Strategic foundations of macroprudential regulation: preventing fire sales externalities

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February 11, 2022

Abstract

We offer a stress test framework in which interaction between regulated banks occurs through pecuniary externalities when they delever. Since banks are constrained to maintain their capital ratio higher than a threshold, the deleveraging problem yields a generalized game in which the solvency constraint of each bank depends upon the decisions of the others. We analyze the game under microprudential but also under macroprudential regulation in which fire sales externalities are banned. We show that a Pareto optimal Nash equilibrium generically exists under macroprudential regulation while the existence under microprudential regulation requires strong conditions. An empirical analysis is also provided.

Keywords: Macroprudential regulation, fire sales, externalities, generalized games

JEL Classification: G21, G28, D43

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1 Introduction

In the new banking regulatory framework called Basel III published in the aftermaths of the subprime financial crisis ([BCBS, 2010]), the Basel Committee points out that they have not only strengthened the classical microprudential regulatory framework (essentially the so-called risk-based capital ratio) but also introduced *a number of macroprudential elements into the capital framework to help contain systemic risks* such as the G-SIB buffer, which concerns institutions classified as systemic institutions (GSIBs) by the Financial Stability Board[1], defined as a capital surcharge (buffer) which depends upon the "systemicness" of the bank based on five public indicators such as size, interconnectedness or complexity[2].

The microprudential regulation focuses on the resilience of depositary institutions (i.e., banks) and its basic aim is to protect depositors by mitigating the incentive of banks to take excessive risk due to government-insured deposits ([Freixas et al., 2015], [Hanson et al., 2011]). By definition, the microprudential regulation adopts a partial equilibrium approach, which means that the impact of asset prices and markets on banks failures, something unrelated to government-insured deposits, is outside of its scope. The minimum capital requirement thus is designed as if each bank were isolated from the financial system and a "one size fits all" framework is adopted, that is, the minimum risk-based capital ratio is 8% for each bank.

The macroprudential regulation adopts a complementary point of view since it focuses on the resilience of the financial system as a whole and its aim is to safeguard it, that is, to ensure the resilience of the financial system to adverse shocks[3]. By definition, the macroprudential regulation adopts a general equilibrium approach, which means that the impact on asset prices due to fire sales on banks failures–negative externalities–falls explicitly within its scope ([Claessens, 2014], [De Nicoló et al., 2012], [Freixas et al., 2015] chapter 9, [Hanson et al., 2011]). While the Basel Committee notes that "greater resilience at the individual bank level reduces the risk of system-wide shocks" ([BCBS, 2010]), solvency of individual depositary institutions does however in general *not imply* the stability of the financial system as a whole (e.g., [Freixas et al., 2015], [Hanson et al., 2011]). As is well-known from the general theory of systems ([Bertalanffy, 1968]), the stability of a system, be it a social, physical, biological etc... critically depends upon the way its elements (agents, particles, cells) interact. For a financial system, its stability property depends thus upon the network of interconnection between financial institutions ([Acemoglu et al., 2015], [Allen and Gale, 2000], [Capponi and Larsson, 2015], [Elliott et al., 2014]), but also upon the various market imperfections (e.g., [Shleifer and Vishny, 2011], [Krishnamurthy, 2010]).

When financial markets are not perfectly competitive, i.e., perfectly liquid to use financial terminology, as in an oligopoly situation ([Vives, 2001]), a financial institution such as a large bank or a large hedge fund may have a positive price impact when selling large quantities of assets and this may result in a lower price. As opposed to other market participants such as hedge funds, banks are particular in that they are heavily regulated and must comply at all times with a minimum

[1] See https://www.fsb.org/wp-content/uploads/P231121.pdf for the list of G-SIBs as of November, 2021.

[2] For European banks, see the European Banking Authority website, https://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions.

[3] As noted in [Hanson et al., 2011], the macroprudential regulation is not related to the existence or not of deposit insurance and thus needs not be restricted to depositary institutions only.

regulatory capital ratio. After an adverse shock, when a given bank does not anymore comply with its regulatory capital requirements, a simple way to restore its capital ratio consists in selling a portion of its assets (deleveraging), possibly at a dislocated price, something called fire sales in finance ([Shleifer and Vishny, 2011]). When more than one bank delever because they are hit with the same adverse shock, at the aggregate level–the financial system–this may lead to an emergent effect called *generalized asset shrinkage* ([Hanson et al., 2011]). As recalled in the foreword of their early paper devoted to the fundamental principles of financial regulations, [Brunnermeier et al., 2009] observe that

> "In trying to make themselves safer, banks, and other highly leveraged financial intermediaries, can behave in a way that collectively undermines the system. Selling an asset when the price of risk increases, is a prudent response from the perspective of an individual bank. But if many banks act in this way, the asset price will collapse, forcing institutions to take yet further steps to rectify the situation." ([Brunnermeier et al., 2009])

Such a phenomenon occurs when many financial institutions such as banks sell a common asset at the same time so that the price of this asset will decrease (fire sales effect). As a result, they will be forced to sell more assets to restore their capital ratios and this will ultimately lead to a running for the exit– a kind of death spiral–observed in August 2007 in which the effect has been *disproportionally larger* than the initial shock ([Pedersen, 2009]). In the stationary state, a number of institutions may be insolvent not because of the initial shock, but because the (equilibrium) price is (much) lower than the price right after the initial exogenous shock ([Braouezec and Wagalath, 2019]). Such a contagion of failures is usually called price-mediated contagion and played an important role in the great financial crisis of 2008 ([Brunnermeier, 2009], [Clerc et al., 2016]). A number of papers document empirically such fire (forced) sales. In [Ellul et al., 2011], the authors document forced sales of corporate bond by insurance companies while [Merrill et al., 2021] document a similar effect for the case of RMBS markets. In the same vein, [Chernenko and Sunderam, 2020] provide empirical evidence of fire sales externalities in the equity mutual fund industry (but see [Choi et al., 2020]).

For banks or insurance companies, it is actually the regulatory constraint together with fair value accounting which leads to forced sales and thus to the (possible) destabilization of the financial through a generalized asset shrinkage, an emergent effect called negative externality and which is *not* taken into account by banks when they delever. The very foundation of macroprudential regulation precisely lies in the correction of these market imperfections–negative externalities– that give rise to systemic risk, in particular externalities related to fire sales ([Freixas et al., 2015], [Hanson et al., 2011]). According to [De Nicoló et al., 2012] and [Claessens, 2014], for the case of externalities related to fire sales, they suggest that they can be addressed either by capital surcharges, liquidity requirements, activities restriction or taxation (see table 1 in [De Nicoló et al., 2012]). Regulators also acknowledge this externality problem posed by large banks (GSIBs) and tackle it, as already discussed, through a GSIB-dependent capital surcharge (buffer) based on quantitative indicators.

> "The selected indicators are chosen to reflect the different aspects of what generates negative externalities and makes a bank critical for the stability of the financial system." (Basel Framework[4], p 17)

[4]See https://www.bis.org/basel-framework.

While such a methodology to design the GSIBs buffer is an interesting novelty[5] of Basel III, it remains unclear from a pure theoretical point of view in what sense the Basel Committee methodology takes into account these negative externalities with the GSIBs buffer. Indicators such as size, interconnectedness or complexity may give information about the possible magnitude of the externality but are not *per se* indicators to address the externality problem.

In this paper, to the best of our knowledge, we are the first to offer a theoretical strategic framework in which we explicitly address the externality problem posed by banks through macro-prudential constraints (or regulation). We consider a fire sales oligopoly model of assets in the spirit of [Braouezec and Wagalath, 2019] but different from [Eisenbach and Phelan, 2022] in which banks compete à la Cournot, that is, through quantities (of assets) sold. In a Cournot oligopoly, and contrary to perfect competitive markets, each bank has a positive impact on the price of a given asset and thus is not a price taker. Our Cournot fire sales approach is novel in three aspects, and these novelties can be summarized as follows.

1. We take into account the solvency constraint of each bank (capital ratio) and this leads to a Cournot oligopoly which is more complex than the classical ones.
2. We consider both micro and macroprudential regulation. Under macroprudential regulation, each bank must take into account the capital ratio of all the other banks and not only its own capital ratio, as in microprudential regulation.
3. We show that under macroprudential regulation, there generically exists a Nash equilibrium that minimizes the total value of the asset sale. Such a result has no equivalent under microprudential regulation.

Due to banking regulation, each bank is constrained to sell a quantity of assets subject to a *solvency constraint*, that is, the deleveraging strategy must be chosen such that its Tier 1 capital ratio is greater than the minimum required, at least 8.5% in Basel III since the capital conservation buffer is 2.5%. Through the price impact, the solvency constraint of each bank turns out to explicitly depend upon the deleveraging decisions of all the other banks. Such a strategic situation, more complex than the classical one in Cournot oligopoly gives rise to a generalized game (see [Facchinei and Kanzow, 2010] or [Fischer et al., 2014] for review papers). As opposed to most game theoretic framework encountered in Economics, in a generalized game, the strategy set of a given bank, its solvency constraint, is *not invariant* with respect to the decisions of the other banks and this feature complicates the analysis. Our framework not only allows us to consider the classical approach to microprudential regulation in which each bank takes into account its own solvency constraint but, and more importantly, it also allows us to consider a macroprudential approach designed to prevent the fire sales externalities (i.e., price-mediated contagion) by explicitly constraining the way banks delever. Quite surprisingly, the analysis of the strategic interaction is easier under macroprudential regulation than under microprudential regulation.

Let us now explain in what sense macroprudential regulation prevents fire sales externalities and consider the point of view of given bank i. Given what banks $j \neq i$ are liquidating, bank i is not

[5] But see [Benoit et al., 2019] for concerns.

allowed to choose a deleveraging strategy such that one (or more than one) bank j would not comply with its regulatory constraint. Put it differently, when a given bank chooses its deleveraging strategy, given what the others are liquidating, it must not only consider its own regulatory constraint but also the regulatory constraint of all the other banks of the banking system. As a result, bank i may not be in a position to choose the cheapest delevaraging strategy because such a strategy might induce the failure of at least one bank. Bank i thus must choose a more expensive deleveraging strategy in order to avoid the failure of some banks, that is, bank i is explicitly constrained to internalize the externality it generates.

The organization of the paper is as follows. In the second section, we discuss the link of our paper with the various strands of literature. In the third section, we present the Cournot fire sale oligopoly while the fourth and fifth section are devoted to the microprudential and the macroprudential analysis. In the sixth section, we apply our model to the French systemic banks and calibrate our model to public data.

2 Related literature

The present paper, which deals with fire externalities and macroprudential regulation, is related to several strands of literature in Economics and Finance. For brevity, we do not discuss, in general, the contribution of the papers.

Stress testing banks. Since the financial crisis of 2007-2008, in Europe as in USA, public authority bodies such as the European Banking Authority and the European Central Bank or the Federal Reserve now implement on a regular basis regulatory stress tests under different scenarios. However, a striking feature of these regulatory stress tests is the *static balance sheet assumption*, that is, banks hit with an adverse shock are not allowed to react even if their capital ratio falls below the required minimum (see [Goldstein, 2017] a lucid criticism of regulatory stress tests). On the contrary, the academic literature on the subject explicitly considers such a reaction, fundamental if one wants to forecast systemic risk. An important paper on the subject is [Greenwood et al., 2015] (see also [Braouezec and Wagalath, 2019], [Cont and Schaanning, 2016], [Duarte and Eisenbach, 2021] for related models) in which they explicitly consider one round of deleveraging after an adverse shock. An interesting aspect of their model (as the framework offered in this paper) is that it is easy to calibrate to data and they offer an empirical analysis using data from the European Banking Authority.

Fire sales and pecuniary externalities in finance. In addition to non-market interdependence, the usual definition of an externality in Economics, [Scitovsky, 1954] notes that the concept of external economies also includes interdependence among firms (e.g., banks) through the *market mechanism*, called pecuniary externality. Such a concept of pecuniary externality has now been extensively applied in Economics (e.g., [Greenwald and Stiglitz, 1986]). In finance, when banks are hit with an adverse common shock and sell a portion of their assets at the same time to (try to) restore their capital ratios, this leads to fire sales, an example of *pecuniary externality* and ultimately to a running for the exist ([Pedersen, 2009]), a kind of death spiral. In the recent literature on the subject, see for instance [Bichuch and Feinstein, 2019], [Bichuch and Feinstein, 2022], [Braouezec and Wagalath, 2019], [Caballero and Simsek, 2013], [Chernenko and Sunderam, 2020], [Duarte and Eisenbach, 2021], [Jeanne and Korinek, 2020], [Kara and Ozsoy, 2020], [Kuong, 2021] to cite few papers, all offer a framework in which the concept of pecuniary externality–fire sales– plays a central role. In a recent paper [Eisenbach and Phelan, 2022] consider, as we do here, a strategic model of fire sales called "Cournot fire sales" in which banks that face a liquidity (or productivity) shock need to delever. However, the authors do not consider capital ratios, which leads them to analyze a standard Cournot oligopoly model, that is, without solvency constraint.

Macroprudential regulation as a tool to mitigate pecuniary externalities. The literature on macroprudential is now abundant, both in classical academic journals in Economics and Finance but also in policy-oriented journals such as IMF publication ([Brockmeijer et al., 2011]), BIS papers ([BIS, 2016]), ESRB papers ([(ESRB), 2014]). In [Brockmeijer et al., 2011], they recall that the objective of macroprudential policy is to limit the build-up of systemic risk and offer an interesting distinction between two types of macroprudential instruments, those that address the *time dimension* of systemic risk (which reflects to its buid-up over time) and those that address its *cross-sectional dimension* (which reflects the distribution of risk at one point in time). In [De Nicoló et al., 2012] and in [Claessens, 2014], they implicitly adopt the cross-sectional point of view and consider a number of tools designed to address the fire sales externalities problem; liquidity requirements, capital surcharges (currently implemented in Basel III, see [Benoit et al., 2019] for an appraisal), taxation and restrictions on activities. Both [De Nicoló et al., 2012] and in [Claessens, 2014] share the point of view that the foundation of macroprudential policy lies in the correction of the market failures–externalities–that gives rise to systemic risk.

Generalized games. First developed by [Arrow and Debreu, 1954] under the terminology "abstract economy" (see [Harker, 1991], [Facchinei and Kanzow, 2010] or [Fischer et al., 2014]), in a generalized game, interaction between agents occurs not only through the payoff function but also through their strategy sets. In Economics, most game theoretic oligopoly à la Cournot are indeed not generalized games since each firm chooses a strategy in an *exogenous strategy set* which by assumption does not depend upon the choice of the other firms (e.g., [Vives, 2001], [Tirole, 1988], 5.4, see also [Ruffin, 1971] for an analysis of the competitive limit).

In a generalized game, the condition under which a Nash equilibrium exists are particularly strong (e.g., [Ichiishi, 1983], see also [Dutang, 2013] for a nice review). To somehow circumvent this existence problem, in an influential paper, [Rosen, 1965] introduced the notion of a *shared constraint*. Under a shared constraint, each agent i, given what the others do, denoted x_{-i}, is constrained to pick a strategy x_i such that the profile of strategies (x_i, x_{-i}) lies in an exogenous set S called the shared constraint. [Rosen, 1965] shows that a Nash equilibrium with shared constraint exists under standard conditions (but see [Tóbiás, 2020]) and it is now very common in the generalized games literature to directly start with a shared constraint. However, as observed in [Braouezec and Kiani, 2021a], the "micro foundation" of the shared constraint is missing, that is, the set S is simply postulated without any reference to the basic individual constraints. We shall here follow [Braouezec and Kiani, 2021a] and consider the *endogenous shared constraint*, which is the shared constraint that results from individual constraints. Within our financial model, the individual constraint called microprudential constraint is the solvency constraint of a given bank while the endogenous shared constraint called macroprudential constraint is solvency constraint of all banks. In the second case, each bank considers the constraint of all banks.

3 The Cournot fire sales oligopoly framework

In classic oligopoly models à la Cournot (e.g. [Tirole, 1988], [Vives, 2001]), the multi-product firm i offers a quantity $q_i = (q_{i,1}, ..., q_{i,n})$. In the simplest case of a mono-product firm ($q_i \in \mathbb{R}^+$) in which firms compete à la Cournot, the price of the good is a function of the sum of the quantities offered and is an example of an aggregative game (e.g. [Nocke and Schutz, 2018]). A striking feature of the classic Cournot oligopoly, as in most game theoretic frameworks used in Economics, is that the strategy set of each firm does *not depend* upon the decisions of the other firms. On the contrary, within our framework in which firms are regulated banks, the solvency constraint of a given bank i explicitly depends upon the quantities of assets sold by the other banks through the price mechanism, which gives rise to a negative pecuniary externality. We consider a banking system at a given time t and we assume that each bank complies with its regulatory ratio(s).

3.1 Banks' balance-sheets and regulatory constraints

We consider a banking system $B = \{1, 2, ..., p\}$ with $p \geq 2$ banks where each bank $i \in B$ invests in two types of risky assets; assets subject to credit risk (loans) and assets subject to market risk (traded securities). Banks may also invest a riskless asset, cash, which represents the value of the bank account of that bank at the Central bank.

Consider the asset side of a given bank i and let V_{i0} be the value of the loans (asset 0) and let $V_{ij} := P_j \times q_{ij} \geq 0$ be the value (in currency) of each risky asset $j \in \{1, 2, ..., n\}$, where q_{ij} is the quantity (in shares) of risky asset j held by bank i and P_j is the price (or value) of the risky asset j at a given date t. Cash is denoted $v_i > 0$. Regarding now the liabilities, let D_i be the sum of the value of deposits and/or debt securities. The balance-sheet of the bank i at time t is as follows.

Balance-sheet of bank i at time t

Assets	Liabilities
Cash: v_i Non-traded assets: V_{i0} Traded assets: $\sum_{j=1}^{n} q_{ij} P_j$	Debt: D_i Equity: E_i
$A_i = v_i + V_{i0} + \sum_{j=1}^{n} q_{ij} P_j$	$E_i + D_i$

In general, loans (e.g., long-term consumers loans subject to credit risk) are illiquid contracts so that their resale value in the short-term is close to zero due to the so-called adverse selection problem. On the contrary, the risky assets subject to market or to counterparty risk (e.g., traded securities such as stocks, ETF, bonds, vanilla derivatives) can be resold in the short-term depending on their market liquidity. Some securities might be very liquid while others might be less liquid. By definition, the total value of the assets at time t is equal to the total value of the liabilities.

$$A_i = v_i + V_{i0} + \sum_{j=1}^{n} q_{ij} P_j = E_i + D_i$$

Assumption 1 *Banks are not directly interconnected through contractual obligation.*

In practice, banks are interconnected through a number of financial debt contracts such as derivatives, repurchase agreements or (long term) bonds. Unfortunately, this network of interconnections can not be retrieved from the observation of the annual reports of each bank. Assuming no contractual obligation between banks means that the debt of a given bank (liability) is either entirely composed with deposits or held by outside investors such as households or non-banks entities. As a result, a given bank A can not *directly* fail because bank B fails, that is, direct contagion (of default) can not arise. Considering such a network of interconnections would actually reinforce the contagion effects analyzed in this paper. From the above balance sheet, the value of equity (or capital) at time t of an operating (i.e., non-failed) bank i is equal to

$$E_i := A_i - D_i = v_i + V_i + \sum_{j=1}^{n} q_{ij} P_j - D_i > 0 \tag{1}$$

and is positive by assumption. Let $\alpha_{ij} > 0$ be the regulatory risk weight of bank i associated to risky asset j. By definition, abstracting operational risk, the total risk-weighted assets of bank i is equal to

$$\mathrm{RWA}_i = \alpha_{i0} V_{i0} + \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j \tag{2}$$

so that the global risk-based capital ratio (RBC) of that bank at time t is equal to

$$\theta_{i,t} := \frac{E_i}{\mathrm{RWA}_i} \tag{3}$$

From Basel III, the total value of equity of bank i at time t (ignoring regulatory adjustments) is equal to Tier 1, capital (going-concern capital) plus Tier 2 capital (gone-concern capital) and the minimum capital ratio is now bank-dependent in that it depends upon the activity of the bank. In this paper, we shall focus on Tier 1 capital ratio and $\theta_{i,t,min}$ denotes this minimum capital ratio at time t. For simplicity, we drop the time index and simply denote it $\theta_{i,min}$. By assumption

$$\theta_{i,t} \geq \theta_{i,min} \quad \text{for each } i = 1, 2, ..., p \tag{4}$$

For simplicity, we shall assume that each bank has only Tier 1 capital, that is, $E_i := \text{Tier } 1_i$ so that the global risk-based capital ratio θ_i is a Tier 1 capital ratio.

3.2 Impact of an exogenous shock on banks' capital ratios

The timing of our model is as follows.

- At time t^+, assets are hit with an adverse shock and banks that do not anymore comply with their regulatory capital ratio sell a portion of their assets.
- At time $t + 1$, equilibrium prices are disclosed and observed.

Banks that are required to react at time t^+ sell a portion of their assets by minimizing the value of the asset sale. More precisely, banks use the known price of each asset at time t^+ to choose their deleveraging strategy in order to be solvent at time $t + 1$. Since no decisions are taken between time t^+ and $t + 1$, the game is static. Throughout this section, we present and discuss the main assumptions of our Cournot fire sales oligopoly model.

Assume that a shock on all risky assets occurs at date t^+ and denote $\Delta = (\Delta_1, ..., \Delta_n) \in [0,1]^n$ the adverse shock vector, where Δ_j represents the size of the shock in percentage of P_j. The price (value) of risky asset j at time t^+ thus is equal to

$$P_{j,t^+} = P_j(1 - \Delta_j) \qquad j = 0, 1, ..., n \tag{5}$$

Let $A^+_{i,t} = v_i + V_{i0}(1 - \Delta_0) + \sum_{j=1}^n q_{ij} P_j (1 - \Delta_j)$ be the value of the asset after the shock. The risk-based capital thus is equal to

$$\theta_{i,t^+}(\Delta) = \frac{\max\{A_{i,t^+} - D_i; 0\}}{\text{RWA}_{i,t^+}} = \frac{\max\{E_{i,t} - V_{i0}\Delta_0 - \sum_{j=1}^n q_{ij}P_j\Delta_j; 0\}}{\alpha_{i0}V_{i0}(1-\Delta_0) + \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)} \tag{6}$$

Let us now define the three following sets

$$\mathcal{Z}_i^{\emptyset} := \{\Delta \in [0,1]^n : \ \theta_{i,t^+}(\Delta) \geq \theta_{i,min}\} \qquad \text{(no reaction)} \tag{7}$$

$$\mathcal{Z}_i^{sale} := \{\Delta \in [0,1]^n : E_{i,t^+}(\Delta) > 0 \text{ and } \theta_{i,t^+}(\Delta) < \theta_{i,min}\} \qquad \text{(asset sale)} \tag{8}$$

$$\mathcal{Z}_i^{fail} := \{\Delta \in [0,1]^n : E_{i,t^+}(\Delta) \leq 0\} \qquad \text{(insolvency and liquidation)} \tag{9}$$

and note that they form a partition of $[0,1]^n$. Of particular interest throughout this paper will be the sets $\mathcal{Z}_i^{sale}$ and $\mathcal{Z}_i^{fail}$. The following fact follows directly from equations (8) and (9).

Fact 1 *For each bank i, the critical sets $\mathcal{Z}_i^{sale}$ and $\mathcal{Z}_i^{fail}$ defined in equations (8) and (9) can be written as follows:*

$$\mathcal{Z}_i^{sale} = \{\Delta \in [0,1]^n : E_i - \theta_{min}\sum_{j=0}^n \alpha_{ij}q_{ij}P_j < \sum_{j=0}^n q_{ij}P_j\Delta_j(1 - \alpha_{ij}\theta_{i,min})\} \tag{10}$$

$$\mathcal{Z}_i^{fail} = \{\Delta \in [0,1]^n : E_i \leq \sum_{j=0}^n q_{ij}P_j\Delta_j\} \tag{11}$$

We shall adopt throughout the paper the following terminology.

Definition 1 *The shock Δ is said to be*

- *small to medium if, for each bank $i \in \mathcal{B}$, $E_i(\Delta) > 0$ but there exists at least one bank i' such that $\theta_j(\Delta) < \theta_{j,min}$, i.e., $\Delta \in \mathcal{Z}_{i'}^{sale}$.*
- *severe if there exists at least one bank $i \in \mathcal{B}$ such that $\theta_i(\Delta) = 0$, i.e., $\Delta \in \mathcal{Z}_i^{fail}$.*

3.3 Deleveraging strategies, market liquidity and endogenous price impact

Since Δ is a common shock, it affects the balance-sheet of *all* banks that hold risky assets and may leave some of them undercapitalized, possibly insolvent. As observed in [Cohen and Scatigna, 2016], there are various channels of adjustment that can be used by an undercapitalized bank to restore its capital ratio. From a regulatory point of view, the best channel is clearly equity issuance although it is frequently considered as the most expensive one. Moreover, issuing new equity takes time, typically several months, which is certainly too long for undercapitalized banks. After an adverse shock, e.g., an event comparable to the failure of Lehman Brothers in 2008, banks liquidate a portion of their assets but do not issue equity ([Brunnermeier and Oehmke, 2014, Cifuentes et al., 2005, Greenlaw et al., 2012, Greenwood et al., 2015]). In the short-run, banks typically use one of the two following deleveraging strategies to reshuffle their assets side[6].

1. Asset shrinking strategy.

2. Risk-reduction strategy

In both cases, the bank will sell a portion of its risky assets. However, in the first case, the bank will use the proceeds to repay a portion of its debt while in the second case, it will use it to invest in cash. As long as markets are perfectly liquid (i.e., perfectly competitive), whether the bank uses the asset shrinking strategy or the risk reduction strategy, its capital ratio will increase since the capital remains constant while the risk-weighted asset decrease. A number of recent papers (e.g., [Gropp et al., 2019], [Juelsrud and Wold, 2020]) document empirically that banks tend to decrease their risk-weighted assets in order to increase their risk-based capital ratio. In what follows, since we essentially focus on the risk-based capital ratio, we make the assumption that undercapitalized (but solvent) banks sell a portion of their risky assets and invest the proceeds in cash in order to (try to) restore their capital ratio. We shall discuss later on the situation in which the bank is required to manage more than one capital ratio.

Assumption 2 *Banks that do not comply with their regulatory risk-based capital ratio reshuffle their risk-weighted assets; they sell a portion of their risky assets and invest the proceeds in the riskless asset (cash).*

Let $x_{ij} \in [0,1]$ be the proportion of risky assets j sold by bank i in reaction to the shock vector Δ at date t^+ and let

$$X_j := \sum_{i \in B} x_{ij} q_{ij} \tag{12}$$

be the total quantity of assets j sold by all banks. Note that $Q_j := \sum_{i \in B} q_{ij}$ is the maximum quantity of asset j that can be liquidated so that $X_j \leq Q_j$. As in [Banerjee and Feinstein, 2021] among others, we consider of a fairly general price impact function denoted $I_j(.)$ that encompasses various price impact functions such as the linear or exponential ones but that only depends upon the quantity of asset j liquidated, that is, there is no cross price impact.

Assumption 3 *The price of the (risky) marketable asset $j = 1, ..., n$ at time $t+1$ is equal to*

$$P_{j,t+1}(\Delta_j, X_j) = \underbrace{P_j \times (1 - \Delta_j)}_{P_{j,t^+}} \times I_j(X_j) \tag{13}$$

where the (price) impact function I_j is once again a twice continuously differentiable and decreasing function of X_j such that

$$I_j(0) = 1 \text{ and } I_j(Q_j) \leq 1 \tag{14}$$

[6] To the best of our knowledge, [Braouezec and Kiani, 2021b] is the unique formal (optimization based) model in which a bank can both issue new equity and/or liquidate assets in order to reach a target capital ratio from the current one.

Without loss of financial generality, we assume that the price impact function is a regular function (continuously differentiable).

Balance-sheet of bank i at date $t+1$ after deleveraging

Assets	Liabilities
Cash: $v_i + \sum_{j=1}^{n} x_{ij} P_{j,t+1}(.) q_{ij}$ Non-tradable assets: $V_{i0}(1-\Delta_0)$ Tradable assets: $\sum_{j=1}^{n}(1-x_{ij}) P_{j,t+1}(.) q_{ij}$	Debt: D_i Equity: $E_{i,t+1}$
$A_{i,t+1} = v_i + V_{i0} + \sum_{j=1}^{n} P_{j,t+1}(.) q_{ij}$	$E_{i,t+1} + D_i$

It will be convenient to write the impact function as follows.

$$I_j(X_j) = 1 - \xi_j(X_j) \tag{15}$$

where $\xi_j(X_j)$ is a continuously differentiable and increasing function consistent with equation (14).

Remark 1 *When $\xi_j(X_j) = a_j X_j$, we wil say as usual that the price impact is linear.*

After the adverse shock Δ_j but before any liquidation, the price of asset j is equal to $P_j^{\text{before}} = P_j(1-\Delta_j)$. When the price impact is linear, after liquidation, the price denoted P_j^{after} is equal to

$$P_j^{\text{after}} := P_j^{\text{before}} \times \left(1 - \frac{\sum_{i\in\mathcal{B}} x_{i,j} q_{i,j}}{\Phi_j}\right) \tag{16}$$

where Φ_j is called a *market depth* and measures the degree of liquidity of the market (of security) j. The greater the market depth, the greater the liquidity of security j. When $\Phi_j = \infty$, the market is *perfectly liquid* in that each market participant considers the price as exogenous. Throughout the paper, we assume that $\Phi_j > Q_j$ where $Q_j = \sum_{i\in\mathcal{B}} q_{i,j}$.

3.4 The deleveraging problem yields a Cournot fire sale oligopoly

Recall that within our framework, loans (i.e., asset 0) are non-tradable assets, that is, they are perfectly illiquid with no resale value. After the shock, banks can only resell assets of the trading book, that is, assets $j = 1, 2, ..., n$. Let

$$\mathcal{E}_i := [0,1]^n \quad \text{and} \quad \mathcal{E} = \prod_i \mathcal{E}_i \tag{17}$$

be respectively the set of liquidation strategies of bank $i \in \mathcal{B}$ and let the set of liquidation strategies of the overall banking system. As usual in game theory, let $x = (x_i, x_{-i}) \in \mathcal{E}_i \times \mathcal{E}_{-i}$ where $\mathcal{E}_{-i} := [0,1]^{n(p-1)}$ is a $n(p-1)$-dimensional vector and assume that x_{-i} is known to bank i. Let $V_{ij} = P_j \times q_{ij}$ be the value of asset j at time t for bank i, i.e., before the shock. Using equations (13) and (15), it is not difficult to show that, for a liquidation strategy x, the total capital of bank i is equal to

$$E_{i,t+1}(\Delta, x_i, x_{-i}) = \max\left\{ E_{i,t} - V_{i0}\Delta_0 - \sum_{j=1}^{n} V_{ij} \times [\Delta_j + \xi_j(X_j)(1-\Delta_j)]\,; 0 \right\} \tag{18}$$

and note that the capital of each bank i depends upon the overall vector of liquidation $x \in \mathcal{E}$, which means that the deleveraging problem is a *strategic problem.* The total capital at time $t+1$ can be decomposed in three different terms, the initial capital of the bank $E_{i,t}$, the depletion of this capital due to an *exogenous* shock Δ_j and the further depletion of this capital due to the *endogenous* banks' reaction which involves the impact function $\xi_j(X_j)$. From the balance sheet, it is easy to see that for a given liquidation strategy x, the risk-weighted assets are equal to

$$\text{RWA}_{i,t+1}(\Delta, x_i, x_{-i}) = \alpha_{i0}V_{i0}(1-\Delta_0) + \sum_{j=1}^{n} \alpha_{ij}V_{ij} \times (1-\Delta_j)\left(1-\xi_j(X_j)\right)(1-x_{ij}) \tag{19}$$

so that the regulatory capital ratio of bank i at time $t+1$ is equal to

$$\theta_{i,t+1}(\Delta, x_i, x_{-i}) = \frac{E_{i,t+1}(\Delta, x_i, x_{-i})}{\text{RWA}_{i,t+1}(\Delta, x_i, x_{-i})} \tag{20}$$

Note that we adopt the natural convention that $\theta_{i,t+1}(\Delta, x_i, x_{-i}) = 0$ when $(x_{i1}, x_{i2}, ..., x_{in}) = (1, 1, ..., 1)$ since $E_{i,t+1} = 0$ (i.e., when bank i is insolvent after the deleveraging process).

Fact 2 *For a given x_{-i}, if for each $i \in B$, $\alpha_{i0}V_{i0}(1-\Delta_0) > 0$, then, the risk-based capital ratio of each bank i as defined in equation (20) is a continuous function on the set $[0,1]^{np}$ and there exists $x_i^{max} \in [0,1]^n$ such that the function θ_i reaches its maximal value*

$$\theta_i^{max} := \sup_{x_i \in [0,1]^n} \theta_i(x_i, x_{-i}) \tag{21}$$

which may be higher or lower than $\theta_{i,min}$.

This continuity property follows from the fact that the price impact function $\xi_j(X_j)$ is assumed to be continuously differentiable for each j and the existence of θ_i^{max} follows from Weierstrass extreme value theorem. Due to the price impact, everything else equal, the numerator of the capital ratio of bank i given by equation (20) is a decreasing function of x_{ij} while its denominator may or may not be a decreasing function of x_{ij}. For the denominator of the capital ratio (i.e., the RWA) to be a decreasing function of x_{ij}, the function $h(x_{ij}) := (1-\xi_j(X_j))(1-x_{ij})$ should be decreasing, that is, $h'(x_{ij}) < 0$. It is easy to see that this depends upon the magnitude of the positive term $x_{ij}\frac{d\xi_j(X_j}{dx_{ij}}$. Assuming even that the RWA is a decreasing function, since the capital is also a decreasing function of x_{ij}, the capital ratio needs not be an increasing function of x_{ij}. We shall come back to that point later on.

Following [Braouezec and Wagalath, 2019], we now introduce the *implied shock* for asset j, that is, the shock which is implied after the liquidation process. This implied shock denoted $\Delta_j(X_j)$ for the asset j is found by solving

$$P_{j,t+1}(\Delta_j, X_j) = P_j \times (1 - \Delta_j(X_j)) \tag{22}$$

and it is easy to show, using equations (13) and (15), that this implied shock for asset j is equal to

$$\Delta_j(X_j) := \Delta_j + \xi_j(X_j)(1-\Delta_j) \tag{23}$$

As long as $X_j \neq 0$, $\Delta_j(X_j) > \Delta_j$ so that the rebalancing process of each asset j at date $t+1$ actually *reinforces* the under performance of asset j caused by the initial shock Δ_j at date t^+. It is precisely because $\Delta_j(X_j)$, an endogenous quantity, is greater than Δ_j that there may be additional failures after the deleveraging process.

Remark 2 *Considering direct contagion within our framework (e.g., [Glasserman and Young, 2016], [Jackson and Pernoud, 2021] for review papers) would actually reinforce indirect contagion.*

Let $f_i(x_i)$ be the total value of the assets sold by bank i at time t^+ in order to restore its capital ratio. For a given bank i, as long as $x_{ij} > 0$, this cost is equal to

$$f_i(x_i) = \sum_{j=1}^{n} x_{ij} q_{ij} \underbrace{P_j(1-\Delta_j)}_{=P_{jt^+}} \tag{24}$$

We make the fairly natural assumption that each bank i tries to minimize the total value of asset sold using the *known prices* of time t^+ subject to the constraint that the capital will be greater than the minimum required at time $t+1$, when the price of each asset j will be equal to $P_{j,t+1}(\Delta_j, X_j)$.

Assumption 4 *Given the shock Δ and what the other banks liquidate, i.e., $x_{-i} \in [0,1]^{n(p-1)}$, each bank i such that $\theta_{i,t^+}(\Delta) < \theta_{min}$ must solve the following constrained optimization problem*

$$\min_{x_i \in [0,1]^n} f_i(x_i) \tag{25}$$

$$\text{subject to } \theta_{i,t+1}(x_i, x_{-i}, \Delta) \geq \theta_{i,min} \tag{26}$$

We do not exclude the vector $x_i = (1, 1, ..., 1)$ which is the situation in which bank i would have to liquidate 100% of its trading book assets, that is, assets $j = 1, 2, ..., n$. Note importantly that given the shock Δ and x_{-i}, the set of solutions of the optimization problem (given by equations (25) and (26)) may be empty. In such a case, the game is *undefined*. From a financial point of view, no solution simply means that bank i is insolvent. We shall allow later on such a possibility of insolvency.

Assumption 5 *Complete information; all the quantities but also the structure of the game are known (indeed common knowledge) to every bank.*

Due to Basel III, banks must now disclose more information than before about their own activity[7] so that it makes sense to assume that each bank is aware of the positions of the other banks in the banking system. Since the structure of the game is itself common knowledge, each bank knows that each bank knows the structure of the game and so on and so forth. As a result, when a given bank i is insolvent after the shock, this failure is known to each bank (i.e., can be perfectly predicted) and this is common knowledge. But the consequences of this failure is also common knowledge. Each bank is perfectly able to predict the consequences of the asset liquidation by bank i on asset prices and the (possible) resulting cascade of failures due to the existence of the price impact (negative fire sales externality).

[7] Systemic banks are constrained to publicly disclose twelve indicators about their activity with the rest of the financial system on the website of the European Banking Authority (EBA). See https://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions.

3.5 Risk-based and non risk-based capital ratios in Basel III

In Basel III, banks must not only comply with the various risk-based capital ratios but also with a non-risk based capital ratio called the leverage ratio defined as Tier 1 capital divided by the total exposure. It turns out that for many European banks, the total value of the assets and the total exposures are very close and differ typically by 5% to 10%, which means that the total value of the assets is a fairly good approximation of the total exposure.

$$L_i \approx \frac{\text{Tier1 capital}}{\text{Total assets}} = \frac{E_i}{A_i} \tag{27}$$

Written as in equation (27), the leverage ratio can be thought of as the particular case of the risk-based ratio when each risk weight is equal to one, including cash. Since the denominator of the leverage ratio incorporates cash, the risk-reduction strategy is now inoperative. To increase its leverage ratio, bank i must adopt an asset shrinking strategy, that is, as already discussed, it must sell a portion of its risky assets and makes use of the proceed to pay back its debt[8]. Everything else equal, with the asset shrinking strategy, both the leverage ratio and the risk-based capital will increase. To see this, assume that the market of asset k is perfectly liquid (no price impact), that is, by selling this asset k at its fair value V_{ik}, the proceeds is equal to V_{ik}. The total value of the assets when asset k has been sold thus is equal to $A_i = \sum_{j \neq k} V_{ij} - V_{ik}$ and the total debt is equal to $D_i - V_{ik}$. Let θ_i and L_i be the risk-based and the leverage ratio before the asset sale.

- The leverage ratio after the sale is now equal to $L_i^{\text{after}} = \frac{E_i}{\sum_{j \neq k} V_{ij} - V_{ik}}$ and is greater than L_i since the denominator has decreased by V_{ik}.
- The risk-based capital ratio after the sale is now equal to $\theta_i^{\text{after}} = \frac{E_i}{\sum_{j \neq k} \alpha_{ij} V_{ij}}$ and is greater than θ_i since the denominator has decreased by $\alpha_{ik} V_{ik}$.

With low price impact, both capital ratios will yet increase. However, when the price impact is significant, depending upon α_{ik} (i.e., whether it is lower or higher than one), the risk-based capital ratio may increase while the leverage decreases or the opposite.

4 Nash Equilibrium analysis under microprudential regulation

4.1 No price impact as a non-strategic problem

The case in which there is no price impact is simple because the problem for each bank is non-strategic. For each $j = 1, 2, ..., n$, the function $\xi_j(.)$ is invariably equal to zero so that, from equation (18) and (19), the capital and the risk-based capital ratio of each bank i depends only upon x_i. For each bank i, the problem reduces to a standard decision problem, that is, $\min_{x_i} f_i(x_i)$ subject to $\theta_i(x_i) \geq \theta_{i,min}$. The next proposition is simple to prove but gives an interesting insight regarding the optimal way to delever when there is no price impact. It also says that depending upon the

[8] This might not always be possible. If a bank is essentially financed with deposits, it is difficult to see how deposits can be repaid...

exposure to loans, the bank might not be able to restore its capital ratio back above the required minimum.

Proposition 1 *Assume no price impact (i.e., for each j, $\mathcal{E}_j(X_j) = 0$ regardless of $X_j \geq 0$) and consider a given bank i such that $\Delta \in \mathcal{Z}_i^{sale}$.*

1. *For each $i \geq 1$ and each $j \geq 1$, the capital ratio $x_{ij} \to \theta_i(x_{ij}, ...)$ is an increasing function of x_{ij}.*
2. *When $V_{i0} = 0$, bank i is always able to restore its capital ratio back above the required minimum and it is optimal to first sell the security k with the highest regulatory weight k (i.e., the security k such that $\alpha_{ik} > \alpha_{ij}$ for $k \neq j$). If the proceeds of the sale is is not enough, it is optimal to sell the asset with next highest risk weight and so on and so forth.*
3. *When $V_{i0} > E_i$, bank i will always be able to restore its capital ratio back above the required minimum through the above optimal deleveraging strategy.*

Proof. See the appendix.

When markets are perfectly liquid (i.e., competitive), the numerator of the capital ratio of each bank is *invariant* with respect to the liquidation decision(s) since each asset can be resold at its current market price, without price impact case, (see equation 18). As a result, the capital ratio of bank i only depends upon its own decision x_i. Since the risk-weighted assets of each asset j decreases with x_{ij}, it thus follows that the capital ratio of bank i increases when x_{ij} increases. When $V_{i0} = 0$, the bank is not exposed to illiquid assets. As a result, by selling an arbitrarily high portion of each asset j, the capital ratio is arbitrarily high since the risk-weighted assets are arbitrarily close to zero. As a result, for a given constraint $\theta_{i,min}$, there exists x_{ij}, $j = 1, 2, ..., n$ possibly close to one) such that $\theta_i(x_{i1}, x_{i2}, ..., x_{in}) = \theta_{i,min}$. While there are possibly many deleveraging strategies to restore its capital ratio, bank i is assumed to choose the cheapest one, which leads to point 2 of the above proposition. To minimize the value of the asset sale, the bank sells first the asset (or security) with the highest risk weight. If this is not enough, it sells the asset with the next highest risk weight and so on and so forth. This liquidation strategy is in sharp contrast with the proportional liquidation rule used in [Greenwood et al., 2015] or in [Cont and Schaanning, 2016] in which the bank sells a proportion of each asset. Such a proportional liquidation strategy is suitable when the bank maintains a non-risk based capital ratio such as the leverage ratio but not a risk-based capital ratio.

4.2 Positive price impact as a generalized game problem

As long as the price impact is positive, from equation equation (26), the capital ratio of each bank i depends upon the decisions of all the other banks. Before discussing the best response properties, let us formulate the capital ratio constraint given in equation (26) as a microprudential (solvency) constraint.

Definition 2 *Given an adverse shock Δ and the liquidation decisions x_{-i} of all banks except bank i, the strategy set that results from the enforcement of the microprudential regulation of bank i defined* *as*

$$X_i(x_{-i}) = \{x_i \in [0,1]^n : \theta_{i,t+1}(x_i, x_{-i}, \Delta) \geq \theta_{i,min}\} \tag{28}$$

is called the microprudential constraint.

Note importantly that the strategy set of bank i denoted X_i for short explicitly depends upon x_{-i} but implicitly depends upon the price impact function $I_1(.), ..., I_n(.)$. It is natural to call the set $X_i(x_{-i})$ the *microprudential constraint* since it is the aim of each bank i to have a capital ratio greater (or equal) than the required minimum. If $X_i(x_{-i})$ is empty, then, bank i is insolvent and must be liquidated, that is, it must sell 100% of its asset, $x_i = (1, 1, ..., 1)$. As long as $X_i(x_{-i})$ is not empty, bank i can restore its capital ratio by choosing a strategy $x_i \in [0,1]^n$ to restore its capital ratio. As we shall see later on, allowing banks to be insolvent raises new issues. The optimization problem given by equations (25) and (26) can now simply be written as

$$\min_{x_i \in [0,1]^n} f_i(x_i) \quad s.t \quad x_i \in X_i(x_{-i}) \tag{29}$$

Within our framework, interaction between banks occurs through the strategy sets but *not* through their objective function.

Fact 3 *For each bank $i \in \mathcal{B}$, since the microprudential constraint $X_i(x_{-i})$ depends upon x_{-i}, it is usual to call $X_i(x_{-i})$ a point-to-set map. As a result, the Cournot fire sales oligopoly defines a generalized game ([Facchinei and Kanzow, 2010], [Fischer et al., 2014] for review papers).*

As already said, generalized games contrast with "classical" games frequently encountered in economic theory (e.g., [Fudenberg and Tirole, 1991], [Moulin, 1986], [Osborne and Rubinstein, 1994]) in which the strategy set $\mathcal{E}_i$ of each agent i does *not* depend upon the decisions of the other agents x_{-i}. In a generalized game, the strategy set of each agent depends upon the decisions of all the other agents and hence is denoted $X_i(x_{-i})$.

Let $BR_i(x_{-i})$ be the best response of bank i. We now provide a geometric characterization of the point-to-set map $X_i(x_{-i})$ when not empty and we show that under linear price impact, the best response is unique.

Lemma 1 *Assume that the price impact is linear for each asset $j \in \{1, 2, ..., n\}$ and consider the situation of bank i for which $X_i(x_{-i}) \neq \emptyset$.*

1. *The point-to set map $X_i(x_{-i})$ is the intersection of a n-dimensional ellipsoid with the unit compact of $\mathbb{R}^n$, $[0,1]^n$.*

2. *The best response $BR_i(x_{-i})$ is either the unique tangency point between the n-dimensional ellipsoid $X_i(x_{-i})$ and a hyperplane or is a corner solution.*

Proof. See the appendix.

When $n = 2$, $X_i(x_{-i}) = \{x_i \in [0,1]^2 : \theta_i(x_i, x_{-i}) \geq \theta_{i,min}\}$ reduces to an *ellipse*. In Fig 1, for simplicity, we make the assumption that the ellipse is contained in the unit square but nothing is changed if we do not make this assumption. Let $\mathcal{F}_{\theta_{i,min}} = \{x_i \in [0,1]^2 : \theta_i(x_i, x_{-i}) = \theta_{i,min}\}$ be a level curve of the capital ratio, that is, all the deleveraging strategies x_i such that the capital ratio is equal to $\theta_{i,min}$. This level curve is depicted in yellow in Fig 1 and is the contour (or the boundary) of the ellipse. Let x_i^{max} be the solution of $\max_{x_i \in X_i(x_{-i})} \theta_i(x_i, x_{-i})$ and note that in general, $\theta_i(x_i^{max}, x_{-i}) := \theta_i^{max} > \theta_{i,min}$. A number of remarks are in order.

1. The point x_i^{max} is (generically) not the center c_i of the ellipse and is such that $x_i^{max} \geq c_i$ component-wise.

2. The two points $x_{i,A}$ and $x_{i,B}$ are located on the same level curve although $x_{i,B} > x_{i,A}$ component-wise.

3. The capital ratio of bank i is not an increasing function of $x_{i,1}$ and $x_{i,2}$. It can be directly seen from Fig 1 that on the north-east of x_i^{max}, the capital ratio of bank i *decreases* when $x_{i,1}$ or $x_{i,2}$ increases.

4. The triangle with a red contour represents the set of point x_i such that the capital of bank i is equal to zero. Depending upon x_{-i} and/or the capitalization of the bank, it may obviously be the case that such a set of points is empty. The equation of the red line follows from the linearity of the capital $E_i(x_i, x_{-i})$ with respect to $x_{i,1}$ and $x_{i,2}$. From equation (18) in the case of linear price impact, it reduces in the two dimensional case to an equation of the form $K - x_{i1}q_{i1} - x_{i2}q_{i2} = 0$ equivalent to $x_{i2} = \frac{-x_{i1}q_{i1}+K}{q_{i2}}$ whose slope is $-\frac{q_{i1}}{q_{i2}} < 0$

Consider point 3. When $x_{i,1}$ and $x_{i,2}$ increase, the risk-weighted assets decrease so that, everything else equal, the capital ratio increases. But everything is not equal. When $x_{i,1}$ and $x_{i,2}$ increase, this also decreases the price of the assets and thus the capital of the bank. By definition, the point $x_{i,A}$ is such that $\theta_i(x_{i,A}, x_{-i}) = \theta_{i,min}$. From Fig 1, since $x_{i,A} < x_i^{max}$ component-wise, by increasing each component of $x_{i,A}$, the capital ratio increases and reaches its maximum in x_i^{max}. Starting from $x_{i,A}$, this means that by selling more of each asset, the drop of the risk-weighted assets is more important than the drop of the capital of the bank so that the capital ratio increases. However, when one starts from the point x_i^{max}, when bank i sells more of each asset, the drop of the capital is more important than the drop of risk-weighted assets so that the capital ratio decreases and this explains point 3. If one continues to increase the sale of each asset after the point $x_{i,B}$, one may reach the region in which the capital of bank i is equal to zero.

Consider now the best response. On Fig 2, the point-to-set map of a given bank $X_i(x_{-i})$ is included in the unit compact of $\mathbb{R}^2$ while it is not on Fig (5). Since the level curves associated to the objective function (i.e., f_i) are lines, in Fig 2, the best response is a tangency point while in Fig 5, it is a corner solution, the best response is located on the boundary of the unit square $[0,1]^2$.

To conclude this paragraph, let us now come back to the multiple ratios situation and let

$$Y_i(x_{-i}) = \{x_i \in [0,1]^n : L_{i,t+1}(x_i, x_{-i}, \Delta) \geq L_{i,min}\} \tag{30}$$

be the microprudential constraint associated with the leverage ratio, where $L_{i,min}$ is the minimum leverage required. Since the leverage ratio is the particular case of the risk-based capital when all the risk weights are equal to one[9] (including cash), the point-to-set map $Y_i(x_{-i})$ also is (when

[9]This is approximately correct when the total exposures is close enough to the total value of the assets.

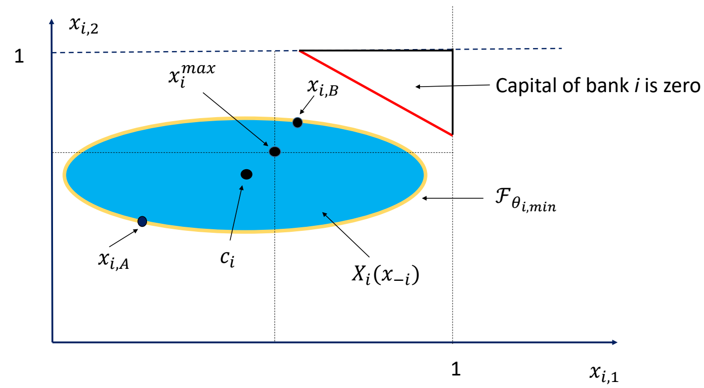

Figure 1: The point-to-set map is an ellipse when the price impact is linear

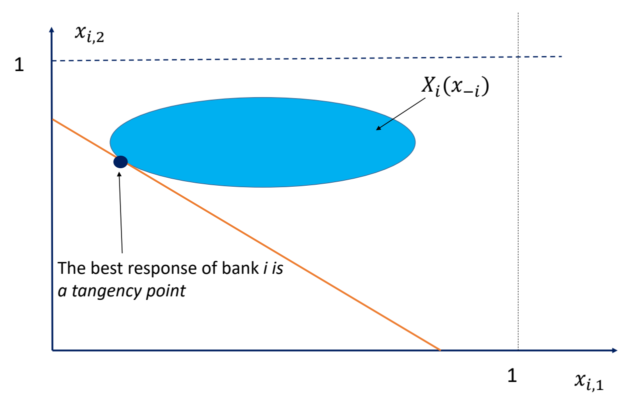

Figure 2: The point-to-set map of a given bank

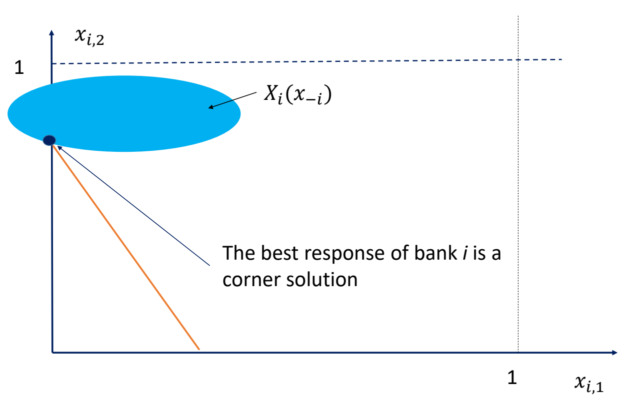

Figure 3: The point-to-set map of a given bank

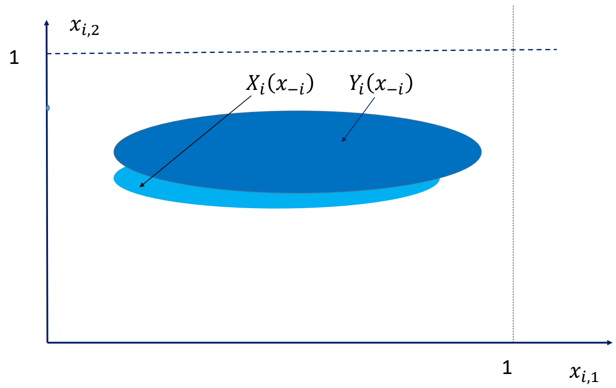

Figure 4: Risk-based capital ratio and leverage ratio

included in the unit compact of $\mathbb{R}^n$) an n-dimensional ellipsoid. Let $H_i(x_{-i}) := X_i(x_{-i}) \bigcap Y_i(x_{-i})$ be the intersection of two n-dimensional ellipsoids and note that H_i is a convex set.

Fact 4 *Assume that price impact is linear and that each bank uses an asset shrinking strategy. When a given bank i manages both the risk-based capital ratio and leverage ratio, then, the overall Tier 1 microprudential constraint results from the intersection of two n-dimensional ellipsoids $H_i(x_{-i})$ with the unit compact of $\mathbb{R}^n$.*

To the best of our knowledge, this is the first paper that offers a description of the management of the two Tier 1 capital ratios in a fairly general framework.

4.3 Existence of a Nash equilibrium under microprudential constraints

Following the terminology introduced in [Braouezec and Kiani, 2021a], let K be a set *admissible strategies* defined as follows

$$K = \{x \in \mathcal{E} : \forall i \in \mathcal{B}, x_i \in X_i(x_{-i})\} \tag{31}$$

and note that if $K = \emptyset$, no equilibrium can exist. The non-vacuity of K thus is necessary for the Nash equilibrium to exist and we shall assume that K is not empty. Given the shock Δ and the price impact functions $I_j(.)$ $j = 1, 2, ..., n$, we make the assumption that the set of admissible strategies K is not empty. Note that when $x \notin K$, the game is *undefined* since at least one bank i does not satisfy its microprudential constraint.

Let $(\mathcal{B}, \mathcal{E}, (\theta_i)_{i\in\mathcal{B}}, (X_i)_{i\in\mathcal{B}})$ define the Cournot fire sale game *with microprudential constraints*. Recall that $\mathcal{E}_i = [0, 1]^n$ and that $\mathcal{E} = \prod_{i\in\mathcal{B}} \mathcal{E}_i$ and note that $K \subset \mathcal{E}$.

Definition 3 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the Cournot fire sale game with microprudential constraints $(\mathcal{B}, \mathcal{E}, (\theta_i)_{i\in\mathcal{B}}, (X_i)_{i\in\mathcal{B}})$ if, for each $i \in \mathcal{B}$ and each $x_i \in \mathcal{E}_i$ such that $x_i \in X_i(x^*_{-i})$, it holds true that $f_i(x^*_i, x^*_{-i}) \leq f_i(x_i, x^*_{-i})$.*

As already said, a necessary but not sufficient for a Nash equilibrium to exist in the game with microprudential constraints is $K \neq \emptyset$. But even when $K \neq \emptyset$, as long as the price impact is positive, the proof of the existence of a Nash equilibrium to the Cournot fire sale game with microprudential constraints $(\mathcal{B}, \mathcal{E}, (\theta_i)_{i\in\mathcal{B}}, (X_i)_{i\in\mathcal{B}})$ remains difficult because virtually nothing is known in general regarding the (topological) properties of the sets $X_i(x_{-i})$. For classical games (e.g., [Dasgupta and Maskin, 1986], theorem 1 and 2), it is well-known that for a Nash equilibrium in pure strategies to exist, the set $\mathcal{E}_i$ must be compact and convex. Within our generalized games framework, for a Nash equilibrium to exist, the point-to-set map $X_i(x_{-i})$ of each bank i must be non-empty, compact and convex *for each* x_{-i} ([Ichiishi, 1983], see the review paper [Dutang, 2013]).

Theorem 1 *([Arrow and Debreu, 1954], [Ichiishi, 1983]) Let $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ be a generalized game with individual constraints and suppose that:*

- *There exists N nonempty convex and compacts sets $C_i \subset \mathbb{R}^{n_i}$ such that for all $x \in \mathbb{R}^n$ with $x_i \in C_i$ for every i, $X_i(x_{-i})$ is nonempty closed and convex, $X_i(x_{-i}) \subset C_i$, and X_i as a point-to-set map, is both upper and lower semi-continuous.*
- *For every player i, the function $\theta_i(, x_{-i})$ is quasiconvex on $X_i(x_{-i})$.*

Then a generalized Nash equilibrium exists.

Coming back to our model, we already know that, when non-empty, $X_i(x_{-i})$ is an ellipsoid when the price impact is linear, so that it is compact and convex. Applying theorem 1 for our model still requires the strategy sets to be non-empty for every x_{-i}. Note also that theorem 1 also requires the point-to-set map $X_i(x_{-i})$ to be both upper and lower semi-continuous, which is a notion of continuity for a set.

Proposition 2 *Assume that the price impact is linear for each asset $j \in \{1, 2, ..., n\}$ and assume that for all i and for all $x_{-i} \in [0,1]^{(p-1)n}$, $X_i(x_{-i})$ is nonempty. Under these assumptions, a generalized Nash equilibrium in microprudential constraints always exist.*

Proof. See the appendix.

The conditions under which a Nash equilibrium exists in microprudential constraints are extremely strong since they require in particular $X_i(x_{-i})$ to be non-empty for all $x_{-i} \in [0,1]^{(p-1)n}$. It requires not only the shock to be small but also the price impact to be low enough. To see this, assume one moment that each bank $k \neq i$ sells 99% of its trading book, that is, $x_k = (0.99, ..., 0.99)$ so that $x_{-i} = ((0.99, ..., 0.99), ..., (0.99, ..., 0.99))$. For the Nash equilibrium to exist in microprudential constraint, given this particular profile of strategies x_{-i}, $X_i(x_{-i})$ must be non-empty for each bank i, which is a very strong assumption. For this profile of strategies, we typically expect bank i to fail due to the price impact but such an insolvency is not yet allowed. Such a possibility will be discussed later on.

5 Nash equilibrium analysis under macroprudential regulation

5.1 How to address externalities related to fire sales ?

According to [Brockmeijer et al., 2011], the aim of the macroprudential regulation (or policy) is to address the two dimensions of systemic risk.

1. The time dimension;
2. The cross-sectional dimension.

The time dimension reflects the procyclical effect that operates over time within the financial system and the real economy. [Brockmeijer et al., 2011] observe that during the boom phase, procyclicality induces excessive leverage from financial institutions and this build up of aggregate risk increases the chance of financial distress. In that sense, the leverage ratio, which by definition limits the leverage of banks, is a time dimension macroprudential tool.

The cross-sectional dimension reflects the distribution of risk in the financial system at a given point in time and depends upon the links financial institutions may have (e.g., through contractual obligations, through identical exposures) but also, as noted in [Brockmeijer et al., 2011], upon the size of the institutions, concentration and substitutability of their activities. In that sense, the various buffers introduced in Basel III as well as the liquidity ratios are cross-sectional macroprudential tools. In this paper, we only consider the cross-sectional dimension of systemic risk seen as a restriction of banks deleveraging decisions.

Up to now, we focused on the microprudential approach to banking regulation, that is, each bank i only considers its own solvency constraint $X_i(x_{-i})$, that is, its set of deleveraging strategies to restore its capital ratio given what the other banks are liquidating (x_{-i}). This solvency constraint is said to be microprudential because bank i takes into account its own solvency constraint only, that is, given x_{-i}, bank i chooses x_i so as to minimize its own cost function. In particular, bank i *does not take into account* the failure externality its decision x_i (i.e., its best response $BR_i(x_{-i})$) may generate. Such a best response of bank i might imply the failure of a subset of banks. In their table 1, [De Nicoló et al., 2012] consider four instruments that may be used to address externalities related to fire sales (see also [Claessens, 2014] for a discussion); capital surcharges, liquidity requirement, taxation and restrictions on activities. However, restrictions on activities is mentioned as a macroprudential tool but is not considered to address the fire sales problem and this is what we want to do here.

Within our model, each bank $i \in \mathcal{B}$ comes up with its solvency constraint $X_i(x_{-i})$ which depends upon its own characteristics and the decisions of other banks x_{-i}. Recall that K (see equation (31)) is the set of admissible deleveraging decisions. If there is a Nash equilibrium, x must lie in K. Following [Braouezec and Kiani, 2021a], assume that K is the endogenous shared constraint; given x_{-i}, the best response of bank i $BR_i(x_{-i})$ must lie in K, which is a restriction of activity. Given x_{-i}, bank i is not anymore allowed to choose $x_i \in X_i(x_{-i})$ if $(x_i, x_{-i}) \notin K$. Such a restriction of activities may typically come from a social planner, indeed regulators or supervisors, and can be interpreted as a systemic constraint that we call a macroprudential regulation.

Definition 4 *Given x_{-i}, the strategy set of bank i that results from the enforcement of the macroprudential regulation, the constraint K, is defined as*

$$K_i(x_{-i}) = \{x_i \in \mathcal{E}_i : x \in K\} \tag{32}$$

It is clear that the macroprudential regulation yields an additional constraint for each bank i since

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \tag{33}$$

When each bank is subject to the macroprudential regulation, given x_{-i}, the choice x_i by bank i is restricted to $K_i(x_{-i})$ and not anymore to $X_i(x_{-i})$ and it is precisely in that sense that the macroprudential regulation introduces a *restriction of activities* compared to the sole microprudential constraint. By definition, as long as $x \in K$, all banks are able to comply with their regulatory constraint. It thus follows that when $x_i \in K_i(x_{-i})$, *all banks* comply with their regulatory capital ratio, i.e., they will all have their capital ratio greater (or equal) than the minimum required. The optimization problem of each bank i can now simply be written as

$$\min_{x_i \in [0,1]^n} f_i(x_i) \quad s.t \quad x_i \in K_i(x_{-i}) \tag{34}$$

Given x_{-i}, under macroprudential regulation, each bank i must now choose x_i to minimize $f_i(x_i)$ subject to $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ *and* subject to $\theta_k(x_i, x_{-i}) \geq \theta_{k,min}$ for $k \neq i$, that is, each bank i takes not only into account its own solvency constraint but also the solvency constraint of *all* the other banks. Under microprudential constraint, it may be the case that the best response $BR_i(x_{-i})$ is such that for some $k \in \mathcal{B}$, $\theta_k(BR_i(x_{-i}), x_{-i}) < \theta_{k,min}$, that is, bank i generates a negative (fire sales) externality on bank k which does not comply with its regulatory constraint. Our macroprudential regulation precisely forbids such negative externalities related to fire sales. We summarize this discussion in the following fact.

Fact 5 *When K is not empty, under the macroprudential regulation, each bank i minimizes $f_i(x_i)$ subject to the constraint that no bank will fail (i.e., solves problem 34), which means that the fire sales externalities problem is addressed.*

From a pure mathematical point of view, our model can be seen as a generalization of the strategic fire sale model introduced in [Braouezec and Wagalath, 2019] in which the number of risky assets is finite and not restricted to one. This mathematical generalization allows us, from a financial point of view, to make a clear distinction between micro and macro prudential regulation. In [Braouezec and Wagalath, 2019], such a distinction is irrelevant since given x_{-i}, each bank has a unique deleveraging strategy to restore its capital ratio, because x_i is scalar. Within our framework in which $n \geq 2$, given x_{-i}, a given bank i may have *several deleveraging strategies* to restore its capital ratio. As a result, a given bank i may not allowed to choose the cheapest one if it generates the failure of other bank(s). As we shall now show, when K is not empty, a Nash equilibrium always exists under macroprudential regulation, and among the Nash equilibria, one minimizes the total value of the asset sale, that is, this Nash equilibrium is Pareto optimal. Contrary to the basic intuition one may have, the analysis is much more easier under macroprudential regulation than under microprudential regulation.

5.2 Existence of a Nash equilibrium that minimizes the value of asset sales

Assume that $x \in K$ and let

$$V(x) := \sum_{i \in \mathcal{B}} f_i(x_i) = \sum_{i \in \mathcal{B}} \sum_j f_i(x_{ij}) \tag{35}$$

be the total value of the asset resold by banks. Before we present an existence result of Nash equilibrium under macroprudential regulation, we offer once again a geometric description of the point-to-set map $K_i(x_{-i})$, when not empty.

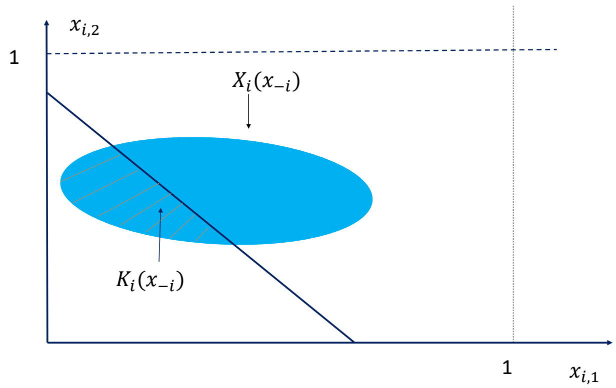

Figure 5: The point-to-set map $K_i(x_{-i}) \subset X_i(x_{-i})$

Lemma 2 *Assume that the price impact is linear for each asset $j \in \{1, 2, ..., n\}$ and consider the situation of bank i for which $K_i(x_{-i}) \neq \emptyset$.*

1. *The point-to-set map $K_i(x_{-i})$ is the intersection of $X_i(x_{-i})$ with $p-1$ affine closed half-spaces. It is therefore the intersection of a n-dimensional ellipsoid with $p-1$ affine closed half-spaces and $[0,1]^n$.*

2. *The best response in macroprudential constraint $BR_i(x_{-i})$ is well defined and is either the unique point of tangency between the ellipsoid delimited by $X_i(x_{-i})$ intersected with $p-1$ closed affine half-spaces (that is, $K_i(x_{-i})$) and an affine hyperplane, or a corner solution.*

In Fig 5, we offer a geometric representation in the two-dimensional case. We give below the definition of a Nash equilibrium of the Cournot fire sale game under macroprudential constraint.

Definition 5 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the Cournot fire sale game with macroprudential constraint $(\mathcal{B}, \mathcal{E}, (\theta_i)_{i\in\mathcal{B}}, (K_i)_{i\in\mathcal{B}})$ if, for each $i \in \mathcal{B}$ and each $x_i \in \mathcal{E}_i$ such that $x_i \in K_i(x^*_{-i})$, it holds true that $f_i(x^*_i, x^*_{-i}) \leq f_i(x_i, x^*_{-i})$.*

As the following result shows, as long as K is not empty, there always exists a Nash equilibrium under macroprudential constraint that minimizes the total value of asset sale, which is thus Pareto optimal.

Proposition 3 *Let Δ be a small to medium shock (i.e., for at least one bank i, $\Delta \in \mathcal{Z}_i^{sale}$). If $K \neq \emptyset$, then, there exists a Nash equilibrium under macroprudential regulation $x^{*,M} \in K$ such that $V(x^{*,M}) := \sum_{i\in\mathcal{B}} f_i(x_i^{*,M})$ is minimized (Pareto optimality) and such that for each $i \in \mathcal{B}$, $\theta_i(x_i^{*,M}, x_{-i}^{*,M}) \geq \theta_{i,min}$.*

Proof. See the appendix.

The Nash equilibrium is obviously Pareto optimal since the total value of the sales is minimized. It is worthwhile to note since the cost functions are all expressed in currency, the Nash equilibrium of interest is the one in which the total value of the asset sale is minimized. Our model is indeed similar to a transferable utility cooperative games for which interpersonal comparisons of utility between agents make sense.

Recall that $H_i(x_{-i}) := Y_i(x_{-i}) \bigcap X_i(x_{-i})$ is the overall Tier 1 microprudential regulatory constraint. When $H_i(x_{-i}) \neq \emptyset$, given what the other banks liquidate, bank i is able to find a strategy such that it complies with the two Tier 1 capital ratios. Let

$$\mathcal{K} = \{x \in \mathcal{E} : \forall i \in \mathcal{B}, x_i \in H_i(x_{-i})\} \tag{36}$$

Fact 6 *If $\mathcal{K}$ is not empty, proposition 3 holds, that is, there exists a Nash equilibrium under macroprudential constraint that minimizes the total value of the resale and such that each bank both complies with the risk-based capital ratio and the leverage ratio.*

At equilibrium, it may obviously be the case that for some bank i, $\theta_i(x_i^{*,M}, x_{-i}^{*,M}) > \theta_{i,min}$. This depends upon the situation of that bank. For instance, a bank i which is extremely well-capitalized may be such that $\theta_i(\mathbf{0}, x_{-i}^{*,M}) > \theta_{i,min}$ where $\mathbf{0} := (0, 0, ..., 0)$ is an $n-$dimensional vector. In such a case, this bank i needs not to delever so that the resulting capital ratio at equilibrium is higher than the required minimum.

5.3 Macroprudential regulation as the natural benchmark

We have shown that a Nash equilibrium which minimizes the total value of asset sales exists in macroprudential constraint. From a theoretical point of view, it would clearly be interesting to compare the properties of a Nash equilibrium under macroprudential constraint $x^{*,M} \in K$ with the Nash equilibrium under microprudential constraint $x^{*,m} \in K$ when it exists. However, as we have seen, the conditions under which a Nash equilibrium $x^{*,m}$ exists in microprudential constraints are extremely strong since the non-vacuity of the strategy set $X_i(x_{-i})$ for each i and each x_{-i} is required. On the contrary, as long as K is not empty, a Nash equilibrium under macroprudential regulation exists, which means that this equilibrium naturally defines the benchmark for regulation.

Definition 6 *The Nash equilibrium under macroprudential constraint $x^{*,M} \in K$ which minimizes the total asset resale is said be microprudentially incentive-compatible if, without any macroprudential constraint, for each bank $i \in \mathcal{B}$, $BR_i(x_{-i}^{*,M}) \in K_i(x_{-i}^{*,M})$.*

When the Nash equilibrium $x^{*,M}$ is microprudentially incentive-compatible, no bank has an incentive to choose a strategy such that another bank would fail to comply with its regulatory constraint. On the contrary, when the Nash equilibrium $x^{*,M}$ is not microprudentially incentive-compatible, at least one bank $i \in \mathcal{B}$ is such that $BR_i(x_{-i}^{**,M}) \notin K_i(x_{-i}^{**,M}$ (i.e., $BR_i^m(x_{-i}^{**,M}) \in X_i(x_{-i}^{**,M}) \backslash K_i(x_{-i}^{**,M})$). As a result, (at least) one bank does not comply with its constraint.

Numerical example. To understand our concepts in a simple framework, let us consider the case of two banks A and B in which each bank can either resell a "small" portion, a "medium" portion, or a "large" portion of each asset. Assume more specifically that 20% is the small portion, 40% is the medium portion and 70% is the large portion. As in the general model, each bank is assumed to be exposed to totally illiquid assets (loans) and to tradable assets. We assume that there are two tradable assets, asset 1 and 2, and we make the further assumption that asset 1 is perfectly liquid while asset 2 is imperfectly liquid. Assuming linear price impact, this means that the market Φ_2 is finite. Within our example, $\Phi_2 = 3000$. For simplicity, prices are normalized to one so that the value is equal to the quantity. Consider the balance sheets of the two banks A and B.

Bank A		**Bank B**	
Assets	Liabilities	Assets	Liabilities
$V_{0,a} = 80$ (credit) $V_{1,a} = 60$ (market) $V_{2,a} = 80$ (market)	$E_a = 10$	$V_{0,b} = 65$ (credit) $V_{2,b} = 30$ (market)	$E_b = 4.7$
$A_a = 220$	220	$A_b = 95$	95

From these two balance sheet, one can see that bank A is exposed to asset 1 and 2 while bank 2 is only exposed to asset 2. From a regulatory point of view, for each bank, the credit risk weight of the loans is equal to $\alpha_0 = 0.5$ while it is equal to $\alpha_2 = 0.6$ for the asset 2. The risk weight of the liquid asset is equal to $\alpha_1 = 0.2$. We also know that the required capital ratios are $\theta_{a,min} = 9\%$ and $\theta_{b,min} = 8\%$. Before the shock, the risk-based capital ratio of bank A is equal to $\theta_a = 10\%$ while the risk-based capital ratio of bank B is equal to $\theta_b = 9.3\%$. Each bank complies with the regulatory risk-based capital ratio.

Assume now that loans (and only loans) are hit with a shock, that is $\Delta = (\Delta_0, 0, 0)$. Assume that $\Delta_0 = 2\%$, which may lead to a large loss (in currency). After the shock, the value of the loans is equal to $V_{0,a} = 78.4$ for bank A and is equal to $V_{0,b} = 63.7$ for bank B. The capital ratios are respectively equal to $\theta_a(\Delta_0) \approx 8.47\%$ and $\theta_b = 6.82\%$, which means that each bank fails to comply with the regulatory capital ratio.

By assumption, under a macroprudential constraint, as long as K is not empty, bank A can not choose a deleveraging strategy so that bank B would not comply with its regulatory constraint. Table 6 provides the overall picture of what can happen. For instance, in the first cell, bank B sells 20% of asset 2 while bank A sells 20% of asset 1 and 2. For such deleveraging strategies, the cost for bank A is equal to 28 and the resulting capital ratio is equal to 8.98%. For bank B, the cost is equal to 6 and its resulting capital ratio is equal to 6.89%. Overall, the total cost is equal to 34.

From table 6, it is easy to see that there is a *unique* profile of strategies such that both banks comply with their regulatory capital ratio, which means that the Nash equilibrium under macroprudential constraint is unique. This profile of strategies corresponds to the cell (in blue in table 6) in which bank A sells 70% and 20% of asset 1 and 2 respectively and ends up with a capital ratio equal to 9.18% while bank B sells 70% of asset 2 and ends up with a capital ratio equal to 8.14%. The total cost for bank A is equal to 58.

Is this Nash equilibrium under macroprudential constraint microprudentially incentive-compatible when $\theta_{a,min} = 9\%$ and $\theta_{b,min} = 8\%$? The answer is negative. Knowing that bank B sells 70% of asset 2, the best response of bank A is to sell 20% of asset 1 and 40% of asset 2 and the resulting capital ratio and costs of bank A are equal to 9.06% and 44 respectively. Without any macroprudential regulation, when bank B sells 70% of asset 2, bank A is able to comply with the regulatory constraint for a cost of 44 instead of 58 with a macroprudential constraint. It thus follows that the

		Bank B					
		20%		40%		70%	
Bank A	(20%,20%)	0.08989	0.06891	0.08813	0.07333	0.08548	0.08149
		28	6	28	12	28	21
		34		40		49	
	(40%,20%)	0.09245	0.06891	0.09063	0.07333	0.08791	0.08149
		40	6	40	12	40	21
		46		52		61	
	(70%,20%)	0.09656	0.06891	0.09467	0.07333	0.09183	0.08149
		58	6	58	12	58	21
		64		70		79	
	(20%,40%)	0.09564	0.06556	0.09364	0.06966	0.09063	0.07724
		44	6	44	12	44	21
		50		56		65	
	(40%,40%)	0.09871	0.06556	0.09664	0.06966	0.09354	0.07724
		56	6	56	12	56	21
		62		68		77	
	(70%,40%)	0.1037	0.06556	0.10153	0.06966	0.09828	0.07724
		74	6	74	12	74	21
		80		86		95	
	(20%,70%)	0.1073	0.0605	0.10476	0.06414	0.10101	0.07087
		68	6	68	12	68	21
		74		80		89	
	(40%,70%)	0.1115	0.0605	0.10892	0.06414	0.10502	0.07087
		80	6	80	12	80	21
		86		92		101	
	(70%,70%)	0.1186	0.0605	0.11581	0.06414	0.11168	0.07087
		98	6	98	12	98	21
		104		110		119	

Figure 6: Two banks example with three rebalancing strategies

profile of strategies $x = (x_a = (70\%, 20\%); x_b = 70\%)$ is *not* a Nash equilibrium when there is no macroprudential constraint. It should be pointed out that this non-existence of a Nash equilibrium in this example critically depends upon the required capital ratios.

Assume now that $\theta_{a,min} = 8.5\%$ and $\theta_{b,min} = 8\%$. From table 6, only three profile of strategies x are in K and in both cases, bank B sells once again 70%. The first one is when bank A sells 20% of each asset for a total cost equal to 28 and the resulting capital ratio is equal to 8.54%. Since the capital ratio of bank B is equal to 8.14%, both banks comply with their regulatory constraint. The second and third profile of strategies (in which both banks comply with their constraints) is when bank A chooses either to sell 40% of asset 1 and 20% of asset 2 for a cost equal to 40 or to sell 70% of asset 1 and 20% of asset 2 for a cost equal to 58. For bank A, the cheapest cost is when it sells 20% of each asset, which means that the profile of strategies $x = (x_a = (20\%, 20\%); x_b = 70\%)$ is the unique Nash equilibrium under macroprudential constraint that is also microprudentially incentive-compatible.

Consider now the more realistic case in which each bank i is able to choose a quantity to resell a percentage of asset j $x_{ij} \in \{0\%, 1\%, 2\%, ..., 99\%, 100\%\}$. Using the numerical values as above with $\theta_{a,min} = \theta_{b,min} = 9\%$, we found numerically that the Nash equilibrium under macroprudential constraint is the profile of strategies $x^{**,M} = (x_a^{*,M}, x_b^{**,M}) = ((72\%; 25\%), 95\%)$. Since $BR_a^m(95\%) = (0\%; 57\%)$, such a Nash equilibrium is not microprudentially incentive-compatible since when bank a chooses to sell 57% of asset 2, bank b does not anymore comply with its constraint.

Policy implications. The foundation of macroprudential regulation is related to the prevention of negative externalities. When K is not empty, an equilibrium under macroprudential regulation that minimizes the total asset sale always exists. Depending upon the situation, this equilibrium prevents some banks to delever in a way that would be detrimental to other banks. By definition, under macroprudential regulation, a bank i is not allowed to delever in a way that a bank k would not anymore comply with its own regulatory constraint. Formally, since $x \in K$ by definition, given x_{-i}, bank i can not choose x_i such that $\theta_k(x_i, x_{-i}) < \theta_{k,min}$. Assume, as in the previous example, that bank a owns perfectly and imperfectly liquid assets while bank b only owns imperfectly liquid assets. After an adverse shock, if bank a finds it cheaper to delever by selling imperfectly liquid assets, the asset price drop may be sufficiently large to generate the insolvency of bank b at equilibrium. Macroprudential regulation prevents such a failure that may arise due to such negative externality related to fire sales.

To analyze more deeply such a situation, let us build on the example given with two banks[10]. We consider the simplest case of two banks a and b in which the trading book contains at most two securities (assets) with different market liquidity.

- Security 1 is perfectly liquid (no price impact).
- Security 2 is imperfectly liquid (positive price impact).

As before, the banking book (loans) of each bank is assumed to be perfectly illiquid, without any resale value in the short-run. Let $\Delta := (\Delta_0, 0, 0)$ be a small to medium shock in the banking book only and Δ is such that for each $i \in \{a, b\}$, $\Delta \in \mathcal{Z}_i^{sale}$. Seen from 2021, such an adverse shock can be interpreted as a consequence of Covid 19 since the lock-down has sharply increased the cost of risk of banks. We make the assumption that after such a small to medium shock, neither bank a nor bank b comply with their regulatory capital ratio.

For the sake of financial interest, banks are assumed to be heterogeneous in terms of diversification, that is, bank b is *less diversified* than bank a. Concretely, bank b is only long security 2 while bank a is long security 1 and security 2. Since bank b has no position in security 1, the only way to delever is to sell asset 2. The situation is different for bank a since it can sell both security 1 and security 2, which means that bank a will always be in a position to increase its capital ratio by selling security 1 since there is no price impact. Throughout the discussion, we make as before the assumption that K is not empty. From proposition 3, we know a Nash equilibrium in macroprudential constraint $x^{*,M}$ that minimizes the total value of the resale exists and may or may not be microprudentially incentive-compatible. Such an incentive-compatible property is a fairly complex function of all the parameters of the models, the regulatory weights, the capital requirements, the positions on securities, the market depth etc...To now understand why, as a function of the market illiquidity of security 2, macroprudential regulation is required, let us make the following assumptions.

1. If market of security 2 were perfectly liquid, bank b would be in a position to restore its capital ratio.
2. Bank a is in a position to restore its capital ratio by selling only security 1, that is, there exists $\overline{x}_{a1} < 1$ such that $\theta_a(\overline{x}_{a1}, 0) = \theta_{a,min}$.

To facilitate the discussion, assume that the price impact is linear for security 2 and let $\Phi_2 := \Phi$ be the market depth. From the above assumptions, since the capital ratio is a continuous function

[10]This paragraph is based on computations that are not reproduced in the paper to focus on policy implication. Computations are available upon request.

of the market depth, there exists a critical market depth (high enough) $\overline{\Phi}$ such that bank b will be able to restore its capital ratio independently of bank a. In such a situation, macroprudential regulation is not required. Assume now that the market depth for security 2 is low enough so that bank b, even alone, is not able to restore its capital ratio due to its own price impact. Let $\underline{\Phi}$ be this market depth[11]. Within such a framework, it thus follows from the discussion that as long as $\Phi \notin [\underline{\Phi}, \overline{\Phi}]$, macroprudential regulation is not justified. Consider now the situation in which $\Phi \in [\underline{\Phi}, \overline{\Phi}]$. Without macroprudential regulation, bank a may find cheaper (say because the regulatory weight of security 1 is much lower than the weight of security 2) to restore its capital ratio by essentially selling security 2. But such a decision may adversely impact bank b which might not be able to restore its capital ratio. In the worse scenario, given that bank b is insolvent (i.e., it must liquidate its position, $x_{b2} = 1$), bank a may still find cheaper to essentially sell security 2. Without macroprudential regulation, only bank a is solvent at equilibrium. On the contrary, with a macroprudential regulation, both bank will be solvent because bank a will not anymore have the possibility to choose the cheapest solution to delever. As a result, the externality related to fire sales disappears. We summarize the above discussion in the following policy implication fact.

Policy implication. *To prevent failures externalities, macroprudential regulation is justified for "intermediate" price impact but is unnecessary when the price impact is either low or high enough.*

In Basel III, banks are not only constrained to have idiosyncratic capital surcharge(s) that may significantly increase their Tier 1 capital but they now also have to maintain two types of liquidity ratios, called LCR and NSFR, that is, banks must have *sufficient high-quality liquid assets (HQLA) to survive a significant stress scenario lasting for 30 days*[12]. While these liquidity ratios have been introduced to avoid (funding) liquidity problems generated by maturity transformation, as observed during the subprime crisis, such liquidity ratios may also play an important role within our framework. By forcing banks to invest in liquid assets, banks are also more resilient after a shock since they can delever using these highly liquid assets. In that sense, liquidity ratios can be thought of as macroprudential instruments.

6 Severe shocks, cascade of failures and Nash equilibrium analysis

6.1 What happens if K is empty ? Forget about the conservation buffer!

Up to now, we made the assumption that K is not empty everything else equal, that is, given the positions of the banks but also the price impact functions and the capital requirement. A striking difference between Basel II and Basel III is precisely the capital requirements. In Basel III, the basic uniform Tier 1 capital requirement is 6% (of the (total) risk-weighted assets). However, all banks must comply with the capital conservation buffer (equal to 2.5%) and the countercyclical buffer among others and systemic banks must moreover comply with the GSIB buffer so the overall tier one capital requirement[13] in Basel III can be higher than 10%. Let β_{ih} be the buffer h for Tier 1 capital of bank i and et $\sum_{h \in H} \beta_{ih} := \beta_i$ be the total capital surcharge. The Tier 1 capital of a

[11] It is easy to compute this critical market depth. For the failure of bank b, it suffices to compute $\underline{\Phi}$ such that $E_b(\Delta, x_{b2} = 1, \underline{\Phi}) = 0$. Note that since $x_{b2} = 1$, the capital ratio is undefined.

[12] See BCBS (2014), "Basel III: the net stable funding ratio", p. 1

[13] Note that some buffers such as the conservation buffer applies to CET 1 capital ratio.

given bank i thus can be in general written as

$$\theta_{i,min} = 6\% + \beta_i \tag{37}$$

As noted on the website of the Bank for International settlements (BIS), the capital conservation buffer is supposed *to ensure that banks have an additional layer of usable capital that can be drawn down when losses are incurred*[14]. During a period of stress, say after a systemic shock, it makes no sense to require the capital conservation buffer of 2.5% so that the overall tier one capital requirement is equal to

$$\theta_{i,min} = 3.5\% + \beta_i \tag{38}$$

When banks are allowed to forget about the capital conservation buffer, K may be not empty and this means that we are back to what we did before. If K is still empty, then, under macroprudential constraints, failures can not be avoided, which means that the adverse shock is severe. In such a situation, that is, for some $i \in \mathcal{B}$, $\Delta \in \mathcal{Z}_i^{fail}$, some banks are insolvent right after the shock. Put it differently, given the shock size and/or the severity of the price impact, a subset of banks may be unable to satisfy their microprudential constraint given in equation (28). We thus have to extend the microprudential constraint X_i to the case in which bank i is insolvent.

6.2 Extended microprudential constraint, cascade of failures and Nash equilibrium

Given x_{-i}, let $\overline{X}_i(x_{-i})$ defines the extended microprudential constraint.

$$\overline{X}_i(x_{-i}) = \begin{cases} X_i(x_{-i}) & \text{when } X_i(x_{-i}) \neq \emptyset \\ (1, 1, ..., 1) := \mathbf{1} & \text{when } X_i(x_{-i}) = \emptyset \end{cases} \tag{39}$$

When for some $i \in \mathcal{B}$, $\Delta \in \mathcal{Z}_i^{fail}$ so that $X_i(x_{-i})$ is empty, these banks are insolvent and thus must thus be liquidated. They must thus sell 100% of their assets, that is, $x_i = (1, 1, ..., 1)$. From a resolution point of view, this means that is no regulatory attempt to assist bank(s) in financial distress[15] (e.g., bail out) but also that there are financial institutions such as solvent banks, hedge funds, institutional investors... that have the required "financial muscles" to purchase the assets sold at a discount ([Acharya and Yorulmazer, 2008]). Note that the extended microprudential constraint is the extension of [Braouezec and Wagalath, 2019] with $p \geq 2$ assets.

When for some banks $i \in \mathcal{B}$, the shock $\Delta \in \mathcal{Z}_i^{fail}$, these banks are insolvent right and must liquidate their assets. In a world without price impact, as long as $V_{i0} > E_i$ (see proposition 1 part 3), all the other banks are able to restore their regulatory capital ratio. However, this remains unclear when the price impact is positive. When these banks that are insolvent right after the initial shock sell their assets, this will depress the price of each asset and thus the capital of solvent banks. It may thus be the case that after these liquidation, a new subset of banks becomes insolvent and so on and so forth. Instead of directly considering the Nash equilibrium in which all banks are participating

[14]See the BIS document as of 2019 entitled "The capital buffers in Basel III – Executive Summary".

[15]In the European Union, since the 2014 banking unions, such a bank resolution procedure exists. We refer the reader to the excellent textbook of [Freixas et al., 2015] (paragraph 8.4.3) for an interesting and exhaustive discussion on the bank resolution procedure, see also [Acharya and Yorulmazer, 2007], [Acharya and Yorulmazer, 2008], [Acharya et al., 2011]

to the deleveraging process (as in [Braouezec and Wagalath, 2019]), we here disentangle the "pure" liquidation process (i.e., only banks that are insolvent sell their assets) from the deleveraging process of banks that are solvent after the liquidation process (i.e., banks that are still solvent after the first stage and that sell a portion of their assets in order to remain solvent at equilibrium). The two-stage is as follows.

1. **Pure liquidation process** (see appendix B). Banks that are insolvent after the shock sell all their assets. If after this liquidation, some additional banks are insolvent, these banks also sell all their assets and so on and so forth.

2. **Nash equilibrium of the deleveraging game**. We look at the Nash equilibrium of this deleveraging game for banks that are still solvent after the liquidation process.

Recall that for the initial shock Δ, some banks are insolvent. By assumption, they liquidate all their asset and this will contribute to decrease the price of each asset j. Let $\Delta^{(1)} = (\Delta_1^{(1)}(.), ..., \Delta_n^{(1)}(.))$ be the implied shock after the first round of liquidation. If there are additional banks that are insolvent for the implied shock $\Delta^{(1)}$ (but were solvent for Δ), denoted by $F^{(1)}$, then there is a second round of liquidation and so on and so forth. Our liquidation process is in the spirit similar to the cascade of bankruptcies considered in many networks papers, e.g., [Amini et al., 2016], [Bernard et al.,], [Detering et al., 2021], [Caccioli et al., 2014] to quote few papers, see also [Jackson and Pernoud, 2021] or [Glasserman and Young, 2016] few insightful review papers. In appendix C, we describe precisely the algorithm and we show that this liquidation process ends after l liquidation rounds, with $l \leq p$.

Assume now that the liquidation process stops after $l < p$ rounds and that there is still a subset of banks that are solvent. The implied shock is equal to $\Delta^{(l)} = (\Delta_1^{(l)}(.), ..., \Delta_n^{(l)}(.))$. By definition, at round l, there are no more failure (i.e., $F^{(l)} = \emptyset$). Let $\Delta^{(l)} := \Delta_{Liq}$. Let $\mathcal{S}$ be the subset of solvent banks after the liquidation process (with a positive capital ratio).

$$\mathcal{S} = \{i \in \mathcal{B} : \Delta_{Liq} \notin \mathcal{Z}_i^{fail}\} \tag{40}$$

Note interestingly that the situation in which the set of solvent banks is $\mathcal{S}$ are hit by a shock Δ_{Liq} is equivalent to the initial situation when the set of solvent banks is $\mathcal{B}$ hit by a small to medium shock Δ. Since a number of banks have been liquidated, we only consider the set of banks $\mathcal{S} \subset \mathcal{B}$ as defined in equation (40). Let

$$K' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, x_i \in X_i(x_{-i})\} \tag{41}$$

When K' is not empty, we are back to the previous analysis and a Nash equilibrium under macroprudential constraint that minimizes the total value of the asset sale will exist. Assume now that due to the severity of the price impact, K' is empty. In such a situation, abstracting from the capital conservation buffer, one must now consider the set

$$\overline{K}' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, x_i \in \overline{X}_i(x_{-i})\} \tag{42}$$

By definition, since K' is empty, a number of solvent banks after the shock Δ_{Liq} will be insolvent after the deleveraging process. In the next result, we show that a Nash equilibrium exists.

Proposition 4 *Let Δ be a severe shock. There exists a Nash equilibrium under macroprudential regulation $x^{*,M} \in \overline{K}'$ such that $V(x^{*,M}) := \sum_{i\in\mathcal{B}} f_i(x_i^{*,M})$ is minimized on $\overline{K}'$ (Pareto optimality) and such that for each $i \in \mathcal{B}$, either $\theta_i(x_i^{*,M}, x_{-i}^{*,M}) \geq \theta_{i,min}$ or $x_i^{*,M} = 1$.*

Proof. See the appendix.

From a regulatory point of view, the "best" criterion to be used in such a situation of failures remains unclear. Should one minimize the number of failures or should one minimize the total value of the asset sales? This raises new issues that are beyond the scope of the paper.

7 Cascade of failures: application to French systemic banks

In this section, to illustrate how our model can be calibrated on data, we focus on the four French GSIBs as of 2020; BNP Paribas, BPCE, Crédit Agricole and Société Générale. Our approach boils down to manually collecting data and consists in extracting from the annual report the relevant disclosed quantities. The calibration of the model to real data raises however new issues as some inputs are not directly disclosed in annual reports, which means that one must find proxies. As in [Braouezec and Wagalath, 2018], we need to recover the banking book, the trading book and their associated risk weights.

7.1 Descriptive statistics and calibration methodology

Let θ and L respectively be the observed risk-capital capital ratio and leverage ratio as of December 2020. In the following table, except the total exposure, all the quantities come from annual reports. Except ratios, all the quantities are expressed in billion.

Risk-based capital ratio and leverage ratio (December, 2020)

Bank	Tier 1	RWA	θ	θ_{min}	E_{xp}	L	L_{min}	A	$\frac{E_{xp}}{A}$
BNP Paribas	98.8	695.52	0.142	0.1096	2266.86	0.0436	0.03	2488.49	0.915
Société Générale	56.18	351.85	0.160	0.1052	1188.5	0.047	0.03	1461.9	0.813
Crédit Agricole	50.02	336.04	0.149	0.0964	1861.5	0.027	0.03	1861	0.95
BPCE	68.98	431.22	0.160	0.12	1374.3	0.050	0.03	1446.26	0.95

In the above table, the total exposure reported comes from the banks individual templates available on the website of European Banking Authority[16] (EBA). Few remarks are in order.

- The total exposure incorporates on balance sheets items but also off balance sheets items. It is somehow surprising that this total exposure E_{xp} is always *lower* than the total value of the assets A, although fairly close to it. This means that A is a fairly good approximation of E_{xp}.

[16] See https://www.eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions. The reason lies in the fact that for some banks, Crédit Agricole, the total exposure reported in the annual report differs from the one found on the website of the EBA by 800 billion. This also means that the leverage ratio is not around 5%, as stated in the annual report, but around 2.8%.

- As opposed to few years ago, the minimum capital required for each bank depends upon its own characteristics. Since the capital conservation is equal to 2.5% in Basel III, one can clearly see that the minimum required for Tier 1 risk-based capital ratio is higher than 8.5%, which means that these banks are subject to additional capital surcharges such as the GSIBs buffer.

Consider a given bank i and let $V_{i,BB}$ and $V_{i,TB}$ be the value of the banking and trading book and let and $\alpha_{i,BB}$ $\alpha_{i,TB}$ be their respective risk weight. We consider cash, denoted v in the model, as a separate item since it does not entail any capital. It thus follows that for each bank i, we have

$$A_i = v_i + V_{i,BB} + V_{i,TB} \tag{43}$$

In the same vein, the (total) risk-weighted assets RWA_i is equal to the risk-weighted asset of the banking book plus the risk-weighted assets of the trading book for each bank i, that is

$$\text{RWA}_i = \text{RWA}_{i,BB} + \text{RWA}_{i,TB} \tag{44}$$

Once $V_{i,BB}, V_{i,TB}$ and $\text{RWA}_{i,BB}, \text{RWA}_{i,TB}$ are recovered, the two risk weights $\alpha_{i,BB}$ and $\alpha_{i,TB}$ are also known since

$$\text{RWA}_{i,BB} = \alpha_{i,BB} V_{i,BB} \qquad \text{RWA}_{i,TB} = \alpha_{TB} V_{i,TB} \tag{45}$$

Banking versus trading book. To construct the banking book and the trading book from the consolidated balance sheet, we make the assumption that the banking book is subject to credit risk while the trading book is subject to market risk and counterparty risk. For the banking book, we make the assumption that it is equal to the financial assets at amortized costs (loans to customers and to credit institutions). From equation (43), the trading book thus is equal to the total value of the assets minus the banking book (loans) and the cash. We thus implicitly incorporate in the trading book a number of items such as intangible assets and goodwill that are not traded assets and which might be difficult to resell in the short-run. Fortunately, as their value is small in percentage, around 1% or 2% of the total assets, this assumption is innocuous.

Partial risk-weighted assets assignment. The credit risk-weighted assets, by far the most important of all the risk-weighted assets, is assigned to the banking book risk-weighted assets as well as the securitization exposures in the banking book risk-weighted assets. On the contrary, the market risk-weighted assets is assigned to the trading book and we decided to also assign the counterparty risk-weighted assets. The difficulty concerns the operational risk-weighted assets and the settlement risk-weighted assets (although this last quantity is negligible). To reflect the fact that operational risk is both present in the banking book and in the trading book, we assign the operational risk-weighted assets as a proportion of the credit risk-weighted assets divided by the market risk-weighted assets plus the credit risk-weighted assets.

Proxied quantities as of December, 2020, (Phased In)

Bank	v	V_{BB}	V_{TB}	RWA_{BB}	RWA_{TB}	α_{BB}	α_{TB}	α_{Avg}
BNP Paribas	308.7	946.8	1232.96	625.32	70.2	0.660	0.057	0.319
Société Générale	168.18	502.14	791.6	306.63	45.22	0.611	0.057	0.272
Crédit Agricole	194.3	953.9	812.9	302.79	33.25	0.317	0.041	0.190
BPCE	153.4	836.82	456	402.74	28.48	0.481	0.062	0.334

One can clearly see from the table that the banking book risk weight is much higher than the trading book risk weight. While this in part depends upon our methodology, it essentially follows from the fact that the credit risk-weighted assets is by far the most important. Its contribution in the total risk-weighted assets varies from 70% (Société Générale) to 81% (BPCE), and this explains why the the banking book risk weight is much higher than the trading book risk weight. We also report the usual average risk weight α_{Avg}, used by regulators to calibrate the leverage ratio, defined here as the risk-weighted assets (RWA) divided by the total assets minus cash $(A - v)$.

7.2 Cascades of failures as a function of the severity of the stress test

We now consider the cascade of failures that may result after a given shock in the banking book, given a price impact, measured by Φ. We call a price impact of x% if, when all banks sell 100% of their trading book, the price decreases by x%. Within our framework, the severity of the stress-test both depends upon the shock Δ and the price impact x%. As before, we consider a shock in the banking book only to eliminate model risk, here wrong assignment of items of the balance sheet. Overall, the severity of the stress test both depends upon the shock Δ and the price impact measured by x, which implicitly depends the market depth Φ. Since it is difficult to assess the accuracy of the market depth Φ, in part because the trading book contains very different type of assets, this parameter can be actually seen as a measure of the severity of the stress test, just like the shock. Recall that when one bank fails, it sells 100% of its assets and this may generate new failures and so on and so forth.

Linear price impact: 1%.

- $\Delta = 6\%$: Crédit Agricole fails and there is no cascade of failures.
- $\Delta = 7\%$: idem
- $\Delta = 8\%$: idem
- $\Delta = 9\%$: Crédit Agricole and BPCE fail and there is no cascade of failures.

Linear price impact: 2%.

- $\Delta = 6\%$: Crédit Agricole fails and there is no cascade of failures.
- $\Delta = 7\%$: idem
- $\Delta = 8\%$: Crédit Agricole fails and BPCE fails after the liquidation of Crédit Agricole.
- $\Delta = 9\%$: Crédit Agricole and BPCE fail and there is no cascade of failures.
- $\Delta = 9.5\%$: Crédit Agricole and BPCE fail. BNP Paribas fails after their liquidation. Société Générale fails after the liquidation of BNP Paribas.

It is interesting to note that for a shock of 9.5%, Crédit Agricole and BPCE fail. After liquidation, BNP Paribas fail and Société Générale in turn fails after the liquidation of BNP.

Linear price impact: 4%.

- $\Delta = 6\%$: Crédit Agricole fails and there is no cascade of failures.
- $\Delta = 7\%$: idem
- $\Delta = 8\%$: Crédit Agricole fails and BPCE fails after the liquidation of Crédit Agricole.
- $\Delta = 9\%$: Crédit Agricole and BPCE fail and BNP Paribas and Société Générale fail after the liquidation.

Consider for instance the case in which the price impact is 2% and the shock is $\Delta = 8\%$. We know that Crédit Agricole fails and BPCE fails after the liquidation of Crédit Agricole. Since BNP Paribas and Société Générale are still solvent (with positive capital), we then consider the Nash equilibrium. It turns out that these two banks alos fail at equlibrium.

It would certainly be interesting to consider the model with at least two risky assets in the trading book. However, due to the difficulty to obtain the various items of the trading book and the difficulty to estimate the price impact, it is our belief that such an comprehensive empirical analysis deserves a full paper.

8 Conclusion

We offer in this paper a new framework in which we make a clear distinction between microprudential and macroprudential regulation. We show that a Pareto optimal Nash equilibrium generically exists under macroprudential regulation while such an existence result under microprudential regulation requires a set of strong conditions. Our results clearly suggest to consider macroprudential regulation as the natural benchmark. An interesting aspect of our framework is that most of the parameters can be calibrated in a fairly easy way. While many theoretical extensions of our model could be done, it is our belief that an interesting and promising work would be to offer a more complete empirical analysis.

9 Appendix: Proofs

Proof of proposition 1. Proof of part 1. When there is no price impact, the total capital of a bank i $E_{i,t+1}(.)$ given in equation (18) is invariant with respect to x_i while the risk-weighted assets $\text{RWA}_{i,t+1}(.)$ given in equation (19) is (for each j but also for each i) a decreasing function of x_{ij}. As a result, the risk-based capital ratio is an increasing function of each x_{ij} □

Proof of part 2. Consider bank i and assume that for all $j \neq k$, $\alpha_{ij} \neq \alpha_{ik}$. One thus can define a permutation such that the asset index are ordered such that $j_1 > j_2 >,.., > j_n$ implies $\alpha_{ij_1} > \alpha_{ij_2} > ... > \alpha_{ij_n}$ (the case in which $\alpha_{ij_1} \geq \alpha_{ij_2} \geq ... \geq \alpha_{ij_n}$ will be discussed later on).

Claim 1 *Consider bank i and in addition to assuming $\Delta \in \mathcal{Z}_i^{sale}$ and no price impact, assume further that $V_{i0} = 0$. Then, there always exists an optimal strategy for bank i to restore its capital ratio back above the minimum required $\theta_{i,min}$. It is optimal for bank i to first sell a portion of the asset j_1 with the highest risk weight. If this is not enough to restore the regulatory capital ratio, it is optimal for the bank to sell 100% of the risky asset j_1 and a portion of risky asset j_2. If selling 100% of the asset j_1 and j_2 is not enough to restore the capital ratio, it is optimal to sell a portion of asset j_3 and so on and so forth.*

Proof of claim 1. For the optimal $(x_{i1}, ..., x_{in})$ it is clear that the constraint is binding, that is $\theta_{i,t+1}(x_i) = \theta_{min}$, that is,

$$\frac{E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j)(1 - x_{ij})} = \theta_{min} \tag{46}$$

which implies that:

$$\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) x_{ij} = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) + \frac{1}{\theta_{min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j) \tag{47}$$

If we rename $X_{ij} = q_{ij} P_j (1 - \Delta_j) x_{ij}$, and $K_i = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) + \frac{1}{\theta_{min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$, this is equivalent to:

$$\sum_{j=1}^{n} \alpha_{ij} X_{ij} = K_i \tag{48}$$

Each bank i seeks to minimize $\sum_{j=1}^{n} X_{ij}$. Therefore, it is optimal to start by selling the asset j_1 with the highest risk weight α_{i,j_1}, then asset j_2, ..., until asset j_k such that the capital ratio is restored □

Remark. In the event of two risky assets j_k and j_{k+1} with the same risk weights, it is equivalent to sell one or the other first, or both at the same time. □

Proof of part 3. Assume now that there are illiquid asset (loans). Since $\Delta \in \mathcal{Z}_i^{sale}$, the total capital after the shock is positive, that is, $E_i - \Delta_0 V_{i0} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j > 0$, which is equivalent to $\Delta_0 < \Delta_c := \frac{E_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{V_{i0}}$. Assume now that the bank resell 100% of the risky assets $j = 1, 2, ...n$, that is, $x_i = \mathbf{1}$. The risk-based capital ratio thus is equal to $\theta_i(\mathbf{1}) = \frac{E_i - \Delta_0 V_{i0} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{\alpha_{i0} V_{i0} (1 - \Delta_0)}$. If $\theta_i(\mathbf{1}) >$

$\theta_{i,min}$, the bank will be able to restore its capital ratio. It is easy to show that $\theta_i(\mathbf{1}) > \theta_{i,min}$ is equivalent to $\Delta_0 < \Delta_i^c := \frac{E_i - \sum_{j=1}^n q_{ij} P_j \Delta_j - \alpha_{i0} V_{i0} \theta_{i,min}}{V_{i0}(1-\alpha_{i0}\theta_{i,min})}$ also easy to show that $\Delta_i^c > \Delta_{i,c}$ is equivalent to $V_{i0} > E_i - \sum_{j=1}^n q_{ij} P_j \Delta_j$ so that bank i will be able to restore its capital ratio. Since $V_{i0} > E_i$, $\Delta_0 < \Delta_{i,c} < \Delta_i^c$ so that bank i will always be in a position to restore its capital ratio □

Proof of lemma 1

Assume that $X_i(x_{-i}) \neq \emptyset$ and note that P_j is the price of asset j at time t (before the shock). We have:

Proof of part 1. Given x_{-i}, the capital ratio of bank i is equal to

$$\theta_i(x_i, x_{-i}) = \frac{E_{i,t} - q_{i0} P_0 \Delta_0 - \sum_{j=1}^n q_{ij} P_j (\Delta_j + \frac{\sum_{k \in \mathcal{B}} x_{kj} q_{kj}}{\Phi_j}(1-\Delta_j))}{\alpha_{i0} q_{i0} P_0 (1-\Delta_0) + \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1-\Delta_j) \left(1 - \frac{\sum_{k \in \mathcal{B}} x_{kj} q_{jk}}{\Phi_j}\right)(1 - x_{ij})}$$

By definition, $X_i(x_{-i})$ is given by the set of points $x_i \in [0,1]^n$ such that: $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$. It is easy to show that $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ is equivalent to equation (49) lower (or equal) than equation (50):

$$\left(\alpha_{i0} q_{i0} P_0 (1-\Delta_0) + \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1-\Delta_j) \left(1 - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}\right)(1 - x_{ij})\right) \theta_{i,min} \tag{49}$$

$$E_{i,t} - q_{i0} P_0 \Delta_0 - \sum_{j=1}^n q_{ij} P_j (\Delta_j + \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}(1-\Delta_j)) \tag{50}$$

which, given x_{-i}, is in turn equivalent to:

$$R_1(x_{i1}) + R_2(x_{i2}) + ... + R_n(x_{in}) \leq C \tag{51}$$

with

$$\begin{aligned} C &= E_{i,t} - q_{i0} P_0 \Delta_0 - \sum_{j=1}^n q_{ij} P_j (\Delta_j + \frac{\sum_{k \neq i} x_{kj} q_{kj}}{\Phi_j}(1-\Delta_j)) - \alpha_{i0} q_{i0} P_0 (1-\Delta_0) \\ &\quad - \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1-\Delta_j) \left(1 - \frac{\sum_{k \neq i} x_{kj} q_{kj}}{\Phi_j}\right) \theta_{i,min} \end{aligned} \tag{52}$$

and for all $j \in [1,..,n]$, $R_j(x_{ij})$ a polynomial of degree 2 with a positive leading coefficient of the form: $R_j(x_{ij}) = d_{ij} \times (x_{ij})^2 + e_{ij} \times (x_{ij})$

with for all $j \in [1,..,n]$,

$d_{ij} = \alpha_{ij} q_{ij} P_j (1-\Delta_j) \frac{q_{ij}}{\Phi_j} \theta_{i,min} > 0$

$e_{ij} = \alpha_{ij} q_{ij} P_j (1-\Delta_j) [-\left(1 - \frac{\sum_{k \neq i} x_{kj} q_{kj}}{\Phi_j}\right) - \frac{q_{ij}}{\Phi_j}] \theta_{i,min} + q_{ij} P_j \frac{q_{ij}}{\Phi_j}(1-\Delta_j)$

This can be rewritten: $R_j(x_{ij}) = d_{ij} \times (x_{ij} + \frac{e_{ij}}{2d_{ij}})^2 - \frac{e_{ij}^2}{4d_{ij}}$.

And (51) is equivalent to:

$$\sum_{j=1}^{n} d_{ij} \times (x_{ij} + \frac{e_{ij}}{2d_{ij}})^2 \leq C + \sum_{j=1}^{n} \frac{e_{ij}^2}{4d_{ij}} \tag{53}$$

If we denote $C' = C + \sum_{j=1}^{n} \frac{e_{ij}^2}{4d_{ij}}$, this is equivalent to:

$$\sum_{j=1}^{n} \frac{(x_{ij} + \frac{e_{ij}}{2d_{ij}})^2}{\frac{C'}{d_{ij}}} \leq 1 (54)$$

And this is equivalent to:

$$\sum_{j=1}^{n} \frac{(x_{ij} - c_{ij})^2}{(a_{ij})^2} \leq 1 \tag{55}$$

with $c_{ij} = \frac{e_{ij}}{2d_{ij}}$ and $a_{ij} = \sqrt{\frac{C'}{d_{ij}}} > 0$. Since the equation of a canonic ellipsoid in dimension n is given by $\sum_{j=1}^{n} \frac{x_j^2}{a_j^2} \leq 1$, it thus follows that equation (55) is the equation of an n-dimensional ellipsoid and this concludes the proof of part 1 $\square$

Proof of part 2. Bank i aims to minimize $f_i(x_i) = \sum_{j=1}^{n} x_{ij} q_{ij} P_j$ subject to $x_i \in X_i(x_{-i})$. Let $\mathcal{F}_i^{(a)}(x_{-i}) = \{x_i \in X_i(x_{-i}) : f_i(x_i) = a\})$ be the level curve associated to the cost function $f_i = a$. Since $f_i(x_i) = \sum_{j=1}^{n} x_{ij} q_{ij} P_j$ is linear in each x_{ij}, each iso cost function defines an hyperplane. By definition of the best response $BR_i(x_{-i})$, it is minimum of the function f_i with respect to x_i subject to $x_i \in X_i(x_{-i})$. It thus follows that the best response $BR_i(x_{-i}) = x_i^*$ is such that the hyperplane is tangent to the ellipsoid delimited by $X_i(x_{-i})$ and thus is unique. When the best response is not a tangency point, it is a corner solution. Let $\mathcal{C}^n := [0,1]^n$ be the unit compact of $\mathbb{R}^n$ and let $\partial\mathcal{C}^n$ be its boundary and int$\mathcal{C}^n$ be its interior so that $\mathcal{C}^n := \partial\mathcal{C}^n \bigcup \text{int}\mathcal{C}^n$. A corner solution is defined as a best response which belongs to $\partial\mathcal{C}^n$ and which can not satisfy the tangency condition. $\square$

Proof of proposition 2

Given i and $x_{-i} \in [0,1]^{(p-1)n}$, consider $X_i(x_{-i})$ and assume that it is nonempty. Since $X_i(x_{-i})$ is a n-dimensional ellipsoid, it is clearly compact and convex.

Let us prove that for all i, X_i is a lower and upper semi-continuous point-to-set map:

Indeed:

- X_i is lower semi-continuous: let us consider a sequence $(v_l) \in \left([0,1]^{(p-1)n}\right)^{\mathbb{N}}$ that converges to $v_\infty \in [0,1]^{(p-1)n}$. We consider $w \in X_i(v_\infty)$, that is $\theta_i(w, v_\infty) \geq \theta_{i,min}$. Let us prove that there exists a sequence (w_l) with $w_l \in X_i(v_l)$ for all l and such that (w_l) converges to w. We consider $\epsilon > 0$. Let us prove that there exists L_0 such that for all $l \geq L_0$ we have that $B'(w,\epsilon) \cap X_i(v_l) \neq \emptyset$. Indeed, if it was not the case, we would have for all $L_0 > 0$ existence of a $l > L_0$ such that $B'(w,\epsilon) \cap X_i(v_l) = \emptyset$. So we could build a subsequence $(v_{\phi(l)})$ that converges to v_∞ and such that $B'(w,\epsilon) \cap X_i(v_{\phi(l)}) = \emptyset$ for all l. This implies that $\theta_i(x_i, v_{\phi(l)}) < \theta_{i,min}$ for all l and for all $x_i \in S(w,\epsilon) = \partial(B'(w,\epsilon))$, and since θ_i is continuous we would have $\theta_i(x_i, v_\infty) \leq \theta_{i,min}$ for all $x_i \in S(w,\epsilon)$. Let us consider the vector $z_i \in S(w,\epsilon)$ that maximizes the distance to 0, that is $d(z_i, 0) = max_{x_i \in S(w,\epsilon)} d(x_i, 0)$. Since all coordinates of w are strictly lower than the coordinates of z_i, we would have $\theta_i(w, v_\infty) < \theta_i(z_i, v_\infty) \leq \theta_{i,min}$ since $\theta_i(x_i, v_\infty)$

is an increasing function of x_i. And $\theta_i(w, v_\infty) < \theta_{i,min}$ is a contradiction. Therefore, there exists L_0 such that for all $l \geq L_0$ we have that $B'(w,\epsilon) \cap X_i(v_l) \neq \emptyset$, and we can build a sequence $w_l \in B'(w,\epsilon) \cap X_i(v_l)$ that converges to w.

- X_i is upper semi-continuous: let us consider a sequence $(v_l) \in \left([0,1]^{(p-1)n}\right)^{\mathbb{N}}$ that converges to $v_\infty \in [0,1]^{(p-1)n}$, and a sequence $(w_l) \in X_i(v_l)$ for all l, that converges to $w_\infty \in [0,1]^{(p-1)n}$. $\theta_i(v_l, w_l) \geq \theta_{i,min}$ for all l, and since θ is continuous we have that $\theta_i(v_\infty, w_\infty) \geq \theta_{i,min}$, and therefore $w_\infty \in X_i(v_\infty)$.

Moreover, since for all x_{-i}, $f_i(, x_{-i})$ is linear in x_i, it is thus quasiconvex on $X_i(x_{-i})$.

Therefore, the assumptions of Theorem 1 are satisfied and under these assumptions there exists a Nash equilibrium in microprudential constraint. □

Proof of lemma 2

Proof of part 1. $K_i(x_{-i})$ is described by the set $\{x_i \in [0,1]^n$ such that $\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ and for all $l \neq i$, $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}\}$.

$\theta_i(x_i, x_{-i}) \geq \theta_{i,min}$ gives us the equation of a n-dimensional ellipsoid similar to Proposition 1.

Moreover, for $l \neq i$ we have:

$$\theta_l(x_i, x_{-i}) = \frac{E_{l,t} - q_{l0}P_0\Delta_0 - \sum_{j=1}^n q_{lj}P_j(\Delta_j + \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j))}{\alpha_{l0}q_{l0}P_0(1-\Delta_0) + \sum_{j=1}^n \alpha_{lj}q_{lj}P_j(1-\Delta_j)\left(1 - \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}\right)(1-x_{lj})}$$

For all $l \neq i$, $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}$ is equivalent to:

$$\left(\alpha_{l0}q_{l0}P_0(1-\Delta_0) + \sum_{j=1}^n \alpha_{lj}q_{lj}P_j\,(1-\Delta_j)\left(1 - \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}\right)(1-x_{lj})\right)\theta_{l,min} \leq E_{l,t} - q_{l0}P_0\Delta_0 - \sum_{j=1}^n q_{lj}P_j(\Delta_j + \frac{\sum_{k\in B} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j))$$

which, given x_{-i}, is in turn equivalent to:

$\sum_{j=1}^n a_{ij}x_{ij} \leq C$ with for all $j \in [1,..,n]$, $a_{ij} \in \mathbb{R}$ and $C \in \mathbb{R}$.

And this is the equation of a closed affine half-space. □

Proof of part 2. A best response in macroprudential constraint $BR_i^M(x_{-i})$ satisfies $\theta_i(x_i, x_{-i}) = \theta_{i,min}$ and for all $l \neq i$, $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}$. $\theta_i(x_i, x_{-i}) = \theta_{i,min}$ gives us the equation of the frontier of a n-dimensional ellipsoid. $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}$ gives us the equations of $p-1$ closed affine half-spaces.

Bank i is seeking to minimize $L_i(x_i) = \sum_{j=1}^n x_{ij}q_{ij}P_j$ subject to $x_i \in K_i(x_{-i})$.

$\sum_{j=1}^n x_{ij}q_{ij}P_j = a$ is an isocost hyperplane, and the minimum for L_i, which is the best response $BR_i^M(x_{-i})$ is reached for a point of tangency of an affine hyperplane $\sum_{j=1}^n x_{ij}q_{ij}P_j = a_i$ with the frontier of the ellipsoid delimited by $X_i(x_{-i})$ intersected with the $p-1$ closed affine half-spaces defined by $\theta_l(x_i, x_{-i}) \geq \theta_{l,min}, l \neq i$. □

Proof of proposition 3

We shall prove the proposition with two lemma.

Lemma A 1 *K is a compact set.*

Proof. Since $K \subset [0,1]^{n\times k}$, it is clearly a bounded set. To show that K is compact, it remains to prove that K is closed. Let $\vec{\theta}(x) \geq \vec{\theta}_{min}$ be the vectorial notation for $\begin{pmatrix} \theta_1(x) \geq \theta_{1,min} \\ \vdots \\ \theta_p(x) \geq \theta_{p,min} \end{pmatrix}$. Recall that

$$K := \{x \in [0,1]^{n\times p} : \vec{\theta}(x) \geq \vec{\theta}_{min}\} \subset [0,1]^{n\times p} \tag{56}$$

and is not empty by assumption. Since $\Delta_0 < 1$, for all $i \in S$, $\alpha_{i0}V_{i0}(1-\Delta_0) > 0$, the denominator of the capital ratio θ_i is strictly positive so that θ_i is a continuous application on $[0,1]^{n\times p}$ (see fact 2). Therefore, K is the preimage of a closed set $[\theta_{1,min}, +\infty[\times ... \times [\theta_{p,min}, +\infty[$ by the continuous application $\theta = (\theta_1, ..., \theta_p)$, and thus K is closed. It thus follows that K is a closed and bounded set of $[0,1]^{n\times p}$ which means that K is a compact set. □

Consider now the application $V : [0,1]^{n\times p} \to \mathbb{R}$, i.e., for a given $x \in [0,1]^{n\times p}$, $V(x) = \sum_{i\in S} f_i(x_i)$. Since for all $i \in \mathcal{B}$, f_i is continuous, V is also continuous on the compact set K. From Weierstrass extreme value theorem, it admits at least one minimum $x^{*,M} \in K$. Let $\mathcal{M}_K \subset K$ be the set of minimizers of the function V on K, possibly a singleton.

Lemma A 2 *Each element of $\mathcal{M}_K$ is a Nash equilibrium of the game under macroprudential constraint.*

We shall prove that x^*_K is a Nash equilibrium, that is, for all i, $x_i^{*,M} = BR_i(x_{-i}^{*,M})$. For notational simplicity, we remove the subscript M. Let us work by contradiction and assume that this is not the case, i.e., there exists $i_0 \in \mathcal{B}$ such that $x^*_{i_0} \neq BR_{i_0}(x^*_{-i_0})$. By definition, given $x^*_{-i_0}$, $x^*_{i_0}$ is not the cheapest deleveraging strategy. Using the fact that for each i, f_i is continuous, this means that there exists x'_{i_0} such that $f_i(x'_{i_0}) < f_i(x^*_{i_0})$ and $(x'_{i_0}, x^*_{-i_0})$ still in K so that $f_{i_0}(x'_{i_0}) + \sum_{i\neq i_0} f_i(x^*_i) < \sum_{i\in S} f_i(x^*_i)$ and this yields the desired contradiction. Therefore, the minima of V are Nash equilibria □

This concludes the proof of proposition proposition 3 □

Proof of Proposition 4

The proof is quite similar to Proposition 3:

$$\overline{K}' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, x_i \in \overline{X}_i(x_{-i})\} \tag{57}$$

$$\overline{X}_i(x_{-i}) = \begin{cases} X_i(x_{-i}) & \text{when } X_i(x_{-i}) \neq \emptyset \\ (1,1,...,1) := \mathbf{1} & \text{when } X_i(x_{-i}) = \emptyset \end{cases} \tag{58}$$

Therefore:

$$\overline{K}' = \{x \in \mathcal{E} : \forall i \in \mathcal{S}, \theta_i(x_i, x_{-i}) \geq \theta_{i,min} \text{ or } x_i = (1,..,1)\} \tag{59}$$

Let's prove that $\overline{K}'$ is a compact set.

- $\overline{K}' \subset [0,1]^{np}$ is clearly a bounded set.
- We want to prove that $\overline{K}'$ is a closed set. Let $(x_m)_{m\in\mathbb{N}} = (x_{1,m}, ..., x_{p,m})_{m\in\mathbb{N}} \in (\overline{K}')^{\mathbb{N}}$ be a sequence which converges to a given $x_\infty \in [0,1]^{np}$. We will show that $x_\infty \in \overline{K}'$.

  Let $i \in \{1, .., p\}$.

  - either $x_{i,\infty} = (1, 1, ..., 1)$
  - or $x_{i,\infty} \in [0,1]^n \setminus \{(1, 1, ..., 1)\}$ and there exists $\epsilon > 0$ such that $B(x_{i,\infty}, \epsilon) \subset [0,1]^n \setminus \{(1, 1, ..., 1)\}$ and there exists $m_0 \in \mathbb{N}$ such that $\forall m \geq m_0, x_{i,m} \in B(x_{i,\infty}, \epsilon)$. Therefore $\forall m \geq m_0, \theta_{i,t+1}(x_m) \geq \theta_{min}$. And since $\theta_{i,t+1}$ is continuous on $B(x_{i,\infty}, \epsilon)$, we have $\theta_{i,t+1}(x_\infty) \geq \theta_{min}$

  So $\overline{K}'$ is a closed set.

Therefore $\overline{K}'$ is a closed bounded set of $[0,1]^{np}$, so K is a compact set.

Same as in Proposition 3, we look at the minimizers of V on $\overline{K}'$, and these are Nash equilibria of the generalized game with shared constraint $\overline{K}'$.

□

10 Appendix B: liquidation process

We now describe formally the algorithm associated to the liquidation process. Since the banking book has no value, when a given bank sells it, the proceeds is equal to zero.

Algorithm of the liquidation process with linear price impact

1. Let $F^{(1)} := \{i \in \mathcal{B} : \Delta \in \mathcal{Z}_i^{fail}\}$. If $F^{(1)} = \emptyset$, then, the liquidation process stops. If $F^{(1)} \neq \emptyset$, all banks $i \in F^{(1)}$ liquidate all their assets, that is, for each $i \in F^{(1)}$, $x_i = \mathbf{1}$. The resulting implied shock given by equation (23) for each asset $j = 1, 2, ..., n$ is equal to $\Delta_j^{(1)}(\sum_{i\in F^{(1)}} q_{ij}) := \Delta_j + \frac{\sum_{i\in F^{(1)}} q_{ij}}{\Phi_j}(1 - \Delta_j)$ so that the implied vector of shock after the first step is equal to $\Delta^{(1)} := (\Delta_1^{(1)}(.), ..., \Delta_n^{(1)}(.))$.
2. Let $F^{(2)} := \{i \in (\mathcal{B}\backslash F^{(1)}) : \Delta^{(1)} \in \mathcal{Z}_i^{fail}\}$. If $F^{(2)} = \emptyset$, then, the liquidation process stops. If $F^{(2)} \neq \emptyset$, all the bank $i \in F^{(2)}$ liquidate all their assets, that is, for each $i \in F^{(2)}$, $x_i = \mathbf{1}$. The resulting implied shock given by equation (23) for each asset $j = 1, 2, ..., n$ is equal to $\Delta_j^{(1)}(\sum_{i\in(F^{(1)}\bigcup(F^{(2)})} q_{ij}) := \Delta_j + \frac{\sum_{i\in(F^{(1)}\bigcup(F^{(2)})} q_{ij}}{\Phi_j}(1 - \Delta_j)$ so that the implied vector of shock after the first step is equal to $\Delta^{(2)} := (\Delta_1^{(2)}(.), ..., \Delta_n^{(2)}(.))$.
3. Repeat until $F^{(k)} := \{i \in (\mathcal{B}\backslash \cup_{a=1}^{k-1} F^{(a)}) : \Delta^{(k-1)} \in \mathcal{Z}_i^{fail}\}$ is not empty.

Fact 7 *The liquidation process stops after a finite number of liquidation rounds $l \leq p$.*

Proof. Assume that the process does not stop at step $k \geq 1$, which means that for an implied shock $\Delta^{(k-1)}$ at step $k-1$, $F^{(k)} \neq \emptyset$, that is, an additional subset of banks fail. Since $F^{(k)} \neq \emptyset$ is equivalent to $\text{Card}(F^{(k)}) \geq 1$ and since $\sum_{k=1}^{p} \text{Card}(F^{(k)}) \geq p$, the liquidation process must stop in at most $l \leq p$ steps such that $\sum_{k=1}^{l} \text{Card}(F^{(k)}) \leq p$ □

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