Repeated Bargaining with Signals

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This paper analyzes repeated bargaining in a market where a buyer purchases goods from different sellers over time. A buyer with private information about his purchasing power meets sellers sequentially. Before setting the price, each seller has noisy information about the previous purchases of the buyer (e.g., internet cookies). We show that reductions in the tracking accuracy (e.g., privacy regulations) imply both less bargains and lower but more likely accepted normal prices. The informativeness of cookies tends to affect the buyers' surplus and market efficiency non-monotonically: while de-regulating cookies increases trade and lowers prices when they are not precise to begin with, the opposite may occur if cookies are already precise.

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1 Introduction

The presence of cookies affects the tradeoffs faced by internet users.[1] For example, cookies and other internet tracking systems may make users cautious on which adds to click on or which goods to purchase, anticipating that their browsing history will affect the behavior of future sellers. Sellers, in turn, can use the information contained in the user's history to infer his purchasing power and tailor the price or goods they offer. A similar problem is faced by firms and government agencies when they purchase goods and services from different providers: each provider can condition its offer on the information available about previous transactions.

Focussing on internet markets, the information that a tracking system provides is determined in two steps. First, the informativeness of each interaction depends on the price set by the seller and the strategy the user uses to decide to accept it or not. While the acceptance of a low price may not be informative, accepting a high price may indicate high purchasing power. Second, each interaction produces a "signal" stored in the cookie, which is piece of information about the price offered by the seller and the acceptance decision of the user. The signal informativeness depends on the precision of the tracking system. Changes in the precision of the tracking system then affect the users' and sellers' incentives and strategies, and together they determine the information available and the transaction prices and probabilities.

This paper studies how changes in the information precision affect the efficiency and welfare in markets with multiple purchases. We find that regulations that decrease the informativeness of cookies—or permit more opacity in previous transactions—are partially counterbalanced by more informative equilibrium behavior. Less informative cookies imply less frequent discounts, but also that prices intended for high purchasing power buyers are lower and accepted more often. We argue that, while banning cookies—or allowing complete opaqueness of procurement contracts—lowers social welfare, partially limiting the amount of information available to future sellers may improve buyer welfare and increase the market efficiency. Our analysis sheds light, from a game-theoretical perspective, on possible implications of legislation initiatives aiming at regulating the use of personal data for pricing purposes, or transparency laws that require government agencies to release information about previous transactions.[2]

[1] Cookies are small text files located in browser directories, created by the websites a user visits and by the ads and widgets run on these websites. Cookies help developers make navigation through their websites more efficient but also track the user's online activity. The information collected is often sold to third parties, who then tailor ads and price offers to each individual consumer.

[2] The pioneer regulation of cookies and internet privacy was the 2018 European Union's *General Data*

Public-offers case: We present and analyze a continuous-time model of repeated trade with imperfect information about previous sales. A long-lived buyer sequentially meets short-lived sellers, each offering a different good. The buyer has a permanent type/valuation, also interpreted as his purchasing power, which can be either low (ℓ) or high (h), with $0 < \ell < h$. In our base model we consider the public-offers case: a cookie is a history of noisy signals about each of the previous purchasing decisions and also containing previous price offers. At each time t, after observing the cookie, the t-seller offers a price to the buyer, and the buyer decides to accept it or not.

We show that there is a unique Markov perfect equilibrium where prices below ℓ are accepted for sure, with state variable equal to the posterior $\phi \in [0, 1]$ about the valuation of the buyer being high. When the posterior about the buyer having a high valuation is below the threshold $\phi^* \equiv \ell/h$, all sellers set price ℓ, which is accepted by both types of the buyer, and there is no learning in equilibrium.[3] In this region, the h-buyer would accept any price below h, but sellers are pessimistic enough that they prefer offering ℓ and ensuring trade.

When the posterior is above ϕ^*, prices above ℓ must be offered with positive probability. Intuitively, if a seller offers a price higher than ℓ but lower than h, sure rejection is not possible in equilibrium: otherwise, the acceptance signal would be deemed as totally uninformative, but then the h-buyer would have the incentive to accept the offer as to get a positive surplus without changing the continuation play. Similarly, if ℓ was supposed to be offered in a region of posteriors, the continuation value of the buyer would be flat in this region. Then, a seller would benefit from offering a price close to h, which would be accepted for sure by the h-buyer.[4]

We argue that three properties should be satisfied when the posterior is close but above ϕ^*. First, by the previous argument, sellers should be willing to offer a high price weakly higher than ℓ/ϕ, which is close to h. Second, the h-buyer should be willing to accept such a high price, even though doing so gives him a small surplus.

Protection Regulation (GDPR), commonly referred to as "Cookie Law." Afterwards, other countries and states, such as India, Australia, and California, have established their own regulations on cookies. Conversely, government purchases are often subject to "transparency" regulations. For instance, transparency and openness of decision-making procedures are seen as foundational values of the EU, with several articles in the *Treaty on the Functioning of the European Union* (TFEU) on them.

[3] In a one-shot version of our model, the seller offers ℓ if the belief about the buyer's valuation being h is below ϕ^*, and h if such belief is above ϕ^*.

[4] Such arguments illustrate how the noise in the information about previous purchasing decisions severely limits the usual equilibrium multiplicity generated by "belief threats", since deviations by the buyer are not perfectly observed.

Third, the continuation value of the h-buyer should be high, since rejecting offers drives the posterior to ϕ^*, where all offers are equal to ℓ. A *discount mechanism* ensures that such conditions hold: in equilibrium, sellers offer the discounted price ℓ with a high probability, which is accepted by both types of the buyer, and otherwise the high price ℓ/ϕ, which is accepted for sure by the h-buyer. Such discount mechanism slows learning and flattens the continuation value of the h-buyer, providing him with the incentive to accept offers which are highly informative and give him little surplus.

When the posterior is high, convincing future sellers that of a low purchasing power is costly, as it requires rejecting offers for a long time. As a result, in this region, sellers only offer high prices, which are accepted by the h-buyer because the signaling value from rejecting offers is small. The equilibrium high-price offers are then V-shaped in the posterior belief of the sellers: high when the posterior is low or high, and low in between.

If the signal is informative enough, a *rejection mechanism* arises for intermediate posteriors: sellers offer a price that the h-buyer rejects with positive probability. Such equilibrium rejection is necessary to slow learning (note that it makes the acceptance signal less informative), giving the buyer the incentive to accept high prices. The equilibrium rejection of high prices slows learning for intermediate beliefs, reducing the gain the h-buyer obtains from mimicking the ℓ-buyer.

Private-offers case: We compare our results with the "private offers case", where cookies do not carry information about the prices offered in the past. As in the public offers case, we find that all sellers offer ℓ and the signal is uninformative when they are pessimistic enough about the buyer's valuation. Also, as before, sellers randomize between low and high price offers for posteriors above but close to ϕ^*. Now, such randomization makes the signal less informative—since both types of the buyer accept offers with a positive probability—, giving the h-buyer the incentive to accept high prices. Again, for high posteriors, sellers only offer high prices. The fact that deviations by sellers are unobservable implies that all equilibrium offers are accepted for sure by the h-buyer, hence the rejection mechanism does not occur in the private offers case.

Effect of policies limiting information: We study the welfare effects of policies affecting the information available to sellers. In internet markets, new regulations tend to limit the amount of information available to online sellers to protect consumers' privacy rights (see footnote 2). In the case of public procurement, transparency policies tend to make information about previous transactions more accessible to prevent wrongdoing and make such markets more accessible to new sellers.

First, we study the effect of policies reducing the precision of the acceptance signal in our base model. When the signal precision is low to begin with, we show that the buyer is worse off from further reductions in the signal precision. Intuitively, if sellers can observe previous purchasing decisions less precisely, the buyer's incentive to reject a high price is lower. Such effect weakens the discount mechanism, as sellers are less willing to offer ℓ to ensure trade. When the signal precision is high, reducing the signal informativeness weakens both discount and rejection mechanisms. A less informative signal implies that high offers can be accepted with a higher probability, allowing the buyer to signal faster a low valuation by rejecting offers, therefore limiting the sellers' capacity to extract surplus. We obtain that the weakening of the rejection mechanism dominates for some parameters, in which case the buyer benefits from a reduction in the signal informativeness.

Second, we analyze the effect of making price offers private. If the signal precision is low, both buyer surplus and social welfare are larger when prices are observable to sellers than when they are not. The reason is that the equilibrium information carried by the signal is higher if sellers know the actual offer received by the buyer. While acceptance of a high offer is very informative when such an offer is observable, signals about acceptance of low and high offers are mixed when offers are unobservable. When the information about previous purchasing decisions is precise, the previous result may be reversed. In this case, the rejection mechanism keeps the posterior high for a longer time when offers are observable, lowering the h-buyer's value from mimicking the ℓ-buyer. Additionally, since the h-buyer trades less often when prices are observable, the market efficiency may be lower as well.

1.1 Literature review

Most of the bargaining literature with private information studies the purchase of a single good by a buyer facing one or more sellers. Its most influential result is the so-called Coase conjecture (Coase, 1972, see Gul, Sonnenschein, and Wilson, 1986, for a formal proof) that the price offered by a monopolist is competitive when offers are frequent. Kaya and Liu (2015) verify that this result extends to the case where the buyer receives offers from a sequence of short-lived sellers, independently of the observability of the previous price offers. Other papers have shown that the Coase conjecture fails under different assumptions, like for example the presence of adverse selection (Deneckere and Liang, 2006, Hörner and Vieille, 2009, and Daley and Green, 2020), capacity choice (McAfee and Wiseman, 2008), or outside options (Board and Pycia, 2014).

The literature on repeated bargaining is more limited. This is, in part, because previous acceptances and rejections generate a repeated signaling game, complicating the analysis and typically generating a large equilibrium multiplicity. Hart and Tirole (1988) show that Coasian forces imply the existence of an equilibrium where prices are equal to the lowest valuation of the buyer at all times, as the acceptance of a higher offer results in a permanent price increase. Kaya and Roy (2020) show that, in the presence of adverse selection, an upper bound of the buyer's payoff when offers are private is lower than his payoff in some equilibria when offers are public. Their result trivially extends to the private-values case. In our model, imperfect observability of acceptance decisions averts Coasian dynamics, providing unique predictions and rich trade dynamics.

Lee and Liu (2013) study a different repeated bargaining model, where a random outside option serves as signal about the type occurs if players fail to agree on a compensation. Equilibrium offers are rejected for intermediate beliefs, and accepted by either both types or only by the low type for extreme beliefs. In contrast, our model's the outside option is fixed and the signal is about agreement—and hence depends on the type only the through equilibrium behavior. The implied dynamics are significantly different, as our buyer must be willing to accept high offers for intermediate beliefs in equilibrium, so mechanisms to slow equilibrium learning—such as random discounts and offer rejections—take place in this range of beliefs.

Our model is also related to the reputations literature. The closest paper to ours is Faingold and Sannikov (2011), where a firm sells goods at a fixed price to a continuum of buyers, and where information about the previous quality choices is revealed through a diffusion process. The firm's type is either "behavioral", who can only produce high-quality goods, or "normal", for whom producing high-quality goods is costly. In our model, the equilibrium behavior of ℓ-buyer resembles that of a behavioral type, and the h-buyer wants to build reputation for having a low valuation. Importantly, paralleling the approach in the bargaining literature, our buyer interacts which only one seller at a time, whose price offer determines the informativeness of the signal at that time. The resulting pricing game generates price dispersion within each posterior, and such equilibrium randomness of the price offered plays an important role in our analysis.[5]

The rest of the paper is divided as follows. We present the model with observable prices in Section 2, and we analyze it in Section 3. In Section 4 we study the welfare effects of reducing the information observed by sellers. In Section 5, we discuss some policy implications and conclude. An Appendix includes the proofs of all results.

[5] In other reputation models between a firm and a market, the price is determined in a centralized and competitive manner, often exogenously set to be equal to the expected value of the good. See, for example, Mailath and Samuelson (2001), Board and Meyer-ter Vehn (2013), or Dilmé (2019b).

2 The model

Time is continuous. There is a buyer. At each instant $t \in \mathbb{R}_+$, he meets a short-lived seller, the "t-seller". The t-seller offers price p_t, the buyer decides to purchase or not. The buyer values the goods of all sellers the same. His valuation, also referred as his type, is private, and it is either ℓ or h, with $0<\ell<h$. A natural interpretation is that the buyer's type is his willingness to pay, which is correlated with his wealth or his access to alternative purchasing options. The initial probability that the type is h is $\phi_0 \in (0,1)$.

There is a public signal about the previous purchasing decisions of the buyer. More concretely, at each moment in time t, the t-seller observes $(X_{t'})_{t' \in [0,t)}$, where

$$X_t \equiv \mu \int_0^t \alpha_{t'} \, \mathrm{d}t' + B_t \; , \tag{1}$$

where $\alpha_{t'} = 1$ and $\alpha_{t'} = 0$ mean, respectively, that the buyer decided accepted and rejected the price offer at time t', where B_t is a normalized Wiener process, and where $\mu > 0$ is a parameter. Until Section 4, we assume that the t-seller also observes the history of price offers made by previous sellers.

We will use ϕ_t to denote the public belief at time t about the type of the buyer being h (given the signal and price histories). We will focus on Markov strategies. For the buyer with type $\theta \in \{\ell, h\}$, a (Markov) *acceptance strategy* associates to each belief $\phi \in [0,1]$ and (on- or off-path) price p, a probability of acceptance $\alpha_\theta(p;\phi) \in [0,1]$. A (Markov) *offer strategy* associates to each belief ϕ a price distribution $\tilde{\pi}(\phi) \in \Delta(\mathbb{R})$.

Consistent strategy profiles

We now define some standard regularity conditions to permit the use of continuous-time techniques to analyze diffusion processes.

Fix a strategy profile $(\vec{\alpha}, \tilde{\pi})$, where $\vec{\alpha} \equiv (\alpha_\ell, \alpha_h)$. Fix also an acceptance strategy $\hat{\alpha}$. Define

$$\tilde{\mu}(\phi; \hat{\alpha}, \vec{\alpha}) \equiv \mu \, (1-\phi) \, \phi \, (\alpha_h - \alpha_\ell) \, (\hat{\alpha} - \phi \, \alpha_h(\phi) - (1-\phi) \, \alpha_\ell) \; , \tag{2}$$

$$\sigma^2(\phi; \vec{\alpha}) \equiv \mu \, (1-\phi)^2 \, \phi^2 \, (\alpha_h - \alpha_\ell)^2 \; . \tag{3}$$

We say that $\hat{\alpha}$ is *consistent* with $(\vec{\alpha}, \tilde{\pi})$ if both

$$\overbrace{\mathbb{E}_{\tilde{p}}\left[\tilde{\mu}(\phi;\hat{\alpha}(\phi,\tilde{p}),\vec{\alpha}(\phi,\tilde{p}))\middle|\tilde{\pi}(\phi)\right]}^{\equiv\tilde{\mu}(\phi;\hat{\alpha},\vec{\alpha},\tilde{\pi})} \quad \text{and} \quad \overbrace{\mathbb{E}_{\tilde{p}}\left[\sigma^2(\phi;\vec{\alpha}(\phi,\tilde{p}))\middle|\tilde{\pi}(\phi)\right]}^{\equiv\sigma^2(\phi;\vec{\alpha},\tilde{\pi})}$$

are piecewise Lipschitz continuous (as a function of ϕ), where $\mathbb{E}_{\tilde{p}}[\cdot|\tilde{\pi}(\phi)]$ is the expectation operator with respect to the variable $\tilde{p}$, which is distributed according to $\tilde{\pi}(\phi)$. We say that $(\vec{\alpha}, \tilde{\pi})$ is *consistent* if α_θ is consistent with $(\vec{\alpha}, \tilde{\pi})$ for both $\theta = \ell, h$.

Consistency is a standard technical requirement in continuous-time models with diffusion processes. It guarantees that a strategy profile $(\vec{\alpha}, \tilde{\pi})$ and an acceptance strategy $\hat{\alpha}$ generate a unique the belief process, and hence a unique outcome of the game. In particular, if sellers follow $\tilde{\pi}$ and they believe the buyer follows a strategy $\vec{\alpha}$, and if the buyer instead follows the acceptance strategy $\hat{\alpha}$, then the belief follows a diffusion process which satisfies

$$\mathrm{d}\phi_t = \tilde{\mu}(\phi_t;\hat{\alpha},\vec{\alpha},\tilde{\pi})\,\mathrm{d}t + \sigma(\phi_t;\vec{\alpha},\tilde{\pi})\,\mathrm{d}W_t\ .$$

Continuation payoff and equilibrium concept

For a given strategy profile $(\vec{\alpha}, \tilde{\pi})$ and an acceptance policy $\hat{\alpha}$ consistent with $(\vec{\alpha}, \tilde{\pi})$, the θ-buyer's payoff is given by

$$V_\theta(\phi;\vec{\alpha},\tilde{\pi},\hat{\alpha}) \equiv \mathbb{E}\left[\int_0^\infty \mathbb{E}\left[\hat{\alpha}(\tilde{p};\phi_t)\,(\theta-\tilde{p})\middle|\pi(\phi_t)\right] e^{-r\,t}\, r\,\mathrm{d}t \middle| \vec{\alpha},\tilde{\pi},\hat{\alpha},\phi_0=\phi\right],$$

where $r > 0$ is the discount rate of the buyer.[6] Given a strategy profile $(\vec{\alpha}, \tilde{\pi})$ and posterior $\phi \in [0,1]$, the buyer's continuation value $V_\theta(\phi)$ is the payoff he obtains by maximizing the right-hand side of the previous expression with respect to $\hat{\alpha}$. Standard arguments imply that the continuation value is continuously differentiable at all almost all ϕ. The corresponding Bellman equation is

$$\begin{aligned} r\,V_\theta(\phi) = \mathbb{E}\Big[\max_{\hat{\alpha}}\big(r\,\hat{\alpha}\,(\theta-\tilde{p}) + \tilde{\mu}(\phi,\hat{\alpha},\vec{\alpha}(\phi,\tilde{p}))\,V_\theta'(\phi) \\ + \tfrac{1}{2}\,\sigma^2(\phi,\vec{\alpha}(\phi,\tilde{p}))\,V_\theta''(\phi)\big)\Big|\tilde{\pi}(\phi)\Big]\ . \end{aligned} \tag{4}$$

Definition 2.1. A *(regular Markov-perfect) equilibrium* is a consistent $(\vec{\alpha}, \tilde{\pi})$, with corresponding value functions (V_ℓ, V_h), piecewise twice-differentiable, and differentiable at all

[6] It is not difficult to see that only two parameters are relevant to determine equilibrium behavior: ℓ/h and μ/r. Instead of normalizing away parameters (e.g., $h=r=1$), we will keep all of them for clarity.

ϕ such that $\alpha_\ell(\hat{p};\phi) \neq \alpha_h(\hat{p};\phi)$ for some $\hat{p}$,[7] satisfying

1. For all ϕ and $\hat{p}$, $\alpha_\theta(\hat{p};\phi)$ belongs to[8]

$$\arg\max_{\hat{\alpha}} \left(r\,\hat{\alpha}\,(\theta - \hat{p}) + \tilde{\mu}(\phi, \hat{\alpha}, \vec{\alpha}(\hat{p};\phi))\, V'_\theta(\phi) \right) . \tag{5}$$

2. For all ϕ, $\tilde{\pi}(\phi)$ belongs to

$$\arg\max_{\pi} \mathbb{E}\left[\left((1-\phi)\,\alpha_\ell(\tilde{p};\phi) + \phi\,\alpha_h(\tilde{p};\phi) \right) \tilde{p} | \pi \right] .$$

The first condition in Definition 2.1 requires the buyer to act optimally. Equation (5) indicates the tradeoff he faces. If a price $\hat{p}$ is such that $\alpha_\ell(\hat{p};\phi) = \alpha_h(\hat{p};\phi)$ (i.e., if the signal is uninformative and so $\tilde{\mu}(\phi, \hat{\alpha}, \vec{\alpha}(\hat{p};\phi)) = 0$) then the buyer accepts for sure any price below his valuation. If, instead, $\hat{p} \in (\ell, h)$ and $\alpha_\ell(\hat{p};\phi) < \alpha_h(\hat{p};\phi)$, the h-buyer obtains an instantaneous flow payoff equal to $h - \tilde{p} > 0$, but he reveals information about his type to future sellers (as the drift of the posterior is positive). The second condition in Definition 2.1 requires that sellers behave optimally. When the posterior is ϕ, the seller chooses the price to maximize the expected revenue, that is, the price multiplied by the probability it is accepted.

We now present a condition on the equilibrium behavior. From now on, we will focus on equilibria satisfying Condition 1, and we call them just equilibria.

Condition 1. The buyer accepts for sure any offer lower or equal to ℓ.

Condition 1 is intuitive, and it is convenient in continuous time to make the analysis tractable. It is analogous to a result often obtained in discrete-time models. In most bargaining models, the *Diamond's paradox* establishes that, if only monopolist sellers make offers, the lowest equilibrium offer is no lower than the lowest buyer valuation.[9]

Note that Condition 1 effectively transforms the ℓ-buyer into a "behavior" or "action" type, who accepts an offer if and only if it is weakly lower than ℓ. Indeed, it is

[7] Note that if $\alpha_\ell(\hat{p};\phi_t) = \alpha_h(\hat{p};\phi_t)$ for some ϕ_t and all $\hat{p}$ then, for any continuation play, the belief remains equal to ϕ_t. Conversely, if $\alpha_\ell(\hat{p};\phi_t) \neq \alpha_h(\hat{p};\phi_t)$ for some $\hat{p}$, then the differentiability of the continuation values permits computing the buyer's incentive to accept the price offer using in equation (5).

[8] Note that if $\alpha_\ell(\hat{p};\phi) = \alpha_h(\hat{p};\phi)$ then $\tilde{\mu}(\phi, \hat{\alpha}, \vec{\alpha}(\hat{p};\phi)) = 0$ for all $\hat{\alpha}$, hence the second term in the argument of arg max in equation (5) is 0 (even if V_θ is not differentiable at ϕ).

[9] In a repeated-trade setting without noise, prices lower than ℓ can be sustained in equilibrium, for example "punishing" the buyer with prices equal to h if he accepts a higher price. The existence of such possibility is unclear when the information about acceptance decisions is noisy.

suboptimal for sellers to offer prices below ℓ in equilibrium. Such type is, in fact, the "Stackelberg type" for the "normal type" h, as defined in Fudenberg and Levine (1989). As we will see, equilibria under Condition 1 will have the property that it will be optimal for both types of the buyer to behave as prescribed by the condition. Hence, Condition 1 can be seen as an equilibrium refinement rather than a behavioral assumption.

3 Equilibrium analysis

3.1 Preliminary results

We begin presenting some preliminary results. They will help building intuition for our main results.

We first note that, when the signal is totally uninformative (i.e., when $\mu = 0$), the buyer behaves myopically, as he would do in a one-shot game. In a one-shot version of the game, a seller offers ℓ if $\phi < \phi^*$, and h if $\phi > \phi^*$, where $\phi^* \equiv \ell/h$. As the following result establishes, the threshold ϕ^* also plays an important role when the signal is informative.

Lemma 3.1. *The following holds in any equilibrium:*

1. *For each $\phi \leq \phi^*$, the support of $\tilde{\pi}(\phi)$ is $\{\ell\}$.*
2. *For each $\phi \in (\phi^*, 1)$, there is some $p(\phi) \in (\ell, h]$ such that the support of $\tilde{\pi}(\phi)$ is either $\{p(\phi)\}$ or $\{\ell, p(\phi)\}$.*

Lemma 3.1 implies that price offers are always smaller than h. This would be trivial in a model with a unique transaction, since no type of the buyer would accept an offer larger than h, hence each seller would be strictly better off by offering ℓ than by offering a price higher than h. In our model with repeated trade, the result is not obvious because the h-buyer has signaling motives. The proof of Lemma 3.1 shows that the continuation value of the h-buyer is decreasing. As a result, the h-buyer never accepts an offer higher than h, as doing so gives him a negative payoff and decreases, on expectation, his continuation value.

An implication of the previous observation is that, as in the static model, ℓ is offered and accepted for sure in equilibrium when $\phi < \phi^*$. Indeed, an immediate implication of Condition 1 is that no seller offers a price strictly below ℓ, and hence the equilibrium payoff of the ℓ-buyer is 0. This implies that the ℓ-buyer rejects all offers above ℓ. As a result, when $\phi < \phi^*$, offering ℓ (which is accepted by both types of the buyer) gives the seller a larger payoff than offering any price in $(\ell, h]$ (which is only accepted by the h-buyer).

The fact that the support of prices is either $\{p(\phi)\}$ or $\{\ell, p(\phi)\}$ (for some $p(\phi)$) when $\phi > \phi^*$ is obtained as follows. Assume the seller offers $\hat{p} \in (\ell, h)$ (on- or off-path). The h-buyer cannot be strictly willing to reject such price: if he was, the signal would be deemed as uninformative by future sellers, but then the buyer would have the incentive to accept the price. Hence, either the h-buyer is indifferent between accepting the price or not, or has a strict incentive to accept it. The h-buyer is indifferent between accepting the price $\hat{p}$ or not if and only if the following equation holds:

$$\overbrace{r\,(h-\hat{p})}^{\text{surplus from trade}} \quad = \quad \overbrace{\mu\,(1-\phi)\,\phi\,\alpha_h(\hat{p};\phi)\,(-V'(\phi))}^{\text{reputation loss}}\,, \tag{6}$$

where from now on we will use V to denote the h-buyer's continuation value to save notation. The term on the left-hand side of equation (6) is the instantaneous buyer's gain from accepting the offer. Such term is his value minus the price. The term on the right-hand side is the implied loss in terms of continuation value, which can be interpreted as a reputation loss.[10] Keeping the rest equal, this term is larger when the signal is more informative, the equilibrium acceptance probability is larger, or when the continuation value is more sensitive to changes in the posterior. The h-buyer is strictly willing to accept $\hat{p}$ only if the left-hand side of the previous equation is bigger than the left-hand side. This implies that, in equilibrium,

$$\alpha_h(\hat{p};\phi) = \min\left\{ -\frac{r\,(h-\hat{p})}{\mu\,(1-\phi)\,\phi\,V'(\phi)}\,,\,1 \right\}. \tag{7}$$

Such acceptance probability acts as a downward-sloping demand: it is 1 if $\hat{p}$ is low enough, and decreases linearly until reaching 0 when $\hat{p} = h$. We can then compute the price $\hat{p} \in (\ell, h)$ that maximizes the seller's payoff $\hat{p}\,\alpha_h(\hat{p};\phi)$. The price $p(\phi)$ in Lemma 3.1 is the unique maximizer of $\hat{p}\,\alpha_h(\hat{p};\phi)$, which is given by

$$p(\phi) \equiv \max\left\{h/2\,,\,\overbrace{h - r^{-1}\,\mu\,(1-\phi)\,\phi\,(-V'(\phi))}^{(*)}\right\}. \tag{8}$$

Expression $(*)$ is equal to the lowest price which is accepted with probability one by the h-buyer, which corresponds to the kink of $\alpha_h(\hat{p};\phi)$. The seller's optimal offer is then either the maximizer of the linear part of $\alpha_h(\hat{p};\phi)$—that is, $h/2$, which is rejected by the

[10] In our model, the buyer would want to build reputation on having a low willingness to pay. As we will see, this implies that V is decreasing, hence $-V'(\phi)$ is positive.

h-buyer with a positive probability—, or the corner solution $(*)$ if such price is above $h/2$—which is accepted by the h-buyer for sure. In particular, we have

$$\alpha(\phi) < 1 \;\Rightarrow\; p(\phi) = h/2\ , \tag{9}$$

where $\alpha(\phi) \equiv \alpha_h(p(\phi);\phi)$ is the equilibrium probability the probability that the h-buyer accepts $p(\phi)$ (recall that such an offer is rejected for sure by the ℓ-buyer).

To obtain intuition for some arguments below, it is important to note that, in any equilibrium, an optimal strategy for the h-buyer is to mimic the ℓ-buyer; that is, to accept an offer if and only if it is equal or lower than ℓ. This follows from the observation that the high price $p(\phi)$ is either the largest price which the buyer accepts with probability one (hence he is indifferent on accepting it or not), or equal to $h/2$ and the buyer randomizes between accepting it or not; hence (6) holds for $\hat{p} = p(\phi)$.

Princing strategy by the sellers

The gain a seller obtains from offering $p(\phi)$ is $\phi\,\alpha(\phi)\,p(\phi)$. By Lemma 3.1, it is weakly optimal for a seller to offer $p(\phi) > \ell$ for all $\phi > \phi^*$. Also, offering ℓ gives a seller a payoff equal to ℓ (since she sells for sure). Hence, we have that

$$\phi\,\alpha(\phi)\,p(\phi) \geq \ell \tag{10}$$

for all $\phi > \phi^*$. The previous expression holds with equality when ℓ is offered with positive probability. Since, by equation (9) either $\alpha(\phi) = 1$ or $p(\phi) = h/2)$ (or both), we have the following condition for sellers to offer h with positive probability for $\phi > \phi^*$:

$$\pi(\phi) \in (0,1) \quad\Rightarrow\quad p(\phi) = \max\{\ell/\phi, h/2\} \tag{11}$$

where, abusing notation again, $\pi(\phi)$ will indicate the probability the price offer equals to $p(\phi)$. Hence, by Lemma 3.1, ℓ is offered with probability $1-\pi(\phi)$.

3.2 Equilibrium characterization

In this section we characterize the equilibrium behavior for the case where the price offers are observable. We divide the analysis into two cases, depending on how informative us the signal.

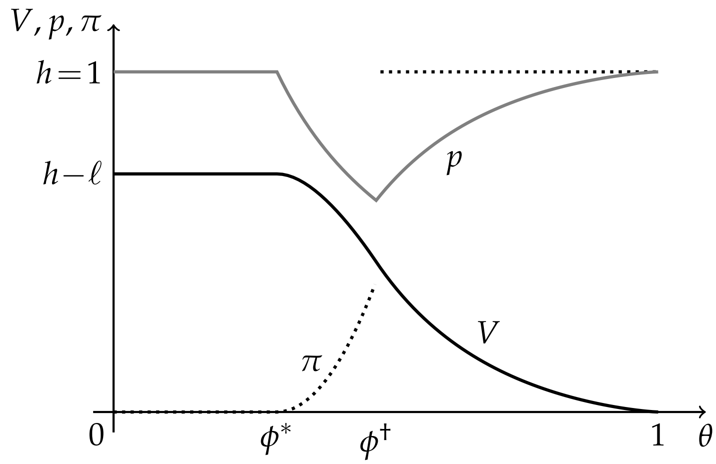

Figure 1: The figure depicts different equilibrium objects for $h=r=1$, $\ell=0.3$, and $\mu=0.8$.[12]

Less informative signal

We first focus on the case that the signal is relatively uninformative; that is, the case where μ is small (in a sense that will be made precise). Equivalent results can be obtained for the case where the buyer is relatively impatient; that is, on the case where r is high.

Proposition 3.1. *There is some largest $\overline{\mu} \in (0, +\infty]$ such that, for all $\mu < \overline{\mu}$, there is an essentially unique equilibrium.*[11] *In such equilibrium, there is some $\phi^\dagger \in (\phi^*, 2\phi^*]$ such that*

1. *On $\phi \in (0, \phi^*]$, π is equal to 0.*
2. *On $(\phi^*, \phi^\dagger)$, π is strictly increasing, α is equal to 1, and p is strictly decreasing.*
3. *On $(\phi^\dagger, 1)$, π and α are equal to 1, and p is strictly increasing.*

Figure 1 illustrates Proposition 3.1. For $\phi < \phi^*$ sellers offer ℓ, the signal is uninformative in equilibrium, and the payoff of the h-buyer is $h-\ell$. As in the static game, even though the h-buyer is willing to accept any price below h, sellers are pessimistic enough about the buyer's valuation that offer a price equal to ℓ.

Take a posterior ϕ higher than, but close to, the threshold ϕ^*. The lowest price above ℓ a seller is willing to offer is ℓ/ϕ, which is strictly larger than $h/2$. Hence, $p(\phi) \geq \ell/\phi$

[11] Essentially unique in the sense that other equilibria differ in a zero-measure set of posteriors that do not affect the outcome of the game.

[12] It is natural to set $p(\phi)=h$ and $\alpha(\phi)=1$ for all $\phi \in (0, \phi^*)$, even though only ℓ is offered in equilibrium in this region. The reason is that when $\phi \in (0, \phi^*)$ it is optimal for the h-buyer accepts with probability one all prices below h, as they give him a positive surplus from trade and no reputation. since $V'(\phi)=0$ (see equation (6)); and also to reject all prices above h.

is accepted for sure by the h-buyer (i.e., $\alpha(\phi) = 1$ by (9)). It may seem contradictory that, when the posterior is close to ϕ^*, the buyer is willing to accept high prices—which give them little surplus—even though rejection would be highly informative and would bring the posterior close to ϕ^*—where the buyer's continuation value is maximal. The apparent contradiction is overcome because, in equilibrium, sellers offer ℓ with a high probability. As we see in Figure 1, such offers flatten the continuation value of the buyer for posterior close to ϕ^*. In consequence, the signaling gain from rejecting the offer is small, making the buyer more willing to accept high prices.

As ϕ increases, the high price $p(\phi) = \ell/\phi$ decreases. Proposition 3.1 establishes that, if the signal is not informative enough, there is a threshold $\phi^\dagger \in (\phi^*, 1)$ where sellers stop making discounted offers. Since $p(\phi) = \ell/\phi > h/2$ for $\phi < \phi^\dagger$, we have

$$\ell/\phi^\dagger = p(\phi^\dagger) \geq h/2 \quad \Rightarrow \quad \phi^\dagger \leq 2\,\phi^* \ .$$

In fact, $\phi^\dagger$ is increasing in μ (see the proof of Proposition 4.1), and $\phi^\dagger = 2\,\phi^*$ if and only if $\mu = \overline{\mu}$. Also, $\overline{\mu} = +\infty$ if and only if $\phi^* \geq 1/2$.

The price increases toward h as the posterior increases toward 1. The reason is that, as we explain above, the h-buyer is indifferent between mimicking the ℓ-buyer (by rejecting the high price) or not (by accepting the high price). This implies that, for a high posterior, the time until the posterior reaches $\phi^\dagger$—that is, the time until the buyer gets offered low-price offers—becomes arbitrary large. The h-buyer then becomes more willing to accept higher prices for high posteriors.

Remark 3.1. In Figure 1 it can be seen that $\pi(\cdot)$ is discontinuous at $\phi^\dagger$. To see why, recall that the h-buyer is indifferent on accepting $p(\phi)$ or not at each posterior $\phi \in (0, 1)$. As we argued, when the posterior is slightly higher than ϕ^*, the h-buyer's incentive to accept a high prices requires that sellers offer ℓ with positive probability, so the high price is ℓ/ϕ and hence decreasing in ϕ.[13] When, instead, ϕ is high, the payoff from mimicking the ℓ-seller is small, so prices are naturally higher than ℓ/ϕ (so $\pi(\phi) = 1$) and the high price $p(\cdot)$ is increasing. The discontinuity in π arises from the necessary discontinuity in the monotonicity of p: while π and p complement each other to provide the incentive to the buyer on $(\phi^*, \phi^\dagger)$, only p provides such an incentive on $(\phi^\dagger, 1)$.

[13] Note thatπ does not directly appear in the h-buyer's indifference condition (8). Nevertheless, it indirectly affects the price through affecting the continuation value (see equation (4), for example).

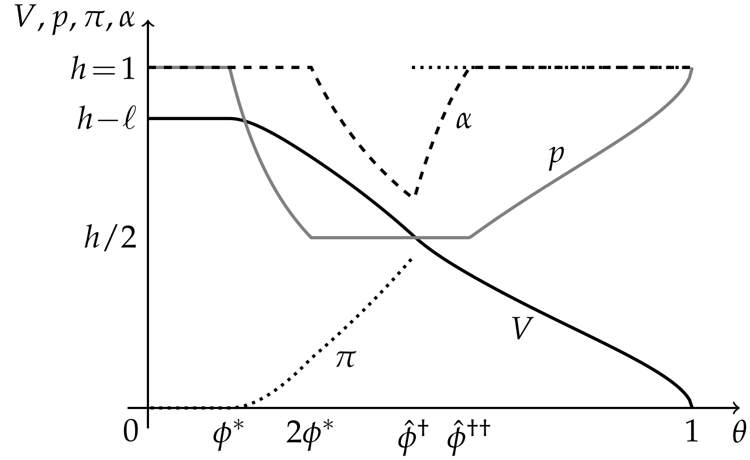

Figure 2: The figure depicts different equilibrium objects for $h=r=1$, $\ell=0.15$, and $\mu=1.5$.

Informative signal

We now study the case where the signal is relatively informative, that is, where $\mu > \overline{\mu}$. This is equivalent to study the case where the buyer is relatively patient. As discussed above, this only occurs if $\phi^* < 1/2$ (since, otherwise, $\overline{\mu} = +\infty$).

The following result establishes that, when the signal is informative, two more equilibrium regions are added on top of the three regions described in Proposition 3.1. Such regions are described below, and depicted in Figure 2.

Proposition 3.2. *Let $\overline{\mu}$ be defined in Proposition 3.1. Then, for all $\mu > \overline{\mu}$, there is an essentially unique equilibrium. For each $\mu > \overline{\mu}$ there are some thresholds $\hat{\phi}^\dagger \in (2\phi^*, 1)$ and $\hat{\phi}^{\dagger\dagger} \in (\hat{\phi}^\dagger, 1)$ such that, in the unique equilibrium:*

1. *On $\phi \in (0, \phi^*)$, π is equal to 0.*
2. *On $(\phi^*, 2\phi^*)$, π is strictly increasing, α is equal to 1, and p is strictly decreasing.*
3. *On $(2\phi^*, \hat{\phi}^\dagger)$, π is strictly increasing, α is strictly decreasing, and p is equal to $h/2$.*
4. *On $(\hat{\phi}^\dagger, \hat{\phi}^{\dagger\dagger})$, π is equal to 1, α is strictly increasing, and p is equal to $h/2$.*
5. *On $(\hat{\phi}^{\dagger\dagger}, 1)$, π and α are equal to 1, and p is strictly increasing.*

As in the case where the signal is not very informative, there is a lower region (equal to $(0, \phi^*)$) where ℓ is offered and accepted for sure. By the same logic as before, for low-intermediate posteriors (posteriors close but above ϕ^*), sellers randomize between offering ℓ, accepted for sure by both types of the buyer, and $p(\phi) = \ell/\phi$, accepted for sure by the h-buyer.

As the posterior increases, the high price $p(\phi) = \ell/\phi$ decreases. Eventually, when the posterior reaches $2\phi^*$, the price is equal to $h/2$, which is the lowest equilibrium price above ℓ by (8). Then, for intermediate posteriors (i.e., $\phi \in (2\phi^*, \hat{\phi}^\dagger)$), additional increases in the posterior do not decrease the price $p(\phi)$ further, as it remains equal to $h/2$. Instead, an increase in the posterior lowers the probability with which $h/2$ is accepted. In fact, the sellers' indifference condition requires that

$$\phi\,\alpha(\phi)\,h/2 = \ell \quad \text{(i.e., } \alpha(\phi) = 2\,\phi^*/\phi)$$

in this region of the beliefs. As the posterior increases, the probability with which sellers offer ℓ decreases. The equilibrium probability with which $h/2$ ensures that the buyer remains indifferent between accepting and rejecting $h/2$.

Similar to the case where the signal is less informative, sellers do not offer ℓ with positive probability at posteriors higher than a given threshold (now denoted $\hat{\phi}^\dagger$). Nevertheless, the price remains equal to $h/2$ for some range of posteriors higher than $\hat{\phi}^\dagger$. Indeed, if the high price would increase immediately after $\hat{\phi}^\dagger$, the h-buyer would accept such price for sure (by (9)). Nevertheless, the right-hand side of (6) (with $\hat{p} = p(\phi)$) would jump up, providing the buyer a strict incentive to reject the high price. Hence, the buyer is indifferent on accepting $h/2$ or not. Such indifference is maintained, in equilibrium, by the fact that the acceptance probability increases—and hence the signal's informativeness—increases with the posterior. Hence, for intermediate-high posteriors (i.e., on $(\hat{\phi}^\dagger, \hat{\phi}^{\dagger\dagger})$), we have that π equals 1 and α is increasing.

Finally, for high posteriors (i.e., on $(\hat{\phi}^{\dagger\dagger}, 1)$) the high price p increases in the posterior, and is accepted for sure by the h-buyer.

4 Effect of privacy policies

We now study the effect of reducing the amount of information each seller has about the previous history. The results give insights on possible effects of policies regulating or banning cookies, which are further discussed in Section 5.

In our model, there are two ways of reducing the information each seller obtains. We will first study the effect of making the acceptance signal less precise (i.e., decreasing μ). We will then analyze a model where sellers can observe the acceptance signal but not the prices offered by previous sellers.

4.1 Limiting the tracking precision

We first study the effect of reducing the signal informativeness. Reductions in μ may correspond to policies limiting the data that can be stored in cookies. On the contrary, increases in μ can be interpreted as improvements in the tracking technology, or regulations requiring transparency in the transactions of government agencies.

Proposition 4.1. *1. For each $\phi \in (\phi^*, 1)$, $V(\phi)$ is strictly increasing in μ on $(0, \overline{\mu})$.*

2. $\lim_{\mu \to \infty} V(\phi) = h - \ell$ for all $\phi \in [0, 1)$.

3. $\lim_{\mu \to 0} V(\phi) = (h - \ell)\, \mathbb{I}_{[0,\phi^*]}(\phi)$ for all $\phi \in [0, 1]$.

Proposition 4.1's first claim establishes that the h-buyer's payoff is increasing in the signal's precision when the signal is not very informative. The intuition for the result is the following. When the acceptance signal is more informative, accepting a high-price offer is more informative about high valuation. In consequence, the buyer's expected loss from accepting a high-price offer (in continuation value terms) should be small. A more informative signal is compensated, in equilibrium, by more frequent discounts, which increase—and therefore flatten—the h-buyer's continuation value. Therefore, an increase in the signal precision when the precision is low makes the buyer better off.

When the signal is informative, more informativeness does not necessarily translate into a higher buyer payoff. In the proof of Proposition 4.1, we show that small increases in μ do increase $V(\phi)$ for all ϕ in $(\phi^*, \hat{\phi}^\dagger]$, but may reduce it for values of ϕ in $(\hat{\phi}^\dagger, 1)$.[14] Intuitively, the buyer's incentive to accept high prices is kept by the slow learning from the h-buyer's random rejection. The implication is that the buyer rejects offers more often when the signal becomes more precise. Such slowing of learning limits the value of mimicking the ℓ-buyer when the posterior is high, which implies that the h-buyer is worse off. Such monotonicity is reminiscent of the result in Liu and Skrzypacz (2014) that, in a model with limited record-keeping, the long-lived player's payoff is non-monotonic in the record length, as too little or too much information make building reputation too costly.

When the signal is very imprecise ($\mu \to 0$), the h-buyer obtains a very low payoff for all $\phi > \phi^*$. As we discussed before, the equilibrium outcome converges to the outcome of an equilibrium of a model without a signal were all sellers offer prices equal to h when $\phi > \phi^*$. Instead, when the signal is very precise ($\mu \to \infty$), the h-buyer obtains a high payoff for all $\phi > \phi^*$. In this case, the outcome converges to an equilibrium of a game where offers and acceptance decisions are perfectly observable. In this equilibrium, all

[14] Numerical examples are provided by the author upon request.

sellers offer ℓ, as the buyer rejects all prices above ℓ/ϕ, since their acceptance would be interpreted as his type being high.

4.2 Private offers case

We now study the case where offers are unobservable to future sellers, sometimes referred to in the literature as the "private offers case" (the model presented and studied in Sections 2 and 3 corresponds to the "public offers case"). Section 4.3 compares the consumer surplus and social welfare of the public and private offers case.

In practice, price offers may be unobservable due to a restrictive regulation on the information content that cookies may contain. For example, cookies may be allowed to contain metrics about the previous user activity, but not data on the actual transactions. Also, search engines may be able to track users until they enter a webpage, and have some information about their behavior in it, but not the actual offer each user obtained.

We now construct a version of the model described in Section 2, where price offers are unobservable to other sellers. Now, the t-seller only observes the public signal $(X_{t'})_{t'<t}$ defined in (1). Markov strategies are defined identically as in Section 2. Given a strategy profile, we define the expected acceptance probability of the θ-buyer for each belief ϕ as

$$\overline{\alpha}_\theta(\phi) = \mathbb{E}_{\tilde{p}}[\alpha_\theta(\tilde{p}|\phi)|\tilde{\pi}(\phi)] \ ,$$

for both $\theta \in \{\ell, h\}$. The value of $\overline{\alpha}_\theta(\phi)$ indicates the believed probability of acceptance of the offer by θ-buyer. Hence, now the Bellman equation is

$$r\, V_\theta(\phi) = \mathbb{E}\Big[\max_{\hat{\alpha}} \big(r\,\hat{\alpha}\,(\theta - \tilde{p}) + \tilde{\mu}(\phi, \hat{\alpha}, \overline{\vec{\alpha}}(\phi))\, V'_\theta(\phi) + \tfrac{1}{2}\,\sigma^2(\phi, \overline{\vec{\alpha}}(\phi))\, V''_\theta(\phi)\big)\Big|\tilde{\pi}\Big] \ . \qquad (12)$$

The crucial difference of the analysis of the unobservable offers case with respect to the observable offers case is that now the drift and variance of the belief process (for a given strategy of the buyer) are independent of the price offer. Hence, while the drift in equation (12) does not depend on the actual offer received by the buyer, the drift in the analogous equation when offers are observable (equation (4)) does depend on it.

The equilibrium concept is analogous to Definition 2.1 with the following differences. First, since now seller deviations are not observable to future sellers, V_θ is differentiable at all ϕ such that $\overline{\alpha}_h(\phi) \neq \overline{\alpha}_\ell(\phi)$. Second, instead of condition (5), α_θ belongs to the following set for all ϕ where V_θ is differentiable:

$$\arg\max_{\hat{\alpha}} \left(r\,\hat{\alpha}\,(\theta - \tilde{p}) + \tilde{\mu}(\phi, \hat{\alpha}, \overline{\overline{\alpha}}(\phi))\, V_\theta'(\phi) \right) . \tag{13}$$

Equilibrium analysis

We begin stating that Lemma 3.1 holds also in the unobservable offers case.

Lemma 4.1. *Lemma 3.1 holds in the unobservable offers model.*

As in the case where offers are observable, the t-seller offers ℓ if $\phi_t \le \phi^*$, and potentially randomizes between offering some price $p(\phi_t)$, with some probability denoted again $\pi(\phi_t)$, and ℓ, with probability $1-\pi(\phi_t)$. Again, $\alpha(\phi_t)$ is the probability with which the h-buyer accepts the price $p(\phi_t)$ when the t-seller offers it.

Even though the support of prices is either singular or binary under both observable and unobservable prices, the logic is quite different. When prices are observable, each price is accepted with a different probability in equilibrium (that is, α_h depends on both $\hat{p}$ and ϕ in equation (6)). This probability affects the informativeness of the signal so that the buyer is indifferent between accepting the price or not. As we saw, a unique price maximizes the acceptance probability multiplied by the price. When prices are unobservable, instead, the reputation loss from accepting an offer $\hat{p}$ is independent of the price offered. Indeed, the h-buyer is indifferent between accepting $\hat{p} \in (\ell, h)$ or not if

$$\overbrace{r\,(h - \hat{p})}^{\text{surplus from trade}} \quad = \quad \overbrace{\mu\,(1-\phi)\,\phi\,(\overline{\alpha}_h(\phi) - \overline{\alpha}_\ell(\phi))\,(-V'(\phi))}^{\text{reputation loss}} , \tag{14}$$

Note that $\overline{\alpha}_\ell(\phi) = 1 - \pi(\phi)$, that is, the ℓ-buyer's acceptance probability coincides with the probability with which ℓ is offered. By the standard take-it-or-leave-it offer argument, a seller offers a price higher than ℓ in equilibrium only if the h-buyer is indifferent on accepting or rejecting it, and he accepts it for sure. We then have that $\overline{\alpha}_h(\phi) = 1$. Hence, in equilibrium, sellers either offer ℓ or

$$p(\phi) = h - \mu / r\,(1-\phi)\,\phi\,\pi(\phi)\,(-V'(\phi)) . \tag{15}$$

Any off-path offer in $(\ell, p(\phi))$ is accepted for sure by the h-buyer, while any offer above $p(\phi)$ is rejected for sure. Unlike the case where price offers are public, the equilibrium probability that the h-buyer accepts $p(\phi)$ (denoted again $\alpha(\phi)$) is always 1, so the rejection mechanism is not present in the private offers case.

Comparing (14) with equation (6), we observe two differences. The first is that, as we argued before, $\alpha_h(p(\phi);\phi) = 1$ when prices are not observable. The second is that we see that the factor $\overline{\alpha}_h(\phi) - \overline{\alpha}_\ell(\phi)$ (which is equal to $\pi(\phi)$) appears in equation (15), but not in equation (6). Intuitively, in the public offers case, future sellers know which offer the buyer received, hence the interpretation of the signal depends only on the price offered, and not the probability with which is offered. In the private offers case, instead, the interpretation of the signal depends on the equilibrium distribution of prices, since sellers do not observe the realization of the price offer.

Proposition 4.2. *Assume price offers are unobservable. Then, there is a unique equilibrium. In such equilibrium, there is some $\phi^\ddagger \in (\phi^*, 1)$ such that*

1. *On $(0, \phi^*]$, π is equal to 0.*
2. *On $(\phi^*, \phi^\ddagger)$, π is strictly increasing, α is equal to 1, and p is strictly decreasing.*
3. *On $(\phi^\ddagger, 1)$, π and α are equal to 1, and p is strictly increasing.*

Proposition 4.2 resembles Proposition 3.1, but now applies to all values of μ. As in the public offers case, sellers offer ℓ with probability one for low posteriors (below ϕ^*). Again, $\pi(\phi) \in (0, 1)$ whenever ϕ is close (but above) ϕ^*. Recall that the t-seller is only willing to offer $p(\phi_t)$ if $p(\phi_t) \geq \ell/\phi_t$. The buyer, in turn, is willing accept a high price only if his continuation value is not very sensitive to the posterior. As a result, learning should be slow in equilibrium when the posterior is higher but close to ϕ^*. Such slow learning occurs because, in equilibrium, sellers randomize on their price offers between offering $p(\phi)$ and ℓ, deeming the acceptance signal less informative. Again, buyers offer the high price for sure for high posteriors.

4.3 Welfare analysis

In this section, we compare the buyer surplus and social welfare of the public and private offers cases analyzed in Sections 3 and 4.2. We will also use the benchmark case where the signal is totally uninformative, interpreted as an online market where cookies are banned, and hence buyers are fully anonymous.[15]

We denote the three cases as follows: "$x =$ no" refers to the case where sellers observe neither prices nor the signal, "$x =$ ob" refers to the case where seller observe

[15] There is a movement aiming at banning the practices that allow advertisers and political parties to track individuals with tailored messages, a practice called "microtargeting". See, for example, https://www.politico.eu/article/targeted-advertising-tech-privacy/

both the prices and the signal, and "$x = \text{un}$" refers to the case where where sellers observe the signal but not the prices. As discussed above, in the case where nothing is observable, sellers offer ℓ for sure when $\phi < \phi^*$ and offer h for sure when $\phi > \phi^*$.

Buyer's surplus

We first analyze how the buyer's surplus is affected by cookies. Given that the ℓ-buyer obtains no surplus under all cases, we focus on the payoff of the h-buyer for our analysis.

Proposition 4.3. *1. If $\mu \le \overline{\mu}$, then $V^{\text{ob}}(\phi) > V^{\text{un}}(\phi) > V^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.*

2. If $\mu > \overline{\mu}$, then $\min\{V^{\text{ob}}(\phi), V^{\text{un}}(\phi)\} > V^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$

The comparison of the three cases is easier when the signal's informativeness is low. In this case, the equilibrium structure is similar in the observable (Proposition 3.1) and unobservable (Proposition 4.2) cases. For low posteriors, sellers offer low prices; for intermediate posteriors, sellers randomize between low and high prices; while for high posteriors, sellers only offer high prices. All offers are accepted by the h-buyer.

For low and high posteriors, the incentives of the buyer and the sellers do not qualitatively depend on the observability of the price offers. In the lower region, sellers offer the discounted price ℓ. In the upper region, sellers offer the highest price which is accepted for sure by the buyer. Fix then some intermediate posterior ϕ. In the public offers case, accepting the high price ℓ/ϕ involves a significant expected increase in the posterior. In equilibrium, the buyer accepts such offers because his continuation value is not sensitive to the posterior. On the contrary, in the unobserved case, other sellers do not know whether the high price ℓ/ϕ or the low price ℓ was offered. Hence, the acceptance of a ℓ/ϕ implies a lower expected increase in the posterior. The continuation value is therefore more steep in the private offers case. Formally, equations (8) and (15) can be written as

$$-\frac{\mathrm{d}}{\mathrm{d}\phi} V^x(\phi) = \gamma^x(\phi)\, \frac{r\,(h-\ell/\phi)}{\mu\,(1-\phi)\,\phi}\,, \tag{16}$$

where $\gamma^{\text{ob}}(\phi) = 1$ and $\gamma^{\text{un}}(\phi) = \frac{1}{\pi^{\text{un}}(\phi)} > 1$. For higher beliefs, V^x solves the same equation for both $x \in \{\text{ob}, \text{un}\}$, since in both cases seller only make high offers and the h-buyer is indifferent between accepting them or not. This is shown to imply that, indeed, the buyer is better off when price offers are observable.

When the signal is informative (i.e., $\mu > \overline{\mu}$), the relative order between V^{ob} and V^{un} may be reversed at some priors. The reason lies on the different mechanisms that slow learning in equilibrium. When prices are unobservable, only the discount mechanism

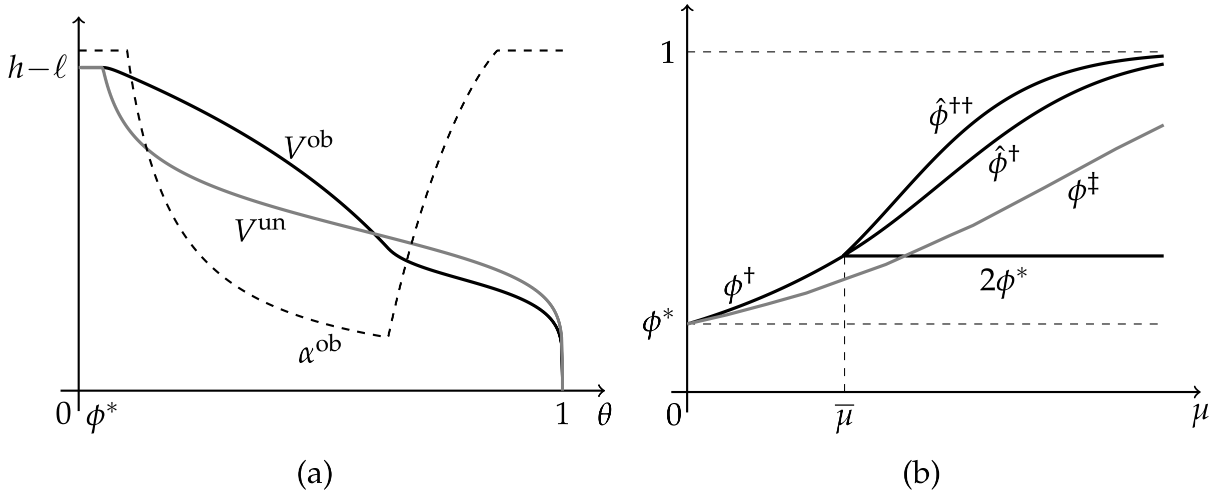

Figure 3: (a) V^{ob} and V^{un} for $h = r = 1$, $\ell = 0.05$, and $\mu = 3$. (b) Thresholds for the public offers case ($\phi^\dagger$, $\hat{\phi}^\dagger$, and $\hat{\phi}^{\dagger\dagger}$) and the private offers case ($\phi^\ddagger$) as a function of μ, for $h = r = 1$ and $\ell = 0.2$.

is present: the positive probability that a low price is offered makes the signal less informative. When prices are observable, on top of the discount mechanism, learning is slowed down by the rejection mechanism. As we saw in Section 3.2, some times the h-buyer rejects high-price offers, and such rejection of high-price offers slows learning for intermediate posteriors. It then takes longer for posterior to reach low values when the h-buyer mimics the ℓ-buyer, if the initial prior is high. The implication is that, if the signal is informative enough, the buyer is worse off at high posteriors when in the public offers case.

Figure 3(a) depicts V^{ob} and V^{un} for parameters where the buyer is not better off (for some posteriors) in the public offers case when the signal is very informative. In the figure, for intermediate posteriors, the buyer receives low-price offers more often in the public offers case, so his payoff is higher. For higher beliefs, nevertheless, the price is higher when offers are observable. As we can see in the picture, the low acceptance probability of high offers at intermediate posteriors allows that the V^{ob} decreases fast while keeping the incentive of the buyer of accepting high prices. It then takes a long time for posterior to become low if it is initially very high, even if the h-buyer mimics the ℓ-buyer. Sellers offer higher prices at high posteriors in the public offers case as a result.

Figure 3(b) depicts the equilibrium thresholds of both public and private offers cases as a function of μ. When the signal is not very informative (i.e., $\mu < \overline{\mu}$), the buyer is better off in the public offers case because the range where discounts are offered, $(0, \phi^\dagger)$, is larger than under the private offers case, $(0, \phi^\ddagger)$. While this is also true when the signal is very informative ($(0, \hat{\phi}^\dagger)$ versus $(0, \phi^\ddagger)$), there is a large region where high offers are rejected in equilibrium (on $(\hat{\phi}^{\dagger}, \hat{\phi}^{\dagger\dagger})$), and this makes the buyer worse off in the public offer case.

Remark 4.1. Proposition 4.3 establishes that, when trade is repeated, the buyer tends to prefer offers to be public rather than private, as rejecting a high offer is a stronger signal of low type. The logic is reversed when there is only one transaction (see Kaya and Liu, 2015): in this case, unobservability enhances the Coasian forces since, when a seller deviates to a higher price, other sellers cannot observe it and update their beliefs faster. Our result, nevertheless, also establishes that unobservability is sometimes beneficial, as prevents the rejection mechanism to slow learning in equilibrium.

Social welfare

We now take the perspective of a social planner that has the same discount rate as the buyer. The social planner values each transaction of the θ-buyer at θ, for both $\theta \in \{\ell, h\}$, independently of the transaction price.

We use $W^x(\phi)$ to denote the social welfare for each case $x \in \{\text{no}, \text{un}, \text{ob}\}$. The welfare is then given, for each $\phi \in (\phi^*, 1)$, by

$$\begin{aligned} W^x(\phi) = (1-\phi)\, & \overbrace{\mathbb{E}^{x,\ell}\big[\textstyle\int_0^\infty \ell\,(1-\pi^x(\phi_t))\, e^{-r\,t}\, r\,\mathrm{d}t \big| \phi_0 = \phi\big]}^{(*)} \\ & + \phi\, \underbrace{\mathbb{E}^{x,h}\big[\textstyle\int_0^\infty h\,(1-\pi^x(\phi_t) + \pi^x(\phi_t)\,\alpha^x(\phi_t))\, e^{-r\,t}\, r\,\mathrm{d}t \big| \phi_0 = \phi\big]}_{(**)}\,, \end{aligned} \tag{17}$$

where $\mathbb{E}^{x,\theta}$ is the expectation in the equilibrium for the x-model conditional on the strategy of the θ-buyer.[16] The term $(*)$ is equal to the social welfare generated by the transactions of the ℓ-buyer. This type of buyer only purchases when the price is ℓ. The term $(**)$ is equal to the social welfare generated by the transactions of the h-buyer. This type of buyer purchases for sure when the price is ℓ, and with probability $\alpha^x(\phi)$ otherwise.

Proposition 4.4. *1. If $\mu \le \overline{\mu}$, then $W^{\text{ob}}(\phi) > W^{\text{un}}(\phi) > W^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.*

2. If $\mu > \overline{\mu}$, then $W^{\text{un}}(\phi) > W^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$.

We observe that the term $(*)$ in equation (17) is 0 when $x = \text{no}$ and $\phi > \phi^*$. Hence, the gain from having cookies (with or without information about prices) comes from the

[16] Note that $\alpha^{\text{no}}(\phi) = \pi^{\text{no}}(\phi) = 1$ and $p(\phi) = h$ for $\phi > \phi^*$; that is, the unique outcome of the game with no information when $\phi_0 > \phi^*$ is one where all sellers offer h and only the h-buyer accepts such offers.

fact that ℓ is offered more frequently than in the no-cookies, case, which implies that the ℓ-buyer purchases more often. Recall that the h-buyer's payoff from mimicking the ℓ-buyer coincides with his equilibrium payoff, since he is always indifferent between accepting $p(\phi)$ or not. Note also that, if the h-buyer mimics the ℓ-buyer, his payoff equals $h-\ell$ multiplied by the discounted times the price ℓ is offered. Since $(*)$ is equal to ℓ multiplied by the discounted times the price ℓ is offered, we have that $(*)$ is equal to $\frac{\ell}{h-\ell} V^x(\phi)$ for all $x \in \{\text{no}, \text{un}, \text{ob}\}$.

The term $(**)$ in equation (17) is equal to h for both $x \in \{\text{no}, \text{un}\}$, since the h-buyer buys with probability one at all times. It is then clear that $W^{\text{un}}(\phi) > W^{\text{no}}(\phi)$ for all $\phi \in (\phi^*, 1)$, since cookies increase the probability of trade of the ℓ-buyer while keeping the same probability of trade of the h-buyer. When $x = \text{ob}$, instead, the h-buyer does not purchase with probability one on a wide range of posteriors when $\mu > \overline{\mu}$. Additionally, given the slow learning owed to the rejection mechanism, the ℓ-buyer purchases less often in the public offers case than in the private offers case if μ is large enough. This implies that some transactions that occur in the other cases are not materialized. As a result, for some parameter values, making prices unobservable improves welfare.

5 Discussion

There is an intense debate over privacy regulation in internet browsing, with internet cookies at its center. Similarly, the desirability of new transparency regulations is often part of the political debate. Our model abstracts from some relevant considerations in these debates, focussing the analysis on pricing and trade efficiency. We here describe some of such relevant considerations and how our model can shed light on them.

Right to privacy: An important objective of some regulations is to guarantee the so-called "right to privacy" to users. For example, a major outcome of the EU's Cookie Law is the requirement that websites allow users to opt-out cookies. We now discuss how adding the possibility of hiding the browsing history would affect our results.

Consider the effect of letting the buyer decide whether to hide or not his history to each seller in our model. In the new model, hiding or revealing the previous history would naturally convey information about the buyer's type. Standard unraveling arguments (á la Milgrom, 1981) would favor the existence of equilibria where the buyer always reveals the previous history, lowering the value of the regulation. There would be, nevertheless, inefficient equilibria, where disclosing the previous history would be perceived as a sign of high type so that no history would be revealed in equilibrium, and hence the buyer would not disclose it.[17]

Efficiency: An important defense of cookies is their use to improve the user's experience, making internet browsing more efficient. In particular, cookies can be used to tailor advertisements that match the interests of the user, reducing frictions in the market for advertisement. Such individualized advertisement is sometimes perceived as harmful as it may enable third parties to (politically) manipulate internet users.

Our model could be adapted to allow for some horizontal differentiation—allowing sellers to offer one of two types of products, for example. The usual tradeoff would then arise: the more information the seller has, the more efficient trade becomes (by offering the good most preferred to the buyer) and the easier the seller can extract surplus from the buyer. It is then plausible that, as in our model, the buyer's signaling motives would induce sellers to offer low prices, but also inefficiencies could arise from the buyer's distortion of his behavior.[18]

Other sources of information: Our model focuses on transactions as the sole source of information about the type (beyond the prior). Cookies, nevertheless, often contain information about all websites visited by the user, such as social media, news websites, blogs, entertainment platforms... Such additional browsing activity may provide information about users—for example, about income, age, gender, or occupation—which sellers may use to determine their price offers.

To study the effect of the additional information, an exogenous signal about the buyer's type á la Daley and Green (2012) could be added to our model.[19] It can be expected that, if such signal were very informative, the buyer's surplus would be close to 0, while if the signal was very imprecise, the outcome would be close to our predictions. Studying the interaction between exogenous and endogenous learning in the

[17] Another discussed effect of requiring each user to report their cookie preferences in each website they visit is costly for the user. It generates the so-called "opt-out fatigue," which diminishes the user's browsing experience quality. Johnson, Shriver, and Du (2020) find that, in the US, even though users express strong privacy concerns, only a very small fraction opt-out from targeted online advertising.

[18] A simple generalization our model consists in assuming that sellers offer one of two types of products. Product 1 is valued at ℓ by ℓ-buyers and at h' by h-buyers. Product 2 is valued at $\ell' \leq \ell$ by ℓ-buyers and $h \geq h'$ by h-buyers (our model would be the case where $h' = h$). Similar to our model, each seller would either offer product 1 at price ℓ (accepted by both types of buyers), or product 2 at a higher price (only accepted by h-buyers).

[19] It could be further assumed that buyers could distort their behavior to affect the information provided by additional browsing activity (similar to our model). We could model such a possibility as in Dilmé (2019a), who studies a dynamic signaling game where the signal depends on effort instead of type.

intermediate case is left for future research.

Observability of cookies. In the model, we assume that the signal history is observed by the buyer. Such assumption keeps the model tractable as it keeps first-order beliefs as the only relevant state variable. A model where the buyer could not observe the information would not be tractable, he would infer information about it from the prices offered by sellers.[20]

[20] Cripps, Mailath, and Samuelson (2007) show that, in a model with private reputation, the reputation of the long-run player eventually vanishes. More recently, Cisternas and Kolb (2021) study signaling with private monitoring in a linear-quadratic model with a Gaussian information structure.

A Proofs of the results

A.1 Proofs of results in Section 3

Proof of Lemma 3.1

Proof. We divide the proof into 3 steps:

Step 1: Preliminary observations. Note that the price is never lower than ℓ, since offering ℓ ensures trade. Then, it is without loss of generality to focus on equilibria where the ℓ-seller accepts the price if and only if is ℓ, and where each seller never offers a price lower than ℓ. As a result, $V_\ell(\phi)=0$ for all $\phi\in(0,1)$.

Note also that, if the h-buyer accepts a price $\hat{p}>\ell$ for sure, then a seller is willing to offer such a price if and only if $\hat{p}\geq\ell/\phi$. Hence, no price in $(\ell,\ell/\phi)$ is offered in equilibrium.

Finally, sometimes, it is convenient to use the log-likelihood instead of the posterior. For each $\phi\in(0,1)$, we define

$$\check{z}(\phi)=\log\left(\tfrac{\phi}{1-\phi}\right)\ \text{ and }\ \check{\phi}(z)=\tfrac{e^z}{1+e^z}\ . \tag{18}$$

Note that $\check{\phi}(\cdot)$ is the inverse of $\check{z}(\cdot)$. Abusing notation, for some function f of ϕ, we will sometimes use $f(z)$ and $f'(z)$ to denote $f(\check{\phi}(z))$ and $\frac{\mathrm{d}}{\mathrm{d}z}f(\check{\phi}(z))$, respectively. Note that, for example, $f'(\check{z}(\phi))=\phi\,(1-\phi)\,f'(\phi)$.

Step 2: Monotonicity of V_h. We begin the proof by stating and proving the following result:

Lemma A.1. *$V_h(\phi)=h-\ell$ for all $\phi\in(0,\phi^*]$ and V_h is strictly decreasing on $(\phi^*,1)$.*

Proof. 1. **Proof that $V_h(\phi)=h-\ell$ for all $\phi\in(0,\phi^*)$.** If only ℓ is offered in $(0,\phi^*)$ then the result follows. Assume then, for the sake of contradiction, that it is optimal for a seller to offer $\hat{p}>\ell$ when the posterior is $\phi\in(0,\phi^*]$. Since, by Step 1, $\hat{p}$ has to be at least ℓ/ϕ—hence strictly higher than h—, hence it must be that $\hat{p}$ is accepted for sure (by the argument laid out in the main text after Lemma 3.1). Then, from equation (8), such price should then satisfy:

$$\ell/\phi\leq\hat{p}=h+\mu/r\,V_h'(\check{z}(\phi))\ . \tag{19}$$

Note that, if $|V_h'(\check{z}(\phi))|$ is small enough, the right-hand side of the previous expression is close to h. This implies that, if $|V_h'(\check{z}(\phi))|$ is small enough, sellers offer

ℓ for sure (since no price above ℓ is optimal), and $V_h(\phi) = h-\ell$. We conclude that either $V_h'(\phi)=0$ for all $\phi\in(0,\phi^*)$, in which case $V_h(\phi)=h-\ell$ for all $\phi\in(0,\phi^*]$, or $V_h'(\phi)\neq 0$ for all $\phi\in(0,\phi^*)$.[21] In other words, when restricted to $(0,\phi^*)$, either V_h is strictly decreasing, or strictly increasing, or equal to $h-\ell$. It is clear V_h cannot be strictly decreasing, since in this case the h-buyer rejects all prices above h, hence it is strictly optimal for each seller to offer ℓ for all $\phi\in(0,\phi^*)$ and $V_h(\phi)=h-\ell$. Assume, for the sake of contradiction, that V_h is strictly increasing. Equation (19) implies that, in this case, $\lim_{z\to-\infty} V_h'(z) = +\infty$ and hence $\lim_{z\to-\infty} V_h(z) = -\infty$, which is a contradiction. Then, the only possibility is that $V_h(\phi)$ is equal to $h-\ell$ for all $\phi\in(0,\phi^*]$.

2. **Proof that V_h is strictly decreasing on $(\phi^*, 1)$.** Assume first $V_h'(\phi_1)=0$ for some $\phi_1\in(\phi^*,1)$. This implies that $\alpha_h(\hat{p};\phi_1)=1$ for all $\hat{p}<h$, and $\alpha_h(\hat{p};\phi_1)=0$ for all $\hat{p}>h$. Hence, at posterior ϕ_1, the seller offers h for sure in equilibrium, since $\phi\, h>\ell$ when $\phi>\phi^*$. Equation (4) then becomes

$$r\, V_h(\phi_1) = \tfrac{1}{2}\,\mu\,(1-\phi_1)^2\,\phi_1^2\, V_h''(\phi_t)\ .$$

   We then have that ϕ_1 it is a minimizer of V. This implies that V is strictly increasing on $(\phi_1,1)$, so all prices offered on $(\phi_1,1)$ are higher than h (by the argument used to obtain equation (8)). But then this implies that $\lim_{\phi\nearrow 1} V_h(\phi)\le 0$, and therefore $V_h(\phi)<0$ for some ϕ, which is a contradiction. We conclude that $V_h'(\phi)\neq 0$ for all $\phi\in(\phi^*,1)$.

   Since V_h cannot be strictly increasing on $(\phi^*,1)$ (because $V_h(\phi^*)=h-\ell$ and equilibrium offers are never lower than ℓ) and $V_h'(\phi)\neq 0$ for all $\phi\in(\phi^*,1)$, we have that the V_h must be strictly decreasing on $(\phi^*,1)$.

   □

(End of the proof of Lemma A.1. Proof of Lemma 3.1 continues.)

Step 3: Proof of the result. The argument in the main text implies that either the seller offers ℓ, or $p(\phi)$ satisfying equation (8), or randomizes between them. An immediate implication of Lemma A.1 is that ℓ is offered with probability one when $\phi\in(0,\phi^*]$. Also, since V_h is strictly decreasing on $(\phi^*,1)$, we have that ℓ is never offered with probability one at posteriors in $(\phi^*,1)$. □

[21] The observations implies that V_h' cannot continuously "approach" 0, since a small value of $V_h'(\phi)$ implies $V_h(\phi)=h-\ell$ when $\phi<\phi^*$.

Proof of Propositions 3.1 and 3.2

Proof. We prove Propositions 3.1 and 3.2 together. We divide the proof into ten steps:

Step 1: Preliminary derivations. We begin the proof by providing a useful equation. The arguments in the main text (following Lemma 3.1) show that, for each posterior $\phi \in (\phi^*, 1)$, the buyer is indifferent between accepting $p(\phi)$ or not (that is, equation (6) holds for $\hat{p} = p(\phi)$). This implies that the buyer's continuation value can be computed *as if* he did reject $p(\phi)$ for all posteriors ϕ. As a result, the continuation value of the buyer satisfies the following equation:

$$r\,V(\phi) = (1-\pi(\phi))\,(h-\ell) - \mu\,(1-\phi)\,\phi^2\,\pi(\phi)^2\,\alpha(\phi)^2\,V'(\phi) \\ + \tfrac{1}{2}\,\mu\,(1-\phi)^2\,\phi^2\,\pi(\phi)^2\,\alpha(\phi)^2\,V''(\phi)\ . \tag{20}$$

The previous expression is convenient as it does not depend on $p(\phi)$.

Step 2: Preliminary results on continuity. We continue by providing a result on the continuity of $\alpha(\cdot)$ and $p(\cdot)$, and the limits of $V(\phi)$ as ϕ tends to ϕ^* and 1.

Lemma A.2. *Both $\alpha(\cdot)$ and $p(\cdot)$ are continuous on $(\phi^*, 1)$. Furthermore, $\lim_{\phi\searrow\phi^*} V(\phi) = h-\ell$ and $\lim_{\phi\nearrow 1} V(\phi) = 0$.*

Proof. The continuity of $\alpha(\cdot)$ and $p(\cdot)$ on $(\phi^*, 1)$ is immediately implied by equations (7) and (8) (recall that $V'(\cdot)$ is continuous and that we defined $\alpha(\phi)$ to be equal to $\alpha_h(p(\phi);\phi)$).

That $\lim_{\phi\searrow\phi^*} V(\phi) = h-\ell$ follows from continuity of the continuation value.

We finally prove that $\lim_{\phi\nearrow 1} V(\phi) = 0$. Recall that, by Lemma A.1, V is strictly decreasing on $(\phi^*, 1)$. We assume, for the sake of contradiction, that $\lim_{\phi\nearrow 1} V(\phi) > 0$. Since it is optimal for the h-buyer to follow the ℓ-buyer's strategy (that is, only accepting offers equal to ℓ), there must exist some increasing sequence $(\phi_n)_n$ converging to 1 such that $(\pi(\phi_n))_n$ is convergent and $\lim_{n\to\infty} \pi(\phi_n) < 1$.[22] We can write equation (8) using log-likelihoods (see equation (18)) as

$$p(\phi) \equiv \max\left\{h/2\,,\, h + r^{-1}\,\mu\,V'(\check{z}(\phi))\right\}\ .$$

Hence, since $\lim_{\phi\nearrow 1} V'(\check{z}(\phi)) = 0$ (because V is strictly decreasing and bounded below

[22] If no such sequence would exist, then there would be a region $(\overline{\phi}, 1)$ where ℓ is never offered. This would imply that mimicking the ℓ-seller would give the h-seller a continuation value converging to 0 as $\phi \to 1$.

by 0), we have that $\lim_{\phi \nearrow 1} p(\phi) = h$. This implies that, if ϕ is close enough to 1, $p(\phi) > \ell/\phi$, hence $\pi(\phi) = 1$. As a result, $\lim_{\phi \nearrow 1} V(\phi) = 0$. □

(End of the proof of Lemma A.2. Proof of Propositions 3.1 and 3.2 continues.)

Step 3: Preliminary results on regimes. We now present a result providing the equations for each of four possible types of regimes on $(\phi^*, 1)$.

Lemma A.3. *The following statements follow for all $\phi^* \leq \phi_1 < \phi_2 \leq 1$:*

1. *If $p(\phi) = \phi/\ell$ for all $\phi \in (\phi_1, \phi_2)$ then $\phi_2 \leq 2\,\phi^*$, $\alpha(\phi) = 1$, $\pi(\phi) \in (0, 1)$,*

$$\pi(\phi) = \frac{2\,\phi\,(h-\ell-V(\phi))}{\phi\,(h-\ell)+2\,\ell\,(1-\phi)}\ , \tag{21}$$

   and

$$\pi'(\phi) = \frac{2\,r\,(\phi-\phi^*)+2\,(1-\phi)\,\phi^*\,\mu\,\pi(\phi)}{(1-\phi)\,\phi\,\mu\,(\phi\,(1-\phi^*)+2\,(1-\phi)\,\phi^*)} > 0\ . \tag{22}$$

2. *If $p(\phi) = h/2$ for all $\phi \in (\phi_1, \phi_2)$ then $\phi_1 \geq 2\,\phi^*$ (hence $\phi^* < 1/2$), $\alpha(\phi) \in [2\,\phi^*/\phi, 1)$, and $V(\phi) > h/4$. Also,*

   (a) *If $\alpha(\phi) = 2\,\phi^*/\phi$ for all $\phi \in (\phi_1, \phi_2)$, then*

$$\pi(\phi) = \frac{2\,\phi\,(h-\ell-V(\phi))}{2\,h-3\,\ell} \tag{23}$$

   and

$$\pi'(\phi) = \frac{r}{2\,\mu\,(1-\phi)\,(2-3\,\phi^*)\,\phi^*} > 0\ . \tag{24}$$

   (b) *If $\alpha(\phi) \in (2\,\phi^*/\phi, 1)$ for all $\phi \in (\phi_1, \phi_2)$, then $\pi(\phi) = 1$,*

$$\alpha(\hat{p}) = -\frac{r\,h}{2\,\mu\,(1-\phi)\,\phi\,V'(\phi)}\ , \tag{25}$$

   and

$$\alpha'(\phi) = \frac{4\,V(\phi)-h\,\alpha(\phi)}{(1-\phi)\,\phi\,h} > 0\ . \tag{26}$$

3. *If $p(\phi) > \max\{h/2, \ell/\phi\}$ for all $\phi \in (\phi_1, \phi_2)$ then $\alpha(\phi) = 1$, $\pi(\phi) = 1$, and*

$$p'(\phi) = \frac{2\,V(\phi)-h+p(\phi)}{(1-\phi)\,\phi}\ . \tag{27}$$

Proof.

1. The fact that $\phi_2 \leq 2\,\phi^*$ follows from the fact that $p(\phi) \geq h/2$ for all ϕ (by equation (8)). The fact that $\alpha(\phi)=1$ follows from equation (9). Equations (21) and (22) follow from equations (8) (with $p(\phi)=\ell/\phi$) and (20) (with $\alpha(\phi)=1$). Finally, $\pi(\phi)\in(0,1)$ because $\pi(\phi)\in[0,1]$ and π is strictly increasing by equation (22).

2. The fact that $\phi_1 \geq 2\,\phi^*$ follows from equation (10). The fact that $\alpha(\phi)\geq 2\,\phi^*/\phi$ follows from the definition of ϕ^* and the optimality condition for the sellers requiring that $\phi\,\alpha(\phi)\,p(\phi) \geq \ell$.

   The fact that $\alpha(\phi)<1$ follows from the fact that, if $\alpha(\phi)>2\,\phi^*/\phi$, then the equality in equation (26) holds (from equations (20) and (25) with $\pi(\phi)=1$), so $\alpha(\phi)=1$ and $p(\phi)=h/2$ only if $V(\phi)=h/4$. The fact that V is strictly increasing on $(\phi^*,1)$ (by Lemma A.1) implies that $\alpha(\phi)<1$ for all $\phi\in(\phi_1,\phi_2)$. Also:

   (a) Equation (24) follows from equations (20) and equation (6) (with $\alpha(\phi) = 2\,\phi^*/\phi$).

   (b) The fact that $\pi(\phi)=1$ follows because the seller strictly prefers offering $h/2$ than offering ℓ (since $\ell < \phi\,\alpha(\phi)\,p(\phi)$). Equation (25) follows from equation (7), and equation (26) follows from equations (20) and (25) (with $\pi(\phi)=1$).

   To prove that $V(\phi)>h/4$, recall the end of the proof of Lemma A.2. It shows that there exists some $\overline{\phi}<1$ such that $\alpha(\phi)=1$ for all $\phi\in[\overline{\phi},1)$. This implies that, if $\alpha(\phi)=1$ for all $\phi\in(\phi_1,\phi_2)$, then there must be some $\overline{\phi}<1$ such that $\alpha'(\overline{\phi})\geq 0$ and so $V(\overline{\phi})\geq h/4$. Since V is strictly decreasing on $(\phi^*,1)$ by Lemma A.1, we have $V(\phi)>h/4$ for all $\phi\in(\phi_1,\phi_2)$.

3. That $\alpha(\phi)=1$ follows from equation (9). That $\pi(\phi)=1$ follows from equation (11). Equation (27) follows from differentiating equation (8) (since the max operator on its right-hand side is larger than $h/2$) and from using equation (20).

$\square$

(End of the proof of Lemma A.3. Proof of Propositions 3.1 and 3.2 continues.)

Step 4: Proof that $p(\phi)=\ell/\phi$ and $\alpha(\phi)=1$ for ϕ close to ϕ^*. Take a sequence $(\phi_n)_n$ strictly decreasing toward ϕ^* such that $\phi_n\in(\phi^*,\min\{1,2\,\phi^*\})$ for all n. From equation (10), we have that $p(\phi_n)\geq \ell/\phi_n > h/2$ for all n, and so $p(\phi_n)\to h$. By Lemma A.3, we have that $\alpha(\phi_n)=1$ for all n. Recall that, from Lemma A.2, $V(\phi_n)$ converges to $h-\ell$. Assume, for the sake of contradiction and taking a subsequence if necessary, that $p(\phi_n) > \ell/\phi_n > h/2$ for all n. Lemma A.3 implies that $p(\phi)$ is increasing when it is strictly larger $\max\{\ell/\phi_n, h/2\}$, hence this implies that $p(\phi) > \max\{\ell/\phi_n, h/2\}$ for all

$\phi \in (\phi^*, 1)$. From equation (27) it is clear that there must be some posterior $\phi' > \phi^*$ such that $p(\phi') > h$, which contradicts Lemma 3.1.

Step 5: Definition of $\phi^\dagger$. From the previous step, there must be some maximal $\phi^\dagger$ such that $p(\phi) = \ell/\phi$ for all $\phi \in (\phi^*, \phi^\dagger)$. Take some $\phi \in (\phi^*, \phi^\dagger)$; that is, we have that $p(\phi) = \ell/\phi$ and $\alpha(\phi) = 1$. From equation (8) we have

$$\ell/\phi = p(\phi) = h + r^{-1}\,\mu\,(1-\phi)\,\phi\,V'(\phi)\ . \tag{28}$$

Additionally, by Lemma A.3, we have $\pi(\phi) \in (0,1)$. We can solve (28) for V (with boundary condition $V(\phi^*) = h - \ell$), and obtain

$$V(\phi) = h - \ell + \tfrac{r\,(\phi\,h-\ell)}{\mu\,\phi} + \tfrac{r\,(h-\ell)}{\mu}\,\log\left(\tfrac{1-\phi}{\phi}\big/\tfrac{1-\phi^*}{\phi^*}\right)\ . \tag{29}$$

Note that we have $V'(\phi^*) = 0$, so V' is continuous at ϕ^*. Then, from equation (20), we have

$$\pi(\phi) = \tfrac{2\,r}{(\phi\,(h-3\,\ell)+2\,\ell)\,\mu}\left(\ell - \phi\,h - \phi\,(h-\ell)\,\log\left(\tfrac{1-\phi}{\phi}\big/\tfrac{1-\phi^*}{\phi^*}\right)\right)\ . \tag{30}$$

Since the expression on the right-hand side of equation (29) tends to $-\infty$ as $\phi \to 1$, it must be that $\phi^\dagger < 1$.

Step 6: Preliminaries for less informative signal. We first focus on the case $\phi^\dagger < 2\,\phi^*$. We want to show that, for all $\phi > \phi^\dagger$, we have $p(\phi) > \max\{h/2, \ell/\phi\}$, and hence $\alpha(\phi) = 1$, $\pi(\phi) = 1$, and equation (27) holds (by Lemma A.3).

We first argue that there is no $\phi \in (\phi^\dagger, \min\{2\,\phi^*, 1\})$ where $p(\phi) = \ell/\phi$. Assume, for the sake of contradiction, that there is an interval (ϕ_1, ϕ_2) with $\phi^\dagger \le \phi_1 < \phi_2 \le 2\,\phi^*$ such that $p(\phi) > \ell/\phi$ for all $\phi \in (\phi_1, \phi_2)$ and $p(\phi_2) = \ell/\phi_2$. Since $p(\phi)$ approaches ℓ/ϕ_2 as $\phi \to \phi_2$ from above the curve ℓ/ϕ, it must be that $\lim_{\phi \nearrow \phi_2} p'(\phi) \le -\ell/\phi_2^2$ which, by equation (27), implies

$$V(\phi_2) \le \tfrac{1}{2}\,(h+\ell-2\,\ell/\phi_2) < \tfrac{1}{2}\,(\ell-h) < 0\ ,$$

which is a contradiction. Hence, there is no $\phi \in (\phi^\dagger, \min\{2\,\phi^*, 1\})$ where $p(\phi) = \ell/\phi$.

We now argue that there is no $\phi \in (\min\{1, 2\,\phi^*\}, 1)$ such that $p(\phi) = h/2$. If $\phi^* \ge 1/2$ the result is clear, so assume that $\phi^* < 1/2$. Assume, for the sake of contradiction, that there is an interval (ϕ_1, ϕ_2) with $2\,\phi^* \le \phi_1 < \phi_2 < 1$ such that $p(\phi) > h/2$ for all $\phi \in (\phi_1, \phi_2)$ and $p(\phi_2) = h/2$. It is clear that $p(\phi) > \max\{\ell/\phi, h/2\}$ for all $\phi \in (\phi_1, \phi_2)$ (since $h/2 > \ell/\phi$ for $\phi > 2\,\phi^*$). This implies, by Lemma A.3, that p is increasing on

(ϕ_1, ϕ_2), hence it is not possible that $p(\phi_2) = h/2$ (recall that p is continuous by Lemma A.2).

We then have proven that for all $\phi > \phi^\dagger$, we have $p(\phi) > \max\{h/2, \ell/\phi\}$, and hence $\alpha(\phi) = \pi(\phi) = 1$ and, by Lemma A.3, equation (27) holds. This implies that equation (20) can be written, for all $\phi \in (\phi^\dagger, 1)$, as

$$\begin{aligned} r\,V(\phi) = (1-\pi(\phi))\,(h-\ell) - \mu\,(1-\phi)\,\phi^2\,\pi(\phi)^2\,\alpha(\phi)^2\,V'(\phi) \\ + \tfrac{1}{2}\,\mu\,(1-\phi)^2\,\phi^2\,\pi(\phi)^2\,\alpha(\phi)^2\,V''(\phi)\ . \end{aligned} \tag{31}$$

The solution to this equation is

$$V(\phi) = C_1\,\left(\tfrac{1-\phi}{\phi}\right)^{\kappa} + C_2\,\left(\tfrac{1-\phi}{\phi}\right)^{-1-\kappa}\ , \tag{32}$$

where

$$\kappa \equiv \tfrac{1}{2}\,(\sqrt{1+8\,r/\mu}-1) > 0\ , \tag{33}$$

and C_1 and C_2 are integration constants. Using that, by Lemma A.2, we have that $\lim_{\phi \nearrow 1} V(\phi) = 0$, we have $C_2 = 0$. We can use the continuity of V' at $\phi^\dagger$ and equations (29) and (32) with $C_2 = 0$ to obtain the value of C_1 as a function of $\phi^\dagger$, so we have that

$$V(\phi) = \tfrac{r}{\mu}\,\tfrac{\phi^\dagger-\phi^*}{\phi^\dagger}\left(\tfrac{\phi}{1-\phi}\Big/\tfrac{\phi^\dagger}{1-\phi^\dagger}\right)^{-\kappa} h \tag{34}$$

for all $\phi \in (\phi^\dagger, 1)$. Finally, the value of $\phi^\dagger$ is obtained using that V is continuous at $\phi^\dagger$. This requirement can be written as

$$-\tfrac{(\kappa-1)\,r\,(\phi^\dagger-\phi^*)}{\phi^\dagger\,\kappa} + r\,(1-\phi^*)\,\log\left(\tfrac{\phi^\dagger}{1-\phi^\dagger}\Big/\tfrac{\phi^*}{1-\phi^*}\right) - \mu\,(1-\phi^*) = 0\ . \tag{35}$$

The left-hand side of the previous expression is $-\mu\,(1-\phi^*)$ when $\phi^\dagger = \phi^*$, and tends to $+\infty$ when $\phi^\dagger \to 1$, so a solution exists. Furthermore, the derivative of the left-hand side of the previous equation with respect to $\phi^\dagger$ is

$$\frac{r\,(\kappa\,(\phi^\dagger-\phi^*) + (1-\phi^\dagger)\,\phi^*)}{\kappa\,(1-\phi^\dagger)\,(\phi^\dagger)^2} > 0\ .$$

Hence, there is exactly one value of $\phi^\dagger$ solving equation (35).

Step 7: Definition of $\overline{\mu}$. We now claim that there is some value $\overline{\mu} \in (0, +\infty]$ such that a solution $\phi^\dagger$ strictly smaller than $2\,\phi^*$ for equation (35) exists if and only if $\mu \le \overline{\mu}$. The result is obviously true if $\phi^* \geq 1/2$, since then $\overline{\mu} = +\infty$ (by the arguments in Step 6). Assume then that $\phi^* < 1/2$.

Note that, differentiating the left-hand side of expression (35) two times with respect to μ (recall that κ depends on μ, see equation (33)), we obtain

$$-\tfrac{7}{8} + \phi^* + \tfrac{\mu+4\,r}{8\,\mu\,(1+2\,\kappa)}\ .$$

Using the value of κ (from (33)) it is easily seen that the previous expression is negative for all $\phi^* < \frac{1}{2}$. Furthermore, the left-hand side of expression (35) is negative when $\phi^\dagger = \phi^*$, and tends to $+\infty$ when $\phi^\dagger \to 1$. It is then clear that there is only only one value of μ for which $\phi^\dagger = 2\,\phi^*$. That is, using Step 6, we have proven that for all $\mu \leq \overline{\mu}$, the unique equilibrium is as described in the state of Proposition 3.1, while for all $\mu > \overline{\mu}$ there is no equilibrium of this form.[23]

Step 8: Preliminaries for the less informative signal. Assume for the rest of the proof that $\mu > \overline{\mu}$. In this case, $\phi^\dagger = 2\,\phi^*$ (where $\phi^\dagger$ is defined in Step 5 as the maximal such that $p(\phi) = \ell/\phi$ for all $\phi \in (\phi^*, \phi^\dagger)$). Since $\alpha(\cdot)$ is continuous on $(\phi^*, 1)$ (by Lemma A.2), an implication of Lemma A.3 is that all equilibria satisfy the characterization provided in Proposition 3.2.[24]

Then, to show existence and uniqueness of equilibria, we have prove the existence and uniqueness thresholds $\hat{\phi}^\dagger$ and $\hat{\phi}^{\dagger\dagger}$ such that the implied continuation value satisfies the smooth pasting conditions and such that $\alpha(\hat{\phi}^{\dagger\dagger}) = 1$.

To prove existence of an equilibrium, we construct the continuation value of the h-buyer. To do so, we use Lemma A.3 to determine the equations governing V for the different regions of beliefs, for some given values of $\hat{\phi}^\dagger$ and $\hat{\phi}^{\dagger\dagger}$.

1. Region $(0, \phi^*)$: In this region we have $V(\phi) = h - \ell$ (by Lemma 3.1).

2. Region $(\phi^*, 2\,\phi^*)$: In this region, equation (29) holds.

3. Region $(2\,\phi^*, \hat{\phi}^\dagger)$: Imposing the smooth pasting conditions at $2\,\phi^*$, we obtain that,

[23] Indeed, as $\mu \to 0$ we have $\phi^\dagger \to \phi^*$, so the value of $\phi^\dagger$ is increasing in μ at $\phi^\dagger = 2\,\phi^*$. (Proposition 4.1 sows that the unique $\phi^\dagger$ solving equation (35) is increasing in μ.)

[24] Indeed, recall that α is continuous by Lemma A.2. By Lemma A.3, it is decreasing only if it is equal to $2\,\phi^*/\phi$. Hence, if an equilibrium exists for $\mu > \overline{\mu}$, there must be some $\hat{\phi}^\dagger$ and $\hat{\phi}^{\dagger\dagger}$ such that $\alpha(\phi) = 2\,\phi^*/\phi$ for all $\phi \in (2\,\phi^*, \hat{\phi}^\dagger)$ (case 2(a) in Lemma A.3), then satisfies equation (25) in $(\hat{\phi}^\dagger, \hat{\phi}^{\dagger\dagger})$ (where it is strictly increasing), and it is equal to 1 in $(\hat{\phi}^{\dagger\dagger}, 1)$.

in this region,

$$V(\phi) = h - \ell + \left(\tfrac{h}{2} + \tfrac{\ell}{4}\,\log\left(\tfrac{1-\phi}{1-2\,\phi^*}\right) - (h-\ell)\log\left(\tfrac{2-2\,\phi^*}{1-2\,\phi^*}\right)\right)\tfrac{r}{\mu}\ . \tag{36}$$

4. Region $(\hat{\phi}^{\dagger}, \hat{\phi}^{\dagger\dagger})$: In this region, V follows equation (20) with $\pi(\phi) = 1$. Using equation (25), we obtain

   $$V(\phi) = \tfrac{1}{2\sqrt{\mu/r}}\,h\,H\big((1-\phi)^{-1}\,c_1 + c_2\big)$$

   for some $c_1 < 0$ (so V is decreasing) and $c_2 \in \mathbb{R}$; where H is the inverse of the integral of the Gaussian distribution multiplied by 2, that is,

   $$H^{-1}(x) \equiv \tfrac{2}{\sqrt{\pi}} \int_0^x e^{-x'^2}\,\mathrm{d}x'\ .$$

   Note that the domain and support of H are, respectively, $(-1, 1)$ and $\mathbb{R}$, and also that H is strictly increasing and strictly convex on $(0, 1)$.

5. Region $(\hat{\phi}^{\dagger\dagger}, 1)$: In this region, V follows equation (32) for some $C_1 > 0$ and $C_2 = 0$.

We have 5 unknown variables to be determined: $\hat{\phi}^{\dagger}$, $\hat{\phi}^{\dagger\dagger}$, c_1, c_2 and C_1. To do so, we have two smooth pasting conditions at $\hat{\phi}^{\dagger}$, two other smooth pasting conditions at $\hat{\phi}^{\dagger\dagger}$, and the requirement that $\alpha(\hat{\phi}^{\dagger\dagger}) = 1$. We have then as many unknown variables as conditions.

Using the smooth pasting conditions at $\hat{\phi}^{\dagger\dagger}$ and that $\alpha(\hat{\phi}^{\dagger\dagger}) = 1$, we obtain

$$C_1 = \tfrac{r\,h}{2\,\kappa\,\mu}\left(\tfrac{\hat{\phi}^{\dagger\dagger}}{1-\hat{\phi}^{\dagger\dagger}}\right)^{\kappa}$$

and also

$$H\big((1-\hat{\phi}^{\dagger\dagger})^{-1}\,c_1 + c_2\big) = \frac{1}{\kappa\,\sqrt{\mu/r}} \tag{37}$$

and

$$H'\big((1-\hat{\phi}^{\dagger\dagger})^{-1}\,c_1 + c_2\big) = -\frac{(1-\hat{\phi}^{\dagger\dagger})}{\hat{\phi}^{\dagger\dagger}\,c_1\,\sqrt{\mu/r}}\ . \tag{38}$$

We can now define the variables $c^{\dagger} \equiv (1-\hat{\phi}^{\dagger})^{-1}\,c_1 + c_2$ and $c^{\dagger\dagger} \equiv (1-\hat{\phi}^{\dagger\dagger})^{-1}\,c_1 + c_2$. Since V is positive and decreasing and H is increasing, it must be that $c^{\dagger} > c^{\dagger\dagger} > 0$. From the two equations (37) and (38), and from the boundary conditions at $\hat{\phi}^{\dagger}$ we obtain the following four equations:

1. The first equation determines the value of $c^{\dagger\dagger}$:

$$c^{\dagger\dagger} = H^{-1}\big(\tfrac{1}{\kappa\sqrt{\mu/r}}\big) = \tfrac{2}{\sqrt{\pi}} \int_0^{\frac{r^{1/2}}{\kappa\mu}} e^{-x^2}\,\mathrm{d}x \;. \tag{39}$$

2. The second and third equations determine the values of $\hat{\phi}^{\dagger}$ and $\hat{\phi}^{\dagger\dagger}$ as a function of $c^{\dagger}$ and $c^{\dagger\dagger}$:

$$\hat{\phi}^{\dagger} = 2\,\phi^{*}\,\frac{H'(c^{\dagger})}{H'(c^{\dagger\dagger})}\,\big(1-\sqrt{\mu/r}\,(c^{\dagger}-c^{\dagger\dagger})\,H'(c^{\dagger\dagger})\big)\;,$$
$$\hat{\phi}^{\dagger\dagger} = \frac{2\,\phi^{*}\,H'(c^{\dagger})}{H'(c^{\dagger\dagger})\,(1+2\,\phi^{*}\,\sqrt{\mu/r}\,(c^{\dagger}-c^{\dagger\dagger})\,H'(c^{\dagger}))}\;.$$

   It is not difficult to see that, as long as $c^{\dagger} > c^{\dagger\dagger}$, we have $\hat{\phi}^{\dagger} < \hat{\phi}^{\dagger\dagger} < 1$.

3. The only value left to determine is $c^{\dagger}$. This is obtained by solving the fourth equation

$$\begin{aligned} 0 = r\,\log\Big((1-2\,\phi^{*})\,\Big(1-2\,\phi^{*}\,H'(c^{\dagger})\big(\overbrace{\tfrac{1}{H'(c^{\dagger\dagger})}-\tfrac{\sqrt{\mu}\,(c^{\dagger}-c^{\dagger\dagger})}{\sqrt{r}}}^{(*)}\big)\Big)\Big)& \\ +\,2\,\phi^{*}\,\big(2\,(1-\phi^{*})\,\mu-2\,(1-\phi^{*})\,r\log\big(1+\tfrac{1}{1-2\,\phi^{*}}\big)-\sqrt{\mu}\,\sqrt{r}\,H(c^{\dagger})+r\big)&\;. \end{aligned} \tag{40}$$

To prove existence of an equilibrium, we have to prove that equation (40) has a solution for $c^{\dagger}$ in $(c^{\dagger\dagger}, 1)$, and to prove uniqueness of an equilibrium that such solution is unique.

Step 9: Existence of an equilibrium for the more informative signal. We first note that if $c^{\dagger}$ is replaced by $c^{\dagger\dagger}$ in equation (40), the resulting equation is equivalent to equation (35) with $\phi^{\dagger} = 2\,\phi^{*}$. This result is intuitive: when $\mu = \overline{\mu}$ (hence equation (35) with $\phi^{\dagger} = 2\,\phi^{*}$ holds), we have $c^{\dagger} = c^{\dagger\dagger}$, which implies that $\hat{\phi}^{\dagger} = \hat{\phi}^{\dagger\dagger} = 2\,\phi^{*}$. Since in this part of the proof we assume that $\mu > \overline{\mu}$ implies that the right-hand side of equation (40) is positive when $c^{\dagger}$ is replaced by $c^{\dagger\dagger}$.

It is easy to see that the term $(*)$ in equation (40) is positive when $c^{\dagger}$ is replaced by $c^{\dagger\dagger}$. Hence, since $H'(c^{\dagger})$ tends to $+\infty$ when $c^{\dagger}$ tends to 1, we have that there exists a value $\overline{c}^{\dagger}$ such that the right-hand side of equation (40) tends to $-\infty$ as $c^{\dagger} \nearrow \overline{c}^{\dagger}$.[25] Then, continuity proves the existence of an equilibrium.

[25] Indeed, the term inside the logarithm on the first line of equation (40) tends to 0 as $c^{\dagger} \nearrow \overline{c}^{\dagger}$.

Step 10: Uniqueness of an equilibrium for the more informative signal. It is only left to prove that the right-hand side of equation (40) is strictly decreasing on $(c^{\dagger\dagger}, \bar{c}^{\dagger})$. To do so, define $\tilde{c}^{\dagger} \equiv H(c^{\dagger})$. Then, the derivative of the right-hand side of equation (40) with respect to $\tilde{c}^{\dagger}$ is

$$0 = -2\,\phi^*\,r\,H''(c^{\dagger})\,\frac{\overbrace{\frac{1}{H'(c^{\dagger\dagger})} - \frac{\sqrt{\mu}\,(c^{\dagger}-c^{\dagger\dagger})}{\sqrt{r}} + \frac{\sqrt{\mu}}{\sqrt{r}}\frac{H'(c^{\dagger})}{H''(c^{\dagger})}}^{(**)}}{1-2\,\phi^*\,H'(c^{\dagger})\left(\frac{1}{H'(c^{\dagger\dagger})} - \frac{\sqrt{\mu}\,(c^{\dagger}-c^{\dagger\dagger})}{\sqrt{r}}\right)} - 2\,\phi^*\,\sqrt{\mu}\,\sqrt{r}\,. \qquad (41)$$

The derivative of the term $(**)$ with respect to $c^{\dagger}$ is $\sqrt{\mu}/(2\,\sqrt{r}\,H(c^{\dagger})) > 0$, and its value at $c^{\dagger} = c^{\dagger\dagger}$ is

$$\frac{e^{-\frac{r}{\kappa^2\mu}}\left(4\,r + \mu - \sqrt{\mu}\,\sqrt{8\,r+\mu}\right)}{2\,\sqrt{\pi}\,r}\,,$$

which is positive. Hence, the term $(**)$ is positive. As a result, the derivative of the right-hand side of equation (40) with respect to $c^{\dagger}$ is negative. We conclude that there is a unique pair of thresholds $(\hat{\phi}^{\dagger}, \hat{\phi}^{\dagger\dagger})$, with $2\,\phi^* < \hat{\phi}^{\dagger} < \hat{\phi}^{\dagger\dagger} < 1$ such that V satisfies all boundary conditions, and therefore a unique equilibrium (which satisfies the characterization of Proposition 3.2) exists. □

A.2 Proofs of results in Section 4

Proof of Proposition 4.1

Proof. The proof is divided into 3 steps.

Step 1. We first show that V is increasing in μ on $(0, \overline{\mu})$. We divide this step of the proof into two sub-steps:

1. We first prove that $\phi^{\dagger}$ is increasing in μ. To do so, recall from the proof of Propositions 3.1 and 3.2 that $\phi^{\dagger}$ is the unique solution of equation (35), which we denote $\phi^{\dagger}_{\mu}$ in this proof.

   We note that the derivative of the left-hand side of expression (35) with respect to μ (recall that κ depends on μ, see equation (33)) is equal to

$$\frac{\mu\,(1+\kappa)+2\,r}{2\,\mu\,(1+2\,\kappa)}\,\frac{\phi^{\dagger}-\phi^*}{\phi^*} - (1-\phi^*)\,.$$

Such expression is strictly increasing in $\phi^\dagger$ and negative for all $\phi^\dagger \in (\phi^*, \overline{\phi}^\dagger)$, where

$$\overline{\phi}^\dagger \equiv \min\left\{1, \phi^* \left(1 - \tfrac{2\,(1+2\,\kappa)\,\mu\,(1-\phi^*)}{2\,r+(1+\kappa)\mu}\right)^{-1}\right\}.$$

If $\overline{\phi}^\dagger = 1$ then we have that the left-hand side of expression (35) is decreasing in μ, and hence $\phi^\dagger$ is increasing in μ. Assume then that $\overline{\phi}^\dagger < 1$. When we evaluate the left-hand side of expression (35) for $\phi^\dagger = \overline{\phi}^\dagger$ we obtain

$$-(1-\phi^\dagger)\left(\mu + \tfrac{2\,(\kappa-1)\,(1+2\,\kappa)\,r\,\mu}{2\,\kappa\,r+\kappa\,(1+\kappa)\,\mu} + r\,\log\left(\tfrac{(1+3\,\kappa)\,\mu-2\,r}{(1+\kappa)\,\mu+2\,r}\right)\right).$$

Using the definition of κ (see equation (33)), it is easy to see that the previous expression is positive. This implies that $\overline{\phi}^\dagger > \phi^\dagger$. As a result, we have that the derivative of the left-hand side of equation (35) at $\phi^\dagger_\mu$ is negative. Recalling that the left-hand side of equation (35) is strictly increasing in $\phi^\dagger$ (see the proof of Propositions 3.1 and 3.2), we have that $\phi^\dagger_\mu$ is increasing in μ on $(0, \overline{\mu})$.

2. Let $z^* \equiv \check{z}(\phi^*)$ and $z^\dagger \equiv \check{z}(\phi^\dagger)$ (recall the definition of $\check{z}$ in (18)). Note that $\phi^\dagger$ for all $z \in [z^\dagger, \infty)$ we have[26]

$$\frac{V(z^\dagger)}{-V'(z^\dagger)} \begin{cases} > \kappa^{-1} & \text{if } z \in (z^*, z^\dagger). \\ = \kappa^{-1} & \text{if } z \in [z^\dagger, +\infty). \end{cases} \tag{42}$$

Take now two values $\mu_1 < \mu_2 < \overline{\mu}$. By part 1, we have that $z^\dagger_{\mu_1} < z^\dagger_{\mu_2}$, where the subindexes μ_1 and μ_2 indicate equilibrium values for each of the values for μ. From equations (8) (with $p(\phi) = \ell/\phi$) and (29), we have $V_{\mu_1}(z) < V_{\mu_2}(z)$ and $-V'_{\mu_1}(z) > -V'_{\mu_2}(z)$ for all $z \in (z^*, z^\dagger_{\mu_1}]$. Assume, for the sake of contradiction, that there is some $z > z^\dagger_{\mu_1}$ such that $V_{\mu_1}(z) = V_{\mu_2}(z)$. There must then be some $\hat{z} > z^\dagger_{\mu_1}$ such that $V_{\mu_1}(\hat{z}) = V_{\mu_2}(\hat{z})$ and $-V'_{\mu_1}(\hat{z}) \le -V'_{\mu_2}(\hat{z})$. But then, this implies,

$$\frac{V_{\mu_2}(\hat{z})}{-V'_{\mu_2}(\hat{z})} \le \frac{V_{\mu_1}(\hat{z})}{-V'_{\mu_1}(\hat{z})} = \kappa^{-1}_{\mu_1} < \kappa^{-1}_{\mu_2} \le \frac{V_{\mu_2}(\hat{z})}{-V'_{\mu_2}(\hat{z})}.$$

This is a contradiction. We then showed that $V_{\mu_1}(\phi) < V_{\mu_2}(\phi)$ for all $\phi \in (\phi^*, 1)$.

Step 2. We now show that $\lim_{\mu\to\infty} V(\phi) = h - \ell$ for all $\phi \in (\phi^*, 1)$. There are two cases:

[26] Indeed, if there is a solution V of (29) and a value of $\phi^\dagger$ such that equation (42) is satisfied, there is an equilibrium as in Proposition 3.1 (since the smooth pasting conditions hold at $\phi^\dagger$ for dome continuation value from the right given in (32) for some C_1 and with $C_2 = 0$).

1. Assume first $\overline{\mu} = +\infty$ (that is, if $\phi^* \geq 1/2$). Take a sequence $(\mu_n)_n$ tending to $+\infty$. Let $(\phi_n^\dagger)_n$ be the sequence of the corresponding equilibrium thresholds, which is increasing by Step 1, and let $\phi_\infty^\dagger \in (\phi^*, 1]$ be its limit.

   From equation (29), we have that $\lim_{n\to\infty} V_n'(\phi) = 0$ for all $\phi \in (\phi^*, \phi_\infty^\dagger)$. Hence, $\lim_{n\to\infty} V_n(\phi) = h - \ell$ for all $\phi \in (\phi^*, \phi_\infty^\dagger)$.

   If $\phi_\infty^\dagger = 1$ then the result holds. If $\phi_\infty^\dagger < 1$ then, given that the drift of the belief process from rejecting offers at each $\phi \in (\phi_\infty^\dagger, 1)$ becomes arbitrarily large as $\mu \to \infty$, we have that $\lim_{n\to\infty} V_n(\phi) = h - \ell$ for all $\phi \in (\phi_\infty^\dagger, \phi^*)$, hence the result holds.

2. Assume that, instead, $\overline{\mu} < +\infty$ (that is, $\phi^* < 1/2$). Then, if μ is large enough, the equilibrium characterization in Proposition 3.2 applies. By the same argument as when $\overline{\mu} = +\infty$, we now have $\lim_{\mu \nearrow +\infty} V'(\phi) = 0$ for all $\phi \in (\phi^*, 2\,\phi^*)$.

   For each $\phi \in (2\,\phi^*, \hat{\phi}^{\dagger\dagger})$, we have (from equation (7))

   $$V'(\phi) = -\frac{r\,h/2}{\mu\,(1-\phi)\,\phi\,\alpha(\phi)} \geq -\frac{r\,h}{4\,\mu\,(1-\phi)\;\phi^*}\;,$$

   where we used $\alpha(\phi) \geq 2\,\phi^*/\phi$ by the optimality of the seller's strategy.

   We can now take a sequence $(\mu_n)_n$ tending to $+\infty$ and let $(\hat{\phi}_n^{\dagger\dagger})_n$ be the sequence of the corresponding equilibrium thresholds. Taking a subsequence if necessary, assume that $(\hat{\phi}_n^{\dagger\dagger})_n$ tends to some value $\hat{\phi}_\infty^{\dagger\dagger} \in [\phi^*, 1]$. Then, we can then use the same argument as for the case where $\overline{\mu} = +\infty$.

Step 3. We now want to show that $\lim_{\mu\to 0} V(\phi) = 0$ for all $\phi \in (\phi^*, 1)$. When μ is small enough, we can use the equilibrium characterization in Proposition 3.1. Take a sequence $(\mu_n)_n$ tending to 0. Let $(\phi_n^\dagger)_n$ be the sequence of the corresponding equilibrium thresholds. Taking a subsequence if necessary, assume that $(\phi_n^\dagger)_n$ tends to some value $\phi_\infty^\dagger \in [\phi^*, 1]$.

Now, for all $\phi \in (\phi^*, \phi_\infty^\dagger)$, we have $\lim_{n\to\infty} V_n'(\phi) = +\infty$ (from equation (29)). This has the implication that, $\phi_\infty^\dagger = \phi^*$. Because, for each n, the buyer is willing to reject all offers for all $\phi > \phi_n^\dagger$ for all n, and because the learning speed tends to 0 for all ϕ, the result holds. □

Proof of Lemma 4.1

Proof. Lemma A.1 is proven analogously for the case where offers are not observable. Additionally, by the usual take-it-or-leave-it offer argument, each seller either offers ℓ (which is accepted for sure by both types of the buyer) or the price $p(\phi) > \ell$ given in equation (15) (which is accepted for sure by the h-buyer and rejected for sure by the ℓ-buyer). □

Proof of Proposition 4.2

Proof. The result follows trivially from Lemma 4.1 when $\phi \leq \phi^* \equiv \ell/h$. The rest of the proof is divided into six steps.

Step 1: Preliminary results on regimes. Note that the continuation value

$$V(\phi) = r\left(\pi(\phi)\,(h - p(\phi)) + (1-\pi(\phi))\,(h-\ell)\right) \\ + \tilde{\mu}(\phi, 1, (1, 1-\pi(\phi)))\,V'(\phi) + \tfrac{1}{2}\,\sigma^2(\phi, (1, 1-\pi(\phi)))\,V''(\phi)\;, \tag{43}$$

where $\tilde{\mu}$ and σ are defined in (2) and (3), respectively (note that the ℓ-buyer accepts an offer with probability $1-\pi(\phi)$). Additionally, the h-buyer's indifference between accepting $p(\phi)$ or not implies that equation (15) holds as well.

We begin with a result characterizing some of the characteristics of the different possible regimes. The following result is analogous to Lemma A.3 in the proofs of Propositions 3.1 and 3.2.

Lemma A.4. *Assume prices are not observable. The following statements follow for all $\phi^* \leq \phi_1 < \phi_2 \leq 1$:*

1. *If $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi_1, \phi_2)$ then $\phi_2 < 1$, $p(\phi) = \ell/\phi$, and*

$$\pi(\phi) = -\tfrac{r\,(h-\ell/\phi)}{(1-\phi)\,\phi\,\mu\,V'(\phi)}\;. \tag{44}$$

2. *If $\pi(\phi) = 1$ for all $\phi \in (\phi_1, \phi_2)$ then*

$$p(\phi) = h + \tfrac{\mu}{r}\,(1-\phi)\,\phi\,V'(\phi)\;. \tag{45}$$

Proof. 1. The indifference of the sellers implies that $p(\phi) = \ell/\phi$. As a result, from equations (15) and (43), we have

$$0 = r\,(\phi-\phi^*)\,\tfrac{V'(\phi)}{h} + \tfrac{r}{2}\,(\phi-\phi^*)^2\,\tfrac{V''(\phi)}{h} + \phi^2\,\mu\left(1-\phi^* - \tfrac{V(\phi)}{h}\right)\tfrac{V'(\phi)^2}{h^2} \tag{46}$$

holds. Equation (44) follows from equations (43) (with $p(\phi) = \ell/\phi$) and (46). Note that equation (44) can be written using log-likelihoods as

$$\pi(z) = -\tfrac{r\,(h-(1+e^{-z})\,\ell)}{\mu\,V'(z)}\ .$$

Since it must be that $V'(z) \to 0$ as $z \to \infty$, we have that $\phi_2 < 1$.

2. **Case** $\pi(\phi) = 1$ **for all** $\phi \in (\phi_1, \phi_2)$**:** Equation (45) follows from the indifference of the h-buyer on accepting $p(\phi)$ or not, that is, equation (15). Additionally, we have the equation (43) becomes

$$r\,V(\phi) = \mu\,(1-\phi)^2\,\phi\,V'(\phi) + \tfrac{1}{2}\,\mu\,(1-\phi)^2\,\phi^2\,V''(\phi)\ . \tag{47}$$

□

(End of the proof of Lemma A.4. Proof of Proposition 4.2 continues.)

Step 2: Continuity of p and π. In this step, we prove that p and π are continuous. We first prove that p is continuous. Note that equation (15) holds for all ϕ. Hence, p is not continuous only if π is not continuous. There are then two cases:

1. Assume first that $\pi(\phi) \in (0, 1)$ for some $\phi \in (0, 1)$ and there is a sequence $(\phi_n)_n$ converging to ϕ such that $\pi(\phi_n) = 1$ for all k. Note that $\pi(\phi) < 1$ only if $p(\phi) = \ell/\phi$. Then, we have that

$$\lim_{n\to\infty} p(\phi_n) = h + \tfrac{\mu}{r}\,(1-\phi)\,\phi\,V'(\phi) = \tfrac{\ell}{\phi} - \tfrac{1-\pi(\phi)}{\pi(\phi)}\,(h - \ell/\phi) < \ell/\phi\ .$$

This is a contradiction.

2. The alternative case is that $\pi(\phi) = 1$ and there is a sequence $(\phi_n)_n$ converging to ϕ such that $\pi(\phi_n) < 1$ for all k and $(\pi(\phi_n))_n$ converges to some $\overline{\pi} < 1$. Note that $\pi(\phi_n) = \ell/\phi_n$ for all n. Then, we have that

$$\ell/\phi = \lim_{n\to\infty} p(\phi_n) = h + \tfrac{\mu}{r}\,(1-\phi)\,\phi\,\overline{\pi}\,V'(\phi) = h - (h - p(\phi))\,\overline{\pi}\ .$$

Since $p(\phi) \ge \ell/\phi$, we have again a contradiction.

Step 3: Determination of the equilibrium structure. Let (ϕ_1, ϕ_2) be a maximal region with $\pi(\phi) \in (0, 1)$ for all $\phi \in (\phi_1, \phi_2)$ (that is, $\pi(\phi_1) \in \{0, 1\}$ and $\pi(\phi_2) = 1$). We aim at proving that there is at most one such interval, and is such that $\phi_1 = \phi^*$. Note that $\phi_2 < 1$

by Lemma A.4. We have that $p(\phi_2)=\ell/\phi_2$. Then, using $\pi(\phi_2)=1$, we have

$$\lim_{\phi \nearrow \phi_2} \pi'(\phi_2) = \tfrac{2\,\ell+\phi_2\,(2\,V(\phi_2)-h-\ell)}{h\,\phi_2\,(1-\phi_2)\,(\phi_2-\phi^*)} \,. \tag{48}$$

We first argue that there is a unique value $\overline{\phi}_2$ such that the right-hand side of the previous equation is positive if $\phi_2 > \overline{\phi}_2$ and negative if $\phi_2 > \overline{\phi}_2$. To see this, recall V is decreasing. Hence, if the numerator is 0 for some value $\overline{\phi}_2$, it must be that $2\,V(\overline{\phi}_2)-h-\ell < 0$. If $\phi_2 > \overline{\phi}_2$, the numerator of the right-hand side of (48) is negative, while if $\phi_2 > \bar{\phi}_2$ the numerator is positive. Now, note that since $\pi'(\phi_2) \geq 0$ and $\phi_2 < 1$, we have that $\phi_2 \leq \overline{\phi}_2$. Also, if $\pi(\phi_1)=1$, we have $\phi_1 \geq \bar{\phi}_2$ since $\pi'(\phi_1) \leq 0$. It then follows that there is at most one interval (ϕ_1,ϕ_2) where $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi_1,\phi_2)$ and $\pi(\phi_1), \pi(\phi_2) \in \{0,1\}$ and, if it exists, such interval is such that $\phi_1=\phi^*$ (and so $\pi(\phi_1)=0$) and $\pi(\phi_2)=1$.

We now prove that there must be some $\hat{\phi}^{\ddagger} \in (\phi^*,1)$ such that $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi^*,\hat{\phi}^{\ddagger})$. To show this, assume not; that is, assume that $\pi(\phi)=1$ for all $\phi \in (\phi^*,1)$. In this case, equation (47) holds for all $\phi \in (\phi^*,1)$. The general solution to this equation is (32) for some $C_1, C_2 \in \mathbb{R}$ and where $\kappa > 0$ is defined in equation (33). Since V is bounded, it must be that $C_2=0$. The value-matching condition imposes that $V(\phi^*)=h-\ell$; that is,

$$V(\phi) = \left(\tfrac{1-\phi}{\phi}\big/\tfrac{1-\phi^*}{\phi^*}\right)^{\kappa}(h-\ell) \,.$$

This implies that

$$\lim_{\phi \searrow \phi^*} p(\phi) = h - \tfrac{\mu}{r}\,\kappa\,(h-\ell) < h \,.$$

This is a contradiction, since $p(\phi) \geq \ell/\phi$.

We let $\phi^{\ddagger} \in (\phi^*,1)$ be the supremum value such that $\pi(\phi) \in (0,1)$ for all $\phi \in (\phi^*,\phi^{\ddagger})$.

Step 4. Equations for existence and uniqueness. From Step 3, we have that there is some $C_1 \in \mathbb{R}_{++}$ such that, for all $\phi \in (\phi^{\ddagger},1)$, we have[27]

$$V(\phi;C_1) = C_1\left(\tfrac{1-\phi}{\phi}\right)^{\kappa} \,. \tag{49}$$

For each C_1, we let $\hat{\phi}(C_1)$ be determined by setting $\pi(\hat{\phi}(C_1)) = 1$ in equation (44)

[27] Note that this coincides with expression (32) with $C_2 = 0$, satisfied by the continuation value on the upper belief regions of beliefs in the observable case.

replacing $V'(\phi)$ by $V'(\hat{\phi}(C_1); C_1)$. We then have

$$C_1 = \frac{r\left(\frac{1-\hat{\phi}(C_1)}{\hat{\phi}(C_1)}\right)^{-\kappa}(\hat{\phi}(C_1)-\phi^*)\,h}{\mu\,\kappa\,\hat{\phi}(C_1)}\,. \tag{50}$$

The derivative of the right-hand side of the previous expression with respect to $\hat{\phi}(C_1)$ is

$$\frac{r\left(\frac{1-\hat{\phi}(C_1)}{\hat{\phi}(C_1)}\right)^{-\kappa}\Big(\phi^*\,(1-\phi^*) - \overbrace{(\phi^*-\kappa)\,(\hat{\phi}(C_1)-\phi^*)}^{(*)}\Big)\,h}{\mu\,\kappa\,(1-\hat{\phi}(C_1))\,\hat{\phi}(C_1)^2}\,.$$

Given that $\hat{\phi}(C_1) > \phi^*$ and $\kappa > 0$, the term $(*)$ is bounded above by $\phi^*\,(1-\phi^*)$. Then, for each $C_1 > 0$, $\hat{\phi}(C_1)$ is uniquely defined, and $\hat{\phi} : \mathbb{R}_{++} \to (\phi^*, 1)$ is a bijective function. Hence, from now on, abusing notation, we use $\hat{\phi} \in (\phi^*, 1)$ instead of C_1 as the free variable coming from the differential equation (47). We also use $V(\cdot;\hat{\phi})$ instead of $V(\cdot; C_1)$.

For any given $\hat{\phi}$, the pair of values of $V(\hat{\phi};\hat{\phi})$ and $V'(\hat{\phi};\hat{\phi})$ can be used as boundary conditions to obtain a unique solution to equation (46) on $(\phi^*, \hat{\phi}]$, which we denote $V(\phi;\hat{\phi})$ without risk of confusion.[28] From the previous expressions we have that

$$V(\hat{\phi};\hat{\phi}) = \frac{(\hat{\phi}-\phi^*)\,r\,h}{\hat{\phi}\,\kappa\,\mu}\,. \tag{51}$$

The right hand side of equation (51) is increasing in $\hat{\phi}$ on $(\phi^*, 1)$, and 0 at $\hat{\phi} = \phi^*$. Additionally, the previous expressions imply

$$V'(\hat{\phi};\hat{\phi}) = -\frac{(\hat{\phi}-\phi^*)\,r\,h}{(1-\hat{\phi})\,\hat{\phi}^2\,\mu}\,. \tag{52}$$

It is easy to see that this is decreasing in $\hat{\phi}$ for all $\hat{\phi} \in (\phi^*, 1)$.

Hence, there is an equilibrium (which must be as specified in the statement of the proposition) for a given value $\phi^\ddagger \in (\phi^*, 1)$ only if there is a solution of equation (46) on $(\phi^*, \phi^\ddagger)$, denoted $V(\cdot;\phi^\ddagger)$, with boundary conditions (51) and (52), and the lower boundary condition holds, that is, if $\lim_{\phi \searrow \phi^*} V(\phi;\phi^\ddagger) = h-\ell$.

Step 5. Change of variables. From now on, we assume an equilibrium with continuation value V exists, and we will establish the necessary and sufficient conditions that

[28] Note that, by the smooth pasting condition, $V(\cdot;\hat{\phi})$ is continuous and differentiable at $\hat{\phi}$. It satisfies equation (46) on $(\phi^*, \hat{\phi}]$ and equation (47) on $[\hat{\phi}, 1)$.

it satisfies, and we will finally establish the existence of a unique equilibrium in Step 6. We change variables defining, for each $\hat{\phi} \in (\phi^*, 1)$,

$$W(y) \equiv \frac{2^{1/2}\sqrt{\mu/r}\,\hat{\phi}}{(\hat{\phi}-\phi^*)\,h}\Big(V\big(\tfrac{\hat{\phi}\,\phi^*\,y}{\hat{\phi}\,(y-1)+\phi^*}\big) - (h-\ell)\Big)$$

for all $y \in \big[1, +\infty\big)$. Note that W is negative and increasing. Note also that the limit $\phi \searrow \phi^*$ corresponds to the limit $y \to \infty$, while $\phi = \hat{\phi}$ corresponds to $y = 1$. Using our definition, equation (46) takes the following simpler form:

$$W''(y) = y^2\, W(y)\, W'(y)^2\ . \tag{53}$$

Since $W(\cdot)$ is negative, it is also concave. The requirement that $\lim_{\phi\searrow\phi^*} V(\phi) = h-\ell$ corresponds to $\lim_{y\to\infty} W(y) = 0$. We will now analyze solutions to equation (53).

We note that the boundary conditions (51) and (52) can be written as boundary conditions on W at $y = 1$ as follows:

$$W(1;\hat{\phi}) = 2^{1/2}\sqrt{\mu/r}\,\big(\tfrac{r}{\kappa\,\mu} - \tfrac{\hat{\phi}\,(1-\phi^*)}{\hat{\phi}-\phi^*}\big)\ , \tag{54}$$

$$W'(1;\hat{\phi}) = \frac{2^{1/2}}{\sqrt{\mu/r}}\,\frac{\hat{\phi}-\phi^*}{(1-\hat{\phi})\,\phi^*}\ . \tag{55}$$

The right-hand side of equation (54) is strictly negative if and only if $\hat{\phi} \in (\phi^*, \hat{\phi}^+)$, where

$$\hat{\phi}^+ \equiv \min\big\{1, \phi^*\,\big(1-(1-\phi^*)\,\kappa\,\mu/r\big)^{-1}\big\} > \phi^*\ .$$

Also, in this range, $W(1;\hat{\phi})$ increases from $-\infty$ to either 0 (if $\hat{\phi}^+ < 1$) or $2^{1/2}\,\big(\kappa^{-1} - \sqrt{\mu/r}\big)$ (if $\hat{\phi}^+ = 1$). The right-hand side of equation (55) is strictly positive and strictly increasing in $\hat{\phi}$ on $(\phi^*, 1)$, and increases from 0 to $+\infty$.

Step 6. Existence and uniqueness. We now aim to show that there is a unique $\hat{\phi} \in (\phi^*, 1)$ such that

$$\lim_{y\nearrow\infty} W(y;\hat{\phi}) = 0\ ,$$

where $W(\cdot;\hat{\phi})$ is the solution to (53) with boundary conditions given by equations (54) and (55). We do it in two parts:

1. **Uniqueness:** Take two different values $\hat{\phi}_1$ and $\hat{\phi}_2$ satisfying $\phi^* < \hat{\phi}_1 < \hat{\phi}_2 < \hat{\phi}^+$. By the previous results, $W(1;\hat{\phi}_1) < W(1;\hat{\phi}_2) < 0$ and $0 < W'(1;\hat{\phi}_1) < W'(1;\hat{\phi}_2)$. We want to show that $W(y;\hat{\phi}_2) - W(y;\hat{\phi}_1)$ is increasing in y, and so $W(\cdot;\hat{\phi})$ tends to 0

for at most one value of $\hat{\phi} \in (\phi^*, \hat{\phi}^+)$. Assume, for the sake of contradiction, there is some value y' such that $W'(y'; \hat{\phi}_1) = W'(y'; \hat{\phi}_2)$, and let y be the infimum with this property. It then has to be that $W(y; \hat{\phi}_1) < W(y; \hat{\phi}_2)$. Then we have

$$W''(y; \hat{\phi}_1) = y^2\, W(y; \hat{\phi}_1)\, W'(y; \hat{\phi}_1)^2 < y^2\, W(y; \hat{\phi}_2)\, W'(y; \hat{\phi}_2)^2 = W''(y; \hat{\phi}_2)\ .$$

This is a contradiction, since $W'(\cdot; \hat{\phi}_1) < W'(\cdot; \hat{\phi}_2)$ on $(1, y)$. Hence, we have that $W'(y; \hat{\phi}_1) < W'(y; \hat{\phi}_2)$ for all $y > 1$. Therefore, if $W(y; \hat{\phi}_1)$ and $W(y; \hat{\phi}_2)$ converge to some value as $y \to \infty$, they converge to different values. A similar argument implies that two solutions of equation (53) cross at most once.

2. **Existence:** Note that a particular solution of (53) is $\hat{W}(y) \equiv -2^{1/2}/y$.[29] Note also that, as $\hat{\phi} \to \phi^*$,

$$W(1; \hat{\phi}) \to -\infty < \hat{W}(1) \ \text{ and } \ W'(1; \hat{\phi}) \to 0 < \hat{W}'(1)\ .$$

Hence, if $\hat{\phi}$ is close enough to ϕ^*, so $\lim_{y\to\infty} W(y; \hat{\phi})$ converges to a value strictly lower than $\lim_{y\to\infty} \hat{W}(y)$, which is equal to 0. We assume, for the sake of contradiction, that $\lim_{y\to\infty} W(y; \hat{\phi}^+) = w^+$, for some $w^+ < 0$. Since $W(1; \hat{\phi}^+) = 0$ if $\hat{\phi}^+ < 1$, then it must be that $\hat{\phi}^+ = 1$ and so $W'(1; \hat{\phi}^+) = +\infty$. Then, any solution of equation (53) with $W(1) < W(1; \hat{\phi}^+)$ is such that $\lim_{y\to\infty} W(y) < w^+$. We let $W(y; 1)$ be defined as $\lim_{\hat{\phi}\to 1} W(y; \hat{\phi})$, which by assumption satisfies $\lim_{y\to\infty} W(y; 1) < 0$. For each ε, let $\tilde{W}_\varepsilon(\cdot)$ be defined as the solution to equation (53) satisfying $\tilde{W}_\varepsilon(2) = W(2; 1)$ and $\tilde{W}'_\varepsilon(2) = W'(2; 1) + \varepsilon$. It is clear that if $\varepsilon > 0$ is chosen strictly larger than 0, then $\tilde{W}_\varepsilon(1) < W(1, 1)$. Since solutions to (53) only cross once, we have $\lim_{y\to\infty} \tilde{W}_\varepsilon(y) > w^+$, but this is a contradiction. Hence, there exists a unique value $\hat{\phi}^\dagger$ such that $W(y; \hat{\phi}^\dagger) = 0$, and hence a unique equilibrium exists.

□

Proof of Proposition 4.3

Proof. Assume $\mu < \overline{\mu}$. We define

$$\hat{V}^{\text{ob}}(\phi) = h - \ell + \tfrac{r\,(\phi\, h - \ell)}{\mu\, \phi} + \tfrac{r\,(h - \ell)}{\mu}\, \log\left(\tfrac{1-\phi}{\phi} \big/ \tfrac{1-\phi^*}{\phi^*}\right)$$

for all $\phi \in [\phi^*, 1)$. Note that this coincides with $V^{\text{ob}}(\phi)$ for all $\phi \in [\phi^*, \phi^\dagger]$ (recall equation (29)). We now define $z^\dagger = \check{z}(\phi^\dagger)$ and $z^\ddagger = \check{z}(\phi^\ddagger)$ (recall the definition of $\check{z}$ in

[29] Such solution is an equilibrium when $\mu = r$, in which case $\hat{\phi}^\dagger = 2\,\phi^*/(1+\phi^*)$.

(18)). Recall also that equation (42) holds for V^{ob}, and we also have

$$\frac{V^{\text{un}}(z^\ddagger)}{-V^{\text{un}\prime}(z^\ddagger)} \begin{cases} > \kappa^{-1} & \text{if } z \in (z^*, z^\ddagger), \\ = \kappa^{-1} & \text{if } z \in [z^\ddagger, +\infty). \end{cases} \tag{56}$$

Since $\hat{V}^{\text{ob}\prime}(\hat{\phi}^\dagger) = V^{\text{un}\prime}(\hat{\phi}^\dagger)$ (because $\pi^{\text{un}}(\hat{\phi}^\dagger) = 1$, and so $\gamma^{\text{ob}}(\hat{\phi}^\dagger) = \gamma^{\text{un}}(\hat{\phi}^\dagger) = 1$ in equation (16)), and hence $\hat{V}^{\text{ob}}(\hat{\phi}^\dagger) > V^{\text{un}}(\hat{\phi}^\dagger)$, we have $z^\ddagger < z^\dagger$. Now the argument proceeds as in the proof of Proposition 4.1: Assume, for the sake of contradiction, that there is some $z > z^\ddagger$ such that $V^{\text{un}}(z) = V^{\text{ob}}(z)$. There must then be some $\hat{z} > z^\ddagger$ such that $V^{\text{un}}(\hat{z}) = V^{\text{ob}}(\hat{z})$ and $-V^{\text{un}\prime}(\hat{z}) \leq -V^{\text{ob}\prime}(\hat{z})$. But then, this implies,

$$\frac{V^{\text{ob}}(\hat{z})}{-V^{\text{ob}\prime}(\hat{z})} \leq \frac{V^{\text{un}}(\hat{z})}{-V^{\text{un}\prime}(\hat{z})} = \kappa^{-1} \leq \frac{V^{\text{ob}}(\hat{z})}{-V^{\text{ob}\prime}(\hat{z})} \, ,$$

that is, $\frac{V^{\text{ob}}(\hat{z})}{-V^{\text{ob}\prime}(\hat{z})} = \kappa^{-1}$. This implies that $\hat{z} \geq z^\dagger$. Nevertheless, we then have that $V^{\text{un}}(z^\dagger) < V^{\text{ob}}(z^\dagger)$, which implies that $V^{\text{un}}(z^\dagger) < V^{\text{ob}}(z^\dagger)$ for all $z > z^\dagger$ (recall that both V^{un} and V^{ob} satisfy equation (49), hence it must be that the value of C_1 is smaller for $x = \text{un}$ than for $x = \text{ob}$). This concludes the proof of the proposition. □

Proof of Proposition 4.4

Proof. We prove each part separately:

1. We first prove that, if $\mu \leq \overline{\mu}$, then $W^{\text{ob}}(\phi_0) > W^{\text{un}}(\phi_0) > W^{\text{no}}(\phi_0)$ for all $\phi_0 \in (\phi^*, 1)$. The last inequality is trivial for the reasons laid out in the main text after the proposition. The first inequality is obtained as follows. Note that, when $\mu \leq \overline{\mu}$, we have that $\alpha^x(\phi) = 1$ for all $x \in \{\text{un}, \text{ob}\}$. Hence, the term $(**)$ in equation (17) is equal to h for all x. As explained after the proposition, the term $(*)$ in equation (17) is equal to $\frac{\ell}{h-\ell} V^x(\phi_0)$ for all $x \in \{\text{un}, \text{ob}\}$. Then, applying Proposition 4.3, the result follows.

2. We now prove that, if $\mu > \overline{\mu}$, then $W^{\text{un}}(\phi_0) > W^{\text{no}}(\phi_0)$ for all $\phi_0 \in (\phi^*, 1)$. This result holds trivially by the arguments after the proposition.

□

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