\tilde{c}_t^n = \left(1-\breve{\Delta}^m\right)\left(\mathrm{E}_t\tilde{c}_{t+1}^n - \breve{r}_t^n + \mathrm{E}_t\widehat{\psi}_{t+1} - \mathrm{E}_t\delta_{t+1}\right) - \breve{\Delta}^m s_t$. If we define the output gap as $x_t = \tilde{c}_t - \tilde{c}_t^n$, we can therefore obtain

$$x_t = \left(1-\breve{\Delta}^m\right)\left(\mathrm{E}_t x_{t+1} - \left(\breve{\imath}_t^m - \mathrm{E}_t\pi_{t+1} - \breve{r}_t^n\right)\right)$$

where the natural rate is

$$\breve{r}_t^n = \mathrm{E}_t \tilde{c}_{t+1}^n - \frac{1}{1-\breve{\Delta}^m} \tilde{c}_t^n + \mathrm{E}_t \widehat{\psi}_{t+1} - \mathrm{E}_t \delta_{t+1} - \frac{\breve{\Delta}^m}{1-\breve{\Delta}^m} s_t$$

In each period, the profit of firm k is

$$\Pi_{k,t} = (1-\tau)\, p_{k,t} Y_{k,t} - W_{k,t} H_{k,t}$$

where τ is a tax rate.

Given the demand schedule

$$Y_{k,t} = \left(\frac{p_{k,t}}{P_t}\right)^{-\theta} Y_t$$

and the production function, the amount of hours needed to produce a given volume of output is

$$H_{k,t} = \left(\frac{Y_{k,t}}{A_t}\right)^{\phi}$$

Substituting these expressions into the profit function gives

$$\Pi_{k,t} = (1-\tau)\, p_{k,t} \left(\frac{p_{k,t}}{P_t}\right)^{-\theta} Y_t - W_{k,t} \left(\frac{1}{A_t}\left(\frac{p_{k,t}}{P_t}\right)^{-\theta} Y_t\right)^{\phi}$$

Given Calvo pricing the expected profits of the firms which get the chance to choose prices at time t are

$$\max_{p_{k,t}} \mathrm{E}_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[(1-\tau)\, p_{k,t} \left(\frac{p_{k,t}}{P_T}\right)^{-\theta} Y_T - W_{k,T} \left(\frac{p_{k,t}}{P_T}\right)^{-\theta\phi} \frac{Y_T^{\phi}}{A_T^{\phi}}\right]$$

with first order condition

$$\left(\frac{p_{k,t}}{P_t}\right)^{1+(\phi-1)\theta} = \phi \frac{\theta}{\theta-1} \frac{\mathrm{E}_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \frac{W_{k,T}}{P_t} \left(\frac{P_t}{P_T}\right)^{-\theta\phi} \frac{Y_T^{\phi}}{A_T^{\phi}}}{\mathrm{E}_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} (1-\tau) \left(\frac{P_t}{P_T}\right)^{-\theta} Y_T}$$

Now use $H_{k,T} = \left(\frac{p_{k,t}}{P_T}\right)^{-\theta\phi} \left(\frac{Y_T}{A_T}\right)^{\phi}$ to substitute out the wage rate

$$W_{k,T} = \bar{C}_T \mu_T^W \gamma \left(\frac{p_{k,t}}{P_T}\right)^{-\theta\phi\nu} \left(\frac{Y_T}{A_T}\right)^{\phi\nu} \bar{H}_T^{-v} \lambda_T^{-1}$$

and obtain, recalling that $Y_T = C_T$,

$$\left(\frac{p_{k,t}}{P_t}\right)^{1+\omega\theta} = \frac{\gamma\phi \frac{\theta}{\theta-1} \mathrm{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{\bar{C}_T}{C_t} \mu_T^W \bar{H}_T^{-v} \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)} \left(\frac{Y_T}{A_T}\right)^{1+\omega}}{\mathrm{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{\bar{C}_T}{C_t} (1-\tau) \left(\frac{P_T}{P_t}\right)^{\theta-1}}$$

where we have defined $(1+\omega) = \phi(1+v)$.

Next, we use the definition of the price-index to obtain

$$\frac{p_{k,t}}{P_t} = \left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}}$$

and write

$$\left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{1+\omega\theta}{1-\theta}} = \frac{\gamma\phi\frac{\theta}{\theta-1}\mathrm{E}_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\frac{\bar{C}_T}{\bar{C}_t}\mu_T^W\bar{H}_T^{-v}\left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)}\left(\frac{Y_T}{A_T}\right)^{1+\omega}}{\mathrm{E}_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\frac{\bar{C}_T}{\bar{C}_t}(1-\tau)\left(\frac{P_T}{P_t}\right)^{\theta-1}}$$

Detrending real variables and following standard derivations we can linearize this condition to obtain $\pi_t = \kappa(\tilde{y}_t - z_t) + \beta\mathrm{E}_t\widehat{\pi}_{t+1}$, for $\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{1+\omega}{1+\omega\theta}$. In the natural equilibrium we obtain $\tilde{y}_t^n = z_t$, so that the Phillips curve can be rewritten as

$$\pi_t = \kappa x_t + \beta\mathrm{E}_t\widehat{\pi}_{t+1}$$

for $x_t \equiv \tilde{y}_t - \tilde{y}_t^n$.

Moreover the detrended natural rate can be solved out explicitly as

$$\breve{r}_t^n = \mathrm{E}_t z_{t+1} - \frac{1}{1-\breve{\Delta}^m}z_t + \psi + \rho_\psi(\psi_t - \psi) - \rho_\delta\delta_t - \frac{\breve{\Delta}^m}{1-\breve{\Delta}^m}s_t$$

In the text we assume that $z_t = 0$ at all times and that $\rho_\psi = 0$ to obtain

$$\breve{r}_t^n = \psi - \rho_\delta\delta_t - \frac{\breve{\Delta}^m}{1-\breve{\Delta}^m}s_t$$

8.2 Second order welfare approximation

Household period utility can be rewritten as

$$U_t = \bar{C}_t\left[\ln\tilde{C}_t + \bar{a}_t + s_t v(0) - \frac{\gamma}{1+v}\int_0^1 H_{k,t}^{1+v}\mathrm{d}k\right]$$

where $\bar{a}_t = \log\bar{A}_t$ and where we used $\tilde{m}_{j,t} = \tilde{m}_t$. Using $H_{k,t} = \left(\frac{Y_{k,t}}{Z_t\bar{A}_t}\right)^{\phi}$ and the demand schedule $Y_{k,t} = \left(\frac{p_{k,t}}{P_t}\right)^{-\theta}Y_t$, we obtain

$$U_t = \bar{C}_t\left[\ln\tilde{Y}_t + \bar{a}_t + s_t v(0) - \frac{\gamma}{1+v}\left(\frac{\tilde{Y}_t}{Z_t}\right)^{\phi(1+v)}d_t\right]$$

for $d_t = \int_0^1 \left(\frac{p_{k,t}}{P_t}\right)^{-\theta(1+\omega)} \mathrm{d}k$. Note that $\bar{C}_t$, $\bar{a}_t$, s_t and $v(0)$ are independent of policy to write

$$U_t = \bar{C}_t \left[\ln \tilde{Y}_t - \frac{\gamma}{1+v} \left(\frac{\tilde{Y}_t}{Z_t}\right)^{\phi(1+v)} d_t \right] + t.i.p.$$

where $t.i.p.$ are terms independent of policy.

Expand to second order noting that $\bar{C} = 1$ and using also $1 + \omega = \phi(1 + v)$:

$$\begin{aligned} U_t - \left(\ln \tilde{Y} - \frac{\gamma}{1+v}\tilde{Y}^{1+\omega} d\right) &\simeq -\frac{1}{2}\gamma\phi^2(1+v)\tilde{Y}^{1+\omega} d\tilde{y}_t^2 - \frac{\gamma}{1+v}\tilde{Y}^{1+\omega} d\hat{d}_t \\ &+ \left(1 - \gamma\phi\tilde{Y}^{1+\omega} d\right)\tilde{y}_t + \left(1 - \gamma\phi\tilde{Y}^{1+\omega} d\right)\bar{c}_t\tilde{y}_t \\ &- \frac{1}{2}\frac{\gamma}{1+v}\tilde{Y}^{1+\omega} d\widehat{d}_t^2 - \frac{\gamma}{1+v}\tilde{Y}^{1+\omega} d\bar{c}_t\widehat{d}_t - \gamma\phi\tilde{Y}^{1+\omega} d\tilde{y}_t\widehat{d}_t \end{aligned}$$

where $\bar{c}_t = \delta_t + \bar{c}_{t-1}$.

The rest of the derivations are standard. Note that

$$d_t = (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{-\frac{\theta(1+\omega)}{1-\theta}} + \alpha\Pi_t^{\theta(1+\omega)} d_{t-1}$$

and that in a zero inflation steady state we obtain $d = 1$. Imposing the steady state subsidy $1 - \tau = \frac{\theta}{\theta-1}$ to ensure that steady state output $\tilde{Y} = \left(\frac{\gamma\phi}{1-\tau}\frac{\theta}{\theta-1}\right)^{-\frac{1}{1+\omega}}$ becomes efficient, so that $1 - \gamma\phi\tilde{Y}^{1+\omega} d = 0$. Temporary utility becomes

$$\begin{aligned} U_t + \frac{1+\ln(\gamma\phi)}{1+\omega} &\simeq -\frac{1}{2}(1+\omega)\tilde{y}_t^2 - \frac{1}{1+\omega}\hat{d}_t \\ &- \frac{1}{2}\frac{1}{1+\omega}\hat{d}_t^2 - \frac{1}{1+\omega}\bar{c}_t\hat{d}_t - \tilde{y}_t\hat{d}_t \end{aligned}$$

Finally, approximate d_t to second order around a zero inflation steady state to obtain

$$\hat{d}_t = \frac{1}{2}(1+\omega)(1+\theta\omega)\frac{\alpha\theta}{1-\alpha}\widehat{\pi}_t^2 + \alpha\hat{d}_{t-1}$$

and integrate this expression backward to the beginning of time t_0 to compute the discounted sum $\sum_{t=t_0}^{\infty}\beta^{t-t_0}\hat{d}_t$ as (assuming $\alpha\beta < 1$)

$$\sum_{t=t_0}^{\infty}\beta^{t-t_0}\hat{d}_t = (1+\omega)(1+\theta\omega)\frac{\alpha\theta}{1-\alpha}\frac{1}{1-\alpha\beta}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{\pi_t^2}{2} + t.i.p.$$

where $t.i.p.$ are proportional to $\widehat{d}_{t_0-1}$ which is independent of policy. It follows that $\hat{d}_t$ is of purely second order in inflation. As a result, the second order terms $\hat{d}_t$, $\bar{c}_t\hat{d}_t$ and $\tilde{y}_t\hat{d}_t$ are of higher order in inflation and we can write intertemporal utility as of the beginning of time t_0 as

$$\frac{1-\alpha}{\alpha\theta}\frac{1-\alpha\beta}{1+\theta\omega}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left[U_t + \frac{1+\ln(\gamma\phi)}{1+\omega}\right] \simeq \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(-\frac{(1-\alpha)(1-\alpha\beta)}{\alpha\theta}\frac{1+\omega}{1+\theta\omega}\tilde{y}_t^2 - \pi_t^2\right)$$

so that the period utility to maximise for the CB can be written as

$$U_t^{CB} = -\pi_t^2 - \lambda \widehat{y}_t^2$$

for $\lambda = \frac{(1-\alpha)(1-\beta\alpha)}{\alpha\theta} \frac{1+\omega}{1+\theta\omega}$.

8.3 Optimal steady state policy

Consider the steady state of our economy and assume purely deterministic growth, i.e. $\bar{A}_{t+1}/\bar{A}_t = \Psi$.

Note that in a steady state with generic (gross) inflation rate Π, steady state price dispersion is

$$d = \frac{1-\alpha}{1-\alpha\Pi^{\theta(1+\omega)}} \left(\frac{1-\alpha\Pi^{\theta-1}}{1-\alpha} \right)^{-\frac{\theta}{1-\theta}(1+\omega)},$$

the steady state nominal interest rate

$$I^m = (1-\Delta^m) \frac{\Psi}{\beta} \Pi$$

or, for a steady state natural rate $R^n = (1-\Delta^m) \frac{\Psi}{\beta}$,

$$I^m = R^n \Pi$$

and steady state output (including the subsidy $1-\tau = \frac{\theta}{\theta-1}$)

$$\tilde{Y} = \left(\frac{1-\alpha\Pi^{\theta-1}}{1-\alpha} \right)^{\frac{1+\omega\theta}{(1-\theta)(1+\omega)}} \left(\frac{1-\alpha\beta\Pi^{\theta-1}}{1-\alpha\beta\Pi^{\theta(1+\omega)}} \gamma\phi \right)^{-\frac{1}{1+\omega}}$$

These expressions can be used to evaluate steady state utility

$$U \propto \ln \tilde{Y} - \frac{\gamma}{1+v} \tilde{Y}^{(1+\omega)} d$$

More specifically, if we consider the limiting case $\beta \to 1$, we can choose steady state inflation Π to maximise the resulting utility

$$U = \frac{\theta}{1-\theta} \ln \left(1 - \alpha\Pi^{\theta-1} \right) + \frac{1}{1+\omega} \ln \left(1 - \alpha\Pi^{\theta(1+\omega)} \right) + t.i.p.$$

subject to the ZLB constraint $I^m \geq 1$ or, equivalently,

$$s.t. \qquad \Pi \geq \frac{1}{R^n}$$

The first order condition require, using $\pi = \log \Pi$ and $r^n = \log R^n$, either

$$\pi = 0, \qquad \text{if} \qquad r^n \geq 0$$

or

$$\pi = -r^n, \qquad \text{if} \qquad r^n < 0$$

8.4 Numerical methods

This section describes the numerical procedure by focusing on the model with optimal policy (sections 2.4 and 2.5). We apply the same method to solve the model with price level targeting (section 2.6).

8.4.1 Stationary natural rate and the ELB

Exogenous shocks

We assume that the rate of change in preferences δ_t follows an AR(1) process. As a result, the high frequency component of the natural rate, i.e., $-\mathrm{E}_t\delta_{t+1}$, follows an AR(1) process:

$$-\mathrm{E}_t\delta_{t+1} = \bar{\delta}_t = \rho_\delta\bar{\delta}_{t-1} + \sigma_\delta\varepsilon_{\delta,t}$$

which we calibrate as in Adam & Billi (2007). Productivity growth is constant, i.e., $\Xi_{t+1} = \frac{\bar{A}_{t+1}}{\bar{A}_t} = \Psi$, and we denote by ψ the log of productivity growth.

System of equilibrium equations

$$\pi_t = \kappa x_t + \beta \mathrm{E}_t\pi_{t+1} \tag{7}$$

$$x_t = \left(1-\breve{\Delta}^m\right)\left[\mathrm{E}_t x_{t+1} - (\breve{i}_t^m - \mathrm{E}_t\pi_{t+1} - \breve{r}_t^n)\right] \tag{8}$$

$$\breve{r}_t^n = \bar{\delta}_t \tag{9}$$

$$2\lambda x_t = -\lambda_{x,t} + \beta^{-1}\left(1-\breve{\Delta}^m\right)\lambda_{x,t-1} + \kappa\lambda_{p,t} \tag{10}$$

$$2\pi_t = \beta^{-1}\left(1-\breve{\Delta}^m\right)\lambda_{x,t-1} - \lambda_{p,t} + \lambda_{p,t-1} \tag{11}$$

$$\lambda_{x,t}(\breve{i}_t^m + \ln\left(1-\breve{\Delta}^m\right) + \psi - \ln\beta) = 0 \tag{12}$$

$$\breve{i}_t^m \geq \log(\beta) - \psi - \log\left(1-\breve{\Delta}^m\right) \tag{13}$$

$$\lambda_{x,t} \geq 0 \tag{14}$$

Solution algorithm

We use a projection approach. To discretize the state space $\mathcal{S} \subset R^3$, we form a grid defined by three N-vectors of evenly spaced points, namely λ_{p-1}, λ_{x-1}, and $\bar{\delta}$. The initial range of values considered for each state variable is $[-0.005, 0.005]$, $[0, 0.005]$, and +/- 5 unconditional standard deviations of $\bar{\delta}_t$ respectively. Then, we proceed iteratively: we solve the model, simulate it, update the boundaries for λ_{p-1} and the upper bound for λ_{x-1} so as to cover all possible values, and solve the model again until both the solution and the grid converge. In our application, we set N=50.

We use piecewise linear interpolation for approximating $\widetilde{x}(s)$ and $\pi(s)$ off the grid, where $s = (\lambda_{p,t-1}, \lambda_{x,t-1}, \bar{\delta}_t)$ denotes the vector of state variables at time t, and fixed-point iteration for solving the system on the grid.

Define $s_{+1} = (\lambda_{p,t}, \lambda_{x,t}, \breve{r}_{t+1}^n)$ the vector of state variables at time $t+1$, and $f^c(.)$ the local polynomial approximating the control variable $c \in \{\widetilde{x}, \pi\}$. Expectation terms are of the form: $\mathrm{E}_t[f^c(s_{+1})] = \int_{-\infty}^{\infty} g^c(\varepsilon_{\delta,t+1}) \exp(-\varepsilon_{\delta,t+1}^2) d\varepsilon_{\delta,t+1}$, which we approximate using a 9 node Gauss-Hermite (GH) quadrature.

The solution algorithm proceeds in three steps.

Step 1: Choose an initial $x_0(s)$ and $\pi_0(s)$, and a tolerance level τ

Step 2: Iteration j. For each possible state s, given $x_{j-1}(s)$ and $\pi_{j-1}(s)$, compute $\lambda_p(s)$ using (11), guess that $\lambda_x(s) = 0$, compute $\mathrm{E}x_{j-1}(s_{+1})$ and $\mathrm{E}\pi_{j-1}(s_{+1})$, and compute $\breve{\imath}^m(s)$ using (8).

If $\breve{\imath}^m(s) \geq \log(\beta) - \psi - \log\left(1 - \breve{\Delta}^m\right)$, retrieve

$$\begin{aligned}
\pi_j(s) &= \kappa x_{j-1}(s) + \beta \mathrm{E}\pi_{j-1}(s_{+1}) \\
x_j(s) &= \frac{1}{2\lambda}\Big[- \lambda_x(s) + \beta^{-1}\left(1 - \breve{\Delta}^m\right)\lambda_{x,t-1} + \kappa\lambda_p(s)\Big]
\end{aligned}$$

If $\breve{\imath}^m(s) < \log(\beta) - \psi - \log\left(1 - \breve{\Delta}^m\right)$, set $\breve{\imath}^m(s) = \log(\beta) - \psi - \log\left(1 - \breve{\Delta}^m\right)$, compute $\lambda_x(s)$ using (10), compute $\mathrm{E}x_{j-1}(s_{+1})$ and $\mathrm{E}\pi_{j-1}(s_{+1})$, and retrieve

$$\begin{aligned}
\pi_j(s) &= \kappa x_{j-1}(s) + \beta \mathrm{E}\pi_{j-1}(s_{+1}) \\
x_j(s) &= \left(1 - \breve{\Delta}^m\right)\left[\mathrm{E}x_{j-1}(s_{+1}) - \left(\breve{\imath}^m(s) - \mathrm{E}\pi_{j-1}(s_{+1}) - \bar{\delta}_t\right)\right]
\end{aligned}$$

Step 3: Let $e_j^\pi(s) = |\pi_j(s) - \pi_{j-1}(s)|$, $e_j^x(s) = |x_j(s) - x_{j-1}(s)|$ and $e_j(s) = e_j^\pi(s) + e_j^x(s)$ denote different measures of approximation error. Stop if $\sum_s e_j(s) < \tau$. Otherwise, update the guess, and repeat step 2.

Our solution requires some initial guess about the true solution to the model. We use the solution to the model in absence of the ELB as an initial guess to solve the model with the Adam & Billi (2007)'s calibration of the natural rate at the steady state. Then, we use this solution as an initial guess to solve the model with our baseline calibration. For lower values, we use the solution to the model with a slightly higher calibration.

Accuracy

To evaluate the accuracy of the solution, we simulate 10000 economies each 1000 periods long. We plug the solution $\left(x^*(s_t), \pi^*(s_t)\right)$ in equations (1) and (3) (if the ELB does not bind), and in equations (1) and (2) (if the ELB does bind), so as to retrieve values for the output gap and the (log) inflation rate implied by these equations $\left(x^{\mathrm{IMP}}(s_t), \pi^{\mathrm{IMP}}(s_t)\right)$. Then, we measure approximation errors either as the residual or as the percentage residual of these equations:

$$e_{1,t}^{x} \equiv \left| x_t^{\text{IMP}} - x_t^* \right| \cdot 100 \tag{15}$$

$$e_{2,t}^{x} \equiv \left| \exp\left(x_t^{\text{IMP}} - x_t^* \right) - 1 \right| \cdot 100 = \left| \frac{\widetilde{Y}_t^{\text{IMP}} - \widetilde{Y}_t^*}{\widetilde{Y}_t^*} \right| \cdot 100 \tag{16}$$

$$e_{1,t}^{\pi} \equiv \left| \pi_t^{\text{IMP}} - \pi_t^* \right| \cdot 400 \tag{17}$$

$$e_{2,t}^{\pi} \equiv \left| \exp\left(4(\pi_t^{\text{IMP}} - \pi_t^*) \right) - 1 \right| \cdot 100 = \left| \frac{\left(\Pi_t^{\text{IMP}}\right)^4 - \left(\Pi_t^*\right)^4}{\left(\Pi_t^*\right)^4} \right| \cdot 100 \tag{18}$$

For each calibration of the steady state value of the natural rate, we report the maximum and the mean approximation errors in columns 4 to 11 of table 4.

Table 4: Stationary natural rate: simulation moments and accuracy indicators

r^n	$\breve{x}$	π	$\max[e_1^x]$	$\mathrm{E}[e_1^x]$	$\max[e_1^\pi]$	$\mathrm{E}[e_1^\pi]$	$\max[e_2^x]$	$\mathrm{E}[e_2^x]$	$\max[e_2^\pi]$	$\mathrm{E}[e_2^\pi]$
3,498	0	0,001	0,063	0	0,007	0	0,063	0	0,007	0
2,998	0	0,002	0,085	0,001	0,01	0	0,085	0,001	0,01	0
2,498	0	0,004	0,116	0,002	0,013	0	0,116	0,002	0,013	0
1,998	0	0,011	0,159	0,004	0,017	0	0,159	0,004	0,017	0
1,498	-0,001	0,025	0,226	0,007	0,023	0,001	0,226	0,007	0,023	0,001
0,998	0	0,058	0,316	0,011	0,03	0,001	0,316	0,011	0,03	0,001
0,498	0,001	0,133	0,451	0,012	0,038	0,001	0,452	0,012	0,038	0,001
-0,002	0,006	0,293	0,621	0,01	0,046	0,001	0,623	0,011	0,046	0,001
0,997	0,002	0,266	0,896	0,025	0,075	0,002	0,9	0,025	0,075	0,002

This table reports simulation moments (annualized and in percent) along with accuracy indicators for each calibration of the steady state value of the natural rate. The last row contains the results for an economy with a steady state value of the natural rate at 1%, and a standard deviation of shocks two times larger than the baseline calibration.

The approximation errors in difference or in percentage difference generate very similar results. The maximum approximation error for the output gap and the inflation rate is always below 1% and 10 basis points respectively. The mean approximation error for the inflation rate is negligeable. However, for lower values of the natural rate, the mean approximation error for the output gap is one order of magnitude larger than the mean of this variable.

Robustness

We test the robustness of the results in two ways. On the one hand, in the spirit of Maliar & Maliar (2015), we use an adaptative grid. For each state variable except $\lambda_{x,t-1}$, we place relatively more points in the middle 95% of the distribution of this variable. For $\lambda_{x,t-1}$, given that the distribution is truncated in and concentrated near zero, we place the points using a multiplicative sequence of the form: $\lambda_{x,k} = \frac{\lambda_{x,k-1}}{1-\delta}$ with $0 < \delta < 1$. Figure 9 provides an illustrative exemple of the grid. Table 5 compares the accuracy of the solution when using the adaptative grid instead of the evenly spaced grid.

Figure 9: An adaptative grid

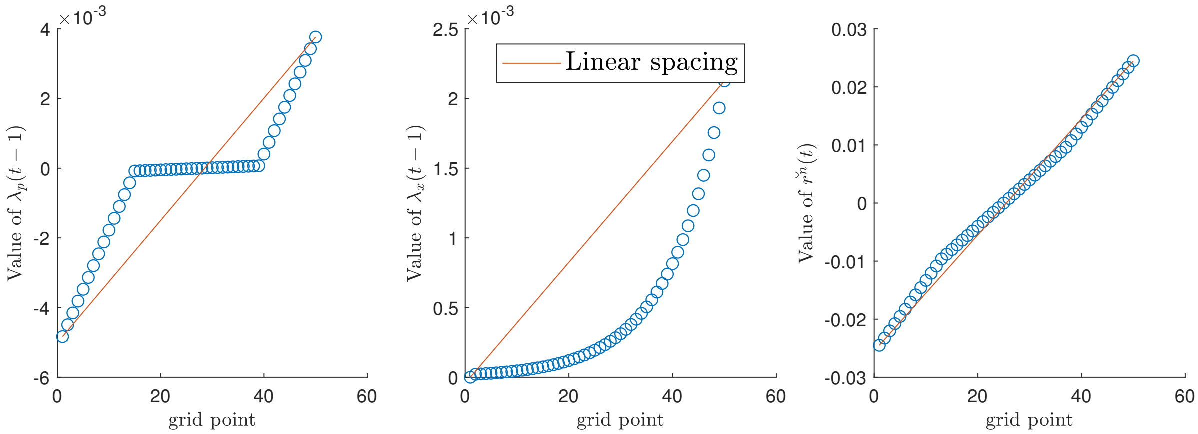

Table 5: Stationary natural rate: evenly spaced vs. adaptative grid

	Evenly spaced grid				Adaptative grid			
r^n	$\max[e_1^x]$	$\mathrm{E}[e_1^x]$	$\max[e_1^\pi]$	$\mathrm{E}[e_1^\pi]$	$\max[e_1^x]$	$\mathrm{E}[e_1^x]$	$\max[e_1^\pi]$	$\mathrm{E}[e_1^\pi]$
3,498	0,063	0	0,007	0	0,033	0	0,005	0
2,998	0,085	0,001	0,01	0	0,041	0	0,005	0
2,498	0,116	0,002	0,013	0	0,055	0,001	0,006	0
1,998	0,159	0,004	0,017	0	0,076	0,001	0,008	0
1,498	0,226	0,007	0,023	0,001	0,106	0,002	0,011	0
0,998	0,316	0,011	0,03	0,001	0,146	0,003	0,014	0
0,498	0,451	0,012	0,038	0,001	0,203	0,003	0,018	0
-0,002	0,621	0,01	0,046	0,001	0,276	0,003	0,022	0
0,997	0,896	0,025	0,075	0,002	0,404	0,006	0,036	0,001

We find that approximation errors diminish significantly when using an adaptative grid. For example, the mean approximation error for the output gap is always below one basis point. But the simulation moments are essentially unchanged (Table 6). From this, we conclude that using a denser grid would certainty improve the accuracy of the solution, but it would not substantially affect the results that are reported in the main text.

On the other hand, we increase the number of GH quadrature nodes from 9 to 30. The results are essentially unchanged.

Table 6: Stationary natural rate: robustness test

	Evenly spaced grid						Adaptative grid					
r^n	$\breve{x}$	π	i	$r-r^n$	ELB freq	ELB dur	$\breve{x}$	π	i	$r-r^n$	ELB freq	ELB dur
3,498	0	0,001	3,498	-0,001	5,515	2,635	0	0,001	3,499	-0,001	5,568	2,708
2,998	0	0,002	2,998	-0,002	9,622	3,141	0	0,002	2,999	-0,001	9,895	3,148
2,498	0	0,004	2,5	-0,003	16,703	3,625	0	0,004	2,501	-0,002	16,778	3,725
1,998	0	0,011	2,004	-0,005	25,722	4,709	0	0,01	2,006	-0,002	26,548	4,653
1,498	-0,001	0,025	1,514	-0,009	39,032	6,098	0,001	0,024	1,519	-0,003	39,034	6,316
0,998	0	0,058	1,044	-0,013	54,463	8,953	0,001	0,056	1,05	-0,004	54,749	9,091
0,498	0,001	0,133	0,618	-0,014	71,885	14,36	0,003	0,13	0,624	-0,004	72,174	14,444
-0,002	0,006	0,293	0,28	-0,012	87,069	28,84	0,007	0,289	0,284	-0,004	87,191	29,36
0,997	0,002	0,266	1,236	-0,028	71,835	14,37	0,006	0,26	1,248	-0,009	72,098	14,484

This table reports simulation moments (annualized and in percent) for each calibration of the steady state value of the natural rate. The last row contains the results for an economy with a steady state value of the natural rate at 1%, and a standard deviation of shocks two times larger than the baseline calibration. The ELB frequency is displayed in percents. The ELB duration corresponds to the average duration of an ELB episode (in quarters).

8.4.2 Drifting natural rate and the ELB

Exogenous shocks

We assume that the rate of change in productivity is integrated and bounded, i.e., $\log(\Xi_{t+1}) = \xi_{t+1} = \psi_{t+1} + \xi_t$ and

$$\xi_{t+1} \in [\xi_L, \xi_H]$$
$$\psi_{t+1} = \sigma_\psi \varepsilon_{\psi,t+1}$$

where ψ_{t+1} denotes the rate of change in productivity growth, and $\varepsilon_{\psi,t+1}$ denotes a realization of the truncated standard normal distribution between $\varepsilon_{\psi,L}(t) = \frac{\xi_L - \xi_t}{\sigma_\psi}$ and $\varepsilon_{\psi,H}(t) = \frac{\xi_H - \xi_t}{\sigma_\psi}$.

System of equilibrium equations

The set of equilibrium conditions includes equations 7 to 8, 10 to 11, and 19 to 22.

$$\breve{r}_t^n = \bar{\delta}_t + \mathrm{E}_t(\psi_{t+1}) \tag{19}$$

$$\mathrm{E}_t(\psi_{t+1}) = \sigma_\psi \frac{\phi(\varepsilon_{\psi,L}(t) - \phi(\varepsilon_{\psi,H}(t)}{\Phi(\varepsilon_{\psi,H}(t) - \Phi(\varepsilon_{\psi,L}(t)} \tag{20}$$

$$\lambda_{x,t}(\breve{\imath}_t^m + \ln\left(1 - \breve{\Delta}^m\right) + \xi_t - \ln\beta) = 0 \tag{21}$$

$$\breve{\imath}_t^m \geq \log(\beta) - \xi_t - \log\left(1 - \breve{\Delta}^m\right) \tag{22}$$

where $\phi(.)$ and $\Phi(.)$ denote the pdf and the cdf of the standard normal distribution respectively.

Solution algorithm

The main change with respect to the procedure described above is threefold. First, there is an additional (exogenous) state variable. We use a grid defined by four N-vectors of evenly spaced points including the rate of productivity growth ξ. The range of values considered for ξ is $[\xi_L, \xi_H]$. Moreover, we set N=40.

Second, we use a combination of Gaussian quadratures to approximate expectation terms. Define $s_{+1} = (\lambda_{p,t}, \lambda_{x,t}, \bar{\delta}_{t+1}, \xi_{t+1})$ the vector of state variables at time t+1. We use the equivalence $\varepsilon_{\psi,t+1} = \frac{\varepsilon_{\psi,H}(t)-\varepsilon_{\psi,L}(t)}{2} y_{t+1} + \frac{\varepsilon_{\psi,H}(t)+\varepsilon_{\psi,L}(t)}{2}$, where y_{t+1} denotes a realization of the truncated standard normal between -1 and 1, to express expectation terms as follows:

$$\mathrm{E}_t[f^c(s_{+1})] = \int_{-\infty}^{\infty} \Big[\int_{-1}^{1} g^c(\varepsilon_{\delta,t+1}, y_{t+1}) dy_{t+1} \Big] \exp(-\varepsilon^2_{\delta,t+1}) d\varepsilon_{\delta,t+1} \tag{23}$$

Then, we use a 20 node Gauss-Legendre (GL) quadrature to approximate the integral in square brackets, and a 9 node Gauss-Hermite quadrature to approximate the first integral.

Third, we use parallel computing to solve the model under different calibrations all in the same time. We proceed in two steps. In the first step, we solve the model under different calibrations of the average value of the natural rate by using fewer grid points (15^4). The running time is about 3 hours. In the second step, we use these solutions as initial guesses to solve each version independently by using more grid points (40^4). All together, these two steps require 52 hours.

Accuracy

Table 7 reports simulation moments over calibrations along with the different measures of accuracy (15)-(18). The maximum approximation error for the output gap and the inflation rate reaches 2% and 28 basis points respectively. The mean approximation error for the inflation rate reaches 1.2 basis points which is low compared to the mean of this variable. However, the mean approximation error for the output gap reaches 3.1 basis points which amounts to about half the mean of this variable in absolute value.

Table 7: Drifting natural rate: simulation moments and accuracy indicators

r^n	$\breve{x}$	π	max$[e_1^x]$	E$[e_1^x]$	max$[e_1^\pi]$	E$[e_1^\pi]$	max$[e_2^x]$	E$[e_2^x]$	max$[e_2^\pi]$	E$[e_2^\pi]$
3,519	-0,001	0,003	0,206	0,003	0,059	0	0,206	0,003	0,059	0
3,019	-0,002	0,008	0,288	0,005	0,071	0,001	0,289	0,005	0,071	0,001
2,519	-0,003	0,018	0,404	0,009	0,093	0,001	0,404	0,009	0,093	0,001
2,019	-0,005	0,04	0,564	0,013	0,121	0,002	0,566	0,013	0,12	0,002
1,519	-0,009	0,088	0,742	0,019	0,154	0,003	0,744	0,019	0,154	0,003
1,019	-0,017	0,18	0,974	0,024	0,192	0,005	0,978	0,024	0,192	0,005
0,519	-0,034	0,333	1,247	0,028	0,237	0,007	1,255	0,028	0,236	0,007
0,019	-0,067	0,561	1,933	0,031	0,283	0,012	1,952	0,031	0,282	0,012

See table 4 for details.

Robustness

We test the robustness of the results in several ways. First, we solve the model by using the adaptative grid described before. As the unconditional distribution of the rate of productivity growth ξ_t looks like a uniform distribution, we keep a vector of N evenly spaced points for this dimension. Table 8 compares the accuracy of the solution when using the adaptative grid instead of the evenly spaced grid. In this case as well, we observe that the adaptative grid is an efficient way of reducing approximation errors. For example, the mean approximation error for the output gap is at least reduced by half.

Table 8: Drifting natural rate: evenly spaced vs. adaptative grid

	Evenly spaced grid				Adaptative grid			
r^n	$\max[e_1^x]$	$\mathrm{E}[e_1^x]$	$\max[e_1^\pi]$	$\mathrm{E}[e_1^\pi]$	$\max[e_1^x]$	$\mathrm{E}[e_1^x]$	$\max[e_1^\pi]$	$\mathrm{E}[e_1^\pi]$
3,519	0,206	0,003	0,059	0	0,112	0,001	0,058	0
3,019	0,288	0,005	0,071	0,001	0,142	0,001	0,071	0
2,519	0,404	0,009	0,093	0,001	0,167	0,003	0,093	0,001
2,019	0,564	0,013	0,121	0,002	0,199	0,004	0,12	0,001
1,519	0,742	0,019	0,154	0,003	0,283	0,007	0,154	0,002
1,019	0,974	0,024	0,192	0,005	0,365	0,01	0,192	0,004
0,519	1,247	0,028	0,237	0,007	0,461	0,013	0,236	0,006
0,019	1,933	0,031	0,283	0,012	1,487	0,017	0,283	0,011

Table 9 compares the simulation moments. Even though the solution is significantly more accurate when using the adaptative grid, the simulation moments do not change substantially, which supports the conclusion that the results are robust and that using a denser grid would be pointless.

Table 9: Drifting natural rate: robustness test

	Evenly spaced grid						Adaptative grid					
r^n	$\breve{x}$	π	i	$r-r^n$	ELB freq	ELB dur	$\breve{x}$	π	i	$r-r^n$	ELB freq	ELB dur
3,519	-0,001	0,003	3,519	-0,003	9,577	3,768	0	0,003	3,521	-0,001	9,766	3,836
3,019	-0,002	0,008	3,022	-0,005	15,16	4,55	0	0,007	3,024	-0,002	15,385	4,625
2,519	-0,003	0,018	2,528	-0,008	22,766	5,684	-0,001	0,017	2,533	-0,003	23,042	5,725
2,019	-0,005	0,04	2,047	-0,013	32,562	7,265	-0,002	0,038	2,053	-0,004	32,78	7,285
1,519	-0,009	0,088	1,589	-0,019	43,902	9,597	-0,005	0,085	1,597	-0,008	44,042	9,574
1,019	-0,017	0,18	1,174	-0,028	55,811	12,92	-0,012	0,175	1,183	-0,014	55,931	12,721
0,519	-0,034	0,333	0,818	-0,042	67,402	17,138	-0,028	0,327	0,828	-0,026	67,377	16,853
0,019	-0,067	0,561	0,527	-0,07	77,831	22,596	-0,062	0,554	0,536	-0,053	77,73	22,141

See table 6 for details.

Second, we increase the number of GL quadrature nodes from 20 to 30. NOT DONE.

Third, instead of using a Gauss-Legendre quadrature to compute the bounded integral in square brackets in (23), we use a Newton-Cotes quadrature scheme. In particular, we use the trapezoid rule which is described in Miranda & Fackler (2002) as "more accurate [than other Newton-Cotes methods] if the integrand exhibits discontinuities in its first derivative". NOT DONE.

8.5 Solution of the linear model with optimal commitment

This appendix presents the analytical solution of the model with optimal commitment in absence of the ELB constraint[2].

The system of equilibrium equations becomes

$$\begin{aligned}
\pi_t &= \kappa x_t + \beta \mathrm{E}_t \pi_{t+1} \\
2\lambda x_t &= \kappa \lambda_{p,t} \\
2\pi_t &= -\lambda_{p,t} + \lambda_{p,t-1} \\
x_t &= \left(1 - \breve{\Delta}^m\right) \left[\mathrm{E}_t x_{t+1} - (\breve{i}_t^m - \mathrm{E}_t \pi_{t+1} - \breve{r}_t^n)\right]
\end{aligned}$$

[2]See Woodford Chapter 7 for a comprehensive discussion.

Rearranging the second and third equations, we obtain $x_t = \frac{\kappa}{2\lambda}\lambda_{p,t}$ and $\pi_t = -\frac{1}{2}\lambda_{p,t} + \frac{1}{2}\lambda_{p,t-1}$. Plugging this into the Phillips curve, we obtain a difference equation for the evolution of the lagrange multiplier

$$\beta \mathrm{E}_t \lambda_{p,t+1} - (1 + \beta + \frac{\kappa^2}{\lambda})\lambda_{p,t} + \lambda_{p,t-1} = 0$$

Using the forward operator L^{-1}, this is equivalent to

$$\mathrm{E}_t[\mathcal{P}(L^{-1})\lambda_{p,t-1}] = 0$$

with $\mathcal{P}(L^{-1}) = \beta(L^{-1})^2 - (1 + \beta + \frac{\kappa^2}{\lambda})L^{-1} + 1$. Let μ_1 and μ_2 denote the roots of this second order polynomial. By identification of $\mathcal{P}(L^{-1})$ with $\mathcal{P}(L^{-1}) = \beta(L^{-1} - \mu_1)(L^{-1} - \mu_2)$, we obtain

$$\beta\mu_1\mu_2 = 1 > 0$$
$$\beta(\mu_1 + \mu_2) = 1 + \beta + \frac{\kappa^2}{\lambda} > 0$$

Moreover

$$\beta(1 - \mu_1)(1 - \mu_2) = -\frac{\kappa^2}{\lambda} < 0$$

The polynomial has two positive and real roots, with one inside and the other outside the unit circle, $0 < \mu_1 < 1 < \mu_2$. The difference equation has only one bounded solution. Rearranging $\mathrm{E}[\beta(L^{-1} - \mu_1)(L^{-1} - \mu_2)\lambda_{p,t-1}] = 0$, we obtain

$$-\mathrm{E}[\beta(1 - \mu_1 L)\mu_2(1 - \mu_2^{-1}L^{-1})\lambda_{p,t}] = 0$$

which is equivalent to $\lambda_{p,t} = \mu_1\lambda_{p,t-1}$. Using the initial condition $\lambda_{p,0} = 0$, the only solution satisfying this equation at any point in time is $\lambda_{p,t} = 0$. This shows that in absence of the ELB constraint, $\widetilde{x}_t = \pi_t = 0 \ \forall$ t.

8.6 The optimal gap adjusted price level target

This appendix derives the expression of the optimal gap adjusted price level target in presence of the ELB constraint on the nominal interest rate[3]. Optimal monetary policy is determined by the following first order conditions.

$$2\lambda x_t = -\lambda_{x,t} + \beta^{-1}\left(1 - \breve{\Delta}^m\right)\lambda_{x,t-1} + \kappa\lambda_{p,t}$$
$$2\pi_t = (\beta)^{-1}\left(1 - \breve{\Delta}^m\right)\lambda_{x,t-1} - \lambda_{p,t} + \lambda_{p,t-1}$$

[3]See Eggertsson and Woodford 2003.

Rearranging, we find

$$\lambda_{p,t} = 2\frac{\lambda}{\kappa}x_t + \frac{1}{\kappa}\lambda_{x,t} - \frac{\beta^{-1}\left(1-\breve{\Delta}^m\right)}{\kappa}\lambda_{x,t-1} \tag{24}$$

$$\pi_t = \frac{\beta^{-1}\left(1-\breve{\Delta}^m\right)}{2}(1+\frac{1}{\kappa})\lambda_{x,t-1} - \frac{\lambda}{\kappa}x_t - \frac{1}{\kappa 2}\lambda_{x,t} + \frac{1}{2}\lambda_{p,t-1} \tag{25}$$

Define the gap adjusted price level (GAPL) $\tilde{P}_t \equiv p_t + \frac{\lambda}{\kappa}x_t$. If the ELB constraint does not bind, $\lambda_{x,t} = 0$. Plugging this into (25), we obtain $\tilde{P}_t = P_t^*$ with

$$P_t^* = p_{t-1} + \frac{\beta^{-1}\left(1-\breve{\Delta}^m\right)}{2}(1+\frac{1}{\kappa})\lambda_{x,t-1} + \frac{1}{2}\lambda_{p,t-1} \tag{26}$$

If the ELB constraint does bind, $\lambda_{x,t} > 0$, and $\tilde{P}_t = P_t^* - \frac{1}{\kappa 2}\lambda_{x,t}$. Define $\Delta_t^{\tilde{P}} \equiv P_t^* - \tilde{P}_t$, we thus have $\Delta_t^{\tilde{P}} = \frac{1}{\kappa 2}\lambda_{x,t}$. Plugging this into (26), we obtain

$$P_t^* = p_{t-1} + \beta^{-1}\left(1-\breve{\Delta}^m\right)(1+\kappa)\Delta_{t-1}^{\tilde{P}} + \frac{1}{2}\lambda_{p,t-1} \tag{27}$$

Replacing (24) into (27), we obtain

$$P_t^* = \tilde{P}_{t-1} + [1+\beta^{-1}\left(1-\breve{\Delta}^m\right)(1+\kappa)]\Delta_{t-1}^{\tilde{P}} - \beta^{-1}\left(1-\breve{\Delta}^m\right)\Delta_{t-2}^{\tilde{P}}$$

and using $\tilde{P}_{t-1} + \Delta_{t-1}^{\tilde{P}} = P_{t-1}^*$, we obtain the expression of the optimal gap adjusted price level target

$$P_t^* = P_{t-1}^* + \beta^{-1}\left(1-\breve{\Delta}^m\right)(1+\kappa)\Delta_{t-1}^{\tilde{P}} - \beta^{-1}\left(1-\breve{\Delta}^m\right)\Delta_{t-2}^{\tilde{P}}$$

8.7 Solution of the linear model with constant price level targeting

This appendix presents the analytical solution of the model with constant (gap adjusted) price level targeting in absence of the ELB constraint on the nominal interest rate. The equilibrium is determined by the following system of equations.

$$\begin{aligned}
&p_t - p_{t-1} = \kappa x_t + \beta(\mathrm{E}_t p_{t+1} - p_t) \\
&x_t = \left(1-\breve{\Delta}^m\right)[\mathrm{E}_t x_{t+1} - (\breve{i}_t^m - (\mathrm{E}_t p_{t+1} - p_t) - \breve{r}_t^n)] \\
&p_t + \frac{\lambda}{\kappa}x_t = P*
\end{aligned}$$

Combining the first and the third equations, we obtain a difference equation for the evolution of the price level.

$$\beta \mathrm{E}_t p_{t+1} - (1+\beta+\frac{\kappa^2}{\lambda})p_t + p_{t-1} = -\frac{\kappa^2}{\lambda}P*$$

Or equivalently

$$\mathrm{E}_t[\mathcal{P}(L^{-1})p_{t-1}] = -\frac{\kappa^2}{\lambda}P* \tag{28}$$

with $\mathcal{P}(L^{-1}) = \beta(L^{-1})^2 - (1+\beta+\frac{\kappa^2}{\lambda})L^{-1} + 1$. This polynomial has two real and positive roots μ_1 and μ_2 with $0 < \mu_1 < 1 < \mu_2$. The proof is provided in appendix 8.5. Rearranging (28), we obtain

$$p_t = \mu_1 p_{t-1} + \frac{1}{\beta\mu_2(1-\mu_2^{-1})}\frac{\kappa^2}{\lambda}P*$$

Using $\frac{1}{\beta\mu_2(1-\mu_2^{-1})} = \frac{1}{\beta(\mu_2-1)} = -\frac{1}{\beta(1-\mu_2)}$ and the fact that $\beta(1-\mu_1)(1-\mu_2) = -\frac{\kappa^2}{\lambda}$, we obtain

$$p_t = \mu_1 p_{t-1} + (1-\mu_1)P* \tag{29}$$

Using the initial condition $p_{-1} = P*$, the only solution satisfying this equation at any point in time is $p_t = P*$. This shows that in absence of the ELB constraint on the nominal interest rate, $\widetilde{x}_t = \pi_t = 0$ for all t.