Improving Matching under Information Constraint: Chinese College Admission Reconsidered*

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February 15, 2022

Abstract

College admissions in China have a distinct feature that the assignment is determined by three factors: test scores, students' preferences, and colleges' preferences. To examine this situation, we extend a standard many-to-one matching model by assuming that colleges have two different types of ordinal rankings over students, i.e., a common score order and individual preferences. Since it takes time and cost for colleges to review each student's application material, colleges have an information constraint; they can only partially figure out their true preferences over students. To alleviate the information problem, the current admission system, called the Chinese parallel (CP) mechanism, implements a so-called "dummy quota policy", allowing each college to receive more applications than its actual quota and to choose the preferred students among the applicants. While the CP mechanism is a variant of the serial dictatorship mechanism, we show that it has various drawbacks. Depending on whether the dummy quota policy is maintained or abandoned, we propose two different modified mechanisms that outperform the CP mechanism.

Keywords: college admission, information constraint, dummy quota, parallel mechanism.

*We would like to thank Shingo Ishiguro, Fuhito Kojima, Shigehiro Serizawa, and seminar participants at CTWZ (Contract Theory Workshop on Zoom) for their helpful comments. All remaining errors are ours. Yasuda gratefully acknowledges research support from Grant-in-Aid for Scientific Research (KAKENHI), 18KK0342.

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1 Introduction

College admissions have been widely discussed in the matching literature. There are two closely related but different approaches to study the matching between students and colleges. The first one is the so-called **college admission model** (Gale and Shapley [6]), where colleges are agents and they have preferences over students. The second one is the **student placement model** (Balinski and Sönmez [2]), where colleges are objects to be consumed and the assignment is determined by students' test scores and their preferences over colleges.

College admissions in China were a typical example of the student placement model. Colleges could not express their preferences and they had no freedom to select students. In 1979, an article on college autonomy published in People's Daily arouse extensive attention. In this famous article, four presidents of universities in Shanghai appealed to the government that colleges should have some degree of autonomy over admissions.[1] Their appeal is understandable, because test scores cannot fully evaluate a student's ability and other admission factors, such as letters of recommendation and extracurricular activities, should be also taken into consideration.

To alleviate the misalignment problem that colleges' preferences do not necessarily align with test scores, the government decided to change its policy in 1980. After several attempted reforms, dummy quota policy was introduced in 1984 and effectively implemented in a new assignment mechanism, which is called "the Chinese sequential (CS) mechanism". The procedure of the CS mechanism with dummy quota policy is described as follows.

1. Dummy quota policy: Each college can receive "student profiles" up to 120% of its real quota during the admission. A student profile is created by her high school and required to include the following materials: basic information, political and moral quality identifications, school performance, graduation test scores, medical test, comprehensive quality evaluation, etc.

2. Chinese sequential (CS) mechanism with dummy quota policy:

   Step 1: Each college considers those students who have ranked it as their 1st choice and

[1] Buqing Su, the president of Fudan university, said that: "If Fudan university can make its own admission decision, then we do not need to follow the unified criterion to admit students [...] However, this is not achievable because currently colleges have no autonomy." Funian Liu, the president of Shanghai normal university, said that: "The current problem is the low level of college autonomy [...] Besides standardized test scores, we hope that colleges are allowed to freely admit their preferred students" (Xiao [16]).

receives student profiles up to its dummy quota, one at a time, following the test score order. Then, each college admits its received students up to its real quota, one at a time, following its preferences over students.

Step t: Each college considers those students who have rejected in Step $t-1$ and ranked it as their t-th choice and receives student profiles up to its remaining dummy quota, one at a time, following the test score order. Then, each college admits its received students up to its remaining real quota, one at a time, following its preferences over students. Note that if a college has space to receive student profiles but has no vacant position, then this college will not be allowed to receive any additional student profile.

The descriptions are based on Articles 6 and 7 of University Recruitment Regulation in 1984. Although test scores still played a central role in the admission, some degree of freedom to select students was allowed. Colleges were able to freely select students among those in their dummy quotas and thus they could partially express their preferences under the CS mechanism. Note that if dummy quotas are infinite, then the CS mechanism is equivalent to the immediate acceptance (IA) mechanism because test scores are irrelevant to determine the applications to each college and the admitted students are immediately confirmed in each step.[2]

The reformed admission system had been supported by colleges and applied for nearly twenty years. Due to university expansion in the late 1990s, college admissions became a widely debated topic in China. Then, several provinces (Hainan, Hunan, and Yunnan) were allowed to administer their own standardized tests and assignment mechanisms (Liu [9]). Since the CS mechanism is very similar to the IA mechanism, it had been heavily criticized by the public for the problem of students' manipulations. In 2003, a new mechanism, which is known as "the Chinese parallel (CP) mechanism", was applied in Hunan province. As we will see later in detail, a good test score is much more important than a good strategy in the ranking of colleges, hence the CP mechanism reduces the incentive for students to manipulate. Consequently, it became popularized in other provinces than Hunan. The procedure of the CP mechanism, which is also based on dummy quota policy, is described as follows.

3. Chinese parallel (CP) mechanism with dummy quota policy:

[2] In this paper, we will mention three existing mechanisms in the literature: the immediate acceptance (IA), the serial dictatorship (SD), and the deferred acceptance (DA) mechanisms. We provide the formal descriptions of these well-known mechanisms in the appendix.

Step 1: Based on students' test scores and their preferences, the clearinghouse sends student profiles to colleges by using the serial dictatorship (SD) mechanism with respect to the given dummy quotas.

Step 2: Each college admits its received students up to its real quota, one at a time, following its preferences over students.

The description is based on Articles 34 and 38 of Hunan's University Recruitment Recruitment Regulation in 2005. Note that if dummy quotas coincide with real quotas, then the CP mechanism is equivalent to the SD mechanism because no student will be rejected in Step 2 above. That is, the CP mechanism finishes in Step 1, which is essentially the same as the SD mechanism, and colleges' preferences become irrelevant to determine the assignment.

Unfortunately, the CP mechanism has a serious limitation: a student profile can be sent to *at most* one college (Li [8]; Zeng [19]). Education examination authority in Hebei province explained this problem as follows: "once a student is rejected by a college, [her student profile will not be sent to any other colleges and as a result] she has to reluctantly enter either the next tier college admission or an supplementary process[3]" (Zhang et al. [18]). Since a student profile can be sent to multiple colleges under the CS mechanism, from the viewpoint of students, being rejected under the CP mechanism is much more painful than that under the CS mechanism.

Colleges are aware of this problem and become hesitant to reject students. Many colleges reduced their dummy quotas (from 120% to 105%-110%) to avoid the trouble with those rejected students (Wu and Gan [15]). At the end, the government also reduced the upper bound of dummy quota in 2015. The new dummy quota policy is described as follows.

4. Dummy quota policy: Each college can receive student profiles up to 105% of its real quota under the CP mechanism, whereas it is still 120% under the CS mechanism.[4]

The description is based on Article 40 of University Recruitment Regulation in 2015. Al-

[3]Colleges in China are categorized into three tiers: tier 1 (national universities), tier 2 (provincial universities), and tier 3 (private universities). Since students usually prefer tier 1 to tier 2 to tier 3, they would like to stay in the current tier rather than fall into the next tier. In the supplementary process, only the students who are unmatched under the CP mechanism are involved. They are asked to submit their new preferences and then the assignment is determined by using the SD mechanism. Since only those positions which are vacant at the end of the CP mechanism are available, students would like to stay in the main round rather than rush into the supplementary process.

[4]Currently, the CS mechanism is mainly applied for tier 0 admission, in which special universities, such as military university and art university, are involved.

though colleges still can partially express their preferences in the admission, the freedom to select students is largely restricted. In several provinces, dummy quota policy is completely abandoned and the CP mechanism works just as the SD mechanism.[5] In these provinces, colleges cannot express their preferences, which is very similar to the situation in 1979. As noted above, the CP mechanism has the following two serious problems.

- Limitation of "one student profile for only one college": Since a student profile can be sent to at most one college, students may suffer a great loss from colleges' rejections.
- Absence of "college autonomy over admissions": Colleges may not be able to express their preferences and tend to admit students based only on test scores.

In this paper, we consider how to solve or alleviate the above problems. To do this, we create an extended matching model in which both test scores and colleges' preferences coexist in the market. In our model, colleges play a hybrid role between agents and objects and the assignment will be determined by three factors: test scores, students' preferences, and colleges' preferences.

In contrast to the traditional matching models, our model applies to a real-life setting that there is a misalignment between test scores and colleges' preferences. For this reason, we consider two stability criteria: **score stability** and **preference stability**. The former requires that students with higher test scores are matched with their better colleges and the latter requires that a student-college pair who prefers each other should be matched together. A matching is said to be **double stable** if it satisfies both score stability and preference stability simultaneously. In addition, respecting stability and manipulability, we consider two notions that allow us to compare different mechanisms. Namely, we say that mechanism A is **more stable** than mechanism B if, (i) at any problem, B produces a stable matching, then A will also produce a stable matching, but (ii) the converse is not always true. Similarly, mechanism A is **more manipulable** than mechanism B if, (i) at any problem, B is manipulable by a student, then A will also be manipulable by a student, but (ii) the converse is not always true. When

[5] Specifically, there are five provinces that dummy quotas are equal to real quotas. See Article 20 of Guangdong's University Recruitment Regulation, Article 29 of Jiangxi's University Recruitment Regulation, Article 4(2) of Shanghai's University Recruitment Regulation, Article 42 of Shanxi's University Recruitment Regulation, and Article 30 of Zhejiang's University Recruitment Regulation in 2020.

neither A is more stable (resp. manipulable) than B nor B is more stable (resp. manipulable) than A, we say that mechanisms A and B are stability (resp. manipulability) **incomparable**.

By using these notions, we show that the CP mechanism is neither score stable, preference stable, nor strategy-proof (Theorem 1) and may not necessarily produce a double stable matching even if such a matching exists. Moreover, the CP mechanism suffers from "the double blocking pair problem": the resulting matching can be both score blocked and preference blocked by the same student-college pair.

In 2015, the Chinese government has demonstrated its commitment to reform the current admission system and encouraged each province to explore an appropriate method of "one student profile for many colleges": a student profile can be sent to multiple colleges (see Article 2 of University Recruitment Recommendation in 2015). Although six years have past, there are a few provinces (Fujian and Zhejiang) use the deferred acceptance (DA) mechanism in their technical school admissions as pilot projects.[6] The method for general college admissions, however, is still an open question to researchers.

The purpose of this paper is to modify the CP mechanism such that the goal of "one student profile for many colleges"can be appropriately achieved. Depending on whether the dummy quota policy is maintained or abandoned, two different scenarios will be discussed in our paper. In the first scenario, we propose the modified parallel with dummy quota policy (MPD) mechanism. Since colleges can only receive a limited number of student profiles, it also suffers from the double blocking pair problem. Despite this drawback, we show that the MPD mechanism is more score stable, more preference stable, and less manipulable than the CP mechanism (Theorem 2). Comparing with the SD (or the DA) mechanism, the MPD mechanism has its own advantage, in the sense that those well-known mechanisms cannot simultaneously improve the CP mechanism in all three directions (Remark 3).

In the second scenario, the dummy quota policy is abandoned (i.e., the upper bound for each college to receive student profiles is removed). We propose the modified parallel with free dummy quota policy (MPF) mechanism. Although the MPF and the CP mechanisms are incomparable

[6]Technical school admissions in China have a feature that test scores are much less important than schools' preferences. If a student want to enter a regular technical school, then since supply is greater than demand, she only has to obtain 33% (250 points) of total test scores (750 points). If a student want to enter a special technical school (e.g. flight attendant training school and nursing school), then the school usually makes its decision based not on her test score, but on other admission factors (e.g. aviation medical exam and moral quality identifications). For these reasons, there is essentially no misalignment problem and we may use the DA mechanism in the technical school admissions.

in both stability and manipulability, the MPF mechanism fixes several drawbacks that the CP mechanism has. Specifically, we show that, the MPF mechanism does not suffer from the double blocking pair problem and produces a double stable matching whenever it exists (Theorem 3). Comparing with the SD (or the DA) mechanism, the MPF mechanism is "balanced", in the sense that it is less score stable but more preference stable than the SD mechanism, and less preference stable but more score stable than the DA mechanism (Remark 4). In light of these results, among the two mechanisms that we propose to alleviate the misalignment problem, we recommend the MPD mechanism if dummy quota policy needs to be maintained, and the MPF mechanism if dummy quota policy can be abandoned.

The remainder of this paper is organized as follows. In the rest of this section, we review the related literature. Section 2 presents the model and preliminary results. Section 3 describes the CP mechanism and explain why we should modify it. Section 4 introduces the MPD and MPF mechanisms and presents the main results. Section 5 concludes. The Appendix contains the formal descriptions of the IA, the SD, and the DA mechanisms and the proofs of the results in Section 4.

1.1 Related literature

The literature on college admissions in China so far has exclusively focused on the transition from the sequential mechanism to the parallel mechanism. Nie [11], Wei [14], and Zhu [19] study a student placement model in which each college has a common test score order over students. They all describe the parallel mechanism as the SD mechanism. Since the SD mechanism is score stable and strategy-proof, they provide a rationale for the use of the parallel mechanism.

Chen and Kesten [3] study a **school choice model** (Abdulkadiroğlu and Sönmez [1]) in which each college has a (independent) priority order over students. They describe the sequential mechanism as the IA mechanism and the parallel mechanism as a variant of the DA mechanism. The latter is expressed as follows:

5. Chinese parallel mechanism described in Chen and Kesten [3]:
   Students are allowed to list at most e colleges within each choice band. For example, if the preferences of a student are given by c_1, c_2, c_3, c_4 and the length of choice band is $e = 2$, then the first choice band contains c_1 (top choice) and c_2 (second choice), and the second

choice band contains c_3 (third choice) and c_4 (fourth choice). The parallel mechanism works in rounds. In each round i, based on students' priorities and their e choices in the i-th choice band, the clearinghouse uses the DA mechanism to determine the assignment. Note that the allocation is finalized for each e choices.

In their framework, the parallel mechanism is equivalent to the IA mechanism if $e = 1$ and the DA mechanism if $e = \infty$. They show that, if the length of choice band e becomes longer, then the parallel mechanism becomes more stable and less manipulable. Given this result, the use of the parallel mechanism is justified and they recommend the government to increase the length of choice band.

Note that the CP mechanism analyzed in our paper is different from the ones described in the existing literature. As previously mentioned, Nie [11], Wei [14], and Zhu [19] describe the parallel mechanism as the SD mechanism. Since we take dummy quota policy into consideration, the CP mechanism, due to this policy, is similar but not identical to the SD mechanism. Chen and Kesten [3] describe the parallel mechanism as a variant of the DA mechanism and thus a student can be rejected multiple times (by priorities). In the current paper, since both test scores and colleges' preferences are relevant, a college may reject a student based on either "test scores"(i.e., her test score is not sufficient) or "colleges' preferences"(i.e., the college prefers some other students). Under the CP mechanism, a student can be rejected multiple times in the former case whereas only once in the latter case, which shows a sharp contrast to Chen and Kesten [3].

Lastly, what is common to all those papers mentioned above is that they consider a standard matching model in which each agent on one side of the market has just one ordinal ranking over the agents on the other side. In the current paper, by contrast, we examine an extended matching model in which each college has two ordinal rankings over the students. The reason why we employ such a model is that we incorporate a real-life complication that colleges are allowed, at least partially, to express their preferences via dummy quota policy. Since colleges' preferences do not necessarily align with test scores, a serious misalignment problem may arise. Nevertheless, this issue has not been properly addressed in the existing literature. In this paper, we aim to fill this gap.[7]

[7]Recently, Chen et al. [4] and Miyazaki and Okamoto [10] study an extended matching model where each agent on one side of the market has multiple ordinal rankings over the agents on the other side. These papers

2 The model

In this paper, we examine an extended college admission model in which colleges have two different types of ordinal rankings over students: a common test score order and individual preferences. We call this model a **Chinese college admissions problem**, which consists of the following.[8]

1. a set of students, $S = \{s_1, \cdots, s_n\}$,
2. a set of colleges, $C = \{c_1, \cdots, c_m\}$,
3. a capacity vector, $q = (q_{c_1}, \cdots, q_{c_m})$,
4. a list of strict student preferences, $P_S = (P_{s_1}, \cdots, P_{s_n})$,
5. a list of strict college preferences, $P_C = (P_{c_1}, \cdots, P_{c_m})$,
6. a strict test score order, $\succ$.

A Chinese college admissions problem or simply a **problem** is a tuple $G = (S, C, q, P_S, P_C, \succ)$, where each element is described above. Since S, C, and q will be fixed, we also denote a problem by $G = (P_S, P_C, \succ)$.

For each student $s \in S$, P_s is a strict preference relation over $C \cup \{\emptyset\}$, where $\emptyset$ is the option of being unmatched. Let R_s be the weak preference relation associated with P_s: for any $c, c' \in C \cup \{\emptyset\}$, cR_sc' if and only if cP_sc' or $c = c'$.

For each college $c \in C$, P_c is a strict preference relation over the set 2^S of all subset of S.[9] Let R_c be the weak preference relation associated with P_c: for any $S', S'' \subseteq S$, $S'R_cS''$ if and only if $S'P_cS''$ or $S' = S''$. We say that preference relation P_c is **responsive** (Roth [13]) if (i)

analyze the hardness of computing stable matchings by discussing a wide range of ranking structures, e.g., how many choices can be listed in each ranking or how many rankings that agents can have. By contrast, the current paper focuses on a specific ranking structure that fits the actual college admission problems in China and studies market design issues; we formalize the current assignment mechanism used in China and propose better-performed alternatives. In Fang and Yasuda [5], based on the same ranking structure, we derive some useful properties of the double stable matching, and also provide its characterization through the existing well-known mechanisms, which complements the current paper.

[8] In Fang and Yasuda [5], we name this problem as a **college admission problem with common priorities**.

[9] Although the colleges' preferences over the students are given in our model, it would be unrealistic to assume that the colleges know their preferences from the beginning, since there are millions of students in the actual market of college admissions in China. Therefore, each college needs to receive students' profiles in order to figure out its preferences. This informational requirement calls for a dummy quotas policy. See also footnote 13.

for any $S' \subseteq S$ with $|S'| \leq q_c$, $s \in S \setminus S'$ and $s' \in S'$, $(S' \cup s) \setminus s' R_c S'$ if and only if $s R_c s'$, (ii) for any $S' \subseteq S$ with $|S'| \leq q_c$ and $s' \in S'$, $S' R_c S' \setminus s'$ if and only if $\emptyset R_c s'$, and (iii) for any $S' \subseteq S$ with $|S'| > q_c$, $\emptyset P_c S'$. Throughout paper, we assume that colleges' preferences are responsive.

The strict test score order $\succ$ is derived from the result of standardized test where $s \succ s'$ means that s has a higher test score than s'. Let $\succeq$ be the weak relation associated with $\succ$: for any $s, s' \in S \cup \{\emptyset\}$, $s \succeq s'$ if and only if $s \succ s'$ or $s = s'$. We assume that the null student has the lowest test score, i.e., for each student $s \in S$, $s \succ \emptyset$.

Now we introduce some notations that are useful for analyzing matching problems. A **matching** μ is a function from $S \cup C$ to subsets of $S \cup C$ such that (i) for any student $s \in S$, $\mu(s) \in C \cup \{\emptyset\}$, (ii) for any college $c \in C$, $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$, and (iii) if $\mu(s) = c$, then $s \in \mu(c)$. A matching μ is **individually rational** for students if for each student $s \in S$, $\mu(s) R_s \emptyset$, and for colleges if for each college $c \in C$, (i) $|\mu(c)| \leq q_c$ and (ii) for each student $s \in \mu(c)$, $s R_c \emptyset$. A matching μ is **score blocked** by a student-college pair (s, c) if (i) $c P_s \mu(s)$ and (ii) either (a) $|\mu(c)| < q_c$ and $s \succ \emptyset$, or (b) for some student $s' \in \mu(c)$, $s \succ s'$; such a pair is called a score blocking pair. A matching μ is **preference blocked** by a student-college pair (s, c) if (i) $c P_s \mu(s)$ and (ii) either (a) $|\mu(c)| < q_c$ and $s P_c \emptyset$, or (b) for some student $s' \in \mu(c)$, $s P_c s'$, such a pair is called a score blocking pair.

Using the blocking conditions mentioned above, we define three essential stability notions.

Definition 1 *A matching μ is* **score stable** *if it is individually rational for students, and there is no score blocking pair. A matching μ is* **preference stable** *if it is individually rational for both students and colleges, and there is no preference blocking pair. A matching μ is called* ***double stable*** *if it is score stable and preference stable.*

In other words, a matching μ is double stable if it satisfies the following three properties: (i) individually rational for students and colleges, (ii) not score blocked by any student-college pair, and (iii) not preference blocked by any student-college pair. Similarly, given some matching μ', a student-college pair (s, c) is called a **double blocking pair** if μ' is both priority blocked and preference blocked by this *same* pair (s, c).

An assignment mechanism or simply a **mechanism** is a systematic procedure that selects a matching for each problem. Let $\phi(P_S, P_C, \succ)$ denote the matching selected by mechanism ϕ for problem $(P_S, P_C, \succ)$. We denote the assignment of student s by $\phi_s(P_S, P_C, \succ)$ and the assignment of college c by $\phi_c(P_S, P_C, \succ)$.

Recall that, in our model, colleges have two different types of ordinal rankings over students. If only one type of rankings can affect assignments, i.e., either test scores or colleges' preferences are completely irrelevant, then one would think such a mechanism is extreme. By contrast, if both types are relevant to decide assignments, then such a mechanism would be moderate. We formally define those concepts as follows.

Definition 2 *A mechanism ϕ is* **extreme** *if for any problem $(P_S, P_C, \succ)$, either (i) $\phi(P_S, P'_C, \succ) = \phi(P_S, P_C, \succ)$ for any P'_C or (ii) $\phi(P_S, P_C, \succ') = \phi(P_S, P_C, \succ)$ for any $\succ'$, in which P'_C is an arbitrary list of college preferences and $\succ'$ is an arbitrary test score order. A mechanism ϕ is* **moderate** *if it is not extreme.*

Remark 1 *Many well-known mechanisms are extreme. For example, the SD mechanism is extreme because test scores are irrelevant. Similarly, the DA mechanism is also extreme because colleges' preferences are irrelevant. By contrast, two mechanisms described in the introduction, the CS and the CP mechanisms, are both moderate.*

A mechanism ϕ is called score (resp. preference) stable at problem $(P_S, P_C, \succ)$ if it produces a score (resp. preference) stable matching at this problem. A mechanism ϕ is score (resp. preference) stable if it always produces a score (resp. preference) stable matching. To compare the degree of stability between two unstable mechanisms, we use the concept of "more stable"introduced by Chen and Kesten [3].

Definition 3 *A mechanism ϕ is* **more score (resp. preference) stable** *than another mechanism ϕ', if (i) at any problem, ϕ' is score (resp. preference) stable, then ϕ is also score (resp. preference) stable, and (ii) at some problem, ϕ is score (resp. preference) stable but ϕ' is not.*

A mechanism ϕ is **strategy-proof** for students if there exist no problem $(P_S, P_C, \succ)$, student s, and preferences Q_s such that $\phi_s(Q_s, P_{-s}, P_C, \succ) P_s \phi_s(P_S, P_C, \succ)$. A mechanism ϕ is manipulable by student s at problem $(P_S, P_C, \succ)$ if there exists Q_s such that $\phi_s(Q_s, P_{-s}, P_C, \succ) P_s \phi_s(P_S, P_C, \succ)$. A mechanism ϕ is manipulable at problem $(P_S, P_C, \succ)$ if there exists some student s such that ϕ is manipulable by her at this problem. To compare the degree of manipulability between two manipulable mechanisms, we use the concept of "more manipulable"introduced by Pathak and Sönmez [12].

Definition 4 *A mechanism ϕ is* **more manipulable** *than another mechanism ϕ', if (i) at any problem, ϕ' is manipulable, then ϕ is also manipulable, and (ii) at some problem, ϕ is manipulable but ϕ' is not.*

We say that two mechanisms, ϕ and ϕ', are **incomparable** with respect to score stability (resp. preference stability, manipulability), if neither ϕ is more score stable (resp. preference stable, manipulable) than ϕ' nor ϕ' is more score stable (resp. preference stable, manipulable) than ϕ.

2.1 Preliminary results on double stable matching

Although double stability is an attractive and natural new stability notion, its existence is not always guaranteed. We use the following example to illustrate this point.[10]

Example 1 *Let $S = \{s_1, s_2\}$, $C = \{c_1, c_2\}$, and $q = (1, 1)$. Students' preferences, colleges' preferences, and test score order are given as follows.*

$$\begin{array}{llllll} P_{s_1}: & c_1, c_2, & P_{c_1}: & s_1, s_2, & \succ: s_2, s_1, \\ P_{s_2}: & c_1, c_2, & P_{c_2}: & s_1, s_2. & \end{array}$$

The unique score stable matching is $\mu := \binom{s_1\ s_2}{c_2\ c_1}$, which is preference blocked by the pair (s_1, c_1). Thus, there is no double stable matching at this problem.

Given this non-existence result, one may want to find a mechanism that produces a double stable matching whenever it exists. Fortunately, there are at least two well-known mechanisms provided to fulfill this purpose. The first one is the SD mechanism.

Proposition 1 —(Fang and Yasuda [5], Proposition 2): The SD mechanism produces a unique score stable matching, hence it always eliminates any double blocking pair. Moreover, the SD mechanism produces a double stable matching whenever it exists.

While the SD mechanism may look attractive, note that the SD mechanism is extreme and it cannot take any colleges' preferences into account. However, the uniqueness of the score stable matching (Proposition 1) implies that, if a mechanism respects colleges' preferences even

[10] The content of this subsection is based on Fang and Yasuda [5]. See the paper for the proofs of the related results. We do not reproduce them here.

a little, it has to give up score stability. That is, incorporating colleges' preferences is *never* compatible with score stability, which is formally expressed as follows.

Proposition 2 —(Fang and Yasuda [5], Remark 2): The SD mechanism is a unique score stable mechanism. Thus, there exists no mechanism which is moderate and score stable.

The second mechanism that fulfills our purpose, i.e., achieving double stability whenever possible, is the student-proposing DA mechanism. The DA mechanisms, other popular extreme mechanisms than the SD mechanism, completely ignore score stability but always achieve preference stability.

Proposition 3 —(Fang and Yasuda [5], Proposition 3): The student-proposing DA mechanism produces a preference stable matching, hence it always eliminates any double blocking pair. Moreover, the student-proposing DA mechanism produces a double stable matching whenever it exists.

Based on Propositions 2 and 3, we obtain the following characterization of the double stable matching. It is widely known that there always exists a so-called student optimal stable matching (SOSM), which Pareto dominates all other stable outcomes; the SOSM is derived by the student-proposing DA mechanism (Gale and Shapley [6]).

Proposition 4 —(Fang and Yasuda [5], Theorem 1): A double stable matching exists if and only if a unique score stable matching coincides with a student optimal stable matching; that matching becomes a unique double stable matching.

Proposition 4 implies that a double stable matching exists if and only if the resulting outcome of the SD mechanism using the common score order coincides with that of the student-proposing DA based only on the agents' preferences. Since both mechanisms are computationally simple, one can easily figure out if a double stable matching exists or not.[11]

Remark 2 *The college-proposing DA mechanism may not produce a double stable matching even if it exists. We can use the following example to verify this point. Let* $S = \{s_1, s_2, s_3\}$*,* $C = \{c_1, c_2, c_3\}$*, and* $q = (1, 1, 1)$*. Students' preferences, colleges' preferences, and test score order are given as follows.*

[11] Borrowing a concept from computer science, the time complexity of the SD mechanism is linear ($O(n)$) and that of the DA mechanism is quadratic ($O(n^2)$), where n is a number of inputs (i.e., students in our case).

$$
\begin{aligned}
&P_{s_1}\colon \quad c_1, c_2, c_3, \quad P_{c_1}\colon \quad s_1, s_2, s_3, \quad \succ\colon s_1, s_2, s_3,\\
&P_{s_2}\colon \quad c_1, c_2, c_3, \quad P_{c_2}\colon \quad s_1, s_3, s_2,\\
&P_{s_3}\colon \quad c_1, c_3, c_2, \quad P_{c_2}\colon \quad s_1, s_2, s_3.
\end{aligned}
$$

The unique double stable matching is $\mu^* := \binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$*, which can be calculated by either the SD or the student-proposing DA mechanism. By contrast, the college-proposing DA mechanism produces a score unstable matching* $\check{\mu} := \binom{s_1\ s_2\ s_3}{c_1\ c_3\ c_2}$*, which is Pareto dominated by* μ^**.*

Although the college-proposing DA mechanism performs differently than the other two mechanisms, all three mechanisms are extreme. The original intention of these extreme mechanisms is to respect either test score order or colleges' preferences, but *not* both. As mentioned in Section 1, the Chinese government wants to take both test scores and colleges' preferences into account. For this reason, we restrict our attention to those moderate mechanisms. In the next section, we will introduce a moderate mechanism which is currently used in Chinese college admissions.

3 Chinese parallel mechanism

In this section, we formally describe the Chinese parallel (CP) mechanism and theoretically show that there is room for improvement in this mechanism. We first introduce the dummy quota policy, which is a constraint for colleges to receive student profiles. For each college c, its dummy quota is denoted by $\bar{d}_c = q_c + \alpha_c$, where α_c is a non-negative integer. Given a problem $G = (P_S, P_C, \succ)$ and a dummy quota vector $\bar{d} = (\bar{d}_{c_1}, \cdots, \bar{d}_{c_m})$, we describe the CP mechanism as follows.

The CP mechanism

Step 1: Based on students' test scores and their preferences, the clearinghouse sends student profiles to colleges by using the SD mechanism with respect to the given dummy quotas.

(1) The student with the highest test score, $s[1]$, is considered. Her profile is received by her top choice. Let $c(s[1])$ be this college and the current received number, $d_{c(s[1])}$, is increased by one.

(k) The student with the k-th highest test score, $s[k]$, is considered. Her profile is received by her top choice among those in the set $\{c \in C : d_c < \bar{d}_c\}$. Let $c(s[k])$ be this college and the current received number, $d_{c(s[k])}$, is increased by one.

Step 2: Colleges make constrained admission decisions. Let $F_c(G)$ be the set of students whose profiles are received by college c at problem G. Each college c admits the students in the set $F_c(G)$, up to its quota q_c, one at a time, following the preferences P_c.

The CP mechanism is a two-step mechanism. In Step 1, test scores play a "cut-off" role and the set of students S is uniquely partitioned into $m+1$ small sets: $F_{c_1}(G), \cdots, F_{c_m}(G), F_{\emptyset}(G)$. In Step 2, each college c can freely select students from the set $F_c(G)$, hence colleges' preferences are partially expressed. We provide an example to illustrate how this mechanism works.

Example 2 *Let $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2\}$, $q = (1, 1)$, and $\bar{d} = (2, 3)$. Students' preferences, colleges' preferences, and test score order are given as follows.*

$$
\begin{array}{llllll}
P_{s_1}: & c_1, c_2, & P_{c_1}: & s_1, s_2, s_3, s_4, & \succ: s_1, s_2, s_3, s_4. \\
P_{s_2}: & c_1, c_2, & P_{c_2}: & s_1, s_2, s_3, s_4. & \\
P_{s_3}: & c_1, c_2, & & & \\
P_{s_4}: & c_1, c_2. & & &
\end{array}
$$

The CP mechanism works as follows. In Step 1, the profiles of s_1 and s_2 are received by their top choice c_1. Since $\bar{d}_{c_1} = 2$, the profiles of s_3 and s_4 are not received by their top choice c_1 but by their second choice c_2. In Step 2, c_1 selects s_1 from $F_{c_1}(G) = \{s_1, s_2\}$ and c_2 selects s_3 from $F_{c_2}(G) = \{s_3, s_4\}$. The mechanism is terminated and the outcome is $CP(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ \emptyset\ c_2\ \emptyset}$.

The above example illustrates that the CP mechanism cannot produce a "desired" matching, even if there is no misalignment problem between test score order and colleges' preferences. First, the CP mechanism is not stable and has a strategic problem. The matching $CP(G)$ in Example 2 is both score blocked and preference blocked by the same pair (s_2, c_2). Moreover, if s_2 reports $Q_{s_2} : c_2, c_1$ while other students report their true preferences, then she becomes better off since s_2 is now assigned to college c_2, which is better than being unmatched. These observations can be summarized and formally stated as the following theorem.

Theorem 1 *The CP mechanism is neither score stable, preference stable, nor strategy-proof for students. Moreover, the CP mechanism also fails to eliminate a double-blocking pair.*

Besides these drawbacks, the CP mechanism fails to produce a double stable matching even if it exists. Note that, in Example 2, $\mu^* := \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ \emptyset}$ is the unique double stable matching. However, the CP mechanism produces the matching $CP(G)$ which is different from μ^*. In general, when colleges' preferences all align with test scores, there always exists a unique double stable matching and it can be obtained by using either the SD or the DA mechanism.[12]

The CP mechanism may also waste dummy quotas in the following sense. In Example 2, when the mechanism is terminated, c_2 has received only two applications while it still has space to receive an additional student profile (note $\bar{d}_{c_2} = 3$). Nevertheless, the profile of s_2 cannot be received by her second choice college c_2, even though s_2 has a higher test score than s_3 (who is currently assigned to c_2). This potential loss is due to the undesirable feature of the CP mechanism that each student profile can be sent to at most one college, which we try to resolve by proposing alternative mechanisms in the next section.

4 Modified parallel mechanisms

Our analysis in the previous section highlighted various drawbacks of the CP mechanism. This section focuses on how to solve or alleviate those drawbacks. Depending on whether the dummy quota policy is maintained or abandoned, we consider two scenarios. For each scenario, we will propose a modified parallel mechanism to improve the allocation outcome of the CP mechanism.

4.1 Scenario 1: The dummy quota policy is maintained

The first scenario is that the dummy quota policy is maintained. In this scenario, each college c, as the same as before, can receive at most $\bar{d}_c$ student profiles throughout the mechanism. Given a problem $G = (P_S, P_C, \succ)$ and a dummy quota vector $\bar{d} = (\bar{d}_{c_1}, \cdots, \bar{d}_{c_m})$, we propose the modified parallel with dummy quota policy (MPD) mechanism, which is described as follows.

The MPD mechanism

Step 1: The CP mechanism (with dummy quota policy $\bar{d}$) is used. The outcome is *tentative* and denoted by $MPD^1(G)$.

[12] If all colleges have identical preferences, then there exists a unique preference stable matching μ^*. Therefore, a matching outcome derived by the student-proposing DA and the college-proposing DA necessarily coincide, i.e., both DA mechanisms produce μ^*. Since the test score order is equal to the colleges' common preferences, the unique preference stable matching μ^* is also priority stable, hence it must be a double stable matching.

Step t, $t \geq 2$: Let $U^t = \{s \in S : MPD_s^{t-1}(G) = \emptyset\}$ be the set of all unmatched students under $MPD^{t-1}(G)$. The clearinghouse sends student profiles to colleges as follows:

(1) The student with the highest test score in the set U^t, $s[t,1]$, is considered. Her profile is received by her top choice among those in the set $A^t_{s[t,1]}$, in which $A^t_{s[t,1]}$ satisfies the following conditions:

- The set does not contain any college that the profile of $s[t,1]$ was received before.
- For each $c \in A^t_{s[t,1]}$, we have (i) $cP_{s[t,1]}\emptyset$, (ii) $d_c < \bar{d}_c$, and (iii) either (a) $|MPD_c^{t-1}(G)| < q_c$, $s[t,1] \succ \emptyset$, or (b) for some $s' \in MPD_c^{t-1}(G)$, $s[t,1] \succ s'$.

Let $c(s[t,1])$ be this college and the current received number, $d_{c(s[t,1])}$, is increased by one.

(k) The student with the k-th highest test score in the set U^t, $s[t,k]$, is considered. Her profile is received by her top choice among those in the set $A^t_{s[t,k]}$. Let $c(s[t,k])$ be this college and the current received number, $d_{c(s[t,k])}$, is increased by one.

Let $F^t_c(G)$ be the set of students whose profiles are received in Step t by college c at problem G. Each college c admits the students in the set $MPD_c^{t-1}(G) \cup F_c^t(G)$, up to its quota q_c, one at a time, following the preferences P_c. The outcome is tentative and denoted by $MPD^t(G)$.

End: The mechanism is terminated when $U^{t'} = \emptyset$ or for each $s \in U^{t'}$, $A_s^{t'} = \emptyset$. Then, $MPD^{t'}(G)$ becomes the final assignment of the mechanism.

The MPD mechanism is a multiple-step mechanism. In Step 1, it *tentatively* produces the same assignment as the CP mechanism, i.e., $MPD^1(G) = CP(G)$. In Step $t \geq 2$, following the test score order, for each (currently) unmatched student $s \in U^t$, we send her profile to her top choice among those in the set A_s^t. As defined in the above description, a college can be included in the set A_s^t, if it (i) hasn't received the profile of s before, (ii) is better than s's unmatched option, (iii) currently can receive an additional profile, and (iv) either has a vacant position or is matched with some student who has a lower score than s. After receiving additional student profiles, colleges admit best students among the group consisting of the additional applicants together with those students who have been tentatively accepted from the previous step. We provide an example to illustrate how the MPD mechanism works.

Example 3 *Let* $S = \{s_1, s_2, s_3, s_4, s_5\}$*,* $C = \{c_1, c_2, c_3, c_4\}$*,* $q = (1, 1, 2, 1)$*, and* $\bar{d} = (3, 2, 3, 2)$*. Students' preferences, colleges' preferences, and test score order are given as follows.*

$$\begin{array}{llllll}
P_{s_1}: & c_1, c_2, c_3, c_4, & P_{c_1}: & s_1, s_2, s_3, s_4, s_5, & \succ: & s_1, s_2, s_3, s_4, s_5, \\
P_{s_2}: & c_1, c_3, c_2, c_4, & P_{c_2}: & s_1, s_2, s_3, s_4, s_5, & & \\
P_{s_3}: & c_1, c_3, c_2, c_4, & P_{c_3}: & s_1, s_2, s_3, s_4, s_5, & & \\
P_{s_4}: & c_3, c_2, c_1, c_4, & P_{c_4}: & s_1, s_2, s_3, s_4, s_5, & & \\
P_{s_5}: & c_3, c_4, c_2, c_1. & & & &
\end{array}$$

The MPD mechanism works as follows. In Step 1*, the CP mechanism produces* $MPD^1(G) = CP(G) = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ \emptyset\ \emptyset\ c_3\ c_3}$*. In Step* 2*,* $U^2 = \{s_2, s_3\}$*. The profile of* s_2 *is received by her second choice* $c_3 \in A^2_{s_2} = \{c_3, c_2, c_4\}$*. Since* $\bar{d}_{c_3} = 3$*, the profile of* s_3 *is not received by her second choice* c_3 *but by her third choice* $c_2 \in A^2_{s_3} = \{c_2, c_4\}$*. Then,* c_2 *selects* s_3 *from* $MPD^1_{c_2}(G) \cup F^2_{c_2}(G) = \{s_3\}$ *and* c_3 *selects* s_2 *and* s_4 *from* $MPD^1_{c_3}(G) \cup F^2_{c_3}(G) = \{s_2, s_4, s_5\}$*. Thus,* $MPD^2(G) = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_3\ c_2\ c_3\ \emptyset}$*. In Step* 3*,* $U^3 = \{s_5\}$*. The profile of* s_5 *is received by her second choice* $c_4 \in A^3_{s_5} = \{c_4, c_2\}$*. Then,* c_4 *selects* s_5 *from* $MPD^2_{c_4}(G) \cup F^3_{c_4}(G) = \{s_5\}$*. The mechanism is terminated and the outcome is* $MPD(G) = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_3\ c_2\ c_3\ c_4}$*.*

In the above example, the MPD mechanism also fails to produce a double stable matching even if it exists. The reason is that colleges can only receive a limited number of student profiles under the dummy quota policy. When s_3 is considered in Step 2, c_3 has reached its dummy quota. Even if s_3 is more preferred and has a higher test score, her profile cannot be sent to c_3. As a result, the MPD mechanism produces the matching $MPD(G)$, which is different from the double stable matching $\mu^* = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_3\ c_3\ c_2\ c_4}$.[13]

Despite this drawback, the MPD mechanism has several distinctive advantages. First, the total number of matched students under the MPD mechanism is weakly higher than that under the CP mechanism. This is because a student profile can be sent to more than one college and those students who were unmatched under the CP mechanism now have a chance to avoid

[13] In Example 3, c_3 tentatively accepts s_5 in Step 1 but later rejects the student in Step 2. Therefore, we may want to design some mechanism such that the chance of s_5 to send her profile to c_3 can be transferred to s_3. However, such redesign is possible only if the clearinghouse can elicit information about colleges' preferences even *before* the colleges receive students' profiles (to make sure that s_5 would be rejected by c_3). In practice, since there are millions of students in the market, it must be hard for colleges to figure out their preferences over a set of students unless they receive those students' profiles. In other words, each college can possibly submit or reveal its preferences only *after* receiving students' profiles, which makes it impossible to resolve the drawback of the MPD mechanism mentioned above. In the next subsection, we show that this drawback can be completely solved once the dummy quota policy is abandoned.

being unmatched. Second, the level of college autonomy under the MPD mechanism is weakly higher than that under the CP mechanism. Since the assignment is finalized at the end of the MPD mechanism, each college now receives a larger set of student profiles and can select more preferred students than under the CP mechanism.

Besides these advantages, the MPD mechanism performs better than the CP mechanism in terms of stability and manipulability. As the following theorem formally establishes, the MPD mechanism has a higher degree of stability and a lower degree of manipulability.

Theorem 2 *Given a dummy quota vector $\bar{d}$, the MPD mechanism is (i) more score stable, more preference stable, and (ii) less manipulable than the CP mechanism.*

Theorem 2(i) indicates that, (a) at any problem in which the CP mechanism produces a score (resp. preference) stable matching, the MPD mechanism also produces a score (resp. preference) stable matching, and that (b) at some problem, the converse is not true. Theorem 2(ii) indicates that, (a) at any problem in which the MPD mechanism is manipulable by some student, the CP mechanism is also manipulable by some (possibly other) student, and that (b) at some problem, the converse is not true.

Before discussing the next scenario, one may want to know whether the existing well-known mechanisms have a similar property to the MPD mechanism. That is, one may curious about whether the SD or the DA mechanism can improve the CP mechanisms in terms of stability and manipulability. We use the following example to consider this question.

Example 4 *Let $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2\}$, $q = (1, 1)$, and $\bar{d} = (2, 2)$. Students' preferences, colleges' preferences, and test score order are given as follows.*

$$\begin{array}{llllll} P_{s_1}\!: & c_1, c_2, & P_{c_1}\!: & s_1, s_2, s_3, s_4, & \succ\!: s_1, s_2, s_3, s_4, \\ P_{s_2}\!: & c_1, c_2, & P_{c_2}\!: & s_3, s_1, s_2, s_4, & \\ P_{s_3}\!: & c_2, c_1, & & & \\ P_{s_4}\!: & c_1, c_2. & & & \end{array}$$

The CP mechanism produces a preference stable matching $\mu = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ \emptyset\ c_2\ \emptyset}$, but the SD mechanism produces a preference unstable matching $\mu' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ \emptyset}$. This means that at this problem, once we switch to use the SD mechanism, preference stability will not be maintained. Next, suppose that students' preferences and colleges' preferences are given as follows.

$$
\begin{array}{llll}
P_{s_1}: & c_1, c_2, & P_{c_1}: & s_4, s_1, s_2, s_3, \quad \succ: s_1, s_2, s_3, s_4, \\
P_{s_2}: & c_2, c_1, & P_{c_2}: & s_2, s_1, s_3, s_4, \\
P_{s_3}: & c_1, c_2, & & \\
P_{s_4}: & c_1, c_2. & &
\end{array}
$$

The CP mechanism produces a score stable matching $\mu'' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ \emptyset}$*, but the DA mechanism produces a score unstable matching* $\mu''' = \binom{s_1\ s_2\ s_3\ s_4}{\emptyset\ c_2\ \emptyset\ c_1}$*. This means that at this problem, once we switch to use the DA mechanism, score stability will not be maintained.*

The above example illustrates that the CP mechanism is (i) not less preference stable than the SD mechanism, and (ii) not less score stable than the DA mechanism. The findings from Example 4 and Theorem 2 are summarized in the following remark.

Remark 3 *Comparing with the SD or the DA mechanism, the MPD mechanism has its own advantage, in the sense that it can improve the CP mechanism in all three directions: more score stable, more preference stable, and less manipulable.*

4.2 Scenario 2: Dummy quota policy is abandoned

The second scenario is that the dummy quota policy is abandoned. In this scenario, the constraint for colleges to receive student profiles is removed and each college can receive a sufficiently large number of student profiles. Given a problem $(P_S, P_C, \succ)$, we propose the modified parallel with free dummy quota policy (MPF) mechanism, which is described as follows.

The MPF mechanism

Step 1: The CP mechanism with *free* dummy quota policy, i.e., for each $c \in C$, $\bar{d}_c = |S|$, is used. The outcome is tentative and denoted by $MPF^1(G)$.

Step t, $t \geq 2$: Let $U^t = \{s \in S : MPF_s^{t-1}(G) = \emptyset\}$ be the set of all unmatched students under $MPF^{t-1}(G)$. The clearinghouse sends student profiles to colleges as follows:

(1) The student with the highest test score in the set U^t, $s[t,1]$, is considered. Her profile is received by her top choice among those in the set $B^t_{s[t,1]}$, in which $B^t_{s[t,1]}$ satisfies the following conditions:

- The set does not contain any college that the profile of $s[t,1]$ was received before.

- For each $c \in B^r_{s[t,1]}$, we have (i) $cP_{s[t,1]}\emptyset$, and (ii) either (a) $|MPF^{t-1}_c(G)| < q_c$, $s[t,1] \succ \emptyset$, or (b) for some $s' \in MPF^{t-1}_c(G)$, $s[t,1] \succ s'$.

(k) The student with the k-th highest test score in the set U^t, $s[t,k]$, is considered. Her profile is received by her top choice among those in the set $B^t_{s[t,k]}$.

Let $F^t_c(G)$ be the set of students whose profiles are received in Step t by college c at problem G. Each college c admits the students in the set $MPF^{t-1}_c(G) \cup F^t_c(G)$, up to its quota q_c, one at a time, following the preferences P_c. The outcome is tentative and denoted by $MPF^t(G)$.

End: The mechanism is terminated when $U^{t'} = \emptyset$ or for each $s \in U^{t'}$, $B^{t'}_s = \emptyset$. Then, $MPF^{t'}(G)$ becomes the final assignment of the mechanism.

The MPF mechanism is very close to the MPD mechanism. In Step 1, the CP mechanism (with free dummy quota policy) produces $MPF^1(G) = CP(G)$. In Step $t \geq 2$, following the test score order, for each student $s \in U^t$, we send her profile to her top choice among those in the set B^t_s. Note that in the definition of A^t_s, colleges can only receive a limited number of student profiles, whereas in the definition of B^t_s, the constraint of dummy quota is removed. We revisit Example 3 below and illustrate how this new mechanism works.

Example 5 *Let $S = \{s_1, s_2, s_3, s_4, s_5\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $q = (1, 1, 2, 1)$. Students' preferences, colleges' preferences, and test score order are given as follows.*

$$\begin{array}{llllll}
P_{s_1}: & c_1, c_2, c_3, c_4, & P_{c_1}: & s_1, s_2, s_3, s_4, s_5, & \succ: & s_1, s_2, s_3, s_4, s_5, \\
P_{s_2}: & c_1, c_3, c_2, c_4, & P_{c_2}: & s_1, s_2, s_3, s_4, s_5, & & \\
P_{s_3}: & c_1, c_3, c_2, c_4, & P_{c_3}: & s_1, s_2, s_3, s_4, s_5, & & \\
P_{s_4}: & c_3, c_2, c_1, c_4, & P_{c_4}: & s_1, s_2, s_3, s_4, s_5, & & \\
P_{s_5}: & c_3, c_2, c_4, c_1. & & & &
\end{array}$$

The MPF mechanism works as follows. In Step 1, the CP mechanism produces $MPF^1(G) = CP(G) = \left(\begin{smallmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ c_1 & \emptyset & \emptyset & c_3 & c_3 \end{smallmatrix}\right)$. In Step 2, $U^2 = \{s_2, s_3\}$. The profile of s_2 is received by her second choice $c_3 \in B^2_{s_2} = \{c_3, c_2, c_4\}$ and the profile of s_3 is received by her second choice $c_3 \in B^2_{s_3} = \{c_3, c_2, c_4\}$. Then, c_3 selects s_2 and s_3 from $MPF^1_{c_3}(G) \cup F^2_{c_3}(G) = \{s_2, s_3, s_4, s_5\}$. Thus, $MPF^2(G) = \left(\begin{smallmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ c_1 & c_3 & c_3 & \emptyset & \emptyset \end{smallmatrix}\right)$. In Step 3, $U^3 = \{s_4, s_5\}$. The profile of s_4 is received by her second choice $c_2 \in B^3_{s_4} = \{c_2, c_4\}$ and the profile of s_5 is received by her second choice $c_2 \in$

$B^3_{s_5} = \{c_2, c_4\}$. *Then,* c_2 *selects* s_4 *from* $MPF^2_{c_2}(G) \cup F^2_{c_2}(G) = \{s_4, s_5\}$. *Thus,* $MPF^2(G) = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_3\ c_3\ c_2\ \emptyset}$. *In Step* 4, $U^4 = \{s_5\}$. *The profile of* s_5 *is received by her third choice* $c_4 \in B^4_{s_5} = \{c_4\}$. *Then,* c_4 *selects* s_5 *from* $MPF^2_{c_4}(G) \cup F^2_{c_2}(G) = \{s_5\}$. *The mechanism is terminated and the outcome is* $MPF(G) = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_3\ c_3\ c_2\ c_4}$.

In the above example, the matching $MPF(G)$ is double stable and not manipulable by any student. One may expect that the MPF mechanism is similar to the MPD mechanism, in the sense that it is also more score stable, more preference stable, and less manipulable than the CP mechanism. Indeed, comparing with the CP mechanism with free dummy quota policy, this is true. However, as the following example shows, the CP mechanism with some positive dummy quotas may produce a better outcome than the MPF mechanism.

Example 6 *Let* $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $q = (1, 1, 1)$, *and* $\bar{d} = (1, 2, 2)$. *Students' preferences, colleges' preferences, and test score order are given as follows.*

$$\begin{array}{llllll}
P_{s_1}: & c_1, c_2, c_3 & P_{c_1}: & s_4, s_1, s_2, s_3, & \succ: s_1, s_2, s_3, s_4, \\
P_{s_2}: & c_2, c_1, c_3 & P_{c_2}: & s_2, s_1, s_3, s_4, & \\
P_{s_3}: & c_1, c_3, c_2 & P_{c_3}: & s_3, s_1, s_2, s_4, & \\
P_{s_4}: & c_1, c_2, c_3. & & &
\end{array}$$

The CP mechanism produces a score stable matching $\mu = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ c_3\ \emptyset}$, *but the MPF mechanism produces a score unstable matching* $\mu' = \binom{s_1\ s_2\ s_3\ s_4}{\emptyset\ c_2\ c_3\ c_1}$. *Moreover, at this problem, the CP mechanism is not manipulable but the MPF mechanism is manipulable by* s_1 *via* $Q_{s_1} : c_3, c_1, c_2$. *Next, suppose that students' preferences and colleges' preferences are given as follows.*

$$\begin{array}{llllll}
P_{s_1}: & c_1, c_2, c_3, & P_{c_1}: & s_1, s_2, s_3, s_4, & \succ: s_1, s_2, s_3, s_4, \\
P_{s_2}: & c_2, c_1, c_3, & P_{c_2}: & s_2, s_1, s_3, s_4, & \\
P_{s_3}: & c_3, c_1, c_2, & P_{c_2}: & s_4, s_3, s_1, s_2, & \\
P_{s_4}: & c_1, c_3, c_2. & & &
\end{array}$$

The CP mechanism produces a preference stable matching $\mu'' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_3}$, *but the MPF mechanism produces a preference unstable matching* $\mu''' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ c_3\ \emptyset}$. *Therefore, the MPF mechanism is neither more score stable, more preference stable, nor less manipulable than the CP mechanism.*

As we show in Examples 5 and 6, the MPF mechanism is score stability incomparable, preference stability incomparable, and manipulability incomparable with the CP mechanism. Therefore, if the CP mechanism is replaced with the MPF mechanism, the effects to the degrees of stability and manipulability are ambiguous.

Given this incomparable result, one may wonder why we propose the MPF mechanism in the current scenario. The main reason is that two drawbacks of the CP mechanism mentioned in Section 3 can be completely solved by the MPF mechanism.

Theorem 3 *Given a problem $(P_S, P_C, \succ)$, the MPF mechanism (i) eliminates any double blocking pair and (ii) produces a double stable matching whenever such a matching exists.*

Using Theorem 3(ii), we can show that, at any problem the SD mechanism is preference stable, the MPF mechanism is also preference stable. We explain the reason as follows. Suppose on the contrary that there is a problem G such that $SD(G)$ is preference stable but $MPF(G)$ is not. Since the SD mechanism produces a (unique) score stable matching, we know that a double stable matching exists at this problem and it coincides with $SD(G)$. However, by Theorem 3(ii), such a double stable matching must also be produced by the MPF mechanism. A contradiction.

Similarly, we can also use Theorem 3(ii) to show that, at any problem the DA mechanism is score stable, the MPF mechanism is also score stable. Moreover, the following example shows that, at some problem the converse of these statements is not always true.

Example 7 *Let $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2\}$, and $q = (1, 1)$. Students' preferences, colleges' preferences, and test score order are given as:*

$$\begin{array}{llllll}
P_{s_1}: & c_1, c_2, & P_{c_1}: & s_4, s_1, s_2, s_3, & \succ: & s_1, s_2, s_3, s_4, \\
P_{s_2}: & c_1, c_2, & P_{c_2}: & s_1, s_2, s_4, s_3, & & \\
P_{s_3}: & c_1, c_2, & & & & \\
P_{s_4}: & c_1, c_2. & & & &
\end{array}$$

The MPF mechanism produces a preference stable matching $\mu = \binom{s_1\ s_2\ s_3\ s_4}{c_2\ \emptyset\ \emptyset\ c_1}$, but the SD mechanism produces a preference unstable matching $\mu' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ \emptyset}$. Next, suppose that students' preferences, colleges' preferences are given as:

$$\begin{aligned}
&P_{s_1}: \quad c_1, c_2, \quad P_{c_1}: \quad s_1, s_2, s_3, s_4, \quad \succ: s_1, s_2, s_3, s_4,\\
&P_{s_2}: \quad c_1, c_2, \quad P_{c_2}: \quad s_4, s_2, s_3, s_1,\\
&P_{s_3}: \quad c_2, c_1,\\
&P_{s_4}: \quad c_1, c_2.
\end{aligned}$$

The MPF mechanism produces a score stable matching $\mu'' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ \emptyset}$*, but the DA mechanism produces a score unstable matching* $\mu''' = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ \emptyset\ \emptyset\ c_2}$*.*

Based on these observations, the MPF mechanism is more preference stable than the SD mechanism and more score stable than the DA mechanism. Therefore, we obtain the following.

Remark 4 *Comparing with the SD or the DA mechanism, the MPF mechanism is "balanced", in the sense that it is (i) less score stable but more preference stable than the SD mechanism, and (ii) less preference stable but more score stable than the DA mechanism.*

5 Conclusion

College admissions in China have experienced a lot of reforms. Although these reforms were effective and improved the allocation outcome, the Chinese government still faces a real-life complication that colleges' preferences does not necessarily align with test score order. In this paper, we focused on such a misalignment problem and addressed the following three questions: what is the formal description of the CP mechanism? what drawbacks does the CP mechanism have? how can we improve the CP mechanism?

To answer these questions, we used an extended matching model in which colleges have two different types of ordinal rankings over students. The CP mechanism, based on the dummy quota policy, is a variant of the SD mechanism but has various drawbacks. We showed that the CP mechanism is neither score stable, preference stable, nor strategy-proof for students. Moreover, the CP mechanism also fails to produce a double stable matching even if it exists.

These drawbacks motivated us to redesign the current mechanism or to propose alternative mechanisms that we expect to perform better than the CP mechanism. Our proposals depend on two different scenarios: whether the dummy quota policy is maintained or abandoned. In the first scenario, we proposed the MPD mechanism and showed that it is more score stable, more preference stable, and less manipulable than the CP mechanism. In the second scenario, we proposed the MPF mechanism and showed that it solves the double blocking pair problem and produces a double stable matching whenever it exists. Comparing with the existing well-known mechanisms such as the SD and the DA mechanism, the MPD and the MPF mechanisms have their own advantages. Therefore, to improve the allocation outcome of the CP mechanism, we recommend the MPD mechanism if the dummy quota policy needs to be maintained, and the MPF mechanism if the dummy quota policy can be abandoned.

Appendix

Three well-known mechanisms. There are three well-known mechanisms mentioned in the introduction. The first one is the serial dictatorship (SD) mechanism, where the assignment is determined by students' test scores and their preferences.

The SD mechanism

Step 1: The student with the highest test score is assigned to her top choice.

Step t, $t \geq 2$: The student with the t-th highest test score is assigned to her top choice among all colleges except the ones whose real quota have been filled. Note that the mechanism is terminated when all students have chosen a college or all colleges have filled their real quotas.

The second one is the immediate acceptance (IA) mechanism, where the assignment is determined by students' preferences and colleges' preferences.

The IA mechanism

Step 1: Each student proposes to their top choice. Each college (i) considers its applicants at this step; (ii) immediately accepts those applicants up to its real quota, one at a time, following its preferences; and (iii) rejects the remaining applicants.

Step t, $t \geq 2$: Each student that has been rejected in the previous step proposes to their t-th choice. Each college (i) considers its applicants at this step; (ii) immediately accepts those applicants up to its remaining real quota, one at a time, following its preferences; and (iii) rejects the remaining applicants. Note that the mechanism is terminated when no student is rejected or all colleges have filled their real quotas.

The third one is the deferred acceptance (DA) mechanism, where the assignment is determined by students' preferences and colleges' preferences.[14]

The student-proposing DA mechanism

Step 1: Each student proposes to their top choice. Each college (i) considers its applicants at this step; (ii) tentatively accepts those applicants up to its real quota, one at a time, following its preferences; and (iii) rejects the remaining applicants.

Step t, $t \geq 2$: Each student that has been rejected in the previous step proposes to their next choice. Each college (i) considers its applicants at this step and all tentatively matched applicants in the previous step; (ii) tentatively accepts those applicants up to its real quota, one at a time, following its preferences; and (iii) rejects the remaining applicants. Note that the mechanism is terminated when no student's proposal is rejected.

Proof of Theorem 2(i).

Proof. We first show that at any problem the CP mechanism is score stable, the MPD mechanism is also score stable. Suppose that there is a problem G such that $CP(G)$ is score stable. Then, $MPD^1(G) = CP(G)$ is (i) individually rational for students and (ii) not score blocked by any pair. By the definition of A_s^t, this means that for each $s \in U^2$, $A_s^2 = \emptyset$. The MPD mechanism is thus terminated in Step 2 and the outcome is $MPD(G) = MPD^2(G) = MPD^1(G) = CP(G)$. Therefore, the MPD mechanism also produces a score stable matching at this problem. Then, we show that the converse is not always true. In Example 2, $MPD(G) = \left(\begin{smallmatrix} s_1 & s_2 & s_3 & s_4 \\ c_1 & c_2 & \emptyset & \emptyset \end{smallmatrix}\right)$ is score stable but $CP(G) = \left(\begin{smallmatrix} s_1 & s_2 & s_3 & s_4 \\ c_1 & \emptyset & c_2 & \emptyset \end{smallmatrix}\right)$ is not.

Next, we show that at any problem the CP mechanism is preference stable, the MPD mechanism is also preference stable. Suppose that there is a problem G such that $CP(G)$ is preference stable. Then, $MPD^1(G) = CP(G)$ is (i) individually rational for both students and colleges and (ii) not preference blocked by any pair. Now consider any student $s \in U^2$ whose profile is additionally received by some college c in Step 2 of the MPD mechanism. If c tentatively accepts s in this step, i.e., $MPD_s^2(G) = c$, then they form a preference blocking pair for $CP(G) = MPD^1(G)$, a contradiction. Thus, s must be rejected by c in this step. As a result,

[14]The college-proposing DA mechanism can be defined in almost the same way by swapping the roles of colleges and students, hence we do not describe it here.

we have $MPD^2(G) = MPD^1(G) = CP(G)$. Continue the argument in this way until the MPD mechanism is terminated in Step T. Since $MPD(G) = MPD^T(G) = MPD^{T-1}(G) = \cdots = MPD^2(G) = MPD^1(G) = CP(G)$, the MPD mechanism also produces a preference stable matching at this problem. Then, we show that the converse is not always true. In Example 2, $MPD(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ \emptyset}$ is preference stable but $CP(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ \emptyset\ c_2\ \emptyset}$ is not. ■

Proof of Theorem 2(ii).

Proof. We start with a useful lemma; it states that if a student s can be tentatively matched with a college c via preferences Q_s under the MPD mechanism, then she also can be matched with this college c by reporting it as her top choice under the CP mechanism.

Lemma 1 *Given a problem $G = (P_S, P_C, \succ)$ and a dummy quota vector $\bar{d}$, if there is a student s and preferences Q_s such that for some number t, $MPD_s^t(Q_s, P_{-s}, P_C, \succ) = c$, then $CP_s(Q_s^c, P_{-s}, P_C, \succ) = c$, where Q_s^c is the preferences that s reports c as her top choice.*

Proof. Let $G(Q_s) := (Q_s, P_{-s}, P_C, \succ)$ and $G(Q_s^c) := (Q_s^c, P_{-s}, P_C, \succ)$. We first show that the profile of s is received by c under the CP mechanism at problem $G(Q_s^c)$, i.e., $s \in F_c(G(Q_s^c))$. Suppose not, then (i) $|F_c(G(Q_s^c))| = \bar{d}_c$ and (ii) for each $s' \in F_c(G(Q_s^c))$, $s' \succ s$. Since any student who has a higher test score than s should be received by the same college under the CP mechanism at both problems $G(Q_s)$ and $G(Q_s^c)$, $F_c(G(Q_s^c)) = F_c(G(Q_s))$. By the description of the MPD mechanism, $F_c(G(Q_s)) = F_c^1(G(Q_s))$. Thus, s cannot be tentatively matched with c under the MPD mechanism at problem $G(Q_s)$, a contradiction.

We then show that s and c are matched under the CP mechanism at problem $G(Q_s^c)$, i.e., $CP_s(G(Q_s^c)) = c$. Suppose not, then (i) $|CP_c(G(Q_s^c))| = q_c$ and (ii) for each $s' \in CP_c(G(Q_s^c))$, $s' P_c s$. We claim that all profiles of the students in the set $CP_c(G(Q_s^c))$ should be also received by c under the CP mechanism at problem $G(Q_s)$, i.e., $CP_c(G(Q_s^c)) \subset F_c(G(Q_s))$. Note that this is true if either (i) $s \in F_c(G(Q_s))$, or (ii) for each $s^1 \in CP_c(G(Q_s^c))$, $s^1 \succ s$. Thus, we only need to consider the situation that (i) $s \notin F_c(G(Q_s))$ and (ii) for some $s^1 \in CP_c(G(Q_s^c))$, $s \succ s^1$ and $s^1 \notin F_c(G(Q_s))$. Let $x(s^1) \neq c$ be such that $s^1 \in F_{x(s^1)}(G(Q_s))$ and we have two cases.

Case 1: $x(s^1) P_{s^1} c$. Since the profile of s^1 is not received by $x(s^1)$ but by c at problem $G(Q_s^c)$,

(i) $|F_{x(s^1)}(G(Q_s^c))| = \bar{d}_{x(s^1)}$ and (ii) for each $s^2 \in F_{x(s^1)}(G(Q_s^c))$, $s^2 \succ s^1$. Define

$$D_{x(s^1)} := \{s^2 \in S : s^2 \in F_{x(s^1)}(G(Q_s^c)); \exists x(s^2) \neq x(s^1), s^2 \in F_{x(s^2)}(G(Q_s))\}$$

If $|D_{x(s^1)}| = 0$, then $F_{x(s^1)}(G(Q_s^c)) = F_{x(s^1)}(G(Q_s))$. This means that the profile of s^1 cannot be received by $x(s^1)$ at problem $G(Q_s)$, a contradiction. Thus, $|D_{x(s^1)}| \geq 1$ and we can find some $s^2 \succ s^1$ in this set. If $x(s^1)P_{s^2}x(s^2)$, then since $s^2 \succ s^1$ and the profile of s^2 cannot be received by $x(s^1)$ at problem $G(Q_s)$, $s^1 \notin F_{x(s^1)}(G(Q_s))$, a contradiction. If $x(s^2)P_{s^2}x(s^1)$, then since the profile of s^2 is not received by $x(s^2)$ but by $x(s^1)$ at problem $G(Q_s^c)$, we can similarly define $D_{x(s^2)}$ and find some $s^3 \succ s^2$ in this set. Continue in this way until either a contradiction or a chain $(c, s^1, x(s^1), s^2, x(s^2), \cdots, x(s^{**}), s^*, x(s^*), s, x(s))$ is obtained. In the former, we are done. In the latter, note that $x(s^*) = c$ because $s \in F_c(G(Q_s^c))$. If $x(s^{**})P_{s^*}x(s^*) = c$, then since $s^* \succ s^{**}$ and the profile of s^* cannot be received by $x(s^{**})$ at problem $G(Q_s)$, $s^{**} \notin F_{x(s^{**})}(G(Q_s))$, a contradiction. If $x(s^*) = cP_{s^*}x(s^{**})$, then since $s^* \succ s^1$ and the profile of s^* cannot be received by c at problem $G(Q_s^c)$, $s^1 \notin F_c(G(Q_s^c))$, a contradiction.

Case 2: $cP_{s^1}x(s^1)$. Since the profile of s^1 is not received by c but by $x(s^1)$ at problem $G(Q_s)$, (i) $|F_c(G(Q_s))| = \bar{d}_c$ and (ii) for each $s^2 \in F_c(G(Q_s))$, $s^2 \succ s^1$. Define

$$E_c := \{s^2 \in S : s^2 \in F_c(G(Q_s)); \exists y(s^2) \neq c, s^2 \in F_{y(s^2)}(G(Q_s^c))\}$$

If $|E_c| < 2$, then there are at least $\bar{d}_c - 1$ students (in the set $F_c(G(Q_s))$) whose profiles are still received by c at problem $G(Q_s^c)$. Since $s \notin F_c(G(Q_s))$ and $s \in F_c(G(Q_s^c))$, the profile of s^1 cannot be received by c at problem $G(Q_s^c)$, a contradiction. Thus, $|E_c| \geq 2$ and we can find two students $s^2, \hat{s}^2$ (with $s^2 \succ s^1$ and $\hat{s}^2 \succ s^1$) in this set. First, we consider the student s^2.

If $cP_{s^2}y(s^2)$, then since $s^2 \succ s^1$ and the profile of s^2 cannot be received by c at problem $G(Q_s^c)$, $s^1 \notin F_c(G(Q_s^c))$, a contradiction. If $y(s^2)P_{s^2}c$, then since the profile of s^2 is not received by $y(s^2)$ but by c at problem $G(Q_s)$, (i) $|F_{y(s^2)}(G(Q_s))| = \bar{d}_{y(s^2)}$ and (ii) for each $s^3 \in F_{y(s^2)}(G(Q_s))$, $s^3 \succ s^2$. Similarly, we can define

$$E_{y(s^2)} := \{s^3 \in S : s^3 \in F_{y(s^2)}(G(Q_s)); \exists y(s^3) \neq y(s^2), s^3 \in F_{y(s^3)}(G(Q_s^c))\}$$

If $|E_{y(s^2)}| = 0$, then $F_{y(s^2)}(G(Q_s)) = F_{y(s^2)}(G(Q_s^c))$. This means that the profile of s^2 cannot be received by $y(s^2)$ at problem $G(Q_s^c)$, a contradiction. Thus, $|E_{y(s^2)}| \geq 1$ and we can find some $s^3 \succ s^2$ who has the lowest test score in this set. If $y(s^2)P_{s^3}y(s^3)$, then since $s^3 \succ s^2$ and the profile of s^3 cannot be received by $y(s^2)$ at problem $G(Q_s^c)$, $s^2 \notin F_{y(s^2)}(G(Q_s^c))$, a contradiction. If $y(s^3)P_{s^3}y(s^2)$, then since the profile of s^3 is not received by $y(s^3)$ but by $y(s^2)$ at problem $G(Q_s)$, we can similarly define $E_{y(s^3)}$ and find some $s^4 \succ s^3$ who has the lowest test score in this set. Continue in this way until either a contradiction or a chain $(c, s^2, y(s^2), s^3, y(s^3), \cdots, y(s^{**}), s^*, y(s^*), s, y(s))$ is obtained. In the former, we are done. In the latter, note that for the student $\hat{s}^2$, we also can obtain either a contradiction or a chain $(c, \hat{s}^2, y(\hat{s}^2), \hat{s}^3, y(\hat{s}^3), \cdots, y(\hat{s}^{**}), \hat{s}^*, y(\hat{s}^*), s, y(s))$. In the former, we are done. In the latter, we have two chains and consider the following two cases.

If there is no student (except s) who belongs to both two chains, then $s^* \neq \hat{s}^*$. If not, then we can find some $s^k = \hat{s}^t$ who has the lowest indexed numbers in these two chains. Suppose that $s^{k-1} \succ \hat{s}^{t-1}$. If $|E_{y(\hat{s}^{t-1})=y(s^{k-1})}| < 2$, then there are at least $\bar{d}_{y(\hat{s}^{t-1})=y(s^{k-1})} - 1$ students (in the set $F_{y(\hat{s}^{t-1})=y(s^{k-1})}(G(Q_s)))$ whose profiles are still received by $y(\hat{s}^{t-1}) = y(s^{k-1})$ at problem $G(Q_s^c)$. Since $s^{k-1} \notin F_{y(\hat{s}^{t-1})=y(s^{k-1})}(G(Q_s))$ and $s^{k-1} \in F_{y(\hat{s}^{t-1})=y(s^{k-1})}(G(Q_s^c))$, the profile of $\hat{s}^{t-1}$ cannot be received by $y(\hat{s}^{t-1}) = y(s^{k-1})$ at problem $G(Q_s^c)$, a contradiction. Thus, $|E_{y(\hat{s}^{t-1})=y(s^{k-1})}| \geq 2$ and we can find two different students, $\hat{s}^t$ and s^k, in this set. By using the similar argument, we can obtain either a contradiction or two new chains: $(y(s^{k-1}) = y(\hat{s}^t), s^k, y(s^k), \cdots, s, y(s))$ and $(y(s^{k-1}) = y(\hat{s}^t), \hat{s}^t, y(\hat{s}^t), \cdots, s, y(s))$. In the former, we are done. In the latter, we can use the similar argument to obtain two chains: $(\cdots, y(s^{**}), s^*, y(s^*), s, y(s))$ and $(\cdots, y(\hat{s}^{**}), \hat{s}^*, y(\hat{s}^*), s, y(s))$, in which there is no student (except s) who belongs to both two chains.

Note that (i) $s^* \neq \hat{s}^*$ and (ii) $y(s^*) = y(\hat{s}^*)$. Suppose that $s^* \succ \hat{s}^*$. Since (i) any student who has a higher test score than s should be received by the same college at both problems $G(Q_s)$ and $G(Q_s^c)$, and (ii) s has the lowest test score in the set $E_{y(s^*)=y(\hat{s}^*)}$, we know that $|E_{y(s^*)=y(\hat{s}^*)}| = 1$. Thus, there are at least $\bar{d}_{y(s^*)=y(\hat{s}^*)} - 1$ students (in the set $F_{y(s^*)=y(\hat{s}^*)}(G(Q_s)))$ whose profiles are still received by $y(s^*) = y(\hat{s}^*)$ at problem $G(Q_s^c)$. Since $s^* \notin F_{y(s^*)=y(\hat{s}^*)}(G(Q_s))$ and $s^* \in F_{y(s^*)=y(\hat{s}^*)}(G(Q_s^c))$, the profile of $\hat{s}^*$ cannot be received by $y(\hat{s}^*) = y(s^*)$ at problem $G(Q_s^c)$, a contradiction.

Thus, $CP_c(G(Q_s^c)) \subset F_c(G(Q_s))$. This means that for each $s' \in CP_c(G(Q_s^c))$ and each $s'' \in CP_c(G(Q_s))$, $s''R_cs'$. Since (i) $|CP_c(G(Q_s^c))| = q_c$ and (ii) for each $s' \in CP_c(G(Q_s^c))$, $s'P_cs$, we have (i) $|CP_c(G(Q_s))| = q_c$ and (ii) for each $s'' \in CP_c(G(Q_s))$, $s''P_cs$. By the description of the MPD mechanism, s cannot be tentatively matched with c under the MPD mechanism at problem $G(Q_s)$, a contradiction. ■

We show that at any problem the MPD mechanism is manipulable, the CP mechanism is also manipulable. Suppose that there is a problem G such that $CP(G)$ is not manipulable but $MPD(G)$ is. There is a student s and preferences Q_s such that $MPD_s(G(Q_s))P_sMPD_s(G)$. Let $x := MPD_s(G(Q_s))$ and we consider two cases.

Case 1: $xP_sCP_s(G)$. Since $MPD_s(G(Q_s)) = x$, s must be tentatively matched with x in some step under the MPD mechanism at problem $G(Q_s)$. Let Q_s^x be the preferences that s reports x as her top choice. By Lemma 1, $CP_s(G(Q_s^x)) = x$. Thus, the CP mechanism is manipulable by s at problem G, a contradiction.

Case 2: $CP_s(G)R_sx$. Let $c(s) := CP_s(G)$. Since $xP_sMPD_s(G)$, we have $c(s)P_sMPD_s(G)$. This means that s is (i) tentatively matched with $c(s)$ in Step 1, but (ii) is rejected by $c(s)$ in a later Step $t(> 1)$ of the MPD mechanism at problem G. Thus, there is a student $s' \neq s$ such that $MPD_{s'}^{t-1}(G) = \emptyset$ and $MPD_{s'}^{t}(G) = c(s)$. We consider two cases. If $MPD_{s'}^1(G) = \emptyset$, then Lemma 1 implies that $CP_{s'}(G(Q_{s'}^{c(s)})) = c(s)$ and the CP mechanism is thus manipulable by s' at problem G, a contradiction. If $MPD_{s'}^1(G) \neq \emptyset$, then s' is (i) tentatively matched with $c(s') := MPD_{s'}^1(G)$ in Step 1, but (ii) is rejected by $c(s')$ in a later Step $t'(1 < t' \leq t-1 < t)$ of the MPD mechanism at problem G. Thus, there is a student $s'' \neq s'$ such that $MPD_{s''}^{t'-1}(G) = \emptyset$ and $MPD_{s''}^{t'}(G) = c(s')$. Continue the argument in this way until either a contradiction or a chain $(s, c(s), s', c(s'), s'', \cdots, s^{**}, c(s^{**}), s^*)$ is obtained. In the former, we are done. In the latter, for the student s^*, $MPD_{s^*}^1(G) = \emptyset$ and $MPD_{s^*}^{t^{**}}(G) = c(s^{**})$. Since Lemma 1 implies that $CP_{s^*}(G(Q_{s^*}^{c(s^{**})})) = c(s^{**})$, the CP mechanism is thus manipulable by s^* at problem G, a contradiction.

Then, we show that the converse is not always true. In Example 2, since $CP_{s_2}(G(Q_{s_2}^{c_2})) = c_2P_{s_2}\emptyset = CP_{s_2}(G)$, the CP mechanism is manipulable by s_2 at the given problem G. However, the MPD mechanism is not manipulable by any student. ■

Proof of Theorem 3(i).

Proof. Suppose that there is a problem G such that $MPF(G)$ is double blocked by a pair (s, c). We assume that the clearinghouse checks whether the profile of s can be sent to c in Step

t of the MPF mechanism. First, we show that the profile of s will be sent to c in this step, i.e., $s \in F_c^t(G)$. If $|MPF_c^{t-1}(G)| < q_c$, then we are done. If $|MPF_c^{t-1}(G)| = q_c$, then let $\underline{s}^{t-1}$ be the student who has the lowest test score among those in the set $MPF_c^{t-1}(G)$. When the MPF mechanism is terminated in Step T, we have $\underline{s}^T \succeq \underline{s}^{T-1} \succeq \cdots \succeq \underline{s}^{t-1}$. Since $MPF(G)$ is score blocked by (s, c), we have $s \succ \underline{s}^T$. This implies that s has enough test score to be received by c in Step t. Thus in either case, $s \in F_c^t(G)$. Then, we show that c will not reject s in any Step $t'(t \leq t' \leq T)$. Suppose not, then (i) $|MPF_c^{t'}(G)| = q_c$ and (ii) for each $s' \in MPF_c^{t'}(G)$, $s' P_c s$. By the description of the MPF mechanism, for each $s' \in MPF_c(G)$ and each $s'' \in MPF_c^{t'}(G)$, $s' R_c s''$. By transitivity, for each $s' \in MPF_c(G)$, we have $s' P_c s$. This means that $MPF(G)$ is not preference blocked by the pair (s, c), a contradiction. ■

Proof of Theorem 3(ii).

Proof. Suppose that there is a problem G such that a double stable matching μ^* exists. Since the SD mechanism produces a unique score stable matching, $\mu^* = SD(G)$. We show that $MPF(G) = SD(G)$. Suppose not, then the set $S^{\neq} := \{s \in S : MPF_s(G) \neq SD_s(G)\}$ is not empty and we can find a student s who has the highest test score in this set. We consider the following two cases.

Case 1: $MPF_s(G) P_s SD_s(G)$. Let $x(s) := MPF_s(G)$. Since $SD_s(G) \neq x(s)$, we have (i) $|SD_{x(s)}(G)| = q_{x(s)}$ and (ii) for each $s' \in SD_{x(s)}(G)$, $s' \succ s$. By the definitions of $S^{\neq}$ and s, any student in the set $SD_{x(s)}(G)$ should be also matched with $x(s)$ under the MPF mechanism. Thus, $MPF_s(G) \neq x(s)$, a contradiction.

Case 2: $SD_s(G) P_s MPF_s(G)$. Let $y(s) := SD_s(G)$. Since $MPF_s(G) \neq y(s)$, we assume that the clearinghouse checks whether the profile of s can be sent to $y(s)$ in Step t of the MPF mechanism. First, we show that her profile will be sent to $y(s)$ in this step, i.e., $s \in F_{y(s)}^t(G)$. Suppose not, then (i) $|MPF_{y(s)}^{t-1}(G)| = q_{y(s)}$ and (ii) for each $s' \in MPF_{y(s)}^{t-1}(G)$, $s' \succ s$. By the description of the MPF mechanism, (i) $|MPF_{y(s)}(G)| = q_{y(s)}$ and (ii) for each $s'' \in MPF_{y(s)}(G)$ and each $s' \in MPF_{y(s)}^{t'-1}(G)$, $s'' \succeq s'$. By the definitions of $S^{\neq}$ and s, any student in the set $MPF_{y(s)}(G)$ should be also matched with $y(s)$ under the SD mechanism. Thus, $SD_s(G) \neq y(s)$, a contradiction.

We then show that $y(s)$ will not reject s in any Step $t(s)$ $(\geq t)$ of the MPF mechanism. Suppose not, then (i) $|MPF_{y(s)}^{t(s)}(G)| = q_{y(s)}$ and (ii) for each $s' \in MPF_{y(s)}^{t(s)}(G)$, $s' P_{y(s)} s$. Thus, there is a student s' such that (i) $MPF_{s'}^{t(s)}(G) = y(s)$, (ii) $s' P_{y(s)} s$, and (iii) $SD_{s'}(G) \neq y(s)$. Let $y(s') := SD_{s'}(G)$. If $y(s) P_{s'} y(s')$, then $(s', y(s))$ form a preference blocking pair for $SD(G)$, a contradiction. If $y(s') P_{s'} y(s)$, then since $MPF_{s'}^{t(s)}(G) = y(s) \neq y(s')$, the clearinghouse checks whether the profile of s' can be sent to $y(s')$ in Step t' $(\leq t(s))$ of the MPF mechanism.

If her profile cannot be sent to $y(s')$ in this step, i.e., $s' \notin F_{y(s')}^{t'}(G)$, then (i) $|MPF_{y(s')}^{t'-1}(G)| = q_{y(s')}$ and (ii) for each $s'' \in MPF_{y(s')}^{t'-1}(G)$, $s'' \succ s'$. Thus, there is a student s'' such that (i) $MPF_{s''}^{t'-1}(G) = y(s')$, (ii) $s'' \succ s'$, and (iii) $SD_{s''}(G) \neq y(s')$. If her profile is received but is rejected by $y(s')$ in any Step $\bar{t}$ $(t' \leq \bar{t} < t(s))$, then (i) $|MPF_{y(s')}^{\bar{t}}(G)| = q_{y(s')}$ and (ii) for each $s'' \in MPF_{y(s')}^{\bar{t}}(G)$, $s'' P_{y(s')} s'$. Note that in either situation, there is an earlier Step $t(s')(< t(s))$ such that we can find a student $s'' \neq s'$ who satisfies that (i) $MPF_{s''}^{t(s')}(G) = y(s')$, (ii) either $s'' \succ s'$ or $s'' P_{y(s')} s'$, and (iii) $SD_{s''}(G) := y(s'') \neq y(s')$. If $y(s') P_{s''} y(s'')$, then $(s'', y(s'))$ form (i) a score blocking pair for $SD(G)$ when $s'' \succ s'$, or (ii) a preference blocking pair for $SD(G)$ when $s'' P_{y(s')} s'$, a contradiction. If $y(s'') P_{s''} y(s')$, then since $MPF_{s''}^{t(s')}(G) = y(s') \neq y(s'')$, the clearinghouse checks whether the profile of s'' can be sent to $y(s'')$ in Step t'' $(\leq t(s') < t(s))$ of the MPF mechanism.

Continue the argument in this way until either a contradiction or a chain $(s, y(s), s', y(s'), \cdots, y^{**}, s^*, y(s^*))$ is obtained. In the former, we are done. In the latter, for the student s^*, we have (i) $MPF_{s^*}^{1}(G) = y(s^{**})$, (ii) either $s^* \succ s^{**}$ or $s^* P_{y(s^{**})} s^{**}$, and (iii) $SD_{s^*}(G) = y(s^*) \neq y(s^{**})$. We consider two cases. If $y(s^{**}) P_{s^*} y(s^*)$, then $(s^*, y(s^{**}))$ form (i) a score blocking pair for $SD(G)$ when $s^* \succ s^{**}$, or (ii) a preference blocking pair for $SD(G)$ when $s^* P_{y(s^{**})} s^{**}$, a contradiction. If $y(s^*) P_{s^*} y(s^{**})$, then since there is no constraint for $y(s^*)$ to receive student profiles, we know that $MPF_{s^*}^{1}(G) \neq y(s^{**})$, a contradiction.

Therefore, the set $S^{\neq}$ is empty and we have $MPF(G) = SD(G)$. Since $SD(G)$ is double stable, $MPF(G)$ should be also double stable. ■

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