Aggregation Across Each Nation: Aggregator Choice and Macroeconomic Dynamics*

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Abstract

We study the implications of trade aggregation in an infinite-horizon economy with multiple countries. Specifically, we ask whether there is a role for alternatives to the Armington aggregator in the workhorse open-economy macroeconomics model. We show analytically that the first-order dynamics of the model are entirely captured by a few sufficient statistics, so that the precise choice of functional form for the trade aggregator is irrelevant. This sufficient-statistics result has the following implications. For given steady-state trade elasticities and home bias, any aggregator that is homogeneous of degree one is equivalent to the Armington aggregator. Similarly, aggregators that are homogeneous of arbitrary degree are equivalent to a simple generalisation of the Armington aggregator for given steady-state trade elasticities and home bias. In models with more than two countries, alternative aggregators can play a role by allowing for steady-state differences in bilateral trade elasticities across different country pairs, which Armington aggregation rules out.

Key Words: International Trade, Open-economy Macroeconomics, Armington Aggregator, Elasticity of Trade.

JEL Codes: F00, F10, F41.

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1 Introduction

Goods trade is a central component of New Open-Economy Macroeconomics (NOEM) models (Corsetti, 2008), playing an important role in the cross-border propagation of macroeconomic shocks. One of the most primitive assumptions in any international macroeconomics model is how domestic and foreign goods are bundled together to form aggregate goods. Because the structure of this aggregation has implications for, *inter alia*, how agents' demand responds to changes in relative prices, it is central to our understanding of many features of the global economy. In this paper, we assess the implications of how trade aggregation is modelled for macroeconomic dynamics and the international transmission of shocks, by deriving sufficient statistics that summarise the impact of the trade aggregator on the first-order dynamics of these models.

The Armington (1969) aggregator, which is a Constant Elasticity of Substitution (CES) aggregator, has been widely applied in the NOEM literature, and is the 'go-to' aggregator in multi-country models.[1] The Armington aggregator is summarised by two types of parameters. The first relate to the share of expenditure on each good, and are typically used to capture the degree of 'home bias' in preferences—in other words the idea that countries tend to spend proportionally more on their domestic goods even if prices are symmetric. The second is a parameter that captures the elasticity of substitution between goods produced in different countries—also known as the 'trade elasticity' or 'Armington elasticity'—governing how relative demand responds to relative prices.

A major reason for the Armington aggregator's wide usage is its tractability and elegant closed-form solutions. However, the flipside of this simplicity is that the value of this single elasticity parameter becomes crucially important for the dynamics of these models. For example, as Corsetti, Dedola, and Leduc (2008) demonstrate, both the sign and size of spillovers in NOEM models depend on the trade elasticity in the Armington aggregator. As such, this elasticity is notoriously difficult to calibrate or estimate and, despite an ever-expanding body of empirical evidence, there remains substantial uncertainty around the appropriate trade elasticity values to apply to different research and policy questions (Feenstra, Luck, Obstfeld, and Russ, 2018). For instance, a low trade elasticity is required to match the empirical Backus-Smith correlation—the negative unconditional correlation between real exchange rates and relative consumption.[2] In contrast, a high trade elasticity helps to replicate micro-evidence around empirically observed patterns of trade substitution.

While the Armington aggregator is commonplace in NOEM models focused on studying the spillovers from macroeconomic shocks, a largely independent literature has put forward a set of alternative aggregators. These alternatives allow for variation in the elasticity of substitution

[1]For instance, the International Monetary Fund's Global Integrated Monetary and Fiscal model (Laxton, Mursula, Kumhof, and Muir, 2010) features layered CES aggregation of domestic and foreign, consumption and investment, and final and intermediate goods.

[2]Corsetti et al. (2008) show that the Backus-Smith correlation can also be matched with high trade elasticities if shocks are assumed to be persistent.

in different ways, and this literature has shown that this variation is key for capturing many empirical facts.

One dimension of variation in the elasticity of substitution is variation over time. Drozd, Kolbin, and Nosal (2017) consider a setup in which varieties are less substitutable in the short-run and more substitutable in the long-run, due to the presence of adjustment costs. They demonstrate that this setup can help to resolve the trade-comovement puzzle in NOEM models, capturing the empirical regularity that countries that trade more with each other tend to have more correlated business cycles.

A second dimension is variation in the elasticity of substitution across firms. This is needed to match the observed heterogeneity in price mark-ups, which is pinned down by the elasticity of substitution in simple price-setting models. Several aggregators have been used in this context, capturing different mechanisms that could account for this heterogeneity.

One common example is the Kimball aggregator, first proposed by Kimball (1995), which allows the elasticity of substitution of a given good to depend on the relative level of consumption of that good. This captures a realistic feature of consumer demand, namely that consumers would be less willing to flock towards relatively cheap goods when their initial level of consumption of that good is high. This aggregator has been applied in both closed- and open-economy settings. In a closed-economy framework, Klenow and Willis (2016) argue that the Kimball aggregator can create a 'micro-founded' real rigidity in consumer demand that helps to generate persistent effects from monetary policy shocks. Harding, Lindé, and Trabandt (2021) show that a nonlinear macroeconomic model with Kimball aggregation helps to explain the "missing deflation puzzle" in the US following the Great Recession. Within an open-economy framework, Gopinath and Itskhoki (2011) show that the variable mark-ups implied by Kimball aggregation can explain the response of reset-price inflation to exchange rate shocks.

Another related alternative aggregator is the Translog aggregator, which is a special case of the Quadratic Mean of Order r, or QMOr, aggregator. This class of aggregators allows the elasticity of substitution between goods, and hence mark-ups, to depend on the number of goods being produced. Bergin and Feenstra (2000) and Feenstra (2003) propose this aggregator to capture the idea that the number of available varieties impacts the degree of competition in the market, and hence the equilibrium mark-ups. A number of papers have used this aggregator to model the 'pro-competitive' effects of trade: by increasing the number of varieties available, increasing trade openness raises the elasticity of substitution and lowers average price mark-ups. Within this setup, Feenstra (2018) emphasises how this mechanism offers an important source of gains from trade, beyond that of traditional comparative advantage. Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2019) further explore the role of variable mark-ups in generating pro-competitive gains from trade.

The aggregators mentioned so far have all been homogeneous of degree one, implying homothetic preferences. More recently, a series of papers have shown that some empirical micro-evidence can only be matched by using aggregators that are not homogeneous of degree one. For instance, Jung, Simonovska, and Weinberger (2019) use preferences that imply a price-elasticity of demand that depends on the consumer's income level. These non-homothetic preferences allow the authors to match pricing-to-market patterns observed in the data, with monopolistically competitive producers setting higher mark-ups and charging higher prices in richer countries.

Despite the known limitations of the Armington aggregator, and this extensive literature showing how alternative aggregators are important for capturing different empirical facts and realistic mechanisms, to date no studies have explored how these aggregators can impact shock transmission in NOEM models. This paper aims to bridge these two literatures and fill this gap. Specifically, we ask how the choice of trade aggregator impacts the dynamics of an otherwise-standard workhorse NOEM model. To this end, we use a generic N-country NOEM model, and characterise the sufficient statistics that capture the impact of the trade aggregator on the first-order dynamics of these models. From there, we derive a series of analytical results about the role of aggregator choice on the model dynamics.[3]

Our first result is that, in a two-country model, any aggregator that is homogeneous of degree one is equivalent to the Armington aggregator to first order. More specifically, within the class of aggregators that are homogeneous of degree one, the first-order dynamics of the two-country NOEM model is determined entirely by the steady-state consumption shares and the steady-state elasticity of substitution—precisely the two quantities that are given parameterically in the Armington aggregator. Hence, for a given calibration of these steady-state objects, the precise form of the aggregator is irrelevant.

We extend this baseline irrelevance result in two main directions. First, we consider a model with more than two countries. In this case, our equivalence result requires that the steady-state elasticity of substitution across every pair of country-goods is the same across all the aggregators being compared. Since the Armington aggregator imposes that this elasticity is the same across all pairs of country-goods, an alternative aggregator can change the first-order dynamics of the model relative to Armington by allowing, in steady state, for different elasticities of substitution across different pairs of goods. In this context, we also show that a nested-CES structure, while allowing for some differences in elasticities across different pairs of goods, is not sufficiently flexible to replicate the dynamics under alternative aggregators.

Second, we consider the case in which the aggregator is not homogeneous of degree one. Here, the first-order dynamics will change relative to the Armington aggregator, due to the difference in the degree of homogeneity. We propose a simple extension of the Armington aggregator, introducing one new parameter, which can parsimoniously replicate any aggregator that is homogeneous of arbitrary degree in a two-country setup. As before, in a setup with more than two countries, differences in steady-state bilateral elasticities of substitution can affect the first-order dynamics of the model in a way that the generalised Armington aggregator cannot replicate.

[3]These results are derived analytically from the linearised model, and so they hold exactly at first order. This means that alternative aggregators may have additional effects at higher order, but by definition these effects will be small unless there is a non-linearity in the model, or if we consider large shocks or shocks to higher moments. These extensions are left for future research.

Our results are similar in spirit to those in Baqaee and Farhi (2019), who investigate the implications of 'Hulten's theorem' (Hulten, 1978) in a multi-sector open-economy setup. Unlike their paper, which focuses on the impact of sectoral shocks propagating through global production networks, our work is focused on the workhorse NOEM model and the international spillovers from country-specific shocks.

The rest of the paper is organised as follows. Section 2 shows the main features of our generic model. Section 3 contains the core sufficient-statistics result, and Section 4 explores the implications of this in different cases. Section 5 concludes.

2 Model Setup

We begin by setting up a generic multi-country NOEM model. For simplicity we consider endowment economies, hence abstracting from production and assuming that only final consumption goods are traded across countries. The results presented below would continue to hold if we introduced a perfectly competitive production sector, which does not take the demand structure into account. In that setting, we could also allow for trade in intermediate inputs or investment goods, and our results would also continue to hold for the aggregation of these goods.

There are N countries, indexed by $n = 1, 2, ..., N$. Time is discrete and infinite. In each time period t, each country n is endowed with a unique tradable good, denoted by $Y_t^{(n)}$, which takes strictly positive values.

The problem of the representative country-n consumer can be split into an *inter*temporal and an *intra*temporal component. The intertemporal aspect of the household problem is independent of the aggregation structure, and defines aggregate quantities. The intratemporal aspect is aggregator-specific, taking the aggregate choices from the intertemporal problem as given.

Intertemporal Problem. The representative consumer in country n has additively separable preferences over time:

$$U_t^{(n)} = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^{\tau} u \left(C_{t+\tau}^{(n)} \right) \right]$$

where $C_t^{(n)}$ denotes aggregate consumption; $u : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable, strictly increasing and strictly concave function, with $\lim_{C \to 0} u'(C) = \infty$; and $\beta \in (0, 1)$ is the discount factor.

Let $P_t^{(n)}$ denote the price of a unit of aggregate consumption in country n and $p_{i,t}^{(n)}$ the price of a unit of the country-i good in period t in country n. The intertemporal budget constraint of the country-n representative consumer is

$$\sum_{\tau=0}^{\infty} \left(P_{t+\tau}^{(n)} C_{t+\tau}^{(n)} - p_{n,t+\tau}^{(n)} Y_{t+\tau}^{(n)} \right) \leq 0$$

Without loss of generality, we will assume complete international capital markets.[4] Equalising the associated optimality conditions for the representative country-n household with the corresponding condition for country-n' yields a risk-sharing condition

$$\kappa^{(n,n')} \frac{u'\left(C_{t+\tau}^{(n')}\right)}{u'\left(C_{t+\tau}^{(n)}\right)} = RER_{t+\tau}^{(n,n')} \qquad n' \neq n \tag{1}$$

where $RER_t^{(n,n')} \equiv P_t^{(n')}/P_t^{(n)}$ denotes the real exchange rate of country n *vis-à-vis* country n', defined such that an increase in its value represents a depreciation for country n, and

$$\kappa^{(n,n')} \equiv \overline{RER}^{(n,n')} \frac{u'\left(\overline{C}^{(n)}\right)}{u'\left(\overline{C}^{(n')}\right)}$$

is a Pareto weight that allows for steady-state asymmetries across countries, and overlines represent steady-state values.

This optimisation problem pins down the sequence of $C_t^{(n)}$ and $RER_t^{(n,n')}$ given the endowment processes.

Intratemporal Problem. The aggregate consumption of households in country n is aggregated from goods produced in all N countries, according to the aggregator function $f : \mathbb{R}^N \to \mathbb{R}$, such that

$$C_t^{(n)} \equiv f\left(\mathbf{c}_t^{(n)}\right) \tag{2}$$

where $\mathbf{c}_t^{(n)} = \left[c_{1,t}^{(n)}, c_{2,t}^{(n)}, ..., c_{N,t}^{(n)}\right]'$ denotes the $N \times 1$ vector of consumption levels, with $c_{i,t}^{(n)}$ denoting the representative country-n household's consumption of goods from country i. We assume that f is twice differentiable and denote by $f_i^{(n)}$ the partial derivative of the function with respect to country-i goods.

The intratemporal problem of the representative household at time t involves minimising total expenditure, taking as given the level of aggregate consumption from the intertemporal optimisation. In other words:

$$\min_{\mathbf{c}_t^{(n)}} \sum_{i=1}^{N} p_{i,t}^{(n)} c_{i,t}^{(n)} \quad \text{subject to} \quad C_t^{(n)} = f\left(\mathbf{c}_t^{(n)}\right)$$

giving the first-order conditions:

$$p_{i,t}^{(n)} = \lambda f_i^{(n)} \quad \forall\, i = 1, \dots N$$

where λ is the Lagrange multiplier on the constraint and $f_i^{(n)} \equiv \partial f(\mathbf{c}_t^{(n)})/\partial c_{i,t}^{(n)}$. These N

[4] Our results would continue to hold if we assumed financial autarky or other forms of incomplete markets. We only require that the comparison across aggregators is done around the same steady state.

optimality conditions can be written as $(N-1)$ relative demand functions:

$$\frac{f_{i,t}^{(n)}}{f_{N,t}^{(n)}} = \frac{p_{i,t}^{(n)}}{p_{N,t}^{(n)}} \quad \text{for } i = 1, 2, ..., N-1 \tag{3}$$

The aggregate consumer price index $P_t^{(n)}$ can be defined simply by:

$$P_t^{(n)} C_t^{(n)} = \sum_{i=1}^{N} p_{i,t}^{(n)} c_{i,t}^{(n)} \tag{4}$$

Given (2) and (3) for all countries $n = 1, ..., N$, world equilibrium in goods markets is given by

$$Y_t^{(n)} \geq \sum_{i=1}^{N} c_{n,t}^{(i)} \quad \text{for } n = 1, 2, ..., N \tag{5}$$

The intratemporal optimisation in each country defines trade quantities $\mathbf{c}_t^{(n)}$ and relative prices $\mathbf{p}_{i,t}^{(n)}/P_t^{(n)}$ given aggregate variable definitions from the intertemporal problem.

3 Sufficient Statistics for the Aggregator

The key question of this paper is how the specific choice of functional form for f affects the model's equilibrium macroeconomic dynamics. Our main result is summarised by the following theorem:

Theorem 1 *The effect of the aggregator function on the first-order dynamics of this model is captured entirely by the following sufficient statistics, where overlines represent the steady-state values of variables and functions thereof:*

(i) the elasticities of substitution between each pairs of goods:

$$\overline{\Phi}_{i,j}^{(n)} \equiv \frac{\partial \ln \left(\overline{c}_i^{(n)} / \overline{c}_j^{(n)} \right)}{\partial \ln \left(\overline{f}_j^{(n)} / \overline{f}_i^{(n)} \right)} \quad \textit{for } i,\ j = 1, 2, ..., N,\ i \neq j$$

(ii) the share of consumption expenditure for each good:

$$\overline{\alpha}_i^{(n)} \equiv \frac{\overline{p}_i^{(n)} \overline{c}_i^{(n)}}{\overline{P}^{(n)} \overline{C}^{(n)}} \quad \textit{for } i = 1, 2, ..., N,$$

(iii) the ratio $\overline{\mathcal{H}}^{(n)}$*, defined as:*

$$\overline{\mathcal{H}}^{(n)} \equiv \mathcal{H}(\overline{\mathbf{c}}^{(n)}) = \frac{\sum_{i=1}^{N} \overline{f}_i^{(n)} \overline{c}_i^{(n)}}{f(\overline{\mathbf{c}}^{(n)})}$$

(iv) the ratios $\overline{\mathcal{H}}_i^{(n)}$ *for each good, defined as:*

$$\overline{\mathcal{H}}_i^{(n)} \equiv \mathcal{H}_i(\overline{\mathbf{c}}^{(n)}) = \frac{\sum_{k=1}^{N} \overline{f}_{ik}^{(n)} \overline{c}_k^{(n)}}{f_i(\overline{\mathbf{c}}^{(n)})} \quad \text{for } i = 1, 2, ..., N,$$

for each country $n = 1, 2, ..., N$*.*

Proof: First, notice that only equations (2)-(5) are directly affected by the aggregator function and the consumption levels $c_{i,t}^{(n)}$. The rest of the model equations are independent of the aggregator by definition. We therefore prove the theorem by showing that the first-order approximation of these four equations only depends on the aggregator function, f, through the steady-state quantities described above. Full derivations are provided in Appendix A. □

The elasticities of substitution, consumption expenditure shares and the ratios $\mathcal{H}(.)$ and $\mathcal{H}_i(.)$ are generically functions of the variables of the model and therefore can vary dynamically. However, Theorem 1 states that the dynamics of the model at first order depend only on the *steady-state* values of these objects.

Before unpacking the implications of this theorem, it is useful to say a few words on $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$. While it is natural to think of an aggregator as being defined by the elasticities of substitution and the consumption shares across goods, the ratios $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$ are less familiar. Even in steady state, these ratios are functions of the individual consumption levels, and their respective functional forms depend on the form of the aggregator. This means that, even for the same steady-state level of individual consumptions, $\overline{\mathbf{c}}^{(n)}$, different aggregators can give rise to different steady-state $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$, and hence different first-order dynamics.

There is one specific sub-class of aggregators for which this is not true: homogeneous functions. The result for these functions is summarised by the following corollary to the theorem:

Corollary 1 *If the aggregator is homogeneous of degree* h*, the first-order dynamics of the model are captured by the following sufficient statistics:* $\overline{\Phi}_{i,j}^{(n)}$, $\overline{\alpha}_i^{(n)}$ *for* $i, j = 1, 2, ..., N,\ i \neq j$*, as defined above, and* h*.*

Proof: Recall, first, that if a function is homogeneous of degree h, then the partial derivatives of that function are homogeneous of degree $(h-1)$. Then, by Euler's theorem, if the function f is homogeneous of degree h, then $\overline{\mathcal{H}}^{(n)} = h$ and $\overline{\mathcal{H}}_i^{(n)} = (h-1)$. Hence, h becomes the sufficient statistic to replace $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$. Full derivations are provided in Appendix B. □

Corollary 1 has implications when comparing across homogeneous aggregators that are summarised in the following corollary:

Corollary 2 *All aggregators that are homogeneous of the same degree will imply the same first-order dynamics, for given* $\overline{\Phi}_{i,j}^{(n)}$ *and* $\overline{\alpha}_{i}^{(n)}$ *for* i, $j = 1, 2, ..., N$, $i \neq j$, *as defined above.*

Proof: This follows directly from Corollary 1. When comparing across aggregators with the same h, then the sufficient statistics collapse to just the elasticities and expenditure shares. □

The following section explores the implications of Theorem 1 and Corollaries 1 and 2 by considering a few separate cases.

4 Implications of the Theorem

4.1 Homothetic Preferences

One of the basic assumptions of most economic models is that preferences are homothetic. A homothetic function is a monotonic transformation of a function that is homogeneous of degree 1, henceforth referred to as HOD(1). Therefore, if the utility function, $u(C_t^{(n)})$, is a monotonic increasing function of aggregate consumption, then utility is homothetic with respect to $\mathbf{c}_t^{(n)}$ if the aggregator function f is HOD(1).

Corollary 2 implies that, within the class of HOD(1) aggregators, the sufficient statistics for the first-order dynamics of the model are just the steady-state elasticities of substitution and the expenditure shares. Since this class includes the Armington aggregator, this means that any alternative HOD(1) aggregator, with the same steady-state elasticities of substitution and expenditure shares, will be equivalent to the Armington aggregator.

In this subsection, we unpack these implications by comparing the Armington aggregator to the Kimball (1995) aggregator, an alternative HOD(1) functional form, which was introduced in Section 1. Our exploration proceeds in three steps: (i) we consider the two-country case, (ii) we extend our analysis to more than two countries, and (iii) we compare these results to a nested-CES framework.

4.1.1 Case 1: Two Countries

If $N = 2$, *all aggregators that are HOD(1) are equivalent at first order to the Armington aggregator with the same steady-state elasticity and home bias.*

Armington Aggregator. The two-country Armington aggregator is given by:

$$C_t^{(n)} \equiv f(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = \left(a_1^{(n)\frac{1}{\phi}} c_{1,t}^{(n)\frac{\phi-1}{\phi}} + \left(1 - a_1^{(n)}\right)^{\frac{1}{\phi}} c_{2,t}^{(n)\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}} \quad \text{for } n = 1, 2$$

and yields the familiar relative demand functions:

$$\frac{c_{1,t}^{(n)}}{c_{2,t}^{(n)}} = \frac{a_1^{(n)}}{1-a_1^{(n)}} \left(\frac{p_{2,t}^{(n)}}{p_{1,t}^{(n)}}\right)^{\phi} \quad \text{for } n = 1, 2$$

where ϕ is the constant elasticity of substitution between the only two goods, and $a_1^{(1)}$ is the degree of home bias in country 1, which maps into the steady-state consumption shares, $\overline{\alpha}_1^{(n)}$ and $\overline{\alpha}_2^{(n)} = (1 - \overline{\alpha}_1^{(n)})$.[5]

The first-order approximation of these equations is given by:

$$\begin{aligned} \widetilde{C}_t^{(n)} &= \overline{\alpha}_1^{(n)} \widetilde{c}_{1,t}^{(n)} + (1 - \overline{\alpha}_1^{(n)}) \widetilde{c}_{2,t}^{(n)} \\ \widetilde{c}_{1,t}^{(n)} - \widetilde{c}_{2,t}^{(n)} &= \phi \left(\widetilde{p}_{2,t}^{(n)} - \widetilde{p}_{1,t}^{(n)} \right) \end{aligned}$$

for $n = 1, 2$, where $\widetilde{x}_t$ is the percentage deviation of variable x from its steady state $\bar{x}$.

These two equations illustrate how the two parameters of the Armington aggregator enter the linearised model. Corollary 2 tells us that any HOD(1) aggregator across two goods can be mapped into an equivalent Armington aggregator, with ϕ set to match the same *steady-state* elasticity of substitution, and $a_1^{(n)}$ set to match the same *steady-state* consumption shares.

To illustrate this property, we compare these linearised equations under CES to the Kimball (1995) aggregator—an alternative HOD(1) specification.

Kimball Aggregator. Consider Kimball (1995)'s aggregator, where aggregate consumption $C_t^{(n)}$ is implicitly defined by:

$$1 = b_1^{(n)} \Upsilon \left(\frac{c_{1,t}^{(n)}}{b_1^{(n)} C_t^{(n)}} \right) + b_2^{(n)} \Upsilon \left(\frac{c_{2,t}^{(n)}}{b_2^{(n)} C_t^{(n)}} \right) \quad \text{for } n = 1, 2$$

where $b_2^{(n)} \equiv (1 - b_1^{(n)})$, and $\Upsilon(.)$ is such that $\Upsilon(1) = 1$, $\Upsilon'(.) > 0$ and $\Upsilon''(.) > 0$. It can be seen from this implicit definition of $C_t^{(n)}$ that aggregate consumption is HOD(1) in consumption of country-specific goods: increasing both $c_{1,t}^{(n)}$ and $c_{2,t}^{(n)}$ by the same factor would require $C_t^{(n)}$ to increase by the same factor for the implicit function to continue to hold.

We follow Klenow and Willis (2016) and specify the function $\Upsilon(.)$ as:[6]

$$\Upsilon(x) = 1 + (\sigma - 1) \exp(\epsilon^{-1}) \epsilon^{\frac{\sigma}{\epsilon} - 1} \left(\Gamma\left(\frac{\sigma}{\epsilon}, \frac{1}{\epsilon}\right) - \Gamma\left(\frac{\sigma}{\epsilon}, \frac{x^{\frac{\epsilon}{\sigma}}}{\epsilon}\right) \right)$$

[5] In a symmetric steady state, in which the prices of the two goods are equal, then $\overline{\alpha}_1^{(1)} = a_1^{(1)}$, but outside of symmetry this mapping will depend on the steady-state relative prices, with $\overline{\alpha}_1^{(n)} = a_1^{(n)} (\overline{p}_1^{(n)} / \overline{P}^{(n)})^{1-\phi}$.

[6] There are multiple formulations of the Kimball (1995) aggregator. For example, Lindé and Trabandt (2018) use a Dotsey and King (2005) specification in their closed-economy analysis. But the specific choice of functional form is irrelevant for our result.

where

$$\Gamma(u,z) = \int_z^{+\infty} s^{u-1}\exp(-s)\mathrm{d}s$$

This specification of $\Upsilon(.)$ yields the following derivative:

$$\Upsilon'(x) = \frac{\sigma-1}{\sigma}\exp\left\{\frac{1-x^{\frac{\epsilon}{\sigma}}}{\epsilon}\right\}$$

This aggregator is defined by three parameters: σ, ϵ and $b_1^{(n)}$ for each country. $b_1^{(n)}$ is a familiar home-bias parameter, which maps into consumption shares, while σ and ϵ pin down the elasticity of substitution.

To see this, consider the relative demand functions from the household's intratemporal problem:

$$\frac{p_{1,t}^{(n)}}{p_{2,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)}{\Upsilon'\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)} = \frac{\exp\left\{\frac{1}{\epsilon}\left(1-\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}}{\exp\left\{\frac{1}{\epsilon}\left(1-\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}} \quad \text{for } n=1,2$$

From this, we can define the consumption shares and elasticity of substitution:

$$\alpha_{1,t}^{(n)} = \frac{p_{1,t}^{(n)}c_{1,t}^{(n)}}{p_{1,t}^{(n)}c_{1,t}^{(n)} + p_{2,t}^{(n)}c_{2,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)c_{1,t}^{(n)}}{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)c_{1,t}^{(n)} + \Upsilon'\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)c_{2,t}^{(n)}}$$

$$\Phi_{1,2,t}^{(n)} = \sigma\left(1+\frac{\alpha_{1,t}^{(n)}}{\alpha_{2,t}^{(n)}}\right)\left[\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{1,t}^{(n)}}{\alpha_{2,t}^{(n)}}\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1}$$

for $n = 1,2$.[7]

This final expression illustrates the key property of Kimball preferences: the elasticity of substitution depends on the relative consumption levels. Notice that as $\epsilon \to 0$, $\Phi_{1,2,t}^{(n)} \to \sigma$, implying that Kimball nests CES, with elasticity σ, as a limit case.

To further explore the properties of the Kimball aggregator, Figure 1 plots the relative demand function, and implied elasticities, for different values of ϵ. To form this plot, we calibrate the three remaining aggregator parameters: $\sigma = 1.5$, $b_1^{(n)} = 0.8$ and $b_2^{(n)} = 0.2$. First, notice that when $p_1^{(n)}/p_2^{(n)} = 1$, we have $c_1^{(n)}/c_2^{(n)} = b_1^{(n)}/b_2^{(n)} = 4$ and $\Phi_{1,2}^{(n)} = \sigma = 1.5$ independently of ϵ. This implies that Kimball is also equivalent to CES at the point of symmetry across good types.[8]

[7] Full derivations are provided in Appendix C.

[8] This point is explored more in Baqaee, Farhi, and Sangani (2021), who highlight the importance of firm heterogeneity when using the Kimball aggregator to aggregate across monopolistically differentiated goods.

More generally, ϵ controls the curvature of the demand function. In the limiting case of CES preferences, as $\epsilon \to 0$, shown in the black dotted lines, the relative demand function is convex and the elasticity of substitution is constant at $\sigma = 1.5$. As ϵ increases, the relative demand curve becomes less convex, and the elasticity of substitution varies with the relative consumption levels. For $\epsilon = \sigma$, the relative demand curve is approximately linear. When $\epsilon > \sigma$, the curve is concave. When this is the case, the concave relative demand curves imply finite "choke prices", above which demand for the relatively more expensive good is 0.

Consider, for example, the concave relative demand at $\epsilon = 5$. Here, as the price of good 1 relative to good 2 in country n rises above 1, relative demand for good 1 falls more than in the CES case. In other words, the elasticity of substitution between the two goods is higher. In contrast, when the relative price of good 1 falls below 1, the relative demand for good 1 rises less rapidly than it does under CES, reflecting a lower elasticity of substitution between the two goods.

Equivalently, when consumption of good 1 is low relative to good 2, then the elasticity of substitution is high, and a decrease in the relative price of good 1 leads to a larger substitution towards good 1. Conversely, when the consumption of good 1 is high relative to good 2, then the elasticity of substitution is lower, and a decrease in the relative price of good 1 leads to a smaller substitution towards good 1.

Within a dynamic model, this relative-demand curvature allows for the elasticity of substitution to vary over time, as the economy is hit by exogenous shocks. This leads to what Klenow and Willis (2016) refer to as "a smoothed version of a kinked demand curve": if a shock drives the relative price of a good up, the elasticity of substitution increases, such that demand declines more than the CES case, while if a shock drives the relative price down, the elasticity decreases, such that demand increases less than the CES case.

Comparing Armington and Kimball. Despite these additional mechanisms in the Kimball aggregator, the application of Corollary 2 to this case tells us that, at first order, Kimball is equivalent to the Armington aggregator. To see why, we take the first-order approximation of the implicit definition of aggregate consumption and the relative demand function under Kimball:

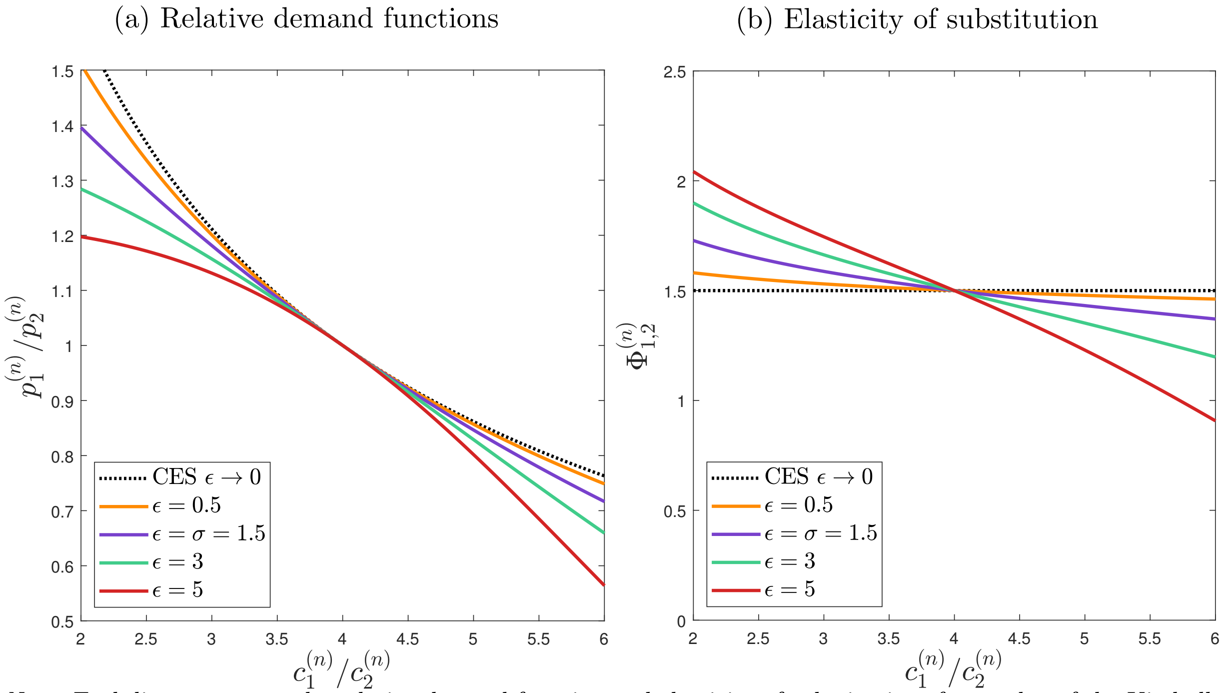
$$\widetilde{C}_t^{(n)} = \overline{\alpha}_1^{(n)} \widetilde{c}_{1,t}^{(n)} + (1 - \overline{\alpha}_1^{(n)}) \widetilde{c}_{2,t}^{(n)}$$
$$\widetilde{c}_{1,t}^{(n)} - \widetilde{c}_{2,t}^{(n)} = \overline{\Phi}_{1,2}^{(n)} \left(\widetilde{p}_{2,t}^{(n)} - \widetilde{p}_{1,t}^{(n)} \right)$$

for $n = 1, 2$, where $\overline{\alpha}_1^{(n)}$ and $\overline{\Phi}_{1,2}^{(n)}$ are the steady-state values of the consumption share and elasticity of substitution as defined above.[9]

From these equations we see that, even under Kimball, the linearised equations only depend on the *steady-state* consumption shares and the *steady-state* elasticity of substitution. The

[9]These expressions can be derived applying the formulas in the proof of Theorem 1 in Appendix A for a generic aggregator.

Figure 1: Kimball (1995) aggregator



Note: Each line represents the relative demand function and elasticity of substitutions for a value of the Kimball 'curvature' parameter ϵ, plotted with $\sigma = 1.5$ and $b_1^{(n)} = 0.8$.

parameters of the Kimball aggregator, including the curvature parameter ϵ, only matter insofar as they pin down these two steady-state values. Importantly, then, despite the fact that $\epsilon > 0$ allowed for the elasticity of substitution to vary dynamically, as described above, these dynamics do not enter the linearised model equations.

Thus, for a given value of the Kimball parameters, we can set the Armington parameters, $a_1^{(n)}$ to match the same $\overline{\alpha}_1^{(n)}$, and $\phi = \overline{\Phi}_{1,2}^{(n)}$, and we see immediately that these linearised equations are exactly equivalent to the linearised equations under CES. In other words, the first-order dynamics of a model specification with the Kimball aggregator are equal to those of a CES specification, for given steady-state consumption shares and steady-state elasticity of substitution.

4.1.2 Case 2: $N > 2$ Countries

If $N > 2$, then the specific form of the aggregator is relevant only to the extent that the bilateral elasticities of substitution across different pairs of goods are different in steady state.

Armington Aggregator. We can define the country-n Armington aggregator over N goods as:

$$C_t^{(n)} = f(c_{1,t}^{(n)}, \dots c_{N,t}^{(n)}) = \left(\sum_{i=1}^{N} a_i^{(n)\,\frac{1}{\phi}} c_{i,t}^{(n)\,\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}} \qquad \text{for } n \in [1, N]$$

where $\sum_{i=1}^{N} a_i^{(n)} = 1$.

The relative demand functions are given by:

$$\frac{c_{i,t}^{(n)}}{c_{N,t}^{(n)}} = \frac{a_i^{(n)}}{a_N^{(n)}} \left(\frac{p_{N,t}^{(n)}}{p_{i,t}^{(n)}}\right)^{\phi} \quad \text{for } i \in [1, N-1],\ n \in [1, N]$$

This leads to the following linearised equations:

$$\widetilde{C}_t^{(n)} = \sum_{i=1}^{N} \overline{\alpha}_i^{(n)} \widetilde{c}_{i,t}^{(n)}$$
$$\widetilde{c}_{i,t}^{(n)} - \widetilde{c}_{N,t}^{(n)} = \phi \left(\widetilde{p}_{N,t}^{(n)} - \widetilde{p}_{i,t}^{(n)}\right) \quad \text{for } i \in [1, N-1]$$

for $n \in [1, N]$.

The pair-wise elasticities of substitution between any two goods is given by the same parameter, ϕ, by definition of the single Armington aggregator. To see how this property affects the comparison with more general aggregators, we go back to the example of the Kimball aggregator considered above.

Kimball Aggregator. In each country n, the implicit definition of the N-good Kimball aggregator is now:

$$1 = \sum_{i=1}^{N} b_i^{(n)} \Upsilon\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right) \qquad \text{for } n \in [1, N]$$

where $\sum_{i=1}^{N} b_i^{(n)} = 1$ and the function $\Upsilon(.)$ is defined as in section 4.1.1.

The resulting relative demand functions are then:

$$\frac{p_{i,t}^{(n)}}{p_{N,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right)}{\Upsilon'\left(\frac{c_{N,t}^{(n)}}{b_N^{(n)} C_t^{(n)}}\right)} = \frac{\exp\left\{\frac{1}{\epsilon}\left(1 - \left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}}{\exp\left\{\frac{1}{\epsilon}\left(1 - \left(\frac{c_{N,t}^{(n)}}{b_N^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}} \quad \text{for } i \in [1, N-1],\ n \in [1, N]$$

As for the two-country case, we can compute the consumption shares and the bilateral elasticities of substitution:

$$\alpha_{i,t}^{(n)} = \frac{p_{i,t}^{(n)} c_{i,t}^{(n)}}{\sum_{j=1}^{N} p_{j,t}^{(n)} c_{j,t}^{(n)}} = \frac{c_{i,t}^{(n)} \Upsilon'\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right)}{\sum_{j=1}^{N} c_{j,t}^{(n)} \Upsilon'\left(\frac{c_{j,t}^{(n)}}{b_j^{(n)} C_t^{(n)}}\right)} \quad \text{for } i \in [1, N]$$

$$\Phi_{i,j,t}^{(n)} = \sigma \left(1 + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}}\right) \left[\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}} \left(\frac{c_{j,t}^{(n)}}{b_j^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1} \qquad \text{for } i, j \in [1, N],\ i \neq j$$

for $n = 1, 2, ..., N$.[10]

As before, the elasticity of substitution depends on the relative consumption levels. As well as allowing the elasticity to vary over time, we see that the elasticity can be different for different pairs of goods, depending on the asymmetries between countries. From the expression for $\Phi^{(n)}_{i,j,t}$, it is easy to see that the elasticities between two pairs of goods, $\{i, j\}$ and $\{i, l\}$, will be equal if and only if one of three conditions holds: (i) $\epsilon = 0$, in which case we are back to the CES aggregator; (ii) $\alpha^{(n)}_{i,t} = 0$, implying that good i is not consumed at all; or, most importantly, (iii) $b^{(n)}_j = b^{(n)}_l$ and $c^{(n)}_{j,t} = c^{(n)}_{l,t}$, such that consumption shares are equal across goods. Ignoring the trivial cases (i) and (ii), we therefore see that the Kimball aggregator implies that the elasticities across different pairs of goods will be different, unless there is perfect symmetry across all countries.

Comparing Armington and Kimball. To see how these differences in elasticities across different country-pairs affect the dynamics of the model, we again take the first-order approximation of the aggregator and relative demand functions under Kimball:

$$\begin{aligned}
\widetilde{C}^{(n)}_t &= \sum_{i=1}^{N} \overline{\alpha}^{(n)}_i \widetilde{c}^{(n)}_{i,t} \\
\widetilde{p}^{(n)}_{i,t} - \widetilde{p}^{(n)}_{N,t} &= \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}^{(n)}_k \sum_{l=1, l\neq k}^{N} \Bigg[\alpha^{(n)}_k \left(\left(\overline{\Phi}^{(n)}_{Nl} \right)^{-1} - \left(\overline{\Phi}^{(n)}_{il} \right)^{-1} \right) + \alpha^{(n)}_l \left(\left(\overline{\Phi}^{(n)}_{ik} \right)^{-1} - \left(\overline{\Phi}^{(n)}_{Nk} \right)^{-1} \right) \\
&\quad + \frac{\alpha^{(n)}_k \alpha^{(n)}_l}{\alpha^{(n)}_N} \left(\left(\overline{\Phi}^{(n)}_{Nl} \right)^{-1} - \left(\overline{\Phi}^{(n)}_{Nk} \right)^{-1} \right) + \frac{\alpha^{(n)}_k \alpha^{(n)}_l}{\alpha^{(n)}_i} \left(\left(\overline{\Phi}^{(n)}_{ik} \right)^{-1} - \left(\overline{\Phi}^{(n)}_{il} \right)^{-1} \right) \Bigg] \\
&\qquad \text{for } i \in [1, N-1]
\end{aligned}$$

for $n \in [1, N]$.[11]

We can see that the presence of the additional countries creates additional terms in the relative demand function, capturing the potential indirect substitution between goods i and N via goods k, l. Importantly, when we have perfect symmetry across countries in steady state, such that $\overline{\Phi}^{(n)}_{i,j} = \overline{\Phi}^{(n)}_{i,l}$ for all i, j and l, then these additional terms disappear from all relative demand functions.[12] This is why these terms were absent for the Armington aggregator. In this symmetric case, therefore, we can again replicate the first-order dynamics from the Kimball aggregator using an Armington aggregator by matching the steady-state consumption shares, and setting ϕ to match this common elasticity of substitution.

However, if we allow for steady-state asymmetries across countries, then these additional terms will create first-order effects that cannot be captured by an Armington aggregator. Notice that

[10] Full derivations are provided in Appendix C.

[11] As before, these expressions are derived within the proof of Theorem 1 in Appendix A for a generic aggregator.

[12] We use the convention that $\Phi_{i,i} = 0$ for all i, so that in the symmetric case, the terms of the equation where such same-good elasticities appear will not simplify away despite the symmetry, and the linearised equation remains valid.

it is again only the steady-state values of the elasticities that enter the linearised equations, and not any dynamic variation in the elasticity. Nonetheless, the Kimball aggregator allows us to map steady-state asymmetries in, say, endowments, into differences in elasticities of substitution, which then impacts the dynamics of the model.

Numerical Exercise with $N = 3$. To illustrate these effects from using the Kimball aggregator, we consider a three-country version of our model. We label the countries as $n = \{H, F, R\}$ and consider our results from the perspective of the Home country, H. For this stylised exercise, we set the discount factor $\beta = 0.99$, and assume the instantaneous utility function $u(.)$ takes the familiar constant relative risk aversion form, with the intertemporal elasticity of substitution equal to 2.

We set the parameters of the Armington and Kimball aggregators so that they would be equivalent in the symmetric steady state. In particular, we set $a_i^{(i)} = b_i^{(i)} = 0.7$ and $a_j^{(i)} = b_j^{(i)} = 0.15$ for all $i, j \in \{H, F, R\}$, $j \neq i$. This implies that in all symmetric steady states, the domestic expenditure share would be 70%, and the remaining expenditure share is split equally across the two foreign countries. Similarly, we set $\phi = \sigma = 1.5$, so that, under symmetry, in all cases $\overline{\Phi}_{i,j}^{(n)} = 1.5$ for all $i, j, n \in \{H, F, R\}$, $j \neq i$.

We depart from symmetry by assuming the endowment of country H is smaller than the endowment of F and R. Normalising these values, we set $\overline{Y}^{(F)} = \overline{Y}^{(R)} = 1$ and $\overline{Y}^{(H)} = 0.5$. In all cases, this reduction in the supply of country-H goods increases its relative price in steady state, though the magnitude of this effect will depend on the parameterisation of the aggregator.

In this asymmetric setting, we keep these parameters fixed and allow the elasticities and expenditure shares to vary. Table 1 shows these implied values across the different values of ϵ, where the $\epsilon = 0$ column corresponds to the Armington. It also reports the implied steady-state relative prices, specifically price of exports relative to imports, alongside the Home consumption of country-H goods relative to country-F goods. Due to the comparative scarcity of the Home good arising from the Home country's position as a small-open economy, its relative international price is higher than it would be in the symmetric case, and Home agents consume a comparative high share of their domestic good. Reflecting this high relative consumption and in line with the "kinked demand curve" mechanism explained with reference to Figure 1, the Home consumer's elasticity of substitution is declining in ϵ and smaller than in the CES case.

Figure 2 shows impulse response functions to a 2% increase in Home endowment. After a positive endowment shock, the aggregate consumption $C^{(H)}$ in the home country always increases, more so with CES or more convex Kimball preferences ($\epsilon < \phi$), while the Home real exchange rate depreciates as the relative price of the Home good decreases.

More interestingly, the Home consumer's consumption responses for goods F and R change qualitatively with the curvature of Kimball preferences. The intuition behind this is as follows. When their relative demand function is more concave (ϵ larger), given that they already consume a large quantity of Home good, the steady-state Home elasticity of substitution for H and F

Table 1: Steady-State Expenditure Shares and Elasticities of Substitution Under Asymmetry

	$\epsilon = 0$	$\epsilon = 0.5$	$\epsilon = 1.5$	$\epsilon = 3$	$\epsilon = 5$
$\bar{p}_H^{(H)}/\bar{p}_F^{(H)}$	1.479	1.459	1.414	1.347	1.279
$\bar{c}_H^{(H)}/\bar{c}_F^{(H)}$	3.837	3.938	4.117	4.307	4.450
$\bar{\alpha}_H^{(H)}$	65.736	66.321	67.306	68.290	68.990
$\bar{\alpha}_F^{(H)}$	17.132	16.839	16.347	15.855	15.505
$\bar{\Phi}_{H,F}^{(H)}$	1.500	1.371	1.172	0.990	0.860
$\bar{\Phi}_{F,R}^{(H)}$	1.500	1.325	1.086	0.891	0.762

Note: Due to the symmetry between F and R, $\bar{p}_F^{(H)} = \bar{p}_R^{(H)}$, $\bar{\alpha}_F^{(H)} \equiv \bar{\alpha}_R^{(H)}$ and $\bar{\Phi}_{H,F}^{(H)} \equiv \bar{\Phi}_{H,R}^{(H)}$.

goods $\overline{\Phi}_{H,F}^{(H)}$ is relatively low. So the Home consumer flocks towards the cheaper Home good more slowly. They will rather use their additional endowment to consume more of the F and R goods, financing it by selling Home goods. This triggers an increase in the imports of F and R goods when ϵ is high enough—$\epsilon = 5$ in Figure 2. Since both foreign countries, F and R, are completely symmetric in this example, the responses of both imports $c_F^{(H)}$ and $c_R^{(H)}$ are identical.

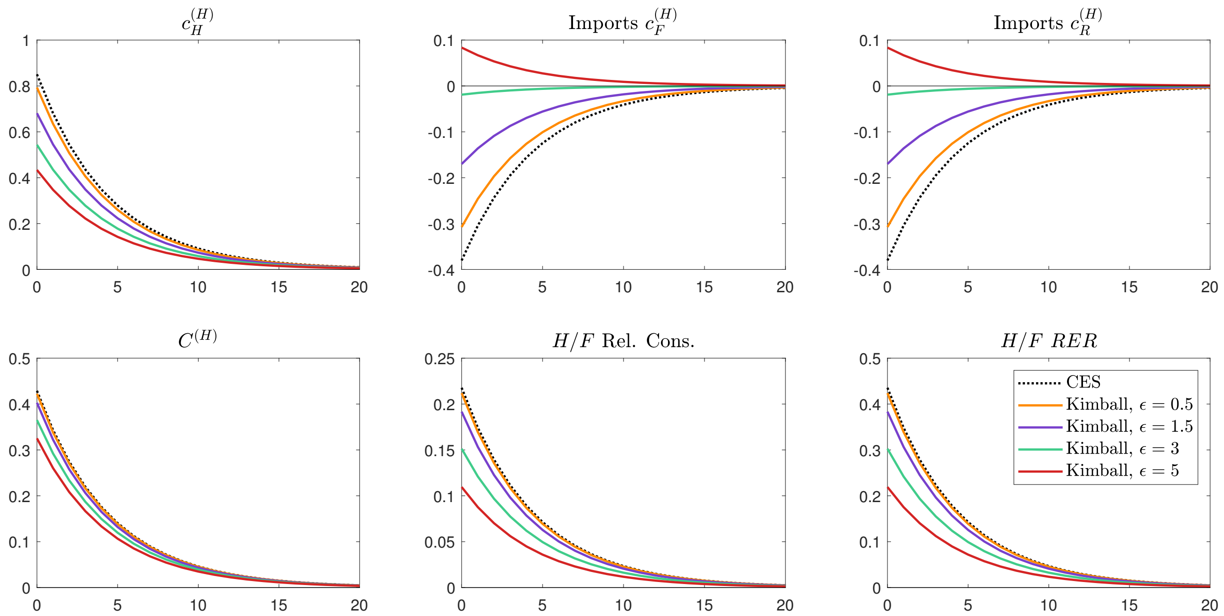
4.1.3 Nested CES with $N > 2$ Countries

If $N > 2$, then a nested CES with $(N-1)$ layers does not generically give enough flexibility to replicate the dynamics of any alternative aggregator.

So far we have compared Kimball to a single-layer CES aggregator, which implied by definition that the bilateral elasticities were the same across all country-good pairs. One way to gain flexibility, while retaining the tractability of the Armington aggregator, is to move to a nested-CES framework. This will imply $N-1$ layers, which allows for $N-1$ elasticity parameters instead of a single one. The question becomes whether this framework can replicate any alternative to the Armington aggregator, by matching all of the bilateral elasticities and expenditure shares.

The answer to this question is generically no, the nested-CES structure does not allow enough degrees of freedom to fully match the first-order dynamics with alternative aggregators when we have more than two countries. As with a single-layer CES, we can easily adjust the nested CES shares parameters to match the steady-state consumption shares of each good in each country. However, we have N bilateral elasticities to match for each country, but only $N - 1$ nested-CES elasticity parameters, and are therefore missing one degree of freedom. With nested-CES, the bilateral elasticities between each country-good pair are combinations of the parameters in each CES layer. This means that we can set the elasticity parameter in each layer recursively to match all steady-state bilateral elasticities but one. While it is true that, for any aggregator, knowing $N - 1$ bilateral elasticities, it is possible to recover the remaining N^{th} bilateral elasticity, the precise relationship that pins this down is specific to the aggregator. Hence, having matched $N - 1$ steady-state bilateral elasticities from the alternative aggregator, with an adequately parameterised nested CES, does not ensure that the remaining N^{th} steady-state bilateral elasticity will be equal to its equivalent with the alternative aggregator. This

Figure 2: IRFs to a 2% endowment shock in the Home country, asymmetric 3-country case



Note: The Home (H) country is assumed to be a small economy with steady-state endowment equal to 0.5, while the two other countries, (F) and (R), are large (endowment equal to 1). Dotted lines represent the CES responses with $\phi = 1.5$, while each solid line represents responses for different values of the Kimball 'curvature' parameter ϵ, with $\sigma = 1.5$. All consumers have symmetric preferences with home bias.

means that we cannot match all of the sufficient statistics given by Corollary 2, and so we cannot match the first-order dynamics.

As a concrete example, let us use again the 3-country setup presented in Section 4.1.2. We consider in each country $n \in \{H, F, R\}$, a nested-CES specification where the aggregate consumption is a CES aggregate of the locally produced good, and a bundle of imported goods. This implies the following specification and characteristics for country H:

$$C_t^{(H)} = f\left(c_{H,t}^{(H)}, c_{F,t}^{(H)}, c_{R,t}^{(H)}\right) = \left(a_H^{(H)\frac{1}{\phi_H}} c_{H,t}^{(H)\frac{\phi_H - 1}{\phi_H}} + \left(1 - a_H^{(H)}\right)^{\frac{1}{\phi_H}} C_{FR,t}^{(H)\frac{\phi_H - 1}{\phi_H}}\right)^{\frac{\phi_H}{\phi_H - 1}}$$

$$\text{where} \qquad C_{FR,t}^{(H)} = \left(a_F^{(H)\frac{1}{\phi_{FR}}} c_{F,t}^{(H)\frac{\phi_{FR} - 1}{\phi_{FR}}} + \left(1 - a_F^{(H)}\right)^{\frac{1}{\phi_{FR}}} C_{R,t}^{(H)\frac{\phi_{FR} - 1}{\phi_{FR}}}\right)^{\frac{\phi_{FR}}{\phi_{FR} - 1}}$$

The bilateral elasticities in country H become:[13]

$$\Phi_{HF}^{(H)} = \frac{\phi_H \phi_{FR} \left(\alpha_F^{(H)} + \alpha_H^{(H)} \alpha_R^{(H)}\right)}{\alpha_F^{(H)} \phi_{FR} + \alpha_H^{(H)} \alpha_R^{(H)} \phi_H}$$

$$\Phi_{HR}^{(H)} = \frac{\phi_H \phi_{FR} \left(\alpha_R^{(H)} + \alpha_H^{(H)} \alpha_F^{(H)}\right)}{\alpha_R^{(H)} \phi_{FR} + \alpha_H^{(H)} \alpha_F^{(H)} \phi_H}$$

$$\Phi_{FR}^{(H)} = \phi_{FR}$$

Suppose we want to set the parameters of these two CES aggregators so as to match the steady-state consumption shares and bilateral trade elasticities from a given parameterisation of the Kimball aggregator, with asymmetries, in order to replicate the first-order dynamics of the model. We can set the share parameters, $a_H^{(H)}$ and $a_F^{(H)}$, to match the steady-state consumption shares obtained from the Kimball aggregator directly. However, we now have two CES elasticity parameters, ϕ_H and ϕ_{FR}, to match the three bilateral elasticities.

In the specific case considered here, with symmetry across countries F and R, their steady-state consumption shares in country H are equal, $\overline{\alpha}_F^{(H)} = \overline{\alpha}_R^{(H)}$, which implies that their bilateral elasticities are also equal, $\Phi_{HR}^{(H)} = \Phi_{HF}^{(H)}$. As this is true in both the Kimball and the nested-CES specifications, this allows us to match the country H first-order relative demand equations using a nested-CES aggregator. However, this is not true any more when turning to country F. The endowment asymmetry across our three countries implies asymmetric steady-state consumption shares and bilateral elasticities in country F, as stated in Table 2. After matching country F's steady-state consumption shares, and two of its bilateral elasticities, we have no degree of freedom left to ensure that the third Kimball bilateral elasticity is matched by the nested-CES specification, and the nested-CES steady-state bilateral elasticity $\overline{\Phi}_{H,F}^{(F)}$ is not equal to the Kimball one. Consequently, a nested CES specification is not flexible enough to match the first-order dynamics of our Kimball 3-country example, due to the endowment asymmetry.

Table 2: Steady-State Expenditure Shares and Elasticities of Substitution for Nested-CES and Kimball Aggregators

	Nested CES	Kimball ($\epsilon = 5$)
$\overline{\alpha}_H^{(F)}$	9.913	9.913
$\overline{\alpha}_F^{(F)}$	74.189	74.189
$\overline{\alpha}_R^{(F)}$	15.898	15.898
$\overline{\Phi}_{H,F}^{(F)}$	1.362	5.590
$\overline{\Phi}_{H,R}^{(F)}$	2.519	2.519
$\overline{\Phi}_{F,R}^{(F)}$	1.109	1.109

[13] See Appendix D for computation details.

4.2 Non-Homothetic Preferences

Theorem 1 and its corollaries also have implications for non-homothetic preferences, meaning if the aggregator is not HOD(1).

First, notice that saying the aggregator is not HOD(1) can mean two things: that it is HOD(h), for $h \neq 1$, or that it is non-homogeneous. We focus on the former case. While Theorem 1 involves all ratios $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_i \quad \forall i = 1, 2, ...N$, recall that Corollary 1, by focusing specifically on homogeneous functions, only depends on h.

It is useful to again consider the two cases.

4.2.1 Case 1: Two Countries

If $N = 2$, then any HOD(h) aggregator, $h \in \mathcal{R}$, is equivalent at first order to a generalised Armington-style aggregator that is HOD(h), with the same steady-state elasticity and consumption shares.

We can define a generalisation of the 2-good Armington aggregator which is HOD(h):

$$F(c_1, c_2) = \left(a_1^{\frac{1}{\phi}} c_1^{\frac{\phi-1}{\phi}} + (1 - a_1)^{\frac{1}{\phi}} c_2^{\frac{\phi-1}{\phi}} \right)^{h \frac{\phi}{\phi-1}}$$

The parameters ϕ and a_1 have the same interpretation as before, and h is a free parameter that determines the degree of homogeneity.[14]

This means that any model that uses an alternative aggregator can be mapped parsimoniously into this generalised Armington aggregator by setting the parameters ϕ and a_1 to match the steady-state elasticity of substitution and consumption shares, as before, and setting $h = \overline{\mathcal{H}}$.

Notice again that, while these results show that deviating from HOD(1) aggregators can affect the first-order dynamics, even with $N = 2$, they also specify that the first-order effect of any HOD(h) aggregator relative to the standard Armington model is determined entirely by a single parameter, h.

4.2.2 Case 2: $N > 2$ Countries

If $N > 2$, then alternative aggregators HOD(h) can create differences with respect to the N-good generalised HOD(h) Armington aggregator, by allowing bilateral elasticities of substitution to be different across different pairs of goods in steady state.

[14] With this generalised aggregator, the relative demand function remains the same as in the CES case:

$$\frac{c_1}{c_2} = \frac{a_1}{1 - a_1} \left(\frac{p_2}{p_1} \right)^{\phi}$$

The trade elasticity is ϕ, and the formula for the consumption share remains $\alpha_1 = a_1 \left(\frac{p_1}{P} \right)^{1-\phi}$, while the ratio $\mathcal{H}$ is equal to h.

The N-good generalised HOD(h) Armington aggregator can be defined as:

$$F(c_1, ... c_N) = \left(\sum_{i=1}^{N} a_i^{\frac{1}{\phi}} c_i^{\frac{\phi-1}{\phi}} \right)^{h\frac{\phi}{\phi-1}}$$

The same reasoning as the HOD(1) case can be applied here, again with the addition that the parameter h is chosen correctly. Note, once again, that the additional mechanism that alternative HOD(h) aggregators bring when $N > 2$ is only through the cross-elasticity differences in steady state. These differences are also the ones that prevent matching a HOD(h) aggregator with a generalised nested-CES specification, using the generalised Armington.

5 Conclusions

We have shown that the first-order dynamics of models that aggregate goods from multiple countries into one consumption bundle can be summarised by sufficient statistics that reflect the characteristics of the aggregation function. These sufficient statistics include the steady-state values of consumption shares, bilateral elasticities and ratios related to the degree of homogeneity of the aggregator. This main result can be unpacked into a number of more specific implications.

First, in a two-country model, the standard Armington aggregator is equivalent at first order to any other aggregator that is homogeneous of degree one, with the same elasticity of substitution and consumption expenditure shares in steady state. We have also put forward a parsimonious generalisation of the Armington aggregator that is homogeneous of arbitrary degree, h. In a two-country setup again, this generalised Armington aggregator is equivalent at first order to any aggregator with the same elasticity of substitution and consumption expenditure shares in steady state, and the same degree of homogeneity.

Second, when the number of countries, N, is larger than two, the Armington aggregator can become restrictive to the extent that it imposes that the bilateral elasticities of substitution of each pair of goods are given by the same parameter. Other aggregators that allow these elasticities to be different in steady state can therefore affect the first-order dynamics of the model. However, again, this implies that the channel through which these aggregators affect the model is captured entirely by the asymmetries in the steady-state pair-wise elasticities of substitution. We also showed that a nested-CES structure, nesting $(N-1)$ Armington aggregators, does not provide enough degrees of freedom to generically replicate alternative aggregators. Similarly, when compared to an alternative aggregator that is homogeneous of degree h, our generalised Armington aggregator can replicate the first-order dynamics under symmetry, but not under asymmetry, due to the differences in the steady-state elasticities of substitution across different pairs of goods.

Notice that throughout the results, only the steady-state elasticities of substitution affected the first-order dynamics of the model. This means that the standard mechanism that many alternative aggregators are used to capture in dynamic models—varying elasticities of substitution across time—does not have a first-order effect in these models.

For clarity, the model we laid out at in Section 2 was a simple endowment economy. However, Theorem 1, and its corollaries, would continue to hold if we introduced a perfectly competitive production sector, which does not take the demand structure into account. This is because the intratemporal consumption-demand block of the model, which is the block which depends on the aggregator function, remains separate to the production side of the model. Moreover, in that case, while we have focused here on the *consumption* aggregator, the same results would hold if we looked at the aggregators used for other types of goods, such as intermediate inputs or investment goods, so long as the optimal composition of these goods, between domestic and foreign goods, remains an intratemporal problem.

Finally, we derived all of these results analytically in a linearised model, and so they hold exactly at first order. This means that, in principle, the alternative aggregators may have further effects on the dynamics of the model at higher orders. However, the standard workhorse NOEM model is very close to being linear, meaning that these higher-order effects are small by definition, especially for the standard size of shocks. We leave it for future research to explore the impact of the trade aggregator choice in different settings in which non-linearities and higher-order effects may matter more.

A Proof of Theorem 1

For each country n, the relevant system of equations that are affected by the aggregator function, f, and the individual consumption levels, $\mathbf{c}_t^{(n)}$, are the definition of aggregate consumption, the definition of the price index, the $N-1$ relative demand functions, and the goods market clearing condition:

$$\begin{aligned}
C^{(n)} &= f(\mathbf{c}^{(n)}) \\
C^{(n)} &= \sum_{i=1}^{N} p_i^{(n)} c_i^{(n)} \\
\frac{p_i^{(n)}}{p_N^{(n)}} &= \frac{f_i^{(n)}}{f_N^{(n)}} \quad \forall\, i = 1, \dots N-1 \\
Y^{(n)} &= \sum_{i=1}^{N} c_n^{(i)}
\end{aligned}$$

where we have dropped the time subscripts for simplicity, and redefined prices to be relative to the CPI. We want to derive the log-linear form of these equations to understand what drives the first-order dynamics, and in particular how it depends on the function f.

To do this, we will apply the general formula for the first-order Taylor expansion. Write each equation in a generic format $F(\mathbf{x}) = 0$, where $\mathbf{x}$ is the vector of all model variables. Then the multivariate first-order Taylor expansion around a point $\overline{\mathbf{x}}$ is given by:

$$\begin{aligned}
F(\mathbf{x}) &\approx \left(F'(\mathbf{x})\,|_{\mathbf{x}=\overline{\mathbf{x}}}\right)' (\mathbf{x} - \overline{\mathbf{x}}) \\
&= \sum_i \left.\frac{\partial F(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} (x_i - \overline{x}_i) \\
&= \sum_i F_i(\overline{\mathbf{x}}) \overline{x}_i \widetilde{x}_i
\end{aligned}$$

where we use the notation $\widetilde{x} \equiv (x - \overline{x})/\overline{x}$, where $\overline{x}$ denotes the steady state. We will apply this formula to each of the equations above.

Aggregate Consumption

$$
\begin{aligned}
C^{(n)} &= f(\mathbf{c}^{(n)}) \\
0 &= C^{(n)} - f(\mathbf{c}^{(n)}) \\
&\approx \overline{C}^{(n)}\widetilde{C}^{(n)} - \sum_{i=1}^{N} \overline{f}_i \overline{c}_i^{(n)} \widetilde{c}_i^{(n)} \\
&= \widetilde{C}^{(n)} - \sum_{i=1}^{N} \frac{\overline{f}_i \overline{c}_i^{(n)}}{\overline{C}^{(n)}} \widetilde{c}_i^{(n)} \\
\widetilde{C}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{f}_i \overline{c}_i^{(n)}}{\overline{C}^{(n)}} \widetilde{c}_i^{(n)}
\end{aligned}
$$

To simplify this equation, recall the FOCs of the cost-minimisation problem defined above:

$$Pp_i^{(n)} = \lambda f_i \quad \forall\, i = 1, ... N$$

We can solve for the Lagrange multiplier using the definition of the aggregate price index:

$$
\begin{aligned}
C^{(n)} &= \sum_{i=1}^{N} p_i^{(n)} c_i^{(n)} \\
&= \sum_{i=1}^{N} \frac{\lambda f_i}{P} c_i^{(n)} \\
&= \frac{\lambda}{P} \sum_{i=1}^{N} f_i c_i^{(n)} \\
\lambda &= \frac{P}{\mathcal{H}(\mathbf{c}^{(n)})}
\end{aligned}
$$

where

$$\mathcal{H}(\mathbf{c}^{(n)}) = \frac{\sum_{i=1}^{N} f_i c_i^{(n)}}{C^{(n)}} = \frac{\sum_{i=1}^{N} f_i c_i^{(n)}}{f(\mathbf{c}^{(n)})}$$

Plugging this into the FOCs:

$$Pp_i^{(n)} = \frac{P}{\mathcal{H}(\mathbf{c}^{(n)})} f_i$$

or

$$p_i^{(n)} = \frac{f_i}{\mathcal{H}(\mathbf{c}^{(n)})} \qquad \Rightarrow \qquad f_i = p_i^{(n)} \mathcal{H}(\mathbf{c}^{(n)})$$

Hence

$$\frac{\overline{f}_i \overline{c}_i^{(n)}}{\overline{C}^{(n)}} = \mathcal{H}(\overline{\mathbf{c}}^{(n)}) \frac{\overline{p}_i^{(n)} \overline{c}_i^{(n)}}{\overline{C}^{(n)}}$$

Define the steady state share of consumption expenditure on good j:

$$\alpha_i^{(n)} \equiv \frac{\overline{p}_i^{(n)}\overline{c}_i^{(n)}}{\overline{C}^{(n)}}$$

Putting these together, denoting $\mathcal{H}(\overline{\mathbf{c}}^{(n)}) \equiv \overline{\mathcal{H}}$, the linearised form of the aggregator is given by:

$$\widetilde{C}^{(n)} \approx \overline{\mathcal{H}} \sum_{i=1}^{N} \alpha_i^{(n)} \widetilde{c}_i^{(n)}$$

This depends on $\overline{\mathcal{H}}$ and $\alpha_i^{(n)}$ for $i = 1, \ldots N$.

Consumer Price Index

$$\begin{aligned}
C^{(n)} &= \sum_{i=1}^{N} p_i^{(n)} c_i^{(n)} \\
0 &= C^{(n)} - \sum_{i=1}^{N} p_i^{(n)} c_i^{(n)} \\
&\approx \overline{C}^{(n)}\widetilde{C}^{(n)} - \sum_{i=1}^{N} \overline{p}_i^{(n)}\overline{c}_i^{(n)}\widetilde{c}_i^{(n)} - \sum_{i=1}^{N} \overline{c}_i^{(n)}\overline{p}_i^{(n)}\widetilde{p}_i^{(n)} \\
&\approx \widetilde{C}^{(n)} - \sum_{i=1}^{N} \frac{\overline{p}_i^{(n)}\overline{c}_i^{(n)}}{\overline{C}^{(n)}}\widetilde{c}_i^{(n)} - \sum_{i=1}^{N} \frac{\overline{p}_i^{(n)}\overline{c}_i^{(n)}}{\overline{C}^{(n)}}\widetilde{p}_i^{(n)} \\
&\approx \widetilde{C}^{(n)} - \sum_{i=1}^{N} \alpha_i^{(n)}\widetilde{c}_i^{(n)} - \sum_{i=1}^{N} \alpha_i^{(n)}\widetilde{p}_i^{(n)} \\
&\approx \widetilde{C}^{(n)} - \frac{1}{\overline{\mathcal{H}}}\widetilde{C}^{(n)} - \sum_{i=1}^{N} \alpha_i^{(n)}\widetilde{p}_i^{(n)} \\
\frac{\overline{\mathcal{H}} - 1}{\overline{\mathcal{H}}}\widetilde{C}^{(n)} &\approx \sum_{i=1}^{N} \alpha_i^{(n)}\widetilde{p}_i^{(n)}
\end{aligned}$$

where the linearised form of the aggregator was used to simplify the equation. Again, this depends on $\overline{\mathcal{H}}$ and $\alpha_i^{(n)}$ for $i = 1, \ldots N$.

Relative Demand Functions

Consider a specific i without loss of generality:

$$
\begin{aligned}
\frac{p_i^{(n)}}{p_N^{(n)}} &= \frac{f_i(\mathbf{c}^{(n)})}{f_N(\mathbf{c}^{(n)})} \\
0 &= \frac{p_i^{(n)}}{p_N^{(n)}} - \frac{f_i(\mathbf{c}^{(n)})}{f_N(\mathbf{c}^{(n)})} \\
0 &\approx \frac{1}{\overline{p}_N^{(n)}}\overline{p}_i^{(n)}\widetilde{p}_i^{(n)} - \frac{\overline{p}_i^{(n)}}{\overline{p}_N^2}\overline{p}_N^{(n)}\widetilde{p}_N^{(n)} - \sum_{k=1}^{N} \left.\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k^{(n)}}\right|_{ss} \overline{c}_k^{(n)}\widetilde{c}_k^{(n)} \\
&= \frac{\overline{p}_i^{(n)}}{\overline{p}_N^{(n)}}\left(\widetilde{p}_i^{(n)} - \widetilde{p}_N^{(n)}\right) - \sum_{k=1}^{N} \left.\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k^{(n)}}\right|_{ss} \overline{c}_k^{(n)}\widetilde{c}_k^{(n)}
\end{aligned}
$$

Consider the partial derivative term:

$$
\begin{aligned}
\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k^{(n)}} &= \frac{1}{f_N}\frac{\partial f_i}{\partial c_k^{(n)}} - \frac{f_i}{f_N^2}\frac{\partial f_N}{\partial c_k^{(n)}} \\
&= \frac{f_{ik}}{f_N} - \frac{f_i f_{Nk}}{f_N^2} \\
&= \frac{f_i}{f_N}\left(\frac{f_{ik}}{f_i} - \frac{f_{Nk}}{f_N}\right)
\end{aligned}
$$

Plugging this back in:

$$
\begin{aligned}
0 &\approx \frac{\overline{p}_i^{(n)}}{\overline{p}_N^{(n)}}\left(\widetilde{p}_i^{(n)} - \widetilde{p}_N^{(n)}\right) - \sum_{k=1}^{N} \frac{f_i}{f_N}\left.\left(\frac{f_{ik}}{f_i} - \frac{f_{Nk}}{f_N}\right)\right|_{ss} \overline{c}_k^{(n)}\widetilde{c}_k^{(n)} \\
&= \frac{\overline{p}_i^{(n)}}{\overline{p}_N^{(n)}}\left(\widetilde{p}_i^{(n)} - \widetilde{p}_N^{(n)}\right) - \frac{\overline{f}_i}{\overline{f}_N}\sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k^{(n)}\widetilde{c}_k^{(n)}
\end{aligned}
$$

Using the fact that $\overline{p}_i^{(n)}/\overline{p}_N^{(n)} = \overline{f}_i/\overline{f}_N$:

$$
\widetilde{p}_i^{(n)} - \widetilde{p}_N^{(n)} \approx \sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k^{(n)}\widetilde{c}_k^{(n)} = \sum_{k=1}^{N} coef_k^{(iN)}\widetilde{c}_k^{(n)}
$$

where $coef_k^{(iN)} \equiv \left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k^{(n)}$.

Consider now the definition of the elasticity of substitution between two different goods x and

y (we consider here the direct partial elasticity as defined by McFadden (1963) or Sato (1967)):

$$\Phi_{xy} = \frac{\partial \ln\left(c_x^{(n)}/c_y^{(n)}\right)}{\partial \ln\left(f_y/f_x\right)} = -\left(\frac{1}{c_x^{(n)}f_x} + \frac{1}{c_y^{(n)}f_y}\right)\left[\left(\frac{f_{xx}}{f_x^2} - \frac{f_{xy}}{f_xf_y}\right) + \left(\frac{f_{yy}}{f_y^2} - \frac{f_{xy}}{f_xf_y}\right)\right]^{-1}$$

In a first step, we derive some relationships between the coefficients of the linearised relative demand function, the steady state bilateral elasticities and the steady state consumption shares.

$$\begin{aligned}
\overline{\Phi}_{iN}^{-1} &= -\left[\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2} - \frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right) + \left(\frac{\overline{f}_{NN}}{\overline{f}_N^2} - \frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right)\right]\left(\frac{1}{\overline{c}_i^{(n)}\overline{f}_i} + \frac{1}{\overline{c}_N^{(n)}\overline{f}_N}\right)^{-1} \\
&= -\left[\frac{1}{\overline{f}_i\overline{c}_i^{(n)}}coef_i^{(iN)} - \frac{1}{\overline{f}_N\overline{c}_N^{(n)}}coef_N^{(iN)}\right]\left(\frac{1}{\overline{c}_i^{(n)}\overline{f}_i} + \frac{1}{\overline{c}_N^{(n)}\overline{f}_N}\right)^{-1} \\
&= -\left[\frac{1}{\overline{f}_i\overline{c}_i^{(n)}}coef_i^{(iN)} - \frac{1}{\overline{f}_N\overline{c}_N^{(n)}}coef_N^{(iN)}\right]\frac{\overline{c}_i^{(n)}\overline{c}_N^{(n)}\overline{f}_i\overline{f}_N}{\overline{c}_N^{(n)}\overline{f}_N + \overline{c}_i^{(n)}\overline{f}_i} \\
&= -\frac{\overline{c}_N^{(n)}\overline{f}_N}{\overline{c}_i^{(n)}\overline{f}_i + \overline{c}_N^{(n)}\overline{f}_N}coef_i^{(iN)} + \frac{\overline{c}_i^{(n)}\overline{f}_i}{\overline{c}_i^{(n)}\overline{f}_i + \overline{f}_N\overline{c}_N^{(n)}}coef_N^{(iN)}
\end{aligned}$$

Using the definition of the steady state expenditure shares:

$$\begin{aligned}
\frac{\overline{c}_i^{(n)}\overline{f}_i}{\overline{c}_i^{(n)}\overline{f}_i + \overline{f}_N\overline{c}_N^{(n)}} &= \frac{\overline{c}_i^{(n)}\frac{\overline{f}_i}{\overline{f}_N}}{\overline{c}_i^{(n)}\frac{\overline{f}_i}{\overline{f}_N} + \overline{c}_N^{(n)}} = \frac{\overline{c}_i^{(n)}\frac{\overline{p}_i^{(n)}}{\overline{p}_N^{(n)}}}{c_i^{(n)}\frac{\overline{p}_i^{(n)}}{\overline{p}_N^{(n)}} + \overline{c}_N^{(n)}} \\
&= \frac{\overline{c}_i^{(n)}\overline{p}_i^{(n)}}{\overline{c}_i^{(n)}\overline{p}_i^{(n)} + \overline{p}_N^{(n)}\overline{c}_N^{(n)}} = \frac{\overline{c}_i^{(n)}\overline{p}_i^{(n)}}{\sum_l \overline{c}_l^{(n)}\overline{p}_l^{(n)}}\frac{\sum_l \overline{c}_l^{(n)}\overline{p}_l^{(n)}}{\overline{c}_i^{(n)}\overline{p}_i^{(n)} + \overline{p}_N^{(n)}\overline{c}_N^{(n)}} \\
&= \frac{\alpha_i^{(n)}}{\alpha_i^{(n)} + \alpha_N^{(n)}}
\end{aligned}$$

And we obtain:

$$\overline{\Phi}_{iN}^{-1} = -\frac{\alpha_N^{(n)}}{\alpha_i^{(n)} + \alpha_N^{(n)}}coef_i^{(iN)} + \frac{\alpha_i^{(n)}}{\alpha_i^{(n)} + \alpha_N^{(n)}}coef_N^{(iN)}$$

i.e.

$$(\alpha_i^{(n)} + \alpha_N^{(n)})\overline{\Phi}_{iN}^{-1} = -\alpha_N^{(n)}coef_i^{(iN)} + \alpha_i^{(n)}coef_N^{(iN)} \tag{6}$$

Equation (6) is the first type of relationship we were aiming for, and is true for every $i = 1, 2, ..., N-1$. Now we derive a second type of relationship, involving two bilateral elasticities.

$$
\begin{aligned}
&\left(\frac{1}{\overline{c}_i^{(n)}\overline{f}_i}+\frac{1}{\overline{c}_k^{(n)}\overline{f}_k}\right)\overline{\Phi}_{ik}^{-1}-\left(\frac{1}{\overline{c}_N^{(n)}\overline{f}_N}+\frac{1}{\overline{c}_k^{(n)}\overline{f}_k}\right)\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\left[\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)+\left(\frac{\overline{f}_{kk}}{\overline{f}_k^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)\right]\\
&\qquad\quad+\left[\left(\frac{\overline{f}_{NN}}{\overline{f}_N^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)+\left(\frac{\overline{f}_{kk}}{\overline{f}_k^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)\right]\\
&\qquad=-\frac{1}{\overline{f}_i}\left(\frac{\overline{f}_{ii}}{\overline{f}_i}-\frac{\overline{f}_{ik}}{\overline{f}_k}\right)-\frac{1}{\overline{f}_k}\left(\frac{\overline{f}_{kk}}{\overline{f}_k}-\frac{\overline{f}_{ik}}{\overline{f}_i}\right)\\
&\qquad\quad+\frac{1}{\overline{f}_N}\left(\frac{\overline{f}_{NN}}{\overline{f}_N}-\frac{\overline{f}_{Nk}}{\overline{f}_k}\right)+\frac{1}{\overline{f}_k}\left(\frac{\overline{f}_{kk}}{\overline{f}_k}-\frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\\
&\qquad=-\frac{1}{\overline{f}_i}\left(\frac{\overline{f}_{ii}}{\overline{f}_i}-\frac{\overline{f}_{ik}}{\overline{f}_k}\right)+\frac{1}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}+\frac{1}{\overline{f}_N}\left(\frac{\overline{f}_{NN}}{\overline{f}_N}-\frac{\overline{f}_{Nk}}{\overline{f}_k}\right)\\
&\qquad=-\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)+\frac{1}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}+\left(\frac{\overline{f}_{NN}}{\overline{f}_N^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)\\
&\qquad=-\frac{\overline{f}_{ii}}{\overline{f}_i^2}+\frac{1}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}+\frac{\overline{f}_{NN}}{\overline{f}_N^2}+\frac{1}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}\\
&\qquad=-\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right)-\left(\frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}-\frac{\overline{f}_{NN}}{\overline{f}_N^2}\right)+\frac{2}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}\\
&\qquad=-\frac{1}{\overline{c}_i^{(n)}\overline{f}_i}coef_i^{(iN)}-\frac{1}{\overline{c}_N^{(n)}\overline{f}_N}coef_N^{(iN)}+\frac{2}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}
\end{aligned}
$$

Now bringing back expenditure shares as above:

$$
\begin{aligned}
&\left(\frac{\sum_l\overline{c}_l^{(n)}\overline{f}_l}{c_i^{(n)}\overline{f}_i}+\frac{\sum_l\overline{c}_l^{(n)}\overline{f}_l}{\overline{c}_k^{(n)}\overline{f}_k}\right)\overline{\Phi}_{ik}^{-1}-\left(\frac{\sum_l\overline{c}_l^{(n)}\overline{f}_l}{\overline{c}_N^{(n)}\overline{f}_N}+\frac{\sum_l\overline{c}_l^{(n)}\overline{f}_l}{\overline{c}_k^{(n)}\overline{f}_k}\right)\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\frac{\sum_l\overline{c}_l^{(n)}\overline{f}_l}{\overline{c}_i^{(n)}\overline{f}_i}coef_i^{(iN)}-\frac{\sum_l\overline{c}_l^{(n)}\overline{f}_l}{\overline{c}_N^{(n)}\overline{f}_N}coef_N^{(iN)}+\frac{2\sum_l\overline{c}_l^{(n)}\overline{f}_l}{\overline{c}_k^{(n)}\overline{f}_k}coef_k^{(iN)}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{1}{\alpha_i^{(n)}}+\frac{1}{\alpha_k^{(n)}}\right)\overline{\Phi}_{ik}^{-1}-\left(\frac{1}{\alpha_N^{(n)}}+\frac{1}{\alpha_k^{(n)}}\right)\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\frac{1}{\alpha_i^{(n)}}coef_i^{(iN)}-\frac{1}{\alpha_N^{(n)}}coef_N^{(iN)}+\frac{2}{\alpha_k^{(n)}}coef_k^{(iN)}\\
&(\alpha_i^{(n)}+\alpha_k^{(n)})\alpha_N^{(n)}\overline{\Phi}_{ik}^{-1}-(\alpha_k^{(n)}+\alpha_N^{(n)})\alpha_i^{(n)}\overline{\Phi}_{Nk}^{-1}\\
&\qquad=-\alpha_k^{(n)}\alpha_N^{(n)}coef_i^{(iN)}-\alpha_i^{(n)}\alpha_k^{(n)}coef_N^{(iN)}+2\alpha_i^{(n)}\alpha_N^{(n)}coef_k^{(iN)} \qquad (7)
\end{aligned}
$$

Equation (7) is our second type of relationship, and is valid for all $i = 1, 2, ..., N-1$ and for all $k \neq i, N$. Now, we can use the relationships obtained in equations (6) and (7) to express the linearised relative demand function as a function of the steady state expenditure shares, elasticities and ratios $\mathcal{H}$ and $\mathcal{H}_j$.

From equation (6), we have:

$$coef_N^{(iN)} = \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} + \frac{\alpha_N^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} \tag{8}$$

And from equation (7), for all $k \neq i, N$:

$$\begin{aligned}
coef_k^{(iN)} =& \frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{\alpha_k^{(n)}}{2\alpha_i^{(n)}} coef_i^{(iN)} + \frac{\alpha_k^{(n)}}{2\alpha_N^{(n)}} coef_N^{(iN)} \\
=& \frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{\alpha_k^{(n)}}{2\alpha_i^{(n)}} coef_i^{(iN)} \\
&+ \frac{\alpha_k^{(n)}}{2\alpha_N^{(n)}} \left(\frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} + \frac{\alpha_N^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} \right) \\
=& \frac{\alpha_k^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} + \frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{2\alpha_N^{(n)}\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1}
\end{aligned} \tag{9}$$

Plugging expressions (8) and (9) in the linearised relative demand function:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& coef_i^{(iN)} \widetilde{c}_i^{(n)} + \sum_{k=1,k\neq i}^{N-1} coef_k^{(iN)} \widetilde{c}_k^{(n)} + coef_N^{(iN)} \widetilde{c}_N^{(n)} \\
=& coef_i^{(iN)} \widetilde{c}_i^{(n)} \\
&+ \sum_{k=1,k\neq i}^{N-1} \left(\frac{\alpha_k^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} + \frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{2\alpha_N^{(n)}\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k^{(n)} \\
&+ \left(\frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} + \frac{\alpha_N^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} \right) \widetilde{c}_N^{(n)} \\
=& coef_i^{(iN)} \widetilde{c}_i^{(n)} + \frac{\alpha_N^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} \widetilde{c}_N^{(n)} + \sum_{k=1,k\neq i}^{N-1} \left(\frac{\alpha_k^{(n)}}{\alpha_i^{(n)}} coef_i^{(iN)} \right) \widetilde{c}_k^{(n)} \\
&+ \sum_{k=1,k\neq i}^{N-1} \left(\frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{2\alpha_N^{(n)}\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k^{(n)} \\
&+ \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \widetilde{c}_N^{(n)} \\
=& \frac{1}{\alpha_i^{(n)}} coef_i^{(iN)} \left(\alpha_i^{(n)} \widetilde{c}_i^{(n)} + \alpha_N^{(n)} \widetilde{c}_N^{(n)} + \sum_{k=1,k\neq i}^{N-1} \alpha_k^{(n)} \widetilde{c}_k^{(n)} \right) \\
&+ \sum_{k=1,k\neq i}^{N-1} \left(\frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{2\alpha_N^{(n)}\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k^{(n)} \\
&+ \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \widetilde{c}_N^{(n)} \\
=& \frac{1}{\alpha_i^{(n)}} \Bigg[coef_i^{(iN)} \left(\sum_{k=1}^{N} \alpha_k^{(n)} \widetilde{c}_k^{(n)} \right) \\
&+ \frac{1}{2} \sum_{k=1,k\neq i}^{N-1} \left(\left(\alpha_i^{(n)} + \alpha_k^{(n)} \right) \overline{\Phi}_{ik}^{-1} - \frac{(\alpha_k^{(n)} + \alpha_N^{(n)})\alpha_i^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k^{(n)} \\
&+ (\alpha_i^{(n)} + \alpha_N^{(n)}) \overline{\Phi}_{iN}^{-1} \widetilde{c}_N^{(n)} \Bigg]
\end{aligned}
$$

From the aggregate consumption linearisation we know that:

$$
\widetilde{C}^{(n)} \approx \overline{\mathcal{H}} \sum_{k=1}^{N} \alpha_k^{(n)} \widetilde{c}_k^{(n)}
$$

So we get:

$$
\begin{aligned}
\alpha_i^{(n)} (\widetilde{p}_i - \widetilde{p}_N) =& coef_i^{(iN)} \frac{\widetilde{C}^{(n)}}{\overline{\mathcal{H}}} \\
&+ \frac{1}{2} \sum_{k=1,k\neq i}^{N-1} \left(\left(\alpha_i^{(n)} + \alpha_k^{(n)} \right) \overline{\Phi}_{ik}^{-1} - \frac{(\alpha_k^{(n)} + \alpha_N^{(n)})\alpha_i^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k^{(n)} \\
&+ (\alpha_i^{(n)} + \alpha_N^{(n)}) \overline{\Phi}_{iN}^{-1} \widetilde{c}_N^{(n)} \qquad (10)
\end{aligned}
$$

With a similar approach, still using equations (6) and (7), we can obtain the following expressions for the coefficients and the linearised relative demand function:

$$coef_i^{(iN)} = \frac{\alpha_i^{(n)}}{\alpha_N^{(n)}} coef_N^{(iN)} - \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1}$$

$$coef_k^{(iN)} = \frac{\alpha_k^{(n)}}{\alpha_N^{(n)}} coef_N^{(iN)} + \frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_k^{(n)}(\alpha_i^{(n)} + \alpha_N^{(n)})}{2\alpha_i^{(n)}\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \quad \forall k \neq i, N$$

implying:

$$\begin{aligned} \alpha_N^{(n)} (\widetilde{p}_i - \widetilde{p}_N) &= coef_N^{(iN)} \frac{\widetilde{C}^{(n)}}{\overline{\mathcal{H}}} \\ &+ \frac{1}{2} \sum_{k=1, k \neq i}^{N-1} \left(\frac{(\alpha_i^{(n)} + \alpha_k^{(n)})\alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} - \left(\alpha_k^{(n)} + \alpha_N^{(n)}\right) \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_k^{(n)}(\alpha_i^{(n)} + \alpha_N^{(n)})}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k^{(n)} \\ &- (\alpha_i^{(n)} + \alpha_N^{(n)}) \overline{\Phi}_{iN}^{-1} \widetilde{c}_i^{(n)} \end{aligned} \tag{11}$$

Using again a similar approach, from equations (6) and (7):

$$\begin{aligned} coef_N^{(iN)} =& \frac{\alpha_k^{(n)} + \alpha_N^{(n)}}{2\alpha_k^{(n)}} \overline{\Phi}_{Nk}^{-1} - \frac{(\alpha_i^{(n)} + \alpha_k^{(n)})\alpha_N^{(n)}}{2\alpha_i^{(n)}\alpha_k^{(n)}} \overline{\Phi}_{ik}^{-1} + \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \\ &+ \frac{\alpha_N^{(n)}}{\alpha_k^{(n)}} coef_k^{(iN)} \quad \forall k \neq i, N \end{aligned} \tag{12}$$

$$\begin{aligned} coef_i^{(iN)} =& \frac{\alpha_i^{(n)}}{\alpha_N^{(n)}} coef_N^{(iN)} - \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \\ =& \frac{\alpha_i^{(n)}}{\alpha_k^{(n)}} coef_k^{(iN)} + \frac{\alpha_i^{(n)}(\alpha_k^{(n)} + \alpha_N^{(n)})}{2\alpha_k^{(n)}\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_i^{(n)} + \alpha_k^{(n)}}{2\alpha_k^{(n)}} \overline{\Phi}_{ik}^{-1} \\ &- \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \quad \forall k \neq i, N \end{aligned} \tag{13}$$

Considering a specific $k \neq i, N$ without loss of generality, we now need to also express $coef_l^{(iN)}$ $(l \neq i, N, k)$ as a function of $coef_k^{(iN)}$, steady state bilateral elasticities and consumption shares. Rewriting equation (13) for any $l \neq i, N, k$:

$$coef_N^{(iN)} = \frac{\alpha_l^{(n)} + \alpha_N^{(n)}}{2\alpha_l^{(n)}} \overline{\Phi}_{Nl}^{-1} - \frac{(\alpha_i^{(n)} + \alpha_l^{(n)})\alpha_N^{(n)}}{2\alpha_i^{(n)}\alpha_l^{(n)}} \overline{\Phi}_{il}^{-1} + \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} + \frac{\alpha_N^{(n)}}{\alpha_l^{(n)}} coef_l^{(iN)}$$

Implying for all $l \neq i, N, k$:

$$\begin{aligned}
coef_l^{(iN)} =& \frac{\alpha_l^{(n)}}{\alpha_N^{(n)}} \left(coef_N^{(iN)} + \frac{(\alpha_i^{(n)} + \alpha_l^{(n)})\alpha_N^{(n)}}{2\alpha_i^{(n)}\alpha_l^{(n)}} \overline{\Phi}_{il}^{-1} - \frac{\alpha_l^{(n)} + \alpha_N^{(n)}}{2\alpha_l^{(n)}} \overline{\Phi}_{Nl}^{-1} - \frac{\alpha_i^{(n)} + \alpha_N^{(n)}}{2\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \\
=& \frac{\alpha_l^{(n)}}{\alpha_k^{(n)}} coef_k^{(iN)} + \frac{\alpha_l^{(n)}(\alpha_k^{(n)} + \alpha_N^{(n)})}{2\alpha_k^{(n)}\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} \\
& - \frac{\alpha_l^{(n)}(\alpha_i^{(n)} + \alpha_k^{(n)})}{2\alpha_i^{(n)}\alpha_k^{(n)}} \overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_l^{(n)})}{2\alpha_i^{(n)}} \overline{\Phi}_{il}^{-1} - \frac{\alpha_l^{(n)} + \alpha_N^{(n)}}{2\alpha_N^{(n)}} \overline{\Phi}_{Nl}^{-1} \qquad (14)
\end{aligned}$$

We can again plug the expressions (12) to (14) into the linearised relative demand function and obtain after some manipulations:

$$\begin{aligned}
\alpha_k^{(n)} (\widetilde{p}_i - \widetilde{p}_N) =& coef_k^{(iN)} \frac{\widetilde{C}^{(n)}}{\overline{\mathcal{H}}} \\
& + \frac{1}{2} \left(\frac{\alpha_i^{(n)}(\alpha_k^{(n)} + \alpha_N^{(n)})}{\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} - \left(\alpha_i^{(n)} + \alpha_k^{(n)}\right) \overline{\Phi}_{ik}^{-1} - \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_i^{(n)} \\
& + \frac{1}{2} \left(\left(\alpha_k^{(n)} + \alpha_N^{(n)}\right) \overline{\Phi}_{Nk}^{-1} - \frac{(\alpha_i^{(n)} + \alpha_k^{(n)})\alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_N^{(n)} \\
& + \frac{1}{2} \sum_{l=1, l\neq i,k}^{N-1} \left(\frac{\alpha_l^{(n)}(\alpha_k^{(n)} + \alpha_N^{(n)})}{\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} \frac{\alpha_l^{(n)}(\alpha_i^{(n)} + \alpha_k^{(n)})}{\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} \right. \\
& \left. + \frac{(\alpha_i^{(n)} + \alpha_l^{(n)})\alpha_k^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{il}^{-1} - \frac{(\alpha_l^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{Nl}^{-1} \right) \widetilde{c}_l^{(n)} \qquad (15)
\end{aligned}$$

Equation (15) is valid for any $k \neq i, N$. Now let's sum equations (10), (11) and all (15) for all $k \neq i, N$ and notice that by definition of the consumption shares:

$$\alpha_i^{(n)} (\widetilde{p}_i - \widetilde{p}_N) + \alpha_N^{(n)} (\widetilde{p}_i - \widetilde{p}_N) + \sum_{k=1, k\neq i}^{N-1} \alpha_k^{(n)} (\widetilde{p}_i - \widetilde{p}_N) = (\widetilde{p}_i - \widetilde{p}_N) \sum_{k=1}^{N} \alpha_k^{(n)} = \widetilde{p}_i - \widetilde{p}_N$$

And we obtain the following expression for the linearised relative demand function:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N = & \frac{\widetilde{C}^{(n)}}{\overline{\mathcal{H}}} \sum_{k=1}^{N} \left(coef_k^{(iN)} \right) \\
& + \widetilde{c}_i^{(n)} \left[\frac{1}{2} \sum_{k=1, k\neq i}^{N-1} \left(\frac{\alpha_i^{(n)}(\alpha_k^{(n)} + \alpha_N^{(n)})}{\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} - \left(\alpha_i^{(n)} + \alpha_k^{(n)} \right) \overline{\Phi}_{ik}^{-1} - \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \right. \\
& \left. - (\alpha_i^{(n)} + \alpha_N^{(n)}) \overline{\Phi}_{iN}^{-1} \right] \\
& + \widetilde{c}_N^{(n)} \left[\frac{1}{2} \sum_{k=1, k\neq i}^{N-1} \left(\left(\alpha_k^{(n)} + \alpha_N^{(n)} \right) \overline{\Phi}_{Nk}^{-1} - \frac{(\alpha_i^{(n)} + \alpha_k^{(n)})\alpha_N^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{iN}^{-1} \right) \right. \\
& \left. + (\alpha_i^{(n)} + \alpha_N^{(n)}) \overline{\Phi}_{iN}^{-1} \right] \\
& + \frac{1}{2} \sum_{k=1, k\neq i}^{N-1} \widetilde{c}_k^{(n)} \left((\alpha_i^{(n)} + \alpha_k^{(n)}) \left(1 + \frac{\alpha_N^{(n)}}{\alpha_i^{(n)}} \right) \overline{\Phi}_{ik}^{-1} - (\alpha_k^{(n)} + \alpha_N^{(n)}) \left(\frac{\alpha_i^{(n)}}{\alpha_N^{(n)}} + 1 \right) \overline{\Phi}_{Nk}^{-1} \right. \\
& \left. + (\alpha_i^{(n)} + \alpha_N^{(n)}) \alpha_k^{(n)} \left(\frac{1}{\alpha_N^{(n)}} - \frac{1}{\alpha_i^{(n)}} \right) \overline{\Phi}_{iN}^{-1} \right) \\
& + \frac{1}{2} \sum_{k=1, k\neq i}^{N-1} \left[\sum_{l=1, l\neq i,k}^{N-1} \widetilde{c}_l^{(n)} \left(\frac{\alpha_l^{(n)}(\alpha_k^{(n)} + \alpha_N^{(n)})}{\alpha_N^{(n)}} \overline{\Phi}_{Nk}^{-1} - \frac{\alpha_l^{(n)}(\alpha_i^{(n)} + \alpha_k^{(n)})}{\alpha_i^{(n)}} \overline{\Phi}_{ik}^{-1} + \frac{(\alpha_i^{(n)} + \alpha_l^{(n)})\alpha_k^{(n)}}{\alpha_i^{(n)}} \overline{\Phi}_{il}^{-1} \right. \right. \\
& \left. \left. - \frac{(\alpha_l^{(n)} + \alpha_N^{(n)})\alpha_k^{(n)}}{\alpha_N^{(n)}} \overline{\Phi}_{Nl}^{-1} \right) \right]
\end{aligned}
$$

Despite a fairly rich expression, this expression depends only on steady state elasticities, consumption shares and the term $\frac{\widetilde{C}^{(n)}}{\overline{\mathcal{H}}} \sum_{k=1}^{N} \left(coef_k^{(iN)} \right)$. Note that:

$$
\begin{aligned}
\sum_{k=1}^{N} coef_k^{(iN)} &= \sum_{k=1}^{N} \left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N} \right) \overline{c}_k^{(n)} \\
&= \sum_{k=1}^{N} \left(\frac{\overline{f}_{ik} \overline{c}_k^{(n)}}{\overline{f}_i} \right) - \sum_{k=1}^{N} \left(\frac{\overline{f}_{Nk} \overline{c}_k^{(n)}}{\overline{f}_N} \right) \\
&= \overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N
\end{aligned}
$$

where $\overline{\mathcal{H}}_l \equiv \frac{\sum_{k=1}^{N} \overline{f}_{lk} \overline{c}_k^{(n)}}{\overline{f}_l}$ for all $l = 1, 2, ..., N$

and therefore, after some rearranging:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N = & \widetilde{C}^{(n)} \frac{\overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N}{\overline{\mathcal{H}}} \\
& + \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k^{(n)} \sum_{l=1, l\neq k}^{N} \left(\alpha_k^{(n)} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \alpha_l^{(n)} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
& \left. + \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_N^{(n)}} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_i^{(n)}} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}
$$

with the convention that $\Phi_{xy} = 0$ if $y = x$.

Recalling that $\widetilde{C}^{(n)}$ can be expressed as a function of the consumptions $\widetilde{c}_l^{(n)}$, the steady state ratio $\overline{\mathcal{H}}$ and the steady state consumption shares, the equation above defines the linearised demand function as depending only on steady state consumption shares $\alpha_l^{(n)}$, steady state bilateral elasticities $\overline{\Phi}_{lm}$ and the steady state ratios $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_l$ ($l = 1, 2, ..., N$; $m = 1, 2, ..., N$).

Considering the above equation for all $i = 1, ... N-1$, we have prooved that the first-order dynamics of all relative demand functions depend only on the steady state values of the sufficient statistics listed in Theorem 1.

Market Clearing Condition

$$\begin{aligned}
Y^{(n)} &= \sum_{i=1}^{N} c_n^{(i)} \\
0 &= Y^{(n)} - \sum_{i=1}^{N} c_n^{(i)} \\
&\approx \overline{Y}^{(n)}\widetilde{Y}^{(n)} - \sum_{i=1}^{N} \overline{c}_n^{(i)}\widetilde{c}_n^{(i)} \\
\widetilde{Y}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{c}_n^{(i)}}{\overline{Y}^{(n)}}\widetilde{c}_n^{(i)}
\end{aligned}$$

Recall that we assumed that in steady state $\overline{C}^{(n)} = \overline{p}_n^{(n)}\overline{Y}^{(n)}$, which implies:

$$\widetilde{Y}^{(n)} \approx \sum_{i=1}^{N} \frac{\overline{p}_n^{(n)}\overline{c}_n^{(i)}}{\overline{C}^{(n)}}\widetilde{c}_n^{(i)}$$

We are going to further assume that there is bilaterally balanced trade between every country-pair, which means that $p_i^{(n)}c_i^{(n)} = p_n^{(i)}c_n^{(i)}$. Plugging this in, assuming that $p_n^{(i)} = p_n^{(n)}$:

$$\begin{aligned}
\widetilde{Y}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{p}_i^{(n)}\overline{c}_i^{(n)}}{\overline{C}^{(n)}}\widetilde{c}_n^{(i)} \\
&= \sum_{i=1}^{N} \alpha_i^{(n)}\widetilde{c}_n^{(i)}
\end{aligned}$$

which again only depends on the $\alpha_i^{(n)}$. □

B Proof of Corollary 1

As shown in Appendix A, the linearised equations characterising the aggregate consumption, the consumer price index and the market clearing conditions already depend only on steady state consumption shares, steady state bilateral elasticities and the steady state ratio $\overline{\mathcal{H}}^{(n)}$. Recall now that the linearised relative demand function equations are defined for all $i = 1, 2, ..., N$ as:

$$\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& \widetilde{C}^{(n)} \frac{\overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N}{\overline{\mathcal{H}}} \\
&+ \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k^{(n)} \sum_{l=1, l\neq k}^{N} \left(\alpha_k^{(n)} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \alpha_l^{(n)} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
&+ \left. \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_N^{(n)}} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_i^{(n)}} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}$$

It is easy to check that a function f homogeneous of degree r has the following property:

$$\frac{\sum_{k=1}^{N} f_{ik} c_k}{f_i(\mathbf{c})} = r - 1$$

This implies that $\overline{\mathcal{H}}_i = r - 1$ for all $i = 1, 2, ..., N$. Hence the first term in the linearised relative demand functions is equal to zero,and the steady state consumption shares, bilateral elasticities and the ratio $\overline{\mathcal{H}}^{(n)}$ are sufficient to characterise the dynamics of the model at first order.

$$\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k^{(n)} \sum_{l=1, l\neq k}^{N} \left(\alpha_k^{(n)} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \alpha_l^{(n)} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
&+ \left. \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_N^{(n)}} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\alpha_k^{(n)} \alpha_l^{(n)}}{\alpha_i^{(n)}} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}$$

□

C Kimball Aggregator Derivations

In this Appendix, we derive the elasticity of substitution between two goods i and j, for $i, j = 1, 2, ..., N$ and $i \neq j$, for a a representative consumer in country n, where $n = 1, 2, ..., N$, implied by the Kimball (1995) aggregator. The elasticity of substitution that we derive is defined as:

$$\Phi_{i,j,t}^{(n)} = \frac{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})}{\mathrm{d}(p_{i,t}^{(n)}/p_{j,t}^{(n)})} \frac{c_{i,t}^{(n)} p_{i,t}^{(n)}}{c_{j,t}^{(n)} p_{j,t}^{(n)}}$$

We note that the first term on the right-hand side of this expression can be written as:

$$\begin{aligned}
\frac{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})}{\mathrm{d}(p_{i,t}^{(n)}/p_{j,t}^{(n)})} &= \left[\frac{\mathrm{d}(p_{i,t}^{(n)}/p_{j,t}^{(n)})}{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})} \right]^{-1} \\
&= \left[\frac{\mathrm{d}(p_{i,t}^{(n)}/p_{j,t}^{(n)})}{\mathrm{d}c_{j,t}^{(n)}} \frac{\mathrm{d}c_{j,t}^{(n)}}{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})} \right]^{-1} \qquad (16)
\end{aligned}$$

We derive this term in two steps.

First, we solve for the final term in equation (16), which can be expressed as:

$$\frac{\mathrm{d}c_{j,t}^{(n)}}{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})} = \left[\frac{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})}{\mathrm{d}c_{j,t}^{(n)}}\right]^{-1}$$
$$= \left[\frac{\partial(c_{j,t}^{(n)}/c_{i,t}^{(n)})}{\partial c_{2,t}^{(n)}} + \frac{\partial(c_{j,t}^{(n)}/c_{i,t}^{(n)})}{\partial c_{i,t}^{(n)}}\frac{\mathrm{d}c_{i,t}^{(n)}}{\mathrm{d}c_{j,t}^{(n)}}\right]^{-1}$$

Within this, we can solve for $\frac{\mathrm{d}c_{i,t}^{(n)}}{\mathrm{d}c_{j,t}^{(n)}}$ by using the total derivative of the aggregator function $C_t^{(n)} = f(\mathbf{c}_t^{(n)})$, where $\mathrm{d}C_t^{(n)} = 0$ and $\mathrm{d}c_{k,t}^{(n)}$ for all $k = 1, 2, ..., N$ where $k \neq i, j$. This yields:

$$\frac{\mathrm{d}c_{i,t}^{(n)}}{\mathrm{d}c_{j,t}^{(n)}} = -\frac{f_{j,t}}{f_{i,t}} = -\frac{p_{j,t}^{(n)}}{p_{i,t}^{(n)}}$$

So then:

$$\frac{\mathrm{d}c_{j,t}^{(n)}}{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})} = \left[\frac{1}{c_{i,t}^{(n)}}\left(1 + \frac{p_{j,t}^{(n)}c_{j,t}^{(n)}}{p_{i,t}^{(n)}c_{i,t}^{(n)}}\right)\right]^{-1}$$

Second, we solve for the first term in equation (16). To do this, we note that the relative demand function can be expressed as:

$$\frac{p_{i,t}^{(n)}}{p_{j,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)}{\Upsilon'\left(\frac{c_{j,t}^{(n)}}{b_j^{(n)}C_t^{(n)}}\right)} \equiv h\left(c_{i,t}^{(n)}, c_{j,t}^{(n)}, C_t^{(n)}\right)$$

So then, when $\mathrm{d}C_t^{(n)} = 0$:

$$\frac{\mathrm{d}(p_{i,t}^{(n)}/p_{j,t}^{(n)})}{\mathrm{d}c_{j,t}^{(n)}} = \frac{\mathrm{d}h}{\mathrm{d}c_{j,t}^{(n)}}$$
$$= \frac{\partial h}{\partial c_{j,t}^{(n)}} + \frac{\partial h}{\partial c_{i,t}^{(n)}}\frac{\mathrm{d}c_{i,t}^{(n)}}{\mathrm{d}c_{j,t}^{(n)}}$$
$$= \frac{1}{\sigma c_{i,t}^{(n)}}\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{p_{i,t}^{(n)}}{p_{j,t}^{(n)}}\frac{1}{\sigma c_{j,t}^{(n)}}\left(\frac{c_{j,t}^{(n)}}{b_j^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}$$

Combining the expressions for the first and second terms in equation (16) yields:

$$\frac{\mathrm{d}(c_{j,t}^{(n)}/c_{i,t}^{(n)})}{\mathrm{d}(p_{i,t}^{(n)}/p_{j,t}^{(n)})} = \left(\frac{1}{\sigma c_{i,t}^{(n)}}\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{p_{i,t}^{(n)}}{p_{j,t}^{(n)}}\frac{1}{\sigma c_{j,t}^{(n)}}\left(\frac{c_{j,t}^{(n)}}{b_j^{(n)}C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right) c_{i,t}^{(n)}\left[1 + \frac{p_{j,t}^{(n)}c_{j,t}^{(n)}}{p_{i,t}^{(n)}c_{i,t}^{(n)}}\right]^{-1}$$

With this, the elasticity of substitution can be written as:

$$\Phi_{i,j,t}^{(n)} = \sigma \left(1 + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}}\right) \left[\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}} \left(\frac{c_{j,t}^{(n)}}{b_j^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} \right]^{-1} \tag{17}$$

D Nested-CES Derivations

Let us recall the formula for the direct partial elasticity between goods x and y:

$$\Phi_{xy}^{-1} = -\left(\frac{1}{c_x f_x} + \frac{1}{c_y f_y}\right)^{-1} \left[\left(\frac{f_{xx}}{f_x^2} - \frac{f_{xy}}{f_x f_y}\right) + \left(\frac{f_{yy}}{f_y^2} - \frac{f_{xy}}{f_x f_y}\right) \right]$$

We can apply it to the 3-country nested CES aggregator defined by :

$$\begin{aligned} C_t^{(H)} = f\left(c_{H,t}^{(H)}, c_{F,t}^{(H)}, c_{R,t}^{(H)}\right) &= \left(a_H^{(H)\,\frac{1}{\phi_H}} c_{H,t}^{(H)\,\frac{\phi_H - 1}{\phi_H}} + \left(1 - a_H^{(H)}\right)^{\frac{1}{\phi_H}} C_{FR,t}^{(H)\,\frac{\phi_H - 1}{\phi_H}}\right)^{\frac{\phi_H}{\phi_H - 1}} \\ \text{where} \qquad C_{FR,t}^{(H)} &= \left(a_F^{(H)\,\frac{1}{\phi_F}} c_{F,t}^{(H)\,\frac{\phi_F - 1}{\phi_F}} + \left(1 - a_F^{(H)}\right)^{\frac{1}{\phi_F}} C_{R,t}^{(H)\,\frac{\phi_F - 1}{\phi_F}}\right)^{\frac{\phi_F}{\phi_F - 1}} \end{aligned}$$

First, we compute the partial derivatives of f.

$$\begin{aligned}
f_H &= a_H^{\frac{1}{\phi_H}} \left(\frac{C}{c_H}\right)^{\frac{1}{\phi_H}} \\
f_F &= (1 - a_H)^{\frac{1}{\phi_H}} a_F^{\frac{1}{\phi_F}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \\
f_R &= (1 - a_H)^{\frac{1}{\phi_H}} (1 - a_F)^{\frac{1}{\phi_F}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \\
f_{HH} &= \frac{1}{\phi_H} f_H \left(-\frac{1}{c_H} + \frac{1}{C} f_H\right) \\
f_{HF} = f_{FH} &= \frac{1}{\phi_H} a_H^{\frac{1}{\phi_H}} (1 - a_H)^{\frac{1}{\phi_H}} a_F^{\frac{1}{\phi_F}} \left(\frac{C}{c_H}\right)^{\frac{1}{\phi_H}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C} = \frac{1}{\phi_H C} f_H f_F \\
f_{HR} &= \frac{1}{\phi_H C} f_H f_R \\
f_{FF} &= f_F \left(-\frac{1}{\phi_F c_F} + \frac{\phi_H - \phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} f_F\right) \\
f_{FR} &= f_F (1 - a_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \left(\frac{\phi_H - \phi_F}{\phi_H \phi_F} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} (1 - a_H)^{\frac{1}{\phi_H}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}}\right) \\
&= f_F \left(\frac{1}{\phi_H C} f_R + \frac{\phi_H - \phi_F}{\phi_H \phi_F} \frac{1}{C_{FR}} (1 - \alpha_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}}\right) \\
f_{RR} &= f_R \left(-\frac{1}{\phi_F c_R} + \frac{\phi_H - \phi_F}{\phi_H \phi_F} (1 - a_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} f_R\right)
\end{aligned}$$

Using the above, we compute the inverse of the bilateral elasticities.

$$
\begin{aligned}
\Phi_{HF}^{-1} =& -\left(\frac{1}{c_H f_H}+\frac{1}{c_F f_F}\right)^{-1}\left[\left(\frac{f_{HH}}{f_H^2}-\frac{f_{HF}}{f_H f_F}\right)+\left(\frac{f_{FF}}{f_F^2}-\frac{f_{HF}}{f_H f_F}\right)\right]\\
=& -\left(\frac{1}{x_H f_H}+\frac{1}{x_F f_F}\right)^{-1}\\
&\times\left[\left(\frac{\frac{1}{\phi_H}f_H\left(-\frac{1}{c_H}+\frac{1}{C}f_H\right)}{f_H^2}-\frac{\frac{1}{\phi_H C}f_H f_F}{f_H f_F}\right)\right.\\
&\left.+\left(\frac{f_F\left(-\frac{1}{\phi_F c_F}+\frac{\phi_H-\phi_F}{\phi_H\phi_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}+\frac{1}{\phi_H C}f_F\right)}{f_F^2}-\frac{\frac{1}{\phi_H C}f_H f_F}{f_H f_F}\right)\right]\\
=& -\left(\frac{1}{c_H f_H}+\frac{1}{c_F f_F}\right)^{-1}\\
&\times\left[\left(\frac{\left(-\frac{1}{\phi_H c_H}+\frac{1}{\phi_H C}f_H\right)}{f_H}-\frac{1}{\phi_H C}\right)\right.\\
&\left.+\left(\frac{\left(-\frac{1}{\phi_F c_F}+\frac{\phi_H-\phi_F}{\phi_H\phi_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}+\frac{1}{\phi_H C}f_F\right)}{f_F}-\frac{1}{\phi_H C}\right)\right]\\
=& -\left(\frac{c_F f_F+c_H f_H}{c_H f_H c_F f_F}\right)^{-1}\\
&\times\left[-\frac{1}{\phi_H c_H f_H}+\frac{1}{\phi_H C}-\frac{1}{\phi_H C}-\frac{1}{\phi_F c_F f_F}+\frac{\phi_H-\phi_F}{\phi_H\phi_F f_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}+\frac{1}{\phi_H C}-\frac{1}{\phi_H C}\right]\\
=& -\frac{c_H f_H c_F f_F}{c_F f_F+c_H f_H}\times\left[-\frac{1}{\phi_H c_H f_H}-\frac{1}{\phi_F c_F f_F}+\frac{\phi_H-\phi_F}{\phi_H\phi_F f_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{1}{C_{FR}}\right]\\
=& -\left[-\frac{c_F f_F}{c_F f_F+c_H f_H}\frac{1}{\phi_H}-\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{1}{\phi_F}+\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{\phi_H-\phi_F}{\phi_H\phi_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}}\right]\\
=& \frac{c_F p_F}{P_{FR}C_{FR}}\left(\frac{c_F p_F+c_H p_H}{P_{FR}C_{FR}}\right)^{-1}\frac{1}{\phi_H}+\frac{c_H p_H}{PC}\frac{PC}{c_F p_F+c_H p_H}\frac{1}{\phi_F}\\
&-\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{\phi_H-\phi_F}{\phi_H\phi_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}}\\
=& \frac{c_F p_F}{PC}\frac{PC}{P_{FR}C_{FR}}\left(\frac{c_F p_F+c_H p_H}{PC}\frac{PC}{P_{FR}C_{FR}}\right)^{-1}\frac{1}{\phi_H}\\
&+\frac{c_H p_H}{PC}\frac{PC}{c_F p_F+c_H p_H}\frac{1}{\phi_F}-\frac{c_H f_H}{c_F f_F+c_H f_H}\frac{\phi_H-\phi_F}{\phi_H\phi_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}}\\
=& \frac{\alpha_F}{1-\alpha_H}\left(\frac{\alpha_F+\alpha_H}{1-\alpha_H}\right)^{-1}\frac{1}{\phi_H}+\alpha_H\left(\alpha_F+\alpha_H\right)^{-1}\frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{\phi_H-\phi_F}{\phi_H\phi_F}a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}}\frac{c_F}{C_{FR}}\\
=& \frac{\alpha_F}{\alpha_F+\alpha_H}\frac{1}{\phi_H}+\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{\phi_H-\phi_F}{\phi_H\phi_F}\frac{p_F}{P_{FR}}\frac{c_F}{C_{FR}}\\
=& \frac{\alpha_F}{\alpha_F+\alpha_H}\frac{1}{\phi_H}+\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H}\frac{\phi_H-\phi_F}{\phi_H\phi_F}\frac{\alpha_F}{1-\alpha_H}\\
=& \frac{1}{\phi_H\phi_F\left(\alpha_F+\alpha_H\right)\left(1-\alpha_H\right)}\left[\alpha_F(1-\alpha_H)\phi_F+\alpha_H(1-\alpha_H)\phi_H-\alpha_H\alpha_F(\phi_H-\phi_F)\right]\\
\Phi_{HF}^{-1} =& \frac{\alpha_F\phi_F+\alpha_H\alpha_R\phi_H}{\phi_H\phi_F\left(\alpha_F+\alpha_H\alpha_R\right)}
\end{aligned}
$$

Hence:

$$\Phi_{HF} = \frac{\phi_H \phi_F \left(\alpha_F + \alpha_H \alpha_R\right)}{\alpha_F \phi_F + \alpha_H \alpha_R \phi_H}$$

We can compute Φ_{HR} in a similar fashion and obtain symmetrically:

$$\Phi_{HR} = \frac{\phi_H \phi_F (\alpha_R + \alpha_H \alpha_F)}{\alpha_R \phi_F + \alpha_H \alpha_F \phi_H}$$

And it is easy to check that $\Phi_{FR} = \phi_F$.

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