Government Debt Management and Inflation with Real and Nominal Bonds*

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Abstract

Rising inflation in the wake of unprecedented debt financed stimulus packages raises concerns about a looming return of persistent inflation, as governments may be tempted to monetize debt. In this paper, we ask whether governments can use real (TIPS) bonds as part of the government debt portfolio to commit not to create elevated inflation? We thus examine optimal debt management in a setting where (i) the government can issue long-term nominal and real bonds, (ii) the monetary authority sets short-term interest rates according to a Taylor rule, and (iii) inflation has real costs as prices are sticky. Nominal debt can be inflated away giving ex-ante flexibility, but real bonds constitute a real commitment ex-post. We show that the optimal government debt portfolio includes a substantial allocation to both real and nominal bonds, which lowers inflation levels but increases inflation volatility in equilibrium. The associated lower correlation between inflation risk and government expenditure is reflected in welfare gains through real debt management. Quantitatively, our results are stronger i) the higher the initial debt level, and ii) the longer debt maturity. Our findings suggest that TIPS should be an important tool for debt management in the presence of looming inflation.

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1 Introduction

Inflation has returned.[1] Indeed, the annual inflation rate in the US edged up to a 13-year high of 5.4 percent in September of 2021, with inflation expectations rising alongside. Similarly, after a decade that was dominated by central bankers' fear of deflation, inflation forecasts and long term Treasury yields have been widening recently as well. These concerns reflect not only the potential upward pressure on prices caused by supply and capacity shortages when demand recovers in the aftermath of the pandemic, but also the surge in government debt across the globe following fiscal stabilization programs and stimulus packages both around the Great Recession and the Pandemic. The 1.9 trillion dollar American Rescue Plan Act of 2021, further adds to US government debt, which is projected to reach around 200 percent of GDP in 2050, according to the CBO (as of March 2021). In situations with such unprecedented debt levels, governments and central banks may be tempted to restore budget balance by monetizing debt, thereby strengthening inflationary pressure.

In this paper, we ask how governments can optimally manage their debt portfolios in the presence of inflation concerns and high debt levels. Starting from the simple observation that real or indexed debt (TIPS) cannot be inflated away ex-post, we examine the government's optimal debt portfolios when it has access to both nominal and real bonds. More specifically, we solve for the optimal Ramsey equilibrium in a setting in which the government has to finance an exogenous stochastic expenditure stream either by levying distortionary labor taxes or by issuing real or nominal debt. We allow for multi-horizon debt and assess the implications of short versus long term debt for equilibrium quantities and debt portfolios. Inflation has real costs because of nominal rigidities through sticky prices and is affected by the monetary authority which sets the nominal short-term interest rate by responding to inflationary pressure following a Taylor rule. Our paper therefore contributes to the literature started in the seminal work of Lucas and Stokey (1983) on optimal fiscal and monetary policy, and considers both long-term nominal and real debt in models with nominal rigidities such as Siu (2004), Schmitt-Grohe and Uribe (2004), and Lustig et al. (2008).

When the government cannot issue TIPS, the Ramsey planner faces a trade-off between responding to shocks using distortionary taxes versus inflation. On the one hand, by inflat-

[1] *The Economist*'s issue of December 12, 2020, was titled "will inflation return?".

ing away the nominal liability, the government can finance additional expenditures without increasing labor taxes. On the other hand, by raising expected inflation, the planner reduces the value of household savings and decreases the price of government nominal bonds. Therefore, both current and future prices of nominal bonds are lowered. The addition of inflation protected securities in the government debt portfolio affects this trade-off in two ways. On the one hard, inflation protected securities constitute a real commitment ex-post and cannot be inflated away as the planner needs to compensate real bond holders. On the other hand, higher inflation has smaller impact on the cost of current and future borrowing since inflation does not affect the price of real bonds.

We find that in equilibrium, the Ramsey planner uses both types of bonds and that the optimal government portfolio prescribes a substantial role to real bonds. We derive analytical results showing that the use of inflation allows to implement real and nominal price differences that help to complete the markets and that the investment position in real and nominal bonds depends on the type of shock considered. In the quantitative model we consider an economy with exogenous government expenditure shocks and find that the optimal policy prescribes the allocation to nominal bonds in good times and reallocation to real bonds in bad times. By doing this the planner uses inflation to reduce the nominal liability and at the same time issues real bonds, whose price does not decrease as much in the presence of rising inflation expectations. Quantitatively, in our baseline calibration, inflation is more volatile but on average lower than in the model with only nominal bonds. This implies a welfare gain of 0.223%, which is achieved through better management of inflation risk and bond prices.

We find that inflation response is shaped by (i) the outstanding nominal debt and (ii) the maturity of debt. When the outstanding nominal debt is high, it becomes more tempting to use inflation as the same inflation rate allows to alleviate a larger debt burden, while creating the same misallocation cost due to nominal rigidities. We find that higher nominal debt leads to high inflation, which is optimal as longs the government reallocates to real bonds once the rising inflation begins to affect nominal bond prices. Longer debt maturity, on the other hand, is related to lower inflation rates as a longer planning horizon allows to spread inflation costs across multiple periods. We find that longer maturity implies inflation that is less volatile but more responsive to expenditure shocks, which, overall, improves household welfare.

The nonlinear nature of the equilibrium inflation response in our model requires an accurate global solution. This is computationally challenging in our environment, as the complexity of solving Ramsey problems with multiple maturities increases in the length of the largest maturity and the state space is highly multicollinear. In this paper we exploit a machine learning algorithm based on a neural networks approach to tackle these problems, as proposed in Villa and Valaitis (2019). The method builds on a version of a parameterized expectations algorithm (den Haan and Marcet, 1990) and uses neural networks to project expected value terms on the state space.

1.1 Related Literature

The paper builds on the papers studying the Ramsey problem with non-state contingent government debt (Aiyagari et al., 2002; Angeletos, 2002; Buerra and Nicolini, 2004; Faraglia et al., 2019; Bhandari et al., 2019). Aiyagari et al. (2002) show that when the government can only issue real bonds of one period maturity, the Ramsey planner achieves the complete markets outcome in the long-run by accumulating assets and using government savings to smooth tax distortions. Angeletos (2002) shows that complete markets outcome can be achieved if the number of maturities available is weakly greater than the number of states, while Buerra and Nicolini (2004) argue quantitatively that this requires unrealistically large long and short positions and rebalancing of government debt. Bhandari et al. (2019) study optimal maturity structure in a model with Epstein-Zin preferences and show that such extreme positions are optimal because of counterfactual asset pricing implications. With Epstein-Zin preferences the optimal policy implies moderate portfolio positions with little rebalancing. Faraglia et al. (2019) remove the assumption that government buys back the whole debt in every period and, instead, consider another extreme where bonds cannot be repurchased before the maturity. They show that under this assumption the optimal debt positions are closer to the data and government borrows in both types of bonds. Debt in long bonds is used to smooth taxes over states and short bonds are used to smooth taxes over time.

The paper is most closely related to the literature studying the optimal mix of monetary and fiscal policy with non-state contingent nominal debt (Chari and Kehoe, 1999; Siu, 2004; Schmitt-Grohe and Uribe, 2004; Lustig et al., 2008; Marcet et al., 2013; Leeper and Zhou, Forthcoming). As known since Lucas and Stokey (1983), the Ramsey planner seeks to manage government debt in order to smooth distortionary taxes over time and across states. Chari and Kehoe (1999) show that such smoothing of tax distortions can be achieved with inflation surprises when the Ramsey planner has control over the monetary policy. Chari and Kehoe (1999)'s conclusion is achieved in a model without nominal rigidities, which means that inflation is no real cost. Siu (2004) and Schmitt-Grohe and Uribe (2004) contemporaneously consider an optimal fiscal and monetary policy mix when planner faces a trade-off between distortionary taxes and inflation in the presence of nominal rigidities. In such a setting optimal policy prescribes a very limited role for inflation even when nominal rigidities are small. Lustig et al. (2008) show that inflation's role is larger when the government can issue bonds with long maturities. The idea is that large inflation implies a higher interest rate on new debt and long maturity allows the government to postpone such costly increase. Such idea is reaffirmed in Marcet et al. (2013). In addition, Leeper and Zhou (Forthcoming) show that importance of inflation also depends on the starting level of government debt and Siu (2004) shows that the role of inflation in optimal policy increases with the size of government expenditure shocks. Marcet et al. (2013) show that the optimal use of inflation depends on the independence of the monetary authority and, when it is independent, on the values of Taylor rule coefficients. Overall, the Ramsey planner is more likely to inflate the debt when the monetary authority is independent. Another paper closely related to ours is Equiza-Goni et al. (2020), which studies the role of inflation-indexed debt when the planner issues long-term debt that is nominal and short-term debt that is inflation-indexed. In this paper we study the trade-off between nominal and real bonds with the same maturity.

Solving the Ramsey problem with multiple maturities is computationally challenging because the number of state variables increases in the length of the largest maturity and the state space is highly multicollinear. In this paper we exploit the neural networks approach to tackle these problems, as proposed in Villa and Valaitis (2019). The methods builds on uses a parameterized expectations algorithm (den Haan and Marcet, 1990) and uses neural network to project expected value terms on the state space.[2]

The paper is organized as follows. Section 2 presents stylized facts and inflation and US federal debt. Section 3 presents the model and analytical results. Section 4 shows the main quantitative results and inspects the role of outstanding debt and maturity length. Finally, section 5 concludes.

2 Stylized Facts

We begin by presenting some stylized facts that motivate our analysis. We focus on the evolution of inflation, government debt and real bonds.

[2]Other examples of the use of neural networks to solve economic models include Duarte (2018), Scheidegger and Bilionis (2019), Maliar et al. (2021), Fernández-Villaverde et al. (2020) and Azinovic et al. (2021).

Figure 1 illustrates the evolution of inflation expectations, as captured by the ten-year break-even inflation. The break-even inflation rate stabilizes at a level of about 2.5% from 2004 through 2007. In 2008, the break-even inflation rate sharply fell. After having reached almost a value of zero during the pandemic, inflation expectations recently spiked up sharply to pre-crisis level but remained fairly volatile.

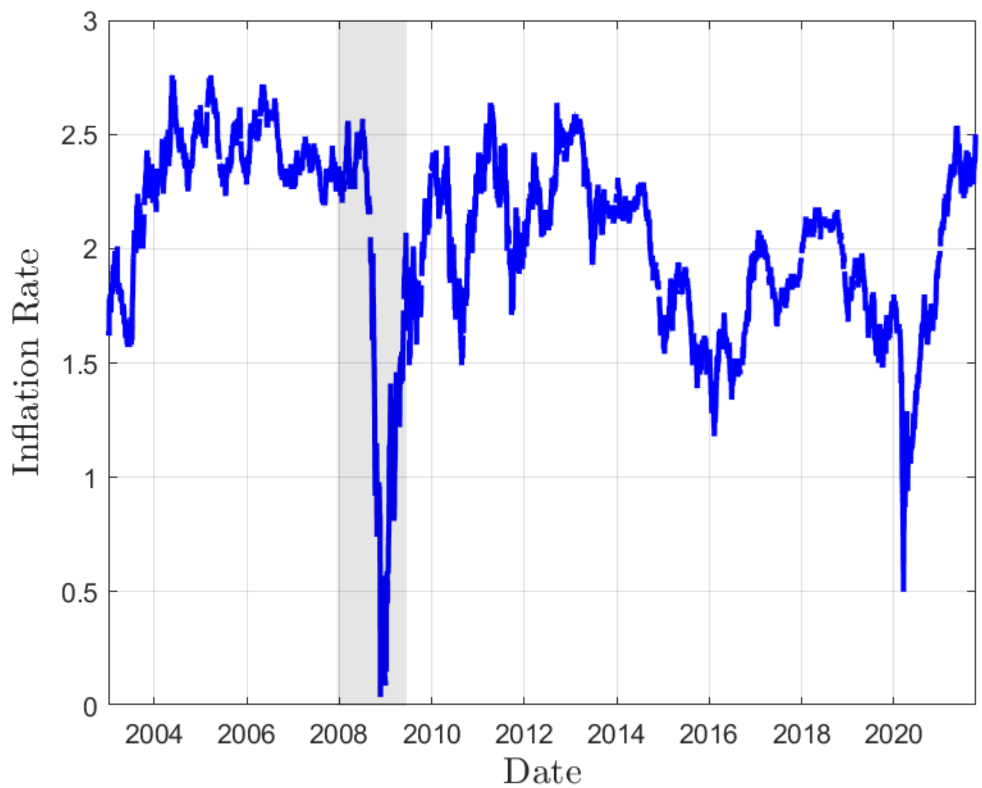


Figure 1: 10 year break-even inflation

Notes: Figure show the US 10-year break-even inflation rate. Break-even inflation is the difference between 10-year nominal and inflation-indexed bond yields. Source: St. Louis Fred database.

Figure 2 depicts the evolution of government debt as measured by the debt-to-gdp ratio. The evolution of government debt exhibits long swings, and hovered between around forty and sixty percent of GDP before the financial crisis. In response to fiscal stimulus packages around the financial crisis and then the pandemic, it has recently reached World War II levels for the first time. Moreover, according to the CBO, under current policies it is projected to reach two hundred percent of GDP by around 2050.

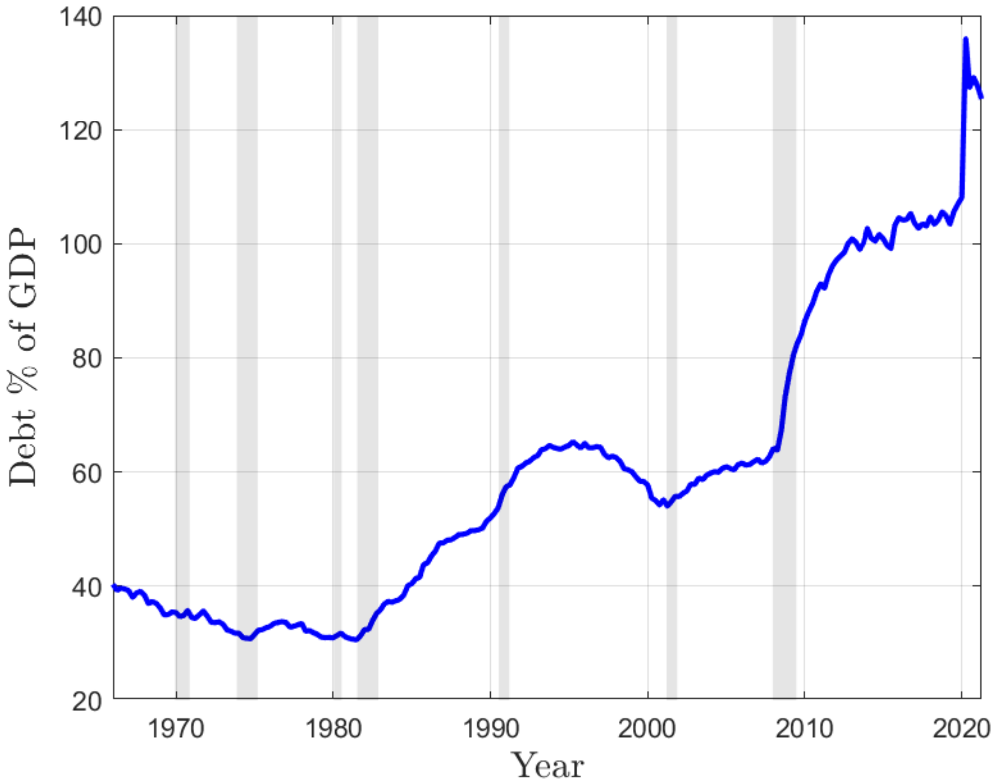


Figure 2: US Debt to GDP

Notes: Figure shows US total public debt to GDP ratio. Data is quarterly and seasonally adjusted. Source: St. Louis Fred database.

Figure 3 plots the evolution of real debt as a fraction of total US government debt. That fraction has grown since the inception of the market for inflation-protected bonds (TIPS) and has stabilized around a modest eight percent in the last ten years.

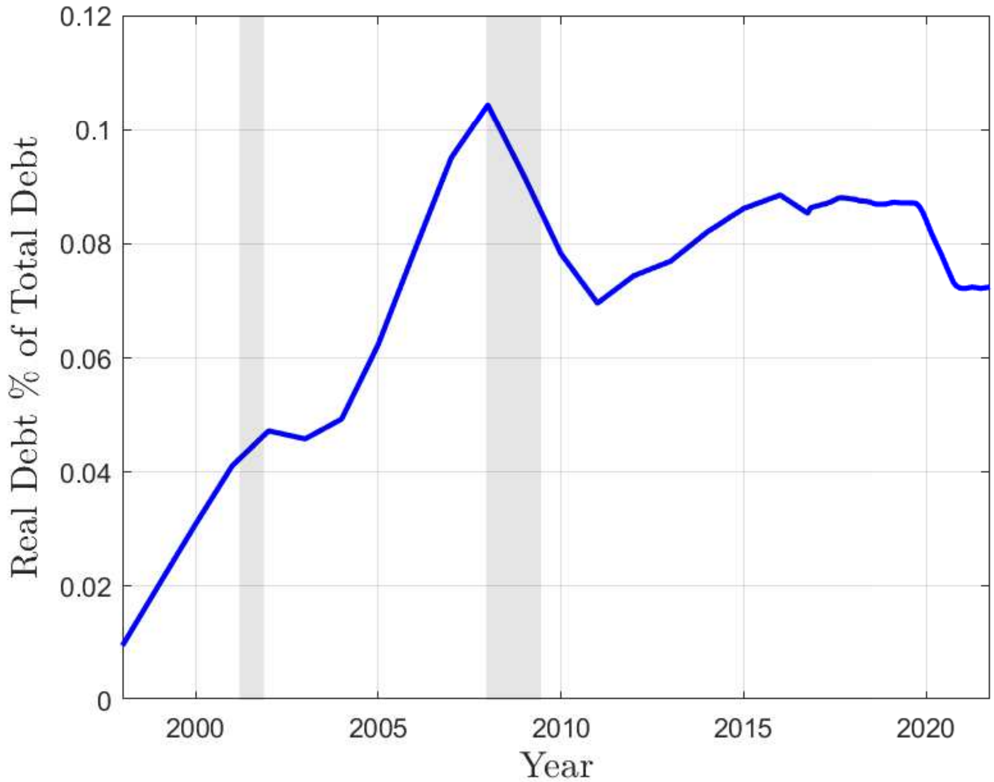


Figure 3: Share of US Real Debt

Notes: Figure shows the share of US inflation-protected securities (TIPS) to US total public debt. Source: US department of Treasury.

We now turn to a general equilibrium model that informs us about the optimal composition of government debt portfolios in the presence of a high fiscal burden and inflation pressure.

3 Model

In this section, we develop a novel DSGE model of fiscal and monetary policy where a government optimally manages debt in the presence of inflation concerns in a setting where: (i) the government can issue long-term nominal and real (TIPS) bonds, (ii) the monetary authority sets short-term interest rates according to a Taylor rule, and (iii) inflation has real costs as prices are sticky. We then proceed to formulate the Ramsey problem and characterize the optimal policy.

3.1 Environment

Time is discrete and the horizon is infinite.

Preferences. There is a continuum of identical infinitely lived households. Each household has preferences represented by the following expected life-time utility:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} U(c_t, l_t) \right],$$

where c_t is its consumption, $\beta \in (0,1)$ is the discount factor, and γ is the inverse elasticity of inter-temporal substitution (or risk aversion). Each household has utility for leisure l_t which is equal to $1 - h_t$, where h_t is hours worked. Households are identical. We assume the utility function U is strictly increasing in both consumption c_t and leisure l_t and concave.

Technology. A continuum of perfectly competitive intermediate firms, indexed by $i \in [0,1]$, produces output through a Cobb-Douglas technology $Y_{i,t} = Ah_{i,t}$ where hours worked is the only input. Intermediate goods are sold at a price $P_{i,t}$ to the final good producer. Aggregate output is given by $Y_t = Ah_t$.

Shocks. Government expenditure g_t is the only source of aggregate risk. We assume g_t follows an AR(1)

$$\log g_{t+1} = (1-\rho) \cdot \mu + \rho_g \log g_t + \xi_{t+1},$$

where ξ_{t+1} is a normally distributed innovation shock with mean zero and variance σ_g^2.

Timing. At the beginning of each period, g_t is realized and each firm produces output according to their specific labor input, distributes dividend and pays wages. Government repays nominal maturing debt at the price of 1 and real maturing debt at the price $\Pi_{j=1}^{N} \pi_{t-j+1}$. Government levies a distortionary labor tax τ_t on labor income. The representative household, conjointly with government financial needs, make savings decisions in nominal and real debt.

Markets. The representative household saves through (i) a N-period non-contingent nominal debt B_t^N traded at a price Q_t^N and (ii) a N-period non-contingent inflation-protected debt b_t^N traded at a price q_t^N. The government issues both types of debt, collects revenues in the current period and repays debt at maturity.

Income. In every period t, the representative household receives labor and investment income according to the following budget constraint

$$c_t + Q_t^N B_t^N + q_t^N b_t^N = (1-\tau_t) w_t A h_t + B_{t-N}^N / \Pi_{j=1}^N \pi_{t-j+1} + b_{t-N}^N.$$

Inflation rate from period $t-1$ to t is calculated as $\pi_t = \frac{P_t}{P_{t-1}}$.

Government. The government finances expenditures g_t by imposing proportional labor taxes τ_t on all labor income and by issuing nominal Q_t^N and real debt q_t^N.

Given the assumption that the government buys back and reissues the entire stock of outstanding debt, the government budget constraint is given by:

$$Q_t^{N-1} \frac{B_{t-1}^N}{\pi_t} + q_t^{N-1} b_{t-1}^N = \tau_t A h_t w_t - g_t + Q_t^N B_t^N + q_t^N b_t^N. \tag{1}$$

Central Bank. We assume the central bank seeks to achieve an inflation target π by setting one-period nominal rate $i_t \equiv 1/Q_t^1$ according to the following Taylor Rule:

$$i_t = \left(\beta \mathbb{E}_t \left[\frac{U_{1,t+1}}{U_{1,t}} \frac{1}{\pi_{t+1}} \right] \right)^{-1} = \frac{1}{\beta} \pi \left(\frac{\pi_t}{\pi} \right)^{\phi_\pi}. \tag{2}$$

3.2 Household

The representative household chooses sequences for: (i) consumption $\{c_t\}_{t=0}^\infty$, (ii) leisure $\{l_t\}_{t=0}^\infty$, (iii) nominal bond demand $\{B_t^N\}_{t=0}^\infty$ and (iv) real bond demand $\{b_t^N\}_{t=0}^\infty$ such that its time-0 expected lifetime utility is maximized and the budget constraint is satisfied $\forall t \geq 0$.

In equilibrium, the household's dynamic demand for nominal bonds is given by

$$Q_t^N = \mathbb{E}_t \left[M_{t,t+N} \cdot \frac{1}{\Pi_{j=1}^N \pi_{t+j}} \right], \tag{3}$$

where $M_{t,t+N} \equiv \beta^N \frac{U_{1,t+N}}{U_{1,t}}$ is the stochastic discount factor. Note that nominal bonds are subject to inflation risk. Given that bonds purchased at period t matures at time $t+N$ the return of the nominal bond is adjusted by the compounded inflation $\Pi_{j=1}^N \pi_{t+j}$.

In equilibrium, the household's dynamic demand for real bonds is given by

$$q_t^N = \mathbb{E}_t \left[M_{t,t+N} \right]. \tag{4}$$

Note that the price of real bonds q_t^N is just the discounted repayment at the par value of 1. Real bonds are inflation-protected; therefore, inflation risk does not enter directly the real price of bonds in equilibrium.

The combination of equations 3 and 4 yields the following equilibrium relationship between nominal and real prices

$$\frac{Q_t^N}{q_t^N} = \frac{\mathbb{E}_t\left[M_{t,t+N}\cdot\frac{1}{\Pi_{j=1}^N \pi_{t+j}}\right]}{\mathbb{E}_t\left[M_{t,t+N}\right]} = \mathbb{E}_t\left[\frac{1}{\Pi_{j=1}^N \pi_{t+j}}\right] + \frac{1}{q_t^N}\mathrm{Cov}_t\left(M_{t,t+N}, \frac{1}{\Pi_{j=1}^N \pi_{t+j}}\right),$$

which states that the ratio between the nominal and the real bonds price can be decomposed in the sum of two components: (i) the expected return on nominal bonds and (ii) the covariance between the stochastic discount factor and the return on nominal bonds. The log-spread between the two prices is equal to the value of the insurance against inflation risk plus an hedging term that accounts for the impact that inflation risk has on the consumption smoothing desire of the household (in our calibrated model, the covariance in the second addendum is typically negative).

Labor supply is standard. In equilibrium, the marginal rate of substitution between consumption and leisure needs to be equal to the wage net of labor tax

$$\frac{U_{2,t}}{U_{1,t}} = (1-\tau_t)Aw_t. \tag{5}$$

3.3 Firms

An intermediate firm i chooses sequences for: (i) prices $\{P_{i,t}\}_{t=0}^{\infty}$ and (ii) labor demand $\{h_{i,t}\}_{t=0}^{\infty}$ in order to maximize the expected time-0 net present value of dividend on behalf of its shareholder (the representative household):

$$\mathbb{E}_0 \sum_{t=0}^{\infty} M_{0,t}\cdot\left[\underbrace{P_{i,t}Y_{i,t} - w_t h_{i,t} P_t - P_t AC_t}_{\text{Dividend}}\right].$$

We assume the firm can set prices incurring the following convex quadratic reduced-form adjustment cost

$$\mathrm{AC}_t = \frac{\varphi}{2}\cdot\left(\frac{P_{i,t}}{P_{i,t-1}} - \pi\right)^2 + \phi_1\cdot\left(\frac{P_{i,t}}{P_{i,t-1}} - \pi\right) + \phi_2.$$

Also, the demand for the intermediate good is given by static profit maximization of the final good producer

$$Y_{it} = \left(\frac{P_{i,t}}{P_t}\right)^{-\frac{1}{\nu}} Y_t.$$

In a symmetric equilibrium ($P_{i,t} = P_t$), the intermediate firm's profit maximization problem yields the new Keynesian Philips curve

$$\frac{1}{\nu}Y_t(\nu - 1 + w_t/A) - \varphi(\pi_t - \pi)\pi_t - \phi_1\pi_t + \mathbb{E}_t[M_{t,t+1}(\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1\pi_{t+1})] = 0. \tag{6}$$

3.4 Government

The government needs to finance spending using labor income taxes and issuing nominal and real non-contingent debts of maturity N. The government budget constraint is

$$Q_t^{N-1}\frac{B_{t-1}^N}{\pi_t} + q_t^{N-1}b_{t-1}^N = \tau_t A h_t w_t - g_t + Q_t^N B_t^N + q_t^N b_t^N, \tag{7}$$

which is 1 expressed in real terms. The resource constraint of the economy is given by

$$c_t + g_t + \mathrm{AC}_t = Ah_t. \tag{8}$$

Combine equations 5 and 8 to define government's surplus as:

$$s_t \equiv w_t A h_t - (1-\tau_t) w_t A h_t - g_t = w_t(c_t + g_t + \mathrm{AC}_t) - \frac{U_{2,t}}{AU_{1,t}}(c_t + g_t + \mathrm{AC}_t) - g_t.$$

Implementability constraint Combining the definition of surplus s_t with equations 3, 4, 5 and 7 gives an intertemporal expression for the government budget constraint

$$\begin{aligned}
&\overbrace{\mathbb{E}_t\left[M_{t,t+N-1}\cdot\frac{1}{\Pi_{j=1}^{N-1}\pi_{t+j}}\right]\frac{B_{t-1}^N}{\pi_t}}^{\text{Buy-back of nominal bonds}} + \overbrace{b_{t-1}^N\mathbb{E}_t\left[M_{t,t+N-1}\right]}^{\text{Buy-back of real bonds}} = \\
&s_t + \underbrace{B_t^N\mathbb{E}_t\left[M_{t,t+N}\cdot\frac{1}{\Pi_{j=1}^{N}\pi_{t+j}}\right]}_{\text{New issuance of nominal bonds}} + \underbrace{b_t^N\mathbb{E}_t\left[M_{t,t+N}\right]}_{\text{New issuance of real bonds}},
\end{aligned}$$

which is the implementability constraint.

Optimal policy Given an exogenous sequence $\{g_t\}_{t=0}^{\infty}$, the *Ramsey planner* seeks sequences of policies $\{\pi_t, \tau_t, B_t^N, b_t^N\}_{t=0}^{\infty}$ and sequences of allocations $\{c_t, l_t, w_t\}_{t=0}^{\infty}$ such that the household's time-0 expected life-time utility is maximized and such that, at every t, (i) the implementability constraint is satisfied, (ii) the new Keynesian Phillips curve holds (equation 6), (iii) the Taylor rule (equation 2) is satisfied and (iv) both nominal and real bonds must lie between a lower bound B^U and an upper bound B^L. Call μ_t, λ_t^{π} and λ_t^T the time-t Lagrange multipliers associated with the implementability constraint, the Phillips curve and the Taylor Rule, respectively. The set of sequences described by the optimal policy further satisfies the following conditions.

The first order condition with respect to nominal bonds is

$$\mu_t = \left[\mathbb{E}_t[U_{1,t+N}/\Pi_{j=1}^N\pi_{t+j}]\right]^{-1}\left[\mathbb{E}_t[\mu_{t+1}U_{1,t+N}/\Pi_{j=1}^N\pi_{t+j}] + \frac{\xi_{U,t}}{\beta^N} - \frac{\xi_{L,t}}{\beta^N}\right], \tag{9}$$

where $\xi_{U,t}$ and $\xi_{L,t}$ are the Lagrange multipliers on the upper and lower bounds, respectively.

The first order condition with respect to real bonds

$$\mu_t = \left[\mathbb{E}_t\left[U_{1,t+N}\right]\right]^{-1}\left[\mathbb{E}_t[\mu_{t+1}U_{1,t+N}] + \frac{\xi^T_{U,t}}{\beta^N} - \frac{\xi^T_{L,t}}{\beta^N}\right], \tag{10}$$

where $\xi^T_{U,t}$ and $\xi^T_{L,t}$ are the Lagrange multipliers on the upper and lower bounds, respectively. Assuming that debt constraints do not bind at time t (i.e. all ξ are zero), equations 9 and 10 give the system

$$\begin{cases} \mu_t = \mathbb{E}_t[\mu_{t+1}] + \left[\mathbb{E}_t\left[U_{1,t+N}/\Pi^N_{j=1}\pi_{t+j}\right]\right]^{-1}\cdot \mathrm{Cov}_t\left[U_{1,t+N}/\Pi^N_{j=1}\pi_{t+j}, \mu_{t+1}\right] \\ \mu_t = \mathbb{E}_t[\mu_{t+1}] + \left[\mathbb{E}_t\left[U_{1,t+N}\right]\right]^{-1}\cdot \mathrm{Cov}_t\left[U_{1,t+N}, \mu_{t+1}\right] \end{cases}$$

which pins down a dynamic for the Lagrange multiplier on the implementability constraint μ_t similar in spirit to the one of Aiyagari et al. (2002). The Lagrange multiplier μ_t follows a risk-adjusted martingale with the the additional condition that links the optimal choices for nominal and real bonds

$$\frac{\mathbb{E}_t\left[U_{1,t+N}\right]}{\mathbb{E}_t\left[U_{1,t+N}/\Pi^N_{j=1}\pi_{t+j}\right]} = \frac{\mathrm{Cov}_t\left[U_{1,t+N}, \mu_{t+1}\right]}{\mathrm{Cov}_t\left[U_{1,t+N}/\Pi^N_{j=1}\pi_{t+j}, \mu_{t+1}\right]}. \tag{11}$$

The optimality condition with respect to wage

$$\mu_t U_{1,t} + \frac{1}{A\nu}\lambda^\pi_t = 0. \tag{12}$$

tights together the dynamic of the lagrange multiplier on the implementability constraint μ_t and on the new Keynesian Phillips curve λ^π_t. The remaining conditions with respect to consumption c_t and inflation π_t can be found in Appendix 6.2.

Special Case We consider a special case with risk-neutral households $U = c_t + v(l_t)$, single period maturity $N = 1$, and no lending constraint $\xi^T_{L,t} = 0$. In this case equation 10 becomes

$$\mu_t = \mathbb{E}_t[\mu_{t+1}] + \mathbb{E}_t\left[\frac{\xi^T_{U,t}}{\beta^N}\right].$$

Since the lagrange multiplier on the borrowing limit is non-negative $\xi^T_{U,t} \geq 0$, then $\mu_t \geq \mathbb{E}_t[\mu_{t+1}]$. We can use the submartingale convergence theorem: μ_t converges almost surely. This last condition and result is equivalent to Aiyagari et al. (2002): in the long-run the government eventually accumulates enough assets that it never needs to tax again. Differently from Aiyagari et al. (2002), the simultaneous presence of both nominal and real debt requires an extra condition to be satisfied. This is given by equation 11, which under risk-neutrality and $N = 1$ is

$$\mathrm{Cov}_t(\pi_{t+1}, \mu_{t+1}) = 0.$$

3.5 Implementing Arrow-Debreu Policies with Non-Contigent Nominal and Real Debt

The Ramsey problem we lay out can be thought of as a portfolio choice problem with incomplete markets in which the planner looks for the optimal allocation in the government debt portfolio of two securities, namely non-contingent nominal and real bonds. To provide some intuition about the determinants of these allocations, we now examine stylized examples in which the objective of the planner is most transparent, namely specifications in which the economy can be in two states only. In such an environment, the planner's objective is choose a portfolio of the two securities that replicates the Arrow-Debreu policies. That is, the planner aims at implementing the complete markets allocation.

We thus ask if we can we use inflation fluctuations to replicate a portfolio of Arrow-Debreu securities? If yes, can we characterize the portfolio of nominal and real non-contingent debt that replicate Arrow-Debreu securities? We consider a simple example in which the government can issue nominal and real debt with one period maturity. The answer to this question depends on the type of shock considered. In the following example we consider a government that faces two types of two-state (L: low, H: high) i.i.d. shocks: (i) inflationary (e.g. increase in government expenditure) and (ii) deflationary (e.g. output or monetary policy shock). The signs of the portfolio positions of nominal and real non-contingent debt that reproduce Arrow-Debreu securities are summarized in table 1.

Debt	Inflationary ($\pi_H > \pi_L$)	Deflationary ($\pi_H < \pi_L$)
Nominal B	$+$	$-$
Real b	$\left(\frac{\pi_{H,t}}{\pi_{L,t}} > \frac{b_L^{CM}}{b_H^{CM}}\right) \implies +$	$+$

Table 1: Signs of the portfolio allocations that reproduce AD securities

Notes: Table reports the signs of the portfolio allocations of non contingent nominal debt B and non contigent real debt b under two different types of shocks. Both types of shocks cause the net present value of surpluses at time 0 to fall. Since, under complete market, state contigent debt equates net present value of debt both type of shocks lead to $b_H^{CM} < b_L^{CM}$, where b_H^{CM} is the value of contigent debt in the high state and b_L^{CM} is the value of contigent debt in the low state.

Inflationary Shock Suppose that the government faces a two-state i.i.d. government expenditure shock $g_t \in \{g_L, g_H\}$. Note also that $U_1(c_{H,t}) > U_1(c_{L,t})$. We can use the Taylor

Rule to back-out an expression for inflation

$$\pi_t = \left(\frac{\pi^{\phi_\pi - 1}}{\mathbb{E}_t \left[\frac{U_{1,t+1}}{U_{1,t}} \frac{1}{\pi_{t+1}} \right]} \right)^{\frac{1}{\phi_\pi}},$$

from which it follows that $\pi_{L,t} < \pi_{H,t}$. Note that this is line with the intuition that when the government faces a high g shock, the planner will optimal choose to increase inflation in correspondance of a monetary expansion period. Moreover, since nominal price is given by $Q^1_{L,t} = \mathbb{E}_t \left[\beta \frac{U_1(c_{t+1})}{U_1(c_{L,t})} \frac{1}{\pi_{t+1}} \right]$ and $Q^1_{H,t} = \mathbb{E}_t \left[\beta \frac{U_1(c_{t+1})}{U_1(c_{H,t})} \frac{1}{\pi_{t+1}} \right]$, it also follows that $Q^1_{L,t} > Q^1_{H,t}$. In word, nominal bonds looses value with high g shock, which also corresponds to high inflation.

Outstanding liabilities $\tilde{b}_t$ at time t are characterized by the left-hand-side of the government budget constraint in real term, equation 7. In particular, with maturity $N = 1$ outstanding liabilities can be expressed as

$$\frac{B_{t-1}(s^{t-1})}{\pi_t(s^t)} + b_{t-1}(s^{t-1}) = \tilde{b}_t(s^t),$$

and they are measurable with respect to s^t thanks to the presence of inflation $\pi(s^t)$.

With complete markets the government insure itself against bad times, hence $b^{CM}_H < b^{CM}_L$. In order to reproduce Arrow-Debreu securities we need

$$\begin{bmatrix} \pi^{-1}_{H,t} & 1 \\ \pi^{-1}_{L,t} & 1 \end{bmatrix} \begin{bmatrix} B_{t-1} \\ b_{t-1} \end{bmatrix} = \begin{bmatrix} b^{CM}_H \\ b^{CM}_L \end{bmatrix}.$$

Note that the portfolio matrix is full-ranked as long as $U_1(c_{H,t}) \neq U_1(c_{L,t})$, which implies that $Q^1_{L,t} \neq Q^1_{H,t}$ and that $\pi_{L,t} \neq \pi_{H,t}$. Hence, the market can be completed.

Solving the linear system yields

$$\begin{bmatrix} B_{t-1} \\ b_{t-1} \end{bmatrix} = \frac{1}{\pi^{-1}_{H,t} - \pi^{-1}_{L,t}} \begin{bmatrix} 1 & -1 \\ -\pi^{-1}_{L,t} & \pi^{-1}_{H,t} \end{bmatrix} \begin{bmatrix} b^{CM}_H \\ b^{CM}_L \end{bmatrix}. \tag{13}$$

Recall that $\pi^{-1}_{L,t} > \pi^{-1}_{H,t}$. Hence, $\frac{1}{\pi^{-1}_{H,t} - \pi^{-1}_{L,t}} < 0$. This and the fact that $b^{CM}_H < b^{CM}_L$ imply that nominal debt is positive $B_{t-1} > 0$. If $\frac{\pi_{H,t}}{\pi_{L,t}} > \frac{b^{CM}_L}{b^{CM}_H}$, then $b_{t-1} > 0$. In words, the higher is the negative effect of the shock on the NPV of surplus the more likely TIPS are to be positive Also, the higher will be inflation in the H shock the more likely TIPS are to be positive.

Deflationary Shock Suppose that the government faces a two-state i.i.d. output shock $z_t \in \{z_L, z_H\}$. Note also that $U_1(c_{H,t}) < U_1(c_{L,t})$. Similarly to before, we can use the

Taylor Rule to back-out an expression for inflation, from which it follows that $\pi_{L,t} > \pi_{H,t}$. Note that this is line with the intuition that when the government faces a high z shock, the planner will optimal choose to decrease inflation in correspondance of a monetary contraction period. Moreover, since nominal price is given by $Q^1_{L,t} = \mathbb{E}_t\left[\beta \frac{U_1(c_{t+1})}{U_1(c_{L,t})}\frac{1}{\pi_{t+1}}\right]$ and $Q^1_{H,t} = \mathbb{E}_t\left[\beta \frac{U_1(c_{t+1})}{U_1(c_{H,t})}\frac{1}{\pi_{t+1}}\right]$, it also follows that $Q^1_{L,t} < Q^1_{H,t}$. In word, nominal bonds reevaluates with high z shock, which also corresponds to low inflation.

The net present value of surpluses decline, hence $b^{CM}_H < b^{CM}_L$. The portfolio positions that reproduce Arrow-Debreu securities are given by equations 13.

Recall that $\pi^{-1}_{L,t} < \pi^{-1}_{H,t}$. Hence, $\frac{1}{\pi^{-1}_{H,t} - \pi^{-1}_{L,t}} > 0$. This and the fact that $b^{CM}_H < b^{CM}_L$ imply that nominal debt is positive $B_{t-1} < 0$. The sign of real debt is determined by the sign of $-\pi^{-1}_{L,t} b^{CM}_H + \pi^{-1}_{H,t} b^{CM}_L$. Since $\pi^{-1}_{L,t} < \pi^{-1}_{H,t}$ and $b^{CM}_H < b^{CM}_L$, we can conclude that $\pi^{-1}_{L,t} b^{CM}_H < \pi^{-1}_{H,t} b^{CM}_L$. Hence, $b_{t-1} > 0$.

4 Quantitative Results

In this section we present the quantitative model results. First we describe the calibration strategy and then present the dynamics of the baseline model comparing it to a counterfactual without TIPS bonds. We then move on to analyze the role of outstanding nominal debt and the length of bond maturity.

4.1 Calibration

The model is calibrated to yearly frequency and the discount factor β is set to 0.96. Household preferences are additively separable in consumption and leisure: $U(c, l) = u(c) + v(l)$, where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $v(l) = B \cdot \frac{l^{1-\eta}}{1-\eta}$. We set γ and η to standard values (see Table 2) and look for B such that households allocation 2/3 of their time to leisure in the steady state. We assume that g_t follows an AR(1) process

$$\log(g_{t+1}) = (1-\rho) \cdot \mu + \rho \log g_t + \epsilon_{t+1}.$$

We estimate the parameters of ρ and σ_ϵ using yearly government expenditure data from BEA from 1947 to 2018, after extracting the cyclical component and linear trend. This gives the estimates for ρ and σ_ϵ of 0.977 and 0.0161, respectively. We then set μ so that the unconditional mean of g_t matches the average government expenditure to GDP ratio in the postwar sample, which is 20%.

In the production sector we set the Rotemberg adjustment cost to 4.375, consistent with the estimate in Sbordone (2002). In addition, our specification of adjustment cost has two extra parameters. First, we introduce parameters ϕ_1 and ϕ_2 to have a well defined steady state in the deterministic version of the model.[3] We set parameter ν, controlling the price elasticity of demand, to 0.1, which is a standard value used in the literature. The Taylor rule responds only to deviations from the steady state inflation rate. We set the steady state inflation rate to 2%, which is the Fed target level. We then set the maturity of government debt N equal to 5 for both nominal and real bonds. This is close to the average maturity of US federal debt (∼5.5 years). Table 2 summarizes all parameter values.

Parameter	Value	Description, source
β	0.96	Discount factor
γ	2	Relative risk aversion
η	1.8	Leisure utility parameter
A	1.0	Technology level
B	4.3276	Relative weight of leisure
$-\frac{1}{\nu}$	-10	Price elasticity of demand
φ	4.375	Rotemberg adjustment cost, Sbordone (2002)
ϕ_π	1.2	Taylor rule response to inflation
Π	1.02	SS inflation, Fed target
ρ, σ_ϵ	0.977, 0.0161	g_t persistence and std, BEA
$\mu(1-\rho)$	0.2	Ratio of government expenditure to GDP, BEA
N	5	Maturity of government debt
ψ	0	TIPS adjustment cost
ϕ_1, ϕ_2	$0.00001, 5.7143 \times 10^{-7}$	Adjustment cost

Table 2: Parameter Values

4.2 Baseline Results

We begin by comparing our calibrated model to a counterfactual scenario where the government can only issue nominal bonds. When the government cannot issue TIPS, the Ramsey planner faces a trade-off between responding to shocks using distortionary taxes versus inflation. On the one hand, by inflating away nominal debt, the government can finance the

[3] We use the equilibrium dynamics of the deterministic model to initialize the neural network. See appendix section 6.3 for a detailed description of the solution method.

additional expenditure without increasing labor taxes. On the other hand, by raising expected inflation, the planner reduces the value of household savings and decreases the price of government nominal bonds. Therefore, both the current and the future price of nominal bonds fall. In addition to that, inflation distorts firms' production decisions as price adjustment is costly. The presence of TIPS in the government debt portfolio affects this trade-off in two ways.

First, higher inflation has less impact on the cost of current and future borrowing, since it does not affect the price of inflation protected bonds. Second, the use of inflation becomes more costly because the planner needs to compensate households holding real bonds.

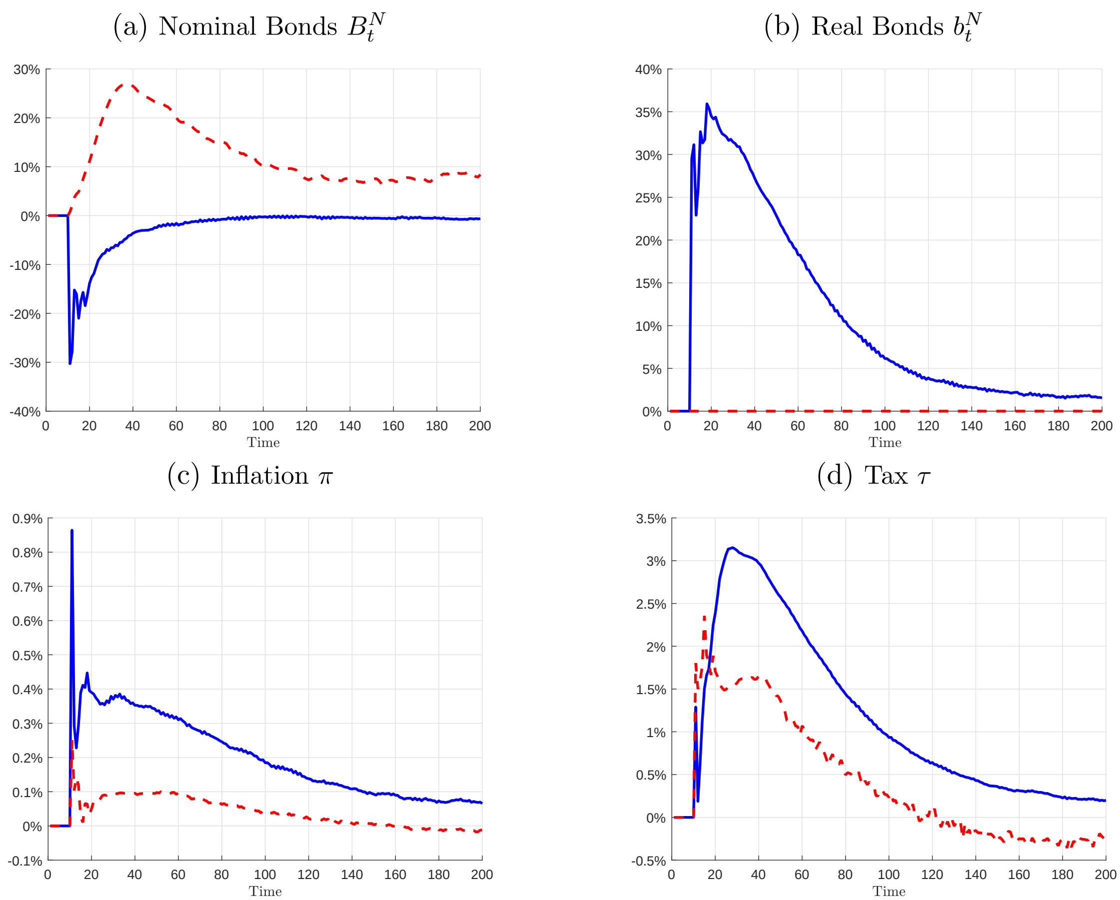


Figure 4: Impulse Response Functions

Notes: Figures shows impulse response functions to a government expenditure shock equal to 3% of GDP. Solid blue line - baseline model, dashed red line - model without TIPS bonds. Panels for inflation and taxes show percentage point difference. Panels for bonds show percentage point difference expressed as a ratio of GDP.

We investigate the workings of the model using impulse response functions in figure 4 where we shock the economy with a one-time government expenditure shock equal to 3% of GDP. We find that real bonds play a substantial role in shaping the optimal policy. The optimal policy prescribes: (i.) the accumulation of nominal liabilities and real assets in good times, and, (ii.) inflating away nominal liabilities and financing government expenditures using real assets in bad times. This stands in contrast to a counterfactual model without TIPS bonds, where government accumulates nominal liabilities in bad times and decumulates it otherwise. Because the government chooses to borrow in nominal bonds in response to shocks, it tried to keep the current nominal bond price high and, therefore, inflation plays a minor role in this counterfactual economy.

Reallocation to TIPS bonds in bad times is supported by moments from model simulation reported in table 3. It shows that TIPS bonds are countercyclical and nominal bonds are procyclical, while the total debt portfolio is countercyclical in both models.[4] On average, the optimal policy features lower levels of tax, inflation and short rates, but a higher responsiveness of these policy tools to government expenditure shocks.

[4]An increase in government expenditure indicates economic downturn. Therefore negative correlation with g_t means that a variable is procyclical.

	No TIPS	Baseline
$\mathbb{E}(\pi_t)$, %	1.986	1.381
$\mathbb{E}(\tau_t)$, %	23.397	21.638
$\mathbb{E}(i_t)$, %	6.232	5.477
$\mathbb{E}(b_t^N/GDP)$	-	-0.321
$\mathbb{E}(B_t^N/GDP)$	0.362	0.09
$\sigma(\pi_t)$	0.001	0.004
$\sigma(\tau_t)$	0.07	0.098
$\rho(\pi_t, g_t)$	0.693	0.885
$\rho(\tau_t, g_t)$	0.883	0.85
$\rho(B_t^N, g_t)$	0.674	-0.703
$\rho(b_t^N, g_t)$	-	0.846
$\rho(B_t^N + b_t^N, g_t)$	0.674	0.829
$\rho(\sigma_t(\pi_{t+1}), g_t)$	0.564	-0.283

Table 3: Main moments

Notes: Table reports sample moments from simulating model equilibrium dynamics for 5000 periods. Simulation is initialized at $b^N, B^N = 0$ and we drop the first 100 periods before calculating moments.

4.3 Example: Simulation With Prolonged Period of High Government Expenditures

Next, we present an example from the model simulation with a prolonged period of high government expenditure in figure 5. Top left panel shows the exogenous process for government expenditure, which starts to increase around period 100 and remains high for around 100 periods. Other three panels show policy variables in the baseline model (solid blue line) and the model without TIPS bonds (dashed red line). Inflation and taxes are on average lower in the baseline model but more responsive to increases in government expenditure. Because inflation is on average lower, nominal bond prices tend to be higher in the baseline model. By keeping the average inflation below the steady state target of 2%, the Ramsey planner incurs real costs but it happens that marginal benefits of having higher bond prices outweights these costs. Likewise, the Ramsey planner internalizes that higher use of inflation translates into more volatile nominal bond price - it drops by 2 percentage points during the period of high expenditure but then recovers from 0.74 to 0.79. This price volatility has little cost for the planner in the baseline model as it is always possible to relocate the portfolio to TIPS bonds if nominal bonds have to sell at a high discounts. This substitution is impossible in the one bond model and therefore, inflation responds to shows very little in the counterfactual economy.

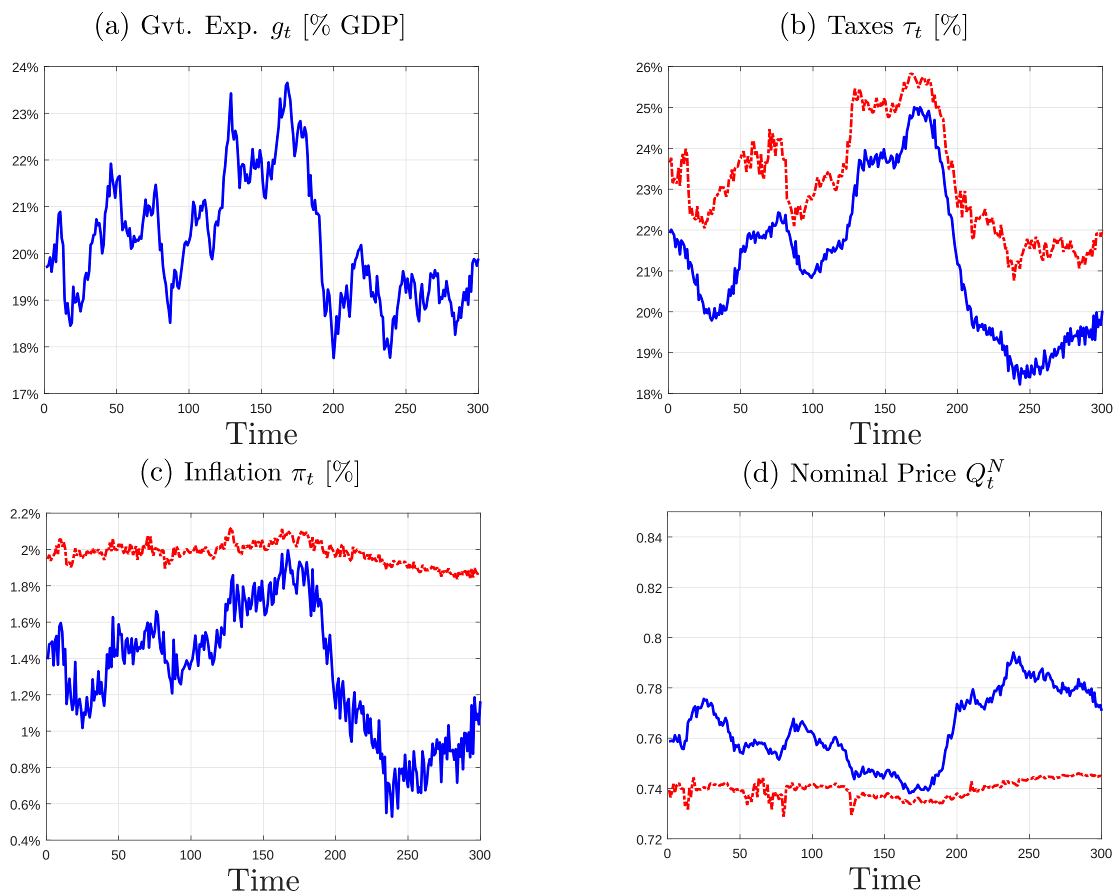


Figure 5: Simulation: policy variables

Notes: Figure shows an excerpt from the simulation of model equilibrium dynamics. Solid blue line - baseline model, dashed red line - model without TIPS bonds. Both models are simulated with the same realization of government expenditure shocks. The same simulation was used to calculate moments in table 3.

As shown by figure 6, higher welfare is achieved through higher consumption and less volatile leisure. Compared to a benchmark model consumption increases by on average 0.8% and leisure volatility falls by 6.64%. In fact, in the baseline model taxes are on average lower and household tends to work more. At the same time, labor supply is less elastic and it does not fluctuate as much even when the labor tax rate is more volatile in the baseline model.

Overall, compared to a benchmark model consumption increases by on average 0.8% and leisure volatility falls by 6.64% and this leads to a consumption equivalent welfare gain of 0.223% compared to a benchmark without TIPS bonds. The next session analyzes the role of outstanding debt in shaping the optimal policy.

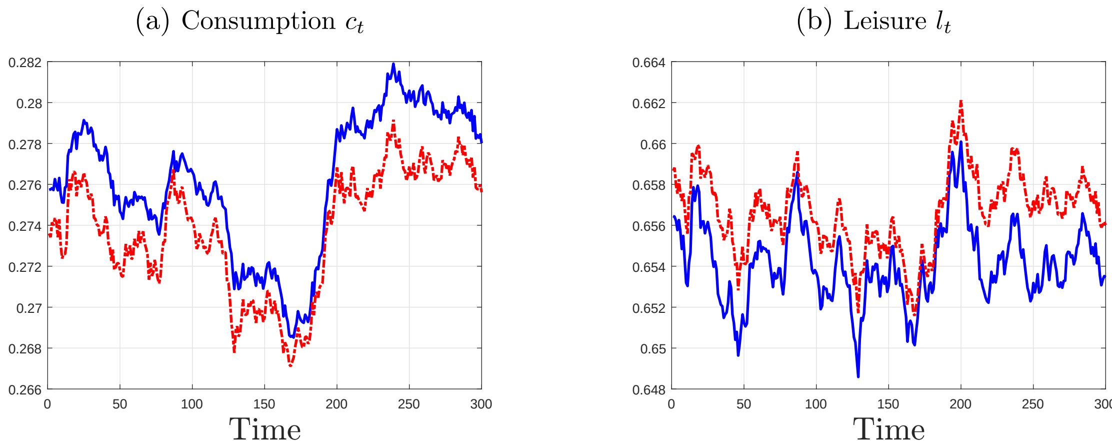


Figure 6: Simulation: allocations

Notes: Figure shows an excerpt from the simulation of model equilibrium dynamics. Solid blue line - baseline model, dashed red line - model without TIPS bonds. Both models are simulated with the same realization of government expenditure shocks. The same simulation was used to calculate moments in table 3.

4.4 Role of Initial Debt

In this section we analyze the relation between outstanding debt and the use of inflation when TIPS bonds are available. Specifically, we ask whether more debt causes more inflation. By using inflation, the Ramsey planner weights the benefits of inflating away nominal liabilities against two types of costs. First, by rational expectations, higher inflation eventually gets reflected in nominal bond prices (equation 3) and new nominal bonds need to sell at a higher discount. Second, inflation has real costs as it distorts firms' pricing decisions (equation 6). The reason that we observe more volatile inflation in the baseline model is because inflations' effect on nominal prices is not relevant for the Ramsey planner when the TIPS bonds are available. In this section we ask if high outstanding nominal debt can lead to high inflation.

The level of outstanding nominal debt changes the trade-off between inflation of nominal liabilities and real distortions. When the outstanding nominal debt is high, the same inflation rate allows to achieve a greater reduction in nominal liability while incurring the same distortion. At the same time, the trade-off between nominal liability effect and inflations' effect on nominal bond prices does not change. The same inflation rate allows to inflate more liabilities but more bonds need to be reissued in the next period. This together suggests that more nominal debt should lead to higher inflation.

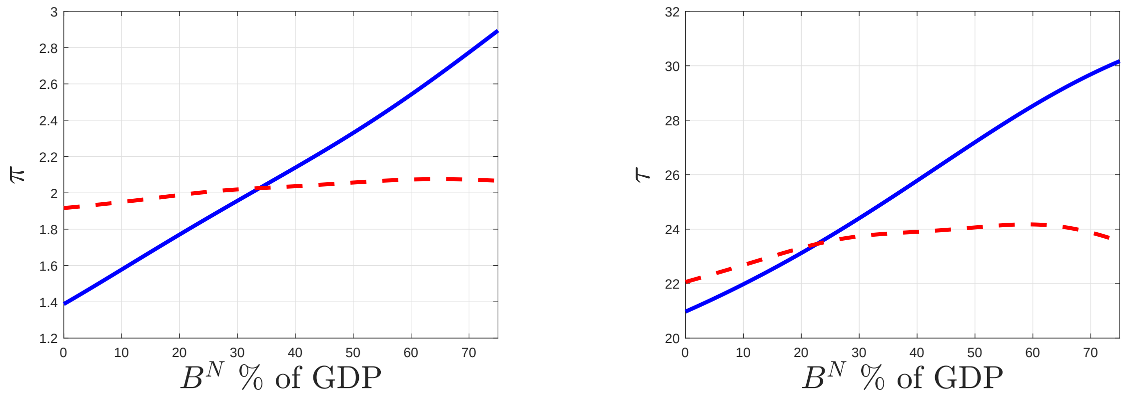


Figure 7: Role of nominal debt

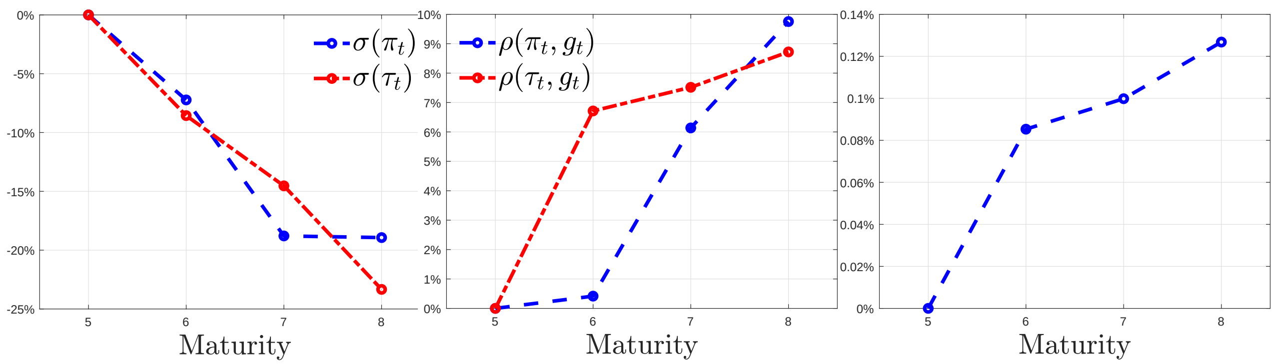
Notes: Figure plots policy functions of inflation and taxes in function of nominal debt. Other state variables are fixed at their mean values. Left - inflation, right - tax rate. Solid blue line show the baseline model, dashed red - model without TIPS bonds.

We investigate the role of nominal debt in models with and without TIPS bonds by looking at the policy functions of inflation and taxes in figure 7, which plots optimal inflation and taxes in function of nominal debt by keeping other state variables at their average levels.[5] The left panel shows that inflation responds positively to nominal debt in both models but the response in the baseline model is much larger. As the outstanding nominal debt increases from 0 to 75% of the GDP, inflation rate increases from 1.4% to 2.9% holding everything else fixed. In contrast, inflation in the one bond model moves from 1.9% to 2.05%. If real misallocation was the main cost of the use of inflation, one would expect that optimal inflation would respond to outstanding nominal debt similarly in both models. However, we observe that inflation responds little to shocks or outstanding debt in a one bond model, consistent with Siu (2004) and Marcet et al. (2013). Yet the reason for this lack of response is that the Ramsey planner mostly cares about the effect that inflation has on nominal bond prices. Since this concern is close to irrelevant in the model with TIPS bond, here the Ramsey planner uses inflation more aggressively.

[5]Since we solve the model using parameterized expectations algorithm, we are not solving for the policy functions explicitly. Instead, we use the model simulated data and the neural network to fit the relation between the policy and the state variables.

4.5 Role of Maturity

In this section we analyze the role of maturity on optimal inflation and taxes. In general, longer maturity brings greater benefits of using inflation. As maturity increases, both inflation and taxes become less volatile, as shown in the left panel of figure 8. Intuitively, longer maturity allows the planner to spread the inflation policy intervention across multiple periods. On the one hand, optimal policy prescribes lower volatility of taxes and inflation as maturity increases. but, on the other hand, higher responsiveness of these policy tools to government expenditures. As shown in the right panel of figure 8, increasing the maturity from five to eight years is associated with the consumption equivalent welfare gain of 0.13%.



(a) Volatility Policy (b) Correlation Policy (c) Welfare Equivalent

Figure 8: Role of maturity

Notes: Figure shows comparative statics when the bond maturity is exogenously increased from five to eight years in our baseline model. Each panel describes the relative values of respective moments relative to the counterpart in the model where maturity is five years. Left panel show the volatility of inflation (dashed blue) and volatility of taxes (dotted red), middle panel shows the correlation of inflation with government expenditure (dashed blue) and correlation between taxes and government expenditure (dotted red). Right panel shows the welfare increase relative to the model where bond maturity is five years.

5 Conclusion

Elevated levels of government debt in the wake of unprecedented stimulus packages increasingly raise concerns about a looming return of inflation, as governments may be tempted to monetize debt. In this paper, we examine optimal government debt management in the presence of inflation concerns in a setting where i) the government can issue long-term nominal and real (TIPS) bonds, ii) the monetary authority sets short-term interest rates according to a Taylor rule, and iii) inflation has real costs as prices are sticky. Nominal debt can be inflated away, but bond prices reflect elevated inflation expectations. Real bond prices are higher, but such debt constitutes a real commitment ex post. We show that the optimal government debt portfolio includes a substantial allocation to real assets and nominal liabilities, which lowers inflation levels but increases inflation volatility in equilibrium. The associated lower correlation between inflation risk and government expenditure is reflected in welfare gains through real debt management. Quantitatively, our results are stronger i) the higher the initial debt level, and ii) the longer debt maturity. Our findings suggest that TIPS should be an important tool for debt management in the presence of looming inflation.

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6 Appendix

6.1 Two-Period Model

Households

A representative household wants to maximize its expected lifetime utility,

$$U(c_0, l_0) + \beta \mathbb{E}_0 U(c_1, l_1)$$

where $l_t = 1 - h_t$, subject to budget constraint at time 0:

$$P_0 c_0 + Q_0 \bar{B}_0 + q_0 \bar{b}_0 = (1 - \tau_0) P_0 w_0 A h_0 + \bar{B}_{-1} + \pi_0 \bar{b}_{-1}$$

and time 1

$$P_1 c_1 = (1 - \tau_1) P_1 w_1 A h_1 + \bar{B}_0 + \pi_1 \bar{b}_0$$

where Q_t and q_t are prices of 1-period nominal and real bonds, $\bar{B}_t$ and $\bar{b}_t$, which are monetary values of 1-period nominal and real bonds, TIPS, at period t, and finally $\pi_t = \frac{P_t}{P_{t-1}}$ is an inflation rate from period $t-1$ to t.

It can be re-written in real terms by dividing by P_t:

$$c_t + Q_t B_t + q_t b_t = (1 - \tau_t) w_t A h_t + B_{t-1}/\pi_t + b_{t-1}$$

where $B_t = \frac{\bar{B}_t}{P_t}$ and $b_t = \frac{\bar{b}_t}{P_t}$ denote the real-value of nominal bonds and TIPS at time t.

Solving for B_0, b_0, and l_t give the following optimality conditions:

- Optimal nominal bonds (FOC_B):

$$Q_0 = \beta \mathbb{E}_0 \left[\frac{U_{1,1}}{U_{1,0}} \frac{1}{\pi_1} \right]$$

- Optimal TIPS (FOC_b):

$$q_0 = \beta \mathbb{E}_0 \left[\frac{U_{1,1}}{U_{1,0}} \right]$$

- Optimal labor supply (FOC_h):

$$\frac{U_{2,t}}{U_{1,t}} = (1 - \tau_t) A w_t$$

Firms

Aggregate output $Y_t = Ah_t$, Intermediate outputs: $Y_{it} = Ah_{it}$

Intermediate firms problem:

$$\max_{P_{i,0},P_{i,1}} P_{i,0}Y_{i,0} - w_0 h_{i,0} P_0 - P_0 AC_0 + \mathbb{E}_0 Q_{0,1}\left[P_{i,1}Y_{i,1} - w_1 h_{i,1} P_1 - P_1 AC_1\right]$$

s.t.

$$AC_t = \frac{\varphi}{2}\left(\frac{P_{i,t}}{P_{i,t-1}} - \pi\right)^2 + \phi_1\left(\frac{P_{i,t}}{P_{i,t-1}} - \pi\right) + \phi_2$$

$$Y_{it} = \left(\frac{P_{i,t}}{P_t}\right)^{-\frac{1}{\nu}} Y_t$$

Where

$$\frac{Q_{0,1}}{Q_{0,0}} = Q_{0,1} = q_0$$

Substitute in the demand function, the SDF and the adjustment cost function

$$\max_{\{P_{i,t}\}_{t=0}^{1}} \mathbb{E}_0 \sum_{t=0}^{1} Q_{0,t}\left[\left(\frac{P_{i,t}}{P_t}\right)^{\frac{\nu-1}{\nu}} Y_t - \left(\frac{P_{i,t}}{P_t}\right)^{-\frac{1}{\nu}} \frac{Y_t}{A} w_t - \frac{\varphi}{2}\left(\frac{P_{i,t}}{P_{i,t-1}} - \pi\right)^2 - \phi_1\left(\frac{P_{it}}{P_{it-1}} - \pi\right) - \phi_2\right]$$

The first order condition with respect to $P_{i,0}$

$$\frac{\upsilon-1}{\upsilon}\left(\frac{P_{i,0}}{P_0}\right)^{-\frac{1}{\nu}}\frac{Y_0}{P_0} + \frac{1}{\nu}\left(\frac{P_{i,0}}{P_0}\right)^{-\frac{1}{\nu}-1}\frac{Y_0}{AP_0}w_0 - \varphi\left(\frac{P_{i,0}}{P_{i,-1}} - \pi\right)\frac{1}{P_{i,-1}} - \phi_1\frac{1}{P_{i,-1}} +$$
$$\mathbb{E}_0\left(Q_{0,1}\left(\varphi\left(\frac{P_{i,1}}{P_{i,0}} - \pi\right)\frac{P_{i,1}}{P_{i,0}^2} + \phi_1\frac{P_{i,1}}{P_{i,0}^2}\right)\right) = 0$$

Imposing the symmetric equilibrium condition $P_{i,0} = P_0$ gives the Philips curve

$$\frac{1}{\nu}Y_0(\nu - 1 + w_0/A) - \varphi(\pi_0 - \pi)\pi_0 - \phi_1\pi_0 + \mathbb{E}_0[q_0(\varphi(\pi_1 - \pi)\pi_1 + \phi_1\pi_1)] = 0$$

The first order condition with respect to $P_{i,1}$

$$Q_{0,1}\left[\frac{\upsilon-1}{\upsilon}\left(\frac{P_{i,1}}{P_1}\right)^{-\frac{1}{\nu}}\frac{Y_1}{P_1} + \frac{1}{\nu}\left(\frac{P_{i,1}}{P_1}\right)^{-\frac{1}{\nu}-1}\frac{Y_1}{AP_1}w_1 - \varphi\left(\frac{P_{i,1}}{P_{i,0}} - \pi\right)\frac{1}{P_{i,0}} - \phi_1\frac{1}{P_{i,0}}\right] = 0$$

Imposing the symmetric equilibrium condition $P_{i,1} = P_1$ gives

$$\frac{1}{\nu}Y_1(\nu - 1 + w_1/A) - \varphi(\pi_1 - \pi)\pi_1 - \phi_1\pi_1 = 0$$

Central Bank

The Central bank follows this Taylor Rule:

$$i_0 = \frac{1}{Q_0} = \left(\beta \mathbb{E}_0 \left[\frac{U_{1,1}}{U_{1,0}} \frac{1}{\pi_1}\right]\right)^{-1} = \frac{1}{\beta} \pi \left(\frac{\pi_0}{\pi}\right)^{\phi_\pi}$$

Ramsey problem

Given the assumption that the government buys back and reissue the entire stock of the outstanding debt, the government budget constraint is given by:

$$\bar{B}_{t-1} + \pi_t \bar{b}_{t-1} = \tau_t A P_t w_t h_t - P_t g_t + Q_t \bar{B}_t + q_t \bar{b}_t$$

which can be re-written in real terms for time 0

$$\frac{B_{-1}}{\pi_0} + b_{-1} = \tau_0 A h_0 w_0 - g_0 + Q_0 B_0 + q_0 b_0$$

Note that at time 1 this becomes

$$\frac{B_0}{\pi_1} + b_0 = \tau_1 A h_1 w_1 - g_1$$

Technology is $c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2 = Ah_t$, recalling that $w_t(1 - \tau_t) = U_{l,t}/(AU_{c,t})$ and $l_t = 1 - h_t$ we can define surplus as:

$$s_t = w_t(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2) - \frac{U_{2,t}}{AU_{1,t}}(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2) - g_t$$

Combining all these information gives the following intertemporal expression for the government budget constraint:

$$\frac{B_{t-1}}{\pi_t} + b_{t-1} = s_t + B_t \mathbb{E}_t \left[\beta \frac{U_{1,t+1}}{U_{1,t}} \frac{1}{\pi_{t+1}}\right] + b_t \mathbb{E}_t \left[\beta \frac{U_{1,t+1}}{U_{1,t}}\right]$$

Sequential formulation

$$\begin{aligned} \max_{\{B_t, b_t, c_t, \pi_t, w_t\}_{t=0}^1} \mathcal{L} =& U(c_0, l_0) + \mu_0 \left(U_{1,0} s_0 + B_0 \mathbb{E}_0 \left[\beta U_{1,1} \frac{1}{\pi_1}\right] + b_0 \mathbb{E}_0 \left[\beta U_{1,1}\right] - U_{1,0} \frac{B_{-1}}{\pi_0} - U_{1,0} b_{-1}\right) \\ &+ \beta \mathbb{E}_0 \left[U(c_1, l_1) + \mu_1 \left(s_1 - \frac{B_0}{\pi_1} - b_0\right)\right] \end{aligned}$$

Subject to

- New Keynesian Phillips Curves:

$$\frac{1}{\nu}Y_0(\nu - 1 + w_0/A) - \varphi(\pi_0 - \pi)\pi_0 - \phi_1\pi_0 + \mathbb{E}_0[q_0(\varphi(\pi_1 - \pi)\pi_1 + \phi_1\pi_1)] = 0$$
$$\frac{1}{\nu}Y_1(\nu - 1 + w_1/A) - \varphi(\pi_1 - \pi)\pi_1 - \phi_1\pi_1 = 0$$

- Taylor Rule:

$$\mathbb{E}_0\left[U_{1,1}\frac{1}{\pi_1}\right] - \frac{1}{\pi}U_{1,0}\left(\frac{\pi_0}{\pi}\right)^{-\phi_\pi} = 0$$

And call λ_t^π and λ_t^T the time-t Lagrange multipliers associated with the Phillips curve and the Taylor Rule, respectively.

1. FOC_{B_0}

$$\mu_0\mathbb{E}_0[U_{1,1}/\pi_1] = \mathbb{E}_0[\mu_1/\pi_1]$$

2. FOC_{b_0}

$$\mu_0\mathbb{E}_0\left[U_{1,1}\right] = \mathbb{E}_0[\mu_1]$$

3. FOC_{c_t}

$$\begin{aligned}&\frac{dU(c_0, l_0)}{dc_0} + \mu_0\left(\frac{dU_{1,0}}{dc_0}\left(s_0 - \frac{B_{-1}}{\pi_0} - b_{-1}\right) + \frac{ds_0}{dc_0}U_{1,0}\right)\\&+ \lambda_0^\pi\left(\frac{\nu - 1 + w_0/A}{\nu} - \frac{dU_{1,0}}{dc_0}\frac{1}{U_{1,0}^2}\beta\mathbb{E}_0\left[U_{1,1}\varphi(\pi_1 - \pi)\pi_1 + \phi_1\pi_1)\right]\right)\\&- \lambda_0^T\frac{1}{\pi}\left(\frac{\pi_0}{\pi}\right)^{-\phi_\pi}\frac{dU_{1,0}}{dc_0} = 0\end{aligned}$$

$$\frac{dU(c_1, l_1)}{dc_1} + \mu_1\left(\frac{dU_{1,1}}{dc_1}s_1 + \frac{ds_1}{dc_1}U_{1,1}\right) + \lambda_1^\pi\left(\frac{\nu - 1 + w_1/A}{\nu}\right) = 0$$

4. FOC_{π_t}

   At time $t = 0$

$$\begin{aligned}&\frac{dU(c_0, l_0)}{d\pi_0} + \mu_0U_{1,0}\frac{ds_0}{d\pi_0} - \mu_0U_{1,0}\frac{B_{-1}}{\pi_0^2}\\&+ \lambda_0^\pi\left(\frac{v - 1 + w_0/A}{v}[\varphi(\pi_0 - \pi) + \phi_1] - \varphi(2\pi_0 - \pi) - \phi_1\right) +\\&+ \lambda_0^T\phi_\pi U_{1,0}\left(\frac{\pi_0}{\pi}\right)^{-\phi_\pi - 1}\frac{1}{\pi^2} = 0\end{aligned}$$

At time $t = 1$

$$\frac{dU(c_1, l_1)}{d\pi_1} + \mu_1 \frac{ds_1}{d\pi_1} - \mu_1 \frac{B_0}{\pi_1^2}$$
$$+ \lambda_1^\pi \left(\frac{v - 1 + w_1/A}{v} [\varphi(\pi_1 - \pi) + \phi_1] - \varphi(2\pi_1 - \pi) - \phi_1 \right)$$
$$+ \lambda_0^\pi \beta^{-1} \beta \frac{U_{1,1}}{U_{1,0}} \left(\varphi(2\pi_1 - \pi) + \phi_1\right) - \lambda_0^T \frac{1}{\beta} \frac{U_{1,1}}{\pi_1^2} = 0$$

5. FOC_{w_t}

$$\mu_t U_{1,t} + \frac{1}{A\nu} \lambda_t^\pi = 0$$

Solution

Consider the case $\phi_1 = 0$ and $U(c, l) = c - \frac{h^2}{2}$, with $h = 1 - l$.

Household optimality conditions imply

$$Q_0 = \beta \mathbb{E}_0 \left[\frac{1}{\pi_1} \right]$$
$$q_0 = \beta$$
$$h_t = (1 - \tau_t) A w_t$$

Firms optimality conditions imply

$$\frac{Ah_0}{\nu}(\nu - 1 + w_0/A) - \varphi(\pi_0 - \pi)\pi_0 + \beta \mathbb{E}_0[\varphi(\pi_1 - \pi)\pi_1] = 0$$
$$\frac{Ah_1}{\nu}(\nu - 1 + w_1/A) - \varphi(\pi_1 - \pi)\pi_1 = 0$$

The Taylor rule becomes

$$1 = \pi \left(\frac{\pi_0}{\pi} \right)^{\phi_\pi} \mathbb{E}_0 \left[\frac{1}{\pi_1} \right]$$

The optimal policy requires to find $\{h_0, h_1, \tau_0, \tau_1, B_0, b_0, c_0, c_1, \pi_0, \pi_1, w_0, w_1, \mu_0, \mu_1, \lambda_0^T, \lambda_0^\pi, \lambda_1^\pi\}$

such that

$$
\begin{aligned}
&c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 = Ah_t \\
&h_t = (1-\tau_t)Aw_t \\
&\mu_0 \mathbb{E}_0[1/\pi_1] = \mathbb{E}_0[\mu_1/\pi_1] \\
&\mu_0 = \mathbb{E}_0[\mu_1] \\
&1 + \mu_0 \left(w_0 - \frac{h_0}{A} \right) + \lambda_0^\pi \left(\frac{\nu - 1 + w_0/A}{\nu} \right) = 0 \\
&1 + \mu_1 \left(w_1 - \frac{h_1}{A} \right) + \lambda_1^\pi \left(\frac{\nu - 1 + w_1/A}{\nu} \right) = 0 \\
&h_0 \frac{\varphi}{A}(\pi_0 - \pi) + \mu_0 \frac{ds_0}{d\pi_0} - \mu_0 \frac{B_{-1}}{\pi_0^2} + \lambda_0^\pi \left(\frac{v - 1 + w_0/A}{v} \varphi(\pi_0 - \pi) - \varphi(2\pi_0 - \pi) \right) + \lambda_0^T \phi_\pi \left(\frac{\pi_0}{\pi} \right)^{-\phi_\pi - 1} \frac{1}{\pi^2} = 0 \\
&h_1 \frac{\varphi}{A}(\pi_1 - \pi) + \mu_1 \frac{ds_1}{d\pi_1} - \mu_1 \frac{B_0}{\pi_1^2} + \lambda_1^\pi \left(\frac{v - 1 + w_1/A}{v} \varphi(\pi_1 - \pi) - \varphi(2\pi_1 - \pi) \right) + \lambda_0^\pi \varphi(2\pi_1 - \pi) - \lambda_0^T \frac{1}{\beta} \frac{1}{\pi_1^2} = \\
&\mu_t + \frac{1}{A\nu} \lambda_t^\pi = 0 \\
&\frac{B_{-1}}{\pi_0} + b_{-1} = \tau_0 A h_0 w_0 - g_0 + Q_0 B_0 + q_0 b_0 \\
&\frac{B_0}{\pi_1} + b_0 = \tau_1 A h_1 w_1 - g_1 \\
&1 = \pi \left(\frac{\pi_0}{\pi} \right)^{\phi_\pi} \mathbb{E}_0 \left[\frac{1}{\pi_1} \right] \\
&\frac{Ah_0}{\nu}(\nu - 1 + w_0/A) - \varphi(\pi_0 - \pi)\pi_0 + \beta \mathbb{E}_0[\varphi(\pi_1 - \pi)\pi_1] = 0 \\
&\frac{Ah_1}{\nu}(\nu - 1 + w_1/A) - \varphi(\pi_1 - \pi)\pi_1 = 0
\end{aligned}
$$

Solution with exogenous inflation

$$
\begin{aligned}
& c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 = A_t h_t \\
& h_t = (1-\tau_t) A_t w_t \\
& \mu_0 \mathbb{E}_0[1/\pi_1] = \mathbb{E}_0[\mu_1/\pi_1] \\
& \mu_0 = \mathbb{E}_0[\mu_1] \\
& 1 + \mu_0 \left(w_0 - \frac{h_0}{A_0} \right) + \lambda_0^\pi \left(\frac{\nu - 1 + w_0/A_0}{\nu} \right) = 0 \\
& 1 + \mu_1 \left(w_1 - \frac{h_1}{A_1} \right) + \lambda_1^\pi \left(\frac{\nu - 1 + w_1/A_1}{\nu} \right) = 0 \\
& \mu_t + \frac{1}{A_t \nu} \lambda_t^\pi = 0 \\
& \frac{B_{-1}}{\pi_0} + b_{-1} = \tau_0 A_0 h_0 w_0 - g_0 + Q_0 B_0 + q_0 b_0 \\
& \frac{B_0}{\pi_1} + b_0 = \tau_1 A_1 h_1 w_1 - g_1 \\
& \frac{A_0 h_0}{\nu}(\nu - 1 + w_0/A_0) - \varphi(\pi_0 - \pi)\pi_0 + \beta \mathbb{E}_0[\varphi(\pi_1 - \pi)\pi_1] = 0 \\
& \frac{A_1 h_1}{\nu}(\nu - 1 + w_1/A_1) - \varphi(\pi_1 - \pi)\pi_1 = 0
\end{aligned}
$$

At time 1. From the NKPC

$$
h_1 = \frac{\nu \varphi(\pi_1 - \pi)\pi_1}{A_1(\nu - 1 + w_1/A_1)}
$$

From the budget constraint

$$
\begin{aligned}
& \frac{B_0}{\pi_1} + b_0 = h_1(A_1 w_1 - h_1) - g_1 \\
& \left(\frac{B_0}{\pi_1} + b_0 + g_1 \right) (A_1\nu - A_1 + w_1)^2 = (\nu\varphi(\pi_1 - \pi)\pi_1)(A_1 w_1 \left(A_1 \nu - A_1 + w_1\right) - \nu\varphi(\pi_1 - \pi)\pi_1)
\end{aligned}
$$

Which can be re-arranged as

$$
\begin{aligned}
& \left(\frac{B_0}{\pi_1} + b_0 + g_1 \right) A_1^2 (\nu - 1)^2 + \left(\frac{B_0}{\pi_1} + b_0 + g_1 \right) w_1^2 + \left(\frac{B_0}{\pi_1} + b_0 + g_1 \right) 2 w_1 A_1 (\nu - 1) = \\
& \nu\varphi(\pi_1 - \pi)\pi_1 A_1^2 w_1 \left(\nu - 1\right) + \nu\varphi(\pi_1 - \pi)\pi_1 A_1 w_1^2 - \left(\nu\varphi(\pi_1 - \pi)\pi_1\right)^2
\end{aligned}
$$

Call $\mathcal{B}_1 = \frac{B_0}{\pi_1} + b_0 + g_1$ and $\mathcal{K}_1 = \nu\varphi(\pi_1 - \pi)\pi_1$, this further becomes

$$(\mathcal{B}_1 - \mathcal{K}_1 A_1)\, w_1^2 + (2\mathcal{B}_1 - \mathcal{K}_1 A_1)\, A_1(\nu - 1)w_1 + \mathcal{B}A_1^2(\nu - 1)^2 + \mathcal{K}^2 = 0$$

If $\nu = 1$. At time 1

$$\begin{aligned}
w_1(B_0, b_0, g_1) &= \frac{\mathcal{K}_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}} \\
h_1(B_0, b_0, g_1) &= \sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1} \\
\tau_1(B_0, b_0, g_1) &= \frac{\mathcal{B}_1}{\mathcal{K}_1 A_1} \\
c_1(B_0, b_0, g_1) &= A_1\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1} - g_1 - \frac{\varphi}{2}(\pi_1 - \pi)^2 \\
\mu_1(B_0, b_0, g_1) &= \frac{A_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}} \\
\lambda_1^\pi(B_0, b_0, g_1) &= -\frac{A_1^2}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}
\end{aligned}$$

At time 0, you need find $B_0, b_0, \mu_0, h_0, w_0$ such that

$$\begin{aligned}
&\mu_0 \mathbb{E}_0\left[\frac{1}{\pi_1}\right] = \mathbb{E}_0\left[\frac{A_1}{\pi_1\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}\right] \\
&\mu_0 = \mathbb{E}_0\left[\frac{A_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}\right] \\
&h_0 w_0 - \varphi(\pi_0 - \pi)\pi_0 + \beta\mathbb{E}_0[\varphi(\pi_1 - \pi)\pi_1] = 0 \\
&\frac{B_{-1}}{\pi_0} + b_{-1} = A_0 h_0 w_0 - h_0^2 - g_0 + Q_0 B_0 + q_0 b_0 \\
&\mu_0 h_0 = A_0
\end{aligned}$$

The budget constraint becomes

$$\frac{B_{-1}}{\pi_0} + b_{-1} = A_0\mathcal{K}_0 - A_0\beta\mathbb{E}_0[\mathcal{K}_1] - \frac{A_0^2}{\mu_0^2} - g_0 + Q_0 B_0 + q_0 b_0$$

We want to seek portfolio B_0 and b_0 such that

$$\begin{aligned}
\mathcal{B}_0 &= A_0\mathcal{K}_0 - A_0\beta\mathbb{E}_0[\mathcal{K}_1] - \frac{A_0^2}{\left(\mathbb{E}_0\left[\frac{A_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}\right]\right)^2} + Q_0 B_0 + q_0 b_0 \\
\mathrm{Cov}_0&\left(\frac{1}{\pi_1}, \frac{A_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}\right) = 0
\end{aligned}$$

Note that the second expression can be rewritten as

$$\mathrm{Cov}_0\left(\frac{1}{\pi_1}, \frac{A_1}{h_1}\right) = 0$$

Or

$$\mathrm{Cov}_0\left(\frac{1}{\pi_1}, \frac{1}{(1-\tau_1)w_1}\right) = 0$$

Define

$$f \equiv A_0\mathcal{K}_0 - A_0\beta\mathbb{E}_0[\mathcal{K}_1] - \frac{A_0^2}{\left(\mathbb{E}_0\left[\frac{A_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}\right]\right)^2} + Q_0 B_0 + q_0 b_0 - \mathcal{B}_0 = 0$$

Then

$$\frac{dB_0}{db_0} = -\frac{\frac{\partial f}{\partial b_0}}{\frac{\partial f}{\partial B_0}} = -\frac{q_0 + 2\frac{A_0^2}{\mu_0^3}\frac{\partial \mu_0}{\partial b_0}}{Q_0 + 2\frac{A_0^2}{\mu_0^3}\frac{\partial \mu_0}{\partial B_0}} < 0$$

To take derivative wrt to an exogenous parameters define

$$f(B_0, b_0, p) \equiv A_0\mathcal{K}_0 - A_0\beta\mathbb{E}_0[\mathcal{K}_1] - \frac{A_0^2}{\left(\mathbb{E}_0\left[\frac{A_1}{\sqrt{\mathcal{K}_1 A_1 - \mathcal{B}_1}}\right]\right)^2} + Q_0 B_0 + q_0 b_0 - \mathcal{B}_0 = 0$$

$$m(B_0, b_0, p) \equiv \mathbb{E}_0\left[\frac{\mu_1}{\pi_1}\right] - \mathbb{E}_0\left[\mu_1\right]\mathbb{E}_0\left[\frac{1}{\pi_1}\right] = 0$$

$$\frac{dB_0}{dp} = \frac{f_2 m_3 - m_2 f_3}{f_1 m_2 - f_2 m_1}$$

$$\frac{db_0}{dp} = \frac{m_1 f_3 - f_1 m_3}{f_1 m_2 - f_2 m_1}$$

$$f_1 = Q_0 + \frac{A_0^2}{\mu_0^3}\frac{\partial \mu_0}{\partial B_0} > 0$$

$$f_2 = q_0 + \frac{A_0^2}{\mu_0^3}\frac{\partial \mu_0}{\partial b_0} > 0$$

$$m_1 = \mathbb{E}_0\left[\frac{\frac{\partial \mu_1}{\partial B_0}}{\pi_1}\right] - \mathbb{E}_0\left[\frac{\partial \mu_1}{\partial B_0}\right]\mathbb{E}_0\left[\frac{1}{\pi_1}\right] = \mathrm{Cov}_0\left(\frac{\partial \mu_1}{\partial B_0}, \frac{1}{\pi_1}\right)$$

$$m_2 = \mathbb{E}_0\left[\frac{\frac{\partial \mu_1}{\partial b_0}}{\pi_1}\right] - \mathbb{E}_0\left[\frac{\partial \mu_1}{\partial b_0}\right]\mathbb{E}_0\left[\frac{1}{\pi_1}\right] = \mathrm{Cov}_0\left(\frac{\partial \mu_1}{\partial b_0}, \frac{1}{\pi_1}\right)$$

$$\frac{\partial \mu_1}{\partial B_0} = \frac{1}{2}\frac{A_1}{(\mathcal{K}_1 A_1 - \mathcal{B}_1)^{\frac{3}{2}}}\frac{1}{\pi_1} > 0$$

$$\frac{\partial \mu_1}{\partial b_0} = \frac{1}{2}\frac{A_1}{(\mathcal{K}_1 A_1 - \mathcal{B}_1)^{\frac{3}{2}}} = \frac{1}{2A_1^2}\mu_1^3 > 0$$

Let's start with $p = g_0$

$$f_3 = -1 < 0$$
$$m_3 = 0$$

$$\frac{dB_0}{dp} = \frac{m_2}{f_1 m_2 - f_2 m_1}$$
$$\frac{db_0}{dp} = -\frac{m_1}{f_1 m_2 - f_2 m_1}$$

Consider that $f_1 < f_2$. If $m_1 > m_2$ then $f_1 m_2 - f_2 m_1 < 0$.

6.2 Model

Households

A representative household wants to maximize its expected lifetime utility,

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} U(c_t, l_t) \right]$$

where $l_t = 1 - h_t$, subject to budget constraint

$$P_t c_t + Q_t^N \bar{B}_t^N + q_t^N \bar{b}_t^N = (1 - \tau_t) P_t w_t A h_t + \bar{B}_{t-N}^N + \Pi_{j=1}^N \pi_{t-j+1} \bar{b}_{t-N}^N$$

where Q_t^N and q_t^N are prices of N-period nominal and real bonds, $\bar{B}_t^N$ and $\bar{b}_t^N$, which are monetary values of N-period nominal and real bonds, TIPS, at period t, and finally $\pi_t = \frac{P_t}{P_{t-1}}$ is an inflation rate from period $t-1$ to t.

It can be re-written in real terms by dividing by P_t

$$c_t + Q_t^N B_t^N + q_t^N b_t^N = (1 - \tau_t) w_t A h_t + B_{t-N}^N / \Pi_{j=1}^N \pi_{t-j+1} + b_{t-N}^N$$

where $B_t^N = \frac{\bar{B}_t^N}{P_t}$ and $b_t^N = \frac{\bar{b}_t^N}{P_t}$ denote the real-value of nominal bonds and TIPS at time t.

Also we add liquidity adjustment cost for trading real bonds

$$c_t + Q_t^N B_t^N + q_t^N b_t^N + \frac{\psi}{2}(b_t^N - b_{t-N}^N)^2 = (1 - \tau_t) w_t A h_t + B_{t-N}^N / \Pi_{j=1}^N \pi_{t-j+1} + b_{t-N}^N$$

Solving for B_t^N, b_t^N, and l_t give the following optimality conditions:

- Optimal nominal bonds (FOC_{B^N}):

$$Q_t^N = \beta^N \mathbb{E}_t \left[\frac{U_{1,t+N}}{U_{1,t}} \frac{1}{\Pi_{j=1}^N \pi_{t+j}} \right]$$

- Optimal TIPS (FOC_{b^N}):

$$q_t^N = \beta^N \mathbb{E}_t \left[\frac{U_{1,t+N}}{U_{1,t}} \right] - \psi \left\{ (b_t^N - b_{t-N}^N) - \beta^N \mathbb{E}_t \left[\frac{U_{1,t+N}(b_{t+N}^N - b_t^N)}{U_{1,t}} \right] \right\}$$

- Optimal labor supply (FOC_h):

$$\frac{U_{2,t}}{U_{1,t}} = (1-\tau_t) A w_t$$

Firms

Aggregate output $Y_t = Ah_t$, Intermediate outputs: $Y_{it} = Ah_{it}$

Intermediate firms problem:

$$\max_{\{P_{i,t}\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{t=0}^{\infty} Q_{0,t} \left[P_{i,t} Y_{i,t} - w_t h_{i,t} P_t - P_t AC_t \right]$$

s.t.

$$AC_t = \frac{\varphi}{2} \left(\frac{P_{i,t}}{P_{i,t-1}} - \pi \right)^2 + \phi_1 \left(\frac{P_{i,t}}{P_{i,t-1}} - \pi \right) + \phi_2$$

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{1}{\nu}} Y_t$$

$$\frac{Q_{0,t+1}}{Q_{0,t}} = q_t^1$$

Substitute in the demand function, the SDF and the adjustment cost function

$$\max_{\{P_{i,t}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} Q_{0,t} \left[\left(\frac{P_{i,t}}{P_t} \right)^{\frac{\nu-1}{\nu}} Y_t - \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{1}{\nu}} \frac{Y_t}{A} w_t - \frac{\varphi}{2} (\frac{P_{i,t}}{P_{i,t-1}} - \pi)^2 - \phi_1 (\frac{P_{it}}{P_{it-1}} - \pi) - \phi_2 \right]$$

The first order condition for $P_{i,t}$

$$\mathbb{E}_0 \left(Q_{0,t} \left[\frac{v-1}{v} \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{1}{\nu}} \frac{Y_t}{P_t} + \frac{1}{\nu} \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{1}{\nu}-1} \frac{Y_t}{AP_t} w_t - \varphi \left(\frac{P_{i,t}}{P_{i,t-1}} - \pi \right) \frac{1}{P_{i,t-1}} - \phi_1 \frac{1}{P_{i,t-1}} \right] \right) +$$
$$\mathbb{E}_0 \left(Q_{0,t+1} \left(\varphi \left(\frac{P_{i,t+1}}{P_{i,t}} - \pi \right) \frac{P_{i,t+1}}{P_{i,t}^2} + \phi_1 \frac{P_{i,t+1}}{P_{i,t}^2} \right) \right) = 0$$

Imposing symmetric equilibrium $P_{i,t} = P_t$ gives the Philips curve

$$\frac{1}{\nu} Y_t(\nu - 1 + w_t/A) - \varphi(\pi_t - \pi)\pi_t - \phi_1 \pi_t + \mathbb{E}_t[q_t^1(\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1})] = 0$$

Central Bank

The Central bank follows this Taylor Rule:

$$i_t = \frac{1}{Q_t^1} = \left(\beta \mathbb{E}_t\left[\frac{U_{1,t+1}}{U_{1,t}}\frac{1}{\pi_{t+1}}\right]\right)^{-1} = \frac{1}{\beta}\pi\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi}$$

Ramsey problem with N-Period TIPS and non TIPS bonds

Given the assumption that the government buys back and reissue the entire stock of the outstanding debt, the government budget constraint is given by:

$$Q_t^{N-1}B_{t-1}^N + \pi_t q_t^{N-1}b_{t-1}^N = \tau_t A P_t w_t h_t - P_t g_t + Q_t^N B_t^N + q_t^N b_t^N$$

which can be re-written in real terms:

$$Q_t^{N-1}\frac{B_{t-1}^N}{\pi_t} + q_t^{N-1}b_{t-1}^N = \tau_t A h_t w_t - g_t + Q_t^N B_t^N + q_t^N b_t^N$$

Technology is: $c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2 = Ah_t$, recalling that $w_t(1-\tau_t) = U_{l,t}/(AU_{c,t})$ and $l_t = 1 - h_t$ we can define surplus as:

$$s_t = w_t(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2) - \frac{U_{2,t}}{AU_{1,t}}(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2) - g_t$$

Combining all these information gives the following intertemporal expression for the government budget constraint:

$$\begin{aligned}
&\mathbb{E}_t\left[\beta^{N-1}\frac{U_{1,t+N-1}}{U_{1,t}}\frac{1}{\Pi_{j=1}^{N-1}\pi_{t+j}}\right]\frac{B_{t-1}^N}{\pi_t} + b_{t-1}^N\mathbb{E}_t\left[\beta^{N-1}\frac{U_{1,t+N-1}}{U_{1,t}}\right] \\
&- \psi b_{t-1}^N\Big\{(b_t^N - b_{t-N+1}^N) - \mathbb{E}_t\left[\beta^{N-1}\frac{U_{1,t+N-1}(b_{t+N-1}^N - b_t^N)}{U_{1,t}}\right]\Big\} \\
&= s_t + B_t^N\mathbb{E}_t\left[\beta^N\frac{U_{1,t+N}}{U_{1,t}}\frac{1}{\Pi_{j=1}^N\pi_{t+j}}\right] + b_t^N\mathbb{E}_t\left[\beta^N\frac{U_{1,t+N}}{U_{1,t}}\right] \\
&- \psi b_t^N\Big\{(b_t^N - b_{t-N}^N) - \mathbb{E}_t\left[\beta^N\frac{U_{1,t+N}(b_{t+N}^N - b_t^N)}{U_{1,t}}\right]\Big\}
\end{aligned}$$

Sequential formulation

$$\max_{\{B_t^N, b_t^N, c_t, \pi_t, w_t\}_{t=0}^{\infty}} \mathcal{L} = \sum_{t=0}^{\infty} \mathbb{E}_t \beta^t \Bigg\{ U(c_t, l_t) + \mu_t \Bigg(U_{1,t} s_t + B_t^N \mathbb{E}_t \left[\beta^N U_{1,t+N} \frac{1}{\Pi_{j=1}^N \pi_{t+j}} \right]$$
$$+ b_t^N \mathbb{E}_t \left[\beta^N U_{1,t+N} \right] - \psi b_t^N \Big\{ U_{1,t}(b_t^N - b_{t-N}^N) - \mathbb{E}_t \left[\beta^N U_{1,t+N}(b_{t+N}^N - b_t^N) \right] \Big\}$$
$$- \mathbb{E}_t \left[\beta^{N-1} U_{1,t+N-1} \frac{1}{\Pi_{j=1}^{N-1} \pi_{t+j}} \right] \frac{B_{t-1}^N}{\pi_t}$$
$$- \mathbb{E}_t \left[\beta^{N-1} U_{1,t+N-1} \right] b_{t-1}^N + \psi b_{t-1}^N \Big\{ U_{1,t}(b_t^N - b_{t-N+1}^N) - \mathbb{E}_t \left[\beta^{N-1} U_{1,t+N-1}(b_{t+N-1}^N - b_t^N) \right] \Big\} \Bigg)$$
$$+ \xi_{U,t}(B^U - B_t^N) + \xi_{L,t}(B_t^N - B^L) + \xi_{U,t}^T(B^U - b_t^N) + \xi_{L,t}^T(b_t^N - B^L) \Bigg\}$$

Subject to

- New Keynesian Phillips Curve:

$$\frac{1}{\nu}(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2)(\nu - 1 + w_t/A) - \varphi(\pi_t - \pi)\pi_t - \phi_1 \pi_t +$$
$$\mathbb{E}_t \left[\Big\{ \varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1} \Big\} \Big\{ \beta \frac{U_{1,t+1}}{U_{1,t}} \left[1 + \psi(b_{t+1}^N - b_t^N) \right] - \psi(b_t^N - b_{t-1}^N) \Big\} \right] = 0$$

- Taylor Rule:

$$\mathbb{E}_t \left[U_{1,t+1} \frac{1}{\pi_{t+1}} \right] - \frac{1}{\pi} U_{1,t} \left(\frac{\pi_t}{\pi} \right)^{-\phi_\pi} = 0$$

And call λ_t^π and λ_t^T the time-t Lagrange multipliers associated with the Phillips curve and the Taylor Rule, respectively.

1. $FOC_{B_t^N}$

$$\mu_t = \left[\mathbb{E}_t [U_{1,t+N}/\Pi_{j=1}^N \pi_{t+j}] \right]^{-1} \left[\mathbb{E}_t [\mu_{t+1} U_{1,t+N}/\Pi_{j=1}^N \pi_{t+j}] + \frac{\xi_{U,t}}{\beta^N} - \frac{\xi_{L,t}}{\beta^N} \right]$$

2. $FOC_{b_t^N}$

$$\mu_t \mathbb{E}_t \left[\beta^N U_{1,t+N} \right] + \psi \mu_t \mathbb{A} = \mathbb{E}_t [\beta^N \mu_{t+1} U_{1,t+N}] + \xi_{U,t}^T - \xi_{L,t}^T - \psi \mathbb{B} + \mathbb{C}$$

where

$$\mathbb{A} = -U_{1,t}(b_t^N - b_{t-N}^N) + \mathbb{E}_t[\beta^N U_{1,t+N}(b_{t+N}^N - b_t^N)] - \\ b_t^N U_{1,t} - b_t^N \mathbb{E}_t[\beta^N U_{1,t+N}] + b_{t-1}^N U_{1,t} + b_{t-1}^N \mathbb{E}_t[\beta^{N-1} U_{1,t+N-1}]$$

$$\mathbb{B} = \mathbb{E}_t[\mu_{t+N} b_{t+N}^N \beta^N U_{1,t+N}] + \mu_{t-N} b_{t-N}^N U_{1,t} + \\ \mathbb{E}_t[\mu_{t+1} U_{1,t+1} \beta (b_{t+1}^N - b_{t-N+2}^N) - \mu_{t+1} \beta^N U_{1,t+N}(b_{t+N}^N - b_{t+1}^N)] - \\ \mathbb{E}_t[\beta^{N-1} \mu_{t+N-1} b_{t+N-2}^N U_{1,t+N-1}] - \mu_{t-N+1} b_{t-N}^N U_{1,t}$$

$$\mathbb{C} = \lambda_t^\pi \psi \mathbb{E}_t \left[\left(\beta \frac{U_{1,t+1}}{U_{1,t}} + 1 \right) (\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1}) \right] \\ - \lambda_{t-1}^\pi \psi \frac{U_{1,t}}{U_{1,t-1}} \{\varphi(\pi_t - \pi)\pi_t + \phi_1 \pi_t)\}] \\ - \psi \mathbb{E}_t \big[\lambda_{t+1}^\pi \beta \{\varphi(\pi_{t+2} - \pi)\pi_{t+2} + \phi_1 \pi_{t+2})\} \big]$$

3. FOC_{c_t}

$$\frac{dU(c_t, l_t)}{dc_t} + \mu_t \left(\frac{dU_{1,t}}{dc_t} s_t + \frac{ds_t}{dc_t} U_{1,t} \right) \\ + \frac{B_{t-N}^N}{\Pi_{j=1}^N \pi_{t-j+1}} \frac{dU_{1,t}}{dc_t} (\mu_{t-N} - \mu_{t-N+1}) + b_{t-N}^N \frac{dU_{1,t}}{dc_t} (\mu_{t-N} - \mu_{t-N+1}) \\ + \lambda_t^\pi \left(\frac{\nu - 1 + w_t/A}{\nu} - \frac{dU_{1,t}}{dc_t} \frac{1}{U_{1,t}^2} \beta \mathbb{E}_t \big[U_{1,t+1} \varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1}) \big] \big\{ 1 + \psi(b_{t+1}^N - b_t^N) \big\} \right) \\ + \lambda_{t-1}^\pi \frac{dU_{1,t}}{dc_t} \frac{1}{U_{1,t-1}} \bigg(\varphi(\pi_t - \pi)\pi_t + \phi_1 \pi_t \bigg) \bigg(1 + \psi(b_t^N - b_{t-1}^N) \bigg) \\ - \lambda_t^T \frac{1}{\pi} \left(\frac{\pi_t}{\pi} \right)^{-\phi_\pi} \frac{dU_{1,t}}{dc_t} + \lambda_{t-1}^T \frac{dU_{1,t}}{dc_t} \frac{1}{\beta \pi_t} \\ + \psi \frac{dU_{1,t}}{dc_t} \Big[- \mu_t b_t^N (b_t^N - b_{t-N}^N) + \mu_t b_{t-1}^N (b_t^N - b_{t-N+1}^N) + b_{t-N}^N \mu_{t-N} (b_t^N - b_{t-N}^N) \\ - b_{t-N+1}^N \mu_{t-N+1} (b_t^N - b_{t-N+1}^N) \Big] = 0$$

4. FOC_{π_t}

$$
\begin{aligned}
&\frac{dU(c_t,l_t)}{d\pi_t}+\mu_t U_{1,t}\frac{ds_t}{d\pi_t}\\
&+\frac{1}{\pi_t}\sum_{k=1}^{N}B_{t-k}\beta^{N-k}\mathbb{E}_t\left[U_{1,t+N-k}\Big(\Pi_{j=0}^{N-1}\pi_{t-k+j+1}\Big)^{-1}\right]\Big(\mu_{t-k+1}-\mu_{t-k}\Big)\\
&+\lambda_t^{\pi}\left(\frac{v-1+w_t/A}{v}[\varphi(\pi_t-\pi)+\phi_1]-\varphi(2\pi_t-\pi)-\phi_1\right)\\
&+\lambda_{t-1}^{\pi}\beta^{-1}\left(\left\{\beta\frac{U_{1,t}}{U_{1,t-1}}\left[1+\psi(b_t^N-b_{t-1}^N)\right]-\psi(b_{t-1}^N-b_{t-2}^N)\right\}\Big\{\varphi(2\pi_t-\pi)+\phi_1\Big\}\right)\\
&+\lambda_t^{T}\phi_\pi U_{1,t}\left(\frac{\pi_t}{\pi}\right)^{-\phi_\pi-1}\frac{1}{\pi^2}-\lambda_{t-1}^{T}\frac{1}{\beta}\frac{U_{1,t}}{\pi_t^2}=0
\end{aligned}
$$

5. FOC_{w_t}

$$
\mu_t U_{1,t}+\frac{1}{A\nu}\lambda_t^{\pi}=0
$$

6.3 Solution Algorithm

At every instant t the information set is $\mathcal{I}_t = \{g_t, \{B^N_{t-k}\}_{k=0}^{N-1}, \{b^N_{t-k}\}_{k=0}^{N-1}, \{\mu_{t-k}\}_{k=1}^{N}\}$. Consider projections of the forward looking terms in the model onto $\mathcal{I}_t$. We model these relationships using one single-layer artificial neural network $\mathcal{ANN}(\mathcal{I}_t)$ with the characteristics described in Table 4. In particular, if the maturity is $N > 2$, then the terms to approximate are the following:

$$\mathcal{ANN}_1 = \mathbb{E}_t\Big[\frac{U_{1,t+N}}{\Pi_{j=1}^N \pi_{t+j}}\Big]$$
$$\mathcal{ANN}_2 = \mathbb{E}_t\Big[\frac{\mu_{t+1} U_{1,t+N}}{\Pi_{j=1}^N \pi_{t+j}}\Big]$$
$$\mathcal{ANN}_3 = \mathbb{E}_t[U_{1,t+N}]$$
$$\mathcal{ANN}_4 = \mathbb{E}_t[U_{1,t+N-1}]$$
$$\mathcal{ANN}_5 = \mathbb{E}_t[U_{1,t+N} b^N_{t+N}]$$
$$\mathcal{ANN}_6 = \mathbb{E}_t[\mu_{t+1} U_{1,t+N}]$$
$$\mathcal{ANN}_7 = \mathbb{E}_t[\mu_{t+1} U_{1,t+1} b^N_{t+1}]$$
$$\mathcal{ANN}_8 = \mathbb{E}_t[\mu_{t+1} U_{1,t+N} b^N_{t+1}]$$
$$\mathcal{ANN}_9 = \mathbb{E}_t[\mu_{t+1} U_{1,t+N} b^N_{t+N}]$$
$$\mathcal{ANN}_{10} = \mathbb{E}_t[\mu_{t+N} U_{1,t+N} b^N_{t+N}]$$
$$\mathcal{ANN}_{11} = \mathbb{E}_t[\mu_{t+N-1} U_{1,t+N-1} b^N_{t+N-2}]$$
$$\mathcal{ANN}_{12} = \mathbb{E}_t[U_{1,t+1}\{\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1}\}]$$
$$\mathcal{ANN}_{13} = \mathbb{E}_t[\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1}]$$
$$\mathcal{ANN}_{14} = \mathbb{E}_t[\lambda^\pi_{t+1}\{\varphi(\pi_{t+2} - \pi)\pi_{t+2} + \phi_1 \pi_{t+2}\}]$$
$$\mathcal{ANN}_{15} = \mathbb{E}_t[U_{1,t+1}\{\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1}\}]$$
$$\mathcal{ANN}^k_{16} = \mathbb{E}_t\Big[U_{1,t+N-k}\Big(\Pi_{j=1}^{N-1} \pi_{t-k+j+1}\Big)^{-1}\Big], \quad \text{for} \quad k \in \{1, 2, \ldots, N-1\}$$
$$\mathcal{ANN}_{17} = \mathbb{E}_t\Big[U_{1,t+1} \frac{1}{\pi_{t+1}}\Big]$$
$$\mathcal{ANN}_{18} = \mathbb{E}_t[\{\varphi(\pi_{t+1} - \pi)\pi_{t+1} + \phi_1 \pi_{t+1}\} b^N_{t+1}]$$

The solution procedure is summarized by the following algorithm.
Given starting values $\mu_{t-1} = 0$ and initial weights for $\mathcal{ANN}$, simulate a sequence of the set of endogenous variables $\{c_t, \mu_t, B^N_t, b^N_t, \pi_t, \lambda^T_t, \lambda^\pi_t, w_t\}$ as follows.[6]

[6]The network can be initially trained imposing $\{b_t\} = 0$.

1. Impose the Maliar moving bounds, see Maliar and Maliar (2003), on debt (these bounds are particularly important and need to be tight and open slowly since the ANN at the beginning can only make accurate predictions around zero debt - that is our initialization point). Proper penalty functions are used instead of the ξ terms to avoid out of bound solutions, see Faraglia et al. (2014) for more details. Using forward-states on the optimality conditions, solve for $c_t, \mu_t, B_t^N, b_t^N, \pi_t, \lambda_t^T, \lambda_t^\pi$, and w_t.[7]

$$\mu_t = \mathcal{ANN}_1(\mathcal{I}_t)^{-1}\left[\mathcal{ANN}_2(\mathcal{I}_t) + \frac{\xi_{U,t}}{\beta} - \frac{\xi_{L,t}}{\beta}\right]$$

$$\begin{aligned}
&\mu_t\beta\mathcal{ANN}_3(\mathcal{I}_t) + \psi\mu_t\mathbb{A}\Big(\mathcal{ANN}_4(\mathcal{I}_t), \mathcal{ANN}_5(\mathcal{I}_t)\Big) \\
&= \beta^N\mathcal{ANN}_6(\mathcal{I}_t) + \xi_{U,t} - \xi_{L,t} - \psi\mathbb{B}\Big(\mathcal{ANN}_7(\mathcal{I}_t), \ldots, \mathcal{ANN}_{11}(\mathcal{I}_t)\Big) \\
&+ \mathbb{C}\Big(\mathcal{ANN}_{12}(\mathcal{I}_t), \mathcal{ANN}_{13}(\mathcal{I}_t), \mathcal{ANN}_{14}(\mathcal{I}_t)\Big)
\end{aligned}$$

$\ldots$

$$\mathcal{ANN}_{17}(\mathcal{I}_t) = \frac{1}{\pi}U_{1,t}\Big(\frac{\pi_t}{\pi}\Big)^{-\phi_\pi}$$

$$\begin{aligned}
&\frac{1}{\nu}(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2)(\nu - 1 + w_t/A) - \varphi(\pi_t - \pi)\pi_t - \phi_1\pi_t + \\
&\frac{\beta\psi}{U_{1,t}}\mathcal{ANN}_{18}(\mathcal{I}_t) + \frac{\beta(1 - b_t^N)}{U_{1,t}}\mathcal{ANN}_{12}(\mathcal{I}_t) - \psi(b_t^N - b_{t-1}^N)\mathcal{ANN}_{13}(\mathcal{I}_t) = 0
\end{aligned}$$

Note that μ_t is now over identified. We tackle this problem by using the Forward-States approach as described in Faraglia et al. (2014). This involves approximating the expected value terms with the state variables that are relevant at period $t + 1$ and invoking the law of iterated expectations.[8]

[7] We also find that including ξ terms explicitly in the training set improves prediction accuracy.

[8] For a detailed description of the procedure using polynomial regressions see Faraglia et al. (2019) or Faraglia et al. (2014). Here we follow the same logic using the neural network.

The equations to solve are:

$$\mu_t = [\mathbb{E}_t\mathcal{ANN}_1(\mathcal{I}_{t+1})]^{-1}\left[\mathbb{E}_t\mathcal{ANN}_2(\mathcal{I}_{t+1}) + \frac{\xi_{U,t}}{\beta} - \frac{\xi_{L,t}}{\beta}\right]$$

$$\mu_t\beta E_t\mathcal{ANN}_3(\mathcal{I}_{t+1}) + \psi\mu_t E_t\mathbb{A}\Big(\mathcal{ANN}_4(\mathcal{I}_{t+1}), \mathcal{ANN}_5(\mathcal{I}_{t+1})\Big)$$
$$= \beta^N\mathbb{E}_t\mathcal{ANN}_6(\mathcal{I}_{t+1}) + \xi_{U,t} - \xi_{L,t} - \psi\mathbb{E}_t\mathbb{B}\Big(\mathcal{ANN}_7(\mathcal{I}_{t+1}), \ldots, \mathcal{ANN}_{11}(\mathcal{I}_{t+1})\Big)$$
$$+ E_t\mathbb{C}\Big(\mathcal{ANN}_{12}(\mathcal{I}_{t+1}), \mathcal{ANN}_{13}(\mathcal{I}_{t+1}), \mathcal{ANN}_{14}(\mathcal{I}_{t+1})\Big)$$

...

$$\mathcal{ANN}_{17}(\mathcal{I}_{t+1}) = \frac{1}{\pi}U_{1,t}\Big(\frac{\pi_t}{\pi}\Big)^{-\phi_\pi}$$

$$\frac{1}{\nu}(c_t + g_t + \frac{\varphi}{2}(\pi_t - \pi)^2 + \phi_1(\pi_t - \pi) + \phi_2)(\nu - 1 + w_t/A) - \varphi(\pi_t - \pi)\pi_t - \phi_1\pi_t +$$
$$\frac{\beta\psi}{U_{1,t}}\mathcal{ANN}_{18}(\mathcal{I}_{t+1}) + \frac{\beta(1-b_t^N)}{U_{1,t}}\mathcal{ANN}_{12}(\mathcal{I}_{t+1}) - \psi(b_t^N - b_{t-1}^N)\mathcal{ANN}_{13}(\mathcal{I}_{t+1}) = 0$$

2. If the solution error is large, or a reliable solution could not be found, the algorithm automatically restores the previous period ANN and tries to proceed with a reduced Maliar bound.[9]

3. If the solution calculated shrinking the bound at iteration $i-1$ is not satisfactory, the algorithm does not go back another iteration but uses the same ANN and tries to lower the $Bound_{i-1}$ again towards $Bound_{i-2}$. Once a reliable solution is found, the algorithm proceeds to calculate the solution for iteration i again, but with $Bound_i = Bound_{i-1} + (Bound_{i-1} - Bound_{i-2})$. In this way, if an error is detected multiple times we guarantee that both $Bound_i$ and $Bound_{i-1}$ keep shrinking toward $Bound_{i-2}$ and there must exist a point close enough to $Bound_{i-2}$ such that the system can be reliably solved with both $Bound_{i-1}$ and $Bound_i$.

4. If the solution found at iteration i is satisfactory, the ANN enters the learning phase supervised by the implied model dynamics, the Maliar bounds are increased and a new iteration starts again.

Keep repeating until the ANN prediction errors converge below a certain small threshold and the simulated sequences of $c_t, \mu_t, B_t^N, b_t^N, \pi_t, \lambda_t^T, \lambda_t^\pi$, and w_t do not change.

[9] If the unreliable solution has been detected in iteration i the algorithm restore the $i-1$ environment and tries to proceed with $Bound_{i-1} = \alpha Bound_{i-1} + (1-\alpha)Bound_{i-2}$.

Parameter	Value
Hidden layers	1
Neurons	10
Activation function	Hyperbolic tangent sigmoid
Training algorithm	Levenberg-Marquardt backpropagation
Blending Factor (μ)	0.01
μ Decrease factor	0.01
μ Increase factor	10
Max num. epochs	1000

Table 4: ANN structure and parameters

Notes: Table reports specification of the neural network.

6.4 Robustness

6.4.1 Changing the Seed

To see how our results depend on the specific realization of the g_t process we solve the model with 20 different seeds using the same staring point as in the main body of the paper. Overall, the main result is robust. Correlation between real and nominal bonds is on average -0.7904 and is negative for all realizations of g_t. Correlation between the difference of B^N and b^N is also negative on average and is only positive in two realizations. We also find that government issues nominal debt and holds real assets most of the time. The mean difference between B^N and b^N is 34.01% of GDP and has been on average negative for only one realizations. The results are summarized in table 5.

	$\rho(b_t^N, B_t^N)$	$\rho(B_t^N - b_t^N, g_t)$	$\mathbb{E}(b_t^N/Y_t)$	$\mathbb{E}(B_t^N/Y_t)$	$\mathbb{E}((B_t^N - b_t^N)/Y_t)$
Mean	-0.7904	-0.3733	-0.1465	0.1936	0.3401
Minimum	-0.9698	-0.8164	-0.3433	-0.2153	-0.0667
Maximum	-0.1315	0.5964	-0.0275	0.6289	0.697

Table 5: Average moments across multiple realizations of g_t

Notes: Table shows the mean, minimum and maximum of selected moments when the model is solved with using different realizations of g_t.

6.4.2 Variance of g_t Process

In this subsection we analyze how the results depend on the variance of government expenditure. Specifically, we solve the model with the same seed but changing the variance of the shock process. We mainly find that the main result of accumulating nominal debt and real assets in good times is stronger when the government expenditure is more volatile. As shown in figure 9, the correlation between nominal bonds and g_t and the correlation between real bonds and g_t increases in absolute value as g_t becomes more volatile. Also, the government debt position becomes more levaraged as shown in the right panel.

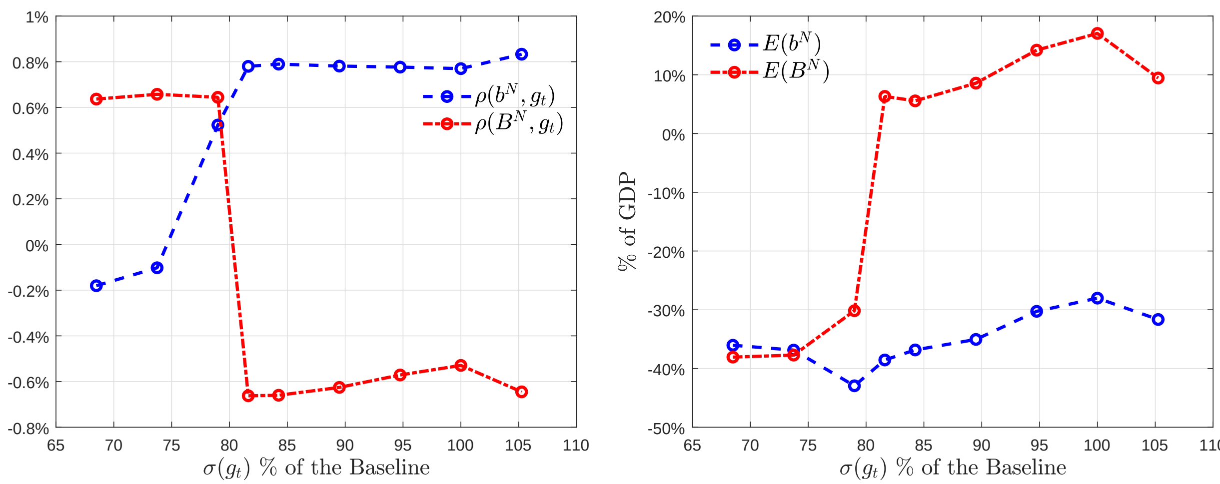

Figure 9: Role of variance of g_t

Notes: Figure shows correlation of real and nominal bonds of g_t and average values of real and nominal bonds in function of the variance of g_t.

In addition to above, we find that 1. volatility of inflation is invariant and volatility of taxes increases in variance of g_t. 2. Correlation of total portfolio and g_t and the correlation between nominal and real bonds are stable. 3. Average inflation increases.