Nonparametric Identification and Estimation of Contests with Uncertainty*

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February 15, 2022

Abstract

Real-world contests are inherently uncertain since the player who exerts the highest effort can still lose. In this paper, I consider a general asymmetric incomplete information contest model with a nonparametric distribution of uncertainty in the contest success function. It generalizes all-pay auctions, Tullock contests, and rank-order tournaments with two asymmetric players. Uncertainty in the contest success function summarizes other factors that influence the contest win outcome apart from the efforts of the players, such as, for example, players' reputation or luck. First, I nonparametrically identify and estimate the distribution of uncertainty using the information on contest win outcomes and efforts. Next, I nonparametrically identify and estimate the distributions of the players' costs of exerting effort. The model provides a method to disentangle two sources of player's advantage: asymmetry in the costs' distributions and the effect of the uncertainty distribution on the winning probability. As an empirical example, I apply the model to the U.S. House of Representatives elections.

Keywords: Contest, Nonparametric Identification, Nonparametric Estimation, Incomplete Information

*I am indebted to my advisers John Asker, Francesco Decarolis, and Rosa Liliana Matzkin for their support and encouragement. I would like to thank the audiences at UCLA, Bocconi, and the CEPR/JIE Conference on Applied Industrial Organization for helpful discussions.

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1 Introduction

A contest is a natural model of costly competition. It describes situations when all players spend resources (for example, exert efforts, spend money, or time) in order to affect their probability of getting an object (or a prize). The effort is a sunk cost, since no matter whether a player wins or loses the cost is always incurred. A broad variety of real-world situations falls under this description. For example, electoral competitions were modeled using contest theory since the 1980s as all the candidates raise and spend money for their campaigns, but only one candidate wins the election and obtains the seat (see, for example, Snyder (1989), Baron (1994) and Skaperdas and Grofman (1995)). The effort in this case is represented by campaign expenditures. In research and development, firms incur R&D expenses but only one gets the patent (Taylor (1995), Che and Gale (2003), Dasgupta (1986)). Applications have also been made to numerous other scenarios including marketing and advertising by firms, litigation, sport events, arm races and rent-seeking activities, such as lobbying.[1]

Most of these real-world situations are inherently uncertain since the player who exerts the highest effort can still lose. For example, the candidate with the highest campaign spending can still lose the election; the company with the highest R&D expenditures can still not get the patent. In contest models, the uncertainty is captured through the contest success function that maps the efforts into probabilities of winning for participating players. As an example, in the case of the Tullock contest, the probability of winning is determined by the relative efforts of the players, and thus uncertainty takes a very particular parametric form.

In this paper, instead, I account for uncertainty in a much more flexible way. I consider a general model of imperfectly discriminating contests, which was introduced by Hillman and Riley (1989), and which generalizes the rank-order tournament model of Lazear and Rosen (1981). In this model, the probability of winning is determined not only by the efforts of the players but also by a stochastic variable that summarizes all other factors that influence the uncertain result of the contest. I call this variable uncertainty. Its distribution is known to the players, but unknown to the researcher. Instead of imposing any type of parametric assumptions, I will nonparametrically estimate the distribution of uncertainty. Uncertainty also determines one possible advantage of one player over the other: even if players exert the same efforts, one might have higher chances of winning. For example, one political candidate might simply be more popular than the other given campaign expenditures; among two equally well-trained athletes, the one who performs better under stress might win more tournaments.

Importantly, this model generalizes many common contests such as all-pay auctions (in which

[1] See, for example, Bell et al. (1975), Farmer and Pecorino (1999), Bernardo et al. (2000), Hirshleifer and Osborne (2001), Baye et al. (2005), Tullock (1980), Moldovanu and Sela (2001), Krueger (1974), Baye et al. (1993), Kang (2015).

there is no uncertainty given the efforts since the bidder with the highest effort wins), Tullock contests, logit-form contests, and several others.[2]

The contest model that I study is of incomplete information and with asymmetric players. Each player has a cost type (or ability) that is drawn from different distributions. Cost types describe how costly it is to exert the efforts for the players, which is a second key source of possible advantage. For example depending on the candidates' abilities, for one candidate it is less costly to raise campaign financing than for the other; for one firm R&D is less costly than for the other. The model is a game of incomplete information in the sense that the players do not observe the other players' cost types, but the distributions of the cost types are common knowledge. Each player exerts effort in order to win a prize, knowing his own cost type, the distribution of the other player's cost type, and the distribution of uncertainty. Fey (2008), Ryvkin (2010), Ewerhart (2014), and Wasser (2013) are a few papers providing the existence of equilibrium results for incomplete information contests.[3]

The literature on nonparametric identification and estimation of incomplete information auctions and contests is very scarce even though the nonparametric methods received growing attention in recent years. The only paper that studies the identification of an imperfectly discriminating contest as a game with incomplete information is the one by He and Huang (2020).[4] In that paper, the authors focus on the Tullock contest, which is a particular case of the general model that I consider.

I first prove the identification of the model. I provide a method to disentangle two sources of the possible advantage of one of the players: asymmetry in the cost types' distributions and the effect of the uncertainty distribution on the winning probability. The important difference between these two advantages is that in most contexts only the former can be influenced by the policymakers, whereas the latter can not (as it captures the broad variety of inherently uncertain factors). Thus it is crucial for policy implications to be able to distinguish and quantify these two sources of advantage. I propose a method to identify both the distribution of players' cost types as well as the distribution of uncertainty from observed efforts and win outcomes, which does not require solving for the Bayesian Nash Equilibrium. The novelty of the paper is that I do not impose any parametric assumption on the uncertainty distribution of the contest success function.

[2]See Jia (2008), Jia et al. (2013), Ryvkin and Drugov (2020) for stochastic derivations of several contest models.

[3]In contrast to incomplete information setting, the literature on the equilibrium characterisation in complete information setting is larger. See, for example, Cornes and Hartley (2005), Yamazaki (2008), Siegel (2009), Siegel (2010).

[4]The first-price auctions, instead, were studied in detail in the block of papers originated from Guerre et al. (2000). Moreover, Shakhgildyan (2019) studies the identification and estimation of the all-pay auctions.

Second, I propose a two-step estimation procedure. In the first step, I estimate the distribution of uncertainty using the information on win outcomes and efforts. Next, I estimate the distributions of the players' cost types. I prove the consistency and asymptotic normality of the proposed estimators and use Fréchet derivatives to find the asymptotic distributions. Given the growing availability of big data, nonparametric estimation becomes more and more desirable, and the proposed model provides a tool that can be applied to a wide range of applications.

As an empirical application, I consider the U.S. House of Representatives elections. I chose the application also considered in He and Huang (2020) to better illustrate how the proposed nonparametric method compares to the existing state-of-the-art method proposed by them. Using the model, I disentangle and estimate two potential advantages of the Incumbent. The first source of advantage comes from the asymmetry in cost types since Incumbent might have a better ability in raising money for the campaign than the Challenger. The second comes from the effect of the uncertainty distribution on winning probability since even if Incumbent and Challenger spend the same amounts on their campaigns, the Incumbent might have higher chances of winning (due to better reputation, name recognition, and other possible non-monetary factors). This separate identification is particularly important for policy counterfactual analysis since only campaign financing can be regulated by the authority.

A large body of empirical work studies the effect of campaign spending on Congressional elections outcomes starting from the pioneering work of Jacobson (1978). My paper contributes to the literature by providing a method of recovering the incumbency advantage in campaign financing, as well as the advantage of the Incumbent due to his reputation in a nonparametric way. This is done using the information on the observed spending as well as winning outcomes, and the nonparametric structural contest model. Results of the model suggest that the incumbency advantage was prevalent throughout the sample period 1972-2016. Incumbents won 93.9% of contests. Moreover, on average Incumbents spent 2.5 times as much as the average Challenger. Using the structural model, I estimate that if the Incumbents were to spend as much as the Challengers they would instead win only 85% of the elections.

The knowledge of the distributions of cost types allows policymakers to quantify the effect of different policy changes. I consider two different policy counterfactual analyses aimed at limiting the incumbency advantage: a public campaign financing of Challengers and a limit on Incumbents' expenditure. I show that the latter is more effective in terms of lowering both the Incumbents' winning probability as well as the total campaign spending.

The rest of the paper is organized as follows. In Section 2, I introduce the contest model with uncertainty. Section 3 discusses the nonparametric identification of the model. Section 4 considers the nonparametric estimation. The application to the U.S. House Elections is presented in Section 5. Section 6 concludes. All the proofs are omitted from the main text and are presented in the Appendix.

2 Contest Model with Uncertainty

2.1 Notations and Definitions

In this paper, I consider a contest of incomplete information with two asymmetric risk-neutral players. This is motivated by the nature of the application in which two candidates are competing for a seat in the U.S. House of Representatives: one is the Incumbent and the other is the Challenger.[5] The general asymmetric model allows the econometrician to study and disentangle two sources of the possible advantage of the players: asymmetry in the cost type distributions and the effect of the uncertainty distribution on the winning probability.

Assumption 1. *Each player has a cost type c_i, $i = 1, 2$, which is his private information. Player draws these costs c_i, $i = 1, 2$ independently from commonly known distributions $F_i(\cdot)$ with supports $[\underline{c}_i, \bar{c}_i]$, densities f_i and quantile functions $q_i = F_i^{-1}$, $i = 1, 2$.*

Assumption 2. *The players exert efforts b_i simultaneously.*[6]

Assumption 3. *The efforts are sunk, regardless of whether or not the player wins a prize.*

Moreover, the impact of the efforts (campaign spending in the application) on the winning probability is uncertain. To incorporate this uncertainty in the model, I assume that the probability of winning is determined not only by the efforts of the players but also by nonparametric stochastic components ϵ_i, $i = 1, 2$. Those components summarize other factors apart from the efforts that are not under the control of the player at the time of exerting the effort, but that could also influence the winning probability. For example, in the application to the elections, both campaign expenditures, as well as other factors such as the Incumbent's reputation, determine voters' preferences over the candidates. The goal is to disentangle and estimate these two potential advantages of the Incumbent. The first source of advantage is due to the fact that the Incumbent often has a better reputation and is more experienced than the Challenger. The other source of advantage is the Incumbent's better skills in raising money for the campaign.

I denote the ratio of epsilons ξ and call it uncertainty. Depending on uncertainty ξ, players' efforts have different effects on the winning probability. Formally,

Assumption 4. *The real impact is $x_i = b_i \cdot \epsilon_i$, $i = 1, 2$, where ϵ_i is assumed to be independent of b_i and c_i. $H_\xi(\cdot)$ is the CDF of $\epsilon_2/\epsilon_1 := \xi$. I refer to ξ as uncertainty. Each ϵ_i, $i = 1, 2$*

[5] The model with arbitrary N can be nonparametrically identified and estimated under the assumption of the identically distributed uncertainty terms.

[6] In case when the players can observe the efforts of each other, the model becomes simpler since there would be no need to integrate over all possible efforts of the other player.

has a positive support, since only positive x_i can lead to victory. $h_\xi(\cdot)$ is corresponding density function.

In this paper, I consider the case when the higher expenditures have a multiplicative effect on the political impact: as in Hillman and Riley (1989), where the model was introduced. By applying the logarithm, it is straightforward to switch from the multiplicative model to the additive model usually assumed in the rank-order tournaments originated from Lazear and Rosen (1981) in which real impact is given by $x_i = b_i + \epsilon_i$. Note that in contrast to the standard assumption in the rank-order tournaments that ϵ_i are i.i.d, the model in the paper is more general as it does not require ϵ_i to be identically distributed. In contrast, a difference in the distribution of ϵ_i would allow quantifying the advantage of one of the players due to the factors that are independent of the spending.

Assumption 5. *At the time of exerting the effort, each player i knows his own cost type c_i, as well as $F_j(\cdot)$ and the distributions of uncertainty ξ.*

Moreover, let $w_i = 1$ if player i wins and $w_i = 0$ otherwise. Then the probability of winning of the first player given the efforts is:

$$P(w_1 = 1|\ b_1, b_2) = P(x_1 > x_2 \mid b_1, b_2) = P(b_1\epsilon_1 > b_2\epsilon_2 \mid b_1, b_2) = P(b_1 > b_2\xi \mid b_1, b_2), \tag{1}$$

where ϵ_1 and ϵ_2 are preferences for player 1 and player 2 respectively.

The expected payoff to player i participating in the contest is given by:

$$E[U_i|c_i, F_j, H_\xi] = P[w_i = 1|c_i, F_j, H_\xi] - c_ib_i = P(b_i\epsilon_i > b_j\epsilon_j|c_i, F_j, H_\xi) - c_ib_i, \tag{2}$$

where $i = 1, 2,\ j = -i$.[7] The final payoff to the player i is $1 - c_ib_i$ if he obtains a good, and $-c_ib_i$ if he does not obtain a good.

It is worth noting that:

Proposition 1. *In a specific case when both ϵ_i and ϵ_j have an exponential distribution with parameter $\lambda = 1$, the contest described above is equivalent to the Tullock contest.*

That proposition shows that a widely used in applications Tullock contest is a particular case of the general model presented in the paper. Moreover, under different exact distributions of ϵ_i, the model reduces to the generalized Tullock contest, logit contest, probit contest, and several others. For example, the inverse exponential distribution of ϵ_i's yields exactly the ratio form of the contest success function. For further discussions see Jia (2008), Jia et al. (2013), Ryvkin and Drugov (2020).

[7]This model can be extended to account for the observables by assuming:

$$P[w_i = 1|c_i, F_j, H_\xi] = P[(b_i + m(X_i))\epsilon_i \geq (b_j + m(X_j))\epsilon_j|c_i, F_j, H_\xi],$$

where both function m and distribution of ϵ_2/ϵ_1 can be identified in the first step.

2.2 Equilibrium Characterization

I consider the strictly monotonic Bayesian Nash equilibrium (BNE) in this incomplete information game. Using the results of Athey (2001), I prove the existence of equilibrium.

Proposition 2. *Given Assumptions 1-5 are satisfied, there exists a pure strategy decreasing BNE of the incomplete information game formulated above.*

The formal proof is presented in the Appendix, but the main intuition is that in this game the single-crossing property holds with strict inequality, and thus the BNE in decreasing strategies exists.

For each cost type, the corresponding effort is defined by the function $s_i(c_i) = b_i$, $i = 1, 2$ that is the equilibrium effort strategy which maximizes the player i's expected payoff. Since $s_i(c_i)$ is strictly monotonic it is invertible and $s_i^{-1}(b_i) = c_i$.

Given these decreasing strategies, we are able to express the first-order conditions of the game in a way that represents the cost types in terms of the efforts' distributions and equilibrium strategies.

Proposition 3. *Given Assumptions 1-5 as well as the assumption of strict monotonicity of the strategies the first-order conditions of this game can be written as:*

$$c_1 = \int_{\underline{c}_2}^{\bar{c}_2} f_2(c_2) \frac{1}{s_2(c_2)} h_\xi \left(\frac{s_1(c_1)}{s_2(c_2)} \right) dc_2 \tag{3}$$

and

$$c_2 = \int_{\underline{c}_1}^{\bar{c}_1} f_1(c_1) \frac{s_1(c_1)}{s_2^2(c_2)} h_\xi \left(\frac{s_1(c_1)}{s_2(c_2)} \right) dc_1. \tag{4}$$

In our case, given the data, private cost types and the distribution of uncertainty are unobserved for the econometrician, whereas efforts are observed. Thus the goal for the identification would be to rewrite the right-hand sides of the equations (3) and (4) in terms of distribution of efforts. The first complication is that the effort distributions depend on the underlying cost type distributions in two ways: directly through the cost types, and indirectly through the equilibrium strategies that we cannot solve analytically.

Moreover, the right-hand sides also depend on the unobserved distribution of uncertainty which we need to identify separately. The method is described in detail in the Section on Identification.

2.2.1 Representation in Terms of Valuations

The problem can be easily reformulated in terms of the valuations since the model is equivalent to the one in which $v_i = 1/c_i$. The expected payoff to player i, in this case, is given by:

$$E[U_i|v_i, F_j, H_\xi] = v_i P[w_i = 1|v_i, F_j, H_\xi] - b_i = v_i P(b_i \epsilon_i > b_j \epsilon_j|v_i, F_j, H_\xi) - b_i, \tag{5}$$

where $i = 1, 2$, $j = -i$, $v_i = \frac{1}{c_i}$ and F_i is the value distribution function whereas f_i is the corresponding density.

It can be easily seen that equations (3) and (4) can be written in terms of valuations:

$$v_1 = \left(\int_{\underline{v}_2}^{\bar{v}_2} f_2(v_2) \frac{1}{s_2(v_2)} h_\xi \left(\frac{s_1(v_1)}{s_2(v_2)} \right) dv_2 \right)^{-1} \tag{6}$$

and

$$v_2 = \left(\int_{\underline{v}_1}^{\bar{v}_1} f_1(v_1) \frac{s_1(v_1)}{s_2^2(v_2)} h_\xi \left(\frac{s_1(v_1)}{s_2(v_2)} \right) dv_1 \right)^{-1}. \tag{7}$$

3 Nonparametric Identification

In this section, I prove that the unknown elements of the model are nonparametrically identified from available data.

In the presented model there are three unknown structural elements for the econometrician - the distributions of cost types $F_i(\cdot)$, $i = 1, 2$ as well as the distribution $H_\xi(\cdot)$ of uncertainty $\xi = \epsilon_2/\epsilon_1$, whereas the number of players, the efforts themselves b_i, $i = 1, 2$ as well as the win results, are observed.[8] Therefore the identification problem reduces to whether the distributions F_i, $i = 1, 2$ and H_ξ are uniquely determined from observed efforts and win outcomes. Note that the distribution $G_i(\cdot)$ of b_i depends on the underlying distributions $F_i(\cdot)$, $i = 1, 2$ not only through c_i, but also through the equilibrium strategies $s_i(\cdot)$, $i = 1, 2$ and the distribution of uncertainty.

Formally, let $\mathcal{G}$ denote the set of all distributions $G = (G_1, G_2)$ over the space of permitted efforts and let p denote the win probability of one of the players, $F = (F_1, F_2) \in \mathcal{F}$ and $H_\xi \in \mathcal{H}$. Let us call the mapping from the private information to efforts $\gamma \in \Gamma$, where $\gamma : \mathcal{F} \times \mathcal{H} \to \mathcal{G} \times p$. Then,

Definition 1. *(Identification). A model $(\mathcal{F}, \mathcal{H}, \Gamma)$ is identified if for every (F, F'), (H_ξ, H'_ξ) and (γ, γ'), $\gamma(F, H_\xi) = \gamma'(F', H'_\xi) \Rightarrow (F, H_\xi, \gamma) = (F', H'_\xi, \gamma')$.*

[8] Note that we cannot not separately identify the distributions of the ϵ's, only their ratio.

The identification argument is conducted in the following two steps. First:

Proposition 4. *The distribution of uncertainty $\xi = \epsilon_1/\epsilon_2$ is identified from the data on efforts and win outcomes.*

The identification follows from the fact that when the ratio of the efforts varies significantly, but the winner's identity does not change, the ratio of the uncertainty terms should also vary to explain the winning outcome. Formally:

$$P(w_1 = 1) = P(b_1\epsilon_1 > b_2\epsilon_2) = P\left(\frac{\epsilon_2}{\epsilon_1} < \frac{b_1}{b_2}\right) := P\left(\xi < \frac{b_1}{b_2}\right) = H_\xi\left(\frac{b_1}{b_2}\right), \tag{8}$$

where conditioning on efforts is omitted for simplicity of the exposition.

Thus the distribution of $\frac{\epsilon_1}{\epsilon_2}$ can be identified from observed win outcomes on the positive support by varying b_1/b_2.

In the second step, the distribution of ξ is used to recover the cost type distribution.

Proposition 5. [9] *Suppose that functions*

$$\lambda_1(b_1, G, H_\xi) \equiv \int_{\underline{b}_2}^{\bar{b}_2} g_2(b_2)\frac{1}{b_2}h_\xi\left(\frac{b_1}{b_2}\right)db_2$$

and

$$\lambda_2(b_2, G, H_\xi) \equiv \int_{\underline{b}_1}^{\bar{b}_1} g_1(b_1)\frac{b_1}{b_2^2}h_\xi\left(\frac{b_1}{b_2}\right)db_1$$

are strictly decreasing on the supports of efforts $[\underline{b_i}, \bar{b_i}]$, $i = 1, 2$, respectfully, and their inverses are differentiable on the supports of types $[\underline{c_i}, \bar{c_i}]$. If $G_i(\cdot)$, $i = 1, 2$, are absolutely continuous probability distributions with support $[\underline{b_i}, \bar{b_i}]$, then there exist absolutely continuous distributions of players' private cost types $F_i(\cdot)$ corresponding to the distributions of efforts. When $F_i(\cdot)$ exist, they are unique with supports $[\underline{c_i}, \bar{c_i}]$, $i = 1, 2$, respectfully, and satisfy $F_i(c_i) = 1 - G_i((\lambda_i)^{-1}(b_i, G, H_\xi))$ for all $c_i \in [\underline{c_i}, \bar{c_i}]$. In addition, $\lambda_i(\cdot, G, H_\xi)$ are the quasi inverse of the equilibrium strategies in the sense that $\lambda_i(b, G, H_\xi) = s_i^{-1}(b, F, H_\xi)$ for all $b \in [\underline{b_i}, \bar{b_i}]$. Moreover, the identifying equations can be rewritten in terms of quantile functions:

$$q_1(1 - t_1) = \int_0^1 \frac{1}{r_2(t_2)}h_\xi\left(\frac{r_1(t_1)}{r_2(t_2)}\right)dt_2 \tag{9}$$

[9]The formulation of the proposition is similar to Theorem 1 in Guerre et al. (2000).

and

$$q_2(1-t_2) = \int_0^1 \frac{r_1(t_1)}{r_2^2(t_2)} h_\xi \left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_1, \tag{10}$$

where t_1, $t_2 \in (0,1)$.

Similar to Guerre et al. (2000), the identification result is based on the property that together with the distribution $F_i(\cdot)$ and the density $f_i(\cdot)$, the derivative of the strategy $s_i'(\cdot)$ can be canceled out from the differential equation.

For every $b \in [\underline{b_i}, \bar{b_i}] = [s_i(\bar{c_i}), s_i(\underline{c_i})]$, we have $G_i(b) = Pr(b_i \leq b) = Pr(c_i \geq s_i^{-1}(b)) = 1 - F_i(s_i^{-1}(b)) = 1 - F_i(c)$, where $b_i = s_i(c_i)$. Thus the distribution $G_i(\cdot)$ is absolutely continuous, has support $[s_i(\bar{c_i}), s_i(\underline{c_i})]$ and density $g_i(b_i) = -\frac{f_i(c_i)}{s_i'(c_i)}$, where $c_i = s_i^{-1}(b_i)$. The formal proof is presented in the Appendix.

Thus the players' cost types can be represented in terms of the distributions of efforts and the estimated earlier distribution of the uncertainty.

Proposition 5 can be also reformulated in terms of valuations.

Corollary 1. *Suppose that functions*

$$\lambda_1(b_1, G, H_\xi) \equiv \left(\int_{\underline{b}_2}^{\bar{b}_2} g_2(b_2) \frac{1}{b_2} h_\xi \left(\frac{b_1}{b_2}\right) db_2\right)^{-1}$$

and

$$\lambda_2(b_2, G, H_\xi) \equiv \left(\int_{\underline{b}_1}^{\bar{b}_1} g_1(b_1) \frac{b_1}{b_2^2} h_\xi \left(\frac{b_1}{b_2}\right) db_1\right)^{-1}$$

are strictly increasing on the supports of efforts $[\underline{b_i}, \bar{b_i}]$ $i = 1, 2$, *respectfully, and their inverses are differentiable on the supports of valuations* $[\underline{v_i}, \bar{v_i}]$. *If* $G_i(\cdot)$, $i = 1, 2$, *are absolutely continuous probability distributions with supports* $[\underline{b_i}, \bar{b_i}]$, *then there exist absolutely continuous distributions of players' valuations* $F_i(\cdot)$ *corresponding to the distributions of efforts. When* $F_i(\cdot)$ *exist, they are unique with supports* $[\underline{v_i}, \bar{v_i}]$, $i = 1, 2$, *respectfully, and satisfy* $F_i(v_i) = G_i(\lambda_i^{-1}(b_i, G, H_\xi))$ *for all* $v_i \in [\underline{v_i}, \bar{v_i}]$. *In addition,* $\lambda_i(\cdot, G, H_\xi)$ *are the quasi inverse of the equilibrium strategies in the sense that* $\lambda_i(b, G, H_\xi) = s_i^{-1}(b, F, H_\xi)$ *for all* $b \in [\underline{b_i}, \bar{b_i}]$. *Moreover, the identifying equations can be rewritten in terms of quantile functions:*

$$q_1^v(t_1) = \left(\int_0^1 \frac{1}{r_2(t_2)} h_\xi \left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_2\right)^{-1} \tag{11}$$

and

$$q_2^v(t_2) = \left(\int_0^1 \frac{r_1(t_1)}{r_2^2(t_2)} h_\xi \left(\frac{r_1(t_1)}{r_2(t_2)} \right) dt_1 \right)^{-1}, \tag{12}$$

where $t_1,\ t_2 \in (0,1)$.

That is a corollary of Proposition 5 since $v_i = \frac{1}{c_i}$, $i = 1,2$.

4 Nonparametric Estimation

In this section, I propose the asymptotically normal estimators of the density h_ξ and the players' cost types.

Let L be the number of contests, l is the the l-th contest, $\{b_{il}, i = 1,2, l = 1,...,L\}$ are the observations of the efforts, $\{w_{il}, i = 1,2, l = 1,...,L\}$ are the observations of the winning outcomes.[10]

In the first step, I estimate the distribution of uncertainty ξ from the observed efforts and winning outcomes using kernel estimation. Specifically, consider player 1 winning probability:

$$\hat{H}_\xi(b) = \hat{P}(w_1 = 1 | b_1/b_2 = b) = \frac{\sum\limits_{l=1}^{L} w_{1l} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right)}{\sum\limits_{l=1}^{L} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right)}, \tag{13}$$

where $K(\cdot)$ is the kernel function and h is the bandwidth.

By taking derivative with respect to b, we can find the estimator for the corresponding density function:

$$\hat{h}_\xi(b) = \hat{H}'_\xi(b) =$$

$$= \frac{\sum\limits_{l=1}^{L} w_{1l} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right) \cdot \sum\limits_{l=1}^{L} K'\left(\frac{b_{1l}/b_{2l} - b}{h}\right) - \sum\limits_{l=1}^{L} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right) \cdot \sum\limits_{l=1}^{L} w_{1l} K'\left(\frac{b_{1l}/b_{2l} - b}{h}\right)}{h \left[\sum\limits_{l=1}^{L} K\left(\frac{b_{1l}/b_{2l} - b}{h}\right) \right]^2}. \tag{14}$$

I use Fréchet derivatives to find the asymptotic distribution. In terms of the density of the observables:

$$H_\xi(b) = \frac{\int w f(w,b) dw}{f(b)},$$

[10] I assume that in each contest the same two types of players take part. In case when there are some observable characteristics of the players and enough data, the analysis is similar, with the only difference that we can condition on the observables.

where $f(w,b)$ is the density of the vector (w,b) and $b = b_1/b_2$. By taking the derivative with respect to b we get:

$$h_\xi(b) = \frac{f(b)\int w\frac{\partial f(w,b)}{\partial b}dw - \frac{\partial f(b)}{\partial b}\int wf(w,b)dw}{f(b)^2} = \frac{f(b)\int wf'(w,b)dw - f'(b)\int wf(w,b)dw}{f(b)^2}.$$

Next, I introduce assumptions that would allow proving the consistency and asymptotic normality of the proposed estimators.

Assumption 6. *The data on $\{b_i, w_i\}$ is i.i.d.*

Assumption 7. *The density $f(b)$ has compact support, is continuously differentiable of order $m \geq \delta + k$, $k \geq 2$, with derivatives which are uniformly bounded.*

Assumption 8. *The kernel function is of order δ, it has compact support and is continuously differentiable on its support.*

Assumption 9. *As $L \to \infty$, $h \to 0$, $\sqrt{Lh^3} \to \infty$, $\sqrt{Lh^{3+2k}} \to 0$.*

Then the following theorem holds:

Theorem 1. *Given the assumptions about the model as well as Assumptions 6-9 are satisfied, the estimator of the density of the uncertainty is consistent and asymptotically normal:*

$$\hat{h}_\xi(b) \xrightarrow{p} h_\xi(b), \text{ and}$$

$$\sqrt{Lh^3}(\hat{h}_\xi(b) - h_\xi(b)) \to N(0, V_\xi),$$

where

$$V_\xi = \left[\frac{P(w=1|\frac{b_1}{b_2}=b)(1-P(w=1|\frac{b_1}{b_2}=b)))}{f^2(b)}\right]\int\left(\frac{\partial K(u)}{\partial u}\right)^2 du$$

Once the distribution of the uncertainty is estimated, we are ready to use it in the estimation of the cost types. In order to do that, we also need to estimate the density of the efforts' distributions. The effort densities can be estimated using the kernel estimator as follows:

$$\hat{g}_i(b_i) = \frac{1}{Lh}\sum_{l=1}^{L} K\Big(\frac{b_i - b_{il}}{h}\Big), \tag{15}$$

Given the estimators of the distributions of uncertainty (14) and the effort densities (15), the cost types can be estimated using their combination:

$$\hat{c}_1 = \int_{\hat{\underline{b}}_2}^{\hat{\bar{b}}_2} \hat{g}_2(b_2)\frac{1}{b_2}\hat{h}_\xi\left(\frac{b_1}{b_2}\right)db_2 \tag{16}$$

and

$$\hat{c}_2 = \int\limits_{\hat{\underline{b}}_1}^{\hat{\bar{b}}_1} \hat{g}_1(b_1) \frac{b_1}{b_2^2} \hat{h}_\xi \left(\frac{b_1}{b_2} \right) db_1 \tag{17}$$

In the following theorem, I show that the proposed estimators are consistent and asymptotically normal, and derive asymptotic distributions.

Theorem 2. *Given the assumptions about the model as well as Assumptions 6-9 are satisfied the proposed estimators of the cost is consistent and asymptotically normal:*

$$\hat{c_1}(b_1) \xrightarrow{p} c_1(b_1), \text{ and}$$
$$\sqrt{Lh^3}(\hat{c}_1(b_1) - c_1(b_1)) \to N(0, V_{c_1}),$$

where

$$V_{c_1} = \int\limits_{\underline{b}_2}^{\bar{b}_2} g_2^2(b_2) \frac{1}{b_2^2} \left[\frac{P(w=1|\frac{b_1}{b_2})(1 - P(w=1|\frac{b_1}{b_2}))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Similarly:

$$\hat{c_2}(b_2) \xrightarrow{p} c_2(b_2), \text{ and}$$
$$\sqrt{Lh^3}(\hat{c}_2(b_2) - c_2(b_2)) \to N(0, V_{c_2}),$$

where

$$V_{c_2} = \int\limits_{\underline{b}_1}^{\bar{b}_1} g_1^2(b_1) \frac{b_1^2}{b_2^4} \left[\frac{P(w=1|\frac{b_1}{b_2})(1 - P(w=1|\frac{b_1}{b_2}))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Thus, the proposed estimators are consistent and asymptotically normal.[11]

Corollary 2. *Given the assumptions about the model as well as Assumptions 6-9 are satisfied in the model with valuations:*

$$\hat{v_1}(b_1) \xrightarrow{p} v_1(b_1), \text{ and}$$
$$\sqrt{Lh^3}(\hat{v}_1(b_1) - v_1(b_1)) \to N(0, V_1),$$

where

$$\hat{v}_1 = \left(\int\limits_{\hat{\underline{b}}_2}^{\hat{\bar{b}}_2} \hat{g}_2(b_2) \frac{1}{b_2} \hat{h}_\xi \left(\frac{b_1}{b_2} \right) db_2 \right)^{-1}, \tag{18}$$

[11] See Appendix for the estimation using quantile functions.

$$V_1 = v_1^4(b_1) \int\limits_{\underline{b}_2}^{\bar{b}_2} g_2^2(b_2) \frac{1}{b_2^2} \left[\frac{P(w=1|\frac{b_1}{b_2})(1-P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Similarly:

$$\hat{v}_2(b_2) \xrightarrow{p} v_2(b_2), \ \ and$$
$$\sqrt{Lh^3}(\hat{v}_2(b_2) - v_2(b_2)) \to N(0, V_2),$$

where

$$\hat{v}_2 = \left(\int\limits_{\hat{\underline{b}}_1}^{\hat{\bar{b}}_1} \hat{g}_1(b_1) \frac{b_1}{b_2^2} \hat{h}_\xi \left(\frac{b_1}{b_2} \right) db_1 \right)^{-1}, \tag{19}$$

$$V_2 = v_2^4(b_2) \int\limits_{\underline{b}_1}^{\bar{b}_1} g_1^2(b_1) \frac{b_1^2}{b_2^4} \left[\frac{P(w=1|\frac{b_1}{b_2})(1-P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du.$$

That is a simple corollary of Theorem 2 since the valuations $v_i = \frac{1}{c_i}$, $i = 1, 2$.

5 Application: U.S. House of Representatives

As an example of the application of the theoretical model described in the previous sections, I quantify the incumbency advantage in the U.S. 1972-2016 House of Representatives elections. Moreover, the model provides a method to separate the advantage into two parts. The first advantage is due to better reputation of the Incumbent. It is characterized by the fact that even when both the Incumbent and the Challenger spend the same amount of money on their campaign, the probability that the Incumbent wins is estimated to be bigger than that of the Challenger. This probability is given by the $P(\xi < 1)$, which is determined by the distribution of uncertainty. In its turn, the second advantage is due to the difference in campaign financing, which is characterized by the difference in the quantile functions of candidates' cost types, where the cost type describes how costly is it for the candidate to raise money. I show that the Incumbent has a lower cost type and thus is better at campaign financing. The important difference between these two advantages is that only the latter can be influenced by the policy-makers, whereas the reputation can not. Thus it is crucial for policy implications to be able to distinguish and quantify them.

5.1 The U.S. House Elections: Incumbent vs. Challenger

I use the data from the U.S. House of Representatives elections.[12] These elections happen every two years. Currently, there are 435 voting seats; winners serve 2-year terms. To quantify the incumbency advantage, I use the data on 6562 Incumbent-Challenger elections during the 1972-2016 period.[13] All the Incumbent's and the Challenger's expenditures are in \$2016. The summary statistics is presented in Table 1 below. On average, Incumbents spent a lot more, about 2.5 times as much as the Challengers on their campaigns, and what is especially striking is that Incumbents won almost 94 percent of the elections. Throughout the observed period, expenditures were increasing with only a slight decline starting in 2010. Please see Figure 1 below.

Table 1: Summary statistics of the Incumbent-Challenger elections

	Obs	Mean	Std. Dev.	Min	Max
Incumbent's Expenditures	6562	1056.65	1043.15	.198	26859.96
Challenger's Expenditures	6562	400.19	698.55	.002	10839.82
Incumbent winning dummy	6562	.939	.240	0	1

* Expenditures are in thousands of 2016 dollars.

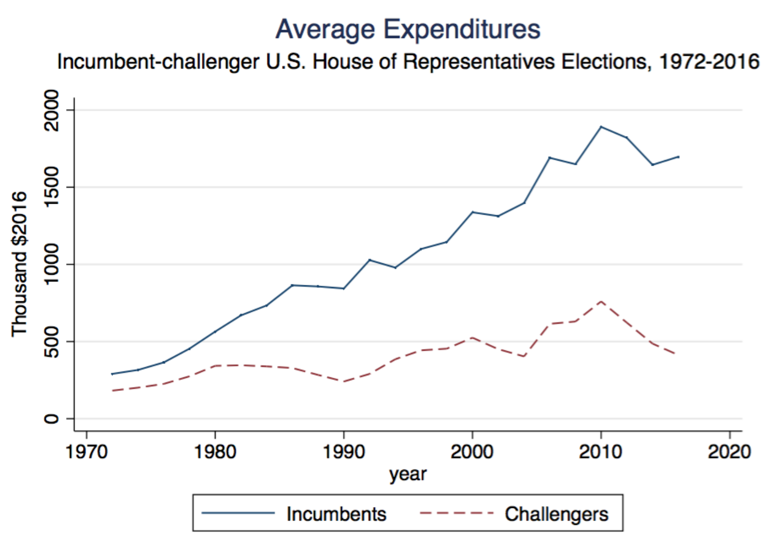


Figure 1: Average expenditures by election cycle

[12] I am very grateful to Gary Jacobson, Professor of Political Science at the University of California, San Diego, for providing me with his data.

[13] I drop the elections in which we have no data on the expenditures as well as one observation for which the ratio of expenditures is greater than 20, but the challenger wins, that is the outlier.

In terms of the theoretical model, we observe the data on 6562 contests with two players each, and winning outcomes, where players are two candidates, and the efforts are their expenditures. Players have different cost types which they draw from different distributions, and they also differ in terms of the distribution of stochastic components of the contest success function.

As a reminder, the expected payoff to player i in this case is given by:

$$\mathbb{E}[U_i|c_i, F_j(c), H_\xi] = P[w_i = 1|c_i, F_j, H_\xi] - c_i b_i = P(b_i\epsilon_i > b_j\epsilon_j|c_i, F_j, H_\xi) - c_i b_i,$$

where $i = 1, 2$, $j = -i$, c_i represents the cost type of each of the candidate, whereas the value of the prize is normalized to one.

The first step is the estimation of the distribution of uncertainty ξ using equations (13) and (14) above. The normal kernel and the optimal bandwidth are used. The results are shown below in Table 2.

Table 2: Cumulative distribution function $H_\xi(\cdot)$

b	1	2	3	4	5	6	7
$\hat{H}_\xi(b)$	0.85	0.88	0.91	0.94	0.97	0.98	1.00

Here b represents the ratio of the Incumbent's and the Challenger's efforts.

If $b = 1$, expenditures are equal, and $H_\xi(1)$ represents the winning probability of the Incumbent in this case. Thus the first incumbency advantage is represented by 85% winning probability even in the case when the expenditures are the same.

The second step is the estimation of the distributions of candidates' cost types using the efforts' distributions and the distribution of uncertainty. Figure 2 represents the results of the model estimation.[14] The first panel represents results across all the data 1972-2016. I also divide all election cycles by decades.[15] The result reflects the Incumbent's advantage in campaign financing as the Challenger's cost type first-order stochastically dominates the Incumbent's cost type distribution. That means that it is easier for the Incumbent to raise money than for the Challenger.

[14]On the lower boundary, the quantile functions were monotonized as the kernel estimators tend to be biased close to the boundaries.

[15]The last period includes 8 elections cycles over the years 2002-2016.

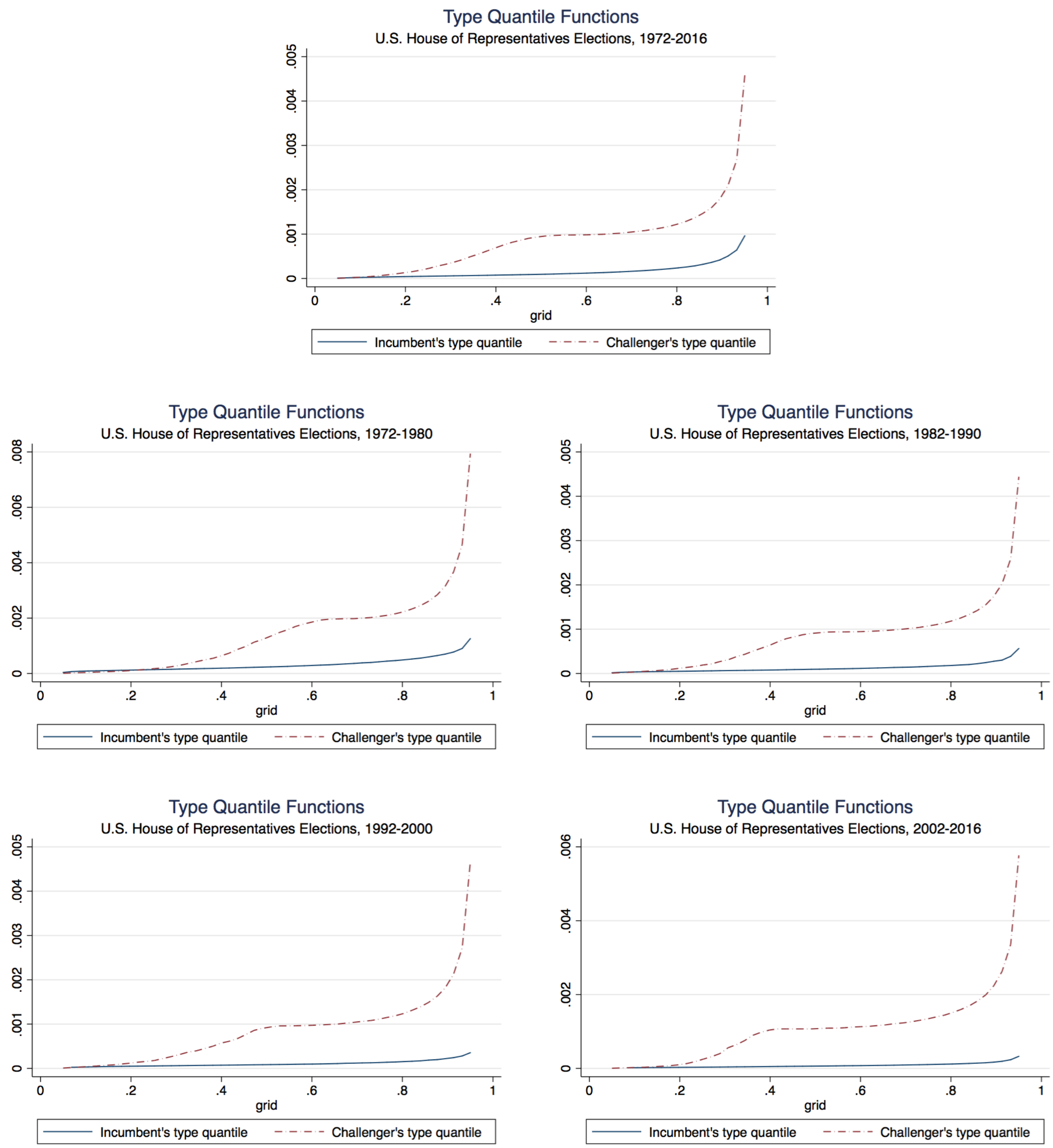
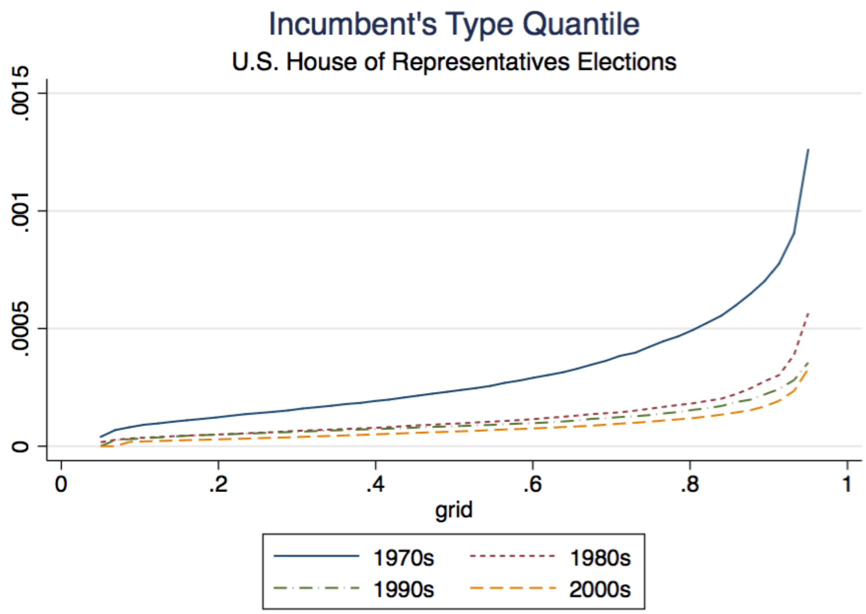


Figure 2: Estimated Quantile Functions of Cost Types by Decade

I also present the change of the quantile functions over the decades in Figure 3 below. For the Incumbent, the cost type distributions in later years first-order stochastically dominate distributions in earlier years. That means that it became easier for the Incumbent to raise funds in recent years. The trend is not monotone for the Challenger.



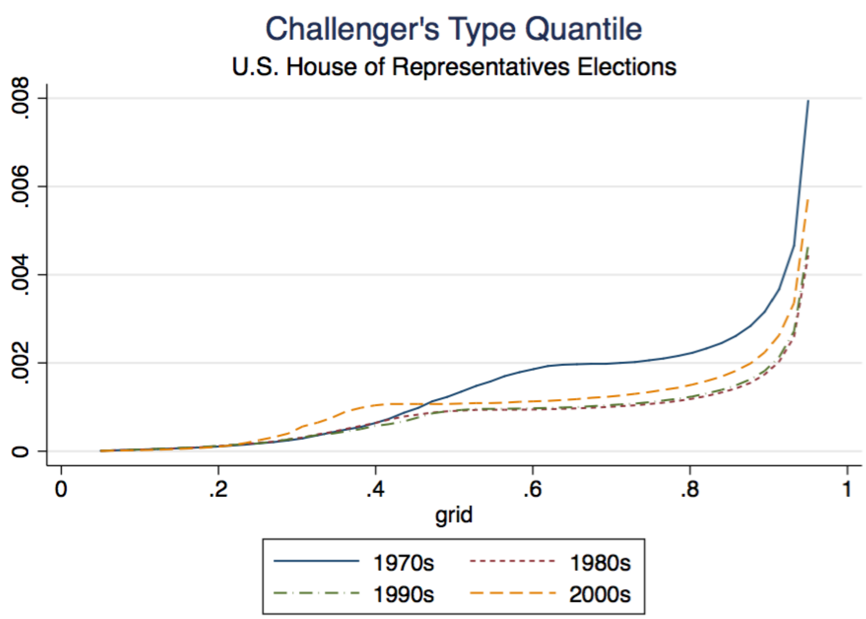


Figure 3: Estimated Quantile Functions of Cost Types over Decades

5.2 Counterfactuals

Once the primitives of the model – such as the distribution of uncertainty and the cost type distributions – are estimated, researchers have the capability to run the counterfactual simulations. In the setting of the election, counterfactuals allow testing different ways to limit the incumbency advantage. Limiting the incumbency advantage is important for the following reasons. First, according to the prevalent opinion in political science, democracy is not possible without sufficient competition as well as the turnover of the seats in Congressional elections. Moreover, the increased total campaign spending is costly for society. Thus it desirable policy is that one that reduces the Incumbent's winning probability, as well as the total campaign spending.

Two well-known policies are the limit on expenditures and public campaign financing. According to Jacobson (1978): "Even though Incumbents raise money more easily from all sources, limits on contributions will not help Challengers because the problem is not equalizing spending between candidates but rather simply getting more money to Challengers so that they can mount competitive races." The reason behind that statement is that the marginal effect of the Challenger's expenditure on the probability to win is greater than that of the Incumbent. Although that is true, this logic doesn't take into account the underlying game between the Incumbent and the Challenger. In reality, as the Challenger increases expenditures, the low-cost type Incumbent also does so, and as a result, the effect on winning probability is uncertain.

Next, I consider two policies, one by one, and compare the conclusions.

5.2.1 Public Campaign Financing

First, I consider public campaign financing for the Challenger, which lowers his cost type's distribution. I quantify the effect of the limit case of the public financing of the Challengers such that the resulting cost type quantile function matches one of the Incumbents. This case eliminates the advantage due to the difference in cost types completely, since now the cost types are assumed to be the same.

I take equal cost type distributions of the Incumbent and the Challenger as given (assume that the Challenger has the same cost type distribution as the Incumbent). The goal is to find the optimal strategies of the players in that case and find the resulting distributions of efforts.[16] After that, I calculate the Incumbent's winning probability knowing the effort strategies and the distribution of uncertainty. Results are presented in Table 3 below.

Table 3: Public campaign financing: resulting winning probability

	Incumbent's probability of winning				
	All	72-80	82-90	92-2000	2002-2016
Original	0.939	0.929	0.953	0.941	0.935
With Challenger's financing	0.899	0.913	0.923	0.904	0.858
Decrease	0.04	0.016	0.03	0.037	0.077

[16]Since it is not possible to solve the model analytically, I do that by approximating the effort distributions by the exponential distributions $\lambda_i e^{-\lambda b_i}$, $i = 1, 2$. See Appendix for further details.

The Incumbent's winning probability decreases by 4 percentage points, from 93.9% to 89.9%.

On the other hand, the reform leads to the increase in expenditures of both candidates, see Table 4 below:

Table 4: Public campaign financing: resulting expenditures

	All	72-80	82-90	92-2000	2002-2016
	Mean of Incumbent's expenditures				
Original	1057	394	792	1110	1650
With challenger's financing	1846	691	1532	2160	4384
Increase	789	297	740	1050	2434
	Mean of challenger's expenditures				
Original	400	243	309	420	557
With challenger's financing	1051	397	860	970	2453
Increase	651	154	551	550	1896

* Expenditures are in thousands of 2016 dollars.

Thus even though the Incumbent's advantage was lowered, this policy would lead to a dramatic increase in expenditures.

5.2.2 Limit on Expenditure

The other popular policy is the limit on expenditure. I consider such a case that the limit is sufficiently low for both candidates to spend the same amount. In this case, $b_1 = b_2$ and the Incumbent's winning probability becomes:

$$P(b_1\epsilon_1 > b_2\epsilon_2) = P(\epsilon_1 > \epsilon_2) = P(\epsilon_2/\epsilon_1 < 1) = H_\xi(1)$$

Using this formula and equation (13), I estimate the winning probability. Results are presented in Table 5 below.

Table 5: Limit on expenditure results

	Incumbent probability of winning				
	All	72-80	82-90	92-2000	2002-2016
Original	0.939	0.929	0.953	0.941	0.935
With the expenditure constraint	0.851	0.873	0.885	0.852	0.789
Decrease	0.088	0.056	0.068	0.089	0.146

It can be seen that the Incumbent's winning probability drops by 8.8 percentage points, from 93.9% to 85.1%, a bigger change than with public campaign financing for the Challenger, and on top of that the expenditures are lower for the Incumbent.

In conclusion, the Challenger's public financing is not as effective as the limit on expenditures in terms of both lowering the Incumbent's winning probability as well as on the total campaign spending. Thus by taking into account the game structure of the model, I have shown that the predictions change once the game-theoretical structure of the interactions between the candidates is taken into account.

6 Conclusion

In this paper, I identified and estimated the incomplete information contest model with non-parametric uncertainty distribution of the contest success function. Uncertainty in the contest success function summarizes other factors that could influence the contest result apart from the efforts of the players. As a result, I recovered the distribution of cost types from the efforts distributions and win outcomes. Here types characterized how costly it is for the player to exert the effort. This model provides the framework that can be applied to the variety of real-life scenarios such as marketing and advertising by firms, litigation, research and development, patent race, procurement of innovative good, research contest, sports event, arms race, rent-seeking activity, such as lobbying, as well as electoral competition. The model proposes a method to disentangle two sources of the possible advantage of one of the players: asymmetry in the cost type distributions and the effect of the uncertainty distribution on the winning probability. As an empirical example, I applied the model to the U.S. House of Representatives elections and recovered the cost type distributions of the Incumbent and the Challenger. The knowledge of the cost types' distributions allows quantifying the effect of different policy changes such as limits on expenditures or funding of Challengers in order to eliminate incumbency advantage. By comparing these two policies, I found the former to be more effective.

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Appendices

A Proof of Proposition 1

In case when both ϵ_i and ϵ_j have exponential distribution with parameter $\lambda = 1$, $f_\epsilon(t) = e^{-t}$ and in its turn $F_\epsilon(t) = 1 - e^{-t}$ and as a result:

$$P(w_1 = 1|\ b_1, b_2) = P(x_1 > x_2 \mid b_1, b_2) = P(b_1\epsilon_1 > b_2\epsilon_2 \mid b_1, b_2) =$$
$$= P(\epsilon_2 < \frac{b_1}{b_2}\epsilon_1 \mid b_1, b_2) = \int_0^{+\infty} F_\epsilon\left(\frac{b_1}{b_2}t\right) f_\epsilon(t)dt = \int_0^{+\infty} \left(1 - e^{-\frac{b_1}{b_2}t}\right) e^{-t}dt =$$
$$= 1 - \frac{1}{1 + \frac{b_1}{b_2}} = \frac{b_1}{b_1 + b_2}$$

which is the contest success function of the well-known Tullock contest.

B Proof of Proposition 2

Let us consider all assumptions required for the Theorem 6 in Athey (2001) to hold. Here I consider the equivalent model in which $v_i = 1/c_i$.

1. $f_i(\cdot)$ - density with respect to Lebesque measure, bounded and atomless.
2. $U_i = p_i(b_1, b_2)(v_i - b_i) + (1 - p_i(b_1, b_2))(-b_i)$ - can be written in the general form considered in the paper.
3. Winner's payoff $v_i - b_i$ and loser's payoff $-b_i$ are continuous in (v_i, b) and bounded as v_i has a finite support $[\underline{v_i}, \bar{v_i}]$ and the players won't find it profitable to exert a higher effort than the valuation.
4. Expected utility $E[U_i] = \int p_i(b_i, s_j(v_j))f_j(v_j)dv_j - b_i$ is bounded and finite.
5. Single-crossing condition $\frac{\partial^2 U_i}{\partial v_i \partial b_i} \geq 0$ is satisfied as:

$$\frac{\partial^2 U_1}{\partial v_1 \partial b_1} = \frac{\partial P_1}{\partial b_1} = \frac{1}{b_1} h_\xi\left(\frac{b_1}{b_2}\right) > 0,$$

$$\frac{\partial^2 U_2}{\partial v_2 \partial b_2} = \frac{\partial P_2}{\partial b_2} = \frac{b_1}{b_2^2} h_\xi\left(\frac{b_1}{b_2}\right) > 0.$$

Thus all the assumptions of Theorem 6 in Athey (2001) are satisfied, hence there exists a pure-strategy Bayesian Nash Equilibrium in nondecreasing strategies. Since the single-crossing property holds with strict inequality, this equilibrium is actually in increasing strategies. Going back to the model in terms of cost types: the equilibrium is in decreasing strategies instead since $c_i = 1/v_i$.

C Proof of Proposition 3

Under the assumptions of strict monotonicity of the strategies and independent cost types, we can write the expected payoff to player 1 when his true cost type is c_1 but he exerts an effort as if it was c as:

$$E[U_1|c_1, F_2, H_\xi] =$$
$$= P[w_1 = 1|b, F_2, H_\xi] - c_1 b = P(b\epsilon_1 > b_2\epsilon_2) - c_1 b = P(b_2\xi < b) - c_1 b =$$
$$= \int_{\underline{b}_2}^{\bar{b}_2} \left[\int_0^{b/b_2} h_\xi(y)dy \right] g_2(b_2)db_2 - c_1 b =$$
$$= \int_{\underline{c}_2}^{\bar{c}_2} \left[\int_0^{s_1(c)/s_2(c_2)} h_\xi(y)dy \right] f_2(c_2)dc_2 - c_1 s_1(c).$$

Using the First order condition (differentiating with respect to c and substituting $c = c_1$ and equating it to zero) we get the following equation for the cost type of player 1:

$$\int_{\underline{c}_2}^{\bar{c}_2} \frac{s_1'(c)}{s_2(c_2)} h_\xi \left(\frac{s_1(c)}{s_2(c_2)} \right) f_2(c_2)dc_2 - c_1 s_1'(c) = 0 \text{ when } c = c_1 \Rightarrow$$

$$c_1 = \int_{\underline{c}_2}^{\bar{c}_2} f_2(c_2) \frac{1}{s_2(c_2)} h_\xi \left(\frac{s_1(c_1)}{s_2(c_2)} \right) dc_2$$

Similarly, for player 2 the expected payoff when his true cost type is c_2 but he exerts an effort as if it was c is:

$$E[U_2|c_2, F_1, H_\xi] =$$
$$= P[w_2 = 1|b, F_1, H_\xi] - c_2 b = P(b\epsilon_2 > b_1\epsilon_1) - c_2 b = P(\xi > b_1/b) - c_2 b =$$
$$= \int_{\underline{b}_1}^{\bar{b}_1} \left[\int_{b_1/b}^{\infty} h_\xi(y)dy \right] g_1(b_1)db_1 - c_2 b =$$
$$= \int_{\underline{c}_1}^{\bar{c}_1} \left[\int_{s_1(c_1)/s_2(c)}^{\infty} h_\xi(y)dy \right] f_1(c_1)dc_1 - c_2 s_2(c).$$

By taking derivative with respect to c and equating it to zero we get the following equation for the cost type of player 2:

$$\int_{\underline{c}_1}^{\bar{c}_1} \frac{s_2'(c)s_1(c_1)}{s_2^2(c)} h_\xi \left(\frac{s_1(c_1)}{s_2(c)}\right) f_1(c_1)dc_1 - c_2 s_2'(c) = 0 \text{ when } c = c_2 \Rightarrow$$

$$c_2 = \int_{\underline{c}_1}^{\bar{c}_1} f_1(c_1)\frac{s_1(c_1)}{s_2^2(c_2)} h_\xi \left(\frac{s_1(c_1)}{s_2(c_2)}\right) dc_1.$$

This proves the proposition.

D Proof of Proposition 5

Similar to Guerre et al. (2000), the identification result is based on the property that together with the distribution $F_i(\cdot)$ and the density $f_i(\cdot)$, the derivative of the strategy $s_i'(\cdot)$ can be canceled out from the differential equation.

Because b_i is a function of c_i, which is random and distributed as $F_i(\cdot)$, b_i is also random. Let's denote its distribution $G_i(\cdot)$ and quantile function $r_i(\cdot) = G_i^{-1}(\cdot)$, $i = 1, 2$.

For every $b \in [\underline{b_i}, \bar{b_i}] = [s_i(\bar{c}_i), s_i(\underline{c_i})]$, we have $G_i(b) = Pr(b_i \leq b) = Pr(c_i \geq s_i^{-1}(b)) = 1 - F_i(s_i^{-1}(b)) = 1 - F_i(c)$, where $b_i = s_i(c_i)$. Thus, the distribution $G_i(\cdot)$ is absolutely continuous, has support $[s_i(\underline{c_i}), s_i(\bar{c}_i)]$ and density $g_i(b_i) = -\frac{f_i(c_i)}{s_i'(c_i)}$, where $c_i = s_i^{-1}(b_i)$.

This allows us to rewrite the differential equation above in terms of the distribution of efforts, that is for the first player:

$$c_1 = \int_{\underline{b}_2}^{\bar{b}_2} g_2(b_2)\frac{1}{b_2} h_\xi \left(\frac{b_1}{b_2}\right) db_2 \tag{20}$$

In its turn, the equation for the second player can be rewritten as:

$$c_2 = \int_{\underline{b}_1}^{\bar{b}_1} g_1(b_1)\frac{b_1}{b_2^2} h_\xi \left(\frac{b_1}{b_2}\right) db_1 \tag{21}$$

Thus, equations now express the individual private cost types c_i as functions of the individual equilibrium efforts b_i, their distributions $G_i(\cdot)$, their densities $g_i(\cdot)$, and the density h_ξ of the uncertainty ξ.

Let us denote $1-t_i = F_i(c_i)$ and $1-t_j = F_j(c_j)$, equivalently $c_i = q_i(1-t_i)$ and $c_j = q_j(1-t_j)$, where $q_i(\cdot)$ and $q_j(\cdot)$ are quantile functions of the distribution of cost types. As a result of monotonicity of the strategies $G_i(s_i(c_i)) = 1 - F_i(v_i)$, applying $r_i(\cdot)$ to both sides of equality, where $r_i(\cdot)$ is quantile function of the effort distribution we get: $s_i(c_i) = r_i(1 - F_i(c_i)) = r_i(t_i)$ and $s_j(c_j) = r_j(1-F_j(c_j)) = r_j(t_j)$. Moreover, $F_j(s_j^{-1}(s_i(c_i))) = 1-G_j(s_i(c_i)) = 1-G_j(r_i(t_i))$, $F_j(\bar{c_j}) = 1$ and $F_j(\underline{c_j}) = 0$. Using these equalities and changing variables we can rewrite the equations (20) and (21) above as:

$$q_1(1-t_1) = \int_0^1 \frac{1}{r_2(t_2)} h_\xi\left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_2 \tag{22}$$

and

$$q_2(1-t_2) = \int_0^1 \frac{r_1(t_1)}{r_2^2(t_2)} h_\xi\left(\frac{r_1(t_1)}{r_2(t_2)}\right) dt_1, \tag{23}$$

where $t_1,\, t_2 \in (0,1)$. This proves the proposition.

E Proof of Theorem 1

First, we would like to estimate $h_\xi(b)$ - the derivative of

$$H_\xi(b) = \frac{\int w f(w,b)dw}{f(b)}$$

By taking the derivative with respect to b we get:

$$h_\xi(b) = \Phi(f) = \frac{f(b)\int w\frac{\partial f(w,b)}{\partial b}dw - \frac{\partial f(b)}{\partial b}\int wf(w,b)dw}{f(b)^2} = \frac{f(b)\int wf'(w,b)dw - f'(b)\int wf(w,b)dw}{f(b)^2}$$

$$\Phi(f+h) = \\ = \frac{[f(b)+h(b)]\int w[f'(w,b)+h'(w,b)]dw - [f'(b)+h'(b)]\int w[f(w,b)+h(w,b)]dw}{[f(b)+h(b)]^2}$$

$$\begin{aligned}
&\Phi(f+h)-\Phi(f)=\\
&\frac{f(b)^2[f(b)+h(b)]\int w[f'(w,b)+h'(w,b)]dw}{f(b)^2[f(b)+h(b)]^2}-\\
&\frac{f(b)^2[f'(b)+h'(b)]\int w[f(w,b)+h(w,b)]dw}{f(b)^2[f(b)+h(b)]^2}+\\
&\frac{-f(b)[f(b)+h(b)]^2\int wf'(w,b)dw}{f(b)^2[f(b)+h(b)]^2}+\\
&\frac{f'(b)[f(b)+h(b)]^2\int wf(w,b)dw}{f(b)^2[f(b)+h(b)]^2}
\end{aligned}$$

$$\begin{aligned}
Num = {} & f^3(b)\int wf'(w,b)dw + f^3(b)\int wh'(w,b)dw + f^2(b)h(b)\int wf'(w,b)dw+\\
&+f^2(b)h(b)\int wh'(w,b)dw - f^2(b)f'(b)\int wf(w,b)dw - f^2(b)f'(b)\int wh(w,b)dw-\\
&-f^2(b)h'(b)\int wf(w,b)dw - f^2(b)h'(b)\int wh(w,b)dw - f^3(b)\int wf'(w,b)dw-\\
&-2f^2(b)h(b)\int wf'(w,b)dw - f(b)h^2(b)\int wf'(w,b)dw + f'(b)f^2(b)\int wf(w,b)dw+\\
&+2f'(b)f(b)h(b)\int wf(w,b)dw + f'(b)h^2(b)\int wf(w,b)dw =\\
&f^3(b)\int wh'(w,b)dw - f^2(b)h(b)\int wf'(w,b)dw + f^2(b)h(b)\int wh'(w,b)dw-\\
&-f^2(b)f'(b)\int wh(w,b)dw - f^2(b)h'(b)\int wf(w,b)dw - f^2(b)h'(b)\int wh(w,b)dw-\\
&-f(b)h^2(b)\int wf'(w,b)dw + 2f'(b)f(b)h(b)\int wf(w,b)dw + f'(b)h^2(b)\int wf(w,b)dw =\\
&Q+P,
\end{aligned}$$

Where:

$$\begin{aligned}
Q = {} & f^3(b)\int wh'(w,b)dw - f^2(b)h(b)\int wf'(w,b)dw - f^2(b)f'(b)\int wh(w,b)dw-\\
&-f^2(b)h'(b)\int wf(w,b)dw + 2f'(b)f(b)h(b)\int wf(w,b)dw
\end{aligned}$$

$$\begin{aligned}
P = {} & f^2(b)h(b)\int wh'(w,b)dw - f^2(b)h'(b)\int wh(w,b)dw - f(b)h^2(b)\int wf'(w,b)dw+\\
&+f'(b)h^2(b)\int wf(w,b)dw
\end{aligned}$$

Moreover:

$$\frac{1}{f^2(f+h)^2} = \frac{1}{f^4} + \frac{1}{f^2(f+h)^2} - \frac{1}{f^4} = \frac{1}{f^4} - \frac{2hf+h^2}{f^4(f+h)^2}$$

As a result:

$$\Phi(f+h)-\Phi(f)=$$

$$=\frac{Q}{f^4(b)}+\frac{P}{f^4(b)}-\frac{Q(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2}-\frac{P(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2}=D+R,$$

Where:

$$D=\frac{Q}{f^4(b)}$$

$$R=\frac{P}{f^4(b)}-\frac{Q(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2}-\frac{P(2h(b)f(b)+h^2(b))}{f^4(b)(f(b)+h(b))^2}$$

$$Q=f^3(b)\int wh'(w,b)dw-f^2(b)h(b)\int wf'(w,b)dw-f^2(b)f'(b)\int wh(w,b)dw-$$
$$-f^2(b)h'(b)\int wf(w,b)dw+2f'(b)f(b)h(b)\int wf(w,b)dw$$

In it's turn

$$f(b)=\int f(w,b)dw$$

Thus

$$Q=f^3(b)\int wh'(w,b)dw-f^2(b)h'(b)\int wf(w,b)dw+the\ rest=$$
$$=f^3(b)\int w(\hat{f}'(w,b)-f'(w,b))dw-$$
$$-f^2(b)\int((\hat{f}'(w,b)-f'(w,b))dw)\int wf(w,b)dw+the\ rest=$$
$$=\int f^2(b)\left[wf(b)-\int wf(w,b)dw\right](\hat{f}'(w,b)-f'(w,b))dw+the\ rest.$$

As a result,

$$D=\int\left[\frac{wf(b)-\int wf(w,b)dw}{f^2(b)}\right](\hat{f}'(w,b)-f'(w,b))dw+the\ rest.$$

Thus

$$\sqrt{Lh^3}(\Phi(f+h)-\Phi(f))\to N\left(0,V\right),$$

where

$$V=\int\left[wf(b)-\int wf(w,b)dw\right]^2\frac{f(w,b)}{f^4(b)}dw.$$

As w can only take 2 values 0 and 1: $\int wf(w,b)dw=f(1,b)$ and

$$\begin{aligned}V=&\left[\frac{f(1,b)^2f(0,b)+f(0,b)^2f(1,b)}{f^4(b)}\right]\int\left(\frac{\partial K(u)}{\partial u}\right)^2du=\\&=\left[\frac{f(1,b)f(0,b)}{f^3(b)}\right]\int\left(\frac{\partial K(u)}{\partial u}\right)^2du\end{aligned}$$

Moreover,

$$\begin{aligned}f(1,b)&=f(b)P(w=1|b)\ \textit{and}\\f(0,b)=f(b)P(w=0|b)&=f(b)(1-P(w=1|b)),\end{aligned}$$

thus

$$V=\left[\frac{P(w=1|b)(1-P(w=1|b)))}{f(b)}\right]\int\left(\frac{\partial K(u)}{\partial u}\right)^2du$$

And as a result,

$$\begin{aligned}&\hat{h}_\xi(b)\to h_\xi(b)\ \textit{in probability, and}\\\sqrt{Lh^3}(\hat{h}_\xi(b)-h_\xi(b))\to N&\left(0,\left[\frac{P(w=1|b)(1-P(w=1|b)))}{f(b)}\right]\int\left(\frac{\partial K(u)}{\partial u}\right)^2du\right)\end{aligned}$$

F Proof of Theorem 2

Now let us denote by $\tilde{f}(w,b_1,b_2)$ the joint density of the vector (w,b_1,b_2) and consider:

$$c_1=1/v_1=\int_{\underline{b}_2}^{\bar{b}_2}g_2(b_2)\frac{1}{b_2}h_\xi\left(\frac{b_1}{b_2}\right)db_2:=\tilde{\Phi}(b_1;f)$$

We also denote:

$$\begin{aligned}h_\xi\left(\frac{b_1}{b_2}\right)&:=\phi(b_1,b_2;\tilde{f})\\g_2(b_2)\frac{1}{b_2}&:=\psi(b_2;\tilde{f})\end{aligned}$$

Then

$$\tilde{\Phi}(b_1;\tilde{f}) = \int_{\underline{b}_2}^{\bar{b}_2} \psi(b_2;\tilde{f})\phi(b_1,b_2;\tilde{f})db_2$$

It follows that:

$$\tilde{\Phi}(b_1;\tilde{f}+\tilde{h}) - \tilde{\Phi}(b_1;\tilde{f}) = \int_{\underline{b}_2}^{\bar{b}_2} \psi(b_2;\tilde{f}+\tilde{h})\phi(b_1,b_2;\tilde{f}+\tilde{h})db_2 - \int_{\underline{b}_2}^{\bar{b}_2} \psi(b_2;\tilde{f})\phi(b_1,b_2;\tilde{f})db_2$$

$$= \int_{\underline{b}_2}^{\bar{b}_2} \phi(b_1,b_2;\tilde{f})D\psi(b_1,b_2;\tilde{f})db_2 + \int_{\underline{b}_2}^{\bar{b}_2} D\psi(b_2;\tilde{f})\phi(b_1,b_2;\tilde{f})db_2 + the\ rest =$$

$$= \int_{\underline{b}_2}^{\bar{b}_2} \int_{w} g_2(b_2)\frac{1}{b_2}\left[\frac{wf(b) - \int wf(w,b)dw}{f^2(b)}\right](\hat{f}'(w,b) - f'(w,b))dwdb_2 + the\ rest,$$

where $b = b_1/b_2$.

$The\ rest$ converges faster as the rate of convergence of $\hat{f}'(w,b)$ is slower than that of $\hat{f}(w,b)$.

Thus:

$$\hat{c_1}(b_1) \xrightarrow{p} c_1(b_1),\ and$$
$$\sqrt{Lh^3}(\hat{c_1}(b_1) - c_1(b_1)) \rightarrow N(0,V_{c_1}),$$

where

$$V_{c_1} = \int_{\underline{b}_2}^{\bar{b}_2} g_2^2(b_2)\frac{1}{b_2^2}\left[\frac{P(w=1|\frac{b_1}{b_2})(1-P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})}\right]db_2 \cdot \int\left(\frac{\partial K(u)}{\partial u}\right)^2 du$$

Similarly:

$$\hat{c_2}(b_2) \xrightarrow{p} c_2(b_2),\ and$$
$$\sqrt{Lh^3}(\hat{c_2}(b_2) - c_2(b_2)) \rightarrow N(0,V_{c_2}),$$

where

$$V_{c_2} = \int_{\underline{b}_1}^{\bar{b}_1} g_1^2(b_1)\frac{b_1^2}{b_2^4}\left[\frac{P(w=1|\frac{b_1}{b_2})(1-P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})}\right]db_2 \cdot \int\left(\frac{\partial K(u)}{\partial u}\right)^2 du$$

And by delta method for the cost types $v_i = \frac{1}{c_i}$:

$$\hat{v}_1(b_1) \xrightarrow{p} v_1(b_1),\ and$$
$$\sqrt{Lh^3}(\hat{v}_1(b_1) - v_1(b_1)) \to N(0, V_1),$$

where

$$V_1 = v_1^4(b_1) \int_{\underline{b}_2}^{\bar{b}_2} g_2^2(b_2) \frac{1}{b_2^2} \left[\frac{P(w=1|\frac{b_1}{b_2})(1 - P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

Similarly:

$$\hat{v}_2(b_2) \xrightarrow{p} v_2(b_2),\ and$$
$$\sqrt{Lh^3}(\hat{v}_2(b_2) - v_2(b_2)) \to N(0, V_2),$$

where

$$V_2 = v_2^4(b_2) \int_{\underline{b}_1}^{\bar{b}_1} g_1^2(b_1) \frac{b_1^2}{b_2^4} \left[\frac{P(w=1|\frac{b_1}{b_2})(1 - P(w=1|\frac{b_1}{b_2})))}{f(\frac{b_1}{b_2})} \right] db_2 \cdot \int \left(\frac{\partial K(u)}{\partial u} \right)^2 du$$

G Estimation using quantile functions

$r_i(\cdot)$ can be estimated from observed efforts:

$$\hat{r}_i(t) = b_i^{(\lceil Lt \rceil : L)}, \tag{24}$$

where $b_i^{(\lceil s \rceil : L)}$ is the s-th lowest order statistic out of L i.i.d. efforts observations; $\lceil \cdot \rceil$ is the ceiling function.

In the second step, the quantile functions of the player's cost types are estimated:

$$\hat{q}_1(t_1) = \frac{1}{\int_0^1 \frac{1}{\hat{r}_2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_2} \tag{25}$$

and

$$\hat{q}_2(t_2) = \frac{1}{\int_0^1 \frac{\hat{r}_1(t_1)}{\hat{r}_2^2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_1}, \tag{26}$$

where $t_1,\, t_2 \in (0,1)$.

Note that the invertibility of the equilibrium strategy is the key for identification as we relied on the assumption that the players use a strictly decreasing effort function.

Similarly, we can estimate the quantile functions of types:

$$\hat{q}_1^c(1-t_1) = \int_0^1 \frac{1}{\hat{r}_2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_2 \tag{27}$$

and

$$\hat{q}_2^c(1-t_2) = \int_0^1 \frac{\hat{r}_1(t_1)}{\hat{r}_2^2(t_2)} \hat{h}_\xi \left(\frac{\hat{r}_1(t_1)}{\hat{r}_2(t_2)} \right) dt_1, \tag{28}$$

where $t_1,\ t_2 \in (0,1)$.

Proposition 6. *(Csorgo (1983)) Let G be a twice differentiable distribution function, having finite support. Assume $\inf\limits_{0<t<1} g(G^{-1}(t)) > 0$ and $\sup\limits_{0<t<1} |g'(G^{-1}(t))| < \infty$. Then $\sup\limits_{0<t<1} |\hat{r}(t) - r(t)| \xrightarrow{a.s.} 0$.*

$\Rightarrow$

$$\sup_{0<t<1} |\hat{r}(t) - r(t)| = o_p(1).$$

It can be proved that:

Proposition 7. *Under the same assumptions as above:*

$$\hat{q}_i(t) - q_i(t) = o_p(1),$$

$i = 1, 2$.

H Analysis of the Public Campaign Financing Counterfactual

I consider equal cost type distributions of the Incumbent and the Challenger by assuming that the Challenger has the same cost type distribution as the Incumbent.

Since it is not possible to solve for the equilibrium efforts analytically, I approximate the effort distributions of each of the players by the exponential distributions $\lambda_i e^{-\lambda_i b_i}$ with unknown parameters λ_i, $i = 1, 2$. Next, I estimate λ_i, $i = 1, 2$, by solving for such λ_i, $i = 1, 2$, that minimize the distance between the actual cost type distributions and the estimated ones from the exponentially distributing bid distributions. The distance between distributions is estimated using the distance between quantile functions on the [0.1,0.9] using the grid of 50 equally spaced points, and calculating the sum of the absolute distances between the values that functions take in these points.