Cautious Belief and Iterated Admissibility*

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Abstract

We define notions of cautiousness and cautious belief to provide epistemic conditions for iterated admissibility in finite games. We show that iterated admissibility characterizes the behavioral implications of "cautious rationality and common cautious belief in cautious rationality" in a terminal lexicographic type structure. For arbitrary type structures, the behavioral implications of these epistemic assumptions are characterized by the solution concept of self-admissible set (Brandenburger, Friedenberg and Keisler 2008). We also show that analogous conclusions hold under alternative epistemic assumptions, in particular if cautiousness is "transparent" to players.

KEYWORDS: Epistemic game theory, iterated admissibility, weak dominance, lexicographic probability systems.

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1 Introduction

The iterated deletion of weakly dominated strategies, called *iterated admissibility* (henceforth IA), is an important and widely applied solution concept for games in strategic form. Shimoji (2004) shows that, in many dynamic games with generic payoffs at terminal nodes, IA is outcome-equivalent to Pearce's (1984) extensive-form rationalizability, a prominent solution concept whose foundations are well understood (Battigalli and Siniscalchi 2002). Applications of IA in games of interest are, for instance, voting (Moulin 1984) and money-burning games (Ben-Porath and Dekel, 1992). Yet, while IA has an independent intuitive appeal, its theoretical foundations have proved to be elusive (see Samuelson 1992). Thus, the decision-theoretic principles and the hypotheses about strategic reasoning that yield IA require careful scrutiny.

A recent literature—starting with the seminal contribution of Brandenburger, Friedenberg and Keisler (2008, henceforth BFK)—has tackled this issue building on two key ideas. The decision-theoretic aspects of the problem have been modeled through the *lexicographic expected*

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utility theory of Blume et al. (1991a). Lexicographic expected utility preferences are represented by *lexicographic probability systems* (henceforth LPS's), i.e., lists of probabilistic conjectures in a priority order, each of which becomes relevant when the previous ones fail to identify a unique best alternative. In games with complete information, opponents' strategies constitute the only payoff-relevant uncertainty. In order to come up with an educated conjecture about opponents' strategies, a player naturally starts reasoning about opponents' beliefs and choice criteria. BFK modeled this aspect with the tools of *epistemic game theory*, the formal, mathematical analysis of how players reason about each other in games.[1]

Inspired by BFK, in this paper we adopt lexicographic expected utility and epistemic game theory for our foundation of IA in finite games. Specifically, we use the formalism of lexicographic type structures to model players' interactive beliefs. However, we start from partially different basic principles. To motivate our analysis, we briefly mention BFK's results.

BFK showed that, for every natural number m, the strategies that survive $m+1$ rounds of iterated elimination of inadmissible (i.e., weakly dominated) strategies are those consistent with the epistemic conditions of *rationality and mth-order assumption of rationality* (henceforth RmAR). Such epistemic conditions are represented as events in a type structure, and the result requires a "richness" property—called completeness—of the type structure. BFK's notion of "rationality" incorporates a full-support requirement, which reflects the idea that nothing is ruled out by the players. As we intuitively explain below, the concept of *assumption* can be thought of as a strong form of "virtually persistent belief" in an event. *Rationality and common assumption of rationality* (henceforth RCAR) is the condition that RmAR holds for every m. BFK considered the natural conjecture that, in complete type structures, RCAR is an epistemic condition for IA—that is, the latter characterizes the behavioral implications of the former. Yet, they show that RCAR is empty in complete and *continuous* type structures.

In this paper we propose an alternative approach to the foundations of IA. Specifically, we provide notions of *rationality*, *cautiousness* and *cautious belief* that justify the choice of iteratively admissible strategies in the following way: in "rich" type structures, IA characterizes the behavioral implications of **cautious rationality and common cautious belief in cautious rationality** (henceforth $\mathrm{R^cCB^cR^c}$). A prominent example of "rich" type structure is the canonical, universal type structure, which represents all collectively coherent hierarchies of lexicographic beliefs. Such a structure is also complete and continuous—see Yang (2015) and Catonini and De Vito (2018). Thus, our first main result (Theorem 1) shows that $\mathrm{R^cCB^cR^c}$, unlike RCAR, is possible in a complete and continuous type structure.

We explain in more detail the main ingredients of our analysis. First, we define **rationality** as lexicographic expected utility maximization. The notion of **cautiousness** requires that all payoff-relevant consequences be deemed possible by the player; so, it describes a cautious attitude of the player towards the opponents' strategies. **Cautious rationality** is given by the conjunction of rationality and cautiousness.

Cautious belief is a strengthening of the notion of *weak belief* (Catonini and De Vito 2020), which is based on the preference-based concept of "infinitely more likely" due to Lo (1999). Intuitively, a player deems an event E infinitely more likely than F if she strictly prefers to bet on E rather than on F regardless of the size of the winning prizes for the two bets (given the same losing outcome). With this, we say that a player *cautiously believes* an event E if she deems each payoff-relevant component of E infinitely more likely than not-E. Put differently, cautious belief requires that: (1) the player deems E infinitely more likely than not-E, and (2) before entertaining the possibility that E does not occur, she takes into account all the possible payoff-relevant consequences of E. Condition (1) corresponds to weak belief; condition (2) says that the player is cautious towards the (weakly) believed event. Thus cautious belief in E is

[1] See Dekel and Siniscalchi (2015) for a recent survey.

stronger than weak belief as it captures cautiousness relative to E.

Cautious belief has many similarities with BFK's notion of assumption. Yet, the two concepts are distinct, and the main difference relies on how players are cautious in "believing" an event E. To gain some intuition, also assumption requires that E be deemed infinitely more likely than not-E; yet, as for cautious belief, this is not enough. Roughly speaking, assumption requires that event E be deemed infinitely more likely that not-E conditional on *every* "virtual observation" consistent with E.[2] This is in line with the full-support requirement in BFK's notion of rationality, which reflects the idea that "everything is possible." By contrast, cautious belief requires that event E be deemed infinitely more likely than not-E conditional on *payoff-relevant* "virtual observations" consistent with E. Thus, it requires a weaker form of "virtual persistence of belief."

Our results are the following. As already mentioned, the main result (Theorem 1) is that IA characterizes the behavioral implications of $\mathrm{R^cCB^cR^c}$ in "rich" type structures. A richness property is necessary, because cautious belief (like assumption) is not a monotone operator, as we will explain below.

For arbitrary type structures, we show (Theorem 2) that the behavioral implications of $\mathrm{R^cCB^cR^c}$ are characterized by the solution concept of *self-admissible set* (henceforth SAS). This concept was introduced by BFK, who note that a finite game may admit many SAS's, and the IA set is one of them. BFK show that, for a fixed type structure associated with a given game, the behavioral implications of RCAR constitute an SAS; the first part of Theorem 2 shows that an analogous conclusion holds for $\mathrm{R^cCB^cR^c}$. It should be noted that, since RCAR and $\mathrm{R^cCB^cR^c}$ are distinct epistemic conditions, they can yield different SAS's in a given type structure.[3] Nonetheless, as BFK show for the case of RCAR, the second part of Theorem 2 shows that every SAS represents the behavioral implications of $\mathrm{R^cCB^cR^c}$ in some type structure.

In addition, we provide alternative epistemic conditions for IA and SAS's whereby it is "commonly believed" that everybody is cautious. We say that a player **certainly believes** an event E if she considers not-E impossible. This notion of certain belief coincides with the standard notion of (probability 1) belief if preferences satisfy Archimedean continuity. We say that event E is **transparent** at some state of the world ω if $\omega \in E$ and there is common certain belief in E at ω. With this, we show that the epistemic condition of **rationality**, **transparency of cautiousness**, **and common cautious belief in both**, provides foundations for IA and SAS's as well. The results are formally stated in Theorems 3 and 4, which are the analogues of Theorems 1 and 2, respectively. As we elaborate in Section 5.3, the above epistemic condition is equivalent to **rationality and common cautious belief in rationality** if we restrict attention to a suitable class of type structures, called cautious type structures. Such results depart from previous justifications of IA —that is, those based on the concept of assumption or weak assumption (BFK, Dekel et. al 2016, Yang 2015). In particular, these justifications require that players' caution (however defined) *cannot* be certainly believed. We discuss in detail this comparison in Section 6.4.

Persistence in the face of non-contradictory (virtual) evidence makes both cautious belief and assumption similar to the notion of "believing whenever possible" (or "strong belief") of Battigalli and Siniscalchi (2002). Thus, like the latter and for similar reasons, cautious belief and assumption are *not monotone*: BFK noticed that it is possible that (i) E is assumed, (ii) E implies F ($E \subseteq F$), and yet (iii) F is not assumed. The same holds for cautious belief. The reason is that, compared to E, there are more pieces of evidence conditional on which

[2] We use the expression "virtual observation" to emphasize that there is no real observation in this static setting. It is just a suggestive language.

[3] For instance, consider the canonical type structure. The behavioral implications of $\mathrm{R^cCB^cR^c}$ and RCAR are characterized by, respectively, the IA set and the empty set. By definition, the empty set is an SAS.

F must be deemed infinitely more likely than not-F. As in Battigalli and Siniscalchi (2002), non-monotonicity entails the need to state results in the context of "rich" type structures. Indeed, considering small ("non-rich") type structures is essentially equivalent to hypothesize that belief-related, non-behavioral events are transparent to the players, and that each epistemic event explicitly mentioned in the statement (e.g., "rationality") is implicitly intersected with the transparency of a belief-event. Because of non-monotonicity, commonly assuming (or cautiously believing) "rationality" within a small type structure need not be more demanding than commonly assuming rationality in a rich type structure, and some strategies consistent with the former may be inconsistent with the latter. This is the reason why the IA set need not contain all other SAS's. See Battigalli and Friedenberg (2012a) for similar considerations and results concerning rationalizability in sequential games, strong belief, and "extensive-form best response sets."

Theorem 1 identifies *terminality* as the relevant richness property of type structures for the justification of IA. In particular, completeness plays no role in the statement and proof. As shown in Friedenberg and Keisler (2020), completeness may not be the appropriate notion of richness to provide epistemic foundations for iterated strict dominance. Leveraging on the ideas and proofs in Friedenberg and Keisler (2020), we show (Theorem 5) that, for any non-degenerate game, there exists a complete type structure such that IA does not characterize the behavioral implications of $\mathrm{R^cCB^cR^c}$. Sections 5.4 and 6.1 elaborate and discuss this result.

Before moving on to the formal analysis, it is worth stressing other features of our approach. First, unlike BFK, we do not restrict the analysis to type structures where players' beliefs are *lexicographic conditional probability systems* (henceforth LCPS's). Loosely speaking, LCPS's are LPS's such that the supports of the component measures are pairwise disjoint. We instead allow for arbitrary LPS-based type structures, as in Dekel et al. (2016). These authors argue that, from the perspective of providing foundations for IA, it is conceptually appropriate to consider unrestricted LPS's (see also Lee 2013). With this, they show that all BFK's results have analogues in the setting of LPS-based type structures.

Second, we adopt Lo's (1999) notion of "infinitely more likely" as building block of cautious belief. An alternative (and stronger) notion is the one proposed by Blume et al. (1989a). We chose Lo's notion mainly for two reasons. First, as pointed out by Blume et al. (1989a), their notion of infinitely more likely may fail to satisfy a natural *disjunction* property, if preferences are not represented by LCPS's. To clarify, consider pairwise disjoint events C, D and E, and suppose that both C and D are infinitely more likely than E. Blume et al. (1989a, p. 70) show that $C \cup D$ (logically: C or D) need not be infinitely more likely than E.

Lo's (1999) notion of "infinitely more likely" is equivalent, in terms of LPS's, to the definition used by Stahl (1995) for the solution concept of *lexicographic rationalizability*—a concept which coincides with IA.[4] Stahl introduced lexicographic rationalizability as a refinement of permissibility (Brandenburger 1992), an iterated elimination procedure for lexicographic beliefs. Stahl's analysis is "pre-epistemic" in the following sense: there is no epistemic apparatus to formally express events such as rationality and some forms of "belief" in rationality. In our view, $\mathrm{R^cCB^cR^c}$ matches very closely the logic of lexicographic rationalizability, exactly as the weaker condition of "cautious rationality and common weak belief of cautious rationality" captures the logic of permissibility, as shown in our previous work (Catonini and De Vito 2020). In a sense, the contribution of this paper can be best seen as providing foundations for lexicographic rationalizability (see Section 6.2).

Finally, we define cautiousness as a full-support condition on the set of strategies. This notion of caution can be found, with some minor differences, in other works (Asheim and Dufwenberg

[4] Stahl (1995) did not provide a preference-based foundation of infinitely more likely. Lo's (1999) definition applies to a wide class of preferences, including the lexicographic expected utility model.

2003, Perea 2012, Heifetz et al. 2019, Lee 2016, Catonini and De Vito 2020). An alternative to cautiousness is the notion that requires the full-support condition on strategies *and* types. Section 6.3 compares the two notions, and discusses the difference between them. Here, we just mention that cautiousness is a condition expressed in terms of belief hierarchies, and it is invariant to details of the type structure that are unrelated to hierarchies (e.g., the topology of type spaces). By contrast, we show (Example 3) that the aforementioned full-support condition is not invariant even to type isomorphisms between type structures. Therefore, we provide a justification of IA and SAS's using *expressible* epistemic assumptions about rationality and beliefs, that is, assumptions which can be expressed in a language describing primitive terms (strategies) and terms derived from the primitives (beliefs about strategies, beliefs about strategies and beliefs of others, etc.)—cf. Battigalli et al. (2021, Section 3.A).

Related literature We have already mentioned some important contributions on the epistemic analysis of IA in games. We provide detailed comments on the closest related literature in Section 6. Other articles with epistemic conditions for IA include Keisler and Lee (2015), Lee (2016a), Heifetz et al. (2019), Halpern and Pass (2019), and, in non-lexicographic frameworks, Barelli and Galanis (2013) and Ziegler and Zuazo-Garin (2020).

Keisler and Lee (2015) construct a *discontinuous* and complete type structure where RCAR is possible. Furthermore, they show that such type structure generates the same set of belief hierarchies as a continuous one. An immediate implication of this findings is that BFK's results hinge on topological details of the type structure that cannot be expressed in terms of belief hierarchies. Keisler and Lee conclude that BFK's negative result stems from the fact that, in a continuous type structure, players are "too cautious" towards the assumed events. Lee (2016a) relaxes the traditional coherency condition on belief hierarchies while maintaining coherency of the represented preferences. With this, he identifies hierarchies of lexicographic beliefs without an upper bound on the length of the LPS's that cannot be represented by any type structure but capture "rationality" and common assumption of "rationality". The notion of "rationality" used by Lee is essentially equivalent to cautious rationality.[5] Heifetz et al. (2019) put forward the solution concept of comprehensive rationalizability, and they give it an epistemic foundation in a universal type structure for LCPS's. Comprehensive rationalizability neither refines nor is refined by IA, but it coincides with IA in many applications. Halpern and Pass (2019) use instead a modal-logic framework to provide an epistemic characterization of IA. They put forward a notion of "generalized belief" which is given semantics in terms of LPS's. Their characterization of IA relies on a corresponding operator ("All I know") which is taken with respect to an appropriate language.

Within a standard Bayesian decision model, Barelli and Galanis (2013) use the idea that each player has a list of preferences which allows her to break ties. With this, they provide an epistemic foundation for IA in an appropriate framework for interactive beliefs. Ziegler and Zuazo-Garin (2020) use instead a decision model of incomplete, but continuous preferences where each player's uncertainty is represented by a set of beliefs. They provide foundations for IA and SAS's in terms of interactive, ambiguous beliefs, rather than LPS's. Both Barelli and Galanis (2013) and Ziegler and Zuazo-Garin (2020) can be regarded as complementary to the LPS-based approach.

Structure of the paper Section 2 introduces IA and self-admissible sets. Section 3 provides formal definitions of LPS's and type structures. Section 4 analyzes cautious rationality and cautious belief. Section 5 contains our epistemic justifications of IA and self-admissible sets.

[5]In a companion paper (Catonini and De Vito 2017), we show that our results can be replicated in the hierarchical space studied by Lee (2016a).

Section 6 discusses certain conceptual aspects of the analysis, and it compares our notions of caution and belief to those derived from BFK's approach. Appendix A provides decision-theoretic foundations for cautiousness and cautious belief. Appendix B and Appendix C collect the proofs omitted from the main text. The Supplementary Appendix contains elaborations on results discussed in Section 6.

2 Iterated admissibility and self-admissible sets

Throughout, we consider finite games. A **finite game** is a structure $G := \langle I, (S_i, \pi_i)_{i\in I}\rangle$, where (a) I is a finite set of players with cardinality $|I| \geq 2$; (b) for each player $i \in I$, S_i is a finite, non-empty set of strategies; and (c) $\pi_i : S \to \mathbb{R}$ is the payoff function.[6]

Each strategy set S_i is given the obvious topology, i.e., the discrete topology. We let $\mathcal{M}(X)$ denote the set of all Borel probability measures on a topological space X. So, given a mixed strategy profile $\sigma \in \prod_{i\in I} \mathcal{M}(S_i)$, we will denote player i's expected utility simply by $\pi_i(\sigma_i, \sigma_{-i})$, i.e.,

$$\pi_i(\sigma_i, \sigma_{-i}) := \sum_{(s_i,(s_j)_{j\neq i})\in S_i\times S_{-i}} \sigma_i(\{s_i\}) \left(\prod_{j\neq i} \sigma_j(\{s_j\})\right) \pi_i(s_i, (s_j)_{j\neq i}).$$

Similarly, given a pure strategy $s_i \in S_i$ and a probability measure $\mu_i \in \mathcal{M}(S_{-i})$, we will denote player i's expected utility by $\pi_i(s_i, \mu_i) := \sum_{s_{-i}\in S_{-i}} \pi_i(s_i, s_{-i})\mu_i(\{s_{-i}\})$. With an abuse of notation, we will also identify the pure strategy $s_i \in S_i$ with the mixed strategy $\sigma_i \in \mathcal{M}(S_i)$ such that $\sigma_i(\{s_i\}) = 1$.

In the remainder of this section, we fix a finite game $G := \langle I, (S_i, \pi_i)_{i\in I}\rangle$. Let $\mathcal{Q}$ be the collection of all subsets of S with the cross-product form $Q = \prod_{i\in I} Q_i$, where $Q_i \subseteq S_i$ for every i.

Definition 1 *Fix a set $Q \in \mathcal{Q}$. A strategy $s_i \in Q_i$ is **weakly dominated with respect to** Q if there exists a mixed strategy $\sigma_i \in \mathcal{M}(S_i)$, with $\sigma_i(Q_i) = 1$, such that $\pi_i(\sigma_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for every $s_{-i} \in Q_{-i}$ and $\pi_i(\sigma_i, s'_{-i}) > \pi_i(s_i, s'_{-i})$ for some $s'_{-i} \in Q_{-i}$. Otherwise, say s_i is **admissible with respect to** Q.*

*If $s_i \in S_i$ is weakly dominated (resp. admissible) with respect to S, say s_i is **weakly dominated** (resp. **admissible**).*

Remark 1 *Fix a set $Q \in \mathcal{Q}$. A standard result (Pearce 1984, Lemma 4) states that a strategy $s_i \in Q_i$ is admissible with respect to Q if and only if there exists $\mu_i \in \mathcal{M}(S_{-i})$, with $\mu_i(Q_{-i}) = 1$, such that $\mu_i(\{s_{-i}\}) > 0$ for every $s_{-i} \in Q_{-i}$, and $\pi_i(s_i, \mu_i) \geq \pi_i(s'_i, \mu_i)$ for every $s'_i \in Q_i$.*

The set of iteratively admissible strategies (henceforth IA set) is defined inductively.

Definition 2 *For every $i \in I$, let $S_i^0 := S_i$, and for every $m \in \mathbb{N}$, let S_i^m be the set of all $s_i \in S_i^{m-1}$ that are admissible with respect to $S^{m-1} := \prod_{j\in I} S_j^{m-1}$. A strategy $s_i \in S_i^m$ is called m-**admissible**. A strategy $s_i \in S_i^\infty := \cap_{m=0}^\infty S_i^m$ is called **iteratively admissible**.*

By finiteness of the game, it follows that $S_i^m \neq \emptyset$ for all $m \in \mathbb{N}$, and, since $S_i^m \supseteq S_i^{m+1}$ for all $m \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $S_i^\infty = S_i^M$. Consequently, the IA set S^∞ is non-empty.

To formally introduce BFK's notion of self-admissible set, we need an additional definition. Say that a strategy $s'_i \in S_i$ **supports** $s_i \in S_i$, if there exists a mixed strategy $\sigma_i \in \mathcal{M}(S_i)$ such that $\sigma_i(\{s'_i\}) > 0$ and $\pi_i(\sigma_i, s_{-i}) = \pi_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

[6] Our notation is standard. For any profile of sets $(X_i)_{i\in I}$, we let $X := \prod_{i\in I} X_i$ and $X_{-i} := \prod_{j\neq i} X_j$ with typical elements $x := (x_i)_{i\in I} \in X$ and $x_{-i} := (x_j)_{j\neq i} \in X_{-i}$.

Definition 3 *A set $Q \in \mathcal{Q}$ is a **self-admissible set (SAS)** if, for every player i,*
(a) every $s_i \in Q_i$ is admissible,
(b) every $s_i \in Q_i$ is admissible with respect to $S_i \times Q_{-i}$,
(c) for every $s_i \in Q_i$ and $s_i' \in S_i$, if s_i' supports s_i then $s_i' \in Q_i$.

Every finite game admits a non-empty SAS—in particular, the IA set is an SAS. But, as shown by BFK, many games possess other SAS's, which can be even disjoint from the IA set. This is illustrated by the following example, which is taken from Brandenburger et al. (2012, Example 5.9).

Example 1 *Consider the following game between two players, Ann (a) and Bob (b):*

$a \backslash b$	ℓ	r
u	$2,2$	$2,2$
m	$3,1$	$0,0$
d	$0,0$	$1,3$

There are three non-empty SAS's: $\{u\} \times \{r\}$, $\{u\} \times \{\ell, r\}$ and $\{m\} \times \{\ell\}$. The SAS $\{m\} \times \{\ell\}$ is the IA set. ♦

A comprehensive analysis of the properties of SAS's in a wide class of games is given by Brandenburger and Friedenberg (2010).[7]

3 Lexicographic beliefs and lexicographic type structures

3.1 Lexicographic probability systems

All the sets considered in this paper are assumed to be Polish spaces (that is, topological spaces that are homeomorphic to complete, separable metrizable spaces), and they are endowed with the Borel σ-field. We let Σ_X denote the Borel σ-field of a Polish space X, the elements of which are called *events*. When it is clear from the context, we suppress reference to Σ_X and simply write X to denote a measurable space.

Given a sequence $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint Polish spaces, the set $X := \cup_{n\in\mathbb{N}} X_n$ is endowed with the *direct sum topology*,[8] so that X is a Polish space. Moreover, we endow each finite or countable product of Polish spaces with the product topology, hence the product space is Polish as well.

Recall that $\mathcal{M}(X)$ denotes the set of Borel probability measures on a topological space X. The set $\mathcal{M}(X)$ is endowed with the *weak**-topology. So, if X is Polish, then $\mathcal{M}(X)$ is also Polish. We let $\mathcal{N}(X)$ (resp. $\mathcal{N}_n(X)$) denote the set of all finite (resp. length-n) sequences of Borel probability measures on X, that is,

$$\begin{aligned} \mathcal{N}(X) \; : &= \bigcup_{n\in\mathbb{N}} \mathcal{N}_n(X) \\ : &= \bigcup_{n\in\mathbb{N}} (\mathcal{M}(X))^n . \end{aligned}$$

[7] Example 1 is the reduced strategic form of the Battle of the Sexes with an Outside Option, which is used by Battigalli and Friedenberg (2012a) to illustrate how "extensive-form best response sets" (EFBR's) are related to Pearce's (1984) notion of "extensive-form rationalizability." In the example, the EFBR's coincide with the SAS's, and Pearce's extensive-form rationalizability coincides with IA.

[8] In this topology, a set $O \subseteq X$ is open if and only if $O \cap X_n$ is open in X_n for all $n \in \mathbb{N}$. The assumption that the spaces X_n are pairwise disjoint is without any loss of generality, since they can be replaced by a homeomorphic copy, if needed (see Engelking 1989, p. 75).

Each $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is called **lexicographic probability system** (**LPS**). In view of our assumptions, the topological space $\mathcal{N}(X)$ is Polish.

For every Borel probability measure μ on X, the support of μ, denoted by $\text{Supp}\mu$, is the smallest closed subset $C \subseteq X$ such that $\mu(C) = 1$. The support of an LPS $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is defined as $\text{Supp}\overline{\mu} := \cup_{l \leq n} \text{Supp}\mu^l$. So, an LPS $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is of **full-support** if $\text{Supp}\overline{\mu} = X$. We write $\mathcal{N}^+(X)$ for the set of full-support LPS's.

For future reference, we also record the following definition. An LPS $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is called **lexicographic conditional probability system** (**LCPS**) if there are events $E_1, ..., E_n$ in X such that, for every $l \leq n$, $\mu^l(E_l) = 1$ and $\mu^l(E_m) = 0$ for $m \neq l$. If X is finite, then an LPS is an LCPS if and only if the supports of the component measures are pairwise disjoint.

Fix Polish spaces X and Y, and a Borel map $f : X \to Y$. The map $\widetilde{f} : \mathcal{M}(X) \to \mathcal{M}(Y)$, defined by

$$\widetilde{f}(\mu)(E) := \mu\left(f^{-1}(E)\right), \quad E \in \Sigma_Y, \, \mu \in \mathcal{M}(X),$$

is called the image (or pushforward) measure map of f. For each $n \in \mathbb{N}$, the map $\widehat{f}_{(n)} : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ is defined by

$$(\mu^1, ..., \mu^n) \mapsto \widehat{f}_{(n)}\left((\mu^1, ..., \mu^n)\right) := \left(\widetilde{f}\left(\mu^k\right)\right)_{k \leq n}.$$

With his, the map $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ defined by

$$\widehat{f}(\overline{\mu}) := \widehat{f}_{(n)}(\overline{\mu}), \, \overline{\mu} \in \mathcal{N}_n(X),$$

is called the **image LPS map of** f. Alternatively put, the map $\widehat{f}$ is the *union* of the maps $\left(\widehat{f}_{(n)}\right)_{n \in \mathbb{N}}$, and it is Borel measurable.[9]

Given Polish spaces X and Y, we let Proj_X denote the canonical projection from $X \times Y$ onto X; in view of our assumption, the map Proj_X is continuous. The marginal measure of $\mu \in \mathcal{M}(X \times Y)$ on X is defined by $\text{marg}_X \mu := \widetilde{\text{Proj}}_X(\mu)$. Consequently, the marginal of $\overline{\mu} \in \mathcal{N}(X \times Y)$ on X is defined by $\overline{\text{marg}}_X \overline{\mu} := \widehat{\text{Proj}}_X(\overline{\mu})$, and the function $\widehat{\text{Proj}}_X : \mathcal{N}(X \times Y) \to \mathcal{N}(X)$ is continuous and surjective.

3.2 Lexicographic type structures

Fix a finite game $G := \langle I, (S_i, \pi_i)_{i \in I} \rangle$. A *type structure* (associated with G) formalizes Harsanyi's (1967-68) implicit approach to model hierarchies of beliefs. The following is a generalization of the standard definition of epistemic type structure with beliefs represented by probability measures (i.e., length-1 LPS's; cf. Mertens and Zamir 1985, Brandenburger and Dekel 1992, Heifetz and Samet 1998).

Definition 4 *An $(S_i)_{i \in I}$-**based lexicographic type structure** is a structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ where*

1. *for each $i \in I$, T_i is a Polish space;*
2. *for each $i \in I$, the function $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ is Borel measurable.*

[9] For details and proofs related to Borel measurability and continuity of the involved maps, the reader can consult Catonini and De Vito (2018).

Each space T_i is called ***type space*** *and each β_i is called* ***belief map.***[10] *Members of type spaces, viz. $t_i \in T_i$, are called* ***types.*** *Each element $(s_i, t_i)_{i \in I} \in \prod_{i \in I} (S_i \times T_i)$ is called* ***state (of the world).***

In what follows, we will omit the qualifier "lexicographic," and simply speak of **type structures** when the underlying strategy sets $(S_i)_{i \in I}$ are clear from the context. Furthermore, if every type in a type structure $\mathcal{T}$ is associated with a probability measure, then we will say that $\mathcal{T}$ is an **ordinary type structure**.

Type structures generate a collection of hierarchies of beliefs for each player. For instance, type t_i's first-order belief is an LPS on S_{-i}, and is given by $\overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i)$. A standard inductive procedure (see Catonini and De Vito 2018, for details) shows how to provide an explicit description of a hierarchy induced by a type.

We will be interested in type structures with one or more of the following features, which do not make reference to hierarchies of beliefs or other type structures.

Definition 5 *A type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is*

- ***finite*** *if the cardinality of each type space T_i is finite;*
- ***compact*** *if each type space T_i is compact;*
- ***belief-complete*** *if each belief map β_i is onto;*
- ***continuous*** *if each belief map β_i is continuous.*

The idea of (belief-)completeness was introduced by Brandenburger (2003) and adapted to the present context.[11] Note that each type space in a belief-complete type structure has the cardinality of the continuum. Finite type structures are compact and continuous, but not belief-complete. No belief-complete lexicographic type structure is also compact and continuous. To see this, observe that if the type structure is compact and continuous, each $\beta_i(T_i)$ is compact but the space $\mathcal{N}(S_{-i} \times T_{-i})$ is not compact,[12] hence β_i is not onto.

We next introduce the notion of type morphism, which captures the idea that a type structure $\mathcal{T}$ is "contained in" another type structure $\mathcal{T}^*$. In what follows, given a type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$, we let T denote the Cartesian product of type spaces, that is, $T := \prod_{i \in I} T_i$. Moreover, for any set X, we let Id_X denote the identity map on X, that is, $\mathrm{Id}_X(x) := x$ for all $x \in X$.

Definition 6 *Fix type structures $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$. For each $i \in I$, let $\varphi_i : T_i \to T_i^*$ be a measurable map such that*

$$\beta_i^* \circ \varphi_i = \left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}} \right) \circ \beta_i.$$

where $\varphi_{-i} := (\varphi_j)_{j \neq i} : T_{-i} \to T_{-i}^*$. The function $(\varphi_i)_{i \in I} : T \to T^*$ is called ***type morphism (from $\mathcal{T}$ to $\mathcal{T}^*$).***

[10] Some authors (e.g., Battigalli and Siniscalchi 1999, Heifetz and Samet 1998) use the terminology "type space" for what is called "type structure" here.

[11] In Dekel et al. (2016), a type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is said to be complete if the range of each belief map β_i is a strict superset of $\mathcal{N}^+(S_{-i} \times T_{-i})$. A belief-complete type structure is complete in the sense of Dekel et al. (2016); the converse does not hold.

[12] The space $\mathcal{M}(X)$ is compact if and only if X is compact, and this in turn implies that the space $\mathcal{N}_n(X)$ is also compact for every *finite* $n \in \mathbb{N}$. But the same conclusion does not hold for the space $\mathcal{N}(X)$. This is an instance of a well-known mathematical fact (see Theorem 2.2.3 in Engelking 1989): If $(X_\theta)_{\theta \in \Theta}$ is an indexed family of non-empty compact spaces with $|X_\theta| > 1$ for all $\theta \in \Theta$, then the direct sum $\cup_{\theta \in \Theta} X_\theta$ is compact if and only if the right-directed set Θ is finite.

The morphism is called ***bimeasurable*** *if the map* $(\varphi_i)_{i\in I}$ *is Borel bimeasurable.*[13] *The morphism is called* ***type isomorphism*** *if the map* $(\varphi_i)_{i\in I}$ *is a Borel isomorphism. Say* $\mathcal{T}$ *and* $\mathcal{T}^*$ *are* ***isomorphic*** *if there is a type isomorphism between them.*

A type morphism requires consistency between the function $\varphi_i : T_i \to T_i^*$ and the induced function $\left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}\left(S_{-i} \times T_{-i}^*\right)$. That is, the following diagram commutes:

$$\begin{array}{ccc} T_i & \xrightarrow{\beta_i} & \mathcal{N}(S_{-i} \times \ T_{-i}) \\ \Big\downarrow \varphi_i & & \Big\downarrow \left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) \\ T_i^* & \xrightarrow{\beta_i^*} & \mathcal{N}(S_{-i} \times \ T_{-i}^*) \end{array} \tag{3.1}$$

Thus, a type morphism maps $\mathcal{T}$ into $\mathcal{T}^*$ in a way that preserves the beliefs associated with types.

The notion of type morphism does not make reference to hierarchies of beliefs. But the important property of type morphisms is that they preserve the explicit description of lexicographic belief hierarchies: the $(S_i)_{i\in I}$-based belief hierarchy generated by a type $t_i \in T_i$ in $\mathcal{T}$ is also generated by its image $\varphi_i(t_i) \in T_i^*$ in $\mathcal{T}^*$. Heifetz and Samet (1998, Proposition 5.1) show this result for the case of ordinary type structures; the generalization to lexicographic type structures is straightforward (see Catonini and De Vito 2018).

Next, we introduce the notion of terminality for a type structure.

Definition 7 *Fix a class* $\mathbb{T}$ *of type structures. A type structure* $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$ *is* ***terminal with respect to*** $\mathbb{T}$ *if for every type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$ *in* $\mathbb{T}$*, there is a type morphism from* $\mathcal{T}$ *to* $\mathcal{T}^*$.

Whenever $\mathcal{T}^*$ is terminal with respect to the class of *all* type structures, we simply say, as customary, that $\mathcal{T}^*$ is *terminal.* In Section 5 we will show that $\mathrm{R^cCB^cR^c}$ justifies IA in every type structure which is terminal with respect to the class of all finite type structures, and that such a type structure exists.

4 Cautiousness and cautious belief

For this section, we fix a finite game $G := \langle I, (S_i, \pi_i)_{i\in I} \rangle$, and we append to G a type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$.

4.1 Rationality and cautiousness

For any two vectors $x := (x_l)_{l=1}^n, y := (y_l)_{l=1}^n \in \mathbb{R}^n$, we write $x \geq_L y$ if either (a) $x_l = y_l$ for every $l \leq n$, or (b) there exists $m \leq n$ such that $x_m > y_m$ and $x_l = y_l$ for every $l < m$; we write $x >_L y$ if condition (b) holds.

Definition 8 *A strategy* $s_i \in S_i$ *is* ***optimal under*** $\beta_i(t_i) := \left(\mu_i^1, ..., \mu_i^n\right) \in \mathcal{N}(S_{-i} \times T_{-i})$ *if, for every* $s_i' \in S_i$,

$$\left(\pi_i(s_i, \mathrm{marg}_{S_{-i}}\mu_i^l)\right)_{l=1}^n \geq_L \left(\pi_i(s_i', \mathrm{marg}_{S_{-i}}\mu_i^l)\right)_{l=1}^n .$$

Say that s_i *is a* ***lexicographic best reply to*** $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$ *if it is optimal under* $\beta_i(t_i)$.

[13] A Borel map $f : X \to Y$ between separable metrizable spaces is bimeasurable if $f(E)$ is Borel in Y provided E is Borel in X.

This is the usual definition of optimality for a strategy, but this time optimality is taken lexicographically. We next introduce the notion of cautiousness.

Definition 9 *A type $t_i \in T_i$ is **cautious** (in $\mathcal{T}$) if $\overline{\text{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$.*

This notion of cautiousness requires that the first-order belief of a type be a full-support LPS. That is, it requires that every payoff-relevant component, viz. $\{s_{-i}\} \times T_{-i}$, be assigned strictly positive probability by at least one of the measures of LPS $\beta_i(t_i)$. For each $i \in I$, we let C_i denote the set of all pairs $(s_i, t_i) \in S_i \times T_i$ such that t_i is cautious.

For strategy-type pairs we define the following notions.

Definition 10 *Fix a strategy-type pair $(s_i, t_i) \in S_i \times T_i$.*

1. *Say (s_i, t_i) is **rational** (in $\mathcal{T}$) if s_i is optimal under $\beta_i(t_i)$.*
2. *Say (s_i, t_i) is **cautiously rational** (in $\mathcal{T}$) if it is rational and t_i is cautious.*

We let R_i denote the set of all rational strategy-type pairs. As one should expect, cautious rationality guarantees admissibility.

Proposition 1 *If strategy-type pair $(s_i, t_i) \in S_i \times T_i$ is cautiously rational, then s_i is admissible.*

The proof of Proposition 1 is in Appendix B.

4.2 Infinitely more likely and cautious belief

We say that player i deems event E infinitely more likely than event F if she prefers to bet on E rather than on F no matter the prizes for the two bets. This preference-based notion of "infinitely more likely" is due to Lo (1999, Definition 1), and it is formalized in Appendix A, where we introduce the appropriate language. Here, we provide the equivalent definition of "infinitely more likely" in terms of the LPS that represents player i's preferences.

Given an LPS $\overline{\mu}_i := (\mu_i^1, ..., \mu_i^n) \in \mathcal{N}(S_{-i} \times T_{-i})$ and an event $E \subseteq S_{-i} \times T_{-i}$, let

$$\mathcal{I}_{\overline{\mu}_i}(E) := \inf \left\{ l \in \{1, ..., n\} : \mu_i^l(E) > 0 \right\},$$

with the convention that $\inf \emptyset := +\infty$. The following definition is from Catonini and De Vito (2020); see also Stahl (1995).

Definition 11 *Fix two disjoint events $E, F \subseteq S_{-i} \times T_{-i}$. Say that E is **infinitely more likely** than F under $\overline{\mu}_i$ if $\mathcal{I}_{\overline{\mu}_i}(E) < \mathcal{I}_{\overline{\mu}_i}(F)$.*

It is straightforward to see that "infinitely more likely" is monotone. That is, if E is infinitely more likely than F under $\overline{\mu}_i$ and G is an event such that $E \subseteq G$, then G is infinitely more likely than F under $\overline{\mu}_i$.

Consider now the following attitudes of player i towards an event E. First, player i deems E infinitely more likely than its complement. Second, player i has a cautious attitude towards the event: Before considering its complement, she considers all the possible payoff-relevant consequences of the event. The notion of cautious belief captures both attitudes.

Definition 12 *Fix a non-empty event $E \subseteq S_{-i} \times T_{-i}$ and a type $t_i \in T_i$ with $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n)$. Event E is **cautiously believed under $\beta_i(t_i)$ at level** $m \leq n$ if the following conditions hold:*

(i) $\mu_i^l(E) = 1$ *for all* $l \leq m$;

(ii) *for every elementary cylinder* $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T_{-i}$*, if* $E \cap \hat{C}_{s_{-i}} \neq \emptyset$ *then* $\mu_i^l\left(E \cap \hat{C}_{s_{-i}}\right) > 0$ *for some* $l \leq m$.

Event E *is* ***cautiously believed under*** $\beta_i(t_i)$ *if it is cautiously believed under* $\beta_i(t_i)$ *at some level* $m \leq n$.

Type $t_i \in T_i$ ***cautiously believes*** E *if* E *is cautiously believed under* $\beta_i(t_i)$.

Condition (i) captures the first attitude. Under condition (i), condition (ii) is equivalent to saying that player i deems all *payoff-relevant parts* of E (i.e., the non-empty intersections of E with each strategy-based cylinder) infinitely more likely than not-E, so it captures the second attitude.

The conceptual consistency between cautiousness and cautious belief is highlighted by the following connection.

Remark 2 *A type* $t_i \in T_i$ *is cautious if and only if* t_i *cautiously believes* $S_{-i} \times T_{-i}$.

Appendix A provides a preference-based foundation for cautious belief in terms of "infinitely more likely," as well as a characterization in terms of infinitesimal nonstandard numbers. An alternative preference-based characterization of cautious belief is provided in Section D.1 of the Supplementary Appendix. Here, we just mention some properties of cautious belief that will be useful for the proofs of our results.

Proposition 2 *Fix a type* $t_i \in T_i$ *with* $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n)$.

1. *Fix non-empty events* $E_1, E_2, ...$ *in* $S_{-i} \times T_{-i}$*. If, for each* k*, type* t_i *cautiously believes* E_k*, then* t_i *cautiously believes* $\cap_k E_k$ *and* $\cup_k E_k$.
2. *A non-empty event* $E \subseteq S_{-i} \times T_{-i}$ *is cautiously believed under* $\beta_i(t_i)$ *if and only if there exists* $m \leq n$ *such that* $\beta_i(t_i)$ *satisfies condition (i) of Definition 12 plus the following condition:*

   (ii') $\cup_{l \leq m} \text{Suppmarg}_{S_{-i}} \mu_i^l = \text{Proj}_{S_{-i}}(E)$.

The proof of Proposition 2 is in Appendix B.

Proposition 2.1 states that cautious belief satisfies one direction of conjunction as well as one direction of disjunction. Proposition 2.2 can be viewed as a "marginalization" property of cautious belief: If E is cautiously believed under $\beta_i(t_i)$, then

(a) $\text{Proj}_{S_{-i}}(E)$ is infinitely more likely than $S_{-i} \backslash \text{Proj}_{S_{-i}}(E)$ under $\overline{\text{marg}}_{S_{-i}} \beta_i(t_i)$; and

(b) *every* strategy in $\text{Proj}_{S_{-i}}(E)$ is infinitely more likely than (every strategy in) $S_{-i} \backslash \text{Proj}_{S_{-i}}(E)$ under $\overline{\text{marg}}_{S_{-i}} \beta_i(t_i)$.

The failure of one direction of conjunction reveals that, although "infinitely more likely" is monotone, cautious belief is not. That is, if t_i cautiously believes E, then t_i may not cautiously believe an event F such that $E \subseteq F$. The reason why this can occur is that player i may not have towards F the same cautious attitude that she has towards E. That is, there may be some payoff-relevant components of $F \backslash E$ which are not deemed infinitely more likely than not-F. This is illustrated by the following example, which is from Catonini and De Vito (2020, Example 1).

Example 2 *Consider a finite game with two players, Ann (a) and Bob (b), where the strategy set of Bob is* $S_b := \{s_b^1, s_b^2, s_b^3\}$*. Append to this game a type structure* $\mathcal{T}$ *such that* $T_b := \{t_b^*\}$*. Consider the LPS* $\overline{\mu}_a := (\mu_a^1, \mu_a^2) \in \mathcal{N}(S_b \times T_b)$ *with* $\mu_a^1(\{(s_b^1, t_b^*)\}) = 1$ *and* $\mu_a^2(\{(s_b^2, t_b^*)\}) = \mu_a^2(\{(s_b^3, t_b^*)\}) = \frac{1}{2}$*. Next, consider the events* $E := \{s_b^1\} \times T_b$ *and* $F := \{s_b^1, s_b^2\} \times T_b$*. Clearly,* $E \subseteq F$*; however,* E *is cautiously believed under* $\overline{\mu}_a$ *at level* 1*, while* F *is not cautiously believed: indeed,* $\mu_a^1(F) = 1$ *and* $\mu_a^2(F) = \frac{1}{2}$*, and, with* $l = 1$*, condition (ii) of Definition 12 is not satisfied for* $\hat{C}_{s_b^2} := \{s_b^2\} \times T_b$*.* ♦

However, it is easy to observe that cautious belief is monotone with respect to events with the same behavioral implications.

Remark 3 *Let* $E_{-i}, F_{-i} \subseteq S_{-i} \times T_{-i}$ *be events such that* $E_{-i} \subseteq F_{-i}$ *and* $\mathrm{Proj}_{S_{-i}}(E_{-i}) = \mathrm{Proj}_{S_{-i}}(F_{-i})$*. If a type* t_i *cautiously believes* E_{-i}*, then* t_i *cautiously believes* F_{-i}*.*

This "quasi-monotonicity" property will play a crucial role in the proof of our main result.

For future reference, it is useful to mention the following notion of belief, called certain belief (Halpern 2010). Fix a non-empty event $E \subseteq S_{-i} \times T_{-i}$ and a type $t_i \in T_i$ with $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n)$. We say that E is **certainly believed under** $\beta_i(t_i)$ if $\mu_i^l(E) = 1$ for all $l \leq n$. In other words, E is certainly believed under $\beta_i(t_i)$ if its complement is deemed subjectively impossible by the player (see Appendix A for a preference-based foundation).

Certain belief satisfies monotonicity: If E is certainly believed under $\beta_i(t_i)$ and F is an event such that $E \subseteq F$, then F is certainly believed under $\beta_i(t_i)$. Furthermore, certain belief is *not* a stronger concept than cautious belief. To see this, consider a type $t_i \in T_i$ such that $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i) \notin \mathcal{N}^+(S_{-i})$. Event $S_{-i} \times T_{-i}$ is certainly believed under $\beta_i(t_i)$, but it is not cautiously believed because t_i is not cautious (see Remark 2). Yet, it is immediate to check that, for cautious types, certain belief implies cautious belief.

5 Epistemic analysis

5.1 Epistemic analysis of IA

In what follows, we fix a finite game $G := \langle I, (S_i, \pi_i)_{i \in I} \rangle$. Given an associated type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$, for each player $i \in I$, we let $R_i^1 := R_i \cap C_i$ denote the set of cautiously rational strategy-type pairs. Let $\mathbf{B}_i^c : \Sigma_{S_{-i} \times T_{-i}} \to \Sigma_{S_i \times T_i}$ be the operator defined by

$$\mathbf{B}_i^c(E_{-i}) := \{(s_i, t_i) \in S_i \times T_i : t_i \text{ cautiously believes } E_{-i}\}, \ E_{-i} \in \Sigma_{S_{-i} \times T_{-i}}.$$

Corollary C.1 in Appendix C shows that the set $\mathbf{B}_i^c(E_{-i})$ is Borel in $S_i \times T_i$ if $E_{-i} \subseteq S_{-i} \times T_{-i}$ is an event; so the operator $\mathbf{B}_i^c$ is well-defined.

For each $m \geq 1$, define R_i^{m+1} recursively by

$$R_i^{m+1} := R_i^m \cap \mathbf{B}_i^c(R_{-i}^m),$$

where $R_{-i}^m := \prod_{j \neq i} R_j^m$. Note that

$$R_i^{m+1} = R_i^1 \cap \left(\bigcap_{l \leq m} \mathbf{B}_i^c\left(R_{-i}^l\right) \right),$$

and each R_i^m is Borel in $S_i \times T_i$ (see Lemma C.2 in Appendix C).

We write $R_i^\infty := \cap_{m \in \mathbb{N}} R_i^m$ for each $i \in I$. If $(s_i, t_i)_{i \in I} \in \prod_{i \in I} R_i^{m+1}$, we say that there is **cautious rationality and** m**th-order cautious belief in cautious rationality** ($\mathbf{R}^{\mathrm{c}} m \mathbf{B}^{\mathrm{c}} \mathbf{R}^{\mathrm{c}}$) at this state. If $(s_i, t_i)_{i \in I} \in \prod_{i \in I} R_i^\infty$, we say that there is **cautious rationality and common cautious belief in cautious rationality** ($\mathbf{R}^{\mathrm{c}} \mathbf{C} \mathbf{B}^{\mathrm{c}} \mathbf{R}^{\mathrm{c}}$) at this state.

With this, we state the first main result of this paper.

Theorem 1 *Fix a type structure $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ which is terminal with respect to the class of all finite type structures. Then:*

(i) *for each $m \geq 1$, $\prod_{i \in I} \mathrm{Proj}_{S_i}\left(R_i^{*,m}\right) = \prod_{i \in I} S_i^m$;*

(ii) $\prod_{i \in I} \mathrm{Proj}_{S_i}\left(R_i^{*,\infty}\right) = \prod_{i \in I} S_i^\infty$.

We point out that type structure $\mathcal{T}^*$ in Theorem 1 exists. In particular, there exists a *universal* type structure for LPS's, that is, a type structure which is terminal and for which the type morphism from every other type structure is unique. Lee (2016b) shows the existence of a universal type structure for a wide class of preferences, which includes those represented by LPS's. Yang (2015) and Catonini and De Vito (2018) construct the canonical type structure for hierarchies of lexicographic beliefs; Catonini and De Vito also show that this type structure is universal. Since the canonical type structure is continuous and belief-complete, it follows from Theorem 1 that $\mathrm{R^cCB^cR^c}$ is possible in a continuous, belief-complete type structure.

The idea of Theorem 1 stems from Theorem 3 in Friedenberg and Keisler (2020), a result concerning epistemic foundations for iterated strict dominance in the context of ordinary type structures. In the spirit of such result, Theorem 1 identifies a "richness" property of the type structure that depends on its ability to capture sufficiently many hierarchies of beliefs (specifically, all those induced by finite type structures).

The proof of Theorem 1, like the proof of Theorem 3 in Friedenberg and Keisler (2020), is based on the following "embedding" argument. We first construct a finite type structure $\mathcal{T}$ such that, for every $m \geq 1$, the behavioral implications of $\mathrm{R^c}m\mathrm{B^cR^c}$ are characterized by the set of m-admissible strategy profiles. Then, by the terminality property of $\mathcal{T}^*$, we map $\mathcal{T}$ in $\mathcal{T}^*$ via type morphism. While doing so, we show that cautious rationality and all orders of belief in cautious rationality are preserved. For this, we need the next three preparatory results. First, we need to show the existence of $\mathcal{T}$.

Lemma 1 *There exists a finite type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ such that, for each $i \in I$ and each $m \geq 1$, $\mathrm{Proj}_{S_i}\left(R_i^m\right) = S_i^m$.*

The proof of Lemma 1 is in Appendix C. Here we just mention that in the finite type structure we construct for Lemma 1 all types are cautious. This fact will be used below (Section 5.3).

Second, we need to claim the invariance of cautious rationality under type morphisms.

Lemma 2 *Fix type structures $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$. Suppose that there exists a type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$, and fix a strategy-type pair $(s_i, t_i) \in S_i \times T_i$. Then:*

(i) t_i is cautious in $\mathcal{T}$ if and only if $\varphi_i(t_i)$ is cautious in $\mathcal{T}^*$;

(ii) (s_i, t_i) is rational in $\mathcal{T}$ if and only if $(s_i, \varphi_i(t_i))$ is rational in $\mathcal{T}^*$.

The proof of Lemma 2 can be found in Catonini and De Vito (2020, Fact C.1).

Third, we need an analogous invariance property for cautious belief.

Lemma 3 *Fix type structures $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$. Suppose that there exists a bimeasurable type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. If a type $t_i \in T_i$ cautiously believes event $E_{-i} \subseteq S_{-i} \times T_{-i}$, then $\varphi_i(t_i)$ cautiously believes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$.*

The proof of Lemma 3 is in Appendix C. For our purpose, it is crucial to observe that, by Remark 3, if $\varphi_i(t_i)$ cautiously believes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$, then $\varphi_i(t_i)$ cautiously believes also every Borel superset E_{-i}^* such that $\mathrm{Proj}_{S_{-i}}\left(E_{-i}^*\right) = \mathrm{Proj}_{S_{-i}}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)$.

Finally, for the proof of Theorem 1, we find it convenient to single out the following fact, whose proof is immediate.

Remark 4 *Fix type structures* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$ *and* $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$*. Suppose that there exists a type morphism* $(\varphi_i)_{i\in I} : T \to T^*$ *from* $\mathcal{T}$ *to* $\mathcal{T}^*$*. Then, for every* $E_i \subseteq S_i \times T_i$,

$$\mathrm{Proj}_{S_i}\left(\left(\mathrm{Id}_{S_i}, \varphi_i\right)(E_i)\right) = \mathrm{Proj}_{S_i}(E_i).$$

With this, we are ready to prove Theorem 1.

Proof of Theorem 1. By Lemma 1, there is a finite type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$ such that $\mathrm{Proj}_{S_i}(R_i^m) = S_i^m$ for each $m \geq 1$ and for each $i \in I$.

Part (i): Fix a type morphism $(\varphi_i)_{i\in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Type structure $\mathcal{T}$ is finite, so $(\varphi_i)_{i\in I}$ is bimeasurable. We will show by induction on $m \geq 1$ that $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R_i^m) \subseteq R_i^{*,m}$ and $\mathrm{Proj}_{S_i}\left(R_i^{*,m}\right) = S_i^m$ for each $i \in I$.

($m = 1$) Fix $i \in I$. It is immediate from Lemma 2 that $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R_i^1) \subseteq R_i^{*,1}$. By Remark 4, $\mathrm{Proj}_{S_i}\left(\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R_i^1)\right) = \mathrm{Proj}_{S_i}\left(R_i^1\right)$, and since $\mathrm{Proj}_{S_i}\left(R_i^1\right) = S_i^1$, we obtain $S_i^1 \subseteq \mathrm{Proj}_{S_i}\left(R_i^{*,1}\right)$. Conversely, Proposition 1 entails $\mathrm{Proj}_{S_i}\left(R_i^{*,1}\right) \subseteq S_i^1$. Therefore, $\mathrm{Proj}_{S_i}\left(R_i^{*,1}\right) = S_i^1$.

($m > 1$) Fix $i \in I$ and $(s_i, t_i) \in R_i^m$. We want to show that $(s_i, \varphi_i(t_i)) \in R_i^{*,m}$. Since $R_i^m \subseteq R_i^{m-1}$, the induction hypothesis yields $(s_i, \varphi_i(t_i)) \in R_i^{*,m-1}$. Hence, it suffices to show that $\varphi_i(t_i)$ cautiously believes $R_{-i}^{*,m-1}$. Since t_i cautiously believes R_{-i}^{m-1} and the type morphism $(\varphi_i)_{i\in I}$ is bimeasurable, it follows from Lemma 3 that $\varphi_i(t_i)$ cautiously believes $(\mathrm{Id}_{S_{-i}}, \varphi_{-i})(R_{-i}^{m-1})$. Next note that

$$\begin{aligned}
\mathrm{Proj}_{S_{-i}}\left(R_{-i}^{*,m-1}\right) &= S_{-i}^{m-1} \\
&= \mathrm{Proj}_{S_{-i}}\left(R_{-i}^{m-1}\right) \\
&= \mathrm{Proj}_{S_{-i}}\left((\mathrm{Id}_{S_{-i}}, \varphi_{-i})(R_{-i}^{m-1})\right),
\end{aligned}$$

where the first equality is the induction hypothesis, the second equality follows from the property of $\mathcal{T}$, and the third equality follows from Remark 4. We also know from the induction hypothesis that $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(R_{-i}^{m-1}) \subseteq R_{-i}^{*,m-1}$; thus Remark 3 allows us to conclude that $\varphi_i(t_i)$ cautiously believes $R_{-i}^{*,m-1}$.

So, we have shown that $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R_i^m) \subseteq R_i^{*,m}$. Then, by the property of $\mathcal{T}$ and Remark 4, we obtain

$$\begin{aligned}
S_i^m &= \mathrm{Proj}_{S_i}\left(R_i^m\right) \\
&= \mathrm{Proj}_{S_i}\left(\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R_i^m)\right) \\
&\subseteq \mathrm{Proj}_{S_i}\left(R_i^{*,m}\right).
\end{aligned}$$

To show the opposite inclusion, fix $(s_i, t_i^*) \in R_i^{*,m}$. Since $R_i^{*,m} \subseteq R_i^{*,m-1}$, it follows from the induction hypothesis that $s_i \in S_i^{m-1}$. Let $\beta_i^*(t_i^*) := \left(\mu_i^1, ..., \mu_i^n\right)$. Since t_i^* cautiously believes $R_{-i}^{*,m-1}$ at some level l, it follows from Proposition 2.2 and the induction hypothesis that

$$\bigcup_{k\leq l} \mathrm{Suppmarg}_{S_{-i}} \mu_i^k = S_{-i}^{m-1}.$$

So, by Proposition 1 in Blume et al. (1991b), we can form a nested convex combination of the measures $\mathrm{marg}_{S_{-i}} \mu_i^k$, for $k = 1, ..., l$, to get a probability measure $\nu_i \in \mathcal{M}(S_{-i})$, with $\mathrm{Supp}\nu_i = S_{-i}^{m-1}$, under which s_i is optimal. Thus, by Remark 1, s_i is admissible with respect to $S_i \times S_{-i}^{m-1}$, and a fortiori with respect to $S_i^{m-1} \times S_{-i}^{m-1}$. Hence $s_i \in S_i^m$.

Part (ii): Fix $i \in I$. Since $(R_i^m)_{m\in\mathbb{N}}$ and $(S_i^m)_{m\in\mathbb{N}}$ are weakly decreasing sequences of finite, non-empty sets, there exists $N \in \mathbb{N}$ such that $R_i^N = R_i^\infty$ and $S_i^N = S_i^\infty$. Then Lemma 1 implies $\mathrm{Proj}_{S_i}(R_i^\infty) = S_i^\infty$. Hence, for every $s_i \in S_i^\infty$, there exists $t_i \in T_i$ such that $(s_i, t_i) \in R_i^m$ for every $m \in \mathbb{N}$. We have shown in the proof of Part (i) that, for every $m \in \mathbb{N}$, $(\mathrm{Id}_{S_i}, \varphi_i)(R_i^m) \subseteq R_i^{*,m}$. So $(\mathrm{Id}_{S_i}, \varphi_i)((s_i, t_i)) \in R_i^{*,m}$ for every $m \in \mathbb{N}$, which implies that $(\mathrm{Id}_{S_i}, \varphi_i)((s_i, t_i)) \in R_i^{*,\infty}$. Therefore, $S_i^\infty \subseteq \mathrm{Proj}_{S_i}(R_i^{*,\infty})$. Conversely, Part (i) entails $\mathrm{Proj}_{S_i}\left(R_i^{*,N}\right) = S_i^N = S_i^\infty$. Hence $\mathrm{Proj}_{S_i}(R_i^{*,\infty}) \subseteq S_i^\infty$. We conclude that $S_i^\infty = \mathrm{Proj}_{S_i}(R_i^{*,\infty})$. ■

5.2 Epistemic analysis of SAS's

Fix a finite game $G := \langle I, (S_i, \pi_i)_{i\in I}\rangle$. The following result states that, for every type structure associated with game G, the behavioral implications of $\mathrm{R^cCB^cR^c}$ constitute an SAS. Conversely, every SAS corresponds to the behavioral implications of $\mathrm{R^cCB^cR^c}$ in some type structure.

Theorem 2 **(i)** *Fix a type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i\rangle_{i\in I}$. *Then* $\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^\infty)$ *is an SAS.*

(ii) *Fix an SAS* $Q \in \mathcal{Q}$. *There exists a finite type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i\rangle_{i\in I}$ *such that, for each* $i \in I$,

$$\mathrm{Proj}_{S_i}(R_i^\infty) = Q_i.$$

The idea of Theorem 2 stems from Theorem 8.1 in BFK, a result concerning the epistemic justification of SAS's with rationality and common assumption of rationality. The proof of Theorem 2, which is similar to that in BFK, is in Appendix C. Here we point out that in the finite type structure we construct for Theorem 2.(ii) *all* types are cautious. We have already mentioned (Section 5.1) that the same property holds for the finite type structure we construct for Lemma 1. This raises the question whether one could incorporate the cautiousness assumption in the definition of type structures for the epistemic analysis of IA and SAS's. That is, could we restrict attention to the class $\mathbb{T}$ of type structures where each type's belief over the opponents' strategies have full support? Are there analogues of Theorems 1 and 2 for such a class? We explore this issue in the next section.

5.3 Alternative epistemic conditions for IA and SAS's

Fix a finite game $G := \langle I, (S_i, \pi_i)_{i\in I}\rangle$ and an associated type structure $\mathcal{T} := \langle S_i, T_i, \beta_i\rangle_{i\in I}$. Say that **type** t_i **certainly believes** a non-empty event $E_{-i} \subseteq S_{-i} \times T_{-i}$ if E_{-i} is certainly believed under $\beta_i(t_i)$. For each player $i \in I$, let $\mathbf{B}_i : \Sigma_{S_{-i}\times T_{-i}} \to \Sigma_{S_i \times T_i}$ be the operator defined by

$$\mathbf{B}_i(E_{-i}) := \{(s_i, t_i) \in S_i \times T_i : t_i \text{ certainly believes } E_{-i}\},\ E_{-i} \in \Sigma_{S_{-i}\times T_{-i}}.$$

As shown in Catonini and De Vito (2020), the set $\mathbf{B}_i(E_{-i})$ is Borel in $S_i \times T_i$ if $E_{-i} \subseteq S_{-i} \times T_{-i}$ is an event; thus, the operator $\mathbf{B}_i$ is well-defined.

Next, fix a non-empty event $E_i \subseteq S_i \times T_i$ for every $i \in I$. Event $E := \prod_{i\in I} E_i$ is **self-evident** (in $\mathcal{T}$) if it satisfies $E \subseteq \prod_{i\in I} \mathbf{B}_i(E_{-i})$; standard results (e.g., Catonini and De Vito 2018, Appendix 5.2) show that E is self-evident if and only if at every state $(s_i, t_i)_{i\in I} \in E$ there is common certain belief in E. With this, we say that there is **transparency of** E at state $(s_i, t_i)_{i\in I}$ if $(s_i, t_i)_{i\in I} \in E$ and E is self-evident.

We let $C^\infty \subseteq \prod_{i\in I} S_i \times T_i$ denote the event corresponding to transparency of cautiousness in $\mathcal{T}$, and we let C_i^∞ denote the corresponding projection on $S_i \times T_i$.[14] With this, for each $i \in I$,

[14] Since the set C_i is a Borel subset of $S_i \times T_i$ (Catonini and De Vito 2020, Corollary D.1), an argument by induction on the iteration of the operator $\mathbf{B}_i$ shows that C_i^∞ is also a Borel subset (event) of $S_i \times T_i$. In particular, if $\mathcal{T}$ is the canonical type structure, then it is possible to show that C_i^∞ is a Polish subset of $S_i \times T_i$.

let $\hat{R}_i^1 := R_i \cap C_i^\infty$. Then, for each $i \in I$ and $m \geq 1$, define $\hat{R}_i^{m+1}$ recursively by

$$\hat{R}_i^{m+1} := \hat{R}_i^m \cap \mathbf{B}_i^c \left(\hat{R}_{-i}^m\right),$$

where $\hat{R}_{-i}^m := \prod_{j \neq i} \hat{R}_j^m$. Write $\hat{R}_i^\infty := \cap_{m \in \mathbb{N}} \hat{R}_i^m$ for each $i \in I$. Therefore, if $(s_i, t_i)_{i \in I} \in \prod_{i \in I} \hat{R}_i^\infty$, we say that at state $(s_i, t_i)_{i \in I}$ there is (a) **rationality and transparency of cautiousness**, and (b) **common cautious belief in (a)**.

Say that $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is a **cautious type structure** if $C_i = S_i \times T_i$ for every $i \in I$. Since event $S \times T := \prod_{i \in I} (S_i \times T_i)$ is self-evident, it is easily seen that $C^\infty = S \times T$ in a cautious type structure. Therefore, if $\mathcal{T}$ is a cautious type structure, then, for each $i \in I$,

$$\hat{R}_i^1 = R_i^1 = R_i;$$

that is, event $\prod_{i \in I} \hat{R}_i^1$ is the set of states where there is rationality. It follows by induction on $m \geq 1$ that $\hat{R}_i^m = R_i^m$ for each $i \in I$. So, if $(s_i, t_i)_{i \in I} \in \prod_{i \in I} \hat{R}_i^{m+1}$, we can say that there is **rationality and mth-order cautious belief in rationality** at this state.

We are now in a position to state a characterization result, which is an analogue of Theorem 1.

Theorem 3 **(i)** *Fix a type structure $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ which is terminal with respect to the class of all finite type structures. Then:*

(i.1) *for each $m \geq 1$, $\prod_{i \in I} \mathrm{Proj}_{S_i} \left(\hat{R}_i^{*,m}\right) = \prod_{i \in I} S_i^m$;*

(i.2) $\prod_{i \in I} \mathrm{Proj}_{S_i} \left(\hat{R}_i^{*,\infty}\right) = \prod_{i \in I} S_i^\infty$.

(ii) *The same conclusions as in (i.1) and (i.2) hold if $\mathcal{T}^*$ is a cautious type structure which is terminal with respect to the class of all finite, cautious type structures.*

To see why Part (i) of Theorem 3 holds, let $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be the finite type structure we construct for the proof of Lemma 1. Since $\mathcal{T}$ is a cautious type structure, $\hat{R}_i^m = R_i^m$ for each $i \in I$ and $m \geq 1$. Finiteness of $\mathcal{T}$ guarantees that the type morphism φ from $\mathcal{T}$ to $\mathcal{T}^*$ is bimeasurable. Lemma 2.(i) entails that $S \times \varphi(T) \subseteq C^*$; that is, the image of self-evident event C^∞ in $\mathcal{T}$ under type morphism φ is a subset of C^*. Proposition 12 in Catonini and De Vito (2018) shows that event $S \times \varphi(T)$ is self-evident in $\mathcal{T}^*$.[15] But then, by the monotonicity property of certain belief, it is immediate to see that at every state in $S \times \varphi(T)$ there is common certain belief in C^*. We therefore conclude that $S \times \varphi(T) \subseteq C^{*,\infty}$. With this, the proof of Theorem 3 is the same as that of Theorem 1—just replace sets such as $R_i^{*,m}$ and $R_{-i}^{*,m}$ with the corresponding sets $\hat{R}_i^{*,m}$ and $\hat{R}_{-i}^{*,m}$.

Theorem 3.(ii) is a characterization result for m-admissible (resp. iteratively admissible) strategies in terms of rationality and mth-order (resp. common) cautious belief in rationality. To see why Theorem 3.(ii) holds, we first point out that a terminal, cautious type structure exists. Let $\mathcal{T}^U$ be the canonical, universal type structure. By Proposition 12 in Catonini and De Vito (2018), the self-evident event $C^{U,\infty}$ identifies a "smaller," cautious type structure $\mathcal{T}^*$. In such a structure, $\hat{R}_i^{*,m} = R_i^{*,m}$ for each $i \in I$ and $m \geq 1$. By the above argument, $\mathcal{T}^*$ is terminal with respect to the class of finite, cautious type structures. With this, the proof of Theorem 1 yields the result.

The following characterization result, pertaining to SAS's, is an analogue of Theorem 2.

[15] Lemma A3 in Battigalli and Friedenberg (2012b) shows an analogous result for conditional type structures, i.e., type structures where types map to conditional probability systems. Catonini and De Vito (2018) adapt the proofs in Battigalli and Friedenberg (2012b) to the case of lexicographic type structures.

Theorem 4 **(i)** *Fix a type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$*. Then* $\prod_{i \in I} \operatorname{Proj}_{S_i}\left(\hat{R}_i^{\infty}\right)$ *is an SAS.*

(ii) *Fix an SAS* $Q \in \mathcal{Q}$*. There exists a finite, cautious type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ *such that, for each* $i \in I$*,*

$$\operatorname{Proj}_{S_i}\left(\hat{R}_i^{\infty}\right) = Q_i.$$

The proof of Theorem 4.(i) is, with some minor modifications, identical to the proof of Theorem 2.(i). Part (ii) of Theorem 4 is essentially Theorem 2.(ii). As noted, in the proof of Theorem 2.(ii) we construct a finite, cautious type structure $\mathcal{T}$ such that $\prod_{i \in I} \operatorname{Proj}_{S_i}\left(R_i^{\infty}\right) = Q$. In such a type structure, $\hat{R}_i^{\infty} = R_i^{\infty}$ for each $i \in I$.

Theorem 4 allows us to obtain an alternative characterization of SAS's. Specifically, if we restrict attention to the class of cautious type structures, we can restate the results of Theorem 4 as follows: *SAS's characterize the behavioral implications of rationality and common cautious belief in rationality across all cautious type structures.*

Some final remarks on the epistemic assumptions considered so far are in order. First, events $\prod_{i \in I} R_i^{\infty}$ and $\prod_{i \in I} \hat{R}_i^{\infty}$ are equivalent in cautious type structures, but they are typically different in non-cautious type structures. In Section D.2 of the Supplementary Appendix we exhibit an example where the game in Example 1 is associated with a finite, *non-cautious* type structure $\mathcal{T}$ such that $\prod_{i \in I} R_i^{\infty}$ and $\prod_{i \in I} \hat{R}_i^{\infty}$ are disjoint. The example shows that the behavioral implications of such epistemic assumptions are characterized by two *disjoint* SAS's—specifically, the sets $\{m\} \times \{\ell\}$ and $\{u\} \times \{r\}$, respectively.

To understand this point, fix an arbitrary type structure $\mathcal{T}$ associated with a finite game. Let us consider the first two steps in the definitions of $\prod_{i \in I} R_i^{\infty}$ and $\prod_{i \in I} \hat{R}_i^{\infty}$. At the first step, we have $\hat{R}_i^1 \subseteq R_i^1$ for each $i \in I$, because transparency of cautiousness implies cautiousness. Does an analogous conclusion hold for the sets $\hat{R}_i^2$ and R_i^2 ($i \in I$)? The answer is no. Recall that, for each $i \in I$,

$$\begin{aligned} \hat{R}_i^2 &: = \hat{R}_i^1 \cap \mathbf{B}_i^c\left(\hat{R}_{-i}^1\right), \\ R_i^2 &: = R_i^1 \cap \mathbf{B}_i^c\left(R_{-i}^1\right). \end{aligned}$$

Since cautious belief is *not* monotonic (see Section 4.2), for some player $i \in I$ we can have $\mathbf{B}_i^c\left(\hat{R}_{-i}^1\right) \nsubseteq \mathbf{B}_i^c\left(R_{-i}^1\right)$ even if $\hat{R}_{-i}^1 \subseteq R_{-i}^1$; this is illustrated by the example in the Supplementary Appendix. Yet, it should be noted that $\prod_{i \in I} \hat{R}_i^2 \subseteq \prod_{i \in I} R_i^2$ holds whenever $\prod_{i \in I} \operatorname{Proj}_{S_i}\left(\hat{R}_i^1\right) = \prod_{i \in I} \operatorname{Proj}_{S_i}\left(R_i^1\right)$. Indeed, under such condition, the "quasi-monotonicity" property of cautious belief (Remark 3) entails that $\mathbf{B}_i^c\left(\hat{R}_{-i}^1\right) \subseteq \mathbf{B}_i^c\left(R_{-i}^1\right)$ for each $i \in I$.

In Section D.2 of the Supplementary Appendix we show that, if $\mathcal{T}$ is a "rich" type structure, then

$$\prod_{i \in I} \hat{R}_i^{\infty} \subseteq \prod_{i \in I} R_i^{\infty} \quad \text{and} \quad \prod_{i \in I} \operatorname{Proj}_{S_i}\left(\hat{R}_i^{\infty}\right) = \prod_{i \in I} \operatorname{Proj}_{S_i}\left(R_i^{\infty}\right).$$

In words, if structure $\mathcal{T}$ is "rich," then the epistemic assumption of "rationality, transparency of cautiousness, and common cautious belief in both" is stronger than $\mathrm{R^cCB^cR^c}$; nonetheless, they are equivalent in terms of behavioral implications.

As one should expect, examples of "rich" type structures satisfying the aforementioned property are terminal structures such as $\mathcal{T}^*$ in the statement of Theorem 1. Other examples are belief-complete type structures (Definition 5): this is formally shown in Theorem D.2, whose proof makes explicit use of the "quasi-monotonicity" property of cautious belief. Thus, if $\mathcal{T}$ is belief-complete, we can conclude, by Theorem 2, that the behavioral implications of both epistemic assumptions are characterized by one specific SAS. But this SAS could be different from the IA set, as the following section illustrates.

5.4 Belief-completeness vs terminality

Theorem 1 identifies a "richness" condition on type structures for the epistemic justification of IA. A related "richness" condition is belief-completeness (Definition 5): a belief-complete type structure induces all possible beliefs about types. In light of this, one might conjecture that the conclusions of Theorem 1 continue to hold for belief-complete type structures. However, this is not the case.

Call a finite game $G := \langle I, (S_i, \pi_i)_{i\in I}\rangle$ **non-degenerate** if $|S_i| \geq 2$ for each $i \in I$. The following result states that, for each non-degenerate, finite game, there exists a continuous, belief-complete type structure where $\mathrm{R}^{\mathrm{c}}\mathrm{CB}^{\mathrm{c}}\mathrm{R}^{\mathrm{c}}$ is not possible at any state. It follows that, in such a structure, the behavioral implications of $\mathrm{R}^{\mathrm{c}}\mathrm{CB}^{\mathrm{c}}\mathrm{R}^{\mathrm{c}}$ constitute the empty SAS (cf. Theorem 2.(i)).

Theorem 5 *Fix a non-degenerate finite game $G := \langle I, (S_i, \pi_i)_{i\in I}\rangle$. There exists a continuous, belief-complete type structure $\mathcal{T} := \langle S_i, T_i, \beta_i\rangle_{i\in I}$ such that*

$$\prod_{i\in I} R_i^\infty = \emptyset.$$

Theorem 5 is inspired by Friedenberg and Keisler (2020, Theorem 1). Specifically, Friedenberg and Keisler consider finite games satisfying a non-triviality condition that is stronger than non-degeneracy. With this, they show the existence of a belief-complete, ordinary type structure in which there is no state consistent with rationality and common belief in rationality.

The proof of Theorem 5 adapts the arguments in Friedenberg and Keisler (2020) to the lexicographic framework, and it can be found in Appendix C. In Section 6 we will further discuss the relationship between Theorem 5 and Theorem 1 in Friedenberg and Keisler (2020). Here, we briefly explain *why* Theorem 5 holds.

Fix a belief-complete type structure $\mathcal{T}$. In such a structure, $\prod_{i\in I} R_i^m \neq \emptyset$ for each $m \geq 1$. In particular, it can be shown that, for each $m \geq 1$, the behavioral implications of $\mathrm{R}^{\mathrm{c}}m\mathrm{B}^{\mathrm{c}}\mathrm{R}^{\mathrm{c}}$ are characterized by the set of m-admissible strategy profiles (see Catonini and De Vito 2021). The reason why the set of states $\prod_{i\in I} R_i^\infty$ can be empty is—conceptually—the same as in Friedenberg and Keisler (2020): While a belief-complete (lexicographic) type structure induces all beliefs about types, it need not induce all possible hierarchies of beliefs. Specifically, Theorems 1 and 5 imply that a belief-complete type structure may not induce all hierarchies of beliefs that can arise in finite type structures.

In the context of ordinary type structures, Friedenberg (2010, Theorem 3.1) shows that a belief-complete type structure is terminal if each type space is compact and each belief map is continuous. An analogue of Friedenberg's result does not exist in the lexicographic framework: as already remarked (see Section 3.2), a belief-complete, lexicographic type structure cannot be compact and continuous. This explains why the terminality property of the type structure is made explicit in the statements (and proofs) of our results (Theorems 1 and 3).

6 Discussion

Given the previous analysis, it is now possible to discuss in detail a set of conceptual issues, some of which were informally addressed in the Introduction. The Supplementary Appendix contains elaborations on some results discussed in this section.

6.1 Weak belief in cautious rationality

A weaker concept than cautious belief (and certain belief) in an event is that of *weak belief.* Formally, an event E is weakly believed if E is infinitely more likely than not-E. As shown in Catonini and De Vito (2020), this is equivalent to requiring that $\mu^1(E) = 1$ for the LPS $(\mu^1, ..., \mu^n)$ representing the agent's beliefs. Using this condition on LPS's, Brandenburger (1992) put forward the solution concept of *permissibility*. This solution concept is weaker than IA because—as shown by Brandenburger—it is equivalent to the so-called Dekel-Fudenberg procedure; that is, one round of elimination of inadmissible strategies is followed by iterated elimination of strictly dominated strategies.

Using arguments similar to those in Section 5.1 and in Catonini and De Vito (2020), it can be shown—as an analogue of Theorem 1—that the behavioral implications of cautious rationality and common weak belief in cautious rationality ($\mathrm{R^cCWBR^c}$) are characterized by permissibility.[16] The same conclusion holds if, additionally, cautiousness is assumed to be transparent (cf. Theorem 3).

A set $Q \in \mathcal{Q}$ is a full weak best response set (full WBRS) if each $s_i \in Q_i$ satisfies conditions (a) and (c) in the definition of SAS, and each $s_i \in Q_i$ is not strictly dominated with respect to $S_i \times Q_{-i}$. Clearly, every SAS is a full WBRS. Every full WBRS is contained in the permissibility set, which is a full WBRS. This is to be contrasted with the existence of SAS's not contained in the IA set. The reason for this difference is that, unlike assumption and cautious belief, weak belief is *monotone*. Theorem 1 in Catonini and De Vito (2020) shows that full WBRS's characterize the behavioral implications of $\mathrm{R^cCWBR^c}$ across all type structures. And the same conclusion holds if cautiousness is transparent. To sum up: If in Theorems 1-4 the notion of cautious belief is replaced by weak belief, then we obtain epistemic characterization results for permissibility and full WBRS's.

An analogue of Theorem 5 holds for $\mathrm{R^cCWBR^c}$. Actually, the result in Theorem 5 can be obtained as a corollary of an impossibility result for $\mathrm{R^cCWBR^c}$ in some belief-complete type structures. Fix a type structure $\mathcal{T}$ associated with a non-degenerated game G. Using the same formalism as in Catonini and De Vito (2020), it is possible to redefine each set $\prod_{i\in I} R_i^{m+1}$ as the set of states consistent with cautious rationality and mth-order weak belief in cautious rationality.[17] With this, the proof of Theorem 5 can be read, verbatim, as a proof of the impossibility of $\mathrm{R^cCWBR^c}$ for some belief-complete type structure—see Remark C.3 in Appendix C. Since $\mathrm{R^cCB^cR^c}$ implies $\mathrm{R^cCWBR^c}$, the result of Theorem 5 follows.

Such impossibility result for $\mathrm{R^cCWBR^c}$ can also be obtained by Theorem 1 in Friedenberg and Keisler (2020) if we assume that the game is non-trivial. Formally (see Friedenberg and Keisler 2020, Section 6), a finite game G is *non-trivial* if, for each $i \in I$ and each $s_i \in S_i$, there exists $\mu_i \in \mathcal{M}(S_{-i})$ such that s_i is not optimal under μ_i. Clearly, if G is non-trivial, then G is non-degenerate. In the context of ordinary type structures, Friedenberg and Keisler show the following result: For any non-trivial game, there exists a continuous, belief-complete type structure for which there is no state consistent with rationality and common belief in rationality. Because weak belief is monotone, a relabeling of the sets in Friedenberg and Keisler's (2020) construction goes almost all the way to a proof for $\mathrm{R^cCWBR^c}$.

6.2 IA and lexicographic rationalizability

In the Introduction, we have informally claimed that $\mathrm{R^cCB^cR^c}$ matches very closely the logic of lexicographic rationalizability (Stahl 1995), an iterated elimination procedure for lexicographic beliefs. Here we make this informal claim precise.

Fix a player $i \in I$ and a non-empty set $Q_{-i} \subseteq S_{-i}$. We let $r_i(\overline{\mu}_i)$ denote the set of player i's

[16] Theorem 2 in Catonini and De Vito (2020) proves the result for a specific terminal type structure, namely the canonical one.

[17] Formally, in the definition of each set R_i^{m+1} the operator $\mathbf{B}_i^c$ must be replaced by the "weak belief" operator $\mathbf{WB}_i$—see Catonini and De Vito (2020).

strategies which are optimal under $\overline{\mu}_i \in \mathcal{N}(S_{-i})$, and

$$\mathcal{B}_c^+(Q_{-i}) := \left\{ \left(\mu_i^1, ..., \mu_i^n\right) \in \mathcal{N}^+(S_{-i}) : \exists m \leq n, \bigcup_{l=1}^{m} \text{Supp}\mu_i^l = Q_{-i} \right\}$$

is the set of all full-support LPS's $\overline{\mu}_i$ such that Q_{-i} is "cautiously believed" under $\overline{\mu}_i$ (cf. Proposition 2.2). In words, $\mathcal{B}_c^+(Q_{-i})$ is the set of all full-support first-order beliefs under which player i deems *every* co-players' strategy profile in Q_{-i} infinitely more likely—in the sense of Lo (1999) or Stahl (1995)—than every profile in $S_{-i} \backslash Q_{-i}$. Note: if $Q'_{-i} \subseteq S_{-i}$ is such that $Q_{-i} \subset Q'_{-i}$ (strict inclusion), then $\mathcal{B}_c^+(Q_{-i}) \nsubseteq \mathcal{B}_c^+(Q'_{-i})$. As explained in Section 4.2, this occurs because there is some $s_{-i} \in Q'_{-i} \backslash Q_{-i}$ which is not deemed—under some $\overline{\mu}_i \in \mathcal{B}_c^+(Q_{-i})$—infinitely more likely than every profile in $S_{-i} \backslash Q'_{-i}$. Furthermore, note that $\mathcal{B}_c^+(S_{-i}) = \mathcal{N}^+(S_{-i})$.

With this, we can define a sequence $\left(\hat{S}^m\right)_{m \geq 0}$ of subsets of S as follows. For each $i \in I$, let $\hat{S}_i^0 := S_i$. Also, let $\hat{S}^0 := \prod_{i \in I} \hat{S}_i^0$ and $\hat{S}_{-i}^0 := \prod_{j \neq i} \hat{S}_j^0$. Recursively define, for $m \geq 1$,

$$\begin{aligned} \hat{S}_i^m \ &: = \left\{ s_i \in S_i : \exists \overline{\mu}_i \in \bigcap_{l=0}^{m-1} \mathcal{B}_c^+(\hat{S}_{-i}^l), s_i \in r_i(\overline{\mu}_i) \right\}, \\ \hat{S}^m \ &: = \prod_{i \in I} \hat{S}_i^m. \end{aligned}$$

Profiles in $\hat{S}^\infty := \cap_{m=0}^{\infty} \hat{S}^m$ are called lexicographic rationalizable. Stahl (1995) shows that $S^m = \hat{S}^m$ for every $m \in \mathbb{N}$, that is, the set of m-admissible strategies coincides with the set of strategies surviving the first m steps of the procedure.

Similarly, SAS's can be given a characterization in terms of justifiability ("best reply to some belief") as follows. Fix a set $Q \in \mathcal{Q}$. Using arguments analogous to those in Catonini and De Vito (2020) and Brandenburger et al. (2012), it can be shown that Q is an SAS if and only if, for each $i \in I$ and each $s_i \in Q_i$, there exists $\overline{\mu}_i \in \mathcal{B}_c^+(Q_{-i})$ such that $s_i \in r_i(\overline{\mu}_i)$ and $r_i(\overline{\mu}_i) \subseteq Q_i$.

Such characterization of SAS is similar to the definition of "extensive-form best response set" (EFBRS, Battigalli and Friedenberg 2012a). Of course, SAS's and EFBRS's are distinct concepts. In particular, EFBRS's are defined in terms of conditional probability systems and strong belief (Battigalli and Siniscalchi 2002). Examples of the difference between SAS's and EFBRS's can be found in Battigalli and Friedenberg (2012b, Section 8.c). Nonetheless, given the analogy between cautious belief and strong belief we highlighted in the Introduction, SAS can be viewed as a strategic-form analogue of EFBRS. In the same way, lexicographic rationalizability, which yields a particular SAS (the IA set), can be viewed as a strategic-form analogue of *strong rationalizability* (Pearce 1984, Battigalli 1997),[18] which yields a particular EFBRS.

6.3 Cautious vs full-support types

We define cautiousness as a full-support condition on the set of strategies. Analogous conditions can be found in other works, such as Perea (2012), Lee (2016), Heifetz et al. (2019) and Catonini and De Vito (2020). A stronger notion than cautiousness is to require that types be associated with full-support LPS's. That is, given a strategy-type pair (s_i, t_i), it is required that $\beta_i(t_i) \in \mathcal{N}^+(S_{-i} \times T_{-i})$.

Differently from cautiousness, the full-support condition on strategies *and* types may crucially depend on modeling details of type structures which are unrelated to belief hierarchies.

[18] Strong rationalizability is also known as "extensive-form rationalizability." We find such terminology ambiguous and hence we avoid it, because this solution concept refers to just one out of several meaningful versions of rationalizability for extensive-form games (see Battigalli et al. 2021).

Example 2 in Catonini and De Vito (2020) shows the existence of two distinct, finite type structures, viz. $\mathcal{T}$ and $\mathcal{T}'$, that induce the same set of belief hierarchies, but the sets of full-support types differ. In particular, every type in $\mathcal{T}$ has full support, while the set of full-support types in $\mathcal{T}'$ is empty. The example shows that this situation occurs because structure $\mathcal{T}'$ is **redundant**; that is, there are distinct types of some player that induce the same hierarchy of beliefs. In light of this, a natural conjecture could be that the set of full-support types does not vary across non-redundant, isomorphic type structures. The following example shows that such a conjecture is wrong.

Example 3 *Consider a finite, non-degenerate game with two players, Ann (a) and Bob (b). Let* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in\{a,b\}}$ *be the canonical, ordinary type structure associated with this game. Structure* $\mathcal{T}$ *is non-redundant, and each belief map is a homeomorphism. Consider Ann, and a type* t_a *associated with a full-support probability measure* $\mu \in \mathcal{M}(S_b \times T_b)$*. We assume that* μ *is a product measure, viz.* $\mu := \mu_1 \otimes \mu_2$*, where both* $\mu_1 \in \mathcal{M}(S_b)$ *and* $\mu_2 \in \mathcal{M}(T_b)$ *have full support. Suppose that there is a non-empty event* $E := F \times G$*, with* $F \subseteq S_b$ *and* $G \subseteq T_b$*, such that* $\mu(E) = 0$*. Since* μ *is a full-support measure,* E *is not an open subset of* $S_b \times T_b$*. Specifically,* F *is (cl)open in* S_b*, while* G *is not open in* T_b*. Hence* $\mu_1(F) > 0$ *and* $\mu_2(G) = 0$*.*

The space $S_b \times T_b$ *is an uncountable Polish space, as it is the topological product of a finite, discrete set (*S_b*) and an uncountable Polish space (*T_b*). So, we can invoke Theorem 13.1 in Kechris (1995) to claim the existence of a finer Polish topology on* T_b *such that (1)* G *is an open set, and (2) it generates the same Borel* σ*-field as the previous topology. Let* T_b^* *denote the Polish space endowed with this finer topology. We have that* $\mathcal{M}(S_b \times T_b) = \mathcal{M}(S_b \times T_b^*)$ *because* $\Sigma_{T_b} = \Sigma_{T_b^*}$*. With this, we can construct a type structure* $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i\in\{a,b\}}$ *as follows. First, set* $\beta_i^* := \beta_i$ *for each player* $i \in \{a, b\}$*. Then let* $T_a^* := T_a$*, while* T_b^* *is the Polish space constructed above.*

Structure $\mathcal{T}^*$ *is isomorphic to* $\mathcal{T}$*, hence both structures induce the same set of belief hierarchies. The unique type isomorphism* $(\varphi_i)_{i\in\{a,b\}} : T^* \to T$ *is the identity map, which is, technically, a Borel isomorphism. It is not a homeomorphism, as spaces* T_b^* *and* T_b *are not homeomorphic. Type* $t_a \in T_a^* = T_a$ *is not a full-support type in* $\mathcal{T}^*$*, since the open set* $E := F \times G \subseteq S_b \times T_b^*$ *is assigned measure* 0 *by* μ*.* ♦

The above argument and Example 3 highlight the critical issues that can arise if we impose the full-support condition. Despite this, it would be worth exploring whether our results continue to hold under the assumption of full-support. That is, are there analogues of the results in Section 5 if the notion of rationality incorporates this full-support requirement (as in BFK and Dekel et al. 2016)?

The answer is a qualified Yes. In the remainder of this section, we explain why this is the case. To formally distinguish the notions of "rationality" and "rationality with full-support," we refer to the latter as BFK-rationality. Let us first consider the case of arbitrary type structures. Fix a finite game G and an associated type structure $\mathcal{T}$. The proof of Theorem 2.(i) shows that the behavioral implications of BFK-rationality and common cautious belief thereof constitute an SAS. Of course, such SAS could be different from the output of $\mathrm{R^cCB^cR^c}$. This is not true if $\mathcal{T}$ is a **full-support type structure**, i.e., a structure where all types of all players have full-support beliefs. In such a case, $\mathcal{T}$ is a cautious type structure,[19] and BFK-rationality coincides with rationality. Next, fix an SAS Q. A minor change in the proof of 2.(ii) (see Remark C.2) yields the existence of a finite, *full-support* type structure such that Q is the set of strategies consistent with (BFK-)rationality and common cautious belief in (BFK-)rationality.

[19] A cautious type structure need not be a full-support type structure, as Example 2 in Catonini and De Vito (2020) shows.

With this, we conclude that analogues of Theorems 2 and 4 hold if cautiousness is replaced by the full-support condition.

As far as the epistemic analysis of IA is concerned, we point out that the methodology and proofs used in Theorems 1 and 3 do *not* work under the full-support condition. The proof of Lemma 1 can be easily adapted to show that structure $\mathcal{T}$ in the statement is a full-support type structure—see Remark C.1. But the "embedding" argument in the proofs of Theorems 1 and 3 fails for a very basic reason: full-support beliefs are not preserved by type morphisms. So, in order to obtain positive results for IA, we need to take a different route. Section F of the Supplementary Appendix shows two positive results, which we briefly illustrate. Fix a finite game $G := \langle I, (S_i, \pi_i)_{i \in I} \rangle$. The first result says that, if G is associated with the *canonical* type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$, then IA characterizes the behavioral implications of BFK-rationality and common cautious belief in BFK-rationality. The proof does not rely on the terminality property of $\mathcal{T}$, but it makes explicit use of the fact that each type is a hierarchy of beliefs. More interestingly, we show the existence of a self-evident event in $\mathcal{T}$ where it is transparent that all types have full-support beliefs. Such self-evident event gives rise to a full-support type structure $\hat{\mathcal{T}} := \langle S_i, \hat{T}_i, \hat{\beta}_i \rangle_{i \in I}$. The second result (Theorem F.1) says that, if G is associated with $\hat{\mathcal{T}}$, then IA still characterizes the behavioral implications of (BFK-)rationality and common cautious belief in (BFK-)rationality. Such a result can be seen as an analogue of Theorem 3, but there is a key difference. Structure $\hat{\mathcal{T}}$ is *not* terminal with respect to finite, full-support type structures,[20] but it is "rich" in the following sense. Consider game $G' := \langle I, (S_i, \pi'_i)_{i \in I} \rangle$, with the same set of players and strategies as in G, but with different payoff functions. Theorem F.1 applies to both G and G', because the construction of $\hat{\mathcal{T}}$ does not hinge on payoff functions. Alternatively put, Theorem F.1 provides a game-independent epistemic foundation for IA—that is, one that can be stated in the same type structure no matter what the game is.

6.4 Comparison to (weak) assumption

In Catonini (2013) and Yang (2015), an event $E \subseteq S_{-i} \times T_{-i}$ is "weakly assumed" under an LCPS $\overline{\mu} := (\mu^1, ..., \mu^n)$ if there exists $m \leq n$ such that

(i) $\mu^l(E) = 1$ for all $l \leq m$,

(ii) $\mu^l(E) = 0$ for all $l > m$,

(iii) for every elementary cylinder $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T_{-i}$, if $E \cap \hat{C}_{s_{-i}} \neq \emptyset$ then $\mu^l\left(E \cap \hat{C}_{s_{-i}}\right) > 0$ for some $l \leq m$.

The difference between weak assumption and BFK's assumption relies on condition (iii): BFK require that, for every open set $O \subseteq S_{-i} \times T_{-i}$, if $E \cap O \neq \emptyset$ then $\mu^l(E \cap O) > 0$ for some $l \leq m$. BFK's assumption is stronger than weak assumption because, technically, every elementary cylinder is an open set. The definition of weak assumption can be extended to all LPS's while preserving its preference-based foundation in the same way as Dekel et al. (2016)

[20] In general, a terminal, full-support type structure does not exist. To see this, consider a finite, non-degenerate game $G := \langle I, (S_i, \pi_i)_{i \in I} \rangle$ and an associated type structure $\mathcal{T}^* := \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$. Suppose that $\mathcal{T}^*$ is terminal with respect to the class of finite type structures. Hence, by non-degeneracy of G, the cardinality of each type set T_i^* is greater than 2—actually, it is infinite. Let $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be a finite, *full-support* type structure such that $T_i := \{\bar{t}_i\}$ for every $i \in I$ (i.e., each type set is a singleton). By terminality of $\mathcal{T}^*$, there is a type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. For every $i \in I$, type $\varphi_i(\bar{t}_i)$ is not a full-support type: using the commutative diagram (3.1), it is easy to check that $\mathrm{Supp}\beta_i^*(\varphi_i(\bar{t}_i)) = S_{-i} \times \{\varphi_{-i}(\bar{t}_{-i})\}$. So $\mathrm{Supp}\beta_i^*(\varphi_i(\bar{t}_i))$ is a proper subset of $S_{-i} \times T_{-i}^*$. With this, we conclude that structure $\mathcal{T}^*$ must necessarily contain non-full-support types.

extend BFK's assumption.[21] So, from now on, we say that event $E \subseteq S_{-i} \times T_{-i}$ is **weakly assumed** under LPS $\overline{\mu} := (\mu^1, ..., \mu^n)$ if there exists $m \leq n$ such that conditions (i) and (iii) above hold, and

(ii)' for each $l > m$, there exists $\left(\alpha_1^l, ..., \alpha_m^l\right) \in \mathbb{R}^m$ such that $\mu^l(F) = \sum_{k=1}^m \alpha_k^l \mu^k(F)$ for each Borel set $F \subseteq E$.

Cautious belief requires only conditions (i) and (iii). Thus, weak assumption implies cautious belief, but the converse does not hold. We show this by means of an example, which is taken from BFK (cf. Dekel and Siniscalchi 2015, Example 12.10).

Example 4 *Consider the following game with two players, Ann (a) and Bob (b):*

$a \backslash b$	L	C	R
U	$4,0$	$4,1$	$0,1$
M	$0,0$	$0,1$	$4,1$
D	$3,0$	$2,1$	$2,1$

The IA set is $S^\infty = \{U, M, D\} \times \{C, R\}$. By Theorem 4.(ii), we can append to this game a cautious*, finite type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in \{a,b\}}$ such that $S^\infty = \mathrm{Proj}_{S_a}(R_a^\infty) \times \mathrm{Proj}_{S_b}(R_b^\infty)$. Consider any type t_a of Ann such that*

(a) strategy D is optimal under $\beta_a(t_a) := (\mu_a^1, ..., \mu_a^n)$, and

(b) t_a cautiously believes Bob's (cautious) rationality.

We show that the event corresponding to Bob's rationality, viz. R_b^1, cannot be weakly assumed under $\beta_a(t_a)$. To ease notation, let $E_{s_b} := \{s_b\} \times T_b$ for each $s_b \in \{L, C, R\}$. There is no rational strategy-type pair of Bob in E_L; hence, $R_b^1 = E_C \cup E_R$. As t_a is cautious, there exists $k \leq n$ such that $\mathrm{marg}_{S_b} \mu_a^k(\{L\}) = \mu_a^k(E_L) > 0$. Let $k^* := \inf\left\{k \leq n : \mu_a^k(E_L) > 0\right\}$. Since R_b^1 is cautiously believed under $\beta_a(t_a)$, it is the case that $\mu_a^1\left(R_b^1\right) = 1$, which implies $\mu_a^1(E_L) = 0$. This in turn yields $k^* \geq 2$.

Next note that, for every $\nu \in \mathcal{M}(S_b)$ such that $\mathrm{Supp}\nu \subseteq \{C, R\}$*, strategy D is optimal under ν if and only if $\nu(\{C\}) = \nu(\{R\}) = 1/2$; this entails that also U and M are optimal under ν. Hence we must have $\mu_a^l(E_C) = \mu_a^l(E_R) = 1/2$ for every $l = 1, ..., k^* - 1$. But D must be optimal also under $\mu_a^{k^*}$, the first component measure of $\beta_a(t_a)$ which assigns strictly positive probability to E_L. It follows that $\mu^{k^*}(E_L) > 0$ and $\mu_a^{k^*}(E_R) > \mu_a^{k^*}(E_C)$.[22] Furthermore, $\mu_a^{k^*}\left(R_b^1\right) < 1$ (for, if $\mu_a^{k^*}\left(R_b^1\right) = 1$, we would have $\mu^{k^*}(E_L) = 0$). With this, we conclude that conditions (i) and (iii) of weak assumption are satisfied for event R_b^1 at level $m := k^* - 1$ of LPS $\beta_a(t_a)$.*

Yet, condition (ii)' of weak assumption does not hold. To see this, note that E_C and E_R are Borel subsets of R_b^1, and $\mu_a^{m+1}(E_R) > \mu_a^{m+1}(E_C)$. Suppose, per contra, that condition (ii)' is satisfied. Then, there exists $(\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$ such that

$$\mu_a^{m+1}(E_R) = \sum_{l=1}^m \alpha_l \mu_a^l(E_R) = \frac{1}{2}\sum_{l=1}^m \alpha_l = \sum_{l=1}^m \alpha_l \mu_a^l(E_C) = \mu_a^{m+1}(E_C),$$

which contradicts $\mu_a^{m+1}(E_R) > \mu_a^{m+1}(E_C)$. We therefore conclude that every rational strategy-type pair (D, t_a) is not consistent with weak assumption of R_b^1. ♦

[21] This is formally shown in Catonini and De Vito (2021), and the proof is essentially identical to that in Dekel et al. (2016). The preference-based definition of BFK's assumption is based on two axioms: *Strict Determination* and *Nontriviality*. Similarly, weak assumption requires Strict Determination and a weaker axiom than Nontriviality.

[22] Let $\nu \in \mathcal{M}(S_b)$ be the marginal of $\mu_a^{k^*}$ on S_b, so that $\nu(\{L\}) > 0$. We need $\nu(\{R\}) > \nu(\{C\})$ for strategy D to be optimal under ν. For, if $\nu(\{R\}) \leq \nu(\{C\})$, then U would be the unique best reply to ν.

Example 4 shows that weak assumption can be strictly stronger than cautious belief. Furthermore, the example also illustrates the difference between our approach to IA and the one based on weak assumption. To clarify, suppose that $\mathrm{R^cCB^cR^c}$ is replaced by the epistemic notion of "cautious rationality and common weak assumption of cautious rationality" ($\mathrm{R^cCA^wR^c}$). Are there analogues of Theorems 1 and 2 under $\mathrm{R^cCA^wR^c}$? Not surprisingly, the answer is Yes; see Catonini and De Vito (2021).

Yet, under weak assumption, it is not possible to provide (game-independent) epistemic foundations for IA if we restrict attention to cautious type structures. Refer back to the type structure of Example 4: in such a structure, events E_C and E_R are proper subsets of R_b^1, and condition (ii)' of weak assumption fails. This is not necessarily true *if the type structure contains both cautious and non-cautious types.* Indeed, it is possible to construct a non-cautious type structure whereby event $R_b^1 := R_b \cap C_b$ ("Bob's cautious rationality") is such that $R_b^1 \nsubseteq E_C \cup E_R$, and any cautiously rational pair (D, t_a) is consistent with weak assumption of R_b^1. For instance, consider LPS $\beta_a(t_a) := (\mu_a^1, ..., \mu_a^n)$ such that $\mu_a^l(R_b) = 1$ for each $l \leq m$, and $\mu_a^l(C_b) = 0$ for each $l > m$. In this case, condition (ii)' of weak assumption is satisfied, because $\mu_a^{m+1}, ..., \mu_a^n$ assign zero probability to Bob's cautious rationality (in particular, we have $\mu_a^{m+1}(E_R) > 0$ and $\mu_a^{m+1}\left(R_b^1\right) = 0$).

Appendix A. Preference basis

We develop preference foundations for cautiousness, certain belief and cautious belief. In so doing, we adopt the following decision-theoretic setup. A *game form* is a structure $\left\langle I, Z, (S_i)_{i\in I}, z\right\rangle$ where (a) I is the finite set of players, (b) each S_i is the finite set of strategies, and (c) $z : S \to Z$ is a surjective outcome function (hence, the set Z is finite). Each player is viewed as a Decision Maker (DM) facing a problem where his co-players' strategies are part of the description of the states, and mixed strategies are the feasible acts. So, fix an $(S_i)_{i\in I}$-based lexicographic type structure $\mathcal{T} := \langle S_i, T_i, \beta_i\rangle_{i\in I}$. We fix a player $i \in I$ (the DM), and, to ease notation, we set $\Omega := S_{-i} \times T_{-i}$. With this, the DM is uncertain about what "state" (strategy-type profiles of the co-players) in Ω will be realized, and he is endowed with a preference relation over all (Borel) measurable functions that assign to each element of Ω an objective randomization on Z. In Blume et al. (1991a), an *act* is a function from an abstract, finite domain of uncertainty Ω to $\mathcal{M}(Z)$.

A *game* is obtained by adding to the game form a profile of von Neumann-Morgenstern utility functions $(v_i)_{i\in I}$, which represent players' preferences over lotteries of consequences, according to expected utility calculations. In what follows, we assume that the codomain of any act on Ω is in utils, i.e., randomizations on material consequences are replaced by their von Neumann-Morgenstern utilities, which take value in the interval $[0, 1]$. With this, an **act** on Ω is a Borel measurable function $f : \Omega \to [0, 1]$.[23] We let ACT(Ω) denote the set of all acts on Ω.

The DM has preferences over elements of ACT(Ω). For $x \in [0, 1]$, we write $\overrightarrow{x}$ for the constant act associated with x, i.e., $\overrightarrow{x}(\omega) := x$ for all $\omega \in \Omega$. Given a Borel set $E \subseteq \Omega$ and acts $f, g \in$ACT(Ω), we define $(f_E, g_{\Omega\setminus E}) \in$ACT($\Omega$) as follows:

$$(f_E, g_{\Omega\setminus E})(\omega) := \begin{cases} f(\omega), & \text{if } \omega \in E, \\ g(\omega), & \text{if } \omega \in \Omega\setminus E. \end{cases}$$

Let $\succsim$ be a preference relation on ACT(Ω) and write $\succ$ (resp. $\sim$) for strict preference (resp. indifference). We assume that preference relation $\succsim$ satisfies the standard axioms of Order and

[23] We omit the formalism for randomizations as all preferences considered below agree on constant acts on Z, hence the utilities are uniquely defined. Moreover, the definition of act used here is also used in BFK and Dekel et al. (2016).

Independence (see BFK for a formal definition). With this, we let $\succsim_E$ denote the *conditional preference* given E, that is, $f \succsim_E g$ if and only if $(f_E, h_{\Omega\backslash E}) \succsim (g_E, h_{\Omega\backslash E})$ for some $h \in$ACT(Ω). Standard results (see Blume et al., 1991a, for a proof) show that, under the axioms of Order and Independence, $(f_E, h_{\Omega\backslash E}) \succsim (g_E, h_{\Omega\backslash E})$ holds for all $h \in$ACT(Ω) if it holds for some h.

An event $E \subseteq \Omega$ is **Savage-null** under $\succsim$ if $f \sim_E g$ for all $f, g \in$ACT(Ω). Say that E is **non-null** under $\succsim$ if it is not Savage-null under $\succsim$. With this, we can introduce the notion of certain belief in terms of the preference relation $\succsim$.

Definition A.1 *Event $E \subseteq \Omega$ is **certainly believed** under $\succsim$ if $f \sim_{\Omega\backslash E} g$ for all $f, g \in$ACT(Ω).*

Throughout, we maintain the assumption that $\overline{\mu} \in \mathcal{N}(\Omega)$ is a lexicographic expected utility representation of $\succsim$, i.e., $\succsim = \succsim^{\overline{\mu}}$.[24] With this, Savage-null events and certain belief can be characterized in terms of LPS's as follows.

Proposition A.1 *Fix $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and event $E \subseteq \Omega$. Then:*
(i) E is Savage-null under $\succsim^{\overline{\mu}}$ if and only if $\mu^m(E) = 0$ for all $m \leq n$;
(ii) E is certainly believed under $\succsim^{\overline{\mu}}$ if and only if it is certainly believed under $\overline{\mu}$.

The proof of Part (i) of Proposition A.1 is quite immediate, and it can be found in Dekel et al. (2016, Remark 2.1). Part (ii) follows from Part (i).

The following definition is due to Catonini and De Vito (2020).

Definition A.2 *Fix $\overline{\mu} \in \mathcal{N}(\Omega)$ and a set of acts* ACT$^*(\Omega) \subseteq$ACT(Ω). *Say that $\overline{\mu}$ exhibits **cautiousness** with respect to* ACT$^*(\Omega)$ *if, for all $f, g \in$*ACT$^*(\Omega)$*, the following condition holds:*
(*) if $f(\omega) \geq g(\omega)$ for each $\omega \in \Omega$ and $f(\omega') > g(\omega')$ for some $\omega' \in \Omega$, then $f \succ^{\overline{\mu}} g$.

Cautiousness is defined with respect to a set of acts that are conceivable given the potential ability of the states to influence utilities. Since the DM is a player i in a game, and the domain of uncertainty is $\Omega := S_{-i} \times T_{-i}$, we find it appropriate to consider ACT$^*(\Omega)$ as the set of acts $f \in$ACT(Ω) such that, for all $s_{-i} \in S_{-i}$, the map $f(s_{-i}, \cdot) : T_{-i} \to [0, 1]$ is constant. We let ACT$^{S_{-i}}(\Omega)$ denote this set of acts. In words, ACT$^{S_{-i}}(\Omega)$ is the set of all acts which are independent of *payoff-irrelevant components* of states $(s_{-i}, t_{-i}) \in \Omega$, i.e., the types of i's co-players. Note: every mixed strategy $\sigma_i \in \mathcal{M}(S_i)$ in a game can be identified with the (feasible) act $f_{\sigma_i} : S_{-i} \times T_{-i} \to [0, 1]$ such that $f(s_{-i}, t_{-i}) = v_i(\sigma_i, s_{-i})$ for all $(s_{-i}, t_{-i}) \in \Omega$; hence, $f_{\sigma_i} \in$ACT$^{S_{-i}}(\Omega)$.

The following result, which is proved in Catonini and De Vito (2020), provides the preference-based foundation for the type-based definition of cautiousness (Definition 9).

Proposition A.2 *Fix a type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and a type $t_i \in T_i$. Then t_i is cautious in $\mathcal{T}$ if and only if $\beta_i(t_i)$ exhibits cautiousness with respect to* ACT$^{S_{-i}}(S_{-i} \times T_{-i})$.

A **bet** (or **binary act**) on Ω is an act of the form $(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E})$, where $x, y \in [0, 1]$ and $E \subseteq \Omega$ is an event.

Definition A.3 *Fix events $E, F \subseteq \Omega$. Event E is **more likely** than F under $\succsim^{\overline{\mu}}$ if for all $x, y \in [0, 1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) \succsim^{\overline{\mu}} (\overrightarrow{x}_F, \overrightarrow{y}_{\Omega\backslash F}).$$

[24] To ease notation, we drop player i's subscript from LPS's $\overline{\mu}_i$ on Ω.

Event E is deemed ***infinitely more likely*** *than F under $\succsim^{\overline{\mu}}$, and write $E \gg^{\overline{\mu}} F$, if for all $x, y, z \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega \backslash E}) \succ^{\overline{\mu}} (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega \backslash F}).$$

In words, E is more likely than F if the DM prefers to bet on E rather than on F given the same prizes for the two bets. Event E is infinitely more likely than F if betting on E is *strictly* preferable to betting on F, and strict preference persists no matter how bigger the prize z for winning the F bet is. This notion of "infinitely more likely" is due to Lo (1999, Definition 1). Note that, if $E \gg^{\bar{\mu}} F$, then E is non-null under $\succsim^{\overline{\mu}}$, while F may, but *need not*, be Savage-null under $\succsim^{\overline{\mu}}$. When $\succsim^{\overline{\mu}}$ has a subjective expected utility representation, $E \gg^{\bar{\mu}} F$ implies that F is Savage-null.

As pointed out by Lo (1999), the likelihood relation $\gg^{\bar{\mu}}$ possesses some natural properties, such as irreflexivity, asymmetry and transitivity. Furthermore, if $E \gg^{\bar{\mu}} F$, then

(P1) E is infinitely more likely than every Borel subset of F; and

(P2) every Borel superset of E is infinitely more likely than F.

The next step is to characterize the likelihood order $\gg^{\bar{\mu}}$ between pairwise *disjoint* events in terms of LPS's representing $\succsim^{\overline{\mu}}$ (see Definition 11 of Section 4.2). Recall that, given $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and non-empty event $E \subseteq \Omega$,

$$\mathcal{I}_{\overline{\mu}}(E) := \inf \left\{ l \in \{1, ..., n\} : \mu^l(E) > 0 \right\},$$

with the convention that $\inf \emptyset := +\infty$. The proof of the following result can be found in Catonini and De Vito (2020).

Proposition A.3 *Fix $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and disjoint events $E, F \subseteq \Omega$. Then, $E \gg^{\overline{\mu}} F$ if and only if $\mathcal{I}_{\overline{\mu}}(E) < \mathcal{I}_{\overline{\mu}}(F)$.*

We now introduce the notion of cautious belief in terms of the likelihood order $\gg^{\bar{\mu}}$. Recall that $\hat{C}_{s_{-i}} \subseteq \Omega$ is called **elementary cylinder** if $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T_{-i}$ for some $s_{-i} \in S_{-i}$. Given s_{-i} and event E, we say that $E_{s_{-i}}$ is a **relevant part** of the event E if $E_{s_{-i}} := E \cap \hat{C}_{s_{-i}} \neq \emptyset$ for some $\hat{C}_{s_{-i}}$. Clearly, every non-empty event E can be written as a finite, disjoint union of all its relevant parts.

Definition A.4 *Fix $\overline{\mu} \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is* ***cautiously believed*** *under $\succsim^{\overline{\mu}}$ if it satisfies the following condition:*

(*) for every relevant part $E_{s_{-i}}$ of E, $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \backslash E$.

In words, event E is cautiously believed under $\succsim^{\overline{\mu}}$ if every relevant part of E is deemed infinitely more likely than $\Omega \backslash E$. Since E can be written as a finite, disjoint union of all its relevant parts, it follows from (P2) that E is deemed infinitely more likely than $\Omega \backslash E$, i.e., $E \gg^{\bar{\mu}} \Omega \backslash E$.

However, the converse need not hold. That is, if $E \gg^{\bar{\mu}} \Omega \backslash E$, then E is non-null under $\succsim^{\overline{\mu}}$, and there exists at least one relevant part $E_{s_{-i}}$ of E such that $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \backslash E$. But this does not rule out the existence of different relevant parts of E that *do not* satisfy this property.

Example A.1 *Refer back to Example 2 in Section 4.2. Event $F := \left\{s_b^1, s_b^2\right\} \times T_b$ is infinitely more likely under $\overline{\mu}_a$ than its complement $\left\{s_b^3\right\} \times T_b$. Yet F is not cautiously believed under*

$\succsim^{\overline{\mu}_a}$: *the relevant part* $F_{s_b^2} := \{s_b^2\} \times T_b$ *is more likely than* $\{s_b^3\} \times T_b$, *but* $F_{s_b^2}$ *is not infinitely more likely than* $\{s_b^3\} \times T_b$. ♦

We say that event $E \subseteq \Omega$ is **weakly believed** under $\succsim^{\overline{\mu}}$ if $E \gg^{\bar{\mu}} \Omega \backslash E$. Event F in Example A.1 is weakly, but not cautiously believed under $\succsim^{\overline{\mu}_a}$.

We next state and prove the characterization result for cautious belief. For the reader's convenience, we restate the LPS-based definition of cautious belief given in the main text, but in terms of relevant parts.

Definition A.5 *Fix* $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. *A non-empty event* $E \subseteq \Omega$ *is* **cautiously believed under** $\overline{\mu}$ **at level** $m \leq n$ *if:*

(i) $\mu^l(E) = 1$ *for all* $l \leq m$*;*

(ii) for every relevant part $E_{s_{-i}}$ *of* E, $\mu^l(E_{s_{-i}}) > 0$ *for some* $l \leq m$.

Event E *is* ***cautiously believed under*** $\overline{\mu}$ *if it is cautiously believed under* $\overline{\mu}$ *at some level* $m \leq n$.

Theorem A.1 *Fix* $\overline{\mu} := (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ *and a non-empty event* $E \subseteq \Omega$. *Then* E *is cautiously believed under* $\succsim^{\overline{\mu}}$ *if and only if* E *is cautiously believed under* $\overline{\mu}$.

Proof. The proof is immediate if $\Omega \backslash E$ is Savage-null under $\succsim^{\overline{\mu}}$, so, in what follows, let $\Omega \backslash E$ be non-null under $\succsim^{\overline{\mu}}$.

Suppose first that E is cautiously believed under $\succsim^{\overline{\mu}}$. Since every relevant part $E_{s_{-i}}$ of E satisfies $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \backslash E$, Proposition A.3 yields $\mathcal{I}_{\overline{\mu}}(E_{s_{-i}}) < \mathcal{I}_{\overline{\mu}}(\Omega \backslash E)$. Hence, $\mathcal{I}_{\overline{\mu}}(\Omega \backslash E) \geq 2$. Let $m := \mathcal{I}_{\overline{\mu}}(\Omega \backslash E) - 1$. Then $\mathcal{I}_{\overline{\mu}}(E_{s_{-i}}) \leq m$. Moreover, for every $k \leq m$, we have $\mu^k(\Omega \backslash E) = 0$, hence $\mu^k(E) = 1$. Therefore conditions (i)-(ii) of Definition A.5 are satisfied.

Conversely, if E is cautiously believed under $\overline{\mu}$ at level m, then condition (i) of Definition A.5 implies $\mathcal{I}_{\overline{\mu}}(\Omega \backslash E) > m$. With this, condition (ii) yields that each $E_{s_{-i}}$ satisfies $\mathcal{I}_{\overline{\mu}}(E_{s_{-i}}) < \mathcal{I}_{\overline{\mu}}(\Omega \backslash E)$. Hence, by Proposition A.3, $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \backslash E$. ■

Cautious belief in E can be given an alternative axiomatic treatment. Section D.1 of the Supplementary Appendix proposes two axioms: *Relevance* says that, conditional on every relevant part of E, the DM can have strict preferences. *Weak Dominance Determination* says that, for any pair of acts $f, g \in \mathrm{ACT}^{S_{-i}}(\Omega)$, whenever f "weakly dominates" g on E, the DM prefers f to g unconditionally. This notion of weak dominance is preference-based, as it corresponds to the notion of *P-weak dominance* of Dekel et al. (2016) for acts in $\mathrm{ACT}^{S_{-i}}(\Omega)$.[25] The result is formally stated in Theorem D.1 of the Supplementary Appendix; as we discuss, such a result also shows that cautious belief is weaker the PWD-assumption—see Definition 4.3 in Dekel et al. (2016).

We conclude this section by providing a characterization of cautious belief in terms of infinitesimal nonstandard numbers. A preference relation $\succsim$ on Ω that admits a lexicographic expected utility representation can be equivalently described by an $\mathbb{F}$-valued probability measure on Ω. Here, $\mathbb{F}$ is a non-Archimedean ordered field which is a strict extension of the set of real numbers $\mathbb{R}$ (see Blume et al. 1991a, Section 6). For instance, the LPS $\overline{\mu} := (\mu^1, \mu^2)$ can be represented by a nonstandard real valued probability $\nu := (1 - \varepsilon)\mu^1 + \varepsilon\mu^2$, where $\varepsilon > 0$ is an infinitesimal nonstandard real such that $x > n\varepsilon$ for each real number $x > 0$ and each $n \in \mathbb{N}$.

Given nonstandard reals x and y, we say that x is **infinitely greater** than y if $x > ny$ for each $n \in \mathbb{N}$. As discussed in Catonini and De Vito (2020), the notion of infinitely more likely in Definition 11 corresponds exactly to the "infinitely greater" relation between the nonstandard

[25] P-weak dominance is defined by Dekel et al. (2016) for any pair of acts belonging to the set ACT(Ω) (the set of *all* acts).

probability values that provide an equivalent representation of preferences. With this in mind, we show that cautious belief can be given an easy, nonstandard characterization.

To this end, we first recall the notion of "standard part" of a nonstandard real number. Fix a nonstandard real x such that $-r < x < r$ for some real number $r > 0$. The standard part of x, which is denoted by $\mathrm{st}(x)$, is the unique real number y such that $|y - x|$ is an infinitesimal. It is easy to check that, given positive nonstandard numbers x and y, if x is infinitely greater than y, then $\mathrm{st}\left(\frac{y}{x}\right) = 0$; the reverse implication is also true—see Halpern (2010, p. 159). Next, fix a non-empty event $E \subseteq \Omega$ and an $\mathbb{F}$-valued probability measure ν representing $\succsim$. Event E is cautiously believed under ν if, for every relevant part $E_{s_{-i}}$ of E, it is the case that $\mathrm{st}\left(\frac{\nu(\Omega\backslash E)}{\nu\left(E_{s_{-i}}\right)}\right) = 0$. Finally note that, as each $\nu\left(E_{s_{-i}}\right)$ is infinitely greater than $\nu(\Omega\backslash E)$, so is $\nu(E)$. This in turn implies $\mathrm{st}(\nu(E)) = 1$, i.e., event E is weakly believed under ν (see Halpern 2010, and Catonini and De Vito 2020).

Appendix B. Proofs for Section 4

We begin with the proof of Proposition 1.

Proof of Proposition 1. By definition, if $(s_i, t_i) \in R_i \cap C_i$, then s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$. Proposition 1 in Blume et al. (1991b) says that for every $\overline{\mu}_i \in \mathcal{N}^+(S_{-i})$ and for every lexicographic best reply s_i' to $\overline{\mu}_i$, there exists a probability measure $\nu_i \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\nu_i = S_{-i}$ and $\pi_i(s_i', \nu_i) \geq \pi_i(s_i'', \nu_i)$ for every $s_i'' \in S_i$. Thus, by Remark 1, s_i is admissible. ■

Next, we prove Proposition 2. To this end, we find it convenient to state and prove an auxiliary result, which is the analogue of Lemma B.1 in BFK.

Lemma B.1 *Fix a type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$. Fix also a type $t_i \in T_i$ with $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n)$ and a non-empty event $E \subseteq S_{-i} \times T_{-i}$. Then, E is cautiously believed under $\beta_i(t_i)$ if and only if there exists $m \leq n$ such that $\beta_i(t_i)$ satisfies condition (i) of Definition 12 plus the following condition:*

(ii") $E \subseteq \left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}$.

Proof. Suppose that E is cautiously believed under $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n)$ at level m. We show that $\beta_i(t_i)$ satisfies condition (ii"). For every $s_{-i} \in \mathrm{Proj}_{S_{-i}}(E)$, we have

$$(\{s_{-i}\} \times T_{-i}) \cap E \neq \emptyset.$$

By condition (ii) of Definition 12, there exists $k \leq m$ such that $\mu_i^k(\{s_{-i}\} \times T_{-i}) > 0$. Thus, $s_{-i} \in \mathrm{Suppmarg}_{S_{-i}}\mu_i^k$. Hence,

$$\begin{aligned} E &\subseteq \mathrm{Proj}_{S_{-i}}(E) \times T_{-i} \\ &\subseteq \left(\bigcup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}. \end{aligned}$$

Conversely, suppose that conditions (i) and (ii") hold. We show that condition (ii) of Definition 12 holds. Fix $s_{-i} \in S_{-i}$ such that $E_{s_{-i}} := (\{s_{-i}\} \times T_{-i}) \cap E \neq \emptyset$. By condition (ii"), $E_{s_{-i}} \subseteq \left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}$. Hence there exists $k \leq m$ such that $s_{-i} \in \mathrm{Suppmarg}_{S_{-i}}\mu_i^k$. Thus, $\mu_i^k(\{s_{-i}\} \times T_{-i}) > 0$. Moreover, by condition (i), $\mu_i^k(E) = 1$. Therefore $\mu_i^k\left(E_{s_{-i}}\right) > 0$, as desired. ■

Proof of Proposition 2. **Part 1**: Let $\overline{\mu}_i := \left(\mu_i^1, ..., \mu_i^n\right)$ and suppose that, for each k, E_k is cautiously believed under $\overline{\mu}_i$ at some level m_k. Let $m_K := \min\{m_k : k = 1, 2, ...\}$. We show that $E := \cap_k E_k$ is cautiously believed at level m_K. For each k, it holds that $\mu_i^l(E_k) = 1$ for all $l \leq m_K$. By the σ-additivity property of probability measures, it follows that $\mu_i^l(E) = 1$ for all $l \leq m_K$. Fix an elementary cylinder $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T_{-i}$ such that $E \cap \hat{C}_{s_{-i}} \neq \emptyset$. Let E_{m_K} be an event in $(E_k)_{k\geq 1}$ which is cautiously believed at level m_K. Obviously, $E_{m_K} \cap \hat{C}_{s_{-i}} \neq \emptyset$. Since E_{m_K} is cautiously believed, by condition (ii) of Definition 12, we have $\mu_i^l\left(E_{m_K} \cap \hat{C}_{s_{-i}}\right) > 0$ for some $l \leq m_K$. Since $\mu_i^l(E) = 1$, we obtain

$$0 < \mu_i^l\left(E_{m_K} \cap \hat{C}_{s_{-i}}\right) = \mu_i^l\left(E_{m_K} \cap \hat{C}_{s_{-i}} \cap E\right) \leq \mu_i^l\left(E \cap \hat{C}_{s_{-i}}\right).$$

Next, let $m_K := \max\{m_k : k = 1, 2, ...\}$. We show that $E = \cup_k E_k$ is cautiously believed at level m_K. Let E_{m_K} be an event in $(E_k)_{k\geq 1}$ which is cautiously believed at level m_K. For each $l \leq m_K$, we have $1 = \mu_i^l(E_{m_K}) \leq \mu_i^l(E)$. For each elementary cylinder $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T_{-i}$ with $E \cap \hat{C}_{s_{-i}} \neq \emptyset$, there is k such that $E_k \cap \hat{C}_{s_{-i}} \neq \emptyset$. By condition (ii) of Definition 12, it follows that $0 < \mu_i^l\left(E_k \cap \hat{C}_{s_{-i}}\right) \leq \mu_i^l\left(E \cap \hat{C}_{s_{-i}}\right)$ for some $l \leq m_k \leq m_K$.

Part 2: Suppose that condition (i) of Definition 12 and condition (ii') are satisfied. Then condition (ii') implies

$$\begin{aligned} E &\subseteq \mathrm{Proj}_{S_{-i}}^{-1}\left(\mathrm{Proj}_{S_{-i}}(E)\right) \\ &= \mathrm{Proj}_{S_{-i}}^{-1}\left(\left(\bigcup_{l\leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l\right)\right) \\ &= \left(\bigcup_{l\leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}, \end{aligned}$$

i.e., condition (ii") in Lemma B.1 holds. Hence E is cautiously believed under $\beta_i(t_i)$.

For the converse, suppose that E is cautiously believed under $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n)$ at level m. By Lemma B.1, it follows that

$$\begin{aligned} \mathrm{Proj}_{S_{-i}}(E) &\subseteq \mathrm{Proj}_{S_{-i}}\left(\left(\bigcup_{l\leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}\right) \\ &= \bigcup_{l\leq m} \mathrm{Suppmarg}_{S_{-i}}\mu_i^l. \end{aligned}$$

To show that this set inclusion holds with equality, let $s_{-i} \notin \mathrm{Proj}_{S_{-i}}(E)$. Then $(\{s_{-i}\} \times T_{-i}) \cap E = \emptyset$. By condition (i) of Definition 12, $\mu_i^l(E) = 1$ for each $l \leq m$, so

$$\mu_i^l(\{s_{-i}\} \times T_{-i}) = \mathrm{marg}_{S_{-i}}\mu_i^l(\{s_{-i}\}) = 0.$$

This implies $s_{-i} \notin \mathrm{Suppmarg}_{S_{-i}}\mu_i^l$. ■

Appendix C. Proofs for Section 5

In this section we first show that, for a given type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$, the sets R_i^m, $m > 1$, are Borel subsets of $S_i \times T_i$. Then we provide the proofs of Lemmas 1-3, as well as the proofs of Theorem 2 and Theorem 5.

We begin by showing that, for a given type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$, the set $\mathbf{B}_i^c(E) \subseteq S_i \times T_i$ is Borel for every event $E \subseteq S_{-i} \times T_{-i}$.

Lemma C.1 *Fix a type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ *and non-empty event* $E \subseteq S_{-i} \times T_{-i}$*. Then the set of all* $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ *under which* E *is cautiously believed is Borel in* $\mathcal{N}(S_{-i} \times T_{-i})$*.*

Proof. Recall that, for any event $E \subseteq S_{-i} \times T_{-i}$, the set of probability measures μ satisfying $\mu(E) = p$ for $p \in \mathbb{Q} \cap [0,1]$ is measurable in $\mathcal{M}(S_{-i} \times T_{-i})$. So the sets of all $\mu \in \mathcal{M}(S_{-i} \times T_{-i})$ satisfying $\mu(E) = 1$ or $\mu(E) = 0$ are Borel in $\mathcal{M}(S_{-i} \times T_{-i})$. Now, fix n and $m \leq n$. By the above argument and by definition of $\mathcal{N}_n(S_{-i} \times T_{-i})$, it turns out that the set

$$\begin{aligned} C^1_{n,m} \quad &: \;= \{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) : \forall l \leq m, \mu^m(E) = 1\} \\ &= \bigcap_{l \leq m} \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) : \mu^l(E) = 1\right\} \end{aligned}$$

is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. Note that $C^1_{n,m}$ is the set of all $\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i})$ for which condition (i) of Definition 12 holds for level m.

By the same argument, it follows that, for every $s_{-i} \in \mathrm{Proj}_{S_{-i}}(E)$, the set

$$\begin{aligned} C^{s_{-i}}_{n,m} \quad &: \;= \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) : \mu^l(\{s_{-i}\} \times T_{-i}) = 0, \forall l \leq m\right\} \\ &= \bigcap_{l \leq m} \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) : \mu^l(\{s_{-i}\} \times T_{-i}) = 0\right\} \end{aligned}$$

is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. Note that the set

$$C^2_{n,m} := \bigcap_{s_{-i} \in \mathrm{Proj}_{S_{-i}}(E)} \left(\mathcal{N}_n(S_{-i} \times T_{-i}) \backslash C^{s_{-i}}_{n,m}\right)$$

is the (measurable) set of all $\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i})$ satisfying condition (ii) of Definition 12 for level m. Define $C_{n,m} := C^1_{n,m} \cap C^2_{n,m}$; clearly, $C_{n,m}$ is a Borel subset of $\mathcal{N}_n(S_{-i} \times T_{-i})$. Hence, the set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ under which E is cautiously believed is given by $\cup_{n \in \mathbb{N}} \cup_{m \in \mathbb{N}} C_{n,m}$, so it is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. ■

By measurability of each belief map in a lexicographic type structure, we obtain the following result.

Corollary C.1 *Fix a type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$*. For every* $i \in I$*, if* $E \subseteq S_{-i} \times T_{-i}$ *is a non-empty event, then* $\mathbf{B}^c_i(E)$ *is a Borel subset of* $S_i \times T_i$*.*

We can state and prove the desired result.

Lemma C.2 *Fix a type structure* $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$*. Then, for each* $i \in I$ *and each* $m \geq 1$*,*

$$R^{m+1}_i = R^1_i \cap \left(\bigcap_{l \leq m} \mathbf{B}^c_i\left(R^l_{-i}\right)\right),$$

and R^m_i *is Borel in* $S_i \times T_i$*.*

Proof. The equality $R^{m+1}_i = R^1_i \cap \left(\cap_{l \leq m} \mathbf{B}^c_i\left(R^l_{-i}\right)\right)$ is obvious. By Corollary D.2 in Catonini and De Vito (2020), it follows that, for each $i \in I$, the set $R^1_i := R_i \cap C_i$ is Borel in $S_i \times T_i$. By Corollary C.1, the set $\mathbf{B}^c_i\left(R^1_{-i}\right)$ is Borel in $S_i \times T_i$. The conclusion follows from an easy induction on m. ■

Proof of Lemma 1. Let $M \geq 1$ be the smallest natural number such that $\prod_{i\in I} S_i^\infty = \prod_{i\in I} S_i^M$.[26] By Lemma E.1 in BFK, for every $n \in \{1, ..., M+1\}$ and $s_i \in S_i^n$, there exists $\mu_{s_i}^n \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\mu_{s_i}^n = S_{-i}^{n-1}$ and

$$\pi_i(s_i, \mu_{s_i}^n) \geq \pi_i(s_i', \mu_{s_i}^n), \forall s_i' \in S_i.$$

We use this result to construct a finite type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i\in I}$ as follows.

For each $i \in I$, let $T_i := S_i^1$, and define each belief map $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ as follows. Pick any $s_i \in T_i$. Fix also an arbitrary $\bar{s}_{-i} \in T_{-i}$, and define $\nu_{s_i}^1 \in \mathcal{M}(S_{-i} \times T_{-i})$ as

$$\nu_{s_i}^1 \left(\{(s_{-i}, \bar{s}_{-i})\}\right) := \mu_{s_i}^1 \left(\{s_{-i}\}\right), \forall s_{-i} \in S_{-i}.$$

Next, let $m := \max\left\{k \leq M+1 : s_i \in S_i^k\right\}$. (Note that if $s_i \in S_i^M$, then $m = M+1$, because $S_i^M = S_i^{M+1}$.) So, if $m = 1$, let $\beta_i(s_i) := \left(\nu_{s_i}^1\right)$. Otherwise, for each $k = 2, ..., m$, define $\nu_{s_i}^k \in \mathcal{M}(S_{-i} \times T_{-i})$ as

$$\nu_{s_i}^k \left(\{(s_{-i}, s_{-i})\}\right) := \mu_{s_i}^k \left(\{s_{-i}\}\right), \forall s_{-i} \in S_{-i}^{k-1},$$

and let

$$\beta_i(s_i) := \left(\nu_{s_i}^m, ..., \nu_{s_i}^1\right).$$

Finiteness of each type set guarantees that each belief map is Borel measurable (in fact, continuous). This completes the definition of the type structure $\mathcal{T}$.

We now show that $\mathcal{T}$ satisfies the required properties. To this end, we find it convenient to define, for each $i \in I$ and $k = 1, ..., M$, the following sets:

$$\Delta_{S_i^k \times T_i} := \left\{(s_i, s_i') \in S_i^k \times T_i : s_i = s_i'\right\}.$$

That is, each set $\Delta_{S_i^k \times T_i}$ is homeomorphic to the diagonal of $S_i^k \times S_i^k$.[27] Next note that, for every $s_i \in S_i^2$, all the component measures of $\beta_i(s_i) := \left(\nu_{s_i}^m, ..., \nu_{s_i}^1\right)$ except for $\nu_{s_i}^1$ are concentrated on those "diagonal" sets, namely

$$\mathrm{Supp}\nu_{s_i}^k = \Delta_{S_{-i}^{k-1} \times T_{-i}}, \; k = 2, ..., m,$$

which implies $\mathrm{Supp}\nu_{s_i}^k \subseteq \mathrm{Supp}\nu_{s_i}^{k-1}$ for $k \geq 3$.

The rest of the proof is by induction.

Induction Hypothesis (n): For each $i \in I$**,** $\mathrm{Proj}_{S_i}(R_i^n) = S_i^n$; moreover, $\Delta_{S_i^n \times T_i} \subseteq R_i^n$ if $n \leq M$, and $\Delta_{S_i^M \times T_i} \subseteq R_i^n$ if $n > M$.

Basis Step $(n = 1)$. Fix $i \in I$ and $s_i \in S_i^1$. Type s_i is cautious, because

$$\mathrm{Suppmarg}_{S_{-i}} \nu_{s_i}^1 = \mathrm{Supp}\mu_{s_i}^1 = S_{-i},$$

and the strategy-type pair (s_i, s_i) is rational, in that

$$\overline{\mathrm{marg}}_{S_{-i}} \beta_i(s_i) = \left(\mu_{s_i}^m, ..., \mu_{s_i}^1\right).$$

This shows that $(s_i, s_i) \in R_i^1$. Therefore $\Delta_{S_i^1 \times T_i} \subseteq R_i^1$, which implies $S_i^1 \subseteq \mathrm{Proj}_{S_i}\left(R_i^1\right)$. Conversely, Proposition 1 yields $\mathrm{Proj}_{S_i}\left(R_i^1\right) \subseteq S_i^1$.

Inductive Step $(n+1)$. For each $i \in I$, we have to show that the following properties hold:

(1) $\mathrm{Proj}_{S_i}\left(R_i^{n+1}\right) = S_i^{n+1}$;

[26] Note that, if $S^0 = S^1$, then M is 1 and not 0. This will simplify exposition.

[27] The diagonal of $S_i^k \times S_i^k$ is the set $\left\{(s_i, s_i') \in S_i^k \times S_i^k : s_i = s_i'\right\}$.

(2) $\Delta_{S_i^{n+1}\times T_i} \subseteq R_i^{n+1}$ if $n+1 \leq M$, and $\Delta_{S_i^M \times T_i} \subseteq R_i^{n+1}$ if $n+1 > M$.

Fix $i \in I$ and $s_i \in S_i^{n+1}$. Let $k := \min\{n+1, M+1\}$. Since $(s_i, s_i) \in \Delta_{S_i^{k-1}\times T_i}$, by the induction hypothesis it follows that $(s_i, s_i) \in R_i^n$. We show that $(s_i, s_i) \in \mathbf{B}_i^c\left(R_{-i}^n\right)$; this will yield $(s_i, s_i) \in R_i^{n+1}$. Write $\beta_i(s_i) := \left(\nu_{s_i}^m, ..., \nu_{s_i}^1\right)$, where $m \geq k$ because $s_i \in S_i^k$ and so, by construction, $\beta_i(s_i)$ must have length at least k. To show that R_{-i}^n is cautiously believed under $\beta_i(s_i)$, recall that $\text{Supp}\nu_{s_i}^l = \Delta_{S_{-i}^{l-1}\times T_{-i}} \subseteq \Delta_{S_{-i}^{k-1}\times T_{-i}}$ for each $l = k, ..., m$. Since $\Delta_{S_{-i}^{k-1}\times T_{-i}} \subseteq R_{-i}^n$ (induction hypothesis), it follows that condition (i) of Definition 12 is satisfied at level $l = m-k+1$. Recall also that $\text{Supp}\nu_{s_i}^k = \Delta_{S_{-i}^{k-1}\times T_{-i}}$. By the induction hypothesis, $\text{Proj}_{S_{-i}}\left(R_{-i}^n\right) = S_{-i}^{k-1} = \text{Proj}_{S_{-i}}\left(\Delta_{S_{-i}^{k-1}\times T_{-i}}\right)$. Hence, $\beta_i(s_i)$ satisfies condition (ii') of Proposition 2.2. Thus $(s_i, s_i) \in \mathbf{B}_i^c\left(R_{-i}^n\right)$, as required. So, we have shown that $S_i^{n+1} \subseteq \text{Proj}_{S_i}\left(R_i^{n+1}\right)$. For part (2), note the following fact: If $n+1 \leq M$, then for every $(s_i, s_i) \in \Delta_{S_i^{n+1}\times T_i}$ we have $s_i \in S_i^{n+1}$; analogously, if $n+1 > M$, then for every $(s_i, s_i) \in \Delta_{S_i^M \times T_i}$ we have $s_i \in S_i^{n+1}$. Therefore, by proving that $(s_i, s_i) \in R_i^{n+1}$ for each $s_i \in S_i^{n+1}$, we have proven (2).

Conversely, pick any $(s_i, s_i') \in R_i^{n+1} \subseteq R_i^n$. Then, by the induction hypothesis, $s_i \in S_i^n$. Let $\beta_i(s_i') := (\mu^1, ..., \mu^n)$. Since s_i' cautiously believes R_{-i}^n at some level l, it follows from Proposition 2.2 and the induction hypothesis that

$$\bigcup_{k\leq l} \text{Suppmarg}_{S_{-i}}\mu^k = S_{-i}^n.$$

So, by Proposition 1 in Blume et al. (1991b), there exists $\nu \in \mathcal{M}(S_{-i})$, with $\text{Supp}\nu = S_{-i}^n$, under which s_i is optimal. Therefore, by Remark 1, $s_i \in S_i^{n+1}$. This shows that $\text{Proj}_{S_i}\left(R_i^{n+1}\right) \subseteq S_i^{n+1}$, establishing (1). ■

Remark C.1 *In the proof of Lemma 1, we constructed a finite type structure $\mathcal{T}$ where each type is cautious. We can slightly modify the construction of $\mathcal{T}$ in such a way that all types are associated with full-support LPS's. The required modification, which does not alter the rest of the proof, pertains to the definition of each measure $\nu_{s_i}^1 \in \mathcal{M}(S_{-i}\times T_{-i})$. Fix an arbitrary $\mu^i \in \mathcal{M}(T_{-i})$ such that* $\text{Supp}\mu^i = T_{-i} := S_{-i}^1$*, and define $\nu_{s_i}^1$ as the product measure of $\mu_{s_i}^1 \in \mathcal{M}(S_{-i})$ and μ^i. With this, each type $s_i \in T_i$ is associated with a full-support LPS, because* $\text{Supp}\nu_{s_i}^1 = S_{-i}\times T_{-i}$.

Proof of Lemma 3. Fix a type t_i that cautiously believes E_{-i}, and set $\beta_i(t_i) := \left(\beta_i^1(t_i), ..., \beta_i^n(t_i)\right)$. Let $t_i^* := \varphi_i(t_i)$. Note that bimeasurability of $(\varphi_i)_{i\in I}$ implies that $\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$ is an event in $S_{-i}\times T_{-i}^*$. We show that t_i^* cautiously believes $\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$, that is, $\beta_i^*(t_i^*) = \widehat{\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)}(\beta_i(t_i))$ satisfies conditions (i) and (ii) of Definition 12.

First, note that

$$E_{-i} \subseteq \left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right).$$

Hence, by definition of type morphism, it follows that, for all $l \leq n$,

$$\begin{aligned}\beta_i^l(t_i)(E_{-i}) &\leq \beta_i^l(t_i)\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right)\\ &= \beta_i^{*,l}(t_i^*)\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right).\end{aligned}$$

Since E_{-i} is cautiously believed under $\beta_i(t_i)$, it follows from condition (i) of Definition 12 that there exists $m \leq n$ such that $\beta_i^l(t_i)(E_{-i}) = 1$ for all $l \leq m$. Therefore, we have that $\beta_i^{*,l}(t_i^*)\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right) = 1$ for all $l \leq m$. Hence $\beta_i^*(t_i^*)$ satisfies condition (i) of Definition 12.

Consider now an elementary cylinder $\hat{C}_{s_{-i}} := \{s_{-i}\} \times T^*_{-i}$ satisfying $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i}) \cap \hat{C}_{s_{-i}} \neq \emptyset$. First, note that

$$\begin{aligned}
(\{s_{-i}\} \times T_{-i}) \cap E_{-i} &\subseteq (\{s_{-i}\} \times T_{-i}) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\
&= \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\hat{C}_{s_{-i}}\right)\right) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\
&= \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\hat{C}_{s_{-i}} \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right).
\end{aligned}$$

Hence, by definition of type morphism, it follows that, for all $l \leq n$,

$$\begin{aligned}
\beta^l_i(t_i)\left((\{s_{-i}\} \times T_{-i}) \cap E_{-i}\right) &\leq \beta^l_i(t_i)\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\hat{C}_{s_{-i}} \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\
&= \beta^{*,l}_i(t^*_i)\left(\hat{C}_{s_{-i}} \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right).
\end{aligned}$$

Since E_{-i} is cautiously believed under $\beta_i(t_i)$ at level $m \leq n$, and since $\hat{C}_{s_{-i}} \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i}) \neq \emptyset$ implies $(\{s_{-i}\} \times T_{-i}) \cap E_{-i} \neq \emptyset$, by condition (ii) of Definition 12 there exists $k \leq m$ such that $\beta^k_i(t_i)\left((\{s_{-i}\} \times T_{-i}) \cap E_{-i}\right) > 0$. Therefore, we obtain

$$\beta^{*,k}_i(t^*_i)\left(\hat{C}_{s_{-i}} \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right) > 0.$$

Thus, $\beta^*_i(t^*_i)$ satisfies condition (ii) of Definition 12. ■

Proof of Theorem 2. Part (i): Fix a type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$. If $\prod_{i \in I} \mathrm{Proj}_{S_i}(R^\infty_i) = \emptyset$, then the result is immediate. So in what follows we will assume that this set is non-empty. For each $i \in I$ and $s_i \in \mathrm{Proj}_{S_i}(R^\infty_i)$, there exists $t_i \in T_i$ such that $(s_i, t_i) \in R^\infty_i$. Since $(s_i, t_i) \in R^1_i$, it follows that s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$. Therefore, by Proposition 1, s_i is admissible, hence condition (a) of Definition 3 is satisfied.

Next note that, for each $k \geq 1$, type t_i cautiously believes R^k_{-i}. So, it follows from Proposition 2.1 that R^∞_{-i} is cautiously believed under $\beta_i(t_i) := (\mu^1, ..., \mu^n)$ at some level m. Moreover, Proposition 2.2 entails $\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}}\mu^l = \mathrm{Proj}_{S_{-i}}\left(R^\infty_{-i}\right)$. Since s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$, Proposition 1 in Blume et al. (1991b) yields the existence of some $\nu \in \mathcal{M}(S_{-i})$ under which s_i is optimal and such that $\mathrm{Supp}\nu = \mathrm{Proj}_{S_{-i}}\left(R^\infty_{-i}\right)$. Remark 1 entails that s_i is admissible with respect to $S_i \times \mathrm{Proj}_{S_{-i}}\left(R^\infty_{-i}\right)$, establishing condition (b) of Definition 3.

Finally, by Corollary A1 in Brandenburger and Friedenberg (2010), every s'_i that supports s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$ as well. It follows that $(s'_i, t_i) \in R^\infty_i$, and this in turn implies that $s'_i \in \mathrm{Proj}_{S_i}(R^\infty_i)$, establishing condition (c) of Definition 3.

Part (ii): Let $Q \in \mathcal{Q}$ be a non-empty SAS. Fix $i \in I$ and $s_i \in Q_i$. By conditions (a) and (b) of Definition 3, and by Remark 1, there exist $\nu^2_{s_i}, \nu^1_{s_i} \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\nu^2_{s_i} = S_{-i}$ and $\mathrm{Supp}\nu^1_{s_i} = Q_{-i}$, and such that s_i is optimal under $\nu^2_{s_i}$ and $\nu^1_{s_i}$. Hence, s_i is a lexicographic best reply to $(\nu^1_{s_i}, \nu^2_{s_i}) \in \mathcal{N}^+(S_{-i})$. Moreover, as in BFK (p. 328), we can choose $\nu^2_{s_i}$ and $\nu^1_{s_i}$ in such a way that every strategy s'_i is optimal under $\nu^2_{s_i}$ and $\nu^1_{s_i}$ if and only if s'_i is supported by s_i. Now we construct a finite type structure $\mathcal{T} := \langle S_i, T_i, \beta_i \rangle_{i \in I}$ as follows.

For each $i \in I$, let $T_i := Q_i$. For every $s_i \in T_i$, define $\mu^1_{s_i}, \mu^2_{s_i} \in \mathcal{M}(S_{-i} \times T_{-i})$ as

$$\begin{aligned}
\mu^1_{s_i}\left(\{(s_{-i}, s_{-i})\}\right) &: = \nu^1_{s_i}(\{s_{-i}\}), \forall s_{-i} \in Q_{-i}, \\
\mu^2_{s_i}\left(\{(s_{-i}, \bar{s}_{-i})\}\right) &: = \nu^2_{s_i}(\{s_{-i}\}), \forall s_{-i} \in S_{-i},
\end{aligned}$$

where $\bar{s}_{-i} \in T_{-i}$ is arbitrarily chosen. Let $\beta_i(s_i) := (\mu^1_{s_i}, \mu^2_{s_i})$. Finiteness of each type set guarantees that each belief map is measurable (in fact, continuous). This completes the definition of the type structure $\mathcal{T}$.

We now show that $\mathcal{T}$ satisfies the required properties. Note that each type $s_i \in T_i$ is cautious because $\text{Supp}\nu^2_{s_i} = S_{-i}$; hence $\mathcal{T}$ is a cautious type structure. For every $i \in I$ and $s_i \in Q_i$, strategy-type pair (s_i, s_i) is cautiously rational by construction; for every $s'_i \notin Q_i$, condition (c) of Definition 3 implies that s'_i does not support s_i, so by construction the pair (s'_i, s_i) is not rational. Hence, $\text{Proj}_{S_i}\left(R^1_i\right) = Q_i$. Now, suppose by way of induction that $(s_i, s_i) \in R^m_i$ for each $i \in I$ and each $s_i \in Q_i$. We show that type s_i cautiously believes R^m_{-i}, establishing that $(s_i, s_i) \in R^{m+1}_i$; this will yield $(s_i, s_i) \in R^\infty_i$. Note that $\text{Supp}\mu^1_{s_i} = \{(s_{-i}, s_{-i}) : s_{-i} \in Q_{-i}\} \subseteq R^m_{-i}$, where the inclusion follows from the induction hypothesis. Moreover, since $\text{Suppmarg}_{S_{-i}}\mu^1_{s_i} = \text{Supp}\nu^1_{s_i} = Q_{-i}$, Proposition 2.2 entails that R^m_{-i} is cautiously believed under $\beta_i(s_i)$ at level 1. Therefore, we conclude that $\text{Proj}_{S_i}(R^\infty_i) = Q_i$. ■

Remark C.2 *In the proof of Part (ii) of Theorem 2, we constructed a finite type structure $\mathcal{T}$ where each type is cautious. This is so because each measure $\nu^2_{s_i}$, which is the marginal of $\mu^2_{s_i}$ on S_{-i}, satisfies* $\text{Supp}\nu^2_{s_i} = S_{-i}$. *Note that we can also construct $\mu^2_{s_i}$ in such a way that* $\text{Supp}\mu^2_{s_i} = S_{-i} \times T_{-i}$ *(see Remark C.1). With this, we obtain a type structure $\mathcal{T}$ where all types are associated with full-support LPS's.*

For the proof of Theorem 5, we need an auxiliary technical fact.

Lemma C.3 *Fix two sequences of pairwise disjoint topological spaces $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$. Let $X := \cup_{n\in\mathbb{N}}X_n$ and $Y := \cup_{n\in\mathbb{N}}Y_n$. Suppose that, for each $n \in \mathbb{N}$, there is a map $f_n : X_n \to Y_n$. If each map f_n is continuous (resp. surjective), then the union map $\cup_{n\in\mathbb{N}}f_n : X \to Y$ is continuous (resp. surjective).*

Proof. Let O be open in Y. By definition of direct sum topology, the set O can be written as $O = \cup_{n\in\mathbb{N}}O_n$, where each $O_n := O \cap Y_n$ is open in Y_n (see Engelking 1989, p. 74). Thus

$$\left(\bigcup_{n\in\mathbb{N}} f_n\right)^{-1}(O) = \bigcup_{n\in\mathbb{N}} f_n^{-1}(O_n).$$

So, if each f_n is continuous, then each $f_n^{-1}(O_n)$ is open, and this in turn implies that $(\cup_{n\in\mathbb{N}}f_n)^{-1}(O)$ is open. The conclusion that $\cup_{n\in\mathbb{N}}f_n$ is surjective if each f_n is surjective is immediate by inspection of the definitions. ■

Proof of Theorem 5. The desired type structure $\mathcal{T} := \langle S_i, T_i, \beta_i\rangle_{i\in I}$ is constructed as follows. For each $i \in I$, let T_i be the Baire space $\mathbb{N}_0^{\mathbb{N}_0}$,[28] so that each $t_i \in T_i$ is an infinite sequence of non-negative integers. The set $\mathbb{N}_0$ is endowed with the discrete topology, and $\mathbb{N}_0^{\mathbb{N}_0}$ is endowed with the product topology. The basic open sets of $\mathbb{N}_0^{\mathbb{N}_0}$ are sets of the form

$$O_k := \left\{(n_1, n_2, ...) \in \mathbb{N}_0^{\mathbb{N}_0} : (n_1, ..., n_k) = (o_1, ..., o_k)\right\}$$

for each $k \in \mathbb{N}_0$ and $(o_1, ..., o_k) \in (\mathbb{N}_0)^k$. With this topology, a basic open set is also closed, so sets of the form O_k constitute a clopen basis. The space $\mathbb{N}_0^{\mathbb{N}_0}$ is Polish and uncountable, but not compact.

For each $i \in I$, we partition T_i into a countable family of non-empty Borel subsets. For each $k \geq 0$, let

$$\mathrm{T}^k_i := \left\{(n_1, n_2, ...) \in \mathbb{N}_0^{\mathbb{N}_0} : n_1 = k\right\}.$$

[28] Here $\mathbb{N}_0$ denotes the set $\{0, 1, 2, ...\}$, i.e., $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Baire space is sometimes defined as the set $\mathbb{N}^{\mathbb{N}}$ of all infinite sequences of natural numbers. This difference is immaterial for all the relevant topological properties we are going to use in this proof.

Each T_i^k is a subbasic clopen subset of T_i; moreover, each T_i^k is homeomorphic to the Baire space. It is clear that $T_i = \cup_{k\geq 0}\mathrm{T}_i^k$, and all the T_i^k's are pairwise disjoint.

The next step is to construct the belief maps in such a way that, for all $k \geq 0$, $t_i \in \mathrm{T}_i^k$ and $s_i \in S_i$, the pair (s_i, t_i) *does not* belong to R_i^{k+1}. For each $i \in I$, we construct a countable partition of $\mathcal{N}(S_{-i} \times T_{-i})$ that "mirrors" the above partition of T_i. This is done as follows: For each $i \in I$, let

$$\Lambda_i^0 := \mathcal{N}(S_{-i} \times T_{-i}) \backslash \mathcal{C}_i^0,$$

where $\mathcal{C}_i^0$ is the set of all LPS's $\overline{\mu}_i \in \mathcal{N}(S_{-i} \times T_{-i})$ such that $\overline{\mathrm{marg}}_{S_{-i}}\overline{\mu}_i \in \mathcal{N}^+(S_{-i})$. Since $|S_i| \geq 2$ for each $i \in I$, it follows that $\Lambda_i^0 \neq \emptyset$.

Next, let

$$\Lambda_i^1 := \left\{\overline{\mu}_i \in \mathcal{C}_i^0 : \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^0) > 0\right\},$$

and, for each $k \geq 2$,

$$\Lambda_i^k := \bigcap_{m\in\{1,\ldots,k-1\}} \left\{\overline{\mu}_i \in \mathcal{C}_i^0 : \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{m-1}) = 0\right\} \cap \left\{\overline{\mu}_i \in \mathcal{C}_i^0 : \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0\right\}.$$

In words: Λ_i^1 is the set of all LPS's on $S_{-i} \times T_{-i}$ such that the marginal on S_{-i} has full support and the first component measure assigns strictly positive probability to $S_{-i} \times \mathrm{T}_{-i}^0$; Λ_i^2 is the set of all LPS's on $S_{-i} \times T_{-i}$ such that the marginal on S_{-i} has full support and the first component measure assigns probability 0 to $S_{-i} \times \mathrm{T}_{-i}^0$, and strictly positive probability to $S_{-i} \times \mathrm{T}_{-i}^1$; and so on.

It is immediate to check that $\mathcal{N}(S_{-i} \times T_{-i}) = \cup_{k\geq 0}\Lambda_i^k$ and all the Λ_i^k's are non-empty, pairwise disjoint sets; so the countable family of all Λ_i^k's is a partition of $\mathcal{N}(S_{-i} \times T_{-i})$.

Claim C.1 *For each $k \geq 0$, Λ_i^k is a Borel subset of $\mathcal{N}(S_{-i} \times T_{-i})$.*

Proof. Since $\mathcal{C}_i^0$ is Borel (see Lemma D.2 in Catonini and De Vito, 2020), so is Λ_i^0. For each $k \geq 1$, let

$$\mathrm{P}_i^k := \left\{\overline{\mu}_i \in \mathcal{C}_i^0 : \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0\right\}.$$

Note that $\Lambda_i^1 = \mathrm{P}_i^1$, and for each $k \geq 2$, Λ_i^k is the intersection of P_i^k with the complements of $\mathrm{P}_i^1, ..., \mathrm{P}_i^{k-1}$. Thus, in order to show that each Λ_i^k is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$, it is sufficient to show that each P_i^k is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. Let

$$\mathrm{M}_i^k := \left\{\mu \in \mathcal{M}(S_{-i} \times \mathrm{T}_{-i}) : \mu(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0\right\}.$$

By Theorem 17.24 in Kechris (1995), if X is a Polish space, then the Borel σ-field on $\mathcal{M}(X)$ is generated by sets of the form $\{\mu \in \mathcal{M}(X) : \mu(E) \geq p\}$, where $E \in \Sigma_X$ and $p \in \mathbb{Q} \cap [0,1]$. Hence, for every $E \in \Sigma_X$, the set $\{\mu \in \mathcal{M}(X) : \mu(E) > 0\}$ is Borel, since it can be written as $\cap_{n\in\mathbb{N}}\left\{\mu \in \mathcal{M}(X) : \mu(E) \geq \frac{1}{n}\right\}$. This implies that M_i^k is Borel in $\mathcal{M}(S_{-i} \times \mathrm{T}_{-i})$. Moreover, for each $n \in \mathbb{N}$, the canonical projection map

$$\begin{array}{rccc} \mathrm{Proj}_{1,n}: & \mathcal{N}_n(S_{-i} \times T_{-i}) & \to & \mathcal{M}(S_{-i} \times \mathrm{T}_{-i}), \\ & (\mu_i^1, ..., \mu_i^n) & \mapsto & \mu_i^1, \end{array}$$

is continuous, hence the set $\mathrm{Proj}_{1,n}^{-1}\left(\mathrm{M}_i^k\right)$ is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. With this, the conclusion follows from the observation that P_i^k can be written as

$$\mathrm{P}_i^k = \left(\bigcup_{n\in\mathbb{N}} \mathrm{Proj}_{1,n}^{-1}\left(\mathrm{M}_i^k\right)\right) \cap \mathcal{C}_i^0.$$

□

Recall that every Borel subset of a Polish space is a Lusin space, when endowed with the relative topology. Moreover, every Lusin space is also analytic (see Cohn 2003, Proposition 8.6.13). Thus, by Claim C.1, each Λ_i^k is analytic. Since each T_i^k is homeomorphic to the Baire space, it follows from Corollary 8.2.8 in Cohn (2003; see also Kechris 1995, p. 85) that, for every $k \geq 0$, there exists a surjective continuous map $\beta_i^{[k]} : \mathrm{T}_i^k \to \Lambda_i^k$. For each $i \in I$, let β_i be the union of the $\beta_i^{[k]}$'s, i.e., $\beta_i := \cup_{k \geq 0} \beta_i^{[k]} : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$. The map is well defined because the T_i^k's are pairwise disjoint. By Lemma C.3, β_i is a continuous (and so Borel) surjective map. This completes the definition of the type structure $\mathcal{T}$.

We now show that $\mathcal{T}$ satisfies the required properties.

Claim C.2 *For each $i \in I$ and for each $k \geq 0$,*

$$(S_i \times \mathrm{T}_i^k) \cap R_i^{k+1} = \emptyset.$$

Proof. By induction on $k \geq 0$.

(Basis step: $k = 0$) Fix $i \in I$ and $(s_i, t_i) \in S_i \times T_i$ with $t_i \in \mathrm{T}_i^0$. We clearly have $(s_i, t_i) \notin R_i^1$ because $\beta_i(t_i) \in \Lambda_i^0$, hence t_i is not cautious. Therefore $(S_i \times \mathrm{T}_i^0) \cap R_i^1 = \emptyset$.

(Inductive step: $k \geq 1$) Suppose we have already shown that $(S_i \times \mathrm{T}_i^{k-1}) \cap R_i^k = \emptyset$ for each $i \in I$. Fix $i \in I$ and $(s_i, t_i) \in S_i \times T_i$ with $t_i \in \mathrm{T}_i^k$. Thus $\beta_i(t_i) := (\mu_i^1, ..., \mu_i^n) \in \Lambda_i^k$, hence $\mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0$. Since, by the induction hypothesis, $(S_{-i} \times \mathrm{T}_{-i}^{k-1}) \cap R_{-i}^k = \emptyset$, it must be the case that $\mu_i^1(R_{-i}^k) < 1$. Therefore R_{-i}^k is not cautiously believed under $\beta_i(t_i)$; this implies $(s_i, t_i) \notin \mathbf{B}_i^c(R_{-i}^k)$. Hence $(s_i, t_i) \notin R_i^{k+1}$. □

To conclude the proof, pick any $(s_i, t_i) \in S_i \times T_i$. Then there exists $k \geq 0$ such that $t_i \in \mathrm{T}_i^k$. By Claim C.2, it follows that $(s_i, t_i) \notin R_i^{k+1}$. Since $R_i^\infty := \cap_{m \geq 1} R_i^m$, this shows that $R_i^\infty = \emptyset$, as required. ■

Remark C.3 *The crucial step of the proof of Theorem 5 is the inductive step in Claim C.2. Specifically, it is shown that event R_{-i}^k is not cautiously believed under $\beta_i(t_i)$ because $\mu_i^1(R_{-i}^k) < 1$. Actually, this shows that event R_{-i}^k is not weakly believed under $\beta_i(t_i)$. As discussed in Section 6.1, it is possible to redefine each set $\prod_{i \in I} R_i^{m+1}$ as the set of states consistent with cautious rationality and mth-order weak belief in cautious rationality. With this new definition, event $\prod_{i \in I} R_i^\infty$ is the set of states at which there is cautious rationality and common weak belief of cautious rationality ($R^c CWBR^c$). The proof of Theorem 5 can be read now, verbatim, as a proof of the impossibility of $R^c CWBR^c$ for a some belief-complete type structure. Notice that the set of states consistent with $R^c CB^c R^c$ is a subset of the set of states consistent with $R^c CWBR^c$. Hence the result of Theorem 5 follows.*

References

[1] Aliprantis, C.D., and K.C. Border (1999): *Infinite Dimensional Analysis.* Berlin: Springer Verlag.

[2] Asheim, G., and M. Dufwenberg (2003): "Admissibility and Common Belief," *Games and Economic Behavior,* 42, 208-234.

[3] Barelli, P., and S. Galanis (2013): "Admissibility and Event-Rationality," *Games and Economic Behavior*, 77, 21–40.

[4] Battigalli, P. (1997): "On Rationalizability in Extensive Games," *Journal of Economic Theory*, 74, 40-61.

[5] Battigalli, P., and A. Friedenberg (2012a): "Forward Induction Reasoning Revisited," *Theoretical Economics*, 7, 57-98.

[6] Battigalli, P., and A. Friedenberg (2012b): "Context-Dependent Forward Induction Reasoning," IGIER Working Paper 351, Bocconi University.

[7] Battigalli, P., A. Friedenberg and M. Siniscalchi (2021): *Strategic Uncertainty: An Epistemic Approach to Game Theory.* (Working Title).

[8] Battigalli, P., and M. Siniscalchi (2002): "Strong Belief and Forward Induction Reasoning," *Journal of Economic Theory*, 106, 356-391.

[9] Blume, L., A. Brandenburger, and E. Dekel (1991a): "Lexicographic Probabilities and Choice Under Uncertainty," *Econometrica,* 59, 61-79.

[10] Blume, L., A. Brandenburger, and E. Dekel (1991b): "Lexicographic Probabilities and Equilibrium Refinements," *Econometrica,* 59, 81-98.

[11] Brandenburger, A. (1992): "Lexicographic Probabilities and Iterated Admissibility," in *Economic Analysis of Markets and Games*, ed. by P. Dasgupta, D. Gale, O. Hart, and E. Maskin. Cambridge MA: MIT Press, 282-290.

[12] Brandenburger, A. (2003): "On the Existence of a 'Complete' Possibility Structure," in *Cognitive Processes and Economic Behavior*, ed. by M. Basili, N. Dimitri and I. Gilboa. New York: Routledge, 30-34.

[13] Brandenburger, A., and E. Dekel (1992): "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 59, 189-198.

[14] Brandenburger, A., and A. Friedenberg (2010): "Self-Admissible Sets," *Journal of Economic Theory*, 145, 785-811.

[15] Brandenburger, A., A. Friedenberg, and H.J. Keisler (2008): "Admissibility in Games," *Econometrica*, 76, 307-352.

[16] Brandenburger, A., A. Friedenberg, and H.J. Keisler (2012): "Fixed Points in Epistemic Game Theory," in *Proceedings of the 2008 Clifford Lectures, AMS Proceedings of Symposia in Applied Mathematics.*

[17] Catonini, E. (2013): "Common Assumption of Cautious Rationality and Iterated Admissibility," PhD thesis, Bocconi University.

[18] Catonini, E., and N. De Vito (2017): "A Comment on 'Admissibility and Assumption'," working paper.

[19] Catonini, E., and N. De Vito (2018): "Hierarchies of Lexicographic Beliefs," working paper.

[20] Catonini, E., and N. De Vito (2020): "Weak Belief and Permissibility," *Games and Economic Behavior*, 120, 154-179.

[21] Catonini, E., and N. De Vito (2021): "Infinitely More Likely under Lexicographic Beliefs," working paper.

[22] Cohn, D.L. (2013): *Measure Theory.* Boston: Birkhauser.

[23] Dekel, E., and M. Siniscalchi (2015): "Epistemic Game Theory," in *Handbook of Game Theory with Economic Applications, Volume 4*, ed. by P. Young and S. Zamir. Amsterdam: North-Holland, 619-702.

[24] Dekel, E., A. Friedenberg, and M. Siniscalchi (2016): "Lexicographic Beliefs and Assumption," *Journal of Economic Theory*, 163, 955-985.

[25] Engelking, R. (1989): *General Topology.* Berlin: Heldermann.

[26] Friedenberg, A. (2010): "When Do Type Structures Contain All Hierarchies of Beliefs?," *Games and Economic Behavior*, 68, 108-129.

[27] Friedenberg, A., and H.J. Keisler (2020): "Iterated Dominance Revisited," *Economic Theory*, https://doi.org/10.1007/s00199-020-01275-z.

[28] Halpern, J.Y. (2010): "Lexicographic Probability, Conditional Probability, and Nonstandard Probability," *Games and Economic Behavior*, 68, 155-179.

[29] Halpern, J.Y., and R. Pass (2019): "A Conceptually Well-Founded Characterization of Iterated Admissibility Using an "All I Know" Operator," in *Theoretical Aspects of Rationality and Knowledge, Proceedings Seventeenth Conference*, ed. by L.S. Moss, 221-232.

[30] Harsanyi, J. (1967-68): "Games of Incomplete Information Played by Bayesian Players. Parts I, II, III," *Management Science*, 14, 159-182, 320-334, 486-502.

[31] Heifetz, A., and D. Samet (1998): "Topology-Free Typology of Beliefs," *Journal of Economic Theory*, 82, 324-341.

[32] Heifetz, A., M. Meier and B. Schipper (2019): "Comprehensive Rationalizability," *Games and Economic Behavior*, 116, 185-202.

[33] Kechris, A. (1995): *Classical Descriptive Set Theory.* Berlin: Springer Verlag.

[34] Keisler, H.J., and B.S. Lee (2015): "Common Assumption of Rationality," working paper, University of Toronto.

[35] Lee, B.S. (2013): "Conditional Beliefs and Higher-Order Preferences," working paper, University of Toronto.

[36] Lee, B.S. (2016a): "Admissibility and Assumption," *Journal of Economic Theory*, 163, 42-72.

[37] Lee, B.S. (2016b): "Generalizing Type Spaces," working paper, University of Toronto.

[38] Lo, K.C. (1999): "Nash Equilibrium without Mutual Knowledge of Rationality," *Economic Theory*, 14, 621-633.

[39] Mertens, J.F., and S. Zamir (1985): "Formulation of Bayesian Analysis for Games With Incomplete Information," *International Journal of Game Theory*, 14, 1-29.

[40] Moulin, H. (1984): "Dominance Solvable Voting Schemes," *Econometrica*, 47, 1337-51.

[41] Pearce, D. (1984): "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica,* 52, 1029-1050.

[42] Perea, A. (2012): *Epistemic Game Theory: Reasoning and Choice*, CUP Press.

[43] Samuelson, L. (1992): "Dominated Strategies and Common Knowledge," *Games and Economic Behavior*, 4, 284-313.

[44] Shimoji, M. (2004): "On the Equivalence of Weak Dominance and Sequential Best Response," *Games and Economic Behavior*, 48, 385-402.

[45] Srivastava, S.M. (1998): *A Course on Borel Sets*. New York: Springer-Verlag.

[46] Stahl, D.O. (1995): "Lexicographic Rationalizability and Iterated Admissibility," *Economic Letters*, 47, 155-159.

[47] Yang, C. (2015): "Weak Assumption and Iterative Admissibility," *Journal of Economic Theory*, 158, 87-101.

[48] Ziegler, G., and P. Zuazo-Garin (2020): "Strategic Cautiousness as an Expression of Robustness to Ambiguity," *Games and Economic Behavior*, 119, 197-215.