Can exploding offers beat open offers ?

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Abstract

Exploding offers have often been considered as a tool to allow mediocre participants in a two sided matching market to leverage incomplete information to get a better partner than what they would have in a perfect information setup. The literature on matching market with exploding offers had so far an extensive focus on incomplete information regarding the quality and the distribution of the participants in one or both sides of a matching market. Participants in both sides of the market face a dilemma: send/accept an exploding offer early using imperfect information or wait to acquire more information but run the risk of being poached.

This paper explores the positive side of exploding offers. I present a multi-period two sided matching market with perfect information about the quality of the participants. Exploding offers are used as a way to streamline the process and spread the workload over multiple periods. I show that the use of exploding offers with sufficiently long duration can yield an allocation identical to the use of open offers and allow all participants to reach the maximum ex-ante utility.

Keywords: two sided matching, exploding offers, multiple period model

Introduction

Matching markets are widespread and the literature associated with them is quite broad. However, most of the focus has been placed on algorithmic resolution of the matching problem. Deferred acceptance Gale and Shapley (1962) and top trading cycles Shapley, Lloyd; Scarf, Herbert (1974) [6] being some of the most well known algorithms. In the majority of the papers, it is assumed that either the market or some planner acquires the relevant information from the players in the matching game and then provides an allocation for everyone at the same time. Players receive an unconditional offer or they don't receive anything at all.

Yet, in the real world there are many decentralised matching markets (consulting firms filling multiple vacancies, universities recruiting master and PhD students for example) that operate using exploding offers. Exploding offers are offers with a time limit. A player in a multi-period matching market has only a limited number of period to formally accept an exploding offer and exit the game. Past a set number of periods, the offer is considered rejected. The exploding offers introduce a new problem for economists: they may unravel the market and force the players to issue sub-optimal offers early in the game instead of waiting for the last period to achieve the most efficient matching.

Niederle and Roth (2009) [3] have identified some markets at risk of unraveling that use exploding offers. A well know example is the market for Gastroenterology fellowship [2] in the US. Most of these market involve applicants whose quality in uncertain but can be discovered if given time. Pan (2018) [4] focused on two-period matching games with imperfect information where the quality of players of at least one side is unknown. Moreover, one side of the market (firms for example) can make strategic decision on whether to issue exploding offers earlier or later.

Yet these are not the only type of markets that feature exploding offers. The German DoSV (Grenet et al. 2019) [1] for matching high school graduates to universities is a multi-period process that is partially decentralised and has exploding offers. Large consulting corporations (Accenture, BCG, PwC etc.) use exploding offers as a way to streamline their recruitment. In these cases the quality of the applicant is easy to assess but logistical constraints still make the use of exploding offers a necessity to prevent congestion and the formation of long waiting queues. Does the use of exploding offers affect the quality of the matching market and who is impacted? Does the length of an exploding offer has an impact as well?

In the present paper I will present a model to study the impact of exploding offers where all players know in advance their qualities and the firm/university side of the market is bottleneck-ed by the capacity of the latter. Firm/universities will not be able to issue early exploding offers strategically because of serialised treatment of applicants similar to Roth and Xing (1997) [5]. It is a many-to-one matching model where the are two types of students to allocate to two universities. Unlike previous literature, our model can have any number of periods and accommodates the use of exploding offers of any length. I show that the use of exploding offers with long duration can help streamline the recruitment process without leading to a loss of utility compared to an equilibrium with an open offer.

I find that, assuming students are well informed about their quality, exploding offers length has little to no impact on the composition of the top quality form/university. Moreover, long lasting exploding offers benefit high quality applicants and lower quality firm/university while harming lower quality applicants, leading to a more positive associating matching.

The rest of the paper will be organised as follows. Section 1 will present the the model. Section 2 is a benchmark case where the universities can only send open offers. Section 3 presents the equilibrium solution with exploding offers of any length while section 4 analyses the welfare of applicants and firms/universities. Finally, section 5 extends the base model by introducing heterogeneous preferences for the high quality students.

For the rest of the paper I will use a student/university terminology but one could apply the model to an applicant/firm setup as well.

1 The model

A population of students is to be matched to three universities A (very desirable), B (less desirable) and C (undesirable) through a multi-period procedure. There are two types of students called α (high quality) and β (low quality).

There are N_α students of type α and N_β students of type β in total where $N_\alpha, N_\beta \in \mathbb{N}$ and $N_\alpha \leq N_\beta$. Let $f = \frac{N_\alpha}{N_\alpha+N_\beta}$ be the ratio of students of type α inside the total student population. By construction $f \leq 1/2$. Students are assumed to know their type. The value of the fraction f is common knowledge.

Universities A and B have limited capacities (respectively c_A and c_B) but university C is considered so large it can accommodate all the students regardless of their type.

Matching procedure in detail : All students send their dossiers to apply to all universities. In period 0 university C presents all students with an unconditional offer that never expires. C should be considered an outside option that is always available as a last resort.

To streamline their recruitment process, both universities A and B will spread the processing of all the students' dossiers they receive over multiple periods. Let $T \geq 2$ be the number of periods needed by the universities to process all the received dossiers.

Because the procedure is done in a finite number of periods and universities A and B process dossiers independently we can divide the population of type α and β students into T^2 different states $(i, j) \in \{1; \ldots; T\}^2$. i is the period when the student will be contacted by university A and j is the period when the student will be contacted by university B. For example, a student of type α in state $(3; 1)$ will have his dossier processed by university A in period 3 and processed from university B in period 1. There are exactly $n_\alpha = N_\alpha / T^2$[1] (resp. $n_\beta = N_\beta / T^2$) students of type α (resp. β) in a single state (i, j). Students have no way before the game starts to determinate the state they will find themselves in. Throughout the procedure, students will discover the state they are in by receiving answers from both universities.

At the beginning of each period t university A processes all the dossiers of α students in the states $(t; x)$ $\forall x \in \{1; \ldots; T\}$, and then processes all the dossiers of the β students in the same states once it has received the answers of the

[1] Removing this divisibility assumption would bring technical complications without improving the paper's message

α students whose dossiers has been processed. A university always knows the type (α or β) of each student it interacts with but cannot discriminate between the different states $(t; x)$ $\forall x \in \{1; \ldots; T\}$. In other words you always know the quality of every single applicant but you do not know how an individual applicant interacted with the competing university.

At the beginning of each period t, university B processes all the dossiers in states $(y; t)$ $\forall y \in \{1; \ldots; T\}$ in a similar fashion (α first then β). Like university A, university B always knows the type of each student it interacts with but cannot discriminate between the states $(y; t)$ $\forall y \in \{1; \ldots; T\}$.

Players' payoff The utility function of all universities is the same and is common knowledge. It depends on the type of student they are matched with at the end of the procedure:

$$\mathcal{U}_A = \mathcal{U}_B = \mathcal{U}_C = \sum \text{Students of type } \beta + V_\alpha \sum \text{Students of type } \alpha$$

Where $V_\alpha > 1$ is the premium utility universities get by hiring α students. The utility function $\mathcal{U}_s$ of all the students is the same. It depends on the university the student is matched with at the end of the procedure:

$$\mathcal{U}_s = \begin{cases} 0 \text{ if matched with } C \\ 1 \text{ if matched with } B \\ V_A > 1 \text{ if matched with } A \end{cases}$$

Type of students and capacity constraints : To avoid trivial cases, restrictions on the capacity of both universities A and B will be placed and link these capacity constraints to the number of students of both types. The capacity of university A is such that $(T-1)Tn_\alpha < c_A < N_\alpha$. The upper bound on the capacity implies that not all students of type α will be able to enroll in university A (the desirable one). The capacity of university B is such that $T(T-1)n_\beta + Tn_\alpha \leq c_B \leq N_\beta$[2]. The lower bound ensures the matching procedures detailed below will not be interrupted early and the upper bound eliminate trivial equilibria where each student of type α and β has a guaranteed place in either A or B. The capacity constraint of both universities are common knowledge.

Actions of the universities. After processing the dossier of a student the university learns its type and can choose to either:

- Present the student with an unconditional but exploding offer of duration $d \geq 0$. The notation for playing this strategy is O_t^A for university A (resp. O_t^B for university B) and $t \in \{1; \ldots; T\}$ (resp. $t \in \{1; \ldots; T\}$) is the time period when the offer is issued.
- Reject the student. The notation for playing this strategy is N_t^A for university A (resp. N_t^B for university B) and $t \in \{1; \ldots; T\}$ (resp. $t \in \{1; \ldots; T\}$) is the time period when the student is notified of his/her rejection.

The parameter $d \geq 0$ is exogenous and common knowledge. If $d \geq T-1$ the offers will be called "opened" as they cannot expire before the end of the procedure.

Actions of the students. At each period a student can thus receive a response from either university A or B or both of them or none of them. As soon as a student receives one offer or more, the student can:

- Accept one of the offers (s)he received, "enroll" in the corresponding uni-

[2]The inequality always holds since $n_\alpha \leq n_\beta$

versity and exit the procedure. The notation for playing this strategy is E_t^X where $X \in \{A; B\}$ is the university whose offer has been accepted and $t \in \{1; \ldots; T\}$ is the time period when the offer is issued.

- Wait an extra period to see if a better offer comes up later. The notation for playing this strategy is W_t where $1 \leq t \leq T$ is the time period when the student decides to wait.

At the end of period T, all offers that have not been accepted automatically expire. Students with no offer from A or B automatically enrolls in C at the end of period T. The students cannot observe the interactions between universities and other students throughout the matching procedure, nor can they observe the number of available spots left in any university during the procedure. Universities always know the type (α or β) of students they are interacting with. However, each university has no way of knowing the interactions of a given student with the competing university. When a student exits the procedure, all universities are informed.

Strategic restrictions A few guiding principles restrict the strategies of the players of this game.

- Students of the same type cannot be distinguished from one another. If a university at a given period has more dossiers of the same type to process than it has available capacity, then the university must sends offers randomly to the students of said type[3].

- No backtracking : A university cannot renege an offer made to a student

[3] Example : University A has 4 seats available in period T-1 but has 7 dossiers of type α and 8 dossiers of type β processed. University A will send an offer to 4 α students randomly chosen among the 7. All dossiers processed in the following period will be automatically rejected.

nor transform a rejection into an admission. Students who reject an offer or let an exploding offer go cannot re-apply nor enroll in the university they rejected.

- All offers have to be honored : A university cannot send offers to more students than it has available capacity.

Extensive representation : To give the reader a visual representation of the game played by students, an extensive representation of a case where $T = 3$, $N_\alpha = 4$ and $N_\beta = 5$ is shown below . In this specific case, there are 9 different states, thus each student will play one game out of nine possible different games. Three of these games are represented below. To help the reader get a better grasp of the timing when the actions are played. **Payoff notation:**

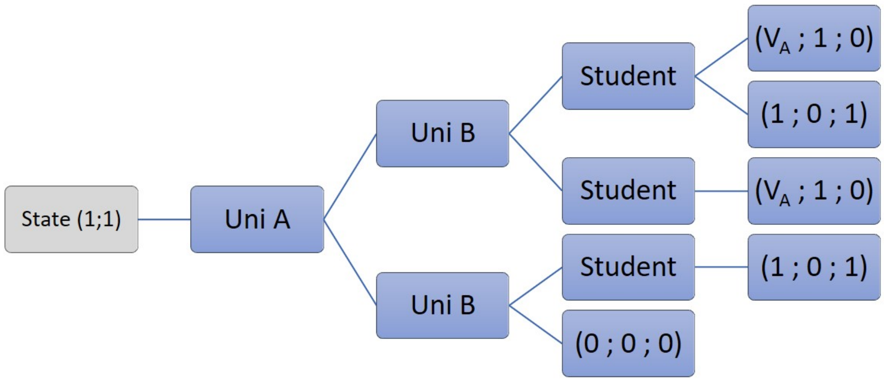
- The cell at the end of each branch is a payoff and is noted as $(x; y; z)$.
- The first number $x \in \{0; 1; V_A\}$ is the student's payoff. The value depends on the university the student enrols in.
- The second number $y \in \{0; 1; V_\alpha\}$ is the extra payoff university A get from the specific student. If the student enrols in university B or C then $y = 0$. If the student enrols in A and is of type α, $y = V_\alpha$. Finally if the student enrols in A and is of type β then $y = 1$.
- The third number $z \in \{0; 1; V_\alpha\}$ is the extra payoff university B get from the specific student. If the student enrols in university A or C then $z = 0$. If the student enrols in B and is of type α, $z = V_\alpha$. Finally if the student enrols in B and is of type β then $z = 1$.

In period 1, university A will play the game $(1; 1)$ 4 times with a different α student each time and 5 times with a different β student each time. The same will happen with games $(1;2)$ and $(1;3)$. A is not able to to differentiate between the games $(1;1)$, $(1;2)$ and $(1;3)$ (A does not know the interactions between a specific student and university B) but can perfectly discriminate games played with an α from games played with a β.

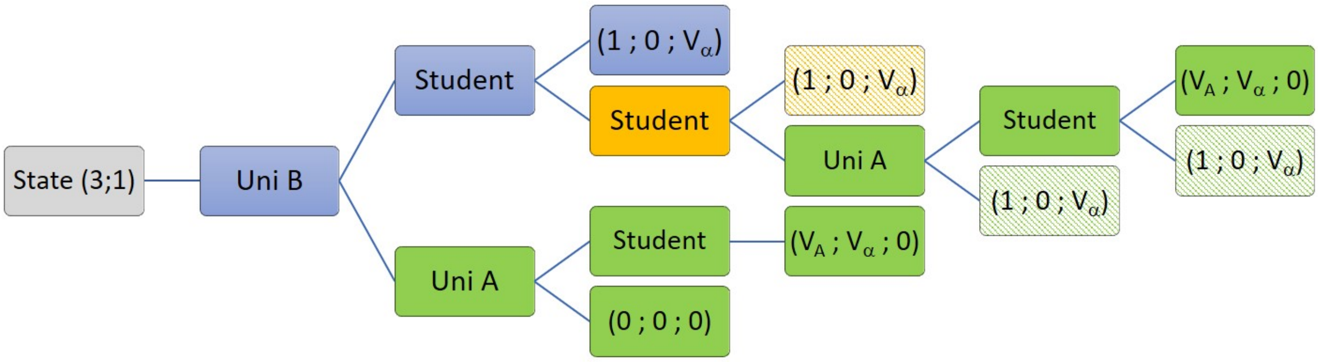
Respectively, university B will play the game $(1;1)$ 4 times with an α student and 5 times with a β student. The same will happen with games $(2;1)$ and $(3;1)$. B is not able to to differentiate between the games $(1;1)$, $(2;1)$ and $(3;1)$ but can perfectly discriminate games played with an α from games played with a β.

In period 2, university A will play the game $(2;1)$ 4 times with an α student and 5 times with a β student. The same will happen with games $(2;2)$ and $(2;3)$. A is not able to to differentiate between the games $(2;1)$, $(2;2)$ and $(2;3)$ but still can perfectly discriminate games played with an α from games played with a β. Moreover, A will automatically know if a student in game $(2;1)$ has enrolled in B and exited the matching procedure. Extend the reasoning to other periods and universities.

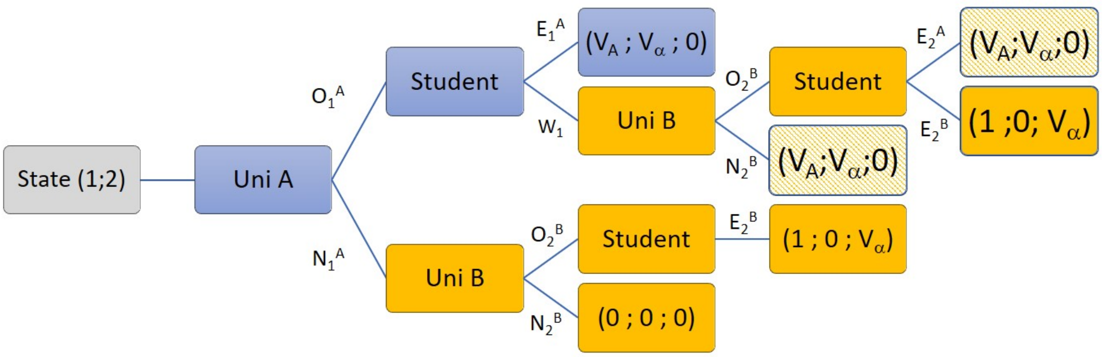
In the trees below, blue cells are played in period 1, orange cells in period 2, green cells in period 3. Cells with a hatching patters **may** be rendered unavailable (i.e replaced by a payoff of $(0;0;0)$) if the duration of the exploding offer is short enough.



(a) The game played by a β student in state (1;1)



(b) The game played by an α student in state (3;1)



(c) The game played by an α student in state (1;2)

Figure 1: The game played by students depending on their state and type

Specific notations

Receiving an offer from A at the last period. Because this specific event will come up very often when solving for an equilibrium it deserves a special notation. Let Ω_A be the event *"A student of type α receives an offer from university A in period 3"*.

Letting an exploding offers expire. Some students may have an incentive to let an exploding offer expire and remain in the matching procedure, hoping to receive an offer from a better university. Given the students' utility function, only an exploding offer from B can be realistically let go. The event "letting an exploding offer from B expire at time t" will be noted W_t^*.

Welfare The aggregated utility of the whole student population will be noted W_s. The aggregated utility of universities A and B will be noted W_u.

Unraveling In the past literature on exploding offers, unraveling happens when firms issue exploding offers early instead of waiting to get the complete information about applicants. In this model, universities cannot strategically time their offers. However, α students can choose to enroll in the less desirable university B early instead of waiting for an answer from A. The matching procedure will "unravel at period t" if such an outcome happens at period t.

Equilibrium concept

Throughout the paper we will be looking for the behaviour of α students in a sub-game perfect Nash equilibrium. The equilibrium exists since the game has a finite number of players each having a finite number of strategies.

2 Equilibrium solution and properties

Before comparing the impact of exploding offers duration let us start with a very general result that will be used as a bedrock for the following ones:

Theorem 1. *If the students know their type perfectly, then university A will never in any equilibrium play O_t^A when encountering the dossier of a student of type β unless the number of dossier of α students left to process is lower than the available capacity of university A.*

This result ensures that University A will not send an offer to a student of type β unless it has no other choice. This will not only restrict the set of equilibria but also allow us to write down the expected payoff of α students in a way that is easy to manipulate. The proof is found in A.1.

The equilibrium with $d \geq T-1$ (open offers): In the case where d is large enough such that the offers can be considered opened, the equilibrium of the game is very straightforward.

- Every period $i \in \{1; \ldots ; T\}$, when processing the dossier of a type α student, university A will play O_i^A as long as it is not at full capacity. Once A is full, A only plays N_i^A. As long as A has not treated all the dossiers of type α students, it will play N_i^A when processing the dossier of a type β student, and play O^A once all the α students are treated.
- Each period, University B plays O_i^B for all students regardless of type if not already full. Afterwards, B plays N_i^B.
- When receiving offers from B in period i, students of type α play W_t $t \geq i$ until they have the opportunity to play O_i^A. At the last period, the students play E_3^B if they have not received an offer from A.

- Students of type β play E^B.

The proof is found in A.2.

Theorem 2. *The only equilibrium that maximises W_s and W_u for all values of V_A and c_A is the one where universities issue open offers.*

This simple result allows the use of the equilibrium with open offers as a welfare benchmark. Open offers allows the market to correctly allocate the maximum number of type α students to the best university (A) and give the second best option to the α students who could not fit into A. The proof is found in A.3.

3 Exploding offers and student behaviour

In this section, universities will only issue exploding offers with duration $d < T - 1$ (a.k.a true exploding offers). Any equilibrium has the following form:

Theorem 3. *If a strategy profile $\mathcal{S}$ is an equilibrium of the model with exploding offers then there exist only one $T^* \in \{1; \ldots; T\}$ such that :*

- *If $t < T^*$ then all α students who face the choice between playing E_t^B and W_t^* will play W_t^*.*
- *If $t > T^*$ then all α students who face the choice between playing E_t^B and W_t^* will play E_t^B.*
- *α students may only mix between E_t^B and W_t^* if and only if $t = T^*$*

Moreover at the equilibrium all students play E_t^A as soon as they have the opportunity to do so. All β students that received an offer from university B will play W_t unless they are facing the choice between W_t^* and E_t^B. In this case they play E_t^B.

University A plays O_t^A whenever it encounters an α student or if it encounters a β student and has more capacity left than there are dossiers of α students left to process. University B plays O_t^B all the time. The proof is found in B.1.

Theorem 4. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two equilibrium strategy profiles. Then $T^* = T^{*\prime}$*

This simple result is a key step in proving the equilibrium uniqueness. The proof can be found in B.2

Theorem 5. *The equilibrium of the game is unique. Moreover, if $V_A \geq 2$, then $T^* = T$ or $t^* = T - 1$*

This theorem wraps up section 3. The equilibrium uniqueness allows comparative statics on both students and universities welfare. The proof can be found in B.3.

3.1 Properties of the solution

Theorem 6. *$T^* = T$ and there is no unraveling if and only if $V_A > \frac{T^2 n_\alpha - C_A}{T n_\alpha}$.*

Proof: $\mathbb{P}(\Omega_A)$ is bounded from below by the lowest probability for an α student to get an offer from A. The lowest probability is reached when the number of open seat for A is minimized and the number of applicants for these seats is maximised. This probability is equal to $\frac{T^2 n_\alpha - C_A}{T n_\alpha}$. If $V_A < \frac{T^2 n_\alpha - C_A}{T n_\alpha}$ then $\mathcal{U}(E^B_{T-1}) = 1 > \mathbb{E}(W^*_{T-1})$ so there is an incentive to at least randomise between enrolling in B or gambling for a seat in A.

Theorem 7. *If $T > 6$ then the game cannot fully unravel.*

This theorem shows that one does not need an asymmetry of information about the quality of an applicant to have the market unravel. The proof can be found in B.4.

Theorem 8. *Let $\Delta C_A = T^2 n_\alpha - C_A$ be the difference between the capacity of A*

and the total number of α students. Let T be the number of period of the matching game. The difference between the critical period T^* and T is constrained by:

$$(T - T^*) \leq Min\left\{\frac{1 + \sqrt{1 + 8\Delta C_A/n_\alpha}}{2}; T\right\}$$

Theorem 8 is a complement to theorem 7 that acts as an empirical check to see if the initial hypothesis holds. Given the capacity constraint of university A one can find a maximum difference between the critical period T^* and the maximum number of period T. If an α students plays E_t^B when $t < T^*$ then either the student is acting irrationally or the student does not know his/her type. The proof can be found in B.5

4 Student and university welfare

Since the equilibrium of the game is unique and follows a specific structure we can analyse the ex-ante welfare of both universities and both groups of students.

Theorem 9. *The ex-post utility of university B is bounded from below by C_B. The ex-post utility of university A is bounded from below by $2C_A - (T - T^*)n_\alpha$. Both universities are always filled to capacity.*

This result is quite straightforward and shows there is always a modicum of positive assortative matching even with exploding offers. The worst case scenario for B happens when all the α students who let their offer from B expire fail to get an offer from A. In this case A is filled to capacity with αs thus B can recruit from the entire pool of β students.

The worst case scenario for A is an extreme event where all the α students who could randomise end up enrolling in B. The welfare of B is maximised. A still has access to the pool of β students to fill its remaining seats.

Theorem 10. *Let $\mathcal{S}$ be the equilibrium of a game with parameters T, c_A, c_B, V_A, n_α, n_β and an exploding offer of duration $d = 0$. Let T^* be the critical period associated with equilibrium $\mathcal{S}$.*

Let $\mathcal{S}'$ be the equilibrium of the game with parameters T, c_A, c_B, V_A, n_α, n_β and an exploding offer of duration $d' \geq 1$. Let T'^* be the critical period associated with the equilibrium $\mathcal{S}'$.

If $d' \leq T^* - 1$ then $T^* = T'^*$ and if there is a randomisation, $p = p'$.

If $d' > T^* - 1$ then $T'^* = d$ and the equilibrium must be played in pure strategies.

The proof can be found in C.1. Below is a graphical illustration for the case when $T = 7$. We will assume w.l.o.g that $T^* = 4$

(1;7)	(2;7)	(3;7)	(4;7)	(5;7)	(6;7)	(7;7)
(1;6)	(2;6)	(3;6)	(4;6)	(5;6)	(6;6)	(7;6)
(1;5)	(2;5)	(3;5)	(4;5)	(5;5)	(6;5)	(7;5)
(1;4)	(2;4)	(3;4)	(4;4)	(5;4)	(6;4)	(7;4)
(1;3)	(2;3)	(3;3)	(4;3)	(5;3)	(6;3)	(7;3)
(1;2)	(2;2)	(3;2)	(4;2)	(5;2)	(6;2)	(7;2)
(1;1)	(2;1)	(3;1)	(4;1)	(5;1)	(6;1)	(7;1)

Figure 2: Base equilibrium $\mathcal{S}$ with $d = 0$

In the graph above, the α students in the green cell (state $(7;7)$) will compete for a seat in A for sure as their offer from B has not expired. α students in the yellow cells enroll in B just before the offer expires. Students in the blue cell may randomise with probability p in period 4. If the length of the exploding offer increases to 1, the illustration changes to the one below:

(1;7)	(2;7)	(3;7)	(4;7)	(5;7)	(6;7)	(7;7)
(1;6)	(2;6)	(3;6)	(4;6)	(5;6)	(6;6)	(7;6)
(1;5)	(2;5)	(3;5)	(4;5)	(5;5)	(6;5)	(7;5)
(1;4)	(2;4)	(3;4)	(4;4)	(5;4)	(6;4)	(7;4)
(1;3)	(2;3)	(3;3)	(4;3)	(5;3)	(6;3)	(7;3)
(1;2)	(2;2)	(3;2)	(4;2)	(5;2)	(6;2)	(7;2)
(1;1)	(2;1)	(3;1)	(4;1)	(5;1)	(6;1)	(7;1)

Figure 3: New equilibrium $\mathcal{S}'$ with $d = 1$

In this case the critical period is the same but students in a different state will randomise. Graphically, one row of students in red (let their offer from B expire to compete for a seat in A) became a row of green (compete for a seat in A while the offer from B is still up). Below is an other illustration when $d = 3 = T^* - 1$:

(1;7)	(2;7)	(3;7)	(4;7)	(5;7)	(6;7)	(7;7)
(1;6)	(2;6)	(3;6)	(4;6)	(5;6)	(6;6)	(7;6)
(1;5)	(2;5)	(3;5)	(4;5)	(5;5)	(6;5)	(7;5)
(1;4)	(2;4)	(3;4)	(4;4)	(5;4)	(6;4)	(7;4)
(1;3)	(2;3)	(3;3)	(4;3)	(5;3)	(6;3)	(7;3)
(1;2)	(2;2)	(3;2)	(4;2)	(5;2)	(6;2)	(7;2)
(1;1)	(2;1)	(3;1)	(4;1)	(5;1)	(6;1)	(7;1)

Figure 4: Base equilibrium $\mathcal{S}''$ with $d = 3$

If the duration of the exploding offers exceeds the critical period, the row of students who could randomise is converted into a row of students who still benefit from an offer from B. The rest of the students who will still have a choice to make between W^* and E^B will systematically pick the latter: the competition for the remaining seats in A is too fierce.

(1;7)	(2;7)	(3;7)	(4;7)	(5;7)	(6;7)	(7;7)
(1;6)	(2;6)	(3;6)	(4;6)	(5;6)	(6;6)	(7;6)
(1;5)	(2;5)	(3;5)	(4;5)	(5;5)	(6;5)	(7;5)
(1;4)	(2;4)	(3;4)	(4;4)	(5;4)	(6;4)	(7;4)
(1;3)	(2;3)	(3;3)	(4;3)	(5;3)	(6;3)	(7;3)
(1;2)	(2;2)	(3;2)	(4;2)	(5;2)	(6;2)	(7;2)
(1;1)	(2;1)	(3;1)	(4;1)	(5;1)	(6;1)	(7;1)

Figure 5: Base equilibrium $\mathcal{S}'''$ with $d = 4$

Theorem 10 demonstrate how small of an impact long lasting exploding offers have on the welfare of both students and universities. The only significant change in terms of welfare happens when d goes from 3 to 4: the welfare of A, B and α students is maximised and the welfare of β is now minimised.

5 Exploding offers with heterogeneous preferences

Now that we understand the behaviour of a model featuring students with homogeneous preferences let us see what happens when heterogeneous preferences are introduced. In this section we will modify the base model by splitting up the α students into q subgroups α_1 to α_q. There is a total of $T^2 n_{\alpha_q}$ students of type α_q for all q such that $\sum_{i=1}^{q} n_{\alpha_i} \leq n_\beta$. The students of type α_i $\forall i$ are identical to each other in every way except their preferences. The utility of α_i students is the following :

$$\forall\ i\ \mathcal{U}_{\alpha_i} = \begin{cases} 0 \text{ if matched with } C \\ 1 \text{ if matched with } B \\ V_{\alpha_i} > 1 \text{ if matched with } A \end{cases}$$

The utility of being unmatched is -∞ like everyone else. The valuations V_{α_i} are such that $1 < V_{\alpha_1} < V_{\alpha_2} < \ldots < V_{\alpha_q}$. The β students have a valuation $V_\beta = V_{\alpha_1}$

Universities cannot distinguish between any of the subgroups of α students before, during or after the recruitment process and receive the same utility from enrolling any of them. The capacity constraint of university A is now $(T-1)T(\sum_{i=1}^{q} n_{\alpha_i}) < c_A < T^2(\sum_{i=1}^{q} n_{\alpha_i})$ while the capacity constraint of university B is $T(T-1)n_\beta + T(n_\alpha + n_\gamma) \leq c_B \leq T^2 n_\beta$[4].

The matching procedure is mostly unchanged. At the beginning of each period university A processes $T(\sum_{i=1}^{q} n_{\alpha_i} + n_\beta)$ dossiers and university B processes $T(\sum_{i=1}^{q} n_{\alpha_i} + n_\beta)$ dossiers of students of type α_i and β.

Theorem 11. *If a strategy profile $\mathcal{S}$ is an equilibrium of the model with exploding offers then there exist q critical periods $T_1^* \leq T_2^* \leq \ldots \leq T_q$ such that $\forall i \in \{1; \ldots; q\}$:*

- $T_i^* \in \{1; \ldots; T\}$
- *If $t < T_i^*$ then all α_i students who face the choice between playing E_t^B and W_t^* will play W_t^*.*
- *If $t > T_i^*$ then all α_i students who face the choice between playing E_t^B and W_t^* will play E_t^B.*
- *α_i students may only mix between E_t^B and W_t^* if and only if $t = T_i^*$. If they don't mix they play W_t^**

Only one subgroup α_i of students may play mixed strategies in $\mathcal{S}$.

The proof can be found in D.1

[4]This always holds since $n_\alpha + n_\gamma \leq n_\beta$

Corollary: Eliciting relative preferences of students If one sees two α students and one plays E_t^B while the other plays W_{t+i}^* $i > 0$ then they belong to two different subgroups and the former has a lower valuation V_α than the latter.

This result enables a researcher to partially extract the relative valuation of α students by looking at their behaviour when facing the choice between letting an exploding offer go or taking it. It can be crossed referenced with a survey before or after the game has occurred to assess if the student has a rational behaviour.

6 Concluding remarks

In this paper I have presented a multi-period model to assess the impact of exploding offers of different length on the welfare of the players in a two sided matching market with complete information about the quality of the players.

I conclude that there provided the students do not value the high quality university too much compared to the low quality one, using exploding offers with sufficiently long durations will result in the same outcome as open offers while still allowing universities to spread the workload. If the exploding offers are too short, the low quality universities and high quality applicants will be the biggest losers.
The equilibrium of such a model has a precise structure that makes it easy to identify. The incentives driving the equilibrium are similar to a Stackelberg oligopoly where the players who tie their hands early are able to keep some of the competition at bay. The model suggests that exploding offers with short duration only marginally increase the welfare of high quality applicants and low quality firms/universities at the detriment of the low quality applicants.

Finally, introducing heterogeneous preferences to the model can enable economists to partially elicit the relative preferences of the high quality applicants based on the period at which they decide to go with the low quality firm/university.

A Proofs for section 2

A.1 Proof of theorem 1

The proof will be by contradiction. Let us assume without loss of generality that there exists an equilibrium in which university A plays at least once O_t^A when facing a β student with probability one. We will show that an α student can profitably deviate to take the place of a β student.

1. University A is certain to get all the α students who are in state $(x; y)$ such that $y \in \{1; \ldots; T\}$ and $x+d \leq y$ where d is the duration of an exploding offer is there is one[5]. All the students (α or β) in these states will be contacted by university A before or at the same time as university B.

2. The number of students in these specific states is $\sum_{t=1}^{t=T} \text{Max}\{(t+d); T\}q > (1+T) * T/2 * 2n_\alpha > T^2 n_\alpha > c_A$. Thus University A is ensure to be filled to capacity with students. University A will have spare students to send offers to in the final period T.

3. Because we are in a trembling hand setup, university A weakly prefers an equilibrium in which the capacity is filled as late as possible to equilibria in which capacity is filled earlier. This allows to "catch" any α student deviating and applying to A instead of accepting an offer from B. As such in no equilibrium will university A be filled before period T.

4. Type α students know that A will always play O^A when encountering them unless the university is already full. As shown earlier, A is never full before the last period. Because the game is structured in such a way that β students are always processed after the α, an α student who deviates will always receive an offer from A.

[5] Recall : $d = +\infty$ is the offer is open

5. Thus this is not an equilibrium. □

A.2 Proof of the equilibrium with open offers.

The proof is very intuitive. Students will never lose the opportunity to play E^B since the offers are opened. As such there is no risk to play W for students until the last period. However, $N_\alpha > C_A$ thus β students are fully aware they have zero chances to receive an offer from A and thus enroll in B immediately. As for α students they know they cannot be turned down by B as B will have enough spare capacity to host all the α who will not receive an offer from A because of capacity constraints. They can safely play W and hope a better opportunity shows up without taking any risk.

University A can confidently turn down all the β students, knowing no α student will bail out and enroll in B before receiving an answer from A first. □

A.3 Proof of theorem 2

The maximum combined utility for the universities A and B with an equilibrium involving open offers is $2c_A + 2(T^2 n_\alpha - c_A) + (c_B - ((T^2 n_\alpha - c_A))$. University A is filled to capacity with α students and university B enrolls the leftover α students then fills the spare capacity up with β students.

The rest of this proof will be by contradiction. Let us assume there exist an equilibrium with exploding offers that gives the maximum combined utility presented above.

- The equilibrium cannot involve mixed strategies, as this will cause ex-post mismatches with non-zero probabilities.
- Assume the existence of an equilibrium with exploding offers the gives this

level of utility. Such an equilibrium require that $(T^2 n_\alpha - c_A)$ α students play E_t^B while the others wait for an offer from A. One can increase V_A to an arbitrarily large number such that the expected payoff of playing W_t^* is larger than playing E_t^B. At least one α will deviate. This alpha will have a positive probability to be rejected. University B will be forced to recruit a β student, lowering its utility. $\square$

B Proofs for the model with exploding offers

B.1 Proof of theorem 3

The proof has three steps: First we derive the formula of the expected payoff for α students who decide to let an offer from B expire. Then we show that this payoff is decreasing as the matching procedure goes on. Lastly, we use a proof by contradiction to show that the value of T^* is unique.

B.1.1 Expected payoff of students α

Let $k \in \{1; \ldots; T\}$. At the equilibrium, the expected payoff of α students who plays W_{T-k}^* is equal to :

$$\mathbb{E}(\mathcal{U}(W_{T-k}^*)) = \sum_{i=1}^{k-1} \left(\frac{1}{k} V_A \right) + \frac{1}{k} \mathbb{P}(\Omega_A) V_A$$

Proof of step B.1.1 An α student will only play W_{T-k}^* is (s)he has received an offer from B at an earlier period (that we will call y). Because the student is still waiting for a response from A, the student does not know his/her state perfectly. The student can be in any state $(x; y)$ where $T - k = y + d < x \leq T$. Because of theorem 1, in every potential state except state $(T; y)$ the student is assured to receive an offer from A.

α students are uniformly distributed among all possible states thus the probability to be in a specific state $(x;y)$ where $T-k < x \leq T$ is $\frac{1}{k}$ $\square$.

B.1.2 The expected payoff of α students is decreasing

The expected payoff of playing W_t^* is decreasing in t.

Proof : Let $1 < j < k < T$. Both j and k are integers.

$$\begin{gathered}
\mathbb{E}(\mathcal{U}(W_{T-k}^*)) > \mathbb{E}(\mathcal{U}(W_{T-j}^*)) \\
\sum_{i=1}^{k-1}\left(\frac{1}{k}V_A\right) + \frac{1}{k}\mathbb{P}(\Omega_A)V_A > \sum_{i=1}^{j-1}\left(\frac{1}{j}V_A\right) + \frac{1}{j}\mathbb{P}(\Omega_A)V_A \\
\frac{k-1}{k} + \frac{1}{k}\mathbb{P}(\Omega_A) > \frac{j-1}{j} + \frac{1}{j}\mathbb{P}(\Omega_A) \\
1 - \frac{1}{k}(1-\mathbb{P}(\Omega_A)) > 1 - \frac{1}{j}(1-\mathbb{P}(\Omega_A)) \\
k > j \quad \square
\end{gathered}$$

At any equilibrium the probability of the event Ω_A is the same for all students. Thus at any equilibrium whenever a student of type α plays W_t^* then all the α students facing a choice between $E_{t'}^B$ and $W_{t'}^*$ for all $t' < t$ will play the later strategy. Following the same logic if E_t^B is played by an alpha student instead of W_t^*, then it will be played by all α for every future time period. $\square$

B.2 Proof of theorem 4

Proof by contradiction. Let us assume $T^* \neq T^{*\prime}$. Without loss of generality let $T^* < T^{*\prime}$. In equilibrium $\mathcal{S}'$, all α students who have to choose between $E_{T^*}^B$ and $W_{T^*}^*$ will play the latter as per Theorem 3. Their expected utility is greater than 1. In equilibrium $\mathcal{S}$, the α students who have to choose between $E_{T^*}^B$ and $W_{T^*}^*$ will either mix or only play $W_{T^*}^*$. By definition, in equilibrium $\mathcal{S}$,

$\mathcal{U}(E^B_{T^*}) = 1 = \mathbb{E}(\mathcal{U}(W^*_{T^*}))$.

However, because the strategy profile $\mathcal{S}'$ is an equilibrium the payoff of playing $W^*_{T^*}$ is strictly greater than 1 even when playing the strategy profile $\mathcal{S}$. Thus in $\mathcal{S}$ all α students can deviate and only play $W^*_{T^*}$ and force the other players to play equilibrium $\mathcal{S}'$. Thus the strategy profile $\mathcal{S}$ is not an equilibrium. □

B.3 Proof of theorem 5

The proof is in three steps :

- The first focuses on the behaviour of α students when students have a sufficiently high valuation of university A ($V_A \geq 2$).
- The second step extends the reasoning to the general case.
- The third step focuses on the behaviour of β students.

B.3.1 Step 1: The simple case when $V_A \geq 2$

Starting with $V_A \geq 2$ greatly simplifies the equilibrium structure. Elements of the proof for the simple case will be re-used for the general case. For the simple case I will first show that the critical period can only the last period or the penultimate one.

Lemma: *If $V_A \geq 2$, then $T^* = T$ or $t^* = T - 1$*

Proof: As per lemma B.1.1, the payoff of playing W^*_{T-2} is equal to

$$\mathbb{E}(\mathcal{U}(W^*_{T-2})) = \left(\frac{1}{2}V_A + \frac{1}{2}\mathbb{P}(\Omega_A)V_A\right)$$

Since $V_A \geq 2$ then $\frac{1}{2}V_A \geq 1$ thus $\mathbb{E}(\mathcal{U}(W^*_{T-2})) > \mathcal{U}(E^B_{T^*})$. Because of property B.1.2, the reasoning can be extended to every period $T-k$ where $k \geq 2$. Thus $\forall t \leq T-2: \ \mathbb{E}(\mathcal{U}(W^*_t)) > \mathcal{U}(E^B_t)$. As such the critical period can only be T or $T-1$. $\square$.

If the critical period is the last period then all α students will let their offers from B expire. The equilibrium becomes trivial (and unique). As per theorem 3 at most one group of student will randomise strategies. Moreover as per lemma B.3.1 if students randomise, they will only do it during period $T-1$. All students play the same randomization between W_{T^*} with probability p and $E^B_{T^*}$ with probability $(1-p)$. It goes without saying that if the equilibrium involves mixed strategies then when $p=0$, $\mathbb{E}(\mathcal{U}(W^*_{T^*})) > 1$ and if $p=1$, $\mathbb{E}(\mathcal{U}(W^*_{T^*})) < 1$.

Lemma: $\mathbb{E}(\mathcal{U}(W^*_{T^*}))$ is a strictly decreasing function of p

Proof: If randomising happens in period T-1, the number of students still applying to university A (named A_A) at the last period is a random variable follows a binomial distribution $\mathcal{B}(n_\alpha; p) + \kappa$ where κ is a constant that includes the number of students who let offers from B expire in previous periods as well as the students who have an offer from B that has not expired yet. Let Δ_{C_A} the spare capacity of university A at the beginning of the last period. It is a fixed number if $V_A \geq 2$. The probability of a student of type α to receive an offer from A at the last period given the number of remaining applicants is:

$$\mathbb{P}(\Omega_A|A_A) = \text{Min}\left\{\frac{\Delta_{C_A}}{A_A}; 1\right\}$$

$\mathbb{P}(\Omega_A|A_A)$ is a strictly decreasing function of A_A. Because students are mixing, by definition $n_\alpha + \kappa > \Delta_{C_A}$. If all the randomising students decide to play $W^*_{T^*}$ the probability of each getting an offer from A cannot be one.

The expected utility of playing $W^*_{T^*}$ is:

$$\mathbb{E}(\mathcal{U}(W^*_{T^*})|p) = V_A \sum_{k=1}^{n_\alpha} \binom{n_\alpha}{k} p^k (1-p)^{n_\alpha - k} \mathbb{P}(\Omega_A|\kappa + k)$$

Where $\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}$ is the probability distribution function of a random variable following the binomial distribution $\mathcal{B}(n_\alpha; p)$. By definition $\sum_{k=1}^{n_\alpha} \binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k} = 1$ and $\frac{\partial}{\partial p} \sum_{k=1}^{n_\alpha} \binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k} = 0$.

$\forall\ 0 < p < 1$ there exist a constant k^* such that :

$$\forall k \leq k^* \ ; \ \frac{\partial}{\partial p}\left[\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right] < 0$$
$$\forall k \geq k^* \ ; \ \frac{\partial}{\partial p}\left[\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right] > 0$$

It follows that:

$$\forall k \leq k^* \ ; \ \frac{\partial}{\partial p}\left[\sum_{k=1}^{k^*}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right] < 0$$
$$\frac{\partial}{\partial p}\left[\sum_{k=1}^{k^*}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\mathbb{P}(O^A_T|\kappa+k)\right] \leq \mathbb{P}(O^A_T|\kappa+k^*)\frac{\partial}{\partial p}\left[\sum_{k=1}^{k^*}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right] < 0$$

and

$$\forall k \geq k^* \ ; \ \frac{\partial}{\partial p}\left[\sum_{k=k^*}^{n_\alpha}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right] > 0$$
$$0 < \frac{\partial}{\partial p}\left[\sum_{k=k^*}^{n_\alpha}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\mathbb{P}(O^A_T|\kappa+k)\right] < \mathbb{P}(O^A_T|\kappa+k^*)\frac{\partial}{\partial p}\left[\sum_{k=k^*}^{n_\alpha}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right]$$

When combining the two parts of the sum of partial derivatives one obtains:

$$\frac{\partial}{\partial p}\left[\sum_{k=1}^{n_\alpha}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\mathbb{P}(O^A_T|\kappa+k)\right] < \mathbb{P}(O^A_T|\kappa+k)\frac{\partial}{\partial p}\left[\sum_{k=1}^{n_\alpha}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\right]$$
$$\frac{\partial}{\partial p}\left[\sum_{k=1}^{n_\alpha}\binom{n_\alpha}{k} p^k (1-p)^{n_\alpha-k}\mathbb{P}(O^A_T|\kappa+k)\right] < 0$$

Thus $\mathbb{E}(\mathcal{U}(W^*_{T^*}))$ is a strictly decreasing function of p. Thus the equation $\mathbb{E}(\mathcal{U}(W^*_{T^*})) = 1$ has a unique solution for p. As such there can be only one equilibrium in which a group of α students mix W^* with probability p and $E^B_{T^*}$ with probability $(1-p)$ if $V_A \geq 2$.

B.3.2 Step 2: The general case with any value of V_A

When relaxing the values of V_A the critical period T^* may be lower than $T-1$. If it is not, the step 1 proof applies. If T^* is lower than $T-1$ but the equilibrium is played in pure strategies then it is unique (trivial). If T^* is lower than $T-1$ and the equilibrium is played in mixed strategies then α students randomise between W^* with probability p and $E^B_{T^*}$ with probability $(1-p)$. In this case the expected utility of letting an offer from B expire at time T^* becomes:

$$\mathbb{E}(\mathcal{U}(W^*_{T^*})) = \sum_{i=1}^{T-T^*-1} \left(\frac{1}{T-T^*} V_A \right) + \frac{1}{T-T^*} \mathbb{P}(\Omega_A) V_A$$

Let Δ_{C_A} the spare capacity of university A at the beginning of the last period. It is not a fixed number anymore as there may be some students who will randomise in period T^* who will receive an offer from A before the last period. The probability of a student of type α to receive an offer from A at the last period given the number of remaining applicants and the spare capacity of A is:

$$\mathbb{P}(\Omega_A | \Delta_{C_A};\ A_A) = \text{Min} \left\{ \frac{\Delta_{C_A}}{A_A}; 1 \right\}$$

where

$$\Delta_{C_A} \sim \eta - \mathcal{B}((T-T^*-1)n_\alpha; p)$$
$$A_A \sim \mathcal{B}(n_\alpha; p) + \kappa$$

η is a constant and represents the leftover capacity of A at time T^* minus the number of α students who will be contacted by A before the last period and have either let their offer from B expire *before* T^* or will still have an offer from B that has not expired yet. κ is a constant that includes the number of students who will be contacted by A during the last period and who let offers from B expire in previous periods as well as the students who have an offer from B that has not expired yet.

Lemma: $\mathbb{P}(\Omega_A)(p)$ is a strictly decreasing function of p Let us write $\mathbb{P}(\Omega_A)$ as a function of two randomisation parameters p and ρ:

$$\begin{cases} \mathbb{P}(\Omega_A)(p;\rho) = \left[\sum_{k=1}^{n_\alpha} \binom{n_\alpha}{k} p^k (1-p)^{n_\alpha - k} \mathbb{E}(\mathbb{P}(O_T^A | \kappa + k)) \right] \\ \mathbb{E}(\mathbb{P}(O_T^A|\kappa+k)) = \left[\sum_{i=1}^{(T-T^*-1)n_\alpha} \binom{(T-T^*-1)n_\alpha}{i} \rho^i (1-\rho)^{(T-T^*-1)n_\alpha - i} \mathbb{P}(O_T^A | \eta - i;\ \kappa + k) \right] \end{cases}$$

Let $0 < p < p' < 1$ and $0 < \rho < \rho' < 1$ without loss of generality. From lemma B.3.1, $\mathbb{E}(\mathbb{P}(O_T^A|\kappa + k))$ is a weakly decreasing function of ρ. Moreover, $\mathbb{E}(\mathbb{P}(O_T^A|\kappa+k)) \geq \mathbb{E}(\mathbb{P}(O_T^A|\kappa+k'))$ while $\mathbb{P}(\Omega_A)(p;\rho)$ is a strictly decreasing function of p. It follows that:

$$\mathbb{P}(\Omega_A)(p;\rho) \geq \mathbb{P}(\Omega_A)(p;\rho') > \mathbb{P}(\Omega_A)(p';\rho')$$

To conclude all one has to do it to equate p and ρ to get that $\mathbb{P}(\Omega_A)(p)$ is a strictly decreasing function of p. Thus $\mathbb{E}(\mathcal{U}(W_{T^*}^*))$ is a strictly decreasing function of p and the equation $\mathbb{E}(\mathcal{U}(W_{T^*}^*)) = 1$ has a unique solution. and the equilibrium in mixed strategies is unique.

B.3.3 Step 3: β students never play W^*

Proving the uniqueness of strategy for β students is much more straightforward. If α students play using only pure strategies, then the probability of A being full is 1, thus βs immediately accept the offer from B as they have a zero probability of ever receiving an offer from A. The same reasoning applies for equilibrium where α play with mixed strategies but the probability of A being full is 1.

If there is a non-zero probability of A ending up not full (because C_A and V_A are too small), we need to check if β students have a profitable deviation by playing W^* instead of E_t^B. The expected utility of a β student of playing W^*_{T-1} (the penultimate period).

$$\mathbb{E}(\mathcal{U}(W^*_{T-1})) = V_A \sum_{k=1}^{n_\alpha} \binom{n_\alpha}{k} p^k (1-p)^{n_\alpha - k} \mathbb{P}(O_T^A | \kappa + k \ ; \ \beta)$$

Notice that A only send an offer to a β if there is some spare capacity left after going through all the α applicants. The maximum spare capacity available for β students in the last period is n_α which occurs when all the α students randomising between W^*_{T-1} and E^B_{T-1} pick E^B_{T-1}[6]. Moreover, the β student who played W^*_{T-1} is in competition with all the β students in state $(T;T)$. As such : $\forall k \in \{0; \ \dots \ n_\alpha\}$ $\mathbb{P}(O_T^A|\kappa + k \ ; \ \beta) < (\mathbb{P}(O_T^A|\kappa + k)$. Thus playing W^*_{T-1} as a β student at time $T-1$ will yield a strictly lower payoff than playing W^*_{T-1} as an α student i.e $\mathbb{P}(O_T^A|\beta) < \mathbb{P}(\Omega_A)$.

Now let us extend to a more general case of playing W_t^* $\forall t$

$$\mathbb{E}(\mathcal{U}(W_t^*)) = \sum_{i=1}^{T-t-1} \left(\frac{1}{T-t} V_A \mathbb{P}(O^A_{T-t+i}|\beta) \right) + \frac{1}{T-t} \mathbb{P}(O_T^A|\beta) V_A$$

[6]In the event that α students start randomising before time $T-1$ any spare capacity would be immediately filled with β students in earlier periods

Notice that if $T-t+i<T^*$ then $\mathbb{P}(O^A_{T-t+i}|\beta)=0$. Before the critical period T^* all the α students play W^* and A knows it. As such there is no reason to send an offer to a β student. The β students can only hope to receive an offer from A once the critical period T^* is reached, and if a higher than expected number of αs randomise in favour of E^B_t. Thus $\mathbb{P}(O^A_t|\beta)<1$. Knowing that $\mathbb{P}(O^A_T|\beta)<\mathbb{P}(\Omega_A)$ we can conclude that playing $W^*_t\ \forall t\geq T^*$ as a β student at time $T-1$ will yield a strictly lower payoff than playing $W^*_t\ \forall t\geq T^*$ as an α student.

To conclude, in en equilibrium where α students play mixed strategies in any period where α students mix between E^B_t and W^*_t, β students prefer to play E^B_t. In any period where all αs play W^*_t, the expected utility of playing W^*_t for βs is even lower than when αs start mixing. As such β students never play $W^*_t\ \forall t$.

□

B.4 Proof of theorem 7

$\forall t>T^*$ α student plays W^*_t and thus will never end up with university A. The number n_{miss} of α students who will never apply to A is an algebraic series:

$$\begin{aligned} &if:\ T^*<T\\ n_{miss}&=\sum_{i=1}^{T-T^*}(i-1)n_\alpha\\ n_{miss}&=\frac{T-T^*}{2}(T-T^*-1)n_\alpha \end{aligned}$$

Because in any equilibrium the probability of A being fully filled with α students cannot be zero, this implies that the number of missing A students cannot bring the number of A applicants below C_A. Since $(T-1)Tn_\alpha<C_A<T^2n_\alpha$, n_{miss} is bounded from above by Tn_α. Notice that if $T=6$ and $T^*=2$, $n_{miss}=6n_\alpha$. So

the game with 6 periods does not fully unravel. Increasing the number of period T by one while keeping $T^* = 2$ violates the inequality as well. By recurrence, the partial unraveling holds for all values of T. □

B.5 Proof of theorem 8

This is a generalization of theorem 7 and parts of its proof (B.4). Instead of using the upper bound for n_{miss}, the true difference ΔC_A between the number of α and the capacity of A is used.

$\forall t > T^*$ α student plays W_t^* and thus will never end up with university A. The number n_{miss} of α students who will never apply to A is an algebraic series:

$$\begin{aligned} if:\ T^* &< T \\ n_{miss} &= \sum_{i=1}^{T-T^*} (i-1) n_\alpha \\ n_{miss} &= \frac{T-T^*}{2}(T-T^*-1) n_\alpha \end{aligned}$$

Because in any equilibrium the probability of A being fully filled with α students cannot be zero, this implies that the number of missing A students cannot bring the number of A applicants below C_A. Thus:

$$\begin{aligned} n_{miss} &\leq \Delta C_A \\ \frac{T-T^*}{2}(T-T^*-1) n_\alpha &\leq \Delta C_A \\ (T-T^*)^2 - (T-T^*) - 2\Delta C_A / n_\alpha &\leq 0 \\ \iff \frac{1-\sqrt{1+8\Delta C_A/n_\alpha}}{2} &\leq (T-T^*) \leq \frac{1+\sqrt{1+8\Delta C_A/n_\alpha}}{2} \end{aligned}$$

Since $0 < T^* \leq T$ we can integrate these constraints back into the inequality above:

$$(T - T^*) \leq \text{Min}\left\{\frac{1 + \sqrt{1 + 8\Delta C_A/n_\alpha}}{2}; T\right\}$$

$\square$

C Proofs for the players' welfare

C.1 Proof of theorem 10

The proof is by construction. Let $\mathcal{S}$ be the equilibrium of a game with parameters T, c_A, c_B, V_A, n_α, n_β and an exploding offer of duration $d = 0$. Let T^* be the critical period associated with equilibrium $\mathcal{S}$.

The notations κ and η from the proof of theorem 5 will be reused in this proof. In equilibrium $\mathcal{S}$ the number η of α students who will enroll in B before the last period is $\sum_{i=1}^{T-T^*}(i-1)n_\alpha$. In equilibrium $\mathcal{S}$ there are κ students of type α who will be competing for sure for a seat in A in the last period. These are αs in state $(T;T)$, all the ones in states $(T;x)$ $\forall x < T^*$ and either all the α from state $(T;T^*)$ (equilibrium is played in pure strategies) or a random number of them. If the number of α is random this implies a randomisation with probability p such that $\mathcal{U}(E^B_{T-1}) = 1 = \mathbb{E}(W^*_{T-1})$ where $\mathbb{E}(W^*_{T-1})$ is a function of κ and η.

Let $\mathcal{S}'$ be an equilibrium candidate for the game with parameters T, c_A, c_B, V_A, n_α, n_β and an exploding offer of duration $d' \geq 1$. If $T'^* = T^*$, then there are η' students of type α who will enroll in B before the last period. $\eta' = \sum_{i=1+d}^{T-T^*-d}(i-d-1)n_\alpha$. Notice that $\eta = \eta'$.

Following the same logic, there are κ' students of type α who will be competing for sure for a seat in A in the last period. These are all the α in state $(T; x)$ $\forall x \geq T-d$, all the ones in states $(T; y)$ $\forall y < T^* - d$ and either all the α from state $(T; T^* - d)$ (equilibrium is played in pure strategies) or a random number of them.

This means that in the equilibrium candidate $\mathcal{S}'$ in period $T'^* = T^*$, the α students face the exact same randomization problem than in $\mathcal{S}$. Thus they behave identically. $\square$

D Proofs of the extended model

D.1 Proof of theorem 11

It is a straightforward reuse of the proof of theorem 3. Since $\mathbb{P}(\Omega_A)$ is the same for all α_i the behaviour of each individual α_i subgroup of students is similarly structured:

- Let the offer from B expire before a critical period
- Enroll in B instead of waiting for an answer from A after the critical period.
- Either let the offer expire or randomise during the critical period.

Because the utility of each subgroup of α_i is different the critical periods may be different. However, because of lemma B.1.2, the critical periods will be ordered. $\square$

D.2 Proof of the corollary

Let there be two α students named i and j. Let us assume without loss of generality that student i played E_t^B while student j played W_{t+k}^* with $k > 0$. Either:

1. Students i and j have the same sub-type
2. Student i has a higher sub-type than student j
3. Student j has a higher sub-type than student i

If both students have the same subtype then this is a contradiction as once at least one student of a give subtype has played E^B then all students of the same subtype in a later period must play E^B as well.

If student i has a higher subtype than student j then at every point in the game $\mathbb{E}(\mathcal{U}_{\alpha_i}(W_t^*)) > \mathbb{E}(\mathcal{U}_{\alpha_j}(W_t^*))$. Thus if the student i plays E_t^B then all students that have the same subtype as student j will play E^B at time t. As stated above, if at least one student of a given sub-type plays E^B all the students of the same sub-type must play E^B in every subsequent period. We reach a contradiction.

The only option left is student j has a higher sub-type than student i. Which is possible. You simply need to have a V_{α_i} sufficiently high such that student i prefers to enrol while student j prefers to let the offer explode. □

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