Asymmetric optimal auction design with loss-averse bidders

Akitoshi Muramoto*and Takeharu Sogo†

February 21, 2022

Abstract

We study optimal auctions with expectation-based loss-averse bidders, who incur a psychological loss in expectation when there is uncertainty in auction outcome. When bidders are ex-ante identical, although symmetric designs are optimal for bidders with expected-utility preferences, expected revenues are higher in an optimal mechanism with a single buyer than in any symmetric mechanism with multiple bidders if the degree of loss aversion is sufficiently large relative to the variation in valuations. Furthermore, with certain conditions, optimal mechanisms are *necessarily asymmetric*. When bidders differ in valuation distributions, in the optimal auction with certain conditions, if the degree of loss aversion is sufficiently large relative to the variation in valuations, one of the bidders always wins; if it is sufficiently small, it is optimal to favor the (almost) weak bidder as prescribed in Myerson (1981) but different extents. Greater degrees of loss aversion magnify the difference from the Myerson levels.

JEL Classification: D03; D44; D81; D82; D84
Keywords: Asymmetric Auctions; Optimal Auctions; Reference Dependence; Expectation-based Loss Aversion

*Faculty of Business Administration, Osaka University of Economics, Japan (a.muramoto@osaka-ue.ac.jp)
†SKEMA Business School—Université Côte d'Azur, France (takeharu.sogo@skema.edu)

1 Introduction

We study the optimal auction design when bidders have expectation-based loss averse preferences à la Kőszegi and Rabin (2006, 2007) (more precisely, we adopt the Choice Acclimating Personal Equilibrium), and we demonstrate the optimality of asymmetric designs.

Kőszegi and Rabin (2006, 2007) assume that people form expectations about outcomes, which in turn form their reference point, and people are loss averse relative to this expectation, i.e., more sensitive to losses than gains of equal size relative to their expectations.[1] A series of studies finds support for their framework.[2]

Although we are unaware of direct evidence, there are reasons to believe that not only individuals but also firms have expectation-based loss-averse preferences in auctions. For instance, employees who are in charge of buying a good or service through an auction may need to be briefed within the firm before the auction on their expectations of how likely they are to win the auction and how much they will have to pay, or their promotion or bonus may depend on the auction outcome. In such cases, the expectations that the employees have before an auction form a reference point, and that reference point may influence the firm's bidding behavior as if it were loss averse.

We first establish that when the degree of loss aversion is sufficiently large relative to the variation in bidder valuations, expected seller revenues are higher in an optimal mechanism with a single buyer—an extreme form of asymmetric mechanisms—than in any symmetric optimal mechanism with multiple bidders.

There are two important implications of expectation-based loss aversion. First, loss-averse bidders dislike risk because they suffer psychological losses, on average, when there is uncertainty about auction outcomes. Second, because they dislike risk, bidders are unhappier when they expect greater uncertainty in auction outcomes, even if the outcomes are the same ex post, which is the key difference from risk aversion based on a standard concave utility.

With these bidders, sellers usually face the tension between efficiency and rent extraction. Sellers can design auction mechanisms whose outcomes are less uncertain so that bidders do not have to suffer psychological losses. As our result suggests, for example, a seller facing potential bidders A, B, C can invite only bidder A so that bidders B and C do not have to form expectations about winning or losing and suffer any psychological loss. However, the attempt to reduce bidder uncertainty leads to reduce competition among bidders and the seller extracts less bidder rents. The result that eliminating competition completely is optimal holds only if the efficiency concern outweighs the rent extraction concern, i.e., when the degree of loss aversion is sufficiently large relative to the variation in bidders' valuations.

This finding indicates that the optimal design takes an extreme form of asymmetric designs even when bidders are ex-ante identical and it highlights the difference from risk aversion based on a standard concave utility. When bidders are ex-ante identical and expected utility maximizers, we can always find symmetric optimal designs whenever there are optimal designs (Maskin and Riley, 1984).

Moreover, it indicates that it is *not* without loss of generality to focus on symmetric

[1] Kahneman and Tversky (1979) first provide the theory of reference dependence. See also, for example, Bell (1985); Loomes and Sugden (1986) for earlier models of expectation-based reference dependence.

[2] This includes Abeler, Falk, Goette and Huffman (2011); Crawford and Meng (2011); Marzilli Ericson and Fuster (2011); Gill and Prowse (2012).

designs with expectation-based loss-averse bidders. Although there have been many recent studies on auctions among expectation-based loss-averse bidders, to the best of our knowledge, this study is the first to analyze asymmetric designs. Eisenhuth (2019) characterizes optimal mechanisms within the class of symmetric mechanisms for ex-ante identical bidders. All the other papers study standard auctions with symmetric rules. Thus, our analysis provides new implications as to how sellers should design auctions when they face expectation-based loss-averse bidders.

We also find that the benefit of asymmetric designs extends to standard auctions (not just optimal mechanisms). In the first- or second-price auctions (when there is no loss aversion in the money dimension), the seller's expected revenue decreases in the number of bidders if loss aversion in the good dimension is large and valuations are distributed according to power distribution $F(v) \equiv \theta^{\kappa}$, with sufficiently large κ.

For general distributions, it is obviously not optimal to eliminate competition completely. However, the optimality of more general asymmetric mechanisms, in which bidders win with different and positive probabilities, extends to more general distributions. We establish that with two bidders if certain conditions (on valuation distribution and the loss-gain weight in the good dimension) are satisfied, optimal mechanisms are *necessarily asymmetric*. In particular, if valuations are uniformly distributed, optimal mechanisms are necessarily asymmetric.

The key intuition is that since loss-averse bidders dislike uncertainty about auction outcomes, the seller also benefits from reducing uncertainty, thereby saving rents required to satisfy bidders' participation constraints. Indeed, when a bidder wins with probability $\frac{1}{2}$, it faces the greatest uncertainty and hence incurs the greatest psychological disutility as to gain and loss. More in general, bidders incur greater loss-gain disutility as their winning probabilities get closer to $\frac{1}{2}$. Thus, it may be better for the seller to favor a certain bidder and disfavor the other bidders or even completely shut down some bidders from participating because it reduces the expected loss-gain disutility: it increases the winning probability of the favored bidder away from $\frac{1}{2}$ and decreases the winning probabilities of the disfavored bidders away from $\frac{1}{2}$. Thus, for example, an asymmetric mechanism under which one bidder always wins, regardless of its type, may be uniquely optimal if bidders suffer excessively large psychological loss.

We further establish that when bidders differ in their valuation distributions, the degree of favoritism must be modified from the level prescribed by Myerson (1981) and the optimal extent depends on the degree of loss aversion. When the degree of loss aversion is sufficiently large compared to the variation in bidders' valuations, the optimal mechanism sells the item with probability one to one bidder. Such a way of selling is optimal because it eliminates the uncertainty perfectly and causes no psychological loss. When the degree of loss aversion is not so large, stochastically selecting the auction winner is optimal even though such stochastic selection exposes bidders to uncertainty and causes loss gain. Roughly speaking, the optimal mechanism is the Myerson mechanism modified to strengthen the degree of favoritism of the bidder who wins more likely than the other bidder under the Myerson mechanism to reduce the uncertainty and psychological loss caused by it.

Our work contributes to the growing literature on auctions with expectation-based loss-averse bidders (Lange and Ratan, 2010; Eisenhuth, 2019; Rosato and Tymula, 2019; Balzer and Rosato, 2021; von Wangenheim, 2021; Rosato, 2014; Balzer, Rosato and von Wangenheim, 2021).[3] All of these papers focus on symmetric designs and study standard auctions, with the exception of Eisenhuth (2019), which studies optimal auctions. As important recent contributions, von Wangenheim (2021); Rosato (2014); Balzer, Rosato and von Wangenheim (2021) deal with dynamic auctions and study how expectations evolve dynamically as reference points to influence bidding behavior. Our static approach has the disadvantage that it cannot make such dynamic arguments, but it can analyze the optimal mechanism and provide guidance on what expectations should be formed ultimately.

Our analysis relates to research on asymmetric designs or discrimination in auctions. Myerson (1981) and McAfee and McMillan (1989) provide a theoretical foundation for favoritism, based on ex ante heterogeneity in bidder characteristics. Researchers have identified situations in which asymmetric designs are optimal even when bidders are identical ex ante. See, for example, Lu (2009); Celik and Yilankaya (2009); Bernhardt, Liu and Sogo (2020) for settings in which bidders incur participation costs, and Lewis and Yildirim (2002); Iossa and Rey (2014); Barbosa and Boyer (2021) for dynamic settings with effects such as reputation and learning-by-doing. The key insight from the literature is that the trade-off between efficiency and rent extraction makes asymmetric designs optimal. This study also deals with the same trade-offs, but differs in that it arises from the utility loss associated with loss aversion, and the magnitude of this utility loss is endogenously determined through expectations and auction outcomes, which makes the analysis more demanding.

2 The Model

We follow the model environment of Rosato and Tymula (2019), except that we introduce bidder heterogeneity and consider optimal mechanisms. A risk-neutral seller auctions off an indivisible item to $n \geq 1$ risk-neutral potential bidders. Each bidder i's valuation $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$ is private information and is drawn independently from a distribution $F_i(\cdot)$, where the density $f_i(\cdot)$ is strictly positive and differentiable over its support $[\underline{\theta}_i, \overline{\theta}_i]$. We assume that each bidder's virtual valuation is increasing , i.e., $\psi_i'(\theta_i) > 0$ where $\psi_i(\theta_i) \equiv \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$ (Myerson, 1981). The value of the object is normalized to 0 for the seller.

Bidders have expectation-based reference-dependent preferences as in Kőszegi and Rabin (2006, 2007). To analyze the decision of whether to buy a single item, we adopt the simplification made in Heidhues and Kőszegi (2014), Eisenhuth (2019), and Rosato and Tymula (2019). The utility of bidder i of valuation θ_i is given by

$$u_i(c^g, p \mid K^g, K^p, \theta_i) \equiv \underbrace{\theta_i c^g - p}_{\text{intrinsic utility}} \underbrace{+ \theta_i \int_{r^g} \mu_i(c^g - r^g) dK^g + \int_{r^p} \mu_i(r^p - p) dK^p}_{\text{gain-loss utility}},$$

where $c^g, r^g \in \{0, 1\}$ signify the good dimension and r^g distributes according to K^g; $p, r^p \geq 0$

[3] Research on tournaments with expectation-based loss-averse agents includes Gill and Stone (2010) and Dato, Grunewald and Müller (2018).

signify the money dimension and r^p distributes according to K^p; and for $l \in \{g, p\}$

$$\mu_i^l(x) \equiv \begin{cases} \eta_i^l x & \text{if } x \geq 0 \\ \eta_i^l \lambda_i^l x & \text{if } x < 0. \end{cases}$$

If bidder i of valuation θ_i wins the item ($c^g = 1$) and pays p, it receives intrinsic utility of $\theta_i - p$. Moreover, it derives gain-loss utility in both the good and money dimensions. In the good dimension, it derives gain-loss utility from comparing its actual consumption value $c^g\theta_i$ to reference point r^g in the good dimension. Similarly, in the money dimension, it derives gain-loss utility from comparing its actual payment p to reference point about payment r^p. For $l \in \{g, p\}$, we assume $\eta_i^l \lambda_i^l > \eta_i^l > 0$ capturing loss aversion (Kahneman and Tversky, 1979; Tversky and Kahneman, 1991). To ensure that no bidder puts more weighs on gain-loss utility than intrinsic utility in the good dimension, we assume $\Lambda_i^g \equiv \eta_i^g(\lambda_i^g - 1) \leq 1$ for any bidder i (Herweg et al., 2010).

After learning their valuations, bidders submit bids to maximize their interim expected utility. Given the distribution of the reference points $K = (K^g, K^p)$ and the distribution of outcomes (c^g, p) $L = (L^g, L^p)$, the interim expected utility of bidder i with valuation θ_i is

$$U_i(L \mid K, \theta_i) = \int_{\{c^g, p\}} u_i(c^g, p \mid K, \theta_i) dL.$$

We adopt the Choice Acclimating Personal Equilibrium (CPE). That is, for any choice set D_i for bidder i, a selection $K \in D_i$ is CPE for bidder i of valuation θ_i if $U_i(K \mid K, \theta_i) \geq U_i(K' \mid K', \theta_i)$ for all $K' \in D_i$ (Kőszegi and Rabin, 2007). This is well suited for situations in which bidders submit bids sufficiently long before actually deriving utility from an auctioned item, so that each bidder's bidding strategy determines not only the distribution of auction outcomes, but also the distribution of the reference points.

Eisenhuth (2019, Proposition 1) establishes that it is without loss of generality to restrict attention to direct mechanisms. This result extends directly to our setting, because its proof does not depend on bidder homogeneity. Each bidder i reports a value $\theta_i \in \Theta_i \equiv [\underline{\theta}_i, \overline{\theta}_i]$ and a direct mechanism (Q, M) consists of a profile of allocation rules $Q = (Q_1, \ldots, Q_n)$ and a profile of payment rules $M = (M_1, \ldots, M_n)$ with $M_i : \times_i \Theta_i \to \mathbb{R}$, where $Q_i : \times_i \Theta_i \to [0, 1]$ is the probability that bidder i obtains the item and $\sum Q_i(\theta) \leq 1$.

Let $\theta_{-i} \equiv (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \in \times_{j \neq i} \Theta_i$, $f_{-i}(\theta_{-i}) \equiv \Pi_{j \neq i} f_j(\theta_j)$, and $f(\theta) \equiv \Pi_i f_i(\theta_i)$. When all other bidders truthfully bid, if bidder i instead reports $\theta_i' \in \Theta_i$, then i's expected probability of winning the item is

$$q_i(\theta_i') \equiv \int_{\theta_{-i}} Q_i(\theta_i', \theta_{-i}) f_{-i}(\theta_{-i}) \, d\theta_{-i} \tag{1}$$

and i's expected payment is

$$m_i(\theta_i') \equiv \int_{\theta_{-i}} M_i(\theta_i', \theta_{-i}) f_{-i}(\theta_{-i}) \, d\theta_{-i}. \tag{2}$$

The interim expected utility of bidder i with valuation θ_i when reporting θ_i' while all other bidders report truthfully is

$$\pi_i\left(\theta_i'|\theta_i\right) \equiv \underbrace{\theta_i q_i(\theta_i') - m_i(\theta_i')}_{\text{intrinsic utility}} + \underbrace{\theta_i \gamma_i^g(\theta_i') - \gamma_i^p(\theta_i')}_{\text{gain-loss utility}}$$
$$= \theta_i[q_i(\theta_i') + \gamma_i^g(\theta_i')] - [m_i(\theta_i') + \gamma_i^p(\theta_i')], \tag{3}$$

where the interim expected gain-loss utility in the money dimension $\gamma_i^p(\theta_i')$ is given by

$$\gamma_i^p(\theta_i') = \eta_i^p \lambda_i^p \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) > m_i(\theta_i')\right\}} [M_i(\theta_i', \theta_{-i}) - m_i(\theta_i')]\, f_{-i}\left(\theta_{-i}\right) d\theta_{-i}$$
$$- \eta_i^p \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) < m_i(\theta_i')\right\}} [m_i(\theta_i') - M_i(\theta_i', \theta_{-i})]\, f_{-i}\left(\theta_{-i}\right) d\theta_{-i}, \tag{4}$$

and the interim expected gain-loss utility in the good dimension is $\theta_i \gamma_i^g(\theta_i')$ and $\gamma_i^g(\theta_i')$ is given by

$$\gamma_i^g(\theta_i') = \eta_i^g \int_{\theta_{-i}} Q_i(\theta_i', \theta_{-i})(1 - q_i(\theta_i')) f_{-i}\left(\theta_{-i}\right) d\theta_{-i} - \eta_i^g \lambda_i^g \int_{\theta_{-i}} \left(1 - Q_i(\theta_i', \theta_{-i})\right) q_i(\theta_i') f_{-i}\left(\theta_{-i}\right) d\theta_{-i}$$
$$= -\Lambda_i^g q_i(\theta_i')(1 - q_i(\theta_i')). \tag{5}$$

Letting $\alpha_i^g(q_i) \equiv (1 - \Lambda_i^g) q_i + \Lambda_i^g {q_i}^2$, we can rewrite $q_i(\theta_i) + \gamma_i^g(\theta_i)$ as follows:

$$q_i(\theta_i) + \gamma_i^g(\theta_i) = q_i(\theta_i) - \Lambda_i^g q_i(\theta_i)(1 - q_i(\theta_i)) = (1 - \Lambda_i^g) q_i(\theta_i) + \Lambda_i^g q_i(\theta_i)^2 = \alpha_i^g(q_i(\theta_i)). \tag{6}$$

Observe that $\Lambda_i^g > 0$ makes $\gamma_i^g(\theta_i)$ negative, and $\gamma_i^g(\theta_i)$ is minimized at $q_i(\theta_i) = \frac{1}{2}$. That is, loss aversion induces gain-loss disutility in expectation, and this expected gain-loss disutility is maximized when a bidder expects to win with probability $\frac{1}{2}$, which is when the bidder faces the greatest uncertainty as to winning and losing the auction. In standard auction formats, Rosato and Tymula (2019) shows that loss aversion pushes bidders who expect to win with more (resp. less) than probability $\frac{1}{2}$ to overbid (resp. underbid). In contrast, we analyze optimal auction mechanisms and study how loss aversion affects the optimality of asymmetric designs.

Individual rationality requires

$$\pi_i\left(\theta_i\right) \equiv \pi_i\left(\theta_i|\theta_i\right) \geq 0, \ \forall \theta_i, \tag{7}$$

where $\pi_i\left(\theta_i\right)$ denotes i's equilibrium utility. Incentive compatibility requires

$$\pi_i\left(\theta_i\right) = \max_{\theta' \in \Theta_i} \pi_i\left(\theta_i'|\theta_i\right), \ \forall \theta_i. \tag{8}$$

The seller's problem is to find a direct mechanism (Q, M) that maximizes its expected revenue $\Pi = \int_\theta \sum_i M_i(\theta) f\left(\theta\right) d\theta$ such that (7) and (8).

2.1 Optimal mechanisms

We first characterize incentive compatibility. By treating $q_i(\theta_i) + \gamma_i^g(\theta_i) \geq 0$ as the interim expected winning probability and $m_i(\theta_i) + \gamma_i^p(\theta_i)$ as the interim expected payment in (3) and (8), we can use the standard characterization of incentive compatibility for bidders with expected-utility preferences (Krishna, 2002, Section 5.1.2). We obtain the following lemma as a straightforward extension of Eisenhuth (2019) to possibly asymmetric bidders:

Lemma 1 (Eisenhuth 2019, Proposition 1)**.** *Incentive compatibility (8) holds if and only if (i) $q_i(\theta_i)$ is nondecreasing and (ii) the expected payment of bidder i conditional on its type θ_i when all bidders report their types truthfully is given by*

$$m_i(\theta_i) = \theta_i \alpha_i^g(q_i(\theta_i)) - \int_{\underline{\theta}_i}^{\theta_i} \alpha_i^g(q_i(t_i))dt_i - \pi_i\left(\underline{\theta}_i\right) - \gamma_i^p(\theta_i). \tag{9}$$

Moreover, when (8) is satisfied, the expected utility of bidder i conditional on its type θ_i is given by

$$\pi_i\left(\theta_i\right) = \pi_i\left(\underline{\theta}_i\right) + \int_{\underline{\theta}_i}^{\theta_i} \alpha_i^g(q_i(t_i))dt_i. \tag{10}$$

and the seller's expected revenue is given by

$$\int_\theta \sum_i \psi_i(\theta_i)\alpha_i^g(q_i(\theta_i))f\left(\theta\right)d\theta - \sum_i \pi_i\left(\underline{\theta}_i\right) - \int_\theta \sum_i \gamma_i^p(\theta_i) f\left(\theta\right)d\theta. \tag{11}$$

We now characterize optimal auctions:

Lemma 2. *In any optimal auction, the following are satisfied:*

1. *$\gamma_i^p(\theta_i) = 0$ for any θ_i and any i.*
2. *$\pi_i\left(\underline{\theta}_i\right) = 0$ for all i.*
3. *The seller's expected revenue is given by*

$$\int_\theta \sum_i \psi_i(\theta_i)\alpha_i^g(q_i(\theta_i))f(\theta)d\theta. \tag{12}$$

4. *Suppose bidders are ex-ante identical: $F_i = F$, $\Lambda_i^g = \Lambda^g$, and $\Lambda_i^p = \Lambda^p$. Within symmetric mechanisms, the highest expected revenue is given by*

$$n\int_{\hat{\theta}}^{\overline{\theta}} \psi(\theta_i)\left[\left(1-\Lambda^g\right)F(\theta_i)^{n-1} + \Lambda^g F(\theta_i)^{2(n-1)}\right]dF\left(\theta_i\right), \tag{13}$$

where

$$\hat{\theta} = \begin{cases} \underline{\theta} & \text{if } \psi\left(\underline{\theta}\right) \geq 0 \\ \theta' & \text{if } \psi\left(\underline{\theta}\right) < 0 \text{ and } \psi\left(\theta'\right) = 0. \end{cases}$$

First, $\gamma_i^p(\theta_i) = 0$ can be implemented through all-pay auctions because actual payments do not depend on other bidders' valuations. Eisenhuth (2019) restricts attention to symmetric mechanisms among ex-ante identical bidders. Within symmetric mechanisms, it is optimal to allocate the item to a bidder who has the highest virtual valuation $\psi(\theta_i)$ whenever it is above the seller's reservation value 0, which yields the expected revenue of (13). This can be implemented by all-pay auctions with the identical optimal reserve price. However, as shown below, even with ex-ante identical bidders, restricting attention to symmetric mechanisms is *not* without loss of generality, i.e., asymmetric designs may yield strictly higher expected revenues than (13).

The seller revenue in optimal auctions (12) is a simple generalization of Myerson (1981). The only difference is $\alpha_i^g(q_i(\theta_i))$, which reduces to the case of Myerson (1981), i.e., $\alpha_i^g(q_i(\theta_i)) = q_i(\theta_i)$, when $\Lambda_i^g = 0$ for all i.

Observe that when all bidders receive zero gain-loss utility in the money dimension ($\gamma_i^p(\theta_i) = 0,\ \forall i$), social surplus is the sum of bidders' utility net of the sum of gain-loss disutility in the good dimension, which is given by

$$\int_\theta \sum_i \theta_i \alpha_i^g(q_i(\theta_i)) f(\theta)\, d\theta. \tag{14}$$

With $\Lambda_i^g = 0$ for all i, since $\alpha_i^g(q_i(\theta_i)) = q_i(\theta_i)$ holds, the efficient mechanism that maximizes social surplus is to award the good to a bidder with the highest valuation θ_i. However, with $\Lambda_i^g > 0$, since gain-loss disutility borne by bidders reduces social surplus, one needs to take into account of the gain-loss disutility to achieve efficiency.

3 Ex-ante identical bidders

In this section, we consider ex-ante identical bidders: $F_i = F$, $\Lambda_i^g = \Lambda^g$, and $\Lambda_i^p = \Lambda^p$.

3.1 Ex-ante identical bidders with expected-utility preferences

As a benchmark, we first establish that it is without loss of generality to restrict to symmetric mechanisms when we consider ex-ante identical bidders with expected-utility preferences:

Theorem 3 (Maskin and Riley 1984, Footnote 11)**.** *Suppose that bidders are ex-ante identical and have expected-utility preferences. If there is an asymmetric optimal mechanism, then there is also a symmetric optimal mechanism.*

3.2 Ex-ante identical expectation-based loss-averse bidders

With expectation-based loss-averse bidders, even when they are ex-ante identical, we show that asymmetric designs can yield higher expected revenues than any symmetric design.

We first consider an extreme form of asymmetric designs in which all potential bidders except for one are completely excluded from the auction. The following proposition states that when the variation in valuations is sufficiently small relative to the degree of loss aversion, restricting to only one bidder is optimal.

Proposition 4. *Suppose $n \geq 2$, $\Lambda^g \leq 1$, and $\overline{\theta} = \underline{\theta} + \varepsilon$. If $\frac{\varepsilon}{\underline{\theta}} < \frac{\Lambda^g\left(\frac{n-1}{2n-1}\right)}{1-\Lambda^g\left(\frac{n-1}{2n-1}\right)}$; that is, if the variation in valuations is sufficiently small relative to the degree of loss aversion in the good dimension, expected seller revenues are higher in the optimal mechanism with a single buyer than in any symmetric mechanism with $n \geq 2$ bidders.*

The intuition is simple. In general there are two types of benefits of having more bidders for sellers. One is greater selection: the highest valuation increases with the number of potential bidders. The other benefit is rent extraction: competition tends to reduce bidder surplus. These benefits increase as the variation in bidder valuations increases.

However, if bidders are loss averse, there is a benefit for the seller to lower competition to reduce gain-loss disutility borne by bidders because the seller's expected revenue is social surplus minus the sum of bidder payoffs. Hence, when the degree of loss aversion is sufficiently large relative to the variation in valuations, the benefit of reducing gain-loss disutility outweighs the benefit of having more bidders, making it optimal to sell to a single buyer.

We now consider more general asymmetric mechanisms in which bidders win with different and positive probabilities. For simplicity we consider two bidders ($n = 2$). The highest revenue within symmetric designs is given by (13) and it can be achieved when bidder 1 wins if and only if its valuation is greater than bidder 2's valuation, i.e., $\theta_1 > \theta_2$ and $\theta_1 \geq \hat{\theta}$.

On the other hand, the asymmetric mechanism that we consider is the following: Given interval $(\theta_L, \theta_L + \varepsilon) \subset (\hat{\theta}, \overline{\theta})$, when a pair of bidder valuations (θ_1, θ_2) falls onto region $(\theta_L, \theta_L + \varepsilon) \times (\theta_L, \theta_L + \varepsilon)$, bidder 1 wins; when it does not, bidder 1 wins if and only if $\theta_1 > \theta_2$. The pink and blue regions in the left figure of Figure 1 indicate a pair of valuations such that bidder 1 wins. The next proposition provides a sufficient condition under which this asymmetric mechanism yields strictly higher expected revenue than any symmetric mechanism for sufficiently small $\varepsilon > 0$.

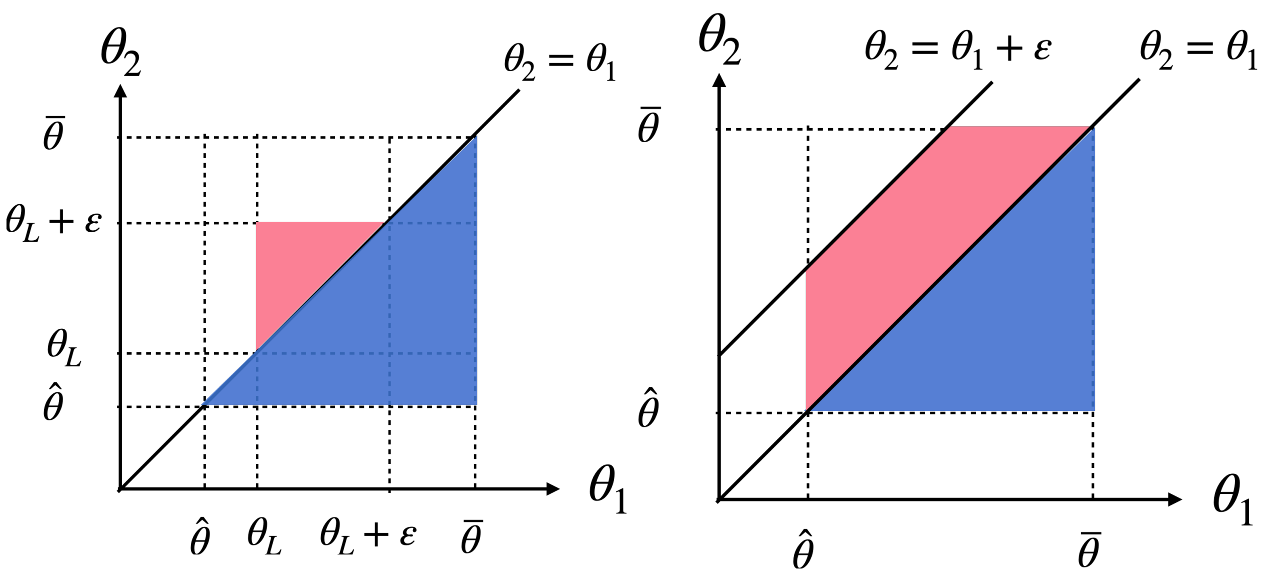


Figure 1: A pair of valuations (θ_1, θ_2) for which bidder 1 wins in the two asymmetric designs

Proposition 5. *Suppose $F_i = F$, $\Lambda_i^g = \Lambda^g$, $\Lambda_i^p = \Lambda^p$, and $n = 2$. If there exits an open interval $I \subset (\hat{\theta}, \overline{\theta})$ such that for any $\theta \in I$*

$$\psi(\theta)f(\theta) - \psi'(\theta)F(\theta) > \frac{1-\Lambda^g}{2\Lambda^g}\psi'(\theta), \tag{15}$$

then optimal mechanisms are asymmetric. In particular, if $\psi(\underline{\theta}) \geq 0$ (i.e., $\hat{\theta} = \overline{\theta}$) and Λ^g is sufficiently large that

$$\lim_{\theta \searrow \underline{\theta}} \psi(\theta) > \frac{1-\Lambda^g}{2\Lambda^g} \lim_{\theta \searrow \underline{\theta}} \frac{\psi'(\theta)}{f(\theta)}$$

holds, then optimal mechanisms are asymmetric.

We next show that if the density of valuations does not decrease too quickly (e.g., uniform distributions), then optimal mechanisms are necessarily asymmetric. In doing so, we consider the following asymmetric mechanism: bidder 1 wins if $\theta_1 + \varepsilon > \theta_2$ and $\theta_1 \geq \hat{\theta}$ while bidder 2 wins if $\theta_2 > \theta_1 + \varepsilon$ and $\theta_2 \geq \hat{\theta}$. The pink and blue regions in the right figure of Figure 1 indicate a pair of valuations such that bidder 1 wins. The following proposition shows that this asymmetric mechanism yields a higher expected revenue than any symmetric mechanism if $\varepsilon > 0$ is sufficiently small.

Proposition 6. *Suppose $F_i = F$, $\Lambda_i^g = \Lambda^g$, $\Lambda_i^p = \Lambda^p$, $n = 2$. If $f'(\theta) > -\frac{2\Lambda^g f(\theta)^2}{(1-\Lambda^g)+2\Lambda^g F(\theta)}$; that is, if the density $f(\theta)$ does not decrease too quickly for $\theta \geq \hat{\theta}$, then optimal mechanisms are asymmetric. In particular, if valuations are uniformly distributed, then optimal mechanisms are asymmetric.*

3.3 First- and second-price auctions

In the previous section we showed the optimality of asymmetric designs even when bidders are ex-ante identical, in which optimal auctions must satisfy $\gamma_i^p(\theta_i) = 0$ for all bidders; that is, actual payments do not depend on other bidders' valuations. In practice, implementing auctions similar to all-pay auctions may be difficult.

However, the benefit of asymmetric designs extends to standard auctions. We now establish that when the degree of loss aversion in the good dimension is large ($\Lambda^g = 1$), in the first- or second-price auctions, expected revenues may decrease with the number of bidders:

Proposition 7. *Suppose $\Lambda_i^g = 1$, $\Lambda_i^p = 0$, and valuations are distributed over $[0, 1]$ according to $F_i(\theta) = F(\theta) = \theta^\kappa$, with $\kappa > 0$. If κ is sufficiently large, then expected revenues in the first- and second-price auctions are decreasing in the number of potential bidders n.*

As explained earlier, there are two types of benefits of having more bidders from the perspective of sellers: greater selection and rent extraction. With loss averse bidders, the seller needs to compare these benefits of having more bidders with the demerit of increasing gain-loss disutility borne by bidders from introducing more competition. When κ is sufficiently large, the value of greater selection and rent extraction is so small that the demerit of having more bidders outweighs, making it beneficial for the seller to reduce bidder participation.

4 Heterogeneous valuation distributions

We have established the optimality of asymmetric designs even when bidders are ex-ante identical. We now study how bidder asymmetry in valuation distributions affects the optimal extent of favoritism. To simplify our analysis, we consider a case of $n = 2$ with $F_1 \neq F_2$. We further assume that θ_2 is deterministic, so bidder 2's virtual value is $\psi_2(\theta_2) = \theta_2$.

By Lemmas 1 and 2, the expected revenue in the optimal mechanism is given by

$$\int_{\theta_1} \psi_1(\theta_1)\alpha^g(q_1(\theta_1))dF_1(\theta_1) + \psi_2(\theta_2)\alpha^g(q_2(\theta_2))$$

where $q_1(\theta_1)$ is nondecreasing. We first establish that it is without loss of generality to restrict our analysis to cutoff mechanisms.

Lemma 8. *The optimal mechanism takes the cutoff form: bidder 2 wins with probability* $q_2 \in [0,1]$ *and there exists a threshold* $\theta^* \in [\underline{\theta}, \overline{\theta}]$ *such that*

- *(i) Bidder 1 with valuation* θ_1 *wins if* $\theta_1 > \theta^*$ *and loses if* $\theta_1 < \theta^*$*;*
- *(ii) The ex-ante expected losing probability of bidder 1 is no less than the winning probability of bidder 2, i.e.,* $F_1(\theta^*) \geq q_2$*.*

To focus on the effect of loss aversion, we make the following assumption:[4]

Assumption A1. $\max\{\psi_1(\underline{\theta}), 0\} < \psi_2(\theta_2) = \theta_2 < \psi_1(\overline{\theta})$.

Under **A1**, the capacity constraint should bind, i.e., $q_2 = F_1(\theta^*)$ in the optimal mechanism. Thus, by the lemma, the optimal auction solves

$$\max_{\theta^* \in [\underline{\theta}, \overline{\theta}]} \Pi(\theta^*) = \int_{\theta^*}^{\overline{\theta}} \psi_1(\theta_1)dF_1(\theta_1) + \theta_2\alpha^g(F_1(\theta^*)). \tag{16}$$

Note that when $\Lambda_2^g = 0$, the optimal cutoff is set so that $\psi_1(\theta^*) = \psi_2(\theta_2)$, i.e., $\theta^* = \psi_1^{-1}(\theta_2)$. We refer this as the Myerson level and denote by θ_M. Under **A1**, $\theta_M \in (\underline{\theta}, \overline{\theta})$. We establish how the optimal threshold should be modified from the Myerson level θ_M:

Proposition 9. *Consider the optimal mechanism under **A1**. Suppose that the degree of bidder 2's loss aversion* Λ_2^g *is sufficiently small. Then the optimal threshold* $\theta^*(\Lambda_2^g)$ *is unique, satisfies* $\theta^*(0) = \theta_M \in (\underline{\theta}, \overline{\theta})$*, and depends on* Λ_2^g *but not on* Λ_1^g*. Moreover,* $\theta^*(\Lambda_2^g)$ *is differentiable and satisfies the following:*

- $\theta^{*\prime}(\Lambda_2^g) > 0$ *if bidder 2 wins more often than losing in the Myerson mechanism.*
- $\theta^{*\prime}(\Lambda_2^g) < 0$ *if bidder 2 loses more often than winning in the Myerson mechanism.*

[4]When $0 \geq \max\{\theta_2, \psi_1(\underline{\theta})\}$, the seller should never award the good to bidder 2; thus, bidder 1 wins if and only if $\theta_1 \geq \psi_1^{-1}(0)$ regardless of $\Lambda_2^g \in [0,1]$. Moreover, when $\psi_1(\underline{\theta}) \geq \max\{0, \theta_2\}$, bidder 1 should always win regardless of $\Lambda_2^g \in [0,1]$.

First, the optimal threshold depends on only Λ_2^g because bidder 1 faces no uncertainty at the interim stage. Second, the optimal threshold is the Myerson level θ_M when bidder 2 are not loss averse, i.e., $\Lambda_2^g = 0$. Hence, $\theta^*(0) = \theta_M$.

Third, the optimal threshold is continuously adjusted from the Myerson level so that the winning probability of bidder 2 moves away from 1/2 (i.e., less uncertain). Thus, the direction of adjustment depends on whether the winning probability of bidder 2 is higher than 1/2 in the Myerson mechanism. When Λ_2^g is not too large, the seller must raise the threshold from the Myerson level to increase bidder 2' winning probability to reduce bidder 2's uncertainty. The similar logic applies for the second bullet point.

However, if Λ_2^g is sufficiently large, the optimal threshold may not change as stated in Proposition 9. The threshold may jump from below the Myerson level to above, or vice versa. Such a jump may occur depending on F_1. For example, suppose that bidder 2 loses more often than winning at the Myerson level (i.e., $F_1(\theta_M) < 1/2$) and that bidder 1's virtual value $\psi_1(\theta_1)$ rapidly increases in θ_1 for $\theta_1 < \theta_M$ but hardly increases for $\theta_1 > \theta_M$. Then, we can adjust the threshold downward from the Myerson level θ_M to reduce the uncertainty, but the adjustment required for optimization becomes larger as Λ_2^g rises. With this virtual valuation, a large downward adjustment lowers virtual valuation much. On the other hand, the loss in virtual valuation is small if the adjustment is upward so that bidder 2 wins with probability 1 to eliminate the uncertainty. Thus, as Λ_2^g rises, the threshold becomes smaller than the Myerson level θ_M when Λ_2^g is small, but it jumps up to $\overline{\theta}$ when Λ_2^g is large enough.

The following proposition explicitly computes the optimal threshold when bidder 1's valuation is uniformly distributed over $[\underline{\theta}, \overline{\theta}]$. In this case, bidder 1's virtual valuation increases in θ_1 at a constant rate: $\psi_1(\theta_1) = 2\theta_1 - \overline{\theta}$. Thus, the above jump does not occur.

Proposition 10. *Suppose **A1**. Suppose that θ_1 is uniformly distributed over $[\underline{\theta}, \overline{\theta}]$ and θ_2 is deterministic, so* $\max\{2\underline{\theta} - \overline{\theta}, 0\} < \theta_2 < \overline{\theta}$ *holds. In the optimal auctions,*

1. *When $\theta_2 > \underline{\theta}$ and $\Lambda_2^g < \frac{\overline{\theta}-\theta_2}{\theta_2}$, bidder 1 wins if $\theta_1 \geq \theta^*(\Lambda_2^g)$ and bidder 2 wins if $\theta_1 < \theta^*(\Lambda_2^g)$, where*

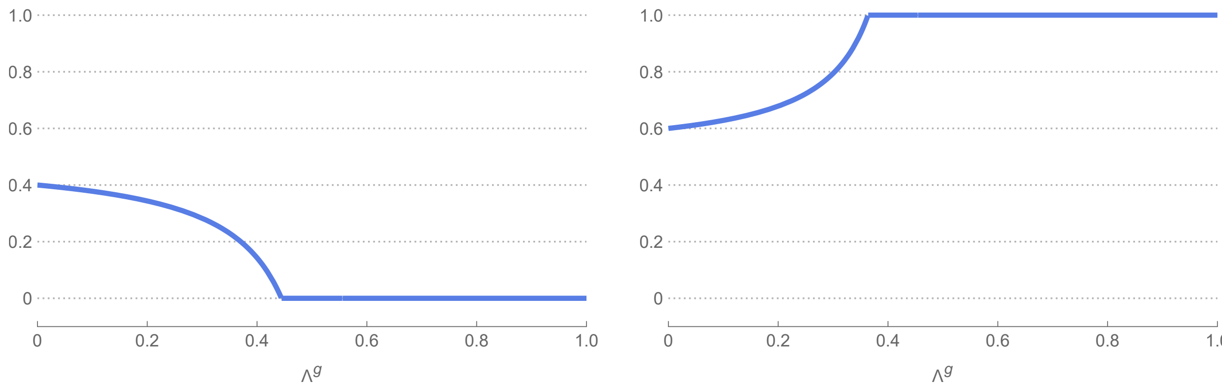
$$\theta^*(\Lambda_2^g) = \frac{1}{2}\frac{(\overline{\theta} - \underline{\theta})[\overline{\theta} + (1 - \Lambda_2^g)\theta_2] - 2\Lambda_2^g\underline{\theta}\theta_2}{\overline{\theta} - \underline{\theta} - \Lambda_2^g\theta_2}. \tag{17}$$

   $\theta^*(\Lambda_2^g) > \psi_1^{-1}(\theta_2) = \frac{1}{2}(\overline{\theta} + \theta_2)$ and $\theta^*(\Lambda_2^g)$ is increasing in Λ_2^g, i.e., the optimal mechanism favors bidder 2, but to the greater degree than the Myerson level, and greater Λ_2^g makes it even greater.

2. *When $\theta_2 > \underline{\theta}$ and $\Lambda_2^g \geq \frac{\overline{\theta}-\theta_2}{\theta_2}$, bidder 2 always wins.*

3. *When $\theta_2 < \underline{\theta}$ and $\Lambda_2^g < \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$, bidder 1 wins if $\theta_1 \geq \theta^*(\Lambda_2^g)$ and bidder 2 wins if $\theta_1 < \theta^*(\Lambda_2^g)$. $\theta^*(\Lambda_2^g) \in (\underline{\theta}, \psi_1^{-1}(\theta_2))$ and $\theta^*(\Lambda_2^g)$ is decreasing in Λ_2^g, i.e., the optimal mechanism favors bidder 2, but to the lesser degree than the Myerson level, and greater Λ_2^g makes it even lesser.*

4. *When $\theta_2 < \underline{\theta}$ and $\Lambda_2^g \geq \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$, bidder 1 always wins.*



(a) $\theta_2 = 0.9$ (b) $\theta_2 = 1.1$

Figure 2: Ex-ante probabilities of bidder 2 winning as a function of Λ_2^g for different bidder 2 valuations θ_2 when θ_1 is uniformly distributed over $(1, 1.5)$.

To gain the intuition for items 2 and 4, let us consider when $\Lambda_2^g > \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$ is satisfied. $\Lambda_2^g > \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$ implies that $\Lambda_2^g > \frac{\overline{\theta}-\theta_2}{\theta_2}$ when $\theta_2 > \underline{\theta}$ and $\Lambda_2^g > \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$ when $\theta_2 < \underline{\theta}$. This corresponds to the case where the degree of loss aversion is sufficiently large relative to the variation in valuations. In this case, regardless of whether $\theta_2 > \underline{\theta}$, the benefit of reducing gain-loss disutility borne by bidder 2 (the only bidder who faces uncertainty about winning in our setting) is so great that the benefits of rent extraction and greater selection can be ignored. This makes it optimal to always select one bidder as the winner, leaving us with two candidates for the optimal mechanism: selling to bidder 2 at price θ_2, or selling to bidder 1 with probability one at price $\underline{\theta}$ (the highest price that satisfies incentive compatibility for bidder 1 of any valuation). Thus it is optimal to sell to bidder 1 if and only if $\underline{\theta} > \theta_2$.

To illustrate this, suppose that $\Lambda_2^g > 0.5$, θ_1 is uniformly distributed over $(1, 1.5)$, and $\theta_2 \in (1-\varepsilon, 1+\varepsilon)$ where $\varepsilon > 0$ is arbitrary small. Then, without loss aversion, bidder 1 wins if and only if $\theta_2 < \psi_1(\theta_1) = 2\theta_1 - 1.5$, i.e., $\theta_1 > 1.25 \pm \frac{\varepsilon}{2}$, where each bidder wins with probability almost $\frac{1}{2}$. However, with sufficiently large loss aversion ($\Lambda_2^g > 0.5$), one of the bidders wins with probability 1: if $\theta_2 \in (1-\varepsilon, 1]$, then bidder 1 always wins, and if $\theta_2 \in [1, 1+\varepsilon)$, then bidder 2 always wins.

We now turn to the case of $\Lambda_2^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$, in which the variation in valuations may be too large for the seller to ignore the benefit of competition: rent extraction and greater selection. Thus, the optimal mechanism may select the winner stochastically, but such a selection creates uncertainty and imposes psychological costs on bidder 2.

Without loss aversion, the optimal balance between rent extraction and greater selection can be achieved by awarding the good to the bidder with highest virtual valuation. With loss aversion, the seller must also consider mitigating the psychological costs incurred with bidder 2, who is the only bidder that faces uncertainty as to winning in our setting.

When $F_1(\psi_1^{-1}(\theta_2)) = (\frac{\overline{\theta}+\theta_2}{2} - \underline{\theta})\frac{1}{\overline{\theta}-\underline{\theta}} < \frac{1}{2}$, i.e., $\theta_2 < \underline{\theta}$, the probability that $\psi_2(\theta_2) > \psi_1(\theta_1)$ is less than $\frac{1}{2}$. In this case, without loss aversion, the expected winning probability of bidder 2 is less than $\frac{1}{2}$. Therefore, as in Myerson (1981), the optimal mechanism favors bidder 2, but the degree of optimal favoritism must be reduced in order to reduce its winning probability away from $\frac{1}{2}$ and mitigate the psychological cost to the bidder. The larger Λ_2^g is, the larger the required correction from the Myerson level becomes.

On the other hand, when $\theta_2 > \underline{\theta}$, the expected winning probability of bidder 2 is greater than $\frac{1}{2}$ without loss aversion. Thus, the optimal mechanism again favors bidder 2, but the degree of optimal favoritism is increased in order to increase its winning probability away from $\frac{1}{2}$ and mitigate the gain-loss disutility borne by bidder 2.

Figure 2 illustrates Proposition 10, where θ_1 is uniformly distributed over $(1, 1.5)$. When $\theta_2 = 0.9$, $F_1(\psi_1^{-1}(\theta_2)) = 0.4$. Thus, when there is no loss aversion (i.e., $\Lambda_2^g = 0$), bidder 2 wins with probability 0.4. As Λ_2^g becomes larger, the optimal mechanism continuously *reduces* the probability of bidder 2 winning to mitigate the gain-loss disutility. Once Λ_2^g reaches $\frac{\bar{\theta}-2\underline{\theta}+\theta_2}{\theta_2} \approx 0.444$, bidder 2 always loses. In contrast, when $\theta_2 = 1.1$, bidder 2 wins with probability 0.6 when $\Lambda_2^g = 0$. As Λ_2^g rises, the optimal mechanism continuously *increases* the probability of bidder 2 winning. Once Λ_2^g reaches $\frac{\bar{\theta}-\theta_2}{\theta_2} \approx 0.363$, bidder 2 always wins.

5 Conclusion

We showed that with expectation-based loss-averse bidders, the seller trades off rent extraction with the inefficiency of forcing bidders to incur psychological disutility when choosing the degree of competition. We first established that selling to a single buyer—an extreme form of asymmetric designs—is optimal even when bidders are ex-ante identical if the value of rent extraction is relatively small. We then established that optimal mechanisms are necessarily asymmetric for more general valuation distributions. Furthermore, we found that when bidders have different valuation distributions, the degree of favoritism must be modified from the level prescribed by Myerson (1981) based on the degree of loss aversion.

Appendix

Proof of Lemma 1. Let $\tilde{q}_i(\theta_i) \equiv q_i(\theta_i) + \gamma_i^g(\theta_i)$ and $\tilde{m}_i(\theta_i) \equiv m_i(\theta_i) + \gamma_i^p(\theta_i)$. (3) is rewritten by

$$\pi_i\left(\theta_i'|\theta_i\right) = \theta_i \tilde{q}_i(\theta_i') - \tilde{m}_i(\theta_i').$$

Then, by treating $\tilde{q}_i(\theta_i)$ and $\tilde{m}_i(\theta_i)$ as the interim expected winning probability and payment of bidder i of type θ_i, it is routine to follow the standard procedure for bidders with expected-utility preferences (Krishna, 2002, Section 5.1.2) to show that incentive compatibility (8) holds if and only if $\tilde{q}_i(\theta_i)$ is nondecreasing and the interim expected payment $\tilde{m}_i(\theta_i)$ satisfies (9). Moreover, it follows from (6) that $\tilde{q}_i(\theta_i) = \alpha_i^g(q_i(\theta_i))$ and $\alpha_i^g(q_i(\cdot))$ is increasing in $q_i(\cdot)$. Therefore, the first part of the lemma holds.

For the latter part of the lemma, integration by parts and $\tilde{q}_i(\theta_i) = \alpha_i^g(q_i(\theta_i))$ yield

$$\int_{\underline{\theta}_i}^{\bar{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \tilde{q}_i(t_i) dt_i dF_i\left(\theta_i\right) = \int_{\underline{\theta}_i}^{\bar{\theta}_i} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \alpha_i^g(q_i(\theta_i)) dF_i(\theta_i). \tag{18}$$

Then the seller's expected revenue is written by

$$\begin{aligned}
&\sum_i \int_{\underline{\theta}_i}^{\overline{\theta}_i} m_i(\theta_i) dF_i(\theta_i) \\
&= \sum_i \left\{ \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i \tilde{q}_i(\theta_i) dF_i(\theta_i) - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \tilde{q}_i(t_i) dt_i dF_i(\theta_i) \right\} \\
&\quad - \sum_i \pi_i(\underline{\theta}_i) - \sum_i \int_{\underline{\theta}_i}^{\overline{\theta}_i} \gamma_i^p(\theta_i) dF_i(\theta_i) \\
&= \sum_i \int_{\underline{\theta}_i}^{\overline{\theta}_i} \left[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] \alpha_i^g(q_i(\theta_i)) dF_i(\theta_i) - \sum_i \pi_i(\underline{\theta}_i) - \sum_i \int_{\underline{\theta}_i}^{\overline{\theta}_i} \gamma_i^p(\theta_i) dF_i(\theta_i).
\end{aligned}$$

The first equality follows by (9) and the second equality follows by (18). Moreover, substituting (9) into (3) yields (10). □

Proof of Lemma 2. We first show that $\gamma_i^p(\theta_i) \geq 0$. The interim expected loss-gain utility in the money dimension can be written as

$$\begin{aligned}
&\eta_i^p \lambda_i^p \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) > m_i(\theta_i')\right\}} \left[M_i(\theta_i', \theta_{-i}) - m_i(\theta_i')\right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
&= \eta_i^p \lambda_i^p \int_{\theta''_{-i}} \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) > M_i(\theta_i', \theta''_{-i})\right\}} \left[M_i(\theta_i', \theta_{-i}) - M_i(\theta_i', \theta''_{-i})\right] f_{-i}(\theta_{-i}) d\theta_{-i} f_{-i}\left(\theta''_{-i}\right) d\theta''_{-i} \\
&= \eta_i^p \lambda_i^p \int_{\left\{\theta''_{-i}: M_i(\theta_i', \theta_{-i}) > M_i(\theta_i', \theta''_{-i})\right\}} \int_{\theta_{-i}} \left[M_i(\theta_i', \theta_{-i}) - M_i(\theta_i', \theta''_{-i})\right] f_{-i}(\theta_{-i}) d\theta_{-i} f_{-i}\left(\theta''_{-i}\right) d\theta''_{-i} \\
&= \eta_i^p \lambda_i^p \int_{\left\{\theta''_{-i}: m_i(\theta_i') > M_i(\theta_i', \theta''_{-i})\right\}} \left[m_i(\theta_i') - M_i(\theta_i', \theta''_{-i})\right] f_{-i}\left(\theta''_{-i}\right) d\theta''_{-i},
\end{aligned}$$

where the first and last equalities hold by the definition of $m_i(\theta_i')$ and the second equality holds by interchanging the order of integration. Together with (4), it follows that

$$\begin{aligned}
\gamma_i^p(\theta_i') &= \eta_i^p \lambda_i^p \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) > m_i(\theta_i')\right\}} \left[M_i(\theta_i', \theta_{-i}) - m_i(\theta_i')\right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
&\quad - \eta_i^p \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) < m_i(\theta_i')\right\}} \left[m_i(\theta_i') - M_i(\theta_i', \theta_{-i})\right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
&= \eta_i^p \lambda_i^p \int_{\left\{\theta''_{-i}: m_i(\theta_i') > M_i(\theta_i', \theta''_{-i})\right\}} \left[m_i(\theta_i') - M_i(\theta_i', \theta''_{-i})\right] f_{-i}\left(\theta''_{-i}\right) d\theta''_{-i} \\
&\quad - \eta_i^p \int_{\left\{\theta_{-i}: M_i(\theta_i', \theta_{-i}) < m_i(\theta_i')\right\}} \left[m_i(\theta_i') - M_i(\theta_i', \theta_{-i})\right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
&= \Lambda_i^p \int_{\left\{\theta_{-i}: m_i(\theta_i') > M_i(\theta_i', \theta_{-i})\right\}} \left[m_i(\theta_i') - M_i(\theta_i', \theta_{-i})\right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
&\geq 0.
\end{aligned}$$

The strict inequality holds if and only if M is not degenerate (Eisenhuth, 2019, Lemma 1).

By Lemma 1, the seller's revenue is decreasing in both $\gamma_i^p(\theta_i)$ and $\pi_i(\underline{\theta}_i)$. Moreover, reducing $\gamma_i^p(\theta_i)$ relaxes (7) but does not affect (8), yielding $\gamma_i^p(\theta_i) = 0$. The seller can choose m_i according to (9) and $\pi_i(\underline{\theta}_i) = 0$ while satisfying (7) and (8), yielding $\pi_i(\underline{\theta}_i) = 0$. Then, (11) is written by $\int_\theta \sum_i \psi_i(\theta_i)\alpha_i^g(q_i(\theta_i))f(\theta)\,d\theta$. The expected revenue in optimal symmetric mechanisms follows from Eisenhuth (2019, Proposition 4). □

Proof of Theorem 3. Suppose that bidders are ex-ante symmetric (i.e., $F_i = F$) and each bidder i's preference is represented by a Bernoulli utility function $u(q_i, -m_i|\theta_i)$, where $q_i \in \{0, 1\}$ is the amount of its consumption of the good and m_i is its payment.

We focus on a direct mechanism (Q, M), where for each report profile $\theta = (\theta_1, \ldots, \theta_n)$, $Q_i(\theta)$ is the probability that bidder i obtains the item (i.e., $q_i = 1$) and $\sum Q_i(\theta) \leq 1$, and M_i is a random payment rule such that bidder i pays m that is drawn from the cumulative distribution $M_i(\theta)$. The expected payoff of bidder i with type θ_i from reporting $\hat{\theta}_i$ when the other bidders report truthfully is given by

$$U_i(\hat{\theta}_i|\theta_i) = \int_{\theta_{-i}} \int_{m_i} \int_{q_i} u(q_i, -m_i|\theta_i) dQ_i(\theta(\hat{\theta}_i; i)) dM_i(\theta(\hat{\theta}_i; i)) f_{-i}(\theta_{-i})\, d\theta_{-i}$$

where $\theta(\hat{\theta}_i; i) \equiv (\theta_1, \ldots, \theta_{i-1}, \hat{\theta}_i, \theta_{i+1}, \ldots, \theta_n)$.

Definition 11. *A direct mechanism (Q, M) is said to be symmetric if for any two bidders i and j $(i \neq j)$ and any type profile $\theta \equiv (\theta_1, \ldots, \theta_n)$ with $\theta_i = \theta_j$,*

$$Q_i(\theta) = Q_j(\hat{\theta}) \text{ and } M_i(\theta) = M_j(\hat{\theta})$$

hold, where $\hat{\theta} \equiv (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ is the type profile in which θ_i and θ_j are flipped: $\hat{\theta}_j = \theta_i$, $\hat{\theta}_i = \theta_j$, and $\hat{\theta}_k = \theta_k$ for $k \neq i, j$.

Lemma 12. *Suppose that a direct mechanism (Q, M) is individual rational and incentive compatible. For any permutation σ of $\{1, \ldots, n\}$, let (Q', M') be a permutated mechanism defined as $Q'_i(\theta) \equiv Q_{\sigma(i)}(\theta_\sigma)$ and $M'_i(\theta) = M_{\sigma(i)}(\theta_\sigma)$, where $\theta_\sigma \equiv (\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})$. If bidders are symmetric, then (Q', M') yields the same revenue as (Q, M) and is individually rational and incentive compatible.*

Proof. In the new mechanism (Q', M'), each bidder i plays the role of bidder $\sigma(i)$ of the original mechanism (Q, M). Since bidders are symmetric, (IR) and (IC) are satisfied, achieving the same revenue. □

Suppose that there is an asymmetric optimal mechanism. Then, by the revelation principle, there is also a direct optimal mechanism (Q, M). We define a new direct symmetric mechanism (Q^S, M^S) as follows: for each $\theta \equiv (\theta_1, \ldots, \theta_n)$

$$Q_i^S(\theta) \equiv \frac{1}{\#S_n} \sum_{\sigma \in S_n} Q_{\sigma(i)}(\theta_\sigma), \quad M_i^S(\theta) \equiv \frac{1}{\#S_n} \sum_{\sigma \in S_n} M_{\sigma(i)}(\theta_\sigma),$$

where S_n is the set of permutations σ of the set $\{1, \ldots, n\}$ and $\#S_n = n!$ is the number of all permutations σ in S_n.

To show that (Q^S, M^S) is optimal, we show that (i) (Q^S, M^S) yields the same revenue as (Q, M); (ii) (Q^S, M^S) is feasible; (iii) (Q^S, M^S) is individual rational and incentive compatible.

(i): Let Π and Π^S be the expected revenues in (Q, M) and (Q^S, M^S), respectively. Then,

$$\begin{aligned}
\Pi &= \int_\theta \left(\sum_i \int_{m_i} m_i \; dM_i(\theta) \right) f(\theta) \, d\theta \\
&= \int_\theta \left(\frac{1}{\#S_n} \sum_{\sigma \in S_n} \sum_i \int_{m_{\sigma(i)}} m_{\sigma(i)} \; dM_{\sigma(i)}(\theta_\sigma) \right) f(\theta) \, d\theta \\
&= \int_\theta \left(\sum_i \frac{1}{\#S_n} \sum_{\sigma \in S_n} \int_{m_{\sigma(i)}} m_{\sigma(i)} \; dM_{\sigma(i)}(\theta_\sigma) \right) f(\theta) \, d\theta \\
&= \int_\theta \left(\sum_i \int_{m_i} m_i \; dM_i^S(\theta) \right) f(\theta) \, d\theta \\
&= \Pi^S,
\end{aligned}$$

where the second equality holds because $\sum_i M_{\sigma(i)}(\theta_\sigma) = \sum_i M_i(\theta)$ for any permutation σ.

(ii): For any type profile θ,

$$\sum_i Q_i^S(\theta) = \sum_i \left(\frac{1}{\#S_n} \sum_{\sigma \in S_n} Q_{\sigma(i)}(\theta_\sigma) \right) = \left(\frac{1}{\#S_n} \sum_{\sigma \in S_n} \sum_i Q_i(\theta) \right) = \sum_i Q_i(\theta),$$

where the second equality follows because $\sum_i Q_i(\theta) = \sum_i Q_{\sigma(i)}(\theta_\sigma)$ for any permutation σ.

(iii): Let $U_i^S(\hat{\theta}_i|\theta_i)$ be the expected payoff of bidder i with type θ_i from reporting $\hat{\theta}_i$ when the other bidders report truthfully in (Q^S, M^S). Then,

$$\begin{aligned}
&U_i^S(\hat{\theta}_i \mid \theta_i) \\
&= \int_{\theta_{-i}} \int_{m_i} \int_{q_i} u(q_i, -m_i \mid \theta_i) dQ_i^S(\theta(\hat{\theta}_i; i)) dM_i^S(\theta(\hat{\theta}_i; i)) f_{-i}(\theta_{-i}) d\theta_{-i} \\
&= \frac{1}{\#S_n} \sum_{\sigma \in S_n} \int_{\theta_{-i}} \int_{m_i} \int_{q_i} u(q_i, -m_i \mid \theta_i) dQ_{\sigma(i)}(\theta_\sigma(\hat{\theta}_i; \sigma(i))) dM_{\sigma(i)}(\theta_\sigma(\hat{\theta}_i; \sigma(i))) f_{-i}(\theta_{-i}) d\theta_{-i} \\
&= \frac{1}{\#S_n} \sum_{\sigma \in S_n} U_{\sigma(i)}(\hat{\theta}_i \mid \theta_i).
\end{aligned}$$

Thus, $U_i^S(\theta_i|\theta_i) = \frac{1}{\#S_n} \sum_{\sigma \in S_n} U_{\sigma(i)}(\theta_i|\theta_i) \geq 0$ holds because $U_{\sigma(i)}(\theta_i|\theta_i) \geq 0$ by individual rationality in (Q, M). Moreover, $U_i^S(\theta_i|\theta_i) = \frac{1}{\#S_n} \sum_{\sigma \in S_n} U_{\sigma(i)}(\theta_i|\theta_i) \geq \frac{1}{\#S_n} \sum_{\sigma \in S_n} U_{\sigma(i)}(\hat{\theta}_i|\theta_i) = U_i^S(\hat{\theta}_i|\theta_i)$ holds by incentive compatibility in (Q, M). □

Proof of Proposition 4. First note that since $1 > \Lambda^g \left(\frac{n-1}{2n-1} \right)$ with $n \geq 2$ and $\Lambda^g \leq 1$, the statement is well defined. With a single buyer, the seller can make a take-it-or-leave-it offer and receive revenue of $\underline{\theta}$.

In the optimal symmetric mechanism, by Lemma 2, the expected revenue is given by

$$
\begin{aligned}
& n\int_{\hat{\theta}}^{\underline{\theta}+\varepsilon} \psi(\theta_i)F(\theta_i)^{n-1}[1-\Lambda^g+\Lambda^g F(\theta_i)^{n-1}]dF(\theta_i) \\
&= n(1-\Lambda^g)\int_{\hat{\theta}}^{\underline{\theta}+\varepsilon} \psi(\theta_i)F(\theta_i)^{n-1}dF(\theta_i) + n\Lambda^g \int_{\hat{\theta}}^{\underline{\theta}+\varepsilon} \psi(\theta_i)F(\theta_i)^{2n-2}dF(\theta_i) \\
&= (1-\Lambda^g)\int_{\hat{\theta}}^{\underline{\theta}+\varepsilon} \psi(\theta_i)f_1^{(n)}(\theta_i)d\theta_i + \frac{n}{2n-1}\Lambda^g \int_{\hat{\theta}}^{\underline{\theta}+\varepsilon} \psi(\theta_i)f_1^{(2n-1)}(\theta_i)d\theta_i \\
&= (1-\Lambda^g)(1-F_1^{(n)}(\hat{\theta}))\mathbb{E}[\psi(Y_1^{(n)}) \mid Y_1^{(n)} \geq \hat{\theta}] + \frac{n\Lambda^g(1-F_1^{(2n-1)}(\hat{\theta}))}{2n-1}\mathbb{E}[\psi(Y_1^{(2n-1)}) \mid Y_1^{(2n-1)} \geq \hat{\theta}] \\
&\leq (1-\Lambda^g)\psi(\underline{\theta}+\varepsilon) + \frac{n\Lambda^g}{2n-1}\psi(\underline{\theta}+\varepsilon),
\end{aligned}
$$

where $Y_1^{(n)}$ is the highest order statistic of n independent draws from F, $f_1^{(n)}(y) \equiv nF^{n-1}(y)f(y)$ is its density, and $F_1^{(n)}(y) \equiv F(y)^n$ is its distribution. The inequality holds because ψ is increasing.

Taking the difference in the expected revenues yields

$$
\begin{aligned}
\underline{\theta} - \left\{(1-\Lambda^g)\psi(\underline{\theta}+\varepsilon) + \frac{n\Lambda^g}{2n-1}\psi(\underline{\theta}+\varepsilon)\right\} &= \underline{\theta} - \left\{(1-\Lambda^g)(\underline{\theta}+\varepsilon) + \frac{n\Lambda^g}{2n-1}(\underline{\theta}+\varepsilon)\right\} \\
&= \underline{\theta}\Lambda^g\left(\frac{n-1}{2n-1}\right) - \varepsilon\left\{1-\Lambda^g\left(\frac{n-1}{2n-1}\right)\right\},
\end{aligned}
$$

which is strictly positive by our assumption in the proposition. □

Proof of Proposition 5. We first prove the first part of the proposition. Consider the asymmetric mechanism described preceding Proposition 5 such that $\theta_L \in I$. When both θ_1 and θ_2 lie in $(\theta_L, \theta_L+\varepsilon)$, $Q_1(\theta_1,\theta_2) = 1$ and $Q_2(\theta_1,\theta_2) = 0$. When they do not, $Q_1(\theta_1,\theta_2) = 1$ if $\theta_1 > \theta_2$ and $Q_2(\theta_1,\theta_2) = 1$ if $\theta_1 < \theta_2$. Thus, it follows that $q_1(\theta_1) = F(\theta_1)$ and $q_2(\theta_2) = F(\theta_2)$ if $\theta_1, \theta_2 \notin (\theta_L, \theta_L+\varepsilon)$; $q_1(\theta_1) = F(\theta_L+\varepsilon)$ and $q_2(\theta_2) = F(\theta_L)$ if $\theta_1, \theta_2 \in (\theta_L, \theta_L+\varepsilon)$.

By setting payment rule (M_1, M_2) satisfying (9), $\pi_i(\underline{\theta}_i) = 0$, and $\gamma_i^p(\theta_i) = 0$ for $i \in \{1,2\}$, since both q_1 and q_2 are nondecreasing, Lemma 1 implies that this mechanism satisfies incentive compatibility (8) and the seller's expected revenue can be written as

$$
\begin{aligned}
\Pi(\varepsilon) \equiv & \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_1)\alpha^g(F(\theta_L+\varepsilon))dF(\theta_1) + \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_2)\alpha^g(F(\theta_L))dF(\theta_2) \\
& + 2\int_{\theta_L+\varepsilon}^{\bar{\theta}} \psi(\theta_i)\alpha^g(F(\theta_i))dF(\theta_i) + 2\int_{\hat{\theta}}^{\theta_L} \psi(\theta_i)\alpha^g(F(\theta_i))dF(\theta_i),
\end{aligned}
$$

The first term is the expected payment from bidder 1 of valuation $\theta_1 \in (\theta_L, \theta_L+\varepsilon)$; the second term is the expected payment from bidder 2 of valuation $\theta_2 \in (\theta_L, \theta_L+\varepsilon)$; and the third and fourth terms are the expected payment from bidder i of $\theta_i \notin (\theta_L, \theta_L+\varepsilon)$, where coefficient 2 follows by symmetry between the bidders.

In what follows, we use Maclaurin series to conclude $\Pi(\varepsilon) > \Pi(0)$ by showing that

$\Pi'(0) = \Pi''(0) = 0$ and $\Pi'''(0) > 0$. It follows that

$$\begin{aligned}
\Pi'(\varepsilon) &= \psi(\theta_L+\varepsilon)\alpha^g(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon) + \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_1)\alpha^{g\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)dF(\theta_1) \\
&\quad + \psi(\theta_L+\varepsilon)\alpha^g(F(\theta_L))f(\theta_L+\varepsilon) - 2\psi(\theta_L+\varepsilon)\alpha^g(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon) \\
&= \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_1)\alpha^{g\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)dF(\theta_1) + \psi(\theta_L+\varepsilon)\alpha^g(F(\theta_L))f(\theta_L+\varepsilon) \\
&\quad - \psi(\theta_L+\varepsilon)\alpha^g(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon);
\end{aligned}$$

$$\begin{aligned}
\Pi''(\varepsilon) &= \psi(\theta_L+\varepsilon)\alpha^{g\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)^2 \\
&\quad + \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_1)\{\alpha^{g\prime\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)^2 + \alpha^{g\prime}(F(\theta_L+\varepsilon))f'(\theta_L+\varepsilon)\}dF(\theta_1) \\
&\quad + \alpha^g(F(\theta_L))\{\psi'(\theta_L+\varepsilon)f(\theta_L+\varepsilon) + \psi(\theta_L+\varepsilon)f'(\theta_L+\varepsilon)\} \\
&\quad - \psi'(\theta_L+\varepsilon)\alpha^g(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon) - \psi(\theta_L+\varepsilon)\alpha^{g\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)^2 \\
&\quad - \psi(\theta_L+\varepsilon)\alpha^g(F(\theta_L+\varepsilon))f'(\theta_L+\varepsilon) \\
&= \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_1)\{\alpha^{g\prime\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)^2 + \alpha^{g\prime}(F(\theta_L+\varepsilon))f'(\theta_L+\varepsilon)\}dF(\theta_1) \\
&\quad + \{\alpha^g(F(\theta_L)) - \alpha^g(F(\theta_L+\varepsilon))\}\{\psi'(\theta_L+\varepsilon)f(\theta_L+\varepsilon) + \psi(\theta_L+\varepsilon)f'(\theta_L+\varepsilon)\};
\end{aligned}$$

$$\begin{aligned}
\Pi'''(\varepsilon) &= \psi(\theta_L+\varepsilon)\{\alpha^{g\prime\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)^2 + \alpha^{g\prime}(F(\theta_L+\varepsilon))f'(\theta_L+\varepsilon)\}f(\theta_L+\varepsilon) \\
&\quad + \int_{\theta_L}^{\theta_L+\varepsilon} \psi(\theta_1)\frac{d}{d\varepsilon}\{\alpha^{g\prime\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)^2 + \alpha^{g\prime}(F(\theta_L+\varepsilon))f'(\theta_L+\varepsilon)\}dF(\theta_1) \\
&\quad - \alpha^{g\prime}(F(\theta_L+\varepsilon))f(\theta_L+\varepsilon)\{\psi'(\theta_L+\varepsilon)f(\theta_L+\varepsilon) + \psi(\theta_L+\varepsilon)f'(\theta_L+\varepsilon)\} \\
&\quad + \{\alpha^g(F(\theta_L)) - \alpha^g(F(\theta_L+\varepsilon))\}\frac{d}{d\varepsilon}\{\psi'(\theta_L+\varepsilon)f(\theta_L+\varepsilon) + \psi(\theta_L+\varepsilon)f'(\theta_L+\varepsilon)\}.
\end{aligned}$$

Thus, $\Pi'(0) = \Pi''(0) = 0$ follows. Moreover,

$$\begin{aligned}
\Pi'''(0) &= \psi(\theta_L)\{\alpha^{g\prime\prime}(F(\theta_L))f(\theta_L)^2 + \alpha^{g\prime}(F(\theta_L))f'(\theta_L)\}f(\theta_L) \\
&\quad - \alpha^{g\prime}(F(\theta_L))f(\theta_L)\{\psi'(\theta_L)f(\theta_L) - \psi(\theta_L)f'(\theta_L)\} \\
&= \psi(\theta_L)\alpha^{g\prime\prime}(F(\theta_L))f(\theta_L)^3 - \psi'(\theta_L)\alpha^{g\prime}(F(\theta_L))f(\theta_L)^2 \\
&= 2\Lambda^g\psi(\theta_L)f(\theta_L)^3 - \psi'(\theta_L)\{(1-\Lambda^g) + 2\Lambda^g F(\theta_L)\}f(\theta_L)^2 \\
&= \{2\Lambda^g[\psi(\theta_L)f(\theta_L) - \psi'(\theta_L)F(\theta_L)] - \psi'(\theta_L)(1-\Lambda^g)\}f(\theta_L)^2 \\
&> 0,
\end{aligned}$$

where the third equality holds by $\alpha^{g\prime}(q) = (1-\Lambda^g) + 2\Lambda^g q$ and $\alpha^{g\prime\prime}(q) = 2\Lambda^g$, and the inequality holds because $\theta_L \in I$. This proves the first part of the proposition.

For the latter part of the proposition, if $\hat{\theta} = \underline{\theta}$ and $\lim_{\theta \searrow \underline{\theta}} \psi(\theta) > \frac{1-\Lambda^g}{2\Lambda^g} \lim_{\theta \searrow \underline{\theta}} \frac{\psi'(\theta)}{f(\theta)}$ hold, then there exists $\varepsilon > 0$ such that (15) holds for any $\theta \in [\underline{\theta}, \underline{\theta}+\varepsilon)$. The proposition follows. □

Proof of Proposition 6. When bidder 1 wins if $\theta_1 > \max\{\theta_2 - \varepsilon, \hat{\theta}\}$ and bidder 2 wins if $\theta_2 > \max\{\theta_1 + \varepsilon, \hat{\theta}\}$, $q_1(\theta_1) = F(\theta_1 + \varepsilon)$ and $q_2(\theta_2) = F(\theta_2 - \varepsilon)$ for $\theta_1, \theta_2 \geq \hat{\theta}$.

We set payment rule (M_1, M_2) satisfying (9), $\pi_i(\underline{\theta}_i) = 0$, and $\gamma_i^p(\theta_i) = 0$ for $i \in \{1, 2\}$. Then, since both q_1 and q_2 are nondecreasing, Lemma 1 implies that this mechanism satisfies incentive compatibility (8) and the seller's expected revenue can be written as

$$\begin{aligned}\Pi(\varepsilon) &\equiv \int_{\hat{\theta}}^{\overline{\theta}} \psi(\theta_1)\alpha^g(F(\theta_1+\varepsilon))dF(\theta_1) + \int_{\hat{\theta}}^{\overline{\theta}} \psi(\theta_2)\alpha^g(F(\theta_2-\varepsilon))dF(\theta_2) \\ &= \int_{\hat{\theta}}^{\overline{\theta}} \psi(\theta)[\alpha^g(F(\theta+\varepsilon)) + \alpha^g(F(\theta-\varepsilon))]dF(\theta).\end{aligned}$$

In what follows, we prove $\Pi(\varepsilon) > \Pi(0)$ by showing that $\Pi'(0) = 0$ and $\Pi''(0) > 0$. We have

$$\begin{aligned}\Pi'(\varepsilon) &= \int_{\hat{\theta}}^{\overline{\theta}} \psi(\theta)[\alpha^{g\prime}(F(\theta+\varepsilon))f(\theta+\varepsilon) - \alpha^{g\prime}(F(\theta-\varepsilon))f(\theta-\varepsilon)]dF(\theta); \\ \Pi''(\varepsilon) &= \int_{\hat{\theta}}^{\overline{\theta}} \psi(\theta)\frac{d}{d\varepsilon}[\alpha^{g\prime}(F(\theta+\varepsilon))f(\theta+\varepsilon) - \alpha^{g\prime}(F(\theta-\varepsilon))f(\theta-\varepsilon)]dF(\theta).\end{aligned}$$

Thus, $\Pi'(0) = 0$. Moreover, since $\alpha^{g\prime}(q) = (1-\Lambda^g) + 2\Lambda^g q$ and $\alpha^{g\prime\prime}(q) = 2\Lambda^g$, it follows that

$$\begin{aligned}&\frac{d}{d\varepsilon}[\alpha^{g\prime}(F(\theta+\varepsilon))f(\theta+\varepsilon) - \alpha^{g\prime}(F(\theta-\varepsilon))f(\theta-\varepsilon)] \\ &\quad = \alpha^{g\prime\prime}(F(\theta+\varepsilon))f(\theta+\varepsilon)^2 + \alpha^{g\prime}(F(\theta+\varepsilon))f'(\theta+\varepsilon) \\ &\qquad + \alpha^{g\prime\prime}(F(\theta-\varepsilon))f(\theta-\varepsilon)^2 + \alpha^{g\prime}(F(\theta-\varepsilon))f'(\theta-\varepsilon) \\ &\quad = 2\Lambda^g f(\theta+\varepsilon)^2 + [(1-\Lambda^g) + 2\Lambda^g F(\theta+\varepsilon)]f'(\theta+\varepsilon) \\ &\qquad + 2\Lambda^g f(\theta-\varepsilon)^2 + [(1-\Lambda^g) + 2\Lambda^g F(\theta-\varepsilon)]f'(\theta-\varepsilon),\end{aligned}$$

which becomes $4\Lambda^g f(\theta)^2 + 2[(1-\Lambda^g) + 2\Lambda^g F(\theta)]f'(\theta)$ as $\varepsilon \to 0$. Since $f'(\theta) > -\frac{2\Lambda^g f(\theta)^2}{(1-\Lambda^g)+2\Lambda^g F(\theta)}$, $4\Lambda^g f(\theta)^2 + 2[(1-\Lambda^g) + 2\Lambda^g F(\theta)]f'(\theta) > 0$ holds, yielding $\Pi''(0) > 0$. □

Proof of Proposition 7. When $\Lambda^p = 0$, first- and second-price auctions are revenue-equivalent by Lange and Ratan (2010). Thus, we consider second-price auctions. By Lange and Ratan (2010), the symmetric equilibrium bidding strategy is given by $\beta(\theta) = \theta\{1 - \Lambda^g[1 - 2F(\theta)^{n-1}]\} = 2\theta F(\theta)^{n-1}$. Thus, the expected revenue is written by

$$\begin{aligned}\mathbb{E}[\beta(Y_2^{(n)})] &= \int_0^1 2\theta F(\theta)^{n-1} f_2^{(n)}(\theta)d\theta \\ &= 2n(n-1)\kappa \int_0^1 (\theta^{\kappa(2n-2)} - \theta^{\kappa(2n-1)})d\theta \\ &= 2n(n-1)\kappa \left(\frac{1}{\kappa(2n-2)+1} - \frac{1}{\kappa(2n-1)+1}\right) \\ &= \frac{h_1(n;\kappa)}{h_2(n;\kappa)},\end{aligned}$$

where $Y_2^{(n)}$ is the second-highest order statistic of n independent draws from F, $f_2^{(n)}(y) \equiv n(n-1)(1-F(y))F^{n-2}(y)f(y)$ is its density, $h_1(n;\kappa) \equiv 2n(n-1)\kappa^2$, and $h_2(n;\kappa) \equiv [\kappa(2n-2)+1] \times [\kappa(2n-1)+1]$.

Denoting $h_1'(n;\kappa) \equiv \frac{\partial}{\partial n}h_1(n;\kappa)$ and $h_2'(n;\kappa) \equiv \frac{\partial}{\partial n}h_2(n;\kappa)$, the log derivative of the expected revenue with respect to n is given by

$$\begin{aligned}\frac{d}{dn}\ln \mathbb{E}[\beta(Y_2^{(n)})] &= \frac{h_1'(n;\kappa)}{h_1(n;\kappa)} - \frac{h_2'(n;\kappa)}{h_2(n;\kappa)} \\ &= \frac{2n-1}{n(n-1)} - \frac{2\kappa[\kappa(2n-1)+1] + 2\kappa[\kappa(2n-2)+1]}{[\kappa(2n-2)+1][\kappa(2n-1)+1]} \\ &= \frac{2n-1}{n(n-1)} - \frac{2(4n-3)+\frac{4}{\kappa}}{[(2n-2)+\frac{1}{\kappa}][(2n-1)+\frac{1}{\kappa}]}.\end{aligned}$$

When κ is sufficiently large, $\mathbb{E}[\beta(Y_2^{(n)})]$ decreases with n because

$$\begin{aligned}\lim_{\kappa\to\infty}\frac{d}{dn}\ln \mathbb{E}[\beta(Y_2^{(n)})] &= \frac{2n-1}{n(n-1)} - \frac{2(4n-3)}{(2n-2)(2n-1)} \\ &= \frac{(2n-1)^2 - n(4n-3)}{n(n-1)(2n-1)} \\ &= \frac{-1}{n(2n-1)} < 0. \quad \square\end{aligned}$$

Proof of Lemma 8. If (i) holds, then (ii) holds by the capacity constraint. Thus, it suffices to show (i). First, we can write allocation rules $(Q_1(\cdot), Q_2(\cdot))$ as a function of only θ_1 because bidder 2's valuation is deterministic. Let $q_1(\theta_1) = Q_1(\theta_1)$ and $q_2 = \mathbb{E}[Q_2(\theta_1)]$ be the associated interim expected winning probabilities of type θ_1 of bidder 1 and bidder 2.

Moreover, for any allocation rule $Q_1(\cdot)$ for bidder 1 and bidder 2's winning probability $q_2 \in [0,1]$, there is an allocation rule $Q_2(\cdot)$ for bidder 2 that satisfies the capacity constraint and $q_2 = \mathbb{E}[Q_2(\theta_1)]$ if and only if $\mathbb{E}[Q_1(\theta)] \leq 1 - q_2$. The "only-if" part follows by the capacity constraint $Q_1(\theta_1) + Q_2(\theta_1) \leq 1$. For the "if" part, suppose $\mathbb{E}[Q_1(\theta)] \leq 1 - q_2$, i.e., $1 - \mathbb{E}[Q_1(\theta_1)] \geq q_2$. Let

$$\hat{Q}_2(\theta_1;\hat{\theta}) \equiv \begin{cases} 1 - Q_1(\theta_1) & \text{if } \theta_1 < \hat{\theta} \\ 0 & \text{if } \theta_1 \geq \hat{\theta}. \end{cases}$$

Then, there exists a cutoff $\hat{\theta}^c \in [\underline{\theta}, \overline{\theta}]$ satisfying $\mathbb{E}[\hat{Q}_2(\theta_1;\hat{\theta}^c)] = q_2$ because $\mathbb{E}[\hat{Q}_2(\theta_1;\hat{\theta})]$ is continuous in $\hat{\theta}$ and satisfies $\mathbb{E}[\hat{Q}_2(\theta_1;\underline{\theta})] = 0$ and $\mathbb{E}[\hat{Q}_2(\theta_1;\overline{\theta})] = 1 - \mathbb{E}[Q_1(\theta_1)] \geq q_2$. It follows that $Q_1(\theta_1) + \hat{Q}_2(\theta_1;\hat{\theta}^c) \in \{Q_1(\theta_1), 1\}$, i.e., the capacity constraint is satisfied. The "if" part follows.

The above argument, with Lemmas 1 and 2, implies that the optimal auction solves

$$\max_{q_1(\cdot),q_2} \mathbb{E}[\psi_1(\theta_1)\alpha^g(q_1(\theta_1))] + \psi_2(\theta_2)\alpha^g(q_2)$$

subject to the two constraints: $\mathbb{E}[q_1(\theta_1)] \leq 1 - q_2$ and $q_1(\cdot)$ is nondecreasing. Let $(q_1^*(\cdot), q_2^*)$ be the solution. Then, $q_1^*(\cdot)$ must be the solution to the following: for some $A \in [0, 1 - q_2^*]$,

$$\max_{q_1(\cdot)} \mathbb{E}[\psi_1(\theta_1)\alpha^g(q_1(\theta_1))] \text{ subject to } \mathbb{E}[q_1(\theta_1)] = A \text{ and } q_1(\cdot) \text{ is nondecreasing.} \quad \text{(P)}$$

Moreover, let $q_1^A(\cdot)$ be the expected winning probability of type θ_1 that satisfies the following:

$$q_1^A(\theta_1) = 1 \text{ if } \theta_1 > F_1^{-1}(1-A) \text{ while } q_1^A(\theta_1) = 0 \text{ if } \theta_1 < F_1^{-1}(1-A). \quad (\heartsuit)$$

Let us also define the following sub-problem (P'):

$$\max_{q_1(\cdot)} \mathbb{E}[\psi_1(\theta_1)q_1(\theta_1)] \text{ subject to } \mathbb{E}[q_1(\theta_1)] = A \text{ and } q_1(\cdot) \text{ is nondecreasing.} \quad \text{(P')}$$

In order to prove (i), it suffices to show that the solution to (P) satisfies ($\heartsuit$). To prove it, our proof proceeds as follows: first, $q_1^A(\cdot)$ is a solution to (P'), second, $q_1^A(\cdot)$ is also a solution to (P), and finally, $q_1^A(\cdot)$ is the unique solution to (P).

For any $q_1(\cdot)$ that satisfies the constraints of (P'), let $F_{q_1}(\theta) \equiv A^{-1}\int_{\underline{\theta}}^{\theta} q_1(\theta_1)dF_1(\theta_1)$ be a cumulative distribution and $A^{-1}q_1(\theta_1)f_1(\theta_1)$ is its density. F_{q_1} is indeed a cumulative distribution over $[\underline{\theta}, \overline{\theta}]$ because F_{q_1} is nondecreasing and continuous with $F_{q_1}(\underline{\theta}) = 0$ and $F_{q_1}(\overline{\theta}) = A^{-1}A = 1$. It follows that

$$\mathbb{E}[\psi_1(\theta_1)q_1(\theta_1)] = \int_{\underline{\theta}}^{\overline{\theta}} \psi_1(\theta)q_1(\theta)dF_1 = A\int_{\underline{\theta}}^{\overline{\theta}} \psi_1(\theta)\cdot A^{-1}q_1(\theta)dF_1 = A\int_{\underline{\theta}}^{\overline{\theta}} \psi_1(\theta)dF_{q_1}. \quad (19)$$

Since

$$F_{q_1}(\theta_1) = A^{-1}\int_{\underline{\theta}}^{\theta_1} q_1(s)f_1(s)ds = 1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} q_1(s)f_1(s)ds \geq 1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s)ds$$

holds, $\max\{1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s)ds, 0\}$ is a lower bound of $F_{q_1}(\theta_1)$.

We show that $F_{q_1^A}(\theta_1)$ coincides with this lower bound for any $\theta_1 \in [\underline{\theta}, \overline{\theta}]$. First observe that $1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s)ds$ is increasing with θ_1 and takes zero at $\theta_1 = F_1^{-1}(1-A)$, implying that $1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s)ds > 0$ if and only if $\theta_1 > F_1^{-1}(1-A)$. Then, it follows that for $\theta_1 \leq F_1^{-1}(1-A)$

$$F_{q_1^A}(\theta_1) = A^{-1}\int_{\underline{\theta}}^{\theta_1} \underbrace{q_1^A(s)}_{=0} f_1(s)ds = 0 = \max\{1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s)ds, 0\};$$

and for $\theta_1 > F_1^{-1}(1-A)$

$$\begin{aligned}
F_{q_1^A}(\theta_1) &= A^{-1}\int_{\underline{\theta}}^{\theta_1} q_1^A(s) f_1(s) ds \\
&= A^{-1}\underbrace{\int_{\underline{\theta}}^{\overline{\theta}} q_1^A(s) f_1(s) ds}_{=A} - A^{-1}\int_{\theta_1}^{\overline{\theta}} \underbrace{q_1^A(s)}_{=1} f_1(s) ds \\
&= 1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s) ds \\
&= \max\{1 - A^{-1}\int_{\theta_1}^{\overline{\theta}} 1\cdot f_1(s) ds, 0\}.
\end{aligned}$$

Since $F_{q_1^A}(\theta_1)$ coincides with this lower bound for any $\theta_1 \in [\underline{\theta}, \overline{\theta}]$, $F_{q_1^A}(\theta_1) \leq F_{q_1}(\theta_1)$, i.e., $F_{q_1^A}(\theta_1)$ first-order stochastically dominates F_{q_1}. This, together with monotonicity of $\psi_1(\cdot)$ and (19), implies that $\mathbb{E}[\psi_1(\theta_1) q_1^A(\theta_1)] \geq \mathbb{E}[\psi_1(\theta_1) q_1(\theta_1)]$. This inequality implies that

$$\mathbb{E}[\psi_1(\theta_1)\alpha^g(q_1^A(\theta_1))] = \mathbb{E}[\psi_1(\theta_1) q_1^A(\theta_1)] \geq \mathbb{E}[\psi_1(\theta_1) q_1(\theta_1)] \geq \mathbb{E}[\psi_1(\theta_1)\alpha^g(q_1(\theta_1))], \tag{20}$$

where the first equality follows by $q_1^A(\theta_1) \in \{0,1\}$ and the last inequality holds by $\alpha^g(q) \leq q$ for any $q \in [0,1]$. Therefore, $q_1^A(\cdot)$ satisfying (♡) is a solution to both (P') and (P).

Finally, we show that $q_1^A(\cdot)$ is the unique solution to (P). By (20), any solution to (P) must be $q_1(\cdot) \in \{0,1\}$ except on a set of measure zero because $\alpha^g(q) < q$ for any $q \in (0,1)$. Since any solution $q_1(\cdot)$ to (P) must be nondecreasing, it must be of the cutoff form (♡). □

Proof of Proposition 9. Let $\Pi(\theta^*, \Lambda_2^g)$ be the expected revenue in the threshold mechanism with threshold θ^* and Λ_2^g in (16) and let $\theta^*(\Lambda_2^g)$ be the solution. Since θ_2 is deterministic, we omit the subscript for F_1 and f_1. We have

$$\begin{aligned}
\Pi_{\theta^*}(\theta^*, \Lambda_2^g) &= -\psi_1(\theta^*) f(\theta^*) + \theta_2 \alpha^{g\prime}(F(\theta^*)) f(\theta^*) \\
&= -\psi_1(\theta^*) f(\theta^*) + \theta_2\{1 - \Lambda_2^g + 2\Lambda_2^g F(\theta^*)\} f(\theta^*), \\
\Pi_{\theta^*\theta^*}(\theta^*, \Lambda_2^g) &= -\psi_1'(\theta^*) f(\theta^*) - \psi_1(\theta^*) f'(\theta^*) + \theta_2\{2\Lambda_2^g f(\theta^*)^2 + (1 - \Lambda^g + 2\Lambda_2^g F(\theta^*)) f'(\theta^*)\}, \\
\Pi_{\theta^*\Lambda_2^g}(\theta^*, \Lambda_2^g) &= \theta_2 f(\theta^*)\{2F(\theta^*) - 1\},
\end{aligned} \tag{21}$$

where subscripts denote partial derivatives.

First, we show that for sufficiently small $\Lambda_2^g > 0$, $\theta^*(\Lambda_2^g)$ satisfies the first-order condition $\Pi_{\theta^*}(\theta^*(\Lambda_2^g), \Lambda_2^g) = 0$. It suffices to show that $\theta^*(\Lambda_2^g)$ is not a corner solution because $\Pi(\theta^*, \Lambda_2^g)$ is differentiable in θ^*. Since the Myerson level θ_M is a unique interior solution under **A1** when $\Lambda_2^g = 0$, $\max\{\Pi(\underline{\theta}, 0), \Pi(\overline{\theta}, 0)\} < \Pi(\theta_M, 0)$ holds. Thus it follows by continuity of Π in Λ_2^g that for sufficiently small $\Lambda_2^g > 0$

$$\max\{\Pi(\underline{\theta}, \Lambda_2^g), \Pi(\overline{\theta}, \Lambda_2^g)\} < \Pi(\theta_M, \Lambda_2^g) \leq \Pi(\theta^*(\Lambda_2^g), \Lambda_2^g),$$

where the last inequality holds by the optimality of $\theta^*(\Lambda_2^g)$.

Second, for sufficiently small $\Lambda_2^g > 0$, $\theta^*(\Lambda_2^g)$ is unique and satisfies

$$\theta^{*\prime}(\Lambda_2^g) = -\frac{\Pi_{\theta^*\Lambda_2^g}(\theta^*(\Lambda_2^g), \Lambda_2^g)}{\Pi_{\theta^*\theta^*}(\theta^*(\Lambda_2^g), \Lambda_2^g)} \tag{22}$$

by the implicit function theorem. Indeed, we can apply the implicit function theorem because $\Pi_{\theta^*}(\theta^*, \Lambda_2^g)$ is continuously differentiable by (21) and $\Pi_{\theta^*\theta^*}(\theta_M, 0) \neq 0$:

$$\Pi_{\theta^*\theta^*}(\theta_M, 0) = -\psi_1'(\theta_M)f(\theta_M) - \psi_1(\theta_M)f'(\theta_M) + \theta_2 f'(\theta_M) = -\psi_1'(\theta_M)f(\theta_M) < 0,$$

where the first equality holds by (21), the second equality holds by $\psi_1(\theta_M) = \theta_2$, and the inequality holds by $\psi_1' > 0$.

Third, $\Pi_{\theta^*\theta^*}(\theta^*(\Lambda_2^g), \Lambda_2^g) < 0$ holds for sufficiently small $\Lambda_2^g > 0$ because $\Pi_{\theta^*\theta^*}(\theta_M, 0) < 0$ and both $\theta^*(\Lambda_2^g)$ and $\Pi_{\theta^*\theta^*}(\theta^*, \Lambda_2^g)$ are continuous.

Finally, we have $\theta^{*\prime}(\Lambda_2^g) > 0$ if $F_1(\theta_M) > \frac{1}{2}$ while $\theta^{*\prime}(\Lambda_2^g) < 0$ if $F_1(\theta_M) < \frac{1}{2}$ because the sign of $\theta^{*\prime}(\Lambda_2^g)$ equals that of $2F(\theta^*(\Lambda_2^g)) - 1$ by (21), (22), and $\Pi_{\theta^*\theta^*}(\theta^*(\Lambda_2^g), \Lambda_2^g) < 0$. □

Proof of Proposition 10. By (16), the expected revenue is written by

$$\Pi(\theta^*) = \frac{1}{\overline{\theta} - \underline{\theta}} \int_{\theta^*}^{\overline{\theta}} (2\theta_1 - \overline{\theta}) d\theta_1 + \theta_2 \alpha^g \left(\frac{\theta^* - \underline{\theta}}{\overline{\theta} - \underline{\theta}} \right). \tag{23}$$

Noting that $d\alpha^g(q)/dq = (1 - \Lambda^g) + 2\Lambda^g q$, differentiation yields

$$\Pi'(\theta^*) = \frac{-1}{\overline{\theta} - \underline{\theta}}(2\theta^* - \overline{\theta}) + \theta_2 \left[(1 - \Lambda^g) + 2\Lambda^g \left(\frac{\theta^* - \underline{\theta}}{\overline{\theta} - \underline{\theta}} \right) \right] \frac{1}{\overline{\theta} - \underline{\theta}}.$$

Equating this to zero gives $(2\theta^* - \overline{\theta})(\overline{\theta} - \underline{\theta}) = (1 - \Lambda^g)(\overline{\theta} - \underline{\theta})\theta_2 + 2\Lambda^g(\theta^* - \underline{\theta})\theta_2$, yielding (17).

Moreover,

$$\Pi''(\theta^*) = \frac{2}{\overline{\theta} - \underline{\theta}} \left(-1 + \frac{\theta_2 \Lambda^g}{\overline{\theta} - \underline{\theta}} \right),$$

which is negative if and only if $\theta_2 \Lambda^g < \overline{\theta} - \underline{\theta}$.

When $\theta_2 \Lambda^g > \overline{\theta} - \underline{\theta}$, since $\frac{\partial \Pi}{\partial \theta^*}$ is convex, the solution is either $\overline{\theta}$ or $\underline{\theta}$. Then, by (23), $\Pi(\theta^* = \overline{\theta}) = \theta_2 \alpha^g(1) = \theta_2$ while $\Pi(\theta^* = \underline{\theta}) = \frac{1}{\overline{\theta} - \underline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (2\theta_1 - \overline{\theta}) d\theta_1 + \theta_2 \alpha^g(0) = \underline{\theta}$. Thus, it is optimal to set $\theta^* = \underline{\theta}$ if $\underline{\theta} > \theta_2$ and $\theta^* = \overline{\theta}$ if $\underline{\theta} \leq \theta_2$.

In what follows, suppose $\theta_2 \Lambda^g < \overline{\theta} - \underline{\theta}$. Since $\frac{\partial \Pi}{\partial \theta^*}$ is concave, it is optimal to set θ^* as close to $\theta^*(\Lambda^g)$ as possible within $[\underline{\theta}, \overline{\theta}]$. It follows that

$$\begin{aligned}
\theta^*(\Lambda^g) - \underline{\theta} &= \frac{1}{2} \frac{(\overline{\theta} - \underline{\theta})[\overline{\theta} + (1 - \Lambda^g)\theta_2] - 2\Lambda^g \underline{\theta} \theta_2}{\overline{\theta} - \underline{\theta} - \Lambda^g \theta_2} - \underline{\theta} \\
&= \frac{(\overline{\theta} - \underline{\theta})[\overline{\theta} + (1 - \Lambda^g)\theta_2] - 2\Lambda^g \underline{\theta} \theta_2 - 2(\overline{\theta} - \underline{\theta} - \Lambda^g \theta_2)\underline{\theta}}{2(\overline{\theta} - \underline{\theta} - \Lambda^g \theta_2)} \\
&= \frac{(\overline{\theta} - \underline{\theta})[\overline{\theta} - 2\underline{\theta} + (1 - \Lambda^g)\theta_2]}{2(\overline{\theta} - \underline{\theta} - \Lambda^g \theta_2)}.
\end{aligned}$$

By $\theta_2\Lambda^g < \overline{\theta} - \underline{\theta}$, it follows that $\theta^*(\Lambda^g) > \underline{\theta}$ if and only if $\overline{\theta} - 2\underline{\theta} + (1-\Lambda^g)\,\theta_2 > 0$, or equivalently $\Lambda^g < \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$. Moreover,

$$\begin{aligned}
\overline{\theta} - \theta^*(\Lambda^g) &= \overline{\theta} - \frac{1}{2}\frac{(\overline{\theta}-\underline{\theta})[\overline{\theta}+(1-\Lambda^g)\theta_2] - 2\Lambda^g\underline{\theta}\theta_2}{\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2} \\
&= \frac{2(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)\overline{\theta} - (\overline{\theta}-\underline{\theta})[\overline{\theta}+(1-\Lambda^g)\theta_2] + 2\Lambda^g\underline{\theta}\theta_2}{2(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)} \\
&= \frac{(\overline{\theta}-\underline{\theta})(\overline{\theta}-(1+\Lambda^g)\theta_2)}{2(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)}.
\end{aligned}$$

By $\theta_2\Lambda^g < \overline{\theta} - \underline{\theta}$, we have $\overline{\theta} > \theta^*(\Lambda^g)$ if and only if $\overline{\theta} - (1+\Lambda^g)\theta_2 > 0$, or $\Lambda^g < \frac{\overline{\theta}-\theta_2}{\theta_2}$.

Consider when $\theta_2 > \underline{\theta}$. In this case, $\frac{\overline{\theta}-\theta_2}{\theta_2} < \frac{\overline{\theta}-\underline{\theta}}{\theta_2} < \frac{\overline{\theta}-\underline{\theta}+\theta_2-\underline{\theta}}{\theta_2} = \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$ holds. By $\Lambda^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$, this leaves us two cases: $\Lambda^g < \frac{\overline{\theta}-\theta_2}{\theta_2}$ and $\frac{\overline{\theta}-\theta_2}{\theta_2} \leq \Lambda^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$. If $\Lambda^g < \frac{\overline{\theta}-\theta_2}{\theta_2}$, then $\theta^*(\Lambda^g) \in (\underline{\theta}, \overline{\theta})$, making it optimal to set $\theta^* = \theta^*(\Lambda^g)$; if $\frac{\overline{\theta}-\theta_2}{\theta_2} \leq \Lambda^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$, then $\theta^*(\Lambda^g) \geq \overline{\theta}$, making it optimal to set $\theta^* = \overline{\theta}$, i.e., always sell to bidder 2.

Consider when $\theta_2 < \underline{\theta}$. In this case, $\frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2} = \frac{\overline{\theta}-\underline{\theta}+\theta_2-\underline{\theta}}{\theta_2} < \frac{\overline{\theta}-\underline{\theta}}{\theta_2} < \frac{\overline{\theta}-\theta_2}{\theta_2}$ holds. By $\Lambda^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$, this leaves us two cases: $\Lambda^g < \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$ and $\frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2} \leq \Lambda^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$. If $\Lambda^g < \frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2}$, then $\theta^*(\Lambda^g) \in (\underline{\theta}, \overline{\theta})$, making it optimal to set $\theta^* = \theta^*(\Lambda^g)$; if $\frac{\overline{\theta}-2\underline{\theta}+\theta_2}{\theta_2} \leq \Lambda^g < \frac{\overline{\theta}-\underline{\theta}}{\theta_2}$, then $\theta^*(\Lambda^g) \leq \underline{\theta}$, making it optimal to set $\theta^* = \underline{\theta}$, i.e., always sell to bidder 1.

We now compare $\theta^*(\Lambda^g)$ with the Myerson level $\psi_1^{-1}(\theta_2) = \frac{1}{2}(\overline{\theta}+\theta_2)$.

$$\begin{aligned}
\theta^*(\Lambda^g) - \psi_1^{-1}(\theta_2) &= \frac{1}{2}\frac{(\overline{\theta}-\underline{\theta})[\overline{\theta}+(1-\Lambda^g)\theta_2] - 2\Lambda^g\underline{\theta}\theta_2}{\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2} - \frac{1}{2}(\overline{\theta}+\theta_2) \\
&= \frac{(\overline{\theta}-\underline{\theta})[\overline{\theta}+(1-\Lambda^g)\theta_2] - 2\Lambda^g\underline{\theta}\theta_2 - (\overline{\theta}+\theta_2)(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)}{2(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)} \\
&= \frac{\Lambda^g\theta_2(\theta_2-\underline{\theta})}{2(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)},
\end{aligned}$$

which implies that $\theta^*(\Lambda^g) > \psi_1^{-1}(\theta_2)$ if and only if $\theta_2 > \underline{\theta}$.

Finally, differentiating (17) yields

$$\begin{aligned}
\frac{d\theta^*(\Lambda^g)}{d\Lambda^g} &= \frac{1}{2}\frac{-(\overline{\theta}+\underline{\theta})\theta_2(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2) + \{(\overline{\theta}-\underline{\theta})[\overline{\theta}+(1-\Lambda^g)\theta_2] - 2\Lambda^g\underline{\theta}\theta_2\}\theta_2}{(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)^2} \\
&= \frac{1}{2}\frac{\theta_2(\overline{\theta}-\underline{\theta})(\theta_2-\underline{\theta})}{(\overline{\theta}-\underline{\theta}-\Lambda^g\theta_2)^2}.
\end{aligned}$$

Thus, the sign of $\frac{d\theta^*(\Lambda^g)}{d\Lambda^g}$ equals that of $\theta_2 - \underline{\theta}$. □

References

Abeler, Johannes, Armin Falk, Lorenz Goette, and David Huffman (2011) "Reference points and effort provision," *American Economic Review*, Vol. 101, No. 2, pp. 470–92.

Balzer, Benjamin and Antonio Rosato (2021) "Expectations-based loss aversion in auctions with interdependent values: Extensive vs. intensive risk," *Management Science*, Vol. 67, No. 2, pp. 1056–1074.

Balzer, Benjamin, Antonio Rosato, and Jonas von Wangenheim (2021) "Dutch vs. First-Price Auctions with Expectations-Based Loss-Averse Bidders." mimeo.

Barbosa, Klenio and Pierre C Boyer (2021) "Discrimination in Dynamic Procurement Design with Learning-by-doing," *International Journal of Industrial Organization*. 102754.

Bell, David E (1985) "Disappointment in decision making under uncertainty," *Operations Research*, Vol. 33, No. 1, pp. 1–27.

Bernhardt, Dan, Tingjun Liu, and Takeharu Sogo (2020) "Costly auction entry, royalty payments, and the optimality of asymmetric designs," *Journal of Economic Theory*. 105041.

Celik, Gorkem and Okan Yilankaya (2009) "Optimal auctions with simultaneous and costly participation," *BE Journal of Theoretical Economics*, Vol. 9, No. 1, pp. 1–33.

Crawford, Vincent P and Juanjuan Meng (2011) "New york city cab drivers' labor supply revisited: Reference-dependent preferences with rational-expectations targets for hours and income," *American Economic Review*, Vol. 101, No. 5, pp. 1912–32.

Dato, Simon, Andreas Grunewald, and Daniel Müller (2018) "Expectation-based loss aversion and rank-order tournaments," *Economic Theory*, Vol. 66, No. 4, pp. 901–928.

Eisenhuth, Roland (2019) "Reference-dependent mechanism design," *Economic Theory Bulletin*, Vol. 7, No. 1, pp. 77–103.

Gill, David and Victoria Prowse (2012) "A Structural Analysis of Disappointment Aversion in a Real Effort Competition," *American Economic Review*, Vol. 102, pp. 469–503.

Gill, David and Rebecca Stone (2010) "Fairness and desert in tournaments," *Games and Economic Behavior*, Vol. 69, No. 2, pp. 346–364.

Heidhues, Paul and Botond Kőszegi (2014) "Regular prices and sales," *Theoretical Economics*, Vol. 9, No. 1, pp. 217–251.

Herweg, Fabian, Daniel Müller, and Philipp Weinschenk (2010) "Binary payment schemes: Moral hazard and loss aversion," *American Economic Review*, Vol. 100, No. 5, pp. 2451–77.

Iossa, Elisabetta and Patrick Rey (2014) "Building reputation for contract renewal: implications for performance dynamics and contract duration," *Journal of the European Economic Association*, Vol. 12, No. 3, pp. 549–574.

Kahneman, Daniel and Amos Tversky (1979) “Prospect Theory: An Analysis of Decision under Risk,” *Econometrica*, Vol. 47, pp. 263–291.

Kőszegi, Botond and Matthew Rabin (2006) “A Model of Reference-Dependent Preferences,” *Quarterly Journal of Economics*, Vol. 121, pp. 1133–1165.

——— (2007) “Reference-Dependent Risk Attitudes,” *American Economic Review*, Vol. 97, pp. 1047–1073.

Krishna, Vijay (2002) *Auction theory*: Academic press.

Lange, Andreas and Anmol Ratan (2010) “Multi-dimensional reference-dependent preferences in sealed-bid auctions—How (most) laboratory experiments differ from the field,” *Games and Economic Behavior*, Vol. 68, No. 2, pp. 634–645.

Lewis, Tracy R and Huseyin Yildirim (2002) “Managing dynamic competition,” *American Economic Review*, Vol. 92, No. 4, pp. 779–797.

Loomes, Graham and Robert Sugden (1986) “Disappointment and dynamic consistency in choice under uncertainty,” *Review of Economic Studies*, Vol. 53, No. 2, pp. 271–282.

Lu, Jingfeng (2009) “Auction design with opportunity cost,” *Economic Theory*, Vol. 38, No. 1, pp. 73–103.

Marzilli Ericson, Keith M and Andreas Fuster (2011) “Expectations as endowments: Evidence on reference-dependent preferences from exchange and valuation experiments,” *Quarterly Journal of Economics*, Vol. 126, No. 4, pp. 1879–1907.

Maskin, Eric and John Riley (1984) “Optimal auctions with risk averse buyers,” *Econometrica*, pp. 1473–1518.

McAfee, R Preston and John McMillan (1989) “Government procurement and international trade,” *Journal of International Economics*, Vol. 26, No. 3-4, pp. 291–308.

Myerson, Roger (1981) “Optimal Auction Design,” *Mathematics of Operations Research*, Vol. 6, pp. 58–73.

Rosato, Antonio (2014) “Loss Aversion in Sequential Auctions: Endogenous Interdependence, Informational Externalities and the ‘Afternoon Effect’.” mimeo.

Rosato, Antonio and Agnieszka A. Tymula (2019) “Loss aversion and competition in Vickrey auctions: Money ain’t no good,” *Games and Economic Behavior*, Vol. 115, pp. 188–208.

Tversky, Amos and Daniel Kahneman (1991) “Loss Aversion in Riskless Choice: A Preference-Dependent Model,” *Quarterly Journal of Economics*, Vol. 106, pp. 1039–1061.

von Wangenheim, Jonas (2021) “English versus Vickrey auctions with loss-averse bidders,” *Journal of Economic Theory*, Vol. 197. 105328.